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Solving Geometric Proportional Analogies with the Analogy Model HDTP

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Abstract

Intelligence tests often use geometric proportional analogies to examine the intelligence quotient of humans. Completing such a series of geometric figures can be a cognitively demanding task, because the solution requires a suitable conceptualization of the geometric figures. Furthermore, there might exist several, equally correct solutions depending on the conceptualization. In this paper, we demonstrate how the symbolic analogy model HDTP solves such analogies: HDTP uses Gestalt principles and qualitative spatial reasoning to compute a psychologically preferred representation of the figures, adapts these representations if necessary, and constructs a solution based on an analogical mapping.

Keywords: analogy; geometric proportional analogies; re-representation

Introduction and Motivation

Analogical reasoning is considered to be fundamental in human cognition and human problem solving (Gust et al., 2008; Hofstadter, 2001). Geometric proportional analogies (GPA) are a special form of analogous problems: A GPA consists of a serious of four geometric figures A, B, C, and D, where the same relation holds between figure A and B as between figure C and D. Such analogies are commonly used in intelligence tests to measure the intelligence quotient. Figure 1 shows an example for a GPA where the figure D is missing. One has to establish an analogous mapping between figures A and C and analyze the relation between A and B, which is afterwards transferred and applied to figure C to construct the missing figure D.

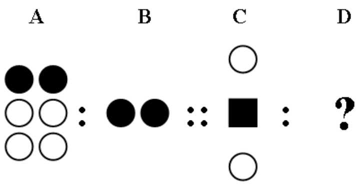

Figure 1: Example for a geometric proportional analogy (GPA) with several possible solutions for figure D.

The difficulty in solving this analogy lies in its ambiguity. It is possible to construct several, equally correct solutions depending on the conceptualization of the geometric figures. We investigated experimentally different solution strategies (Schwering et al., 2008). Figure 2 illustrates three preferred solutions for the running example: Solution 1 can be explained by grouping the black elements in figure A and C. Figure B repeats the black elements of figure A in the middle. Applying the analogous strategy leads to figure D: The black elements of figure C are repeated in the middle. The second solution can be explained by grouping the top elements in figure A (respectively C) and construct B (respectively D) by moving the top elements one unit down along the y-axis. Solution 3 can be explained by grouping the middle elements and repeating them with flipped colors.

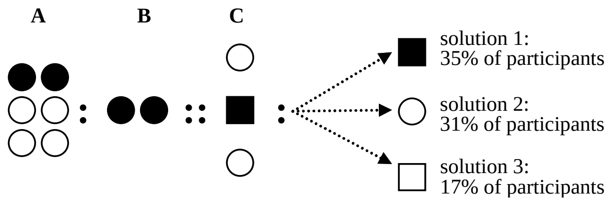

Figure 2: Human subject tests have revealed several different solutions for figure D, of which three preferred solutions are shown in the picture.

In this paper, we extend earlier work (Schwering et al., 2007) and present a computational model to analyze and detect different plausible solutions to GPAs. We show how the analogy model *Heuristic-Driven Theory Projection* (HDTP) uses Gestalt principles and qualitative spatial reasoning to detect cognitively preferred representations of geometric figures, adapts these representations if necessary, and constructs a solution based on an analogical mapping.

The paper is structured as follows: after this introduction, we give an overview of related work on analogy models for GPAs. In section three we explain the basics of HDTP and show how geometric figures are formally represented in HDTP. In section four we give details on the analogy-making process illustrated with an example. We conclude the paper with a discussion and directions for future work.

Related Work

Proportional analogies were studied in various domains such as the natural-language domain (Indurkhya, 1989, 1992), the string domain (Hofstadter & Mitchell, 1995), analogical spatial arrangement at a table top scale (French, 2002), and in the domain of geometric figures.

In (1962), *Evans* developed a heuristic program to solve GPAs. Before the actual mapping process, the program computes meaningful components consisting of several line segments in each figure. Evan's analogy machine determines the relation between A-B, computed a mapping between A-C based on rotation, scaling, or mirroring, and selected an appropriate solution from a list of possible solutions. In contrast to our approach, the representation and the mapping phase are sequentially separated from each other. While we use structural criteria, Evans uses mathematical similarity to detect a suitable mapping between figure A and C. *O'Hara & Indurkhya* (1992; 1993) worked on an algebraic analogy model which is able to adapt the representation of line drawing figures during the analogy-making process. Dastani et al. developed a formal language for this algebraic model to describe elements in geometric figures and compute automatically a structural, Gestalt-based representation (Dastani & Scha, 2003). This approach accounts also for context effects, i.e. figure C has an effect on the conceptualization of figure A (Dastani & Indurkhya, 2001). Both ideas strongly influence our work. We reuse many ideas developed for this algebraic model and apply them to our logic-based framework.

Mullally, O'Donoghue et al. (2006; 2005) investigated GPAs in the context of maps. They used structural commonalities to detect similar configurations in maps and to automatically classify geographic features. Due to the limitation to maps, they did not support the complex spatial analysis required for our GPAs. *Tomai, Forbus* et al. (2004) extended the Structure Mapping Engine to compute spatial relations between geometric figures and solve GPAs. However, this approach does not include re-representation and requires a set of possible solutions to select from. *Davies and Goel* investigate the role of visual analogies in problem solving (Davies et al., 2008). In this approach, they don't focus on GPAs and their special structure.

HDTP as Computational Model for GPA

We describe the basics of HDTP shortly and explain how HDTP is used as a computational model for GPAs. For further details we refer to (Schwering et al., 2009) concerning the syntactic principles of HDTP and (Krumnack et al., 2008) regarding re-representation.

The Analogy Model HDTP

HDTP is a symbolic analogy model where the source and the target domain are formalized as first-order logic theories. HDTP distinguishes between domain knowledge (facts and laws holding for the source or the target domain) and background knowledge, which is true across domains.

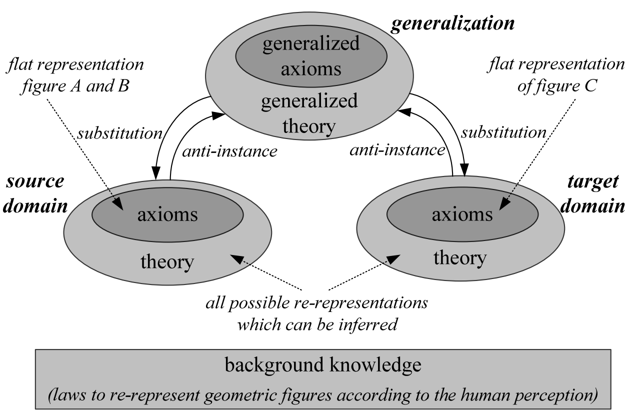

Figure 3: HDTP as computational model for GPA.

Figure 3 shows a rough overview of the architecture: Knowledge about the geometric figures is captured by domain knowledge, while general principles of perception are captured in the background knowledge. An analogy is established by aligning elements of the source with analogous elements of the target domain. In the mapping phase, the source and the target are compared for structural commonalities. HDTP uses anti-unification to identify common patterns in the source and target domain. Anti-Unification is the process of comparing two formulae and identifying the most specific generalization subsuming both formulae.

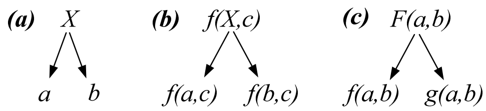

Figure 4: Anti-unification compares two formulae and creates the least general generalization.

We use anti-unification to compare the source theory with the target theory and construct a common, general theory which subsumes possibly many common structures of the source and the target domain. Figure 4 gives examples for anti-unification: formulae are generalized to an anti-instance where different constants or function symbols are replaced by a variable. In (a) and (b), first-order anti-unification is sufficient. The formulae in (c) differ in the function symbols: While first-order anti-unification fails to detect structural commonalities, higher-order anti-unification generalizes function symbols to a variable and retains the structural commonality.

The generalized theory with its substitutions specifies the analogical relation between source and target. Additional information about the source domain (in the case of GPAs, this information is how to construct figure B from A) is transferred to the target domain and applied to figure C to construct figure D.

Formalization of Unstructured Geometric Figures

HDTP starts with an unstructured description of the input geometric figures: all primitive objects are captured and described by their properties.

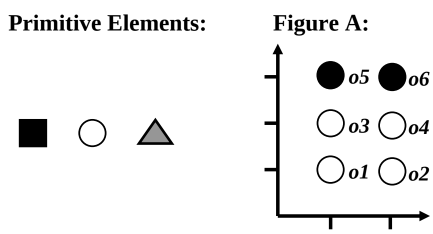

Figure 5: Examples for some primitive elements.

Figure 5 shows primitive elements as used in our GPAs. These primitive elements are described by their shape (circle, square, triangle), their color (black, white, grey), and their position in a grid. It is possible to extend the description by other properties such as size or rotation angle. Table 1 shows the formalization of the left bottom object in figure A of our running GPA example (Figure 1). The description of the complete analogy, i.e. figure A, B, and C, is a list of all primitive objects identified uniquely and described by listing their properties.

Table 1: Formal description of the left bottom object in figure A of the GPA shown in Figure 1.

```
object(o1,[
  shape(circle),
  position(p(1,1)),
  color(white)
]).
```

Formalization of Structural Patters

When the human sensory system observes a geometric figure, it transforms the unstructured information into a structured representation of a pattern or a set of objects. In order to solve a GPA, the observer has to understand the visual information, make sense of the overall pattern she sees. She identifies common structures between figure A and C to establish a mapping between those elements playing the same role in A and C.

Human perception tends to follow a set of Gestalt principles of organization (Wertheimer, 1912). A model for computational cognition of GPAs requires the ability to detect the same structural patterns as humans do. If we restrict the pattern detection strategies to the above mentioned perceptual properties (shape, position, color), the amount of possible structural patterns has a manageable complexity. HDTP captures general rules for pattern detection and human perception in the *background knowledge*, which is general knowledge and not limited to a particular GPA.

Gestalt Perception. Our experiment (Schwering et al., 2008) has shown that many solutions can be explained by Gestalt principles. We identified several dominant pattern detection strategies:

- Grouping based on similarity. People tend to group objects which share one or several common properties, e.g. grouping of black elements as in the GPA example.
- Repeated groups of objects are perceived as iterations of one group, e.g. the three left circles in figure A in the running example are perceived as one group repeated two times.
- Grouping based on proximity. People tend to organize spatially close elements or elements on one (typically horizontal or vertical) axis into units.
- Grouping based on spatial commonalities. Objects with a special position are grouped together, e.g. in the running example people tend to group the top or the middle elements. In this case, spatial nearness as well as common spatial attributes play a role.

Several grouping strategies leading to the same result - in the running example grouping black and grouping top elements result in the same set of objects - usually increase the degree of preference.

Table 2: Grouping based on similarity (common color).

```
group(g1,List):-
filter(figA,color,black,List).
List = [o5,o6].

group(g2,List):-
filter(figA,color,¬black,List).
List = [o1,o2,o3,o4]
```

Table 2 shows a formalization of grouping strategies as they are found in the HDTP background knowledge: First, a group of elements filtered with respect to the property black and then the complement group of non-black objects is formed (we could use white color as equally good grouping criteria).

Qualitative Spatial Relations. Qualitative spatial relations play an important role for the analysis of geometric analogies (Tomai et al., 2004). The absolute position of elements as well as the position relative to other elements in the geometric figure are very important for some pattern detection strategies.

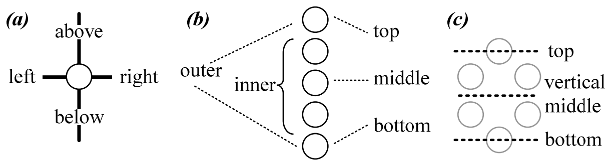

Figure 6: (a) shows the applied spatial calculus, (b) illustrates elements being distinguished by their relative position, and (c) illustrates points with a particular position.

For our analysis we apply a single cross calculus distinguishing the relations of above, below, right and left as shown in Figure 6a. This allows us to distinguish elements by their relative position (Figure 6b), which is important for groupings based on position. Transformations between figure A and B often operate on coordinates of points with a relative position as shown in Figure 6c. Depending on the geometric figure, outstanding points or elements are computed either on a vertical axis (as shown in the figure), a horizontal axis, or in 2 dimensions. The middle element does not necessarily have the same coordinates as the middle point.

Table 3: Gestalt grouping based on position.

```
group(gTop,List):-
filter(figA,position,top,List).
List = [o5,o6]
```

Table 3 shows a formal description of a grouping of elements based on the top position. All other groupings are formalized analogously.

Formalization of Structural Patters in Figure A and C. The most preferred solution in our example is a grouping of black elements and moving these elements to the (vertical) middle. Table 2 shows the formalization of figure A in a structured way: Figure A consists of a group `g1` with black elements and a group `g2` with white elements. `g1` is used in the transformation to construct figure B. At this stage, figure C is formalized as a list of primitive elements `o9`, `o10`, and `o11`. Structural patterns in C are only detected in the re-representation process described later.

Formalization of Transformations from A to B

After a description of structural patterns in A and C, one has to analyze the relation between geometric figure A and B, to transfer the AB-relation and construct D. The AB-relation is represented by describing the transformations applied to either the complete figure A or only a subset of its elements. Based on the findings in our experiment, we implemented the following set of transformations:

- The move transformation changes the position of elements: `moveTo` moves elements to an absolute point determined by coordinates or to a relative point such as top, middle or bottom of the figure and `moveBy` moves elements by a certain vector.
- The `rotate` transformation changes the orientation of an element or a group of elements.
- The `reflect` transformation mirrors an element or a group of elements along an axis.
- The `change attribute` transformation changes the value of an attribute such as color. There is also the possibility to determine two properties being switched: the transformation `colorFlip` flips the colors black and white. This was often used in our experiment.
- The `add` transformation adds new elements to figure B.

In this case, all transformations refer to changes in the position and changes in the attributes, however this set is extendable to other transformations such as topological. Table 4 shows the transformation between figure A and B in our running GPA example. It is applied to a subgroup `g1` in figure A, which contains only the black elements.

Table 4: Transformation between figure A and B.

```
group(figATOfigB,List):-
transform(g1,[moveTo(middle)],List).
List = [o7,o8]
```

The Analogy-Making Process

In the previous chapter, we explained how to represent geometric figures, structural patterns, and transformations between geometric figures. In the following, we discuss the analogy-making process with its different phases: HDTP first determines structural patterns in figure A, computes a structured representation to establish a mapping between figure A and C, and detects the solution of the analogy by transferring the relation between AB to C.

The determination of structural patterns in figure A, the re-representation of figure C and the mapping are not sequentially separated processes, but interact with each other. It may be, that only in the mapping it is recognized, that the current structure does not lead to any good analogy. In this case, HDTP has to search for a new representation.

Determining Structural Patterns

Determining the preferred structured representation of figure A is a highly complex problem determined by the observer's perception. Attribute similarity, spatial nearness and an iteration of identical groups have been the major grouping criteria in our experimental data (compare heuristics described above). In some analogies, all elements of a figure were considered as a single group. That was often the case when figures consisted of a small amount of elements. Furthermore, the perception of figure A is determined by its context, i.e. figure C as you can see in our running example: In the third preferred solution, the middle elements of figure A are grouped, because the middle element in C differs from the other elements in C. Therefore our algorithm of computing groups is also based on the grouping criteria of figure C.

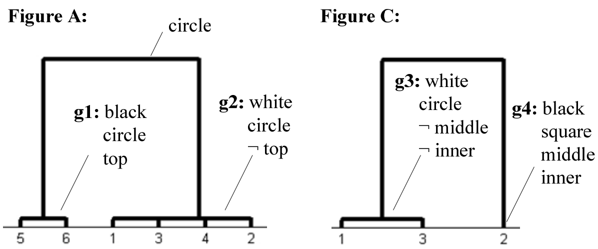

Figure 7: Groupings in figure A and C are determined by clustering the objects according to their similarity.

The groupings as shown in Figure 7 are computed by hierarchically clustering the objects based on their attribute or spatial similarity. Not every grouping is equally preferred[1]. We identify the following criteria to evaluate the degree of preference:

- Grouping is preferred, if the ratio of distinguishing to non-distinguishing criteria is high.
- Grouping is preferred, if the same grouping can be established via different criteria (e.g. based on color and based on position as in our running example).
- Grouping is preferred, if the same grouping is supported in the context (i.e. is preferred in figure A and figure C).

The list of preferred groupings serves as input for the re-representation and mapping phase.

Re-Representation and Mapping

Given one structured representation of figure A, the re-representation process aims at finding a structured

[1] There were developed several complexity measures (Dastani & Scha, 2003; Van der Helm et al., 1992), but they originated rather from computational than from psychological ideas.

representation of figure C that a good match can be established between both figures. HDTP uses rules in the background knowledge[2] to compute alternative representations for a geometric figure. The re-representation benefits from HDTP's logical basis: since the domain descriptions are understood as logical theories, different representations can be inferred from the basic description. It leads to a syntactically different, but semantically equivalent description of the geometric figures.

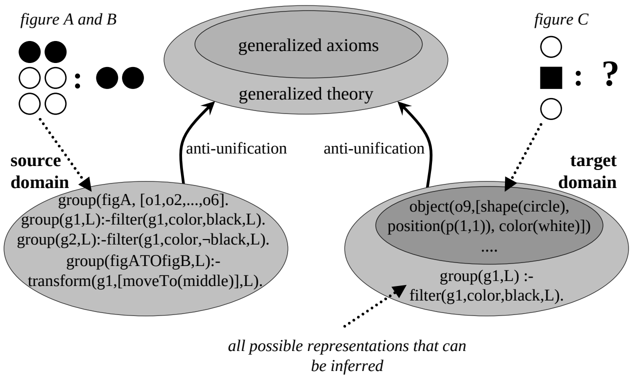

Figure 8: Re-Representation of figure C in HDTP.

The algorithm sketched in Table 5 describes the re-representation and mapping of geometric figures in GPA. It is based on the general HDTP algorithm described in (Krumnack et al., 2008), but is adapted and optimized for GPAs. In the first step, the algorithm selects the preferred groupings as computed in the previous phase determining structural patterns. As Figure 7 shows, in our running example this is a group consisting of objects `o5` and `o6` called `g1` and the complement group `g2` consisting of object `o1`, `o2`, `o3`, `o4`. The first group shares the properties black color, circular shape and the top position. The second group shares the properties white color, circular shape and the "not top" position. Step A3 computes the preference degree of these groups based on the grouping criteria (in this case a high preference degree because the groups share many properties). In step A4, a suitable transformation is computed (in this case, `moveTo(middle)`) and evaluated. Step A6 tries to regroup the objects in figure C. In the example, it succeeds only if the grouping criteria of group `g1` are reduced to either color black or position top. A new group (called `g3`) is created in figure C. In step A7, HDTP establishes a mapping between figure A and figure C via anti-unifying groupings from figure A with groupings from figure C. Higher-order anti-instances and a mapping of different structures with different complexity (e.g. a group mapping on a single element) are generally avoided. Step A9 computes the transfer and stores the solution with its preference degree as well as the generalizations with the substitutions. Particularly in GPAs, all objects in figure A and C should be included in the match. Mappings where objects do not have a counterpart usually lead to non-preferred analogies.

[2] In principle, rules in the domain knowledge can be used as well, however, the descriptions of GPAs do not contain rules.

Table 5: Algorithm for determining structural patterns.

Input: Unstructured representation of figure A, B, and C, list of preferred groupings in A and C.

A1 (select starting group): Select preferred and not yet tested group from list, select all common properties as grouping criteria. If no such group is left, terminate.

A2 (grouping): Form groups according to grouping criteria.

A3 (preference degree): Evaluate each group based on number of properties common to group and distinct to complement group(s) or based on number of common complementary properties of complement group(s).

A4 (transformation): Compute all alternative minimal transformations from figure A to B based on groups. If not all elements in A or B are covered by the transformation, go to step A1.

A5 (preference degree): Evaluate complexity based on number of transformations.

A6 (re-represent): Prove groupings with same criteria on target side. If resulting group is empty, reduce or change grouping criteria without changing the group and go to A2.

A7 (try AU): Find (non-deterministically) the best matching clauses from figure C according to the heuristic

- same operation and same arguments
- else: same operation and different arguments

Mark anti-unified objects as directly covered and objects used in the re-representation process (step A6) as (indirectly) covered.

A8 (preference degree): Evaluate anti-unification based on quality of structural match (i.e. equally complex substitutions on both sides). Furthermore, the mapping should cover all elements in figure A and C.

A9 (transfer): Apply AB-transformation to aligned groups in figure C. If result is non-empty, save solution with degree of preference. Go to step A1.

Output: Possible solutions with degree of preference and structured figure A and C and AB-transformation.

The following mappings are established in the example: `group(g1,L):-filter(figA,color,black,L)` from the source domain and the re-represented group `group(g3,L):-filter(figC,color,black,L)` from the target domain are aligned and generalized to `group(`**`Z`**`,L):-filter(`**`X`**`,color,black,L)` with the substitutions `X`→`figA/fiC` and `Z`→`g1/g3`.

Transfer of Relation AB to Solve GPA

Once there exists a mapping between the structured figures A and C, the AB-transformation must be transferred and applied to figure C (step A9). All existing mappings are re-used: `g1` is replaced by `g3` and the transformation `transform(g3,[moveTo(middle)],List)` is then applied to the target domain. The resulting figure is a black square with the position (1,2) in the middle of the figure. It is one possible solution for figure D.

Conclusions and Future Work

In this paper, we show how the analogy-making framework HDTP can be used to solve geometric proportional analogy problems. GPAs can be considered as special cases of analogies, because of their particular structure A:B::C:D. The analogy is solved by analyzing structural commonalities between figure A and C, transferring the AB-transformation, and applying it to C to construct the missing figure D.

We have shown in detail how to describe geometric figures formally with a limited set of properties, how to detect structural patterns in geometric figures using Gestalt principles and qualitative reasoning mechanisms, and how to detect transformation between figures. Afterwards, we have shown how our framework is used during the analogy-making process, i.e. how figures A and C are re-represented until an analogous structure is found, how a mapping is established and how the transfer works. The algorithm and the heuristics follow the same basic idea as the general version of HDTP. Only small modifications have been necessary to adapt HDTP to solve GPAs. Future work will concentrate on the refinement of suitable heuristics to determine the degree of psychological preference.

References

M. Dastani, & B. Indurkhya. (2001). *Modeling context effect in perceptual domains.* Paper presented at the 3rd International and Interdisciplinary Conference on Modeling and Using Context.

M. Dastani, & R. Scha. (2003). Languages for Gestalts of line patterns. *Journal of Mathematical Psychology, 47*, 429-449.

C. Davies, A. K. Goel, & P. W. Yaner. (2008). Proteus: Visual analogy in problem solving. *Knowledge-Based Systems*.

T. G. Evans. (1962). *A heuristic program to solve geometric analogy problems* (No. Technical Report: AIM-46). Cambridge, MA, USA: Massachusetts Institute of Technology

R. M. French. (2002). The computational model of analogy-making. *Trends in Cognitive Sciences, 6*(5), 200-205.

H. Gust, U. Krumnack, K.-U. Kühnberger, & A. Schwering. (2008). Analogical reasoning: A core of cognition. *Zeitschrift für Künstliche Intelligenz (KI), Themenheft KI und Kognition*(1), 8-12.

D. R. Hofstadter. (2001). Analogy as the Core of Cognition. In D. Genter, K. J. Holyoak & B. Kokinov (Eds.), *The Analogical Mind: Perspectives from Cognitive Science* (pp. 499-538). Cambridge MA: The MIT Press/Bradford Book.

D. R. Hofstadter, & J. C. Mitchell. (1995). The copycat project: A model of mental fluidity and analogy-making. In D. R. Hofstadter & F. A. R. group (Eds.), *Fluid Concepts and Creative Analogies* (pp. 205-267): Basic Books.

B. Indurkhya. (1989). *Modes of analogy.* Paper presented at the International Workshop on Analogical and Inductive Inference (AII).

B. Indurkhya. (1992). *Metaphor and cognition*. Dodrecht: Kluver.

U. Krumnack, H. Gust, K.-U. Kühnberger, & A. Schwering. (2008). *Re-representation in a logic-based model for analogy making.* Paper presented at the 21st Australasian Artificial Intelligence Conference (AI-08), New Zealand.

E.-C. Mullally, & D. P. Donoghue. (2006). Spatial inference with geometric proportional analogies. *Artificial Intelligence Review, 26*(1-2), 129-140.

E.-C. Mullally, D. O'Donoghue, A. J. Bohan, & M. T. Keane. (2005). *Geometric proportional analogies in topographic maps: Theory and application.* Paper presented at the 25th SGAI International Conference on Innovative Techniques and Applications of Artificial Intelligence, Cambridge, UK.

S. O'Hara. (1992). *A model of the "redescription" process in the context of geometric proportional analogy problems.* Paper presented at the International Workshop on Analogical and Inductive Inference.

S. O'Hara, & B. Indurkhya. (1993). *Incorporating (re)-interpretation in case-based reasoning.* Paper presented at the First European Workshop on Topics in Case-Based Reasoning.

A. Schwering, U. Krumnack, K.-U. Kühnberger, & H. Gust. (2007). *Using Gestalt principles to compute analogies of geometric figures.* Paper presented at the 29th Annual Conference of the Cognitive Science Society (CogSci07), Austin: TX.

A. Schwering, U. Krumnack, K.-U. Kühnberger, & H. Gust. (2008). *Investigating Experimentally Problem Solving Strategies in Geometric Proportional Analogies* (Report). Osnabrueck, Germany: University of Osnabrueck http://www.cogsci.uos.de/~ai/analogy_project.html.

A. Schwering, U. Krumnack, K.-U. Kühnberger, & H. Gust. (2009). Syntactic principles of Heuristic-Driven Theory Projection. *Special Issue on Analogies - Integrating Cognitive Abilities. In: Journal of Cognitive Systems Research, 10*(3).

E. Tomai, K. D. Forbus, & J. Usher. (2004). *Qualitative spatial reasoning for geometric analogies.* Paper presented at the 18th International Qualitative Reasoning Workshop, Evanston, Illinois.

P. Van der Helm, R. Van Lier, & E. Leeuwenberg. (1992). Serial pattern complexity: Irregularity and hierarchy. *Perception, 21*, 517-544.

M. Wertheimer. (1912). Experimentelle Studien über das Sehen von Bewegung. *Zeitschrift für Psychologie, 61*, 161-265.