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Mathematical Practices and Arts Integration in an Activity-Based Projective Geometry Course

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Mathematical Practices and Arts Integration in an Activity-Based Projective Geometry Course

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in

Mathematics and Science Education

by

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The dissertation of Jessica Brooke Ernest is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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2016

DEDICATION

To my parents, Judy and Jerry,
for instilling in me a love of knowledge
and for providing me with endless love and support.

To my husband, Dennis,
for the encouragement, advice, love, and laughter...
and for putting up with far too many years of me being away from home.

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ABSTRACT OF THE DISSERTATION

Mathematical Practices and Arts Integration in an Activity-Based
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by

Jessica Brooke Ernest

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2016
San Diego State University, 2016

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It is a general assumption that the mathematical activity of students in school should, at least to some degree, parallel the practices of professional mathematicians (Brown, Collins, Duguid, 1989; Moschkovich, 2013). This assumption is reflected in the Common Core State Standards (CCSSI, 2010) and National Council of Teachers of Mathematics (NCTM, 2000) standards documents. However, the practices included

in these standards documents, while developed to reflect the practices of professional mathematicians, may be idealized versions of what mathematicians actually do (Moschkovich, 2013). This might lead us to question then: "What is it that mathematicians do, and what practices are not being represented in the standards documents?"

In general, the creative work of mathematicians is absent from the standards and, in turn, from school mathematics curricula, much to the dismay of some mathematicians and researchers (Lockhart, 2009; Rogers, 1999). As a result, creativity is not typically being fostered in mathematics students. As a response to this lack of focus on fostering creativity (in each of the science, technology, engineering, and mathematics disciplines – the STEM disciplines), a movement to integrate the arts emerged. This movement, called the STEAM movement – introducing the letter A into the acronym STEM to signify incorporating the arts – has been gaining momentum, yet limited research has been carried out on the efficacy of integrating the arts into mathematics courses.

My experiences as the co-instructor for an activity-based course focused on projective geometry led me to consider the course as a setting for investigating both mathematical practices and arts integration. In this work, I explored the mathematical practices in which students engaged while working to develop an understanding of projective geometry through group activities. Furthermore, I explored the way in which students' learning experiences were enriched through artistic engagement in the course. I discuss *mathematical play* and *acts of imagination* – two mathematical practices in which students engaged, and which emerged from a grounded theory approach to analysis of the classroom data. In addition, I discuss particular ways in which artistic engagement, including creating two mathematically inspired artistic pieces, enriched students' learning experiences in the course. The six themes I address are artistic engagement (a) fostering mathematical play, (b) giving students the opportunity to make sense of pop-up topics, (c) providing students with the opportunity to develop coordination of mathematical tools, (d) allowing students to weave their personal experiences with mathematics, (e) contributing to students' notions of the connections between mathematics and art, and (f) changing students' relationships with art.

Chapter 1

Motivation and Significance

Critical thinking, problem solving, communication, and creativity are frequently cited by employers as attributes necessary for developing a competitive workforce that can succeed in the ever-changing global economy (Lichtenberg, Woock, & Wright, 2008; Casner-Lotto & Barrington, 2006). In the 2008 Ready to Innovate survey report (Lichtenberg et al., 2008), 97 percent of employers surveyed and 99 percent of school administrators surveyed reported that creativity is of increasing significance in today's workplace. And, while approximately 28 percent of employers stated that creativity is not a major concern when staffing, 85 percent of those employers who did state creativity as a major concern, report they struggle to find applicants who are both qualified for the jobs *and* who posses the desired creativity characteristics (Lichtenberg et al., 2008). Thus, while the report from the President's Council of Advisors on Science and Technology (PCAST, 2012) suggests the need to increase the number of students graduating college with degrees in science, technology, engineering, and mathematics (STEM), reports from employers suggest these STEM graduates need to possess the desired critical thinking, communication, and creativity characteristics, as well as developed problem solving abilities.

The National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (NCTM, 2000), as well as the Common Core State Standards Initiative (CCSSI, 2010) include standards that reflect this call for an enlarged set of capabilities. For example, the NCTM standards document (2000) contains sets of standards for each of Problem Solving, Reasoning and Proof, and Communication. Similarly, nearly all of the CCSS mathematical practices reflect notions of problem solving, critical thinking, and communication. The mathematical practices included implicitly in the NCTM standards documents (NCTM, 2000) and explicitly in the CCSSI Mathematical Practices (CCSSI, 2010) reflect the assumption that the mathematical activity of students in school should, at least to some degree, parallel the practices of professional mathematicians (Brown, Collins, Duguid, 1989; Moschkovich, 2013). However, the practices included in these standards documents, while developed to reflect the practices of professional mathematicians, may be idealized versions of what mathematicians do (Moschkovich, 2013). Furthermore, the more creative practices of mathematicians are not reflected in either set of standards, depending upon the interpretation. Creative, as well as aesthetic aspects of the work of mathematicians, then, are often absent from the mathematics curriculum, much to the displeasure of some mathematicians and researchers (Lockhart, 2009; Rogers, 1999).

In response to this lack of focus on fostering creativity in mathematics students, and students in the STEM disciplines more generally, a movement to integrate the arts into the STEM disciplines has emerged (Riley, 2013). This movement, called the STEAM movement – introducing the letter A into the acronym STEM to signify incorporating the arts – has been gaining momentum, to the extent there now exists schools with a STEAM focus (see, for example, Delaney, 2014; Feldman, 2015) Two of the driving principles behind the STEAM movement are that incorporating the arts into the STEM disciplines will foster creativity (e.g., Wallace, Vuksanovich, & Carlile, 2010) and promote transfer between subjects (Catterall, 2002). However, research regarding integrating the arts into the STEM disciplines is limited, including literature supporting the driving principles of fostering creativity and promoting transfer between subjects. This indicates a need to examine the ways in which integrating the arts into the STEM disciplines in general, and mathematics in particular, can benefit students – particularly as the STEAM movement has gained momentum.

This national need for students to develop an enlarged set of capabilities, such as critical thinking, problem solving, communication, and creativity, in conjunction with my experiences as the co-instructor in two activity-based Foundations of Geometry courses led me to consider the range of possibilities that the mathematics and mathematical practices in which students engage while working on projective geometry problems, as well as participating in artistic engagement through arts integration may have for students' development of each of critical thinking, problem solving, communication, and, possibly, creativity. As such, this study aims to examine the mathematical practices in which students engage, as well as the artistic engagement of students in a course focused on projective geometry. I discuss each of the topics, projective geometry and mathematical practices, as well as arts integration, in the following sections.

This chapter is organized into two major sections: Activity-Based Projective Geometry and Artistic Engagement. In each of these major sections, I present two illustrative examples from previous activity-based projective geometry courses. I use these examples to motivate my two research questions for this study. Subsequently, I present rationale, as well my personal motivation, for investigating these two research questions in an activity-based projective geometry course that contained an artistic engagement component. I conclude this chapter by revisiting the ways in which answering my two research questions can contribute to the mathematics education field, as well as the movement to integrate the arts into the STEM disciplines.

1.1 Mathematical Practices and Projective Geometry

In this section I provide motivation, as well as rationale, for pursuing the first of my two research questions. Before I present an illustrative example from prior data, I provide a brief description of Projective Geometry, which is a branch of non-Euclidean geometry, and the focus of the course in which my study took place. In addition, I provide background on the course in which the episode I use as an illustrative example took place, as well as background on the mathematical tool used in the illustration. This background information will orient the reader to the activity that occurs in the forthcoming episode. I use this episode to motivate my first research question. Subsequent to the presentation of my first research question, I discuss the rationale for studying a course in projective geometry.

1.1.1 A Brief Introduction to Projective Geometry

Projective Geometry is a branch of mathematics that originated as an artist's tool during the Renaissance era in an effort to formalize the process by which an artist could create a realistic drawing or painting of a three-dimensional object or scene, thereby representing in two dimensions something that is three-dimensional (Andersen, 2007; Field, 1997; Kline, 1957). As time progressed, these procedures were formalized into a branch of mathematics (Andersen, 2007; Kline, 1957; Speed, 1964), a topic I discuss in the next chapter.

In general, projective geometry in two or three dimensions involves how objects on one line or plane, respectively, project onto a second line or plane, through a center of projection (see *Figure 1*). More specifically, projective geometry is the study of the aspects or properties of mathematical objects that remain *invariant* through projection. For example, the projection of a conic section will result in a conic section, but not necessarily the same type of conic (e.g., a circle may project to a hyperbola). Very generally, the projection of a point, line, image, or object is determined by extending lines from a given point, called the center of projection (your eye, for example) to the points on a line, or to the points on an image or object on a plane (say, a table top in front of you). The *lines of projection* is one phrase to describe these lines extending from the center of projection to the original points, line, image, or object. The intersections of these lines of projection with a second line or plane determines the projection of the points, line, image or object residing on the initial line or plane onto the second line or plane (see *Figures 1 & 2*).

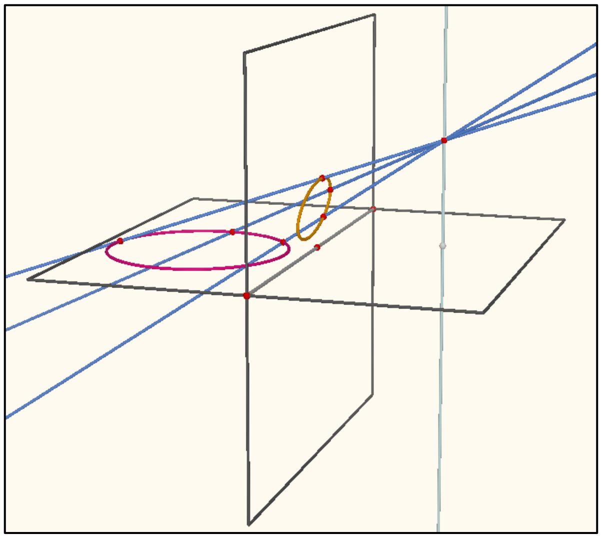


Figure 1. The orange ellipse on the vertical plane is the projection of the magenta circle on the horizontal plane. The blue lines of projection connect points on the circle with a single point, called the center of projection. The points at which the blue lines of projection intersect the vertical plane determine the projection of the magenta circle.

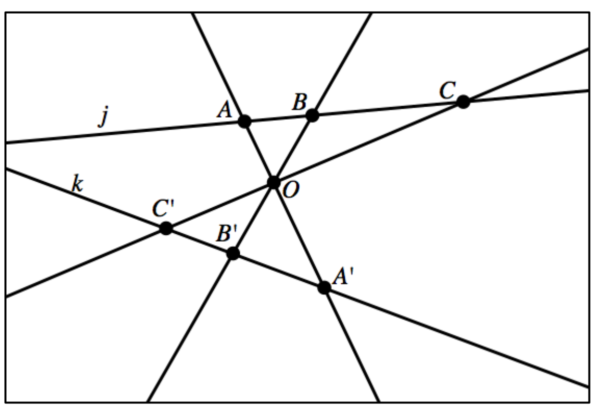


Figure 2. The points *A*, *B*, and *C* on line *j* project through point *O* to the points *A*', *B*', and *C*' on the line *k*. *O* is the center of projection.

Projective geometry is a non-Euclidean geometry, since we take as an axiom that any two lines will intersect somewhere, perhaps at a point at infinity. Thus, the Parallel Postulate does not hold.

1.1.2 Illustrations from Prior Data

1.1.2.1 The Alberti's Window. The forthcoming episodes occurred during the fall semester of 2011 in a Foundations of Geometry course with a focus on projective geometry. In this course, students used a mathematical device called the Alberti's Window (see *Figure 3*) to explore ideas in projective geometry. The Alberti's Window consists of two primary components, a window and an eyepiece. The window is a 12x12 inch rectangular sheet of clear acrylic that stands on a mount perpendicular to the surface on which it sits, generally a tabletop. The second component is an adjustable eyepiece, also constructed from acrylic sheet, through which a person can view drawings or objects.

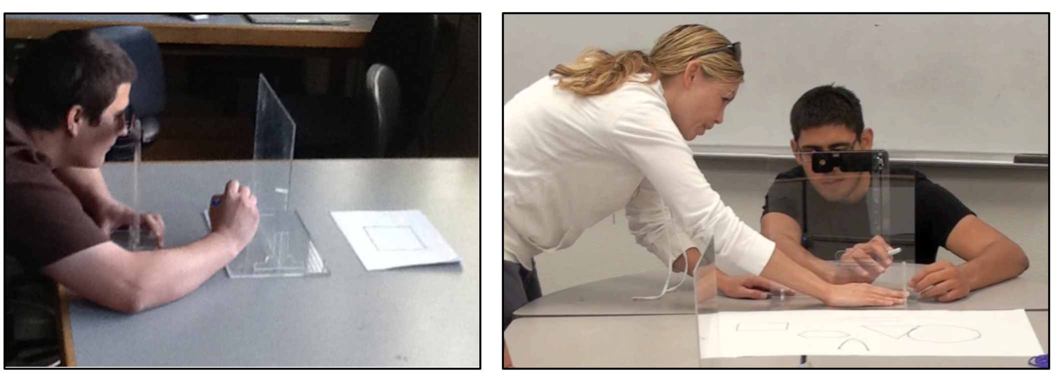


Figure 3. Students use the Alberti's window by looking through the eyepiece and tracing onto the window with a marker the object they see in front of them.

Students in this course used the Alberti's Window by looking through the eyepiece, with one eye, at an image or object that was located on the tabletop. The students then traced onto the window with a dry-erase marker the drawing or object that was located on the table. The image drawn on the window is the projection of the object that resided on the tabletop.

Students in the course engaged in activities involving the Alberti's Window in which the location of the objects on the table would vary. To consider objects that were placed on the table in such a way that the viewer could not see all or part of the object by looking through the eyepiece – for example, if the object was completely behind the viewer – the students often reasoned about the projection by using a piece of string to represent the line of projection, which students frequently referred to as the line of sight. One end of the string would be placed on a point on the object, then stretched through the hole in the eyepiece, and extended. Wherever the string intersected with the window, or in other words, the point on the window where the string touched it, was considered the projection of the point on the object onto the window. Determining projections using the Alberti's Window play a role in each of the following illustrations.

1.1.2.1 "It's never actually going to reach this point. But when it does…" In this episode, two students, Ryan and Veronica, are discussing what happens when you attempt to project onto the Alberti's Window a circle that surrounds the eyepiece – that is, one part of the circle is behind the eyepiece and one part of the circle is between the eyepiece and the window (see *Figure 4*). The students had been instructed to imagine that the window stretches infinitely in both the horizontal and vertical directions.

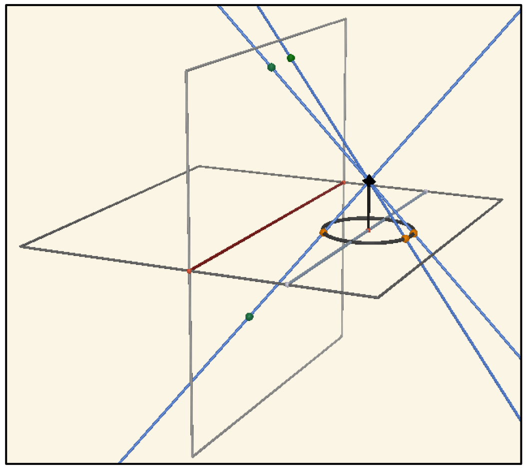
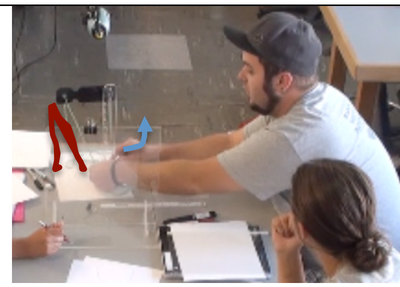
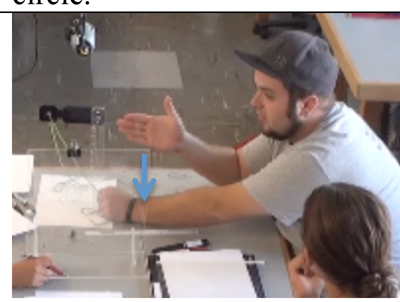
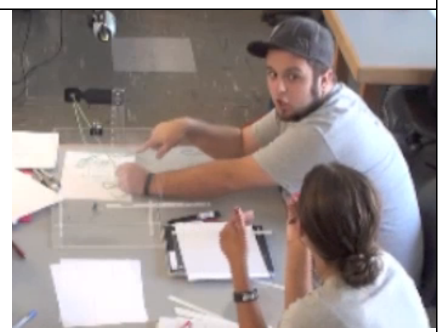


Figure 4. The circle on the horizontal plane is the object to be projected onto the vertical plane. The blue lines of projection connect points on the circle with the center of projection. Where the blue lines of projection intersect the vertical plane determines the projection of the circle (not shown in this figure). Note the points on the circle in front of the center of projection project onto the vertical plane at points below the horizontal plane. Similarly, points behind the center of projection project onto the vertical plane at points above the horizontal plane.

Just prior to this episode, I had asked the students what happens to the projection of the circle at the points that are in line with the eyepiece; that is, the two points that are neither behind nor in front of the eyepiece. The students had agreed the projection of the circle would be a hyperbola, which is correct (although they described the hyperbola as two parabolas, which is incorrect). With the consensus that the projection would be a hyperbola, Ryan and Veronica tried to decide whether the two branches of the hyperbola would end somewhere on the window or extend to the infinite. Veronica suggested that at the two points in line with the eyepiece, it was as if the projection would be located in two places at the same time. They decide the branches of the hyperbolas must end somewhere, but suddenly Ryan entertains the idea that they might actually be infinite. This episode begins as Ryan is using a piece of string to reason about the slopes of the lines extending from the points on the circle, through the eyepiece, and to the window.

1	*Ryan:* These are limits. But I think, I think this slope is gonna get so so so so so so so so steep that it's just going to go on forever.	 Ryan holds a piece of string and moves it slowly toward the eyepiece, as if he is tracing the circle.
2 3	*Ryan:* It's never actually going to reach this point. But, but when it does it'll, it'll be parallel *Veronica:* So it's just like a hole.	 The string becomes flush with the eyepiece. Ryan gestures the plane defined by the eyepiece.
4 5	*Ryan:* to this. So it won't show up on there. (pause) Right? *Veronica:* Yeah.	 Ryan points to the Alberti's Window.

6 7	*Ryan*: I think it is infinite though because this slope will get infinitely close to being equal to that. (pause) Forever. Closer closer closer closer closer closer *Veronica:* Well if it's getting infinitely close -	 Ryan points to the Alberti's Window again.
8 9 10 11 12	*Ryan:* You can always get closer and closer. *Veronica:* That's true. *Ryan:* So there is no limit. *Veronica:* Well if you're getting infinitely closer the limit can exist it just means it won't necessarily- from each side it exists, it just won't necessarily meet. (pause) Do you know what I mean? *Ryan:* Mm hm. (pause) I think it goes on forever. Ryan and Veronica conclude that the branches of the hyperbola do, in fact, extend to the infinite.	 Holding the string in his left hand, Ryan points to the base of the eyepiece with his right hand.

This was a new situation for Ryan and Veronica. While they had already experienced using the Alberti's Window to explore a few projective geometry ideas, this particular situation was novel for them. As such, there are several compelling phenomena during this episode. To arrive at his conclusion that the projection would extend to the infinite, Ryan used the string, extending from the eyepiece to the circle, to think about the lines of projection. He then drew upon mathematical ideas from other branches of mathematics, a standard included in the NCTM document (NCTM, 2000). In Ryan's case, he connected both slopes and limits to his mathematical activity in this projective geometry course.

In the process of using the steepness of the slopes of the lines of projection, Ryan states the line of projection will always be able to get closer to the points that are in line with the eyepiece, but the line will never actually be able to reach that point, since the point is a limit. Despite there having been no talk of limits during this geometry course, Ryan introduces the idea of limits as a way to reason about and imagine the behavior of the projection. Limits present multiple conceptual hurdles for students (Davis & Vinner, 1986, Tall & Vinner, 1981), including the notion of a limit being unattainable (Davis & Vinner, 1986, Tall & Vinner, 1981; Williams, 1991). This episode suggests Ryan may have had conflicting notions of whether or not limits can be attained. However, upon engaging in drawing upon branches of mathematics other than projective geometry, and imagining how the lines of projection become steeper as the lines of projection converge to the points next to the eyepiece, Ryan convinces himself the hyperbola branches will extend to the infinite.

Recall, the episode began just after Ryan and Veronica agreed that the branches of the hyperbola would end somewhere. However, rather than simply stop the discussion there, the two students pushed further to find a justification for the behavior of the curves. Ryan decided to entertain the idea that perhaps the hyperbola branches were in fact endless, and determined this was in fact the case. The CCSS mathematical practice standards (2010) include standards regarding perseverance in solving problems, constructing arguments, and using tools strategically – such as using the string to reason about slopes and limits, then imagine and justify that the projection would be infinite. As such, it is reasonable to expect an activity-based projective geometry course may be an interesting arena for researching students' mathematical practices, and, potentially, their understanding of limits.

1.1.2.2 "It acts like an infinity." This next episode occurred in the same course as the previous episode, during a small group discussion about the justification for why the projection of a parabola must converge at the horizon line. The students had already discussed how the tangent lines on either side of the parabola tend toward parallel as the parabola gets further away from them – keeping in mind the parabola was on a horizontal surface – and that projected parallel lines converge at the horizon line. This brief episode begins just after two students, Veronica and Jay, discussed the tangent lines on either side of the parabola becoming increasingly more parallel, highlighting it as an iterative and continuous process that was taking place.

13	*Veronica:* Well the horizon line is almost like, acts as, acts like an infinity.	 As she says "like an infinity," Veronica produces a quotation gesture with both hands.
14	*Jay:* Which is how you explain that, when something approaches infinity it touches the horizon line.	
15	*Veronica:* Yeah exactly.	

In this episode, Veronica demonstrates an unusual invocation of infinity, suggesting the analogy of the horizon line acting "like an infinity." Jay appears to agree with Veronica, following up with, "when something approaches infinity it touches the horizon line." Similar to the situation with limits, students often have a challenging time with the notion of infinity (Monaghan, 2001; Singer & Voica, 2008; Tall & Tirosh, 2001), yet in this episode, Veronica appears to be imagining the horizon line as a reachable representation of the infinite, or considering the horizon line as displaying characteristics of the infinite. Similar to the previous episode, this illustration indicates an activity-based projective geometry course may be a compelling setting for researching the mathematical practices in which students engage, such as utilizing analogies and imagination.

1.1.3 Research Question 1

The two illustrations above highlight the ways in which students in this activity-based projective geometry course engaged in mathematical practices, such as persisting in solving problems, justifying mathematical thinking, using imagination, and drawing on mathematical ideas from contexts other than geometry, to effectively reason about problems in projective geometry. This leads to my first research question (RQ1):

> In the context of an activity-based projective geometry course, in what mathematical practices do students engage while working on problems in projective geometry?

To provide further insight into my framing of RQ1, for the remainder of this section, I discuss the theoretical perspective from which I approach this study, followed by clarification of what I mean by *mathematical practices*.

1.1.3.1 Theoretical perspective. I approach this study from the theoretical perspective that all learning and knowing is situated (Brown, Collins, & Duguid, 1989; Greeno, 1998; Lave, 1988) and both socially and culturally mediated (Cobb & Yackel, 1996; Forman, 2003). Learners develop understanding through participation in cultural practices (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991). Furthermore, the particular modes of participation in a culture and context are mediated by discourse of social interaction, cultural artifacts and tools, bodily engagement, and symbols. The particular cultural practices are regular patterns of activity, behaviors, and communication within the local community, and which are situated within a broader community. For example, a mathematics classroom community is situated within the broader community of mathematicians.

Particular ways of reasoning develop and emerge through participation in a practice, are embedded in social interactions and activity, and shaped by the cultural practices (Cobb & Yackel, 1996). It is through social interaction, and the particular modes of social interaction in the culture that the learner comes to be a more central member of the community. Learning then is enculturation into the practices of that community (Brown, Collins, Duguid, 1989; Forman, 2003; Goos, 2004), where an individual develops through legitimate peripheral participation (Lave & Wenger,

1991) in activities and interactions within the culture. The practices within a culture of practice may include such things as discourse, beliefs, activities, and dispositions.

It is important to highlight here that learners do not enter every classroom as a clean slate. That is, the individual participation in a community of practice is influenced by the prior experiences and history of the individual, as well as the culture with which the individual identifies. They bring with them their prior experiences and social interactions, developing mathematical understandings, beliefs, and ways of reasoning into a classroom community of practice (Lave & Wenger, 1991). For example, a student may enter a new classroom with an unproductive mathematical disposition (NCTM, 1991) based on prior experiences. However in the particular classroom community, say an activity-based projective geometry course, the student's mathematical disposition may be productive, but overall she may have an unproductive disposition, perhaps due to a change in teacher expectations, perceived difficulty in topics, or a shift in teaching styles.

1.1.3.2 Mathematical Practices. Within this theoretical perspective that learning and knowing is social, situated, and embodied, mathematics then exists as a social activity along with the practices that are appropriated through a process of inquiry, bodily engagement, and collaboration with others. As such, the mathematics involved in the mathematical practices, to which I refer in RQ1, is this kind of mathematics.

Solving mathematical problems is situated within the context in which the problems arise and embedded within the culture of practice. The mathematics and the action and practices associated with those mathematics are coupled. For example, Ryan's use of the string to represent the slopes of the lines of projection was an emergent act embedded within that particular course. This highlights how knowledge is developed though activity within a culture and a context (Brown, Collins, & Duguid, 1989).

Consistent with the theoretical perspective described above, I turn to the framing of mathematical practices by Moschkovich (2007), in which she describes mathematical practices as normative culturally and socially, and historically situated. She builds her framing of mathematical practices on the notion that mathematics is a discursive activity (Gee, 1996; Moschkovich, 2007), and learning is both situated and socially embedded. Furthermore, mathematical practices involve multiple resources, such as artifacts, tools, language, and other social aspects. As such, mathematical practices are embedded within the context in which they occur, and are constituted by the goals and meanings of discourse and purposeful activity (Moschkovich, 2007, 2013). She notes:

> Mathematical practices involve not only meanings for utterances but also focus of attention. [They] are not simply about using a particular meaning for an utterance, but rather using language in the service of goals while coordinating the meaning of an utterance with a particular focus of attention (Moschkovich, 2007, p. 25).

Mathematical practices can thus include discourse, behavior, or activity, in the context of the classroom community. These might include practices such as imagining, justifying, entertaining alternate possibilities, and ways of using mathematical tools.

Thus in analyses, the goal is not to merely categorize mathematical practices, but to examine how they function within this particular projective geometry course.

1.1.4 Rationale: Why study projective geometry?

While the illustrative examples I presented above indicate an activity-based course may be an interesting place to conduct a study about engagement in mathematical practices, the reader may be wondering the rationale for doing so in a projective geometry course, rather than in a course focused on a different mathematical topic. While there is increasing emphasis in undergraduate mathematics education literature on topics such as linear algebra (e.g., Wawro, Sweeney, & Rabin, 2011), abstract algebra (e.g., Larsen, 2009), functions (e.g., Breidenbach, Dubinsky, Hawkes, & Nichols, 1992), calculus and differential equations (e.g., Zandieh, 2000), limits (e.g., Tall & Vinner, 1981; Williams, 1991), and mathematical proof (e.g., Raman, 2003; Weber & Alcock, 2004), there is a noticeable lack of research focused on geometry. This shortage of geometry education research at the undergraduate level may be a result of fewer students at the university level taking geometry courses, or perhaps geometry is seen as less crucial than mathematical branches such as calculus and linear algebra, for example. In particular, there is very limited educational research regarding non-Euclidean geometry, with only a handful of studies turning up upon searching (e.g., Ada & Kurtulus, 2010; Guven & Baki, 2010; Kaisari & Patronis, 2010; Zandieh & Rasmussen, 2010). Of those non-Euclidean studies, I could find no educational research studies on projective geometry.

Research in non-Euclidean geometry, and in particular projective geometry, has potential to prove significant at least five ways. First, projective geometry is frequently employed – although often covertly – in contemporary technology, such as computer graphics and 3-dimensional animation for video games (Herman, Hartmanis, & Goos, 1992; Penna & Patterson, 1986). It is reasonable to expect the trend of using computer technology in the competitive tech-industry will continue, and as such, developing an understating of projective geometry may prove beneficial for those interested in pursuing careers in technology development.

Second, courses in projective geometry, with its significant difference from Euclidean geometry – that is, that the Parallel Postulate does not hold – can create an environment in which students can further develop their argumentation and justification abilities. My experiences observing and working with students over the past several years has informed me that students grapple with the notion there can be a geometry with no parallel lines, essentially thinking that Euclidean geometry is the only geometry. My personal experience in learning mathematics may have been similar in that I earned a bachelor's degree knowing there were multiple forms of geometries, yet I was entirely unfamiliar with any non-Euclidean geometry. In constructing a non-Euclidean geometry, students must, for example, justify why certain properties should be taken as axioms, define mathematical objects, and develop proofs for theorems using these axioms and definitions.

Third, as my illustrative examples indicate, with the inclusion of points at infinity in non-Euclidean geometry, students' reasoning in projective geometry is often influenced by such notions as limits and the infinite, both topics which have been shown to be challenging for students in other courses (Davis & Vinner, 1986, Monaghan, 2001; Singer & Voica, 2008; Tall & Tirosh, 2001; Tall & Vinner, 1981). Notions of limits and infinity are fundamental for multiple areas of mathematics, but in particular calculus and analysis. Research in an activity-based projective geometry course could provide valuable insight into how students' engagement in certain mathematical practices might help them grapple with notions such as infinity and limits.

Fourth, ideas of projective geometry could be useful for various industrial technologies. For example, by visualizing the way in which cows viewed the chutes to slaughter, Temple Grandin created a more humane ramp and chute that allowed cows to remain calm while on the way to slaughter (Grandin, 1980). While Temple Grandin may not have used projective geometry per se to create this new ramp and chute design, it is the alternative visualization of a situation, and imagination, that led to her creative invention. Projective geometry also lends itself to visualization and imagination – two activities that are important for developing an understanding of mathematics, but are rarely addressed directly in mathematics courses.

Fifth, the geometric transformation of projection is a form of mapping, a topic that has not been extensively studied in mathematics education. Research in an activity-based projective geometry course could provide valuable insight into students' reasoning about and mathematical practices involving mappings.

1.2 Artistic Engagement in Mathematics

Similar to the previous section on projective geometry and mathematical practices, in this section, I introduce two illustrations from a prior course focused on projective geometry to begin discussing my motivation for studying artistic engagement, which some may call arts integration. Students in the activity-based Foundations of Geometry course mentioned previously, engaged in creating artistic projects designed by using ideas from projective geometry in conjunction with Geometer's Sketchpad. These art projects are discussed in detail in Chapter 3. Prior to introducing the two illustrative examples, I provide a brief personal background of my relationship to mathematics and art.

1.2.1 Personal Background and Motivation

My interest in the connections between mathematics and the arts began when I was rather young. While I was attending elementary school, which happened to be a K through 8 school, every year, each student was required to prepare a poster for the annual science fair, in which two schools participated. While many students conducted routine experiments such as creating a volcano with vinegar and baking soda volcano, I typically veered from the common path. In seventh grade, my friend and I decided to design and conduct an experiment regarding the effects of listening to various genres of music on mathematics exam performance for different grade levels. While I certainly recall our experiment design, it is too far in the past to recall our results. However, I do recall we were awarded the top prize at the science fair.

Most recently however, my interest in the connection between mathematics and the arts stemmed from my work on the Tangible Math grant (NSF 0816406). One component of this work included acting as the teaching assistant in a Foundations of Geometry course, as well as a Technology in Teaching Mathematics course. In each of these courses, students created artistic pieces using the mathematics with which they engaged in the course. In the Foundations of Geometry course, students created paintings using an airbrush in conjunction with stencils they designed using Geometer's Sketchpad. (I describe these projects in greater detail in Chapter 3, as the students in the course in which my study takes place participated in the same type of artistic project.) During the Technology in Teaching Mathematics course, students first created a string-art design and subsequently designed a quilt square, using Geometer's Sketchpad for both designs. In each of these courses, the students recorded video reflections regarding the art pieces they created in the course. A single set of these reflection videos ultimately sparked my interest in investigating artistic engagement in mathematics courses.

Below, I present examples of two students' artistic designs created using ideas from projective geometry. Rather than introduce transcript from the students' video reflections, I provide an illustrative description with selected quotes.

1.2.2 Illustrations from Prior Data

1.2.2.1 Man on the Moon. Marissa was an exceptionally engaged student in the Foundations of Geometry course. During one of Marissa's first experiences using the Alberti's Window, her group chose to project a circle. The projection of a circle, when traced onto the window is an ellipse. Marissa was particularly surprised by this projection, exclaiming, "That's not a circle!" When it came time to create an artistic design using ideas of projective geometry, she found inspiration in her experience of seeing a projected circle for the first time.

Since Marissa found the projection of a circle to be particularly surprising, even curious and funny, she chose to create an artistic design that brought forth this surprise and curiosity. Marissa chose to create a design she called *The Man on the Moon* (*Figure 5*). Marissa's design demonstrates a playfulness in mathematical engagement not often found in students. Many students discuss mathematics as cold and dry, simply a tool to be used for solving particular problems. That is, playfulness is not an attribute of mathematics that comes to the fore for many students.

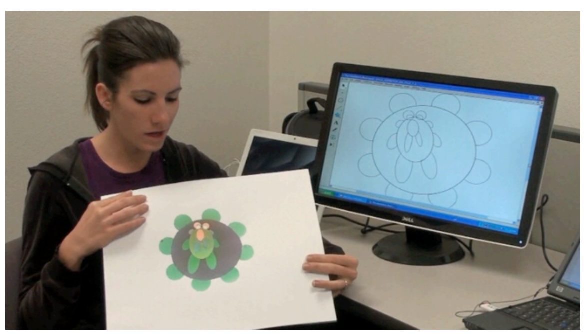


Figure 5. Marissa discusses her art piece, *Man on the Moon.*

Creating this artistic piece appeared to contribute to Marissa's learning experiences in at least two ways. First, Marissa's design was created using all circles and their projections, which is a challenging endeavor. To construct a design where both the original images and the projection come together in one coherent piece is not a trivial task – particularly if the desire is for the projection to take the form of something specific, for example, a comical character. Marissa had to use her understanding of projective geometry that she had developed to attain her desired design.

The second way creation of the artistic piece contributed to Marissa's learning experiences was by giving Marissa an outlet to express her curiosity about a mathematical relationship. This is an uncommon opportunity in the mathematical classroom, and could aid in Marissa's retention of projective geometry understanding by linking her experiences with the mathematics to her emotion of surprise (Rinne, Gregory, Yarmolinskaya, & Hardiman, 2011).

1.2.2.2 Just Keep Swimming. Karryn, unlike Marissa, was not a particularly engaged student in the Foundations of Geometry course. She regularly attended class and turned in her homework assignments, but she earned average grades on her assignments, and in the course overall. When the final project was introduced, which was creating an artistic piece using projective geometry ideas, Karryn stopped me at the end of class to tell me what she wanted to make for her project. She wanted to create a swimming pool from the view of the lifeguard stand, since she worked as a lifeguard and swimming had been a lifelong passion for her. She seemed to know ideas from projective geometry could somehow allow her to create her desired design, but she admitted to me she wasn't sure how to construct it.

Since Karryn was not particularly engaged in the course, I suspect, but have no supportive evidence aside from her painting and her reflection, that she had to delve more deeply into projective geometry ideas than she had previously. To be able to create such a precise design (see *Figure 6*) by using projections is not a trivial task, particularly as the Geometer's Sketchpad software she used to create the design deals strictly with two dimensions, yet her design depicts three dimensions. As such, I expect that Karryn developed a deeper understanding of the projective geometry ideas that had been explored in class. Furthermore, I suspect it was Karryn's personal connection to swimming and lifeguarding that motivated her to delve deeper into the mathematics. In this sense, Karryn's understanding of the projective geometry and her personal connection to swimming are intertwined.

Figure 6. Karryn discusses her art piece, *Just Keep Swimming.*

1.2.3 Research Question 2

Up to the point at which I watched students' reflection videos regarding the creation of their artistic pieces, which was approximately five months after the Foundations of Geometry course had ended, I had not fully appreciated what students might gain from creating their art pieces. After watching the reflection videos, I became convinced the potential of artistic engagement in a mathematics course in general, and in this course in particular, deserved to be investigated.

These experiences lead to my second research question (RQ2):

> In what ways can various means of artistic engagement enrich students' learning experiences and opportunities in an activity-based projective geometry course?

To further explain my second research question, I provide a brief description of what I mean by *artistic engagement*, as well as *enrichment*.

1.2.3.1 Artistic Engagement. In the context of the particular Foundations of Geometry course in which my study takes place, there were multiple forms of artistic engagement. First, the introduction of projective geometry, the content focus of the course, was motivated using ideas from the arts. Second, students in the course participated in creating two artistic pieces using ideas from projective geometry, and wrote reflective essays in which they discussed their experiences creating the artistic pieces. Third, students read two writings related to art, one regarding mathematics as an art, and the other about the emergence of contemporary art. In addition, students wrote reflective essays and participated in whole-class discussions about each of the readings. And fourth, students spent an afternoon on a fieldtrip to a Museum of Contemporary Art.

1.2.3.2 Enrichment. In my second research question, the word *enrich* is meant to indicate a form of *value added* to the mathematics course itself. This might include ways in which the artistic engagement supplemented or augmented the learning experiences and opportunities of the students in ways that would likely not have occurred without participating in the artistic engagement. For example, in the case of Karryn, she likely would not have had the vision or opportunity to create her artistic piece that had deep, personal meaning for her.

1.2.4 Rationale: Why study artistic engagement in mathematics?

In response to reports of employers difficulties assembling a workforce skilled in creativity and innovation (e.g., Lichtenberg et al., 2008), an initiative has emerged to integrate the arts into the learning of the science, technology, engineering, and mathematics (STEM) disciplines – suggesting the acronym STEM be transformed into STEAM, incorporating the ‘A’ to represent art and design (Riley 2013). While STEAM, STEaM, and STE+aM are all used, depending on the source, I will use STEAM for ease of reading. (However, I generally prefer to use STE+aM, as the ‘+a’ is a nice visual suggestion of the integration of the arts into the STEM disciplines.)

In general, the argument put forth by STEAM supporters is that placing an emphasis on the STEM disciplines alone will not result in the development of students’ individual creativity and innovation necessary for competing in today’s global market. By placing an emphasis on art and design – in particular in relation to and in conjunction with the STEM disciplines – the expectation is a fostering of student creativity (e.g., Wallace, Vuksanovich, Carlile, 2010). Furthermore, since art courses and art requirements in the K-12 system continue to be reduced, students have fewer outlets for developing creative and artistic skills, and thus introducing arts into the STEM curriculum could make up for the possibility of diminished creative abilities as a result of fewer artistic outlets (Burnaford, Brown, Doherty, & McLaughlin, 2007).

While the primary fuel for the STEAM movement is employers' needs of finding qualified applicants with the desired creative skills, there are multiple – and perhaps more relevant – reasons for integrating the arts into the STEM curriculum. An important reason for considering alternative rationales for arts integration is that research focused on cultivating creativity varies, partly due to disagreement about the nature of creativity and how it should be measured (Plucker, Beghetto, Dow, 2004).

One reason for integrating the arts into the mathematics classroom is to disrupt the widely held belief that mathematics is not a creative subject, but merely a system of rules and procedures to be implemented when needed. This inaccurate portrayal of the nature of mathematics can result in disenchanted students, bored with mathematics and uninterested in exploring new topics.

A second reason for integrating the arts into the mathematics classroom is that artistic engagement in mathematics may be an entry point into the STEM disciplines for students who otherwise might not pursue a STEM career. These students could be underrepresented or underserved populations in mathematics, students who are not able to situate mathematics with respect their own lives, as well as students who simply don't find mathematics interesting. Arts integration has the potential to help underrepresented and underserved populations by giving these students an avenue to develop personal connections to mathematics, which could serve as an entry point into the appreciation of mathematics. Often students do not see connections between mathematics and their own lives. If students from underrepresented populations in mathematics can bring forth their own culture into the mathematics classroom through engagement with artistic endeavors, the culture to which they belong may become relevant in the mathematics classroom, which in turn may result in the ability to situate the mathematics in relation to their own lives. Additionally, making personal connections to ideas, mathematical or otherwise, aids in long-term retention of content (Rinne et al., 2011).

My intention with this study is to explore the ways in which arts integration, through various forms of artistic engagement, can enrich students learning experiences and opportunities in the mathematics classroom, which may provide insight into alternative reasons for integrating the arts into mathematics courses.

1.3 Chapter Summary

In this chapter I provided the motivation and rationale for investigating my two research questions:

> **Research Question 1:** In the context of an activity-based projective geometry course, in what mathematical practices do students engage while working on problems in projective geometry?
>
> **Research Question 2:** In what ways can various means of artistic engagement enrich students' learning experiences and opportunities in an activity-based projective geometry course?

In Chapter 2, I provide a review of literature related to my study. First I discuss the emergence and mathematical formalization of projective geometry. I then review the literature on mathematical practices, followed by arts integration. In Chapter 3, I discuss the methodology used in my study. I discuss the results of my study in Chapters 4, 5, and 6. And, finally, in Chapter 7, I discuss the implications of my results, the limitations of the study, and the ways in which my study has informed my interests for future research.

Chapter 2
Literature Review

2.1 Projective Geometry

The Projective Geometry branch of mathematics emerged from practices, developments, and discoveries in perspective (Andersen, 2007; Field, 1997; Kline, 1957). In this section, I highlight several influential contributors to the study of perspective, and in turn, projective geometry. While there have been numerous contributors to the study of perspective and projective geometry, I have chosen to highlight those contributors who were influential within the topics presented in the Foundations of Geometry course in which my study takes place.

2.1.1 The Origins of Perspective

Certainly, Greek geometers were aware of particular aspects of perspective theory and projection, as Pappus' theorem, named for Pappus of Alexandria, is one of the core theorems in projective geometry (Shenitzer, 1991; Speed, 1964). However, the more substantial origins of the development of perspective theory, and in turn, projective geometry, came during the time of the Renaissance (Kline, 1957, Speed, 1964). During the Renaissance, artist, scientists, and mathematicians became enamored of the notion that underlying all of nature was mathematical principles, perhaps as a result of the resurgence of studying Greek works (Kline, 1957). This notion of mathematics as a foundation in nature led to a prolonged effort to find a method for representing in two-dimensions a three-dimensional scene, and to discover the mathematical underpinnings of such a method (Kline, 1957).

Perhaps one reason for the desire to draw or paint a reproduction of a visual scene, suggested by Andersen (2007), who refers to this scene as an *instantaneous view*, is that it can be challenging for an artist to find room for all the objects she might want to include in a given scene. Therefore, it would be useful to have a method for dealing with such things as objects at varying distances from the painter, as well as occlusion. A developed method for reproducing a scene would then provide the painter with a way to organize the depictions of all the objects in relation to space (Andersen, 2007).

The method developed for creating a two-dimensional representation of a three-dimensional scene came from the principle of projection and section (Kline, 1957). This principle is based on the idea that what someone sees of a particular scene depends on the location of the viewer with respect to the scene. As such, the principle of projection and section fixes the viewer with respect to the scene. And, to eliminate the issue of the complication of human optics, it is supposed the viewer looks at the scene only thorough one eye. A brief description of the method of projection and section follows.

We can imagine a set of lines extending from each point on the objects in a scene to a viewer's eye. For simplicity, suppose the viewer is looking at a cube. This set of lines extending form the viewer's eye to the points on the cube is called the *projection* of the cube. We can then imagine a glass window, a screen, or a thin veil, placed between the eye and the cube, and the lines of projection passing through the glass. The set of points at which the lines of projection intersect the glass is called a *section* (Andersen, 2007; Kline, 1957). (Note that in the course in which my study took place, students referred to the collection of intersection points as the projection, rather than a section.) Clearly, the image of a given section is dependent upon where the location of the glass is located with respect to the cube and the eye (*Figure 7*).

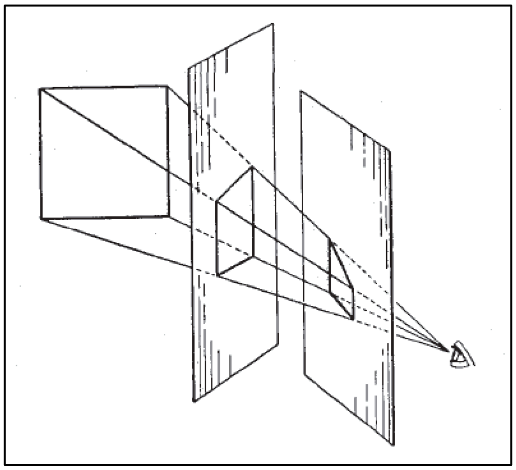


Figure 7. Representation of two different sections of a projection. (Kline, 1957, p. 626)

Mathematicians began to wonder whether there must be particular properties of sections of the same scene that are consistent across section. For example, if one were to compare two sections of the same scene projection, then would there be some consistent geometrical properties of the images of the two sections. Similarly, considering two sections of the same scene, but viewed form different positions, would there be some consistent geometrical properties of the images of two sections

(Kline, 1957). Looking for commonalities across sections gave rise to the mathematization of perspective, and resulted in the field of projective geometry. In the next sections, I highlight some of the influential contributors to projective geometry.

2.1.2 The First Accounts

2.1.2.1 Alberti. While Filippo Brunelleschi (1377-1446) is generally believed to be the first artist to paint compositions in genuine perspective, it is not clear how he constructed those perspectives (Andersen, 2007). Leon Battista Alberti, on the other hand, in his *De Pictura* (Alberti, 1435/1991), was the first to write an account of a method for representing an instantaneous view (Andersen, 2007). Interestingly, the notion of parallel lines meeting at a vanishing point was an accepted practice before the writings of Alberti, so it was not he who introduced that aspect of his construction (Andersen, 2007; Field, 1997). It could be this rule of parallel lines converging at a vanishing point was already a tool of the artist trade, passed down from teacher to apprentice, but no one seems to know how or when it came to be (Andersen, 2007).

Since the convergence rule was implicit knowledge, the challenge then was to determine how to construct the lines that were parallel to the ground or floor. A rule at the time suggested equidistantly spaced lines that were parallel to the ground should be spaced in the ratio of three to two, where the largest of the lines is closest to the bottom of the canvas (*Figure 8*). Alberti showed this construction was incorrect by using the diagonals of the rectangular tiles as a check. That is, since the diagonals of the tiles would line up prior to being put into perspective, then they must also line up when put into perspective (Andersen, 2007; Field, 1997).

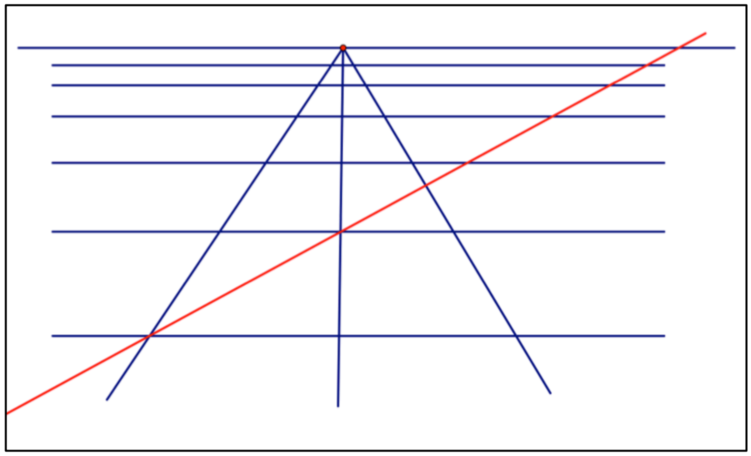


Figure 8. A grid where horizontal lines are spaced in the ratio of three to two. The red line demonstrates the diagonal of the lower-left tile does not pass through the vertices of subsequent tiles.

Alberti took the idea that the eye can be conceptualized as a mathematical point, and that any other point in front of the eye is perceived on, what I referred to in Chapter 1 as a line of projection, also referred to as a line of sight. He imagined a visual pyramid constructed by lines of projection connecting the eye to each of the vertices of a polygon on the ground. He then imagined a thin veil between the polygon and the eye. Where the visual pyramid intersected the veil was the perspective image of the polygon (Alberti, 1435/1991, Andersen, 2007). . It should be evident then why the device students used in the projective geometry course is referred to as the Alberti's Window.

Alberti, inspired by both optics and mathematics, had two methods for creating an instantaneous view. In the first, the viewer would be directly in front of the object to be put into perspective and would utilize a grid of woven threads to guide the construction. In the second, the artist would construct a perspectival grid through a set of procedures (Andersen, 2007).

2.1.2.2 Piero. While Piero della Francesca's (?- 1492) method for constructing perspectives differed from Alberti, one of the major advancements by Piero was utilizing diagonals of perspective tiles to aid in the actual construction, rather than simply as a check for accurate construction (Andersen, 2007). Piero's treatise was called *De prospectiva*, and it was likely written sometime before 1482. Piero, unlike Alberti, had more mathematical and tedious descriptions of how to create perspective images, however this is likely a result of his attempt to explain both how and why the constructions worked. He utilized similar triangles and proportions in legitimizing his constructions, of which he had several, including a construction for anamorphoses (Andersen, 2007; Field, 1997).

2.1.2.3 da Vinci. Certainly Leonardo da Vinci (born-died) was a skilled artist, however his contributions to perspective were not as significant. Perhaps his most interesting contributions to perspective are his three categories of perspective. Da Vinci considered *linear perspective*, the *perspective of color*, and the *perspective of disappearance* as three separate forms of perspective. Perspective of color was related to changes or reduction in color based on distance from the viewer, and perspective of disappearance was related to changes or reduction in preciseness of the look of objects based on distance from the viewer (Leonardo, as cited in Andersen, 2007). Da Vinci wrote the *Trattato della pittura*, where his ideas on perspective were quite varied and tedious. His artistic works however, demonstrated his skill in constructing perspective images.

2.1.3 Mathematical Theory

After Alberti, Piero, and da Vinci, multiple artists – architects, painters, and sculptors – contributed to the theory of perspective. For example, Dominican Egnazio Danti (1536-1586) in the late sixteenth century, articulated several noteworthy definitions regarding vanishing points, such as, diagonal lines all meeting at a vanishing point, called a *distance point* by Danti (Andersen, 2007; Field, 1997). Guidobaldo and Stevin carried out the next major advancements in perspective theory.

2.1.3.1 Guidobaldo. Guidobaldo del Monte (1545-1607), who Andersen (2007) considers to be the father of the mathematical theory of perspective, published six books on perspective in 1600, called *Perspectivae libri sex*. While many before Guidobaldo, such as Danti and Benedetti, attempted to explain the geometrical ideas serving as a foundation for perspective, Guidobaldo himself chose to approach the subject by developing a set of general rules (Andersen, 2007; Field, 1997). These rules included creating a general vanishing point though considering the sets of perspective images of sets of parallel lines.

Included in his six books were numerous methods for putting into perspective images that were situated on the ground. This is similar to the activities that students in the projective geometry course carried out with the Alberti's Window. In addition, one of his books contained methods for putting three-dimensional objects into perspective, and another instructed how to determine the shadow of an object when illuminated by a single-point light source (Andersen, 2007; Field, 1997).

Overall, Guidobaldo considered the perspective image of a given point to be where the perspective line from the given point to the eye intersected the picture plane. However, he explained this in a more complicated fashion by considering a visual pyramid decomposed into the triangles that make up its sides. His constructions then utilized these triangles. For example, he used this to determine that the perspective image of a line segment that is parallel to the picture plane will be parallel to the initial line segment. While others before him had used this idea, as well as others, for constructing perspective images, Guidobaldo felt they required proofs, and so formulated proofs (Andersen, 2007).

One of his most notable proofs was that of the convergence rule – that parallel lines converge to a vanishing point. While this can be derived through mathematics, Guidobaldo was the first to have done so (Andersen, 2007). Other accomplishments of Guidobaldo include using proportions to determine the distance between the ground line and the image of a point, noting that all sets of parallel lines have vanishing points on the horizon line, reasoning about inverse problems of perspective, and reasoning about geometric constructions on the picture plane (e.g., given one line and a point on the picture plane, construct a second line that is perspectively parallel to the first) (Andersen, 2007).

2.1.3.2 Stevin. The mathematization of the perspectival situation, in which a viewer wants to depict an object on a picture plane, was the major contribution of

Simon Stevin (1548-1620) to the theory of perspective. Stevin defined the eye as a point, and the picture plane as an infinite, transparent surface. To construct an object in perspective was to determine how the object would look projected onto the infinite surface (Andersen, 2007). This is similar to the way in which students in the projective geometry course discussed in Chapter 1 were asked to imagine that the Alberti's Window stretched infinitely in both the horizontal and vertical directions.

For Stevin, the theory of perspective was a mathematical discipline. He formed the basis for perspective through the construction of sixteen definitions and two postulates. Rather than using a visual pyramid, or the triangles of a visual pyramid, he defined a projection by an arbitrary point. This approach of defining a projection by an arbitrary point, and imagining the picture plane to be an infinite, transparent surface, allowed Stevin to consider projections with oblique planes, rather than perpendicular planes. While he was not the first to consider oblique planes, his approach allowed for greater ease in doing so (Andersen, 2007). The combination of Guidobaldo's and Stevin's works is what shifted the theory of perspective from merely an artist's tool to an actual, mathematical science (Andersen, 2007).

2.1.3.3 Desargues. The French mathematician Girard Desargues (1591-1661) may be one of the better-known names in the history of projective geometry. Bell (as cited in Speed, 1964), suggested Desargues, in addition to Blaise Pascal were the inventors of projective geometry. Desargues motivation for developing theorems in perspective was to help artists, engineers, and stonecutters (Kline, 1957). His work was not well received by his contemporaries, party due to his introduction of many new obscure mathematical terms, and partly due to the development of analytic geometry by Descartes (Shenitzer, 1991, Speed, 1964).

Desargues is generally credited with the introduction of points at infinity and proving the theorem that would eventually bear his name (Speed, 1964), however in a more convoluted way than would be done today. Desargues Theorem states that *if two triangles are perspective from a point, then they are perspective from a line.* Another significant contribution of Desargues was finding that the cross ratio of four points is invariant under projection (Andersen, 2007; Field, 1997). This property can be used to determine the location of a vanishing point in a sketch or painting, even when limited information is provided.

2.1.3.4 Pascal. Blaise Pascal (1623-1662), another French mathematician, supposedly was urged by Desargues to explore the common geometric properties of sections (Kline, 1957). One of Pascal's major contributions, and one of the most significant theorems in projective geometry (Speed, 1964), is Pascal's Theorem, which states, If a hexagon is inscribed in a conic, the three pairs of opposite sides must intersect in three collinear points. While Pascal's work in projective geometry was not included as a topic in the course in which my study took place, Pascal is worth noting for his significant work in projective geometry and conic sections.

2.1.3.5 Taylor. While Brook Taylor (1685-1731) was known for many different ideas in mathematics, his major contribution in the theory of perspective, as it pertains to this projective geometry course, is the introduction of the notion of a vanishing line, which is a line converging to a vanishing point.

2.1.3.6 Principal of Duality. Several mathematicians contributed to the theory of duality in projective geometry. The basic notion of duality is that within definitions and theorems, points and lines can be interchanged, and the definitions and theorems still hold. For example, given the axiom that *two points determine a line*, the dual is that *two lines determine a point*. The dual of this axiom can be proven to hold through using the axioms of projective geometry. Joseph Gergonne (1771-1859) introduced the principle of duality (Shenitzer, 1991; Speed, 1964), and Jacob Steiner (1796-1863) and Georg Karl Christian von Staudt (1798-1867) further developed the treatment of duality (Speed, 1964). Steiner also changed the meaning of the word *projective*, such that it became a relation between ranges (points on a line) and pencils (a set of concurrent lines) (Speed, 1964).

There were several other notable mathematicians who contributed to the field of projective geometry including Gaspard Monge (1746-1818), Jean-Victor Poncelet (1788-1867), and Arthur Cayley (1821-1895), among others. However, many of these mathematicians' contributions to projective geometry were less relevant to the course in which my study took place.

2.2 Mathematical Practices

Education researchers are shifting from perspectives that place sole focus on the cognitive aspects of learning to perspectives that embrace the local and broader contexts in which learning occurs (Brown, Collins, & Duguid, 1989; Cobb & Yackel, 1996; Lave & Wenger, 1991), thus accounting for both classroom activity and the broader mathematical culture. This shift has resulted in an increased focus on the activities and practices in which students engage in the mathematics classroom. This emphasis on practices is reflected in both the National Council of Teachers of Mathematics (NCTM) Principles and Standards document (NCTM, 2000) and by the introduction of the Eight Mathematical Practices into the Common Core State Standards Initiative (CCSSI) documents (CCSSI, 2010).

The term *mathematical practice* varies among researchers and practitioners (Moschkovich, 2013), and the grain size of what constitutes a mathematical practice is additionally variable. For example, a mathematical practice might be a normative behavior or activity of a professional mathematician (Burton, 1999), or it might be an informal way of reasoning by a child in a marketplace (Nunes, Schliemann, & Carraher, 1993). Distinctions between what constitutes a mathematical practice also includes such aspects as the form of the practice – such as discourse, activity, or ways of reasoning – and whether the practice is locally emergent in a culture or already established in a broader community. In general though, mathematical practices can be classified into three large, not necessarily mutually exclusive categories. These categories are the mathematical practices of (a) professional mathematicians, (b) regular people operating in the everyday world, and (c) students in school. In this section I provide a review of each of these three categories of mathematical practices.

2.2.1 Professional Mathematicians

The mathematical practices of professional mathematicians are often thought of as the normative behaviors, operations and discourse practices – the *authentic activity* – of professional mathematicians while engaging in doing mathematics. As Moschkovich (2013) notes, it has been assumed the mathematical activity of students in school should, at least to some degree, parallel the practices of professional mathematicians (Brown, Collins, Duguid, 1989). This is reflected in the *implicit* practices included in The National Council of Teachers of Mathematics (NCTM) practices and standards (NCTM, 2000) and the eight *explicit* mathematical practices in the Common Core State Standards for Mathematics (CCSSM; CCSSI, 2010).

In the NCTM Principles and Standards for School Mathematics document (NCTM, 2000), are the *process* standards, which are multiple standards that reflect the desire for students to engage in and develop productive habits and activities associated with the authentic practices of mathematicians. These standards include such practices as, making and investigating mathematical conjectures, developing and evaluating mathematical arguments and proofs, analyzing and evaluating the mathematical arguments of others, and communicating thinking to teachers and peers (NCTM, 2000). Similar practices can be found in the CCSSM.

The CCSSM includes a list of eight practices for mathematical behavior and activity that teachers are expected to foster in their students while engaging them in mathematical content (CCSSI, 2010). These eight mathematical practice standards indicate students should be able to:

- make sense of problems and persevere in solving them;
- reason abstractly and quantitatively;

- construct viable arguments and critique the reasoning of others;
- model with mathematics;
- use appropriate tools strategically;
- attend to precision;
- look for and make use of structure, and;
- look for and express regularity and repeated reasoning (CCSSI, 2010).

These eight mathematical practices, largely derived from the National Council of Teachers of Mathematics (NCTM) Process Standards and the National Research Council's (NRC) Adding It Up report (Rasmussen et al., 2011), represent the first instance in which mathematical practice standards have been explicitly discussed and included in a standards document. Prior to the CCSSM, mathematical practices had been woven implicitly through state standards documents and national recommendations such as that from the NCTM.

The NCTM and CCSSM standards reflect what are considered to be the well-established practices of professional mathematicians. The expectation of engaging students in the classroom in these authentic activities, discourses, and behaviors of mathematicians – such as making conjectures and claims, defending those claims, and critiquing the arguments of others – students will develop a means of understanding and approaching practices of mathematicians (Moschkovich, 2013). Moschkovich cautions however that what is often considered to be the authentic activity of mathematicians could in fact be idealized versions of what mathematicians do, since many writings regarding the work mathematicians have been self reports and empirical studies, and not ethnographic in nature.

The research that has been conducted regarding practices of mathematicians has indicated there is no one set of practices for the field, but rather, the practices vary from mathematician to mathematician, and even across time (Burton, 1999). Furthermore, the research has indicated the practices in which mathematicians regularly engage are much more broad, and even sometimes contradictory to the practices considered as established in the mathematical community (Burton, 1999). In a study of 70 professional mathematicians, in which they were interviewed about their profession, beliefs, and practices, Burton found the majority of participants described their work as being a social activity, and that this was a more recent shift in the mathematical culture. Furthermore, while the general perception of mathematics is that it is about *knowing what to do* and then *solving problems with certainty*, the mathematicians described using intuition and having to be accepting uncertainty to a degree (Burton, 1999).

In addition to the general overarching practices of mathematicians, such as the nature of mathematical activity, research has been conducted on more particular practices of mathematicians, such as problem solving (Carlson, Bloom, & Glick, 2008; Schoenfeld, 1985), proof production (Raman, 2003), proof evaluation (Inglis & Alcock, 2012; Weber, 2008; Weber & Mejia-Ramos, 2011), and writing (Burton & Morgan, 2000). While some similar patterns were found in particular realms, such as having similar reasoning cycles while attempting to solve problems (Carlson, Bloom, & Glick, 2008; Schoenfeld, 1985), the general finding was that there is no one way mathematicians do things, and as such, there are no uniformly established set of mathematical practices of all mathematicians.

2.2.2 Everyday People

A second category of mathematical practices is that of the everyday people in the world. I include in this category the group of people Lave (1988) dubbed "Just Plain Folk," as well as people who utilize mathematics for work but are not professional mathematicians. This represents a wide range of degrees to which people use mathematics, and the ways in which they use mathematics. For example, the practices of people in this group could include such activities as ordinary people grocery shopping and preparing meals (Lave, 1998), children calculating total costs and change in a busy marketplace (Nunes, Schliemann, & Carraher, 1993), nurses determining dosages of medication to administer to patients (Hoyles, Noss, & Pozzi, 2001), engineers constructing mathematical models (Gainsburg, 2006), or a child simply locating an address (Aracavi, 2002).

In each of these situations listed, the research highlights the fact there is no one way of doing things, and that rarely do people use school-taught algorithms to solve problems. Furthermore, when asked to solve the same or similar problems with school taught mathematics, the results are generally not as good as when the individuals utilize their own informal ways of reasoning (Hoyles, Noss, & Pozzi, 2001; Nunes, Schliemann, & Carraher, 1993). In a study of hospital nurses for example, Hoyles, Noss, and Pozzi (2001) found that nurses utilized many different proportional-reasoning strategies, often tied to the individual medications and the quantities or volumes of medications, when determining dosages for patients, yet none were school taught algorithms. And, when pressed for mathematical explanation for strategies, the nurses often made arithmetic errors. This gap in performance between informal and formal ways of doing mathematics has prompted a call for a reconciliation between the mathematics done in school and the mathematics done out of school (Carraher, T. N., Carraher & Schliemann, 1985; Richards, 1991).

2.2.3 Students in School

A third category of mathematical practices is that of students in school. That is, the particular ways in which students engage with mathematics in the classroom. This, of course, varies widely from school to school (Boaler, 1998) and classroom to classroom (Cobb, Wood, Yackel, & McNeal, 1992). For example, the practices in a more traditional classroom, in which students receive lectures and then solve problems using the methods presented, will vary significantly from a more inquiry-oriented classroom in which students develop mathematical understanding through a process of logic and discovery (Richards, 1991). Richards (1991) makes a distinction between these two varieties of mathematical activity and the associated discourse practices, calling the first one *school math* and the second one *inquiry math*. He relates these two categories to the mathematical activity of mathematicians, differentiating the mathematics that mathematicians write and the mathematics that they do before they write, calling these two categories *journal math* and *research math*, respectively. He suggests school math is more similar to journal math, in that it is the process of reconstruction of problem-solving activity, while inquiry math is more similar to research math, in that it is the process of discovery and construction of mathematics.

Some researchers have argued the mathematical practices in which students should engage is the authentic activity of mathematicians (Cobb, Wood, & Yackel, 1993; Brown, Collins, & Duguid 1989), while others suggest students should engage in mathematical practices of everyday people (Carraher, T. N., Carraher & Schliemann, 1985; Lave, 1988). Each of these recommendations has implications for the activity and mathematical practices that take place in the classroom. Engaging students in the authentic activity of mathematicians, for example, would include such activities as making conjectures, constructing and justifying arguments, and critiquing the arguments of others. Engaging students in the practices of everyday mathematics on the other hand, may have students developing mathematical understanding directly in the contexts in which the mathematics might be used.

While it appears the mathematical practices of professional mathematicians may seem in opposition to everyday mathematical practices, there are ways of bridging the two (Arcavi, 2002; Moschkovich, 2002). Moschkovich (2002) suggests that legitimating everyday mathematics through engaging students in academic discussions and constructing mathematical arguments about everyday situations – thereby making the informal, everyday practices into something resembling more formal mathematical activity – is one way to incorporate both the academic and the everyday practices such that students come to develop competency in both. Similarly, Arcavi (2002) suggests students should engage in the mathematization of everyday problem situations.

2.2.4 Significance

The framing of mathematical practice I used for this study, as discussed in Chapter 1, allows me the flexibility to examine both the mathematical practices of mathematicians and those of everyday activity as they emerge in the course. Furthermore, it allows me to investigate the interplay between mathematical practices within the two categories and the ways in which they function for students. Mathematical practices in the course in which my study took place included practices such as, imagining, playing, justifying, entertaining alternate possibilities, and adapting mathematical tools.

The course in which my study takes place is a rich setting for the practice of imagining, particularly since certain projections were difficult to predict without engaging the imagination. Imagining is a crucial mathematical practice related to developing mathematical intuition and conjecturing. Furthermore, imagining can lead a student to expand their mathematical horizons, similar to how Ryan, from Chapter 1, utilized the string to imagine how the circle would project onto the window and in turn determined that mathematical projections can be infinite.

2.3 Arts Integration

The terms arts integration, arts-enriched or arts-infused curriculum, learning through or with the arts, as well as similar phrases, can conjure up feelings of trepidation in educators and researchers, perhaps one reason being the lack of established definitions. When people speak of arts integration there is potential for it to mean many different things. There is no universal agreement about what constitutes integration of the arts into other subjects. In general though, arts integration can be described as the blending of one or more forms of the arts with any other non-arts subject for the purposes of teaching and learning. The degree to which the two are blended vary widely (Burnaford, Brown, Doherty, & McLaughlin, 2007). In this section I provide an account of three conceptualizations of arts integration, as well as four ways in which the arts can be integrated. I follow this with a review of the few mathematics and arts integration studies, specific to the visual arts. Finally, I describe the significance of arts integration in this study.

For ease of reading, when referring to the integration of art and another subject, I simply use mathematics as the second subject. The reader should note the word mathematics could be replaced by any other subject indicator, such as physics, reading, or history.

2.3.1 Conceptualizations

In a rather comprehensive review of arts integration literature, Burnaford, Brown, Doherty, and McLaughlin (2007) identified three common ways in which arts integration has been conceptualized in the literature. While the phrasing of the term for which integrating one or more subjects with the arts varies, such as art-infused, learning in and through the arts, learning with the arts, the authors chose to simply use *arts integration* to describe the collection. The three major conceptualizations the authors encountered are arts integration (a) *as learning "through" and "with" the arts*, (b) *as a curricular connections process*, and (c) *as collaborative engagement*.

The focus of arts integration classified as *learning "through" and "with" the arts* is the transfer of learning between the two subjects being integrated. For example, a teacher might incorporate artistic activities involving rotational symmetry in her math course. The intention of this might be for students to glean mathematical understanding of rotational symmetry from the activities. The students' activities of creating an artistic piece with rotational symmetry might include determining how to partition a circle into equal sized slices and determining how many slices would be needed to complete the design. The result may be that students develop an understanding of angle measurement, or, in more advanced settings, students might learn about cyclic groups.

When arts integration is considered *as a curricular connection process*, the focus is on finding a common big idea that unifies the arts with mathematics. For example, in my study, the big idea might be the relationship between linear perspective and the mathematical property that in projective geometry any two lines intersect somewhere. Researchers have cautioned that the curricular connection must be substantive, and not manufactured or superficial (Burnaford et al., 2007).

Finally, arts integration viewed as *collaborative engagement* centers on the partnership between the educating actors in each of the disciplines of art and mathematics. For example, a mathematics instructor could collaborate with a teaching artist to determine the best and potentially effective ways to incorporate arts into the mathematics curriculum.

In my study, I conceptualize arts integration as associated with two of these three categories. First, I view arts integration as a curricular connection process – as the example given is one that stemmed from the Foundations of Geometry course in which my study is set. Second, I view arts integration as learning "through" and "with" the arts. This relates to the example given in Chapter 1 in which Karryn possibly had to engage more deeply with the mathematical material to create her design. This increased engagement may have led to a deeper understanding of the behavior of lines in projective geometry.

2.3.2 Styles

In the preceding section, I discussed three conceptualizations of arts integration, but within each of these conceptualizations, the mode of arts integration can vary (Burnaford et al., 2007). Bressler (as cited in Burnaford et al., 2007) identified (a) *subservient*, (b) *co-equal*, (c) *affective integration,* and (d) *social integration* as four potential modes of arts integration. These modes are different from the three conceptualizations in that the modes address the style in which the arts are integrated.

A *subservient* mode of arts integration incorporates art into a mathematics course in such a way that the art is merely in service to the mathematics, or the art is simply a “handmaiden.” That is, the focus is placed on the main content of mathematics course and the art just sprinkled on top. For example, at the most superficial level of the subservient mode, students often learns songs in their mathematics courses as mnemonic devices for remembering formulas or procedures. Here, the “music” is only there to assist students in memorization.

In the *co-equal* mode of arts integration, the attempt is to give equal value to both the arts and the mathematics. That is, neither the art nor the mathematics is superior in any of its substance or learning objectives. The *affective integration* mode utilizes the arts in mathematics courses to conjure up feelings and emotions in students. In the affective integration mode, students are encouraged to express those feelings and emotions through the arts, and they learn to be creative across both disciplines. Finally, the *social integration* mode has the purpose of interconnecting cultures or communities through projects and partnerships. For example, a history class might engage in creating a scenic mural in a community, depicting aspects of the culture of those living in the community.

Rather than view the *subservient and co-equal* modes as dichotomous, I prefer to conceptualize them as a continuum. A fully subservient mode would be similar to the example I described of students learning a song as a mnemonic for remembering a formula, while fully co-equal would be just as it describes; that is, both subjects are given equal weight and attention. The arts integration in my study falls somewhere on that continuum, where the main focus of the course was projective geometry, but the arts motivated the mathematics in the course, and subsequently, the mathematics inspired the art creation.

2.3.3 Mixed Opinions

A mention of arts integration can receive strong reactions from artists, mathematicians, and educators alike. These reactions vary from exceptional support to vehement opposition, and appear to be informed by limited research and opinion.

2.3.3.1 Support. The initial support of arts integration stemmed from the emphasis placed on holistic learning and aesthetic experience by John Dewey (1934). Research in arts integration has indicated that arts integration provides students with learning experiences that are emotionally and intellectually encouraging (Deasey, 2002; Goldberg, 2011), stimulates more holistic and integrated ways of understanding ideas (Mason, 1996), and promotes meaning making (Efland, 2002) and creativity (Marshall, 2005). Furthermore, many proponents of arts integration cite transfer as a reason to engage students in the arts (Catterall, 2002), with the notion that motivation from and ideas learned through the arts will transfer to other subjects, and vice versa.

In studies of larger-scale arts integration programs, research suggests implementation of arts integration appears to be beneficial to students. In particular, students who are involved in art tended to stay in school longer (Catterall, Chapleau, & Iwanga, 1999) and have better performance in school (Catterall et al. 1999; Deasey, 2002; Smithrim & Upitis, 2005). Similarly, research has indicated students receiving instruction in music tend to perform higher on mathematics testing (Vaughn, 2000). However, these larger studies show correlation, but not causation.

2.3.3.2 Challenges. One of the reasons for opposition to arts integration relates to the subservient style of arts integration and the evidence that it exists (Mishook & Kornhaber, as cited in Burnaford et al., 2007). Artists are concerned that integration of the arts, particularly when the arts are used as merely a handmaiden to other subjects, will devalue the arts and result in further funding cuts in arts education (Horowitz, 2004). And, while I have not come across any literature that explicitly states mathematicians have similar concerns about arts integration, the concerns have arisen in my personal communications with mathematicians – however, mathematicians appear to be worried about the devaluing of mathematics, but not worried so much about funding cuts to mathematics.

Other criticisms of arts integration include teachers' concerns about meeting potential arts integration related curriculum requirements while already having a packed-full curriculum (Horowitz, 2004), and, for others, the serious lack of empirical research associated with arts integration.

2.3.4 Arts Integration in Mathematics Education

Connections between mathematics and the arts are well-accepted, evidenced by such things as the annual International Conference for Mathematics and the Arts (the Bridges conference), a Special Interest Group of the Mathematical Association of America (SIGMAA) on mathematics and art, and the *Journal of Mathematics and the*

Arts that is entirely dedicated to the subject. Some have even touted mathematics *as* an art (e.g. Lockhart, 2009) – however that claim has been met with resistance from other mathematicians (Hickman & Huckstep, 2003). It is somewhat surprising then that the research on arts integration in mathematics courses is so limited.

Much of the research regarding arts integration and mathematics is with respect to larger-scale, school-wide or district-wide arts integration programs that examine the academic performance of students. Several of these studies have shown the implementation of arts integration programs have resulted in increased mathematics exam scores with respect to control schools (Catterall et al. 1999; Deasey, 2002; Smithrim & Upitis, 2005). At least one study however showed that females who participated in an arts integration program scored worse than control groups on mathematics abilities (Luftig, 2000). In each of these studies, only correlation and not causation was able to be determined.

Limited empirical research has been conducted to specifically examine arts integration in mathematics courses. In fact, in the large arts integration literature review by Burnaford et al. (2007), only one instance of visual arts integration in a mathematics classroom was cited. There are several accounts of arts integration efforts in mathematics courses in which, after describing the artistic activity, the author, usually the instructor, anecdotally, by means of instructor observation of student behavior, suggests a benefit to students. Below, I describe the limited arts integration studies pertaining to visual arts in the mathematics classroom, beginning with two accounts of activities that anecdotally suggest students benefit from arts integration. In addition, I describe a visual-arts integration study in a science course.

2.3.4.1 Anecdotal Studies

2.3.4.1.1 Fraction, decimal, and percent equivalents. In an implementation of an artistic method for teaching students about fraction, decimal, and percent equivalents, Scaptura, Suh, and Mahaffey (2007) reported that creating artistic pieces and relating them to mathematics was an effective learning experience for students. Engaging students in an art project inspired by the artwork of Ellsworth Kelly (see *Figure 9*), students attached small colored squares onto a 10 x 10 grid to create an artistic design (see *Figure 10)*. Upon completing their designs, students counted the number of squares of each color on their grid and wrote that number on a provided worksheet. The worksheet contained five columns: color, number, fraction, decimal, and percent. Students were tasked with determining the number to place in each category for each color of square.

The authors suggested this activity is an effective way to help students clearly visualize the relationship between fraction, decimal, and percent representations. They stated the activity gave students a way to verify their answers without the help of the teacher – which seemed to particularly benefit students for whom English was a second language. Additionally, some students were able to recognize numerical pattern shortcuts. All the results presented by the authors were observations from the implementation of the activity, as there was no formal research design.

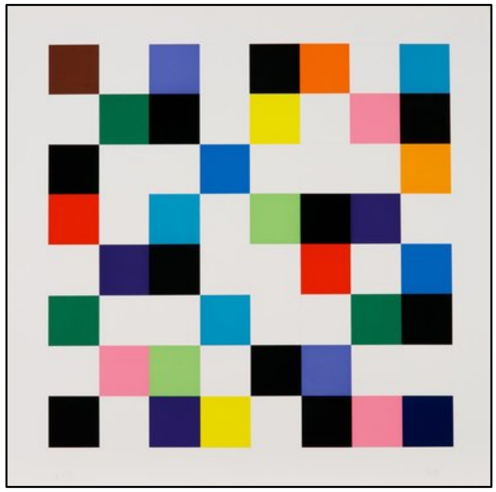


Figure 9. Ellsworth Kelly, *Colors on a Grid (image retrieved from* http://www.lacma.org/art/exhibition/ellsworth-kelly-prints-and-paintings)

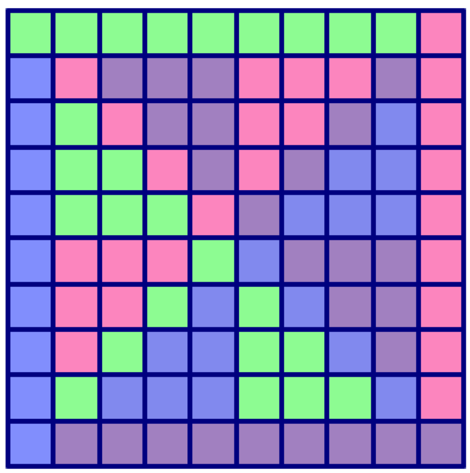


Figure 10. Reproduction of a student fraction grid (Scaptura et al., 2007).

2.3.4.1.2 Geometry and Dance. In this collaboration, a geometry professor and a choreographer developed activities to engage undergraduate students from a math class and from a dance class in learning about Platonic solids (Parsley & Soriano, 2008). The faces of a Platonic solid are all congruent regular polygons; thus, the Platonic solid is a regular polyhedron. First, using pipe cleaners and straws, the students constructed as many Platonic solids as they could and discovered there are only five possible. Subsequently, the students met at the dance studio and were asked to interpret the Platonic solid activity they had engaged in. Groups of students attempted to construct the shapes with their bodies - limbs of the students became edges of the shapes, and whole bodies sometimes became faces or edges.

The activity proved challenging for the students, however, the students appeared to have developed their spatial reasoning skills. Students reported, through discussion and surveys, they felt they could better visualize these solids and how the solids could be rotated in space. One student mentioned how this activity made him realize how important congruency of angles and edges were for Platonic solids. Overall, the students reported enjoying the experience of embodied learning.

Other accounts of courses that integrate the arts include integrating Frank Lloyd Wright's architecture and design into a liberal arts math class (Ashton, 2010), modeling college algebra problems with poetry (Glaz & Liang, 2009), and using graph theory for making Celtic knots and Platonic solids (Hayes, 2007).

2.3.4.2 Empirical Studies

2.3.4.2.1 Escher's World workshops. In the spring and summer of 1995, 12 high school students from Boston public schools were brought to the Escher's World mathematics studio at the Massachusetts Institute of Technology (MIT) for a 12-hour educational experience (Cossentino & Shaffer, 1999; Shaffer, 1997). The students participated in workshops in which they reflexively learned about art and mathematics by creating graphics posters using the mathematical ideas of rotational and reflective symmetries. Students created their poster designs using the dynamic geometry software Geometer's Sketchpad (Jackiw, 1995), and thus students were also learning about technology.

Based on observations of student behavior and structured interviews, Cossentino & Shaffer (1999; Shaffer, 1997) found that all but one of the students were able to explain ideas about symmetry after the workshop, where only one was able to prior to the workshop. Students developed the ability to apply the ideas of reflective and rotational symmetry to both the creation and artistic analysis of images, whether artistic or otherwise. The students' analyses of artistic images became more intricate, demonstrated by their abilities to notice more aspects of symmetry in designs. Finally, more students used visual problem solving strategies, and students reported liking mathematics more.

2.3.4.2.2 Arts integration and instructors. Other studies have examined how arts integration can affect and inform the instructors of a course (Jacobson & Lehrer, 2000). For example, in a study in which second-grade students learned about geometrical transformations by quilt making, Jacobson and Lehrer (2000) examined the ways in which teachers mediated classroom conversations about the mathematical aspects of the quilts. They reported that even when teachers are skilled and experienced, the effectiveness of the teachers' implementations of the quilt lesson depended upon the ways in which the teachers facilitated classroom discussion – with those teachers who were more knowledgeable about student thinking regarding geometry and space, guiding the discussion to refine and extend students' thinking.

A study in physics examined the integration of symmetry ideas in an undergraduate physics course, where students participated in several different arts integration aspects, including creating sketches based on readings and lessons (van der Veen, 2012). The authors suggest student drawings can provide insight for the teacher into what kinds of learners the students are, as well as what concepts they understand. In addition, the authors suggest these student-created sketches can reflect the changing attitudes and perspectives of students' relationships to the subject (van der Veen, 2012). As such, the drawings can be used as an assessment tool. This is consistent with the findings from Katz et al. (2011) who used student sketches to assess changes in student perceptions of themselves at the beginning of and at the end of an informal science education program.

2.3.5 Significance

The purpose of my study was to determine the possible ways in which artistic engagement can enrich students learning experiences and opportunities in a mathematics course. That is, the purpose is not to measure whether arts integration aids in students' mathematical performance. As such, my study will add to the small base of literature in mathematics education regarding arts integration. In particular, my study will add to the literature that is not simply an account of the activities and events that occur in the classroom surrounding the integrated arts.

Chapter 3
Methodology

In this chapter I describe the course in which this study takes place, the data collection methods, and the data analysis methods. In addition, I delineate which of the data components will be primary and secondary sources for analysis. In an effort to streamline this chapter and limit redundancy, I describe the entirety of data collection and then address data analysis.

The setting for this research project was an activity-based Foundations of Geometry course at a large southwestern university that took place in the fall semester of 2012. The students in the course worked in groups on novel problems to develop an understanding of projective geometry. The available data for this course included classroom video data of four groups of students, individual interviews with 15 students, as well as student homework and exams. The subsequent sections of this chapter will provide more detailed descriptions of the course itself and the data collected in the course. I first describe the participants in the study, followed by a detailed description of the course format, setting, and content, including the arts integration aspects of the course. I then provide a description of the available data that was collected in the course, followed by a description of my methods for data analysis. Finally, in the last section of this chapter, I address reliability and validity.

3.1 Study Backdrop

3.1.1 Participants

The participants for this study consisted of 16 students enrolled in a Foundations of Geometry course with 29 students at a large Southwestern university, during the fall semester of 2012. All of the participants volunteered to participate in the study and were selected on the basis of their availability to participate in interviews, for which they were provided with incentives.

The participants were primarily prospective secondary mathematics teachers and were in the process of earning Bachelor of Art degrees in mathematics, in preparation for earning a single-subject teaching credential. In particular, 10 of the 16 participants were in the midst of this course of study, with intentions of earning a single-subject teaching credential. Of the remaining six participants, two were earning a Bachelor of Science or Arts in Mathematics, one was earning a Bachelor of Science in Chemistry, one was earning a Bachelor of Science in Mechanical Engineering, and two had begun their graduate studies towards earning a Master of Arts degree in Teaching Service. Given that 10 participants were prospective secondary mathematics teachers and two participants were earning a degree in Teaching Service, it can be said that at least 12 of the 16 participants intended to teach mathematics in the future. Nine of the participants were female and seven were male, with ages ranging from 19 to 47.

While the overall data includes 16 participants, I focused my analyses on five participants in particular. The selection of the five participants for analyses, as well as a brief description of each of those five participants, is addressed in the data analysis section of this chapter.

3.1.2 Classroom Setting

The setting for this study was a Foundations of Geometry course with 29 students enrolled. The students sat at tables together, facing each other, in groups of three or four. Eight tables were arranged in a U-shape around the classroom, with a large, open floor space in the center of the room (*Figure 11*). At times, the instructor, or groups of students, would utilize this center floor space for demonstrations or explanations of their ideas using the mathematical tools of the course.

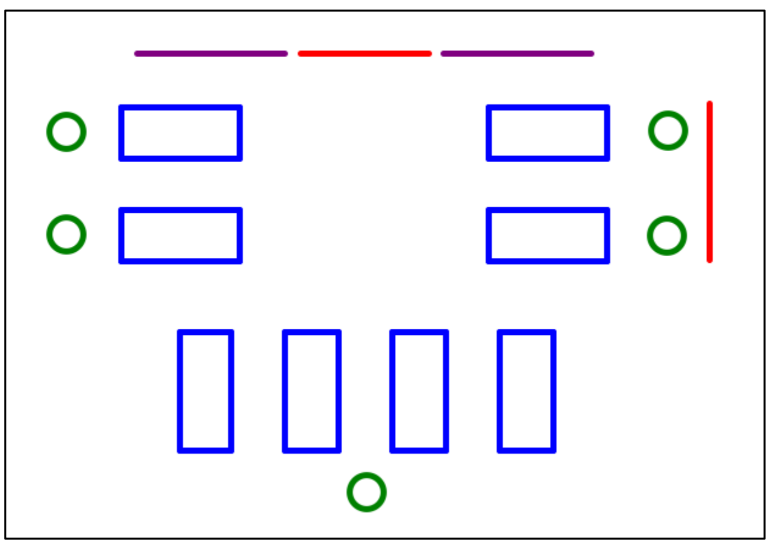


Figure 11. Diagram of the classroom layout, where blue rectangles represent tables, green circles represent camera locations, purple lines represent whiteboards, red lines represent projector screens, one of which is a SmartBoard.

The instructor of the course, Dr. R., often used Geometer's Sketchpad (GSP) (Jackiw, 1995) in conjunction with a SmartBoard located at the front of the classroom. Whiteboards that Dr. R. occasionally utilized were located on either side of the SmartBoard. At times, students utilized the SmartBoard or the whiteboards in the presentations of their ideas to the class. A large screen that displayed an identical image to that of the SmartBoard was located on the right-hand wall of the classroom.

3.1.3 Course Format

This Foundations of Geometry course was an activity-based course, in the sense that students were given specific, directed, in-class activities that they worked on in assigned groups of three or four. As each activity progressed, the Dr. R., and I, as the co-instructor of the course, interacted with the groups of students. At times during the activities, the Dr. R. would call the class back together to discuss how groups were thinking about the activities and the associated mathematical ideas. As a result, the mathematical ideas in the course were generally drawn out from the groups of students, rather than explained by Dr. R. or myself.

As stated above, my personal role in the course was as the co-instructor. This means that I interacted on a daily basis with the groups of students by answering questions, drawing out their ideas, asking probing questions, and guiding their mathematical activity as needed. My role as a researcher was known to the entire class, however my researcher role was secondary to my role as the co-instructor. As such, I acted in a participant-as-observer role (Creswell, 2013; Merriam, 2009), and, consequently, realize that my activity in the course affected the data. I discuss further the consequences of my role as participant-as-observer in the reliability and validity section of this chapter.

3.1.4 Course Content

While the setting for this study took place in a Foundations of Geometry course, the primary topic of focus in the course was Projective Geometry. As discussed in Chapter 2, Projective Geometry is a branch of mathematics that originated as an artists' tool during the Renaissance era in an effort to formalize the process by which an artist could create a realistic drawing or painting of a three-dimensional object or scene, thereby representing in two dimensions something that is three-dimensional (Andersen, 2007; Field, 1997; Kline, 1957). Subsequently, those interested in the governing mathematics involved in this artists' tool developed and extended the tool to become a conventional branch of geometry (Kline, 1957). My reference the artists' tool here is not suggesting a physical object, but a formalized process by which an artist could create an artistic piece true to linear perspective.

In general, projective geometry in two or three dimensions involves how objects on one line or plane, project onto a second line or plane, respectively, through a center of projection (see *Figure 12 & 13*). More specifically, projective geometry is the study of the aspects or properties of objects that remain invariant through projection. The projection is determined by extending lines from a given point, called the center of projection (your eye, for example) to points on an image or object residing on a plane (say a table top in front of you). These lines are called the lines of projection. The image determined by the intersection of the lines of projection with a second plane (say a window in front of you), is the projection of the original image or object (*Figure 12*). (Recall from Chapter 2, what I call *projection* here, historically has been called a *section*.) A similar treatment can be used to determine the projection of points on a line to points on a second line (*Figure 13*). Projective geometry is considered a non-Euclidean geometry, since we take as an axiom that any two lines will intersect. Thus, the Parallel Postulate does not hold.

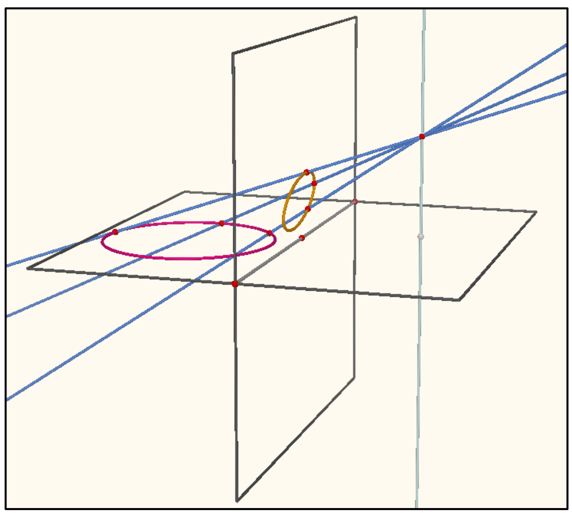


Figure 12. The orange ellipse on the vertical plane is the projection of the magenta circle on the horizontal plane. The blue lines of projection connect points on the circle with a single point, called the center of projection. The places at which the blue lines of projection intersect the vertical plane result in the projection of the magenta circle.

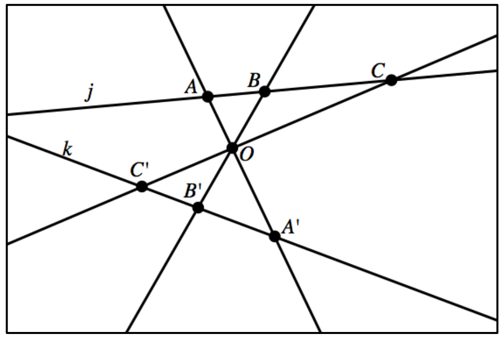


Figure 13. The points *A*, *B*, and *C* on line *j* project through point *O* to the points *A*', *B*', and *C*' on the line *k*. *O* is the center of projection.

The course content was subdivided into three sections: (a) the physical and spatial aspects of projections in three dimensions, (b) synthetic projective geometry in two and three dimensions, and (c) analytic projective geometry. While the data I considered for my analyses only includes the first two sections of the course, I give a brief overview below of each of the three sections.

3.1.4.1 Section One: Physical and Spatial Projective Geometry. Students began their exploration of projective geometry ideas through the use of a mathematical tool called the Alberti's Window (*see Figure 14*), which consists of two primary components. The first component of the Alberti's Window is a 12x12 inch square sheet of clear acrylic that stands on a mount perpendicular to the surface on which it sits, generally a tabletop. The second component is an adjustable eyepiece, also constructed from acrylic sheet, through which the user views drawings or objects. The proximity of the eyepiece to the window can easily be adjusted by simply moving the eyepiece to a new location. The height of the eyepiece is adjustable by loosening a wing nut on a screw and sliding the eyepiece up or down.

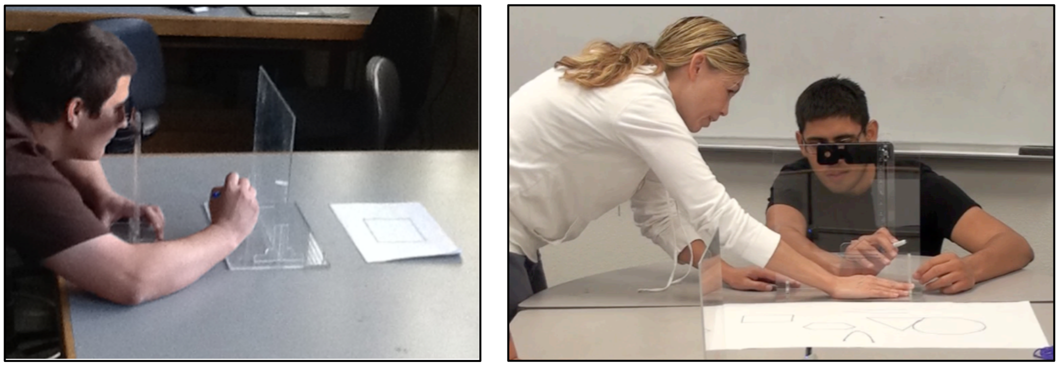


Figure 14. Students use the Alberti's window by looking through the eyepiece and tracing onto the window with a marker the object they see in front of them.

The initial exploration of projective geometry using the Alberti's Window consisted of the students looking through the eyepiece with one eye at an image or object that was located on the tabletop, on the opposite side of the window from the viewer, and tracing onto the window with a dry-erase marker, the drawing or object they saw in front of them (*Figure 14*). In this situation the viewer is tracing the projection of the image, which sits on the tabletop, onto the window, where the center of projection is the viewer's eye. In this scenario, a circle will project to an ellipse and parallel lines will project to lines intersecting at a vanishing point on the horizon line. The horizon line here is the horizontal line on the window that is the same height above the tabletop as is the hole in the eyepiece.

After ample exploration with objects located on the opposite side of the Alberti's Window from the viewer, students were asked to imagine the window stretched infinitely in all directions. The image on the tabletop was then positioned between the viewer and the window. Students were asked to extend the notion of a line connecting the center of projection with a point on the image located on the tabletop. The point at which the line intersects the window is the projection of that particular point on the image. The students were subsequently asked to further extend the notion of the lines of projection to images, such that images located behind the user, as well as surrounding the eyepiece (see *Figure 15*) could be projected. In these cases, students needed to utilize their imaginations, as they were not able to physically trace the projection on the window in the same way they were able to when the image was on the opposite side of the window from the viewer.

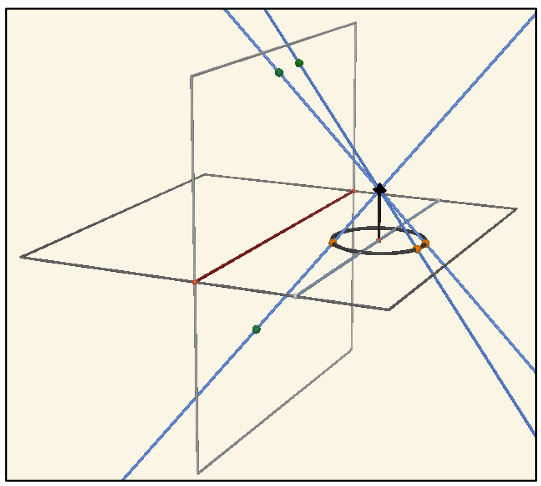


Figure 15. Depiction of lines of projection for points on a circle that surrounds the eyepiece. Points in front of the viewer project onto the window below the horizontal plane. Points behind the viewer project onto the window above the horizon line.

Two particular days of activities with the Alberti's Window deserve special mention here, as in previous classes, students cited these activities as influential in their thinking. On these two days students in the class used the Alberti's Window to imagine the projections of a very large-scale parabola, on the first day, and a single branch of a hyperbola, on the second day. On these two days (one day for each shape), as a class, the students constructed the indicated shape on a soccer field. The students were provided with the coordinates of particular points on the shape, several long measuring tapes, and small, low-profile soccer cones to serve as the coordinate points on the field. Once the given shape had been constructed, the students were asked to determine the projections of the shape in three different scenarios: (a) standing just below the vertex of the parabola, with the window between the viewer and the parabola (*Figure 16*), (b) standing somewhere in the middle of the parabola, with the window directly in front of the viewer (*Figure 17*), and (c) standing on the vertex of the parabola, with the window in front of the viewer, but out of the viewer's physical reach (*Figure 18*) .

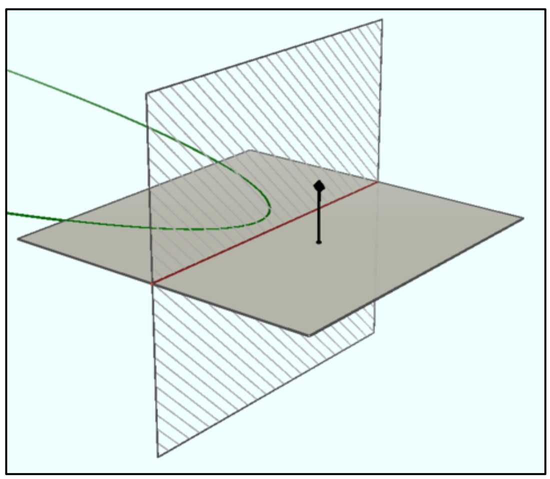


Figure 16. Viewer below the vertex of the parabola, with the window between the viewer and the parabola.

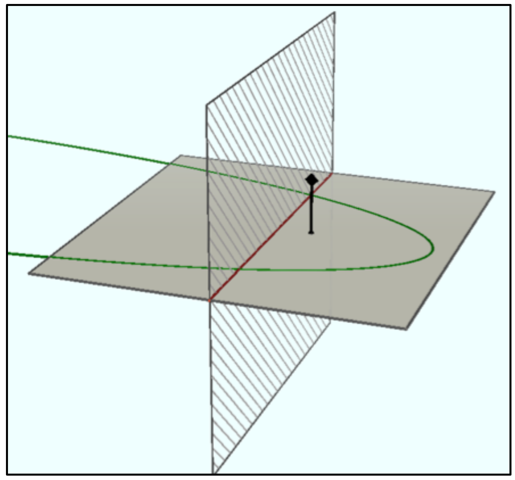


Figure 17. Viewer standing in the middle of the parabola, with the window directly in front of the viewer.

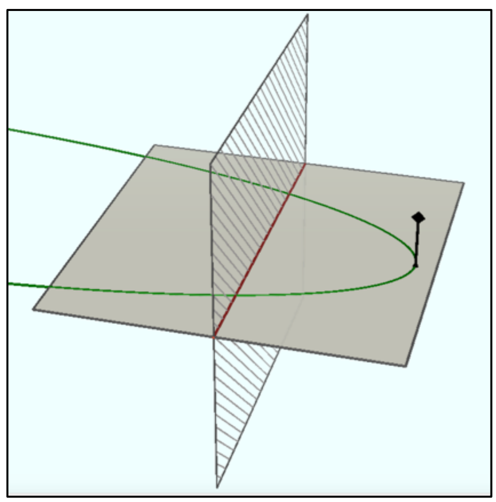


Figure 18. Viewer standing on the vertex of the parabola, with the window in front of the viewer, but out of the viewer's physical reach.

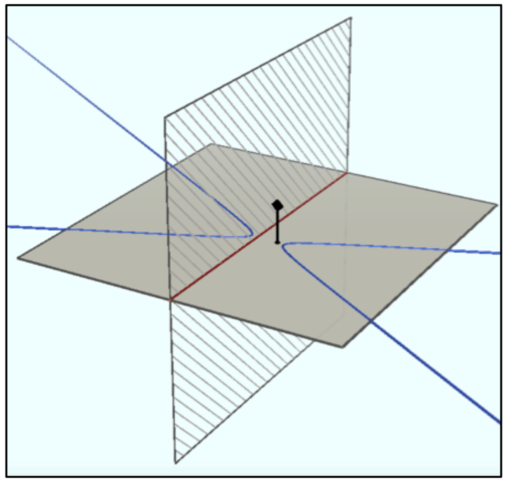


Figure 19. Viewer standing between two hyperbola branches, with the window in front of the viewer.

The students traced on the window the part of the shape they saw in front of them, but then needed to imagine the remaining sections of the shape – specifically, the sections of the shape that were located behind the viewer or between the viewer and the window. Additionally, for the situation of the hyperbola, since the students were asked to only construct one branch of the hyperbola, they needed to imagine the second branch, which was located behind the viewer in each of the three scenarios (*Figure 19*). Not only did the students have to imagine the branch of the hyperbola on the field, they also had to imagine how that branch would project onto the window.

Following a debriefing of the activities on the soccer field, the students were introduced to a Geometer's Sketchpad version of the Alberti's Window. Geometer's Sketchpad (GSP) (Jackiw, 1995) is a dynamic geometry software environment in which various shapes, lines, and other mathematical objects can be made, as well as dragged to other locations in the sketch (on the computer screen). These mathematical objects can also be transformed using a number of transformations, such as rotations, reflections, translations, and dilations. As a two-dimensional representation of the three-dimensional Alberti's Window situation, the GSP sketch shows an overlaying of both the horizontal plane (the tabletop, for example) and the vertical plane (the window) (see *Figure 20*). This is obtained by rotating the horizontal plane 90 degrees about the intersection of the two planes.

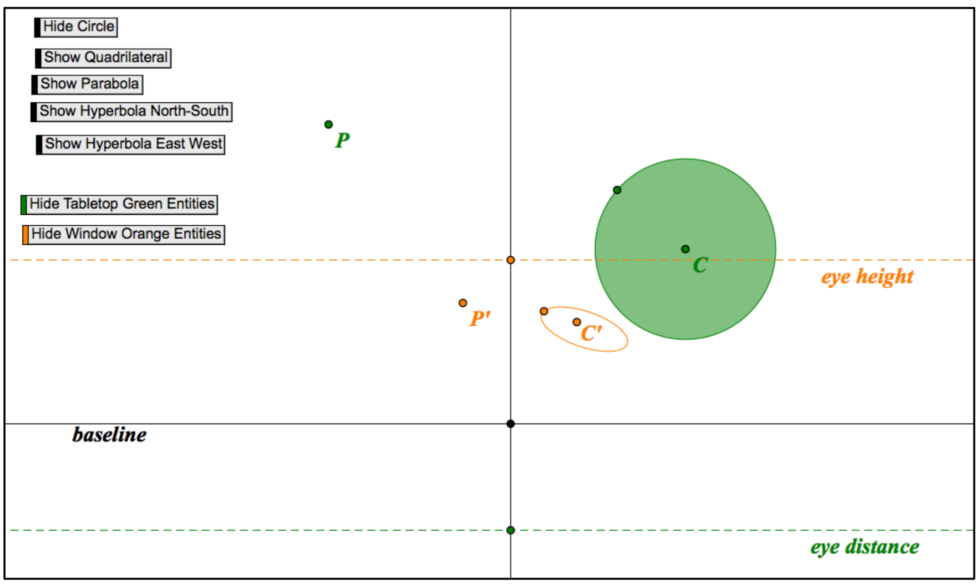


Figure 20. Screen shot of GSP version of Alberti's Window. The green entities are those on the tabletop and the orange entities are those on the window.

Since this GSP sketch of Alberti's Window represented two overlaying planes, objects in the sketch that were located on the horizontal plane (the tabletop) were colored green and objects located on the vertical plane (the window) were colored orange. The sketch contained a set of black perpendicular lines, where the horizontal line represented the intersection of the horizontal and vertical planes, and the vertical line represented the line on which the viewer stands. In addition, since the distance from the eyepiece to the window, as well as the height of the eyepiece, could be adjusted with the physical Alberti's Window, there were adjustable lines in the sketch to represent each of these. That is, in the sketch, there was an eye-distance line, which represented the distance from the eyepiece to the window, and an eye-height line, or horizon line, which represented the height of the eyepiece from the tabletop. Dr. R. built a special transformation into the GSP sketch such that students could project any image they chose.

This GSP sketch of the Alberti's Window was particularly significant, as the students used this sketch during the course to create an artistic design for the midterm and final projects, which I discuss in detail in a subsequent section of this chapter.

3.1.4.2 Section Two: Synthetic Projective Geometry. In this section of the course, students continued to work in groups of four, but began an exploration of the axiomatic aspects of projective geometry, such as definitions, axioms, and theorems. The students discussed their ideas of what the axioms of projective geometry must be, explored the principle of duality, developed proofs for various properties of projection, and presented to the class the reasoning and justifications for their work. Two notable theorems the students explored were Desargues' Theorem and Pappus' Theorem. On most days, groups used a whiteboard, ruler, and dry-erase markers to explore the axioms, the duals of the axioms, and the particular theorems. At other times, groups used GSP to construct particular projections and perspectives.

3.1.4.3 Section Three: Analytic Projective Geometry. In this final section of the course, students were introduced to analytic aspects of projective geometry through homogeneous coordinates. In this section, students used Cabri 3D software (Bainville & Laborde, 2004) for their explorations. Cabri 3D is a dynamic geometry software environment, similar to GSP, however allows for three-dimensional constructions. In this section of the course, groups explored how the axioms of projective geometry continue to hold in a homogeneous coordinate system.

I did not include the data from this section of the course in my analyses for two reasons. First, this section of the course was much shorter and more lecture-based than the other two sections, which resulted in more limited occasions in which students explained their thinking during group work. Second, the designs for the artistic pieces that students created had already been submitted by the time this third section of the course began, and as such, students were not influenced by this section of the course in their design creation.

3.1.5 Artistic Engagement.

The Foundations of Geometry course had several forms of artistic engagement. First, the introduction of projective geometry was motivated using ideas from the arts. Second, students in the course participated in creating two artistic pieces using ideas from projective geometry, and wrote reflective essays in which they discussed their experiences creating the artistic pieces. I discuss these artistic pieces in detail in the next section. Third, students read two writings related to art. The first reading was regarding mathematics as an art, and the way in which traditional school mathematics is stripped of what the author considers to be real mathematics (Lockhart, 2009). The second reading focused on the emergence of modern art (Gompertz, 2012), particularly as it relates to Marcel Duchamp and his widely recognized work, *Fountain* (1917). In addition, students wrote reflective essays and participated in whole-class discussions about each of the readings. And fourth, students spent an afternoon on a fieldtrip to a Museum of Contemporary Art, where they participated in a museum tour and completed worksheets probing their experiences at the museum.

3.1.5.1 Art Project Description. Two times during the course, students participated in creating a personal artistic project. One of these projects was completed approximately in the middle of the course and the second one near the end of the course. For these projects, students used the GSP version of the Alberti's Window to create a design, such as a visual pattern or scene, based on constructing and projecting geometric figures. The intention of these projects was not for students to demonstrate their understanding of projective geometry through their artistic design, but rather to find inspiration in whatever piqued their interest – such as a personal experience, a fascination with projections of particular shapes, or a mathematical theorem – and then use the properties and techniques of projective geometry to create an artistic design for the sake of art creation. As a result, the designs students created do not necessarily look like canonical images of projective geometry.

The process of creating the artistic projects began with the students using the GSP version of the Alberti's Window to create an artistic design, as mentioned previously. It was required the design fit within a 10 inch x 13 inch frame, and it was required projective geometry play a fundamental role in the design. Students were allowed to include both projected and non-projected elements. Once the students completed their designs, a stencil of each design was cut using a Craft Robo Pro, which is a vinyl cutter that functions like a printer, cutting black lines rather than printing them. The students were provided with a sheet of 12-inch x 15-inch airbrush-quality paper and used the stencil in conjunction with an airbrush to paint their design in any way they desired (*see Figure 21* for an example of a stencil design and a finished painting from previous data).

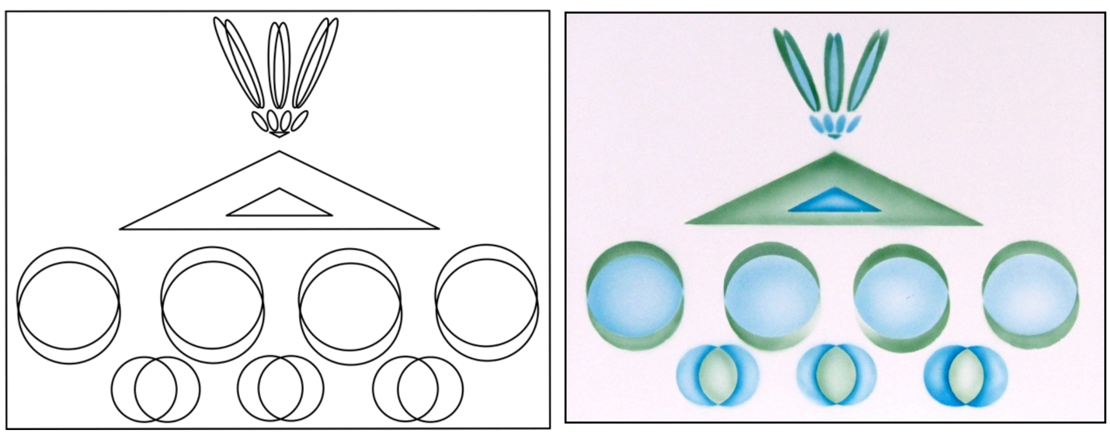


Figure 21. A student's initial stencil design (*left*) and the completed painting (*right*).

Upon finishing their artistic projects, students composed written reflections regarding their experiences creating their artistic pieces. In addition, at the end of the course, students recorded a video reflection about their experiences creating their second artistic project.

A skeptical reader might argue the students in this course were not in fact "creating art," to which I provide a brief response. Suppose you observe a group of college students engaging in a pick-up soccer game, it is quite unlikely you would respond that the individuals are not "playing soccer." While the college students are not professional soccer players, they are still "playing soccer," albeit in the sense of engaging in a *recreational* game of soccer. I would like to encourage the skeptical reader to consider the students' artistic projects in the same way as the recreational soccer game. That is, the students can be considered to be creating art, but in the sense of *recreational* art.

3.2 Data Collection

Data collection for this project consisted of multiple components. In particular, the collected data includes (a) classroom audio and video, (b) audio and video of outdoor class activities, (c) participants' in-class notes, (d) audio and video of individual and group interviews, (e) students' individually created art projects, (f) students' written and video reflections regarding their art projects, and (g) all participants' completed homework assignments and exams. In this section, I provide a description of the data collection processes. As I describe each component of data collection, I indicate whether the data served as a primary or secondary data source.

3.2.1 Classroom Data

Mathematics classrooms, and in particular mathematics classrooms in which students are asked to work in groups on directed activities, are complex environments. At any particular moment, each group in the classroom could be having a very different discussion about the same topic, members in a single group could be engaged in one or more conversations, or an individual group member could be pursuing his or her personal line of reasoning, rather than engaging with the rest of the group. Additionally, the activity within any classroom might include occurrences of conversations, gestures and bodily movements, as well as computer-generated or written inscriptions. To capture the multitude of occurrences within such a complex environment, and in such as way as to capture as many of the occurrences as possible without disrupting the environment, significant data collection, in multiple forms, was carried out. In this particular study, data collection included classroom video of four groups of four students each, audio data for all four groups, and Livescribe SmartPen data from all but one of the 16 participants (One participant, after two classes, stated he normally does not take notes and he found it distracting to use the pen.). In the next sections I discuss in further detail the particulars of the classroom data collection processes.

3.2.1.1 Classroom Video Data. Five high-definition video cameras with wide-angle lenses were positioned in the classroom (refer to *Figure 11* for camera locations) and were used on a daily basis, beginning on the third day of the course and ending on the second to last day of the course. This resulted in a total of 27 consecutive class sessions of collected data. One camera was positioned to capture each of the four groups, at such an angle as to capture as much of the bodily movement of the participants as possible. The remaining camera was placed at the back of the classroom to capture activity in the center, open area of the classroom, as well as activity at the front of the classroom. This final camera placement was necessary as, on a regular basis, particular groups or particular participants from different groups were asked to assist in demonstrations or explain their thinking at the front of the classroom. *Figure 22* shows the view from each of the five cameras.

Figure 22. Approximate view from each of the five cameras used for classroom data collection.

While a total of 27 consecutive class sessions of classroom video were collected, the focus of my analyses, as mentioned previously, was the first and second components of the projective geometry content. The first component of the projective geometry content began on day three of data collection, and the second component concluded on day 23 of data collection. Therefore, 22 days of classroom-video data was analyzed. The classroom video data served as a primary data source for RQ1 and as a secondary data source for RQ2.

3.2.1.2 Classroom Audio Data. Since the classroom was a highly active environment with many people engaging in conversations at the same time, the camera audio was not sufficient for any detailed analysis. Therefore, it was necessary to collect better quality audio data by isolating the audio of each group. This was accomplished through hanging a high-quality microphone above each group's table. The audio streams from each of the four microphones were recorded using a digital audio interface that recorded each line of audio directly onto a MacBook Pro computer using AudioDesk software. Whenever possible, one person on the data collection team listened to the incoming audio at all times to ensure that all four audio streams continued to record. The microphone audio was later synchronized with the camera video using Final Cut Pro.

The audio from the camera at the back of the classroom was determined sufficient, since when the center or front of the classroom was the focus of activity, only the instructor or a few students were talking, and typically only one individual at a time. Therefore, the quality of the audio was such that the utterances from individual students could be heard clearly.

3.2.1.3 Outdoor Class Activity. Data collection for the outdoor activity required additional assistance from fellow researchers. Since the groups of students were mobile during the outdoor activities, it was necessary for the video cameras to be portable. For each group of four participants, one videographer shadowed the group, using the same high-definition camera as was used in the classroom. In an attempt to steady the cameras, for ease of future analysis, each camera was attached to a monopod. This allowed videographers to steady the camera against the ground whenever possible.

To enhance the audio of each group, two of the four participants in each group were equipped with wireless microphones to capture group conversations. The receivers of the wireless microphones ran directly into the cameras. This meant the collected audio was only that from the wireless microphones. Generally this was sufficient, as the group participants typically remained together – thus the two microphones per group were able to capture the speech of all four group participants – however, occasionally, one or more group members would wander out of the range of the two microphones, resulting in no audio for those particular group members. This was a rare occurrence and did not significantly affect data analysis.

3.2.1.4 Smart-Pen data. In a pilot study, students often used handwritten notes to assist in the explanation of ideas of projection to other students. As a result, it was sometimes difficult to analyze how students were considering these ideas of projection. In an attempt to limit the occurrences of this phenomenon, participants in this study were asked to use Livescribe Smart Pens to take notes as they regularly did. The Smart Pen, when used correctly, will generate a real-time reproduction of an individual's writing, as well as the surrounding audio. When used incorrectly, only the final written work is captured. The participants used the pens correctly approximately 80% of the time. However, having the participants' final written work was more informative than having no written work at all. Fifteen of the sixteen participants agreed to use a Smart Pen. The Smart Pen data was used as a secondary data source for each of my two research questions, since participants in this study infrequently referred to their written notes.

3.2.1.5 Screen Captures. During several days of the course, groups engaged in using Geometer's Sketchpad on a MacBook Pro to explore ideas of projection and perspective. Each group was provided with one laptop for use. On these days continuous screen captures of the computer usage was collected using the software program Snag-it. Unfortunately, there were a few instances in which a screen capture was not saved. This occurred when one of the group members closed the computer before the Snag-it program file was saved and closed. This was a rare occurrence and did not significantly affect data analysis. In those instances in which the screen capture was not saved, the computer screen was often visible on the classroom video data. The screen captures served as secondary data sources for each of my two research questions, since, generally group activity could be determined from the classroom video data, and only occasionally was there a need to closely analyze the data from the screen capture.

3.2.2 Interview Data

Five sets of interviews were conducted, and all but one of the 16 participants volunteered to take part in the interviews. Four sets of interviews were individual interviews with participants, and one set consisted of group interviews. While the topics of each of the five sets of interviews varied, the general format was similar, with the exception of the group interview, in which two to five participants were present at one time. Each of the interviews, including the focus-group style interview was semi-structured (Bernard, 1988) in the sense that the interviewer began with a general interview protocol, but participant responses informed follow-up questions, as well as follow-up topics. I served as the interviewer for each of the interviews conducted.

Interviews were video-recorded using one or two high-definition video cameras – the same cameras used for classroom data collection. The purpose of using two cameras was (a) to capture each of the participants in the focus-group-style interview and (b) to capture the participant in front of the computer screen used in the interviews, as well as the facial expressions and gestures of the participant, requiring two cameras for each individual interview. The audio for the interviews was captured either by wireless microphones – one worn by the participant and one worn by the interviewer – or by one of the microphones hanging above one of the group tables in the classroom.

In the following sections I discuss the foci for each of the interviews I conducted with the participants. I address each of the five interviews in the order in which the interviews occurred during the semester. The second of the five interviews was the focus-group-style interview, while the remaining four interviews were with individual participants.

3.2.2.1 Initial interview. The first of the four individual interviews was conducted during the second week of the semester. In these interviews, I asked the participants to address questions related to their experiences with artistic endeavors, Euclidean and non-Euclidean geometries, and non-traditional mathematics instruction. The focus of these interviews was to get to know the participants' backgrounds in and dispositions toward mathematics and art. This initial interview was used as a secondary data source for RQ2. This interview was not used as a data source for RQ1, since the students had not engaged with projective geometry at that time.

3.2.2.2 Group interview. The second of the five interviews was a focus-group-style interview in which two to five participants were present. These focus-group interviews took place during the eighth week of the semester, after the student had visited the Museum of Contemporary Art and read the first chapter from A Mathematicians Lament (Lockhart, 2009), as well as the article regarding Marcel Duchamp and the emergence of contemporary art (Gompertz, 2012).

The central topics of these focus-group interviews were the participants' perceptions of the relationships between mathematics and art, their impressions of the chapter from A Mathematician's Lament, and the class activities and experiences they had found memorable up to that point in the course. These group interviews were used as a secondary data source for both RQ1 and RQ2.

3.2.2.3 Problem-solving interview. During the eighth and ninth weeks of the semester, a semi-structured problem-solving interview was conducted with each of the 15 participants. This interview was conducted after the class had worked with the physical Alberti's Window for several weeks and had worked with the GSP version of the Alberti's Window for a minimum of one week. During the first part of these interviews, using the physical Alberti's Window, participants were asked how the projection of different shapes, in different locations on the tabletop, would project onto the Alberti's Window. The participants were asked to describe, and possibly draw on a piece of paper, how they thought the various projections would look. In the second part of these interviews, participants were asked to describe how they viewed the relationship between the physical and the GSP versions of the Alberti's Window. This problem-solving interview was used as a secondary data source for answering RQ1, since, while the topics of the interview were particular ideas in projective geometry, my focus on mathematical practices were those that generally arose during classroom activity.

3.2.2.4 Midterm Art Project Reflection Interview. After the participants had completed their first of two artistic projects, I conducted the third of four individual interviews. These interviews were conducted during the twelfth week of the semester. In these interviews, participants had access to their completed artistic projects, as well as the GSP sketch of their designs on a computer screen in front of them. In addition, participants had access to a physical Alberti's Window, string, paper, and pens. The foci of these interviews were how the participants created their designs, the mathematical ideas they used in their designs, and what the participants felt they learned from completing the art project. These art project interviews were used as a primary data source for each of my two research questions.

3.2.2.5 Final Art Project Reflection Interview. During the last week of the semester, and the week after, I conducted a final interview with each of the participants. This interview consisted of the same categories of questions as the Midterm Art Project Reflection Interview – specifically, how the participants created their designs, the mathematical ideas they used in their designs, and what the participants felt they learned from completing the art project. In addition, the participants were asked what connections they noticed between the three sections of the course – the physical and spatial aspects of projective geometry, synthetic projective geometry, and analytic projective geometry. These final interviews were used as a primary data source for both of my two research questions.

3.2.3 Assignments and Exams

All homework assignments and exams were collected. Homework assignments and exams were submitted either as Geometer's Sketchpad files or as

Word documents, as specified by the instructor. Some homework assignments were completed in the form of an essay, while other assignments were proofs related to the topics explored in the course. The essay-style homework assignments included personal reflections about experiences with geometry and artistic endeavors, and summaries and reflections about the first chapter from A Mathematician's Lament, as well as the article about Marcel Duchamp and the emergence of contemporary art.

Two take-home exams were given during the course – a midterm exam and a final exam. The two exams had the same structure, where students submitted their completed art projects, their written reflections regarding the art projects, and their solutions to a selection of projective geometry proofs assigned by the instructor. Assignments and exams were used as a secondary data source for RQ2.

3.2.4 Video Reflections

During the final exam period, which occurred just after students had completed the second of two artistic projects, they were asked to record a video reflection regarding their experiences creating their artistic pieces. Each of the participants individually recorded his or her video reflection. Each of the three video cameras used to record these video reflections was operated by one member of the research team. The video cameras used were the same cameras used to collect classroom-video data. During the video reflections, to ensure satisfactory audio quality, the participants wore a wireless lapel microphone that ran directly into the video camera.

During this video reflection, participants answered a set of questions, including how they created their design for the artist project, what they learned from creating their artistic projects, what connections they see between mathematics and art, and whether, as future teachers, they would incorporate artistic engagement into their mathematics courses. In some instances, the videographer asked the participants follow-up questions. The video reflections were used as a primary data source for RQ2, and as a secondary data source for RQ1.

3.3 Participant Selection

While data was collected for sixteen participants, I limited my analyses to five students in particular. These five students were selected on the basis of their clear and thoughtful responses in their video reflections, as well as informed by my lived-experiences in the classroom with the participants. Since my role in the course that served as the setting for this study was as the co-instructor, I was able to develop a rapport with each of the students, particularly as I frequently interacted with them as they reasoned about projective geometry ideas. I selected these five students for my analyses, as I found each of them to be exceptionally articulate, reflective, and expressive when explaining their thoughts and reasoning. As such, the data stemming from these students had the greatest potential of illuminating my research questions, which is a common selection criterion when conducting case studies (Merriam, 2009; Stake, 1995; Yin, 2009).

It is important to note here, with RQ2, I was not looking to determine *whether* artistic engagement enriches students' learning experiences and opportunities – which would suggest I should have analyzed data for all 16 participants – but rather, I set out to determine the ways in which artistic engagement *can* enrich students learning experiences and opportunities. Similarly, with RQ1, I was not attempting to exhaust all of the possible mathematical practices in which students engage during this particular course, but rather, I aimed to explore the realm of possibilities for the mathematical practices in which students might engage while working on problems in projective geometry. Therefore, it is appropriate for my analyses to be limited to those students who were best able to provide the data needed to answer each of my research questions. Below, I provide a brief description of each of the five participants selected for data analysis.

3.3.1 Willow

Willow was a 26 year-old female in the midst of earning her Master of Arts Degree in Teaching Service, with the intention of teaching community college in the future. She held a Bachelor of Science degree in chemistry, with a minor in mathematics. Willow did not recall taking any art classes during her schooling, however she recalled her history teacher incorporating art into certain history projects. She recalled being frustrated by the arts integration into her history class, as she felt she was not necessarily learning the history involved. Willow stated she did not generally engage in art creation, but recalled attempting artistic endeavors once or twice. Despite not regularly engaging in artistic endeavors, Willow stated she enjoyed going to museums to look at artwork.

3.3.2 Jerry

Jerry was a 20 year-old male in the process of earning his Bachelor of Arts Degree in Mathematics. He was on the single-subject track, meaning he aimed to earn his single-subject credential in mathematics after earning his bachelor's degree. He reported liking many subjects – in fact, he briefly considering becoming an art teacher – however, decided that mathematics would be the practical option. From an early age Jerry enjoyed engaging in artistic activities and he mentioned that people often comment on his artistic abilities. He admitted to having some artistic ability, but did not necessarily consider himself to be an artist. He recalled taking one art class in high school, where he learned about one-point and two-point perspective. He noted, while he did not particularly care for the teacher of the art course, as he felt she did not let the students express themselves in their artwork, he generally enjoyed the course.

3.3.3 Alejo

Alejo was a 20 year-old male who began the semester as a double major in philosophy and mathematics, with an emphasis in physics. He changed his major mid-semester in order to earn a Bachelor of Arts Degree in Mathematics, rather than a Bachelor of Science degree. He remained a double major. While he was not on the single-subject track for becoming a secondary teacher, he stated that he was interested in teaching mathematics, science, and logic. Alejo did not particularly care for art, in the sense that he did not consider himself to be an artist. The only times at which he had taken an art class were in middle school. The artistic endeavors in which he previously engaged and enjoyed were activities such as playing the guitar and dancing at family events.

3.3.4 Fiona

Fiona was a 20 year-old female in the process of earning her Bachelor of Arts degree in Mathematics on the single-subject track. Her intention after earning her bachelors degree was to earn a teaching credential and then become a high school mathematics teacher. She recalled being asked to help others with learning mathematics when she was younger, and, at the time of this study, worked as a mathematics tutor. Her artistic background included writing poetry and singing. She expressed that she particularly liked to write poetry, and had a few poems published in her high school newspaper. While she occasionally made sketches to go with her poetry, and doodled when she was bored, she stated that she could not draw particularly well.

3.3.5 Trisha

Trisha was a 26 year-old female in the process of earning her Bachelor of Arts degree in Mathematics on the single-subject track. She had intentions of earning a Master of Arts degree, after she earned her credential, in order to one day become a community college mathematics instructor. Trisha viewed herself as having a strong background in mathematics. Her artistic background included a long history of dancing and teaching dance to children, as well as attending an arts-focused middle school. She mentioned that she had a math class at one time where the teacher had the students engage in art projects that used some geometrical ideas. The performing arts were her artistic outlet, but she claimed she was not skilled at painting or drawing.

3.4 Primary and Secondary Data Sources

At the beginning of data analysis, it was possible any of the collected data would be useful in answering either of my two research questions, since the realms of projective geometry and artistic endeavors – in particular as related to this specific activity-based Foundations of Geometry course – are closely linked. For example, as noted in Chapter 2, the projective geometry branch of mathematics emerged from the realm of art creation, and thus was motivated in the course by linear perspective in artistic sketches and paintings. Similarly, students were required to use ideas of projective geometry, which may or may not have been topics of discussion in the course, to create their art projects, and thus the art projects were, in a sense, motivated by the mathematical ideas. As such, a student's understanding of the mathematical ideas in the course and the student's creation of their art project could be inextricably linked. Consequently, any of the collected data could have been relevant to answering either of my two research questions. However, I anticipated that particular data would be more pertinent to answering Research Question 1, while other data may have been more pertinent to answering Research Question 2. As such, I outline below the data sources that were used as primary and secondary for answering each of my two research questions (see *Figure 23*). The secondary data sources were used for triangulation.

Research questions	Primary Data Sources	Secondary Data Sources
RQ1: In the context of an activity-based projective geometry course, in what mathematical practices do students engage while working on problems in projective geometry?	• Classroom video synced with audio from tables • Problem solving interviews • Midterm and Final Art Project Reflection Interviews	• Front-of-class video • Focus-group interviews • Video Reflections • Homework & Exams • SmartPen data • Screen captures
RQ2: In what ways can various means of artistic engagement enrich students' learning experiences and opportunities in an activity-based projective geometry course?	• Midterm and Final Art Project Reflection Interviews • Video Reflections	• Classroom video synced with table audio • Front-of-class video • Initial interview • Focus-group interviews • Homework & Exams • SmartPen data • Screen captures

Figure 23. Primary and secondary data sources.

3.4.1 Research Question 1

The primary data sources for answering RQ1 were the classroom video from each of the four groups, synced with the audio from the associated table. The classroom video and audio from all four tables were utilized at different times, with three of the four tables being analyzed for each class session. During each class session, the five selected participants were distributed across three of the four groups. In addition to the classroom video data, the problem-solving interviews and the two art project reflection interviews for each of the five selected participants served as primary data sources for RQ1. In each of these interviews, participants engaged in answering questions about projective geometry problems.

The secondary data sources for answering RQ1 included the video of the front of the classroom, the homework and exams for the five selected participants, the focus-group interviews, SmartPen data, and screen captures. The initial interviews with participants were not be used for answering RQ1, as students had not been introduced to projective geometry at that time.

3.4.2 Research Question 2.

The primary data sources for answering RQ2 were the midterm and final art project reflection interviews, as well as the video reflections, for each of the five selected participants. Secondary data sources included all classroom video data, the initial interviews, the focus group interviews, homework and exams, SmartPen data, and screen captures.

3.5 Data Analysis

Data analysis consisted of a three-phase process for each of the two research questions. The three phases, which I discuss in detail in this section, include (a) data preparation & reduction, (b) thematic exploration, and (c) identification of commonalities. I first present a general analytic approach followed by a description of each of the phases of data analysis.

3.5.1 A General Analytic Approach.

In general, I approached my analyses using techniques from grounded theory methodology (Strauss & Corbin, 1990, 1994, 1998). As Merriam (2009) notes, grounded theory methodology lends itself well to many kinds of qualitative studies, in particular due to its inductive and comparative nature. Since the literature surrounding both students' engagement with projective geometry and arts integration is limited, there was little existing theory upon which to build. As such, a grounded approach to data analysis was appropriate (Creswell, 2013), thus allowing the theory to emerge from the data (Strauss & Corbin, 1998).

Using a grounded approach to category identification, including techniques similar to open and axial coding (Strauss & Corbin, 1990, 1994), I developed preliminary categories of mathematical practices and enrichment from artistic engagement for each of the five participants. This was accomplished by (a) compiling the entirety of the reduced set of primary data for each of the five selected participants, (b) creating illustrative summaries, known as thick descriptions (Geertz, 1973), for the episodes of the classroom video that pertained to RQ1 and RQ2, and (c) utilizing the video data, thick descriptions, and interview transcript to develop emergent categories of phenomena. Through multiple examinations of the data, these categories were refined, which included identifying subcategories. For example, a category such as "providing the opportunity" for students to engage in a mathematical practice was partitioned into the subcategories "instructor influence," "nature of task," and "affordances and limitations of tools."

Throughout this analysis, I consistently employed three different analytic techniques. First, I utilized the *constant comparative method* (Strauss & Corbin, 1990, 1994), both within individual students and across the students, which consisted of reviewing various segments of the collected data side by side in order to identify the existing similarities and differences. Second, I considered alternative or rival explanations (Yin, 2009) for the emergent categories of phenomena. Lastly, I triangulated the primary and secondary data sources, as well as conducted investigator triangulation (Stake, 1995; Yin, 2009). Triangulating the primary and secondary data sources consisted of using the secondary data sources to look for regularities in the instances of phenomena within categories that emerged from the primary data sources. Investigator triangulation consisted of other researchers examining categories and associated data to look for general alignment between their interpretations and my own interpretations of phenomena within categories.

3.5.2 Phase 1: Data Preparation and Reduction

Collectively, the primary video-data sources for my two research questions included the set of three interviews for each of the five selected participants, the video reflections, and the classroom video data for three groups of students for the 22 indicated class sessions. At all times during the 22 indicated class sessions, the five selected participants were spread across three groups. With the extensive data that was collected, a data reduction process was necessary (Miles & Huberman, 1994). Below, I outline the first phase of my data analysis, which served as a first pass through the data, as well as a data reduction process. This beginning phase included cataloging and preparation of data through creating content logs and interview transcription.

3.5.2.1 Content Logs. The vast quantities of classroom video data and the scope of my two research questions – in addition to the potential relevance of data to either research question – required a detailed index of the collected data. This process served as a way to catalog the classroom video data, and served as first pass at identifying the mathematical practices in which students engaged during the course. In addition, it served as a means for data reduction (Miles & Huberman, 1994), in the sense that it allowed me to identify video that was central, and that which was peripheral, to answering my research questions.

Creating content logs is a phase of analysis borrowed from interaction analysis methodology (Jordan & Henderson, 1995), yet was appropriate for this data, as the vast quantities of video data required a way to index the data and focus on data that was pertinent to answering my research questions. As Jordan and Henderson (1995) note, content logs are useful for providing an overview of the data and quickly locating particular segments of data. *Figure 24* shows an example of what my particular content logs looked like.

Segment	Start Time	Activity	Mathematical Practices	Artistic Engagement	Notes
S3	05:58	Candace grabs the string and says, so this is what you do. Candace asks Francis to put the eyepiece a little higher. Candace's guess is that you use the string to measure the distance from the eyepiece hole to a point on the square and then transfer that distance to get to the window - like a reflection. Trisha and Mike both say no. Mike and Trisha both say that the window is infinite. Trisha explains that the string would go down to the image and then continue on down and then hit at a point near your feet. Candace questions how you would draw that. Trisha says that you will have to imagine what it will look like. Trisha takes a transparency to hold next to the window. She holds it like the window and indicates that it would continue. Candace says she gets the concept but just doesn't understand how they are supposed to draw it. Trisha starts to suggest using the slope and the amount of string. Mike seems to understand what she is saying and indicates that it will be a rhombus - makes a trapezoid with his hands.	Trisha: -imagining window goes down, extension of line of sight, direct reference to imagination -extension of tool with transparency -explanation - Mike: -imagining the window goes down		Trisha talks about how the string would extend down and hit somewhere down near your feet. When Candace wonders how you would draw that, Trisha replies that they would have to imagine it.

Figure 24. Sample line of content log.

Content logs were composed for the entirety of the classroom video data, for the three groups in which participants were members, for the 22 indicated days. I utilized the video from the front of the classroom as supplemental data, since much of the activity that occurred at the front of the classroom was instructor directions, whole class discussion, presentations of findings from individual groups, and activities in which individual students, or several students, participated. Each of these situations could be identified by the video from the group cameras, and thus the video from the front of the classroom was only necessary when one or more of the five selected participants were at the front of the classroom or when analyzing Dr. R.'s activity. These situations were documented in each individual group's content log.

My particular content logs were accounts of the classroom occurrences and activities of the three groups (at a time) in which the five selected participants were members. Each content log is in the form of a table with columns labeled for date, video segment (since the camera divided long stretches of video into approximately 25 minute segments), start time for the activity, descriptive accounts of group activity, relevance to mathematical practices, relevance to artistic engagement (as a secondary data source), and other notes (see *Figure 24*). The other notes column was used to notate aspects of activity that are particularly interesting or rich.

The content logs were segmented into time increments based upon natural breaks or shifts in activity within the video. In classroom video, the segmentation was based on such occurrences as when the topic of conversation changed, when classroom activity changed from group activity to whole class activity or instructor directions, or when a group's mathematical reasoning shifted. Unless taken up as conversation by more than one group member for more than two exchanges, minor occurrences, such as a single participant making a brief comment about a seemingly unrelated topic, did not result in their own segments. For example, if a participant mentioned how delicious her coffee was that morning, it did not result in a new segment. In each time segment, a detailed description of the activities within the segment was composed.

In the "Mathematical Practices" and "Artistic Engagement" columns of the content logs, I described generally how the particular segment might have been important for answering the indicated research question, in particular as it pertained to the five selected participants. This process was similar to the open coding (Strauss & Corbin, 1990, 1994) procedure of grounded theory methodology, in that as much and as many aspects of the data pertaining to my two research questions were identified with a word or phrase describing the phenomena. For example, in the segment introduced in Chapter 1 (pp. 10-11), in which one student introduces into the conversation a way of reasoning about particular aspects of the projection in terms of limits of a slope when considering the projection of a circle, would be detailed in the "Mathematical Practices" column with phrases such as, "use of tools for reasoning", "reference to limits", and "changing slopes of sightlines". I note here that while the classroom video data is not a primary data source for answering RQ2, it is a secondary data source. As such, I identified the segments of classroom data that may have pertained to RQ2, as it allowed for more fluid triangulation of data.

3.5.2.2 Interview Transcription. The problem-solving interview for each of the five selected participants was a primary data source for answering RQ1, and the two art project reflection interviews for each of the five selected participants were primary data sources for both RQ1 and RQ2. All three sets of these interviews were fully transcribed. In addition, the end-of-term reflection video for each of the five participants was fully transcribed. Similar to the content logs for classroom video data, the transcription allowed me to identify the sections of the interviews and the video reflections that may have been useful in answering each of my two research questions.

3.5.3 Phase 2: Thematic Exploration

While my initial analysis plan was to conduct case studies to answer both RQ1 and RQ2, it quickly became apparent that case studies were not ideal for answering RQ1, as participants' mathematical practices were entangled – meaning, the way in which students worked on problems together made it difficult to determine where one student's engagement with a mathematical practice started and another student's engagement with a mathematical practice ended. As such, I chose to draw from techniques from grounded theory methodology (Strauss & Corbin, 1990, 1994, 1998) to answer RQ1, while I continued to conduct case studies, also guided by grounded theory methodology, to answer RQ2.

3.5.3.1 RQ1: Exploring and Identifying Mathematical Practices. To answer RQ1, I drew from grounded theory methodology (Strauss & Corbin, 1990, 1994, 1998) to identify emergent themes in the data regarding mathematical practices in which the participants engaged while working on problems in projective geometry. Specifically, I used an approach similar to axial coding (Strauss & Corbin, 1990, 1994), in which I carefully analyzed the segments in the classroom video data that I had identified in my content logs as being relevant to answering RQ1. Through analyzing these relevant segments, and using the constant comparative method (Strauss & Corbin, 1990, 1994), I identified multiple themes of mathematical practices. While there were many mathematical practices in which students engaged in the course, and any of these may have been interesting to analyze further, those practices I found particularly intriguing seemed to be related to the unique nature of this activity-based course. As such, rather than focus my analyses on more well-discussed mathematical practices, such as those included in K-12 standards documents (CCSS, 2010; NCTM, 2000), I chose to focus on two practices that I feel are fundamental for mathematicians, yet are often overlooked in the literature: mathematical play and acts of imagination.

I defined *mathematical play* as an exploration of mathematical ideas through individual or group actions that are both *autonomous* and *freeform*. By *autonomous* I mean that the actors have minimal concern with what others around them are doing, or with what others think about what they are doing. By *freeform* I mean the details of the actions are not scripted or prescribed. Mathematical play can include engagement with physical devices, computer programs, acts of imagination, and social interactions, as well as inscriptions. I identified autonomous actions in the data through analyzing an actor's relationship with others in his or her environment, as well as the focus of the actor on his or her own activity. In particular, I identified activity as autonomous when an actor paid little to no attention to others in the environment, or when an actor noticed but disregarded the actions or criticisms of others in the environment. I identified freeform activity in the data when an actor was engaged in self-guided mathematical exploration, which I discuss further in Chapter 4.

It is important to not here, particularly since one of the foci of this research is artistic engagement in a mathematics course, that the characterization of autonomy in my definition of mathematical play has both similarities to and differences from the notion of autonomy within the art fields. Unlike my characterization of autonomy in mathematical play, in which the actor or actors have little concern with whether others are noticing what they are doing, in the art field, the notion of autonomy includes the desire of the artist that others are taking notice of what he or she is doing. Similar to my characterization of autonomy in mathematical play, within the art fields, the actor – the artist, in this case – is not concerned with what others think about his or her actions or work. In summary, within the art fields, autonomy is the desire that others notice your work, but that you, as the artist, find what others think about your work to be inconsequential (B. Stalbaum, personal communication, July 11, 2016). And, within my definition of mathematical play, autonomy is the notion that the actor or actors are not concerned with what others think about what they are doing, nor whether others even notice what they are doing.

I defined an *act of imagination* as a mathematical practice characterized by one or more individuals acting *as if* a mathematical situation or entity were present, despite the entity not being physically present in the current surroundings. To identify acts of imagination in the data, I looked at the way in which the participants used gesture, body positioning, eye gaze, speech, and aspects of mathematical tools to indicate aspects of a mathematical situation or idea that was not physically present (or was only partially present) in their surroundings, yet the students operated and talked in a manner that suggested the aspects of the mathematical situation or idea was present. I further discuss identifying act of imagination in Chapter 5.

3.5.3.2 RQ2: Individual Case Study Analysis. To answer my second research question, I conducted case study analyses for each of the five selected participants. I focused my case study analysis on the particular ways in which in artistic engagement enriched the *individual* learning experiences and opportunities for each of the five selected participants. This consisted of carefully analyzing the video and transcript of the individual interviews with participants during which we discussed their artistic pieces, the participants' video reflections, as well as segments of classroom video identified during Phase 1 that pertained to artistic engagement. In addition, I referred to participants' written reflections regarding their artistic pieces – particularly for data triangulation.

3.5.4 Phase 3: Identifying Commonalities

In the final phase of data analysis for both RQ1 and RQ2, I identified commonalties across the data. For RQ1, I identified commonalities within the two mathematical practices. That is, I looked for commonalities across the instances of each of mathematical play and acts of imagination. For RQ2, I identified commonalities across the five participants. That is, I looked for commonalities in the ways in which students learning experiences and opportunities were enriched through artistic engagement.

3.5.4.1 RQ1 Identifying Commonalities Within Practices. After identifying the instances of mathematical play and acts of imagination in the data for the five participants, I identified commonalities within each of the mathematical practices, in particular, as they pertained to the benefits of mathematical play, the ways in which students used acts of imagination in explanation and justification, and the aspects of the learning environment that provided the opportunity to engage in the two mathematical practices. These were not a priori categories of commonalities, but rather emerged from my analysis of the classroom video data. The analytic strategies for this final stage of analysis were comparable to the strategies used in Phase 2. I then selected particular instances of each of the mathematical practices to analyze more closely and highlight in my results.

My results introduce two emergent mathematical practices not previously elaborated in the literature. Thus, while there were numerous instances of students engaging in mathematical play and in acts of imagination, I selected particular classroom episodes and interview segments based on the potential of the episodes to illustrate and illuminate these practices. These episodes were instances in which the activity of participants, or the instructor, provided a rich illustration or illumination of either mathematical play or an act of imagination. This included ensuring the instance of the practice would be evident to the reader. I wanted the reader to be able to insert herself or himself into the situation occurring in the episode and feel as if they were able to understand the actions of the participants, possibly even giving the reader the ability to recreate the actions of the participants. This meant giving lower priority to episodes in which the participants were partially blocked from the view of the camera or in which the actions of the participants were less distinct and more ambiguous. Higher priority was given to episodes that provided insight into the way in which students benefited from the engaging in mathematical play, the way in which students engaged in acts of imagination during explanation or justification of mathematical situations or ideas, and the particular aspects of the learning environment that provided students the opportunity to engage in the two practices.

3.5.4.2 RQ2: Identifying Common Themes Across Cases. The final phase of data analysis for RQ2 echoed a cross-case analysis, in which each case is one of the five participants. The purpose of this final stage of analysis was to look for the similarities between the ways in which students learning experience and opportunities were enriched through artistic engagement, across the five individual cases. While this is indicated as a separate phase of the data analysis process, the groundwork for this analysis took place during the second phase of data analysis. In particular, as I employed the constant comparative method both within and across cases, similarities and differences between the cases begin to emerge. The analytic strategies for this final stage of analysis were comparable to the individual case study analysis strategies.

3.6 Reliability and Validity

Validity in qualitative research can be characterized as the appropriateness of inferences, interpretations, and claims made about the data (Maxwell, 2005), or in other words, whether the interpretations fit the data. Three analytic approaches included in my analysis design addressed this issue of validity. First, I used the *constant comparative method* (Strauss & Corbin, 1990, 1994) during case studies, both within individual students and across the students. This means, as I interpreted and developed categories of mathematical practices, and enrichments from artistic engagement, along with associated explanations, I compared these emerging categories and explanations back with the data and looked for evidence that refuted my interpretations. Second, I considered alternative interpretations, or rival explanations (Yin, 2009), for phenomena. This consisted of reflecting on the data, attempting to understand the experiences of the participants, and developing other plausible explanations. Lastly, I used secondary data for triangulation. This means I looked to the secondary data for confirming or disconfirming evidence of my interpretations and explanations.

One aspect of my analysis design, conducting investigator triangulation (Stake, 1995; Yin, 2009) is included to address the issue of reliability. This means I engaged other researchers in evaluating my interpretations. That is, I played episodes of video data for other researchers, informing them how I was interpreting the phenomenon in the episode, and asked for feedback regarding the conviction of my interpretations. In addition, I asked these researchers for assistance in developing alternative explanations for phenomena.

I looked to both Dr. Rasmussen and Dr. Nemirovsky for feedback on the mathematical practices that emerged from interpretation of the data. As I interpreted and identified enrichment from artistic engagement, I turned to Dr. Nemirovsky, as well as Natalie Selinski. Natalie Selinski is a doctoral candidate in mathematics education who has a background in both mathematics and art.

Chapter 4
Mathematical Play

"The creation of something new is not accomplished by the intellect but by the play instinct acting from inner necessity. The creative mind plays with the objects it loves"

-Carl Jung (1875–1961)

In this chapter, I discuss one of the results of my first research question:

> In the context of an activity-based projective geometry course, in what mathematical practices do students engage while working on problems in projective geometry?

The emphasis of this chapter is the practice of *mathematical play*. This chapter contains five main sections. In the first section, I provide a brief background into mathematical play in the literature. In the second section, I provide a description and illustration of my definition of mathematical play. In the third section I address two benefits of mathematical play, and in the fourth section I discuss aspects of the learning situations that created the opportunity for mathematical play in the projective geometry course.

4.1 Brief Background

In general, the literature addressing play in the context of learning, and mathematical play in particular, comes from the field of early childhood education. The literature in early childhood studies suggests play is a valuable activity in which children should engage (Piaget, 1951; Singer, Golinkoff, & Hirsh-Pasek, 2006), including as it relates to mathematical development (Ginsburg, 2006; Holton, Ahmed, Williams, & Hill, 2001). As it pertains to mathematics, much of the literature endorsing play as a valid aspect within mathematics learning relates to children's' development of everyday mathematics – such as number sense, shapes, and measurement – through their spontaneous play (Ginsburg, Inoue, & Seo. 1999; Ginsburg, Pappas, & Seo. 2001). In particular, when children engage in spontaneous play, they begin to develop notions of number sense, shapes, and measurement. Others suggest these mathematical aspects do not become mathematical to children until they gain awareness that the aspects are culturally considered mathematical (van Oers, 2010). In this body of literature, it is generally accepted that play is a voluntary activity.

Play related to mathematics learning is not confined to early childhood literature. Holton et al. (2001) argue that play is useful and necessary at all ages and levels, not just during childhood. They suggest research mathematicians engage in play at multiple stages during the research process, and that the mathematical play of children mimics that of research mathematicians (Holton et al., 2001). They define mathematical play as "that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of conclusion." (Holton et al., 2001, p. 403) Within this characterization of mathematical play, the actor must follow through with solving a particular problem, using mathematical rules to arrive at an answer.

In other mathematics education research, playfulness with mathematics resides in the background of studies that foreground other constructs. In some of this research, the focus is the role of the body in mathematical development, for example, as it pertains to the characterization of the role of gesture in collective mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012) and as it pertains to the role of perception and the body in developing mathematical ideas (Nemirovsky, Rasmussen, Sweeney, Wawro, 2012). Other research that backgrounds play while foregrounding other constructs includes research on the aesthetic and the role of emotions and communication in mathematics learning (Sinclair & Heyd-Metzuyanim, 2014), as well as research in mathematical creativity as it relates to mathematical problem posing (Shriki, 2010).

4.2 Mathematical Play As A Mathematical Practice

Within a room abuzz with activity, four students are working at a table with the Alberti's Window. One student sits on a rolling chair a few feet away from the end of the table, holding a piece of paper on which two parallel lines are drawn. Another student leans down on the table, looking through the eyepiece, pen poised, ready to draw on the window. "Okay, move back. Move back," she says. The student sitting on the chair holding the lines rolls slightly backward in his chair, away from the table. "Lower it," she says. "Yeah, right there." She touches the pen to the window and draws two tiny line segments, “Ugh, it’s like points.” Dr. R., the instructor of the course, stands near the student on the chair, smiling, looking back and forth between the student at the window and the student in the chair. Dr. R. says nothing, just watches. “Move back. Lower it.”

After the student at the window traces the next set of projected lines, a third student remarks, “Wait, so that means it’s squishing more this way, than it is coming together,” holding up his hand and indicating with his thumb and forefinger a vertical squishing, then a horizontal squishing. “Oh, that’s true,” says the student in the chair. The third student replies, “So like she said, it would be a horizontal line before it was a point,” likely referring to a previous conversation about how a circle projects at the infinite. The group begins to debate what the behavior of the projected lines, as well as a circle, will be at infinity. Dr. R. walks away, saying nothing, still smiling.

The task during this vignette was to determine what a vanishing point or a horizon line is. The class was told to extrapolate their projected parallel lines to determine where on the window the two lines would converge. Specifically, Dr. R. noted to the class that any lines they could project would be finite, but they could extrapolate by extending their projected lines to see where the lines meet. In addition, he told the class to do this at several different heights of the eyepiece. This particular group of students employed a different strategy, one not mentioned by the instructor. They did not look around the room to see how other groups were carrying out the task, nor did they confer with the instructor about their idea. They simply had an idea of how to extend the non-projected parallel lines and proceeded accordingly. And, as a result, the group came to the discovery that the lines were converging faster vertically than they were horizontally, an important discovery in this context, as it determines how the shape of an image changes when projected onto the window.

As the co-instructor of the course, I recall watching the activity of particular groups, wondering how their seemingly creative engagement arose. Often this creative engagement was in relation to the way in which they were utilizing the mathematical tools of the course. For example, holding an image to be projected in line with the top of the Alberti's Window, or placing the window on the floor. At times, such as in the above vignette, by observing the activity of a group, it looked as if the group may have been unclear on the purpose of a task, as their approach seemed so unusual. However, in watching the classroom video, of course, I came to discover how this creative activity came about for the groups of students for which data was collected. I observed that, typically, the groups were clear on the purpose of the task, only they had other questions they wanted to answer, or they had a different approach than the one suggested by the instructor.

Walking around the classroom and engaging with students in the course, as well as analyzing the classroom video data, it was common to hear students asking, "What if we…," types of questions. For example, "What if we lower the eye height?," "What if we put the window closer to you?," and "What if we had horizontal lines, would we get the same result?" More so than a simple question, these "What if" questions appeared to serve as invitations to inquiry, or invitations to explore, suggestions of what actions to take next. These prompts, serving as invitations to inquiry, were not always "What if" questions, but also came in the form of "How about" questions or "Here, let's" suggestions. Frequently, these prompts led students to exploring an aspect of a mathematical idea or to utilizing a mathematical tool in a slightly different manner, without explicit direction from the instructor. For example, while determining how a square between the window and the viewer would project, two groups chose to move their Alberti's Window to the floor, while keeping the eyepiece on the table – a situation I discuss in more detail in section 4.4.3. This setup with the window on the floor was different, as up to this point the eyepiece and the window were always sitting on the same plane, specifically the tabletop. At other times, a student would introduce such a prompt, however the suggestion for action was not be taken up by the rest of the group.

Occasionally, a group would begin to take up the invitation to inquiry, but the inquiry would be thwarted by a group member observing the activity of another group and realizing that group had a different course of action for working through the task. For example, in one group, during the first instance in which the class was projecting parallel lines, one student proposed projecting a square, since the sides of the square are parallel. Just after she introduces the idea, she and another group member notice the group at the table next to them is projecting a set of two parallel lines. Rather than continue with the square idea, the two students change their course of action by drawing two parallel lines on a piece of paper and then projecting the lines. At other times, a group would notice but disregard the activity of other groups.

The ways in which students prompted these invitations to inquiry, the ways in which these invitations were taken up or not by other group members, as well as the ways in which the activity became derailed at times by noticing another group's activity, suggests an autonomous nature to the explorative activity. The ways in which students engaged in activity that was not indicated by the course instructor, or modified the approach to an activity in their own ways, suggests a freeform nature to the activity. The autonomous and freeform nature of student activity during the exploration of mathematical ideas led to the way in which I defined mathematical play, discussed in the next subsection.

4.2.1 Definition of Mathematical Play

Based on my experiences working with students in the course, a review of the related literature, and through a grounded analysis of the collected data, I developed the following definition of mathematical play:

> *Mathematical play* is an exploration of mathematical ideas through individual or group actions that are both *autonomous* and *freeform*.

By *autonomous* I mean that the actors have minimal concern with what others around them are doing, or with what others think about what they are doing. By *freeform* I mean the details of the actions are not scripted or prescribed. Mathematical play can include engagement with physical devices, computer programs, acts of imagination, and social interactions, as well as inscriptions.

This definition of mathematical play is in contrast to the definition put forth by Holton et al. (2001) in which solving a mathematical problem and coming to some form of conclusion through utilizing formal rules of mathematics is required. Based on my analysis of classroom video data, I observed students engaging in mathematical play without using the formal rules of mathematics to arrive at some form of conclusion while solving a problem. For example, one student used the Alberti's Window to project a star and a crescent moon shape. In doing so, she was simply exploring projections and not trying to solve any particular problem.

At times, students in this course were engaging in play to solve a problem, yet at other times, students engaged in play in a lighthearted manner, seemingly driven by aesthetics. I have therefore reimagined the notion of mathematical play such that solving a mathematical problem, in general, and doing so by using the formal rules of mathematics in particular, is not a requirement. Instead, the actor, or actors, involved in mathematical play may simply be exploring a range of possibilities within a mathematical idea, and so is not necessarily looking to arrive at a conclusion or solution to a particular problem.

Below, I clarify certain subtle differences between instances and non-instances of mathematical play, in particular as it pertains to whether an instance is autonomous and freeform. Following this, I provide an example from the data that illustrates an instance of mathematical play.

4.2.2 Identifying Play

In this subsection, I detail the particular aspects of student activity that indicated autonomous and freeform activity. The features I describe in this section indicate the ways in which I identified mathematical play in the data.

4.2.2.1 Autonomous Activity. Autonomy in mathematical play is characterized by an actor's minimal concern for what others are doing, or for what others think about what the actor is doing. As such, I identified autonomous actions in play through analyzing the actor's relationship with others in his or her environment, as well as the focus of the actor on his or her own activity. In particular, I identified activity as autonomous when an actor paid little to no attention to others in the environment, or when an actor noticed but disregarded the actions or criticisms of others in the environment. For example, in one scenario, which I address in more detail in a subsequent section, Jerry and Alejo's group was engaging in mathematical play as they attempted to determine how a particular projection would appear on the window. A student in a nearby group noticed what Jerry and Alejo's group was doing, and suggested their idea would not work to complete the task at hand. Despite the criticism from the nearby student, rather than abandon their play, the group continued their playful activity to try to determine how the projection would look.

It is important to note here that when an actor observes the activity of others in his or her environment it does not disqualify activity as being autonomous, per se. Rather, when an actor does not consider the actions of others around them, then it can be an indicator of autonomy. However, if an actor observes other actors in the environment, say, from other groups, it cannot be said the observation excludes activity as being autonomous.

4.2.2.2 Freeform Activity. Freeform activity in mathematical play is characterized by the non-scripted and non-prescribed nature of mathematical exploration. As such, I identified freeform activity through two major components. The first component relates to whether the actor is engaged in attempting to work through an assigned task, or if the actor is engaged in self-guided mathematical exploration. These are not mutually exclusive, as an actor may be attempting to work through an assigned task, yet be engaged in a form of exploration while trying to work through the task. The difference is the direction given for the task. For example, in one task, the class was instructed to "play with [projecting] parallel lines in different directions", while in another task, the class was instructed to project parallel lines using different heights of the eyepiece. This difference may seem subtle, however, given the variable aspects of the mathematical tool for the task – the physical Alberti's Window – the students appeared to interpret the instructions rather differently. In particular, when asked to "play with parallel lines in different directions," in addition to varying the orientation of the parallel lines, groups of students varied the eye height, the distance of the eyepiece from the window, and the distance of the lines from the window. On the other hand, when instructed to project parallel lines using different eye heights, in general, groups of students varied the eye height as instructed, yet kept the location of the parallel lines and the location of the eyepiece constant for each projection. In the first of these two situations, varying the eye height may be considered mathematical play. However, in the latter situation, varying the eye height would not be considered mathematical play, since that action was included as part of the task itself. Thus, when considering the freeform nature of mathematical play, the nature of the current task, as well as the current task directions must be taken into consideration.

A second component in identifying freeform activity is the actions and discourse of the actor or actors. The freeform aspect of mathematical play was often accompanied by hypothetical "What if " questions. For example, "What if we lowered the eye height?" These "What if" questions alone were not enough to identify freeform activity, since "what if" was also used to suggest a potential resolution to a difficulty being faced by the actor or actors. For example, in one scenario, when Alejo's group was having difficulty seeing both lines of a set of two parallel lines when looking through the eyepiece, Alejo suggested, "What if we move the lines closer together?" In this scenario, the "What if" question was proposed as a solution to a difficulty the group was having – putting the lines closer together so that both of the two lines could be seen though the eyepiece – rather than as a way to explore the mathematics.

4.2.3 Illustration

In this subsection I illustrate the mathematical play construct through an episode from the data. Since the task instructions are pertinent to determining whether activity is freeform, I first describe the task the students are working on during the episode. I follow this with the illustration and an examination of why this episode constitutes mathematical play. The episode includes Alejo and Jerry, as well as the two other students in their group, Carla and Emily.

4.2.3.1 Episode Background. On the fourth day of working with the physical Alberti's Window, Dr. R. told the class to imagine a set of infinite railroad tracks, and they should determine how the tracks project onto the window. Dr. R. gave no particular indication of how they should determine the projection. The episode occurred as Alejo and Jerry's group were in the process of determining how the section of the railroad tracks located behind the viewer would project onto the window.

In this scenario, the projection of the parallel lines onto the window would be in the shape of a V, such that the bottom point of the V is on the horizon line. Points closer to the viewer will project higher on the window than those points at a greater distance from the viewer. In addition, points that are on the viewer's right-hand side will project to the left side of the window, and similarly, points that are on the viewer's left-hand side will project to the right side of the

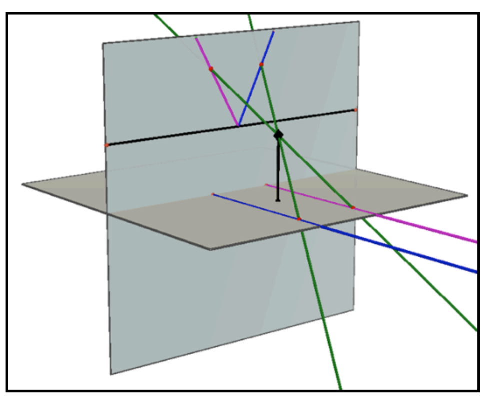
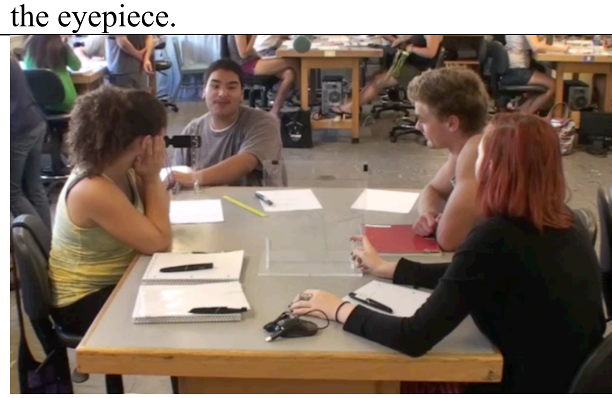


Figure 25. Parallel lines behind the eyepiece being projected onto the window.

window (*Figure 25*), which is the reverse of when points are on the tabletop anywhere in front of the viewer.

Prior to the beginning of the episode, Alejo has mentions the difficulty of projecting elements that are “behind you,” since, when you are facing the window, you are unable to see behind you. He also mentions, both through speech and through a sketch, that after the previous class session, that Dr. R. had told him there is a “flipping” that occurs with the projection when elements are behind the eyepiece. For several minutes, the group attempts to make sense of the situation – in particular, they try to determine a way to create the projection.

16	Alejo: This is our line of sight, and our line of sight goes that way, how do we draw these? This is behind our eye, so how do we draw these?	 Alejo holds a set of parallel lines, drawn on a piece of paper piece of paper, behind the eyepiece. He also holds two strings attached to the eyepiece.
17 18	Alejo: How do we put these on that window? Jerry: (shrugs his shoulders)	 Alejo does a nose point toward the window sitting in the middle of the table.

19 20 21	Alejo: Well I don’t know either. Jerry: Because, how do we perceive what’s behind us? (laughs) Following that line of sight argument, that makes no sense. Alejo: I understand. It’s just what he told me.	 Alejo shrugs his shoulders and lifts his hand into the air.
22	Alejo: How do you draw these	 Alejo indicates to the parallel lines on the paper he holds behind the eyepiece.
23	Alejo: on that one?	 Aljeo draws his hand out in front of him, holding his palm facing him and perpendicular to the table.

24 25	Alejo: Let's forget about our eyes. How would we project these Carla: You don't.	 Alejo indicates to the parallel lines on the paper he holds behind the eyepiece. Carla shrugs her shoulders and shakes her head.
26	Alejo: these onto the window?	 Alejo holds his left hand out in front of him, perpendicular to the table, palm facing him.
27	Carla: Get a mirror.	 Carla puts her head into her hands, sounding frustrated.

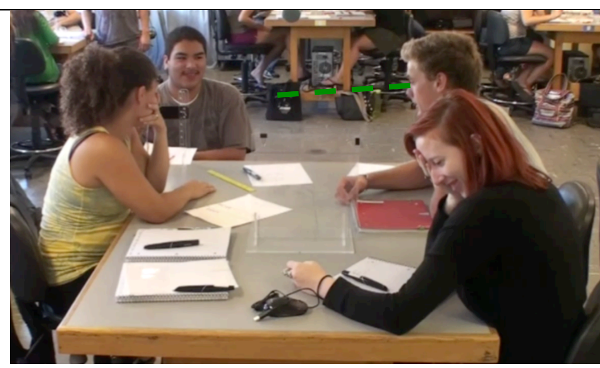
28	Alejo: A mirror?	 Alejo raises his eyebrows and looks at Carla.

Seemingly frustrated by trying to determine a solution to the task, Carla suggests using a mirror to see the lines behind the eyepiece. While it is unclear whether Carla suggests the mirror as a genuine method for projecting the lines, her exasperated tone in suggesting a mirror, as well as the resting of her head into her hands, suggests she may have proposed the mirror out of frustration, and perhaps did not consider whether it could be a useful tool in working through the task. Regardless, Alejo, sounding intrigued by Carla's proposition, adopts the mirror idea as a potential legitimate way to work through the task.

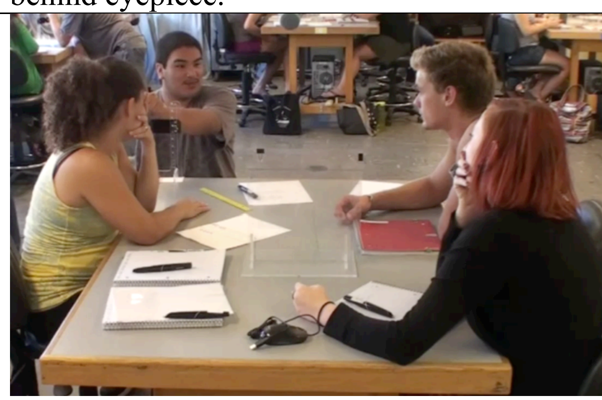
4.2.3.2 The Episode. After Carla proposes using a mirror to project the lines behind the eyepiece and Alejo adopts the idea as a potentially legitimate solution, he tries to convince the rest of the group to consider it with him. This is somewhat prompted by a question from Jerry, in which he wonders if the mirror would flip the projection in some way. After a few exchanges, the group dismisses Alejo's mirror idea, saying that it makes things more confusing.

29 30 31	Jerry: But then a mirror would flip the way that it – Alejo: The mirror would flip it! Carla: Draw a – What?	 Alejo looks at Jerry and points in Jerry's direction with his thumb.
32	Alejo: That way. Right?	 Alejo points down toward the table, at an angle with both index fingers.
33 34 35 36 37	Jerry: But then so is the real image the opposite of what we would see in the mirror? Let's not, let's not talk about mirrors (shaking head from left to right). Alejo: Let's not talk about what? Carla: It just makes it worse. Let's not talk about mirrors. Jerry: Yeah mirrors make it a lot more confusing. All but Alejo: (laughing)	

Alejo attempts to engage the group in considering the mirror as a viable solution for projecting the lines behind the viewer by justifying that the mirror would create a flipping, which is what he stated Dr. R. had mentioned in a conversation after the previous class session. Alejo's group, however, prefers not to consider the mirror idea, stating the mirror makes things more confusing. Despite the resistance of the group, Alejo persists in attempting to consider the mirror idea.

38 39	Alejo: No, let's say you put a mirror here (placing hand at eyepiece hole) Emily: Nuh uh.	 Alejo scoots his chair closer to the eyepiece. Jerry is looking down at the table.
40	Jerry: Okay, let's just think about it then.	 Jerry looks up from the table and toward Alejo.

At this point, with Alejo's persistence in discussing the mirror, while Emily still resists, Jerry agrees to engage with the mirror idea.

41	Alejo: You put a mirror here. Okay so this is kind of like a mirror. You put a mirror here.	 Alejo grabs a transparency from the table and (presumably) holds it next to the eyepiece.
42	Alejo: And I see these lines. They converge too	 Alejo points at the parallel lines on the paper behind eyepiece.
43 44	Alejo: to the horizon Jerry: Mm hm (nodding head up and down).	 Alejo points in front of him toward where the window would be (if it were in front of the eyepiece) at about the height of the eyepiece.

45	Alejo: Would they converge from up here?	 Alejo indicates toward a spot above the eyepiece.
46	Jerry: So then you're drawing it up.	 Jerry traces a line with his index finger and thumb away from him on the table. Alejo does a similar motion as Jerry, but toward himself.
47 48 49	Carla: (to Emily) Do you have a mirror? Jerry: But then, the mirror flips things. Alejo: So then it flips	 Carla looks at Emily.

50	Jerry: The mirror doesn't flip things vertically, it only flips things opposite.	 Jerry makes an upside-down U-shaped arc toward himself with his index finger. Emily reaches down toward her bag.

Carla, in a genuine way, asks Emily if she has a mirror, suggesting that she too is willing to consider the mirror idea. Emily produces a mirror from her bag and Alejo positions himself to look through the eyepiece, holding the mirror in front of the viewing hole. Upon the request of Alejo, Carla stands behind Alejo and holds the parallel lines level with the table (*Figure 26*). They continue this for approximately 40 seconds.

During the time Alejo is holding the mirror up and looking through the eyepiece, a nearby student notices what the group is doing. At the end of those 40 seconds, the nearby student criticizes the group's mirror approach, telling them he does not think their idea will work, and they should simply turn around to see the parallel lines behind them. Alejo and Carla tell the student that turning around is not an option, "since that would change where your window is." Alejo turns away from the student who criticized their approach and back toward his group, insisting he believes the mirror approach will be successful.

At this point, Alejo asks his other group members if they want to try looking with the mirror. Jerry volunteers to try it (*Figure 27*) and positions himself at the eyepiece. The group continues to test and discuss the mirror approach for approximately three and a half more minutes, at which point, I, as the co-instructor of the course, approach the group to see how they are doing on determining how the lines will project. After they explain how they are using the mirror, I redirect their activity by suggesting they use the string as a line of projection to determine where the lines will project.

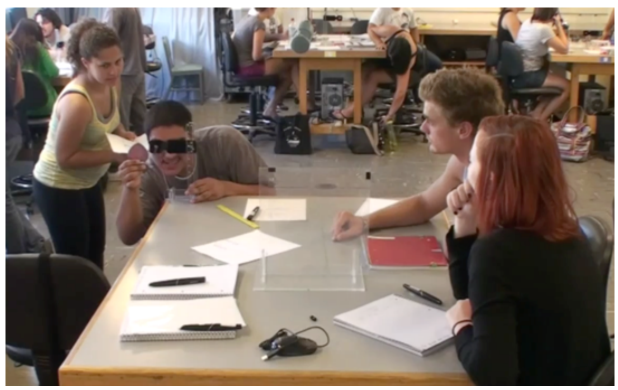


Figure 26. Alejo holds the mirror at the eyepiece while Carla holds the parallel lines behind him.

Figure 27. Jerry holds the mirror at the eyepiece while Alejo holds the parallel lines behind him.

4.2.3.3 Autonomous activity. While it was not Alejo who originally suggested the mirror idea, he adopted the idea as a viable way to find a solution to where the lines would project onto the window and wanted to pursue the idea. Despite the verbal resistance of both Jerry and Carla, Alejo continued to advocate for the mirror idea. Additionally, Alejo did not look to other groups to see how they were carrying out the task, nor did he seek the guidance of either instructor of the course. These components illustrate Alejo's lack of concern for what others thought about this mirror idea, and indicate autonomous activity.

As a result of Alejo's persistence, Jerry expressed a willingness to entertain the mirror idea, followed by Carla. Again, the group did not look around to see how others in the class were attempting to find solutions, now indicating autonomous activity for the rest of the group. Furthermore, despite criticism from a nearby student, who informed the group he did not think the mirror method would work, the group continued to play with the mirror idea for another three and a half minutes. This disregard for the criticism by another student further indicates the autonomous nature of the activity of the group.

It is interesting to note the group's mathematical play with the mirror ended when I approached the group to see how they were coming along with a solution, and reoriented them on how to utilize the string through the eyepiece to find how the lines behind the eyepiece would project. In a sense, it was the criticism of an instructor that affected the continuance of the group's mathematical play. Had I not intervened, this play may have continued.

4.2.3.4 Freeform activity. In the directions given by Dr. R. for the activity in which the group is engaged during this episode, he did not specify in what way the groups should determine how the lines that are behind the eyepiece project onto the window. However, in the previous class session, Dr. R. had discussed one way to think through a similar projection, by using his arms as a line of projection to consider how points behind the eyepiece get projected above the horizon line on the window (*Figure 28*). In addition, the group had been provided with a string to use as a line of projection, which they had used in previous activities in which they considered the projection of lines between the viewer and the window. It is possible, of course, that the group members did not recall either of these methods from the previous class session. Despite having these alternative methods, the group, with persistence from Alejo, turned to the idea of using the mirror to determine the projection, which indicates the freeform nature of the group activity.

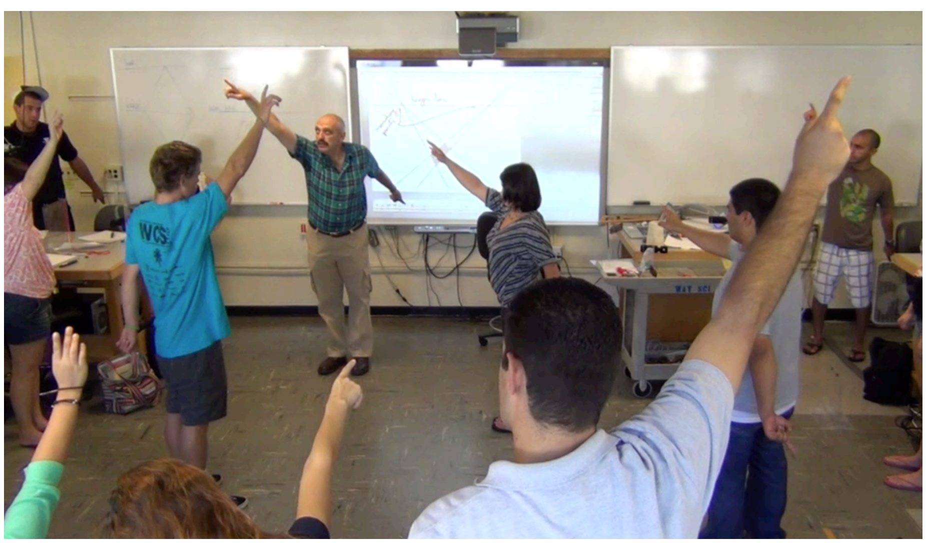


Figure 28. Dr. R. guides the class through an exercise in how to imagine the projection of images behind the viewer.

For Alejo, there was an additional aspect of the freeform nature of the activity. Specifically, Alejo was not the group member to propose using a mirror to determine the projection. Rather, his interest in the mirror was sparked by Carla's suggestion, and then strengthened by Jerry's question about how a mirror might invert the projected image.

4.2.4 Frequency of Instances of Mathematical Play

Recall that this activity-based course with a focus on projective geometry was divided into three components, and that my analyses consisted of two of those three components. The two components on which my analyses were focused were (a) the physical and spatial aspects of projections and the problems that give rise to projective geometry, and (b) synthetic projective geometry in two and three dimensions. These two components were spread across 22 class sessions, with 11 class sessions for each of the two components. Over the course of those 22 class sessions, there were a total of 48 instances of mathematical play that occurred between the five participants. During the first component of the course, there were 31 instances of mathematical play. During the second component there were 17 instances of mathematical play. The greatest number of instances of mathematical play in any one class session was 13. In nine class sessions, three during the first component and 6 during the second component, there were no instances of mathematical play across the five participants.

4.3 Benefits of Mathematical Play

In this section, I describe and illustrate two benefits of mathematical play that arose in this course. First, mathematical play led to students considering *pop-up topics*. By *pop-up topic* I mean a mathematical situation that had not yet been discussed in the course and arose organically from student activity. Often, these mathematical situations were not intended as discussion topics in the course, based on Dr. R. and my discussions regarding course content prior to the class sessions. Second, mathematical play led to argumentation and justification in mathematical situations.

4.3.1 Considering Pop-up Topics

In this course, mathematical play led to students considering pop-up topics – mathematical situations that had not yet been addressed in the course and arose organically from student activity. In this section, I introduce and illustrate an episode in which mathematical play resulted in a group considering a pop-up topic. I first provide the necessary background to assist the reader in understanding the episode in which students engaged in mathematical play and considered the pop-up topic. Next, I illustrate the episode, highlighting the ways in which the episode constitutes mathematical play. Finally, I discuss the aspects of the episode that constitute students considering a pop-up topic.

4.3.1.1 Episode Background. There are three definitions crucial for making sense of this episode: projection, perspective, and invariant. The definitions I provide here are the definitions introduced in the projective geometry course.

> Definition: A *projection* is a correspondence between lines in a pencil and points in a range, where a *pencil* is a set of lines that are concurrent with a certain point, and a *range* is set of points that are collinear. (*Figure 29*)

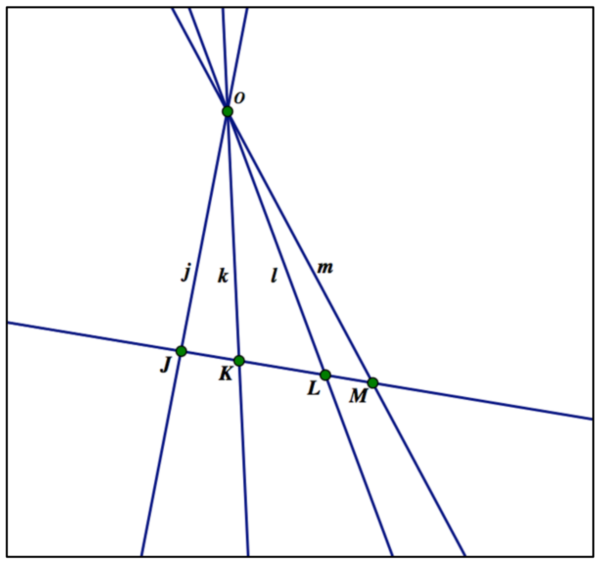


Figure 29. Pencil *jklm* projected into range *JKLM*, through center *O*.

> Definition: A *perspective* is a sequence of two projections. (*Figure 30*)

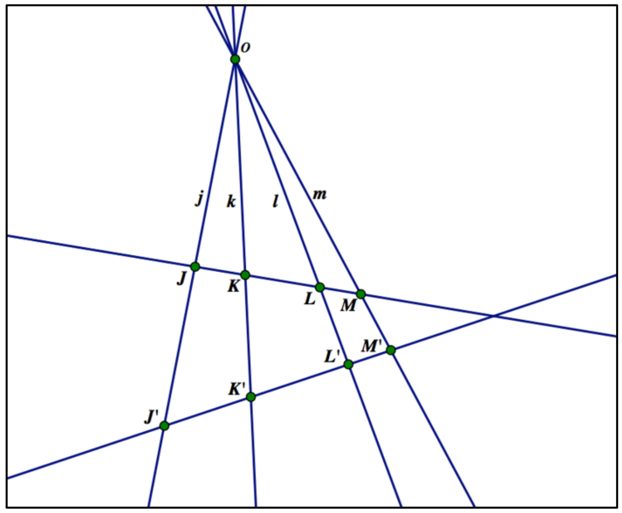


Figure 30. Range *JKLM* projected into pencil *jklm* through center *O*, then *jklm* projected into range *J'K'L'M'*.

Definition: An *invariant* point, *A*, is a point such that after a chain of projections it ends up in the same point, *A*. That is, the point *A* maps back to itself after the chain of projections. (*Figure 31*)

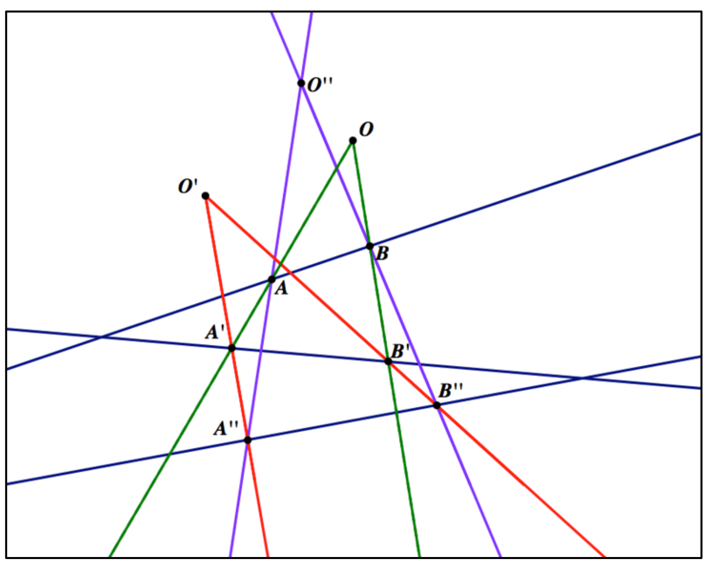


Figure 31. A series of perspectives in which points *A* and *B* are invariant. *A* and *B* project to *A'* and *B'*, respectively, though center *O*. *A'* and *B'* project to *A''* and *B''*, respectively, through center *O'*. Finally, *A''* and *B''* project to *A* and *B*, respectively, through center *O''*.

Prior to this episode, the class was directed to construct a series of perspectives with two points, *A* and *B*, such that the points are invariant. This means that after the series of perspectives, the points *A* and *B* will map back onto themselves. The group begins to construct such a series of perspectives, and Jerry questions whether *A''* and *B''* could be the points where the *O'* pencil intersects the first range line, which is the line with points *A* and *B* (see *Figure 32*, *A** and *B** are the points Jerry is proposing become *A''* and *B''*). Fiona states it could certainly be the case that *A''* and *B''* are on the line with *A* and *B*, at the place where the *O'* pencil intersects the line *AB*. Jerry then questions whether *A* and *B* would be invariant if *A*'' and *B*'' were on the line *AB*, as he is suggesting. Fiona indicates it might work, but that *O*'' (she says "*O* prime") would have to be on the same line as well. Jerry seems intrigued by this, realizing that the points *A*'' and *B*'' could actually map onto the same line as points *A* and *B*. He states, "That's really interesting. Huh." He follows up with, "No, I think that's a cool

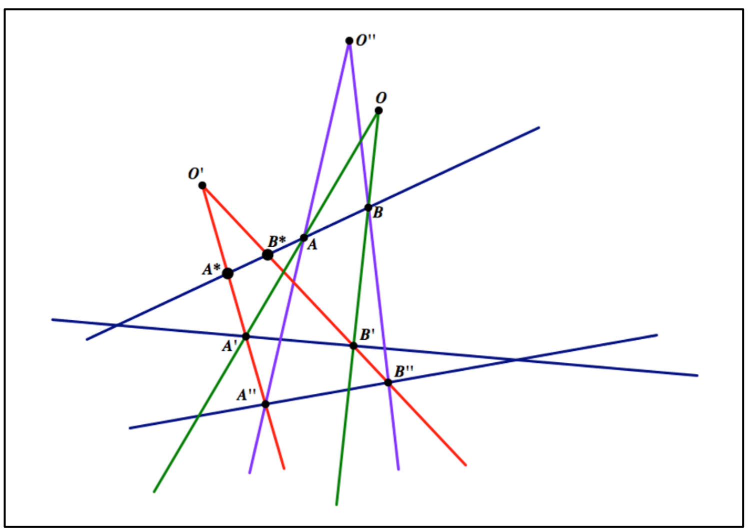
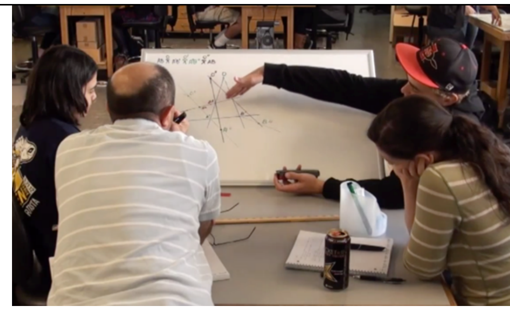
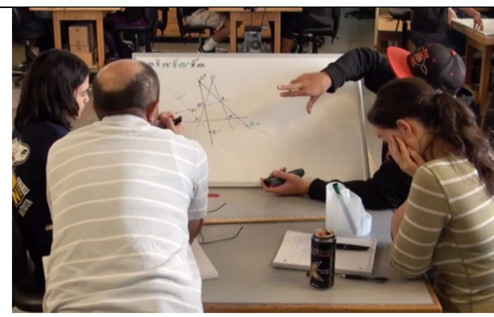


Figure 32. A replication of the diagram drawn on the group's whiteboard.

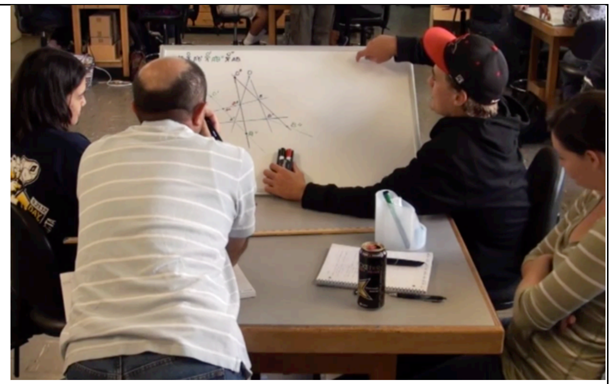
thing to share though." After approximately one minute, Jerry introduces the idea that if the range points, *A*, *B*, *A*'', and *B*'', were on the same line, as he proposed, then the center of projection, *O*'', would also need to be on that same line.

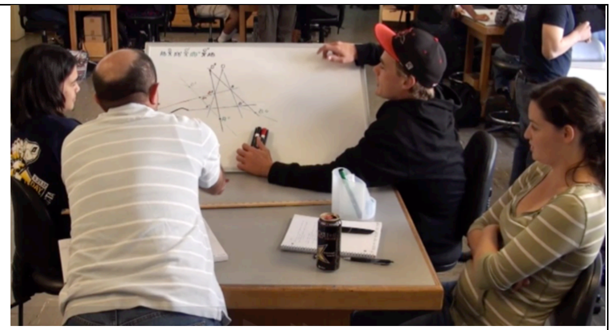
51	Jerry: *whistling* That makes sense because, if you're, that makes sense 'cause they're all on the same line	 Jerry traces a line with his hand over one of the range lines on their whiteboard.
52 53 54 55	Jerry: so of course the O should just be a single line instead of like uh, Fiona: Pencil. Jerry: Yeah. Fiona: A set of pencils? A range of pencils?	 Jerry holds his hand in front of the whiteboard, with two fingers pointing away from himself, fingers slightly spread, then draws his fingers together.
56 57 58	Jerry: It should be, instead of it having the angle between these two lines is like a actual angle, the angle between those two lines would be zero. Candace: It just makes it a, it just makes it a range. Jerry: Making it just one line.	 Jerry points to the first range line on their whiteboard and then the second range line.

Here, Jerry is indicating that for points and their projected images to be on the same line, it would be necessary for the center of projection to be on the same line as well. Jerry mentions the angle between "these two lines" – the lines in the pencil coming from a center of projection – being zero, which makes it one line, rather than two lines. He immediately follows up with wondering what would happen if all the A and B points, including the primes and double primes, as well as the centers of projection, were on the same line.

4.3.1.2 The Episode. In this episode, Jerry poses a question about what would happen if all of the points involved in the projection were on the same line. Specifically, Jerry wants to consider what would happen if *O*, *O* prime, and *O* double prime were the same point, and all the *A* and *B* points, including the primes and double primes, were on the same line, and in particular, on the same line as the centers of projection.

59	Jerry: Cool, sweet, awesome. What if we had them all on the same line?	
60	Candace: *laughing*	
61	Jerry: You know what I mean? O-o-o-o-o	
62	Candace: It would all just be ranges of the pencils.	Jerry leans in slightly and gazes toward the whiteboard.

63	Jerry: What if we just had a O. O double prime er, O, O prime, O double prime.	 Jerry taps the whiteboard with his index finger four times (each time he says O).
64	Jerry: And then A, B is on the same line,	 Jerry moves his hand down slightly on the whiteboard and taps it twice, once when he says “A” and once when he says “B”
65 66 67	Jerry: and then A prime, B prime would be on the same line, and then A double prime, B double prime would be on the same line. Candace: Then they would, then they're all just ranges. Fiona: Mm hm.	 Jerry moves his hand down slightly on the whiteboard and taps the white board four times, one time each when he says “A prime,” “B prime,” “A double prime,” and “B double prime.”

68	Jerry: Let's just draw it.	 Jerry raises his hand back near the top of the whiteboard and taps it once with his finger.
69	Fiona: They wouldn't be perspectives.	
70	Candace: Then they're just ranges on the same pencil.	

In this first part of the episode, Jerry questions what would happen if all of the points involved in a perspective, including the range points and the centers of projection, were collinear. This is the beginning of the instance of Jerry's mathematical play. Jerry quickly eased into his question about having all the points on the same line from his prior realization that if *A*, *A* double prime, *B*, and *B* double prime were to end up on the same range line, then it would be necessary for the center of projection to be on the same line as well, if *A* and *B* were to be invariant. This element, paired with the fact the class had never discussed perspective or projection in which a pencil consisted of only one line, nor points and their projections residing on the same line, indicates Jerry's activity was of a freeform nature.

In his transition from his realization to the posing his question, Jerry's gaze was directed only toward the whiteboard on his group's table. Furthermore, despite Candace and Fiona's attempts to indicate to Jerry his idea does not fit with their notions of a perspective, Jerry continues to pursue the idea, stating he wants the group to construct a sketch of the scenario on the whiteboard. These aspects of Jerry's activity point to his lack of concern for what others in his environment, including his own group members, are doing and what they might think about his actions. Thus, Jerry's activity is autonomous in nature.

At his point Hakan, Jerry's fourth group member, diverts the conversation briefly to question whether a word Fiona has written on the board should be "coincide" or "co-liner." As soon as the distraction is ended, Jerry returns to his idea of having all the range points and centers of projection all on the same line. He asks Candace what she was saying about why his proposal would not constitute a perspective, wondering if she was saying a perspective must consist of ranges and pencils. Candace responds that she was not suggesting Jerry's set up would not be a perspective, but rather his set up would merely consist of ranges.

Hakan again diverts the conversation, this time questioning whether the word for points on the same line should be "co-liner" or "collinear." Jerry confirms the word should be collinear, and again swiftly returns to questioning Candace about why she thinks his setup will not be a perspective.

<table>
<tr><td>71</td><td>Jerry: Um, I'm confused now. Why wouldn't that be a perspective?</td><td rowspan="3">
Jerry leans back slightly in his chair and looks toward Candace.</td></tr>
<tr><td>72</td><td>Candace: I'm not saying it wouldn't, but you have to think of it as, okay,</td></tr>
<tr><td>73</td><td>Jerry: Let's think of it.</td></tr>
</table>

74	Candace: here's, here's *A* and *B*	 Candace places a ruler against and at an angle to a yardstick on the table.
75	Jerry: Let's just pose this question. *O*, *O* prime, *O* double prime. *A*, *B*, *A* prime, *B* prime, *A* double prime, *B* double prime.	 Jerry draws a point near the top of the whiteboard and labels it with *O*, *O*', and *O*''. He t draws a line straight down from that point, and then labels points *A*, *B*, *A*', *B*', *A*'', and *B*'' on the line (*see figure 33*).

Despite the mathematical objection it seems Candace might propose, Jerry persists in trying to convince his group to consider his idea. Rather than allowing Candace to complete her explanation, Jerry interjects with his desire to play with the idea, to the extent that he draws a representation on their whiteboard. This desire to consider the idea, rather than listen to Candace's explanation, is further indicative of autonomous activity, and, accordingly, mathematical play.

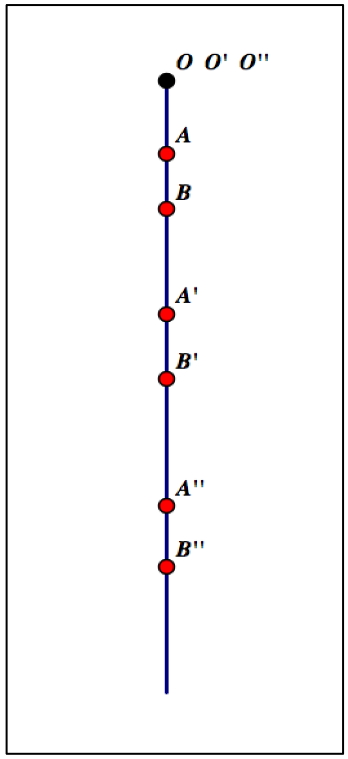


Figure 33. A recreation of the diagram Jerry constructs on the whiteboard.

4.3.1.3 Considering Pop-up Topics. In this episode, Jerry became curious about the properties of perspective and wonders whether the projection of points A and B, through a chain of perspectives, could end up such that the projections are on the same line as the points *A* and *B*. Specifically, he wonders whether *A*' and *B*' could project back to the line with *A* and *B*, through the center *O* (refer to *Figure 32*). This leads him to wonder whether, if *A*' and *B*' do project back to the line with *A* and *B*, then could *A* and *B* be invariant. Up to this point in the discussion about perspectives, the class had not considered whether two points could project back to the line from which they came, after the first perspective was carried out. This is the first instance of Jerry considering a pop-up topic.

The second instance in the episode of Jerry considering a pop-up topic came when Jerry questioned what would happen if all of the A and B points, including the primes and double primes, as well as the centers of projection, were collinear. Up to this point in the course, the class had only discussed the situation in which points in a range projected to points in a range on a separate line, and hence constitutes a pop-up topic.

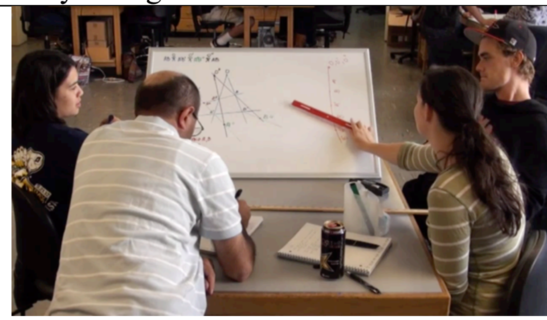
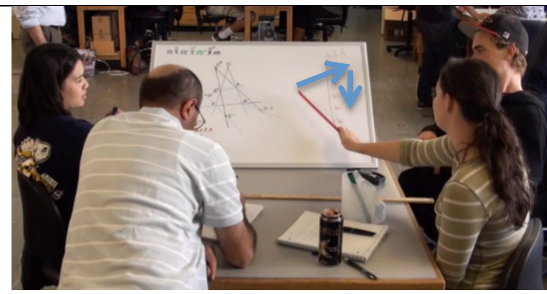
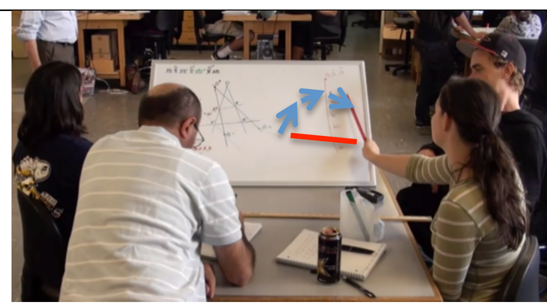
The conversation regarding Jerry's proposal of having all the range points and centers of projection on the same line continues for approximately three and a half more minutes, with short breaks of discussing other topics. In the next subsection, I return to this episode with a focus toward how the group discussion progressed, specifically arguing whether Jerry's suggestion constitutes a perspective, after this pop-up topic was introduced.

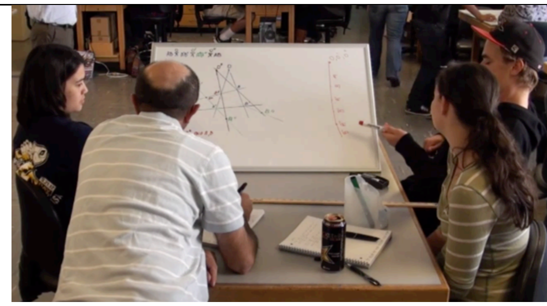
4.3.2 Mathematical Justification and Argumentation

Mathematical play can lead to mathematical justification and argumentation, which are critical aspects of mathematical activity. In this subsection, I expand on the episode introduced in the previous subsection, with a focus toward how the group discussion progressed after Jerry's proposal. In particular, the group begins to mathematically argue whether the particular situation Jerry's has suggested, with the centers of projection and all of the range points collinear, constitutes a perspective.

4.3.2.1 The Justification and Argumentation. In the previous subsection, the episode ended with Jerry drawing his proposed scenario on the whiteboard. At this point, the episode turns toward mathematical argumentation, with Fiona explicitly stating, for the second time, that she does not believe Jerry's sketch will constitute a perspective. Fiona and Candace both attempt to provide justification for the argument that Jerry's scenario does not constitute a perspective.

76 77 78	Fiona: I don't think it'd be a perspective. Candace: No, cause you're just picking – Fiona: It's a range.	 Jerry finishes drawing his scenario. Fiona and Candace toward the whiteboard.
79 80 81	Fiona: So that would be listed *AB*. Candace: 'Cause this is one pencil. This is one pencil. It's not two pencils. Fiona: Yeah.	 Fiona indicates to Jerry's sketch of his scenario with her hand. Candace taps the line in Jerry's sketch with a ruler, and then holds the ruler next to his line.
82	Candace: So it can't be *B*,	 Candace taps next to the letter B in

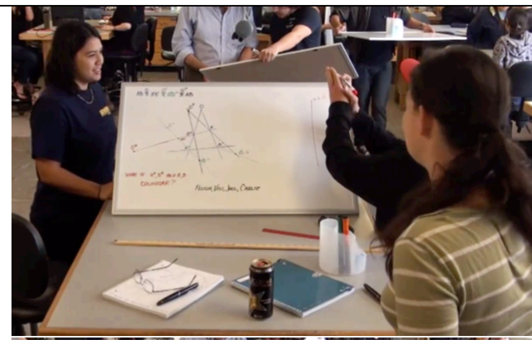
		Jerry’s diagram.
83	Candace: unless you want to do another [line]	 Candace places the ruler on the whiteboard crossing the line in Jerry’s sketch.
84	Candace: like this, and then that could be a point, and then those could be pencils.	 Candace taps the board three times with the ruler – once to the left of Jerry’s diagram, then slighly higher on the right of his diagram, then lower on the right of his diagram.
85 86	Candace: Or, you know what I mean? Like, those could be pencils. Jerry: But are you su- *The group goes quiet for a few moments.*	 Candace places the ruler on the whiteboad, perpendicular to Jerry’s diagram. She then taps the white board in three spots – twice high on the left side of Jerry’s diagram, and once high on the right side of the diagram.

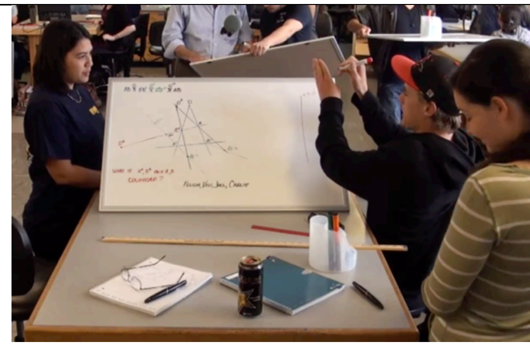
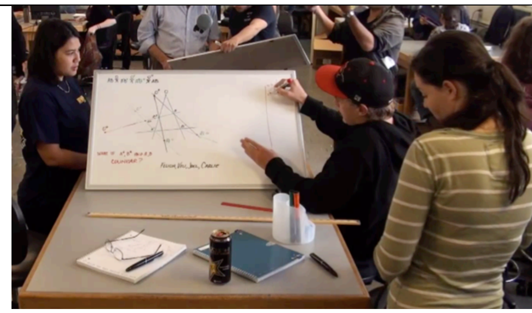
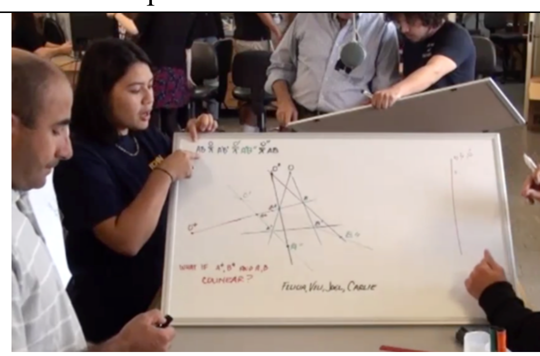
87	Jerry: “It's interesting. We'll leave it there and we'll ask about it.”	 Jerry taps the side of the whiteboard with a marker in his hand.

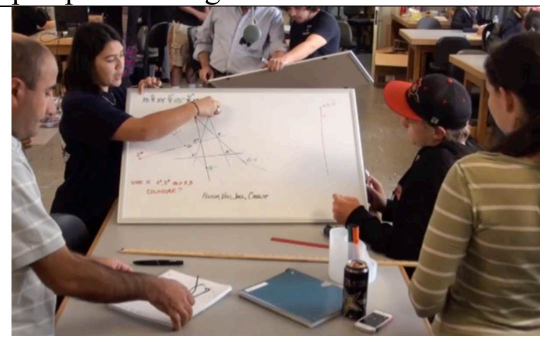
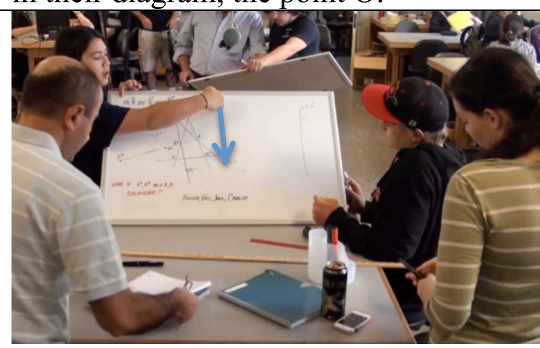
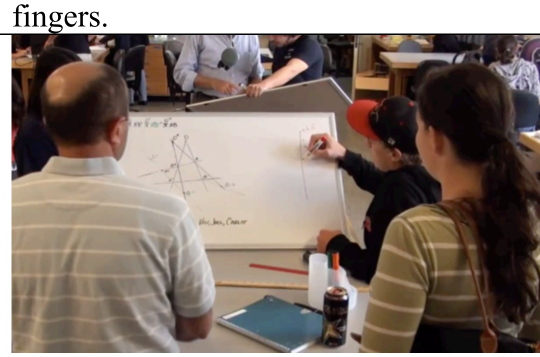
From the beginning of the episode, Jerry is implicitly suggesting that he believes having all the range points and the centers of projection on the same line will constitute a viable projection, if not a perspective. This is evidenced by his multiple attempts at questioning Candace about why his idea would not be a perspective. Fiona and Candace both hold that Jerry’s proposal does not constitute a viable perspective. In this brief passage, Fiona and Candace argue that Jerry’s diagram represents only one range (Fiona) or one pencil (Candace) – in essence, suggesting there is only one line present and thus cannot constitute a perspective. Candace suggests for the mathematical situation to constitute a perspective, Jerry’s diagram would need to include more lines, amounting to additional lines in a pencil and an additional line for a range.

Jerry does not seem convinced by his group members’ arguments, and does not agree or disagree with Fiona and Candace. Instead, he suggests they leave the sketch on their whiteboard and ask about it next time. This suggests Jerry is not convinced the sketch is not a viable perspective, and hence not convinced by Fiona and Candace's arguments.

The class session starts to come to an end, so the conversation wanes. To Jerry's disapproval, Candace erases the sketch of his proposed scenario. After close to one minute, Jerry decides to redraw his scenario on the whiteboard. Upon finishing sketching his scenario for a second time, Jerry moves to clarify arguments put forth by Fiona and Candace.

88	Jerry: Wait! The only thing is like, what, what you're saying is if they were spread, if they came together it would be	  Jerry places his palms together and them spread the heels of his hands apart, keeping his fingertips together.

89	Jerry: *A*, *B* would be the same point?	 Jerry points with a marker in his right hand to a place on the side of his left hand.
90	Jerry: Like *A*, *B* would be, right there. Is that what you guys are arguing?	 Jerry writes an A and a B next to his line, near the top.
91	Fiona: No, but this is called a range, when they're on	 Fiona points to the first part in their notation for a series of perspectives.
92	Fiona: a line like this, and they	

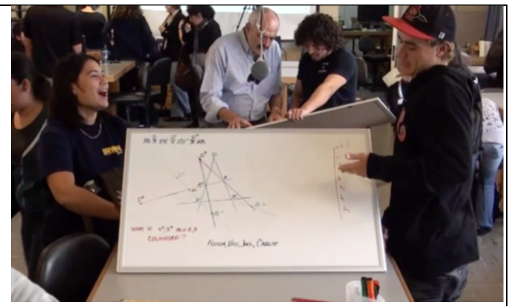
		Fiona traces a line out and back with her hand, over the first range line in their perspective diagram.
93	Fiona: meet at this point.	 Fiona taps the whiteboard with her fingers over the first center of projection in their diagram, the point O.
94 95 96	Fiona: So of they're all on the same line- Jerry: Then it's just a range, it's not even a perspective. Fiona: It's just a range. It's no longer, exactly, it's no longer a perspective.	 Fiona traces a downward line in the middle of the whiteboard with her fingers.
97 98	Jerry: Well let's just, let's just have the idea. Candace: So it can't be a perspective cause they're all on the same pencil, which makes them all ranges of each other.	 Jerry reconstructs his diagram by drawing the A and B points onto his line.

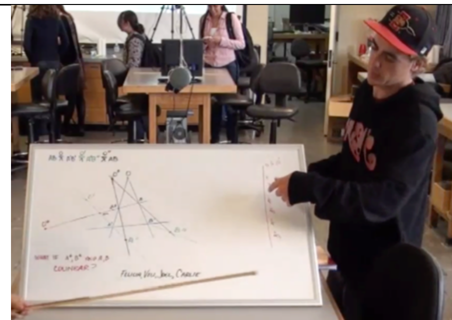
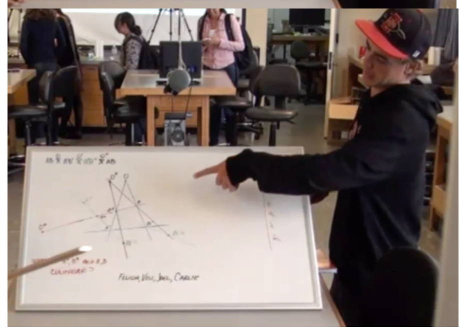
99	Fiona: Exactly. So then you can't say there's a perspective cause a perspective deals with two different lines, meeting at a point, and a point.	 Fiona indicates to the first range line in their perspective diagram.
100	Jerry: Well whatever, we can just make that point next time. For people like me.	 Jerry raises his right hand slightly into the air, then scoots back in his chair.

Jerry questions the argument put forth by Fiona and Candace, wondering if his interpretation of their argument is correct. Fiona says his interpretation is not accurate, and explains her reasoning again. Fiona tries to convince Jerry that what he will be dealing with in his sketch is only a range, with no pencils, since all the points are on the same line. She does this by indicating to the perspective notation located on their whiteboard, and then indicating to the line corresponding to the part of the notation she highlighted with her gesture.

As Fiona is explaining, Jerry suggests that they “just have the idea” and continues to draw his scenario. Candace chimes in to explain why they only have a range in Jerry’s sketch, but he continues to draw. Fiona argues that they cannot say they have a perspective, since a perspective deals with two different lines. Jerry replies, suggesting he still wants to think about the situation.

In the final piece of the episode, Fiona makes a joke, suggesting Jerry is not listening to logic, and Jerry concedes that likely the situation he has proposed does not constitute a perspective.

101	Fiona: Just listen to your logic.	 Fiona packs up her things. Jerry indicates to his diagram on the whiteboard.
102	Jerry: No, I listened to it. Let's just maybe point that out to other people.	
103	Fiona: (laughing) I'm just kidding.	
104	Jerry: I understand what you're saying.	
105	Fiona: I'm just kidding. I think it would be a good question to ask anyway.	
106	Jerry: Yeah.	
107	Fiona: Only because, that's just our opinions, that's not what we understand. Maybe they'll understand something different.	

108	Jerry: That would make sense though, that wouldn't be a perspective. Cause if it was a perspective then that would confuse what a perspective is. If that's a perspective and that's also a perspective, that's like, "Well, which one is it?" You know what I mean?	
109	Fiona: Yeah that gets confusing.	 Jerry indicates to his diagram and then to the group's perspective diagram. Fiona walks away.
110	Jerry: So I'm pretty sure that that one's not.	 Jerry collects his things.

Here, Jerry indicates he suspects Fiona is correct that the situation he proposed would not constitute a perspective. He expresses that if both their first diagram of a chain of perspectives and his diagram of all the points on the same line constitute perspectives, then the definition of perspective is unclear. In this case, it appears, through argumentation, Fiona provided sufficient justification, by Jerry's sense, to convince him his diagram would not constitute a perspective.

4.3.2.2 Summary. In this subsection, I illustrated an episode of mathematical argumentation that emerged from the mathematical play of a student. Jerry introduced a mathematical situation he wanted to consider, which he seemed to believe constituted a perspective, and two of his group members disagreed, stating his situation would not be a perspective. Through the argumentation, with both Fiona and Candace providing justification, Jerry came to reconsider that the situation he proposed might not be a perspective.

4.3.3 Summary of the Benefits of Mathematical Play

In this section, I examined an episode in Jerry and Fiona's group in which Jerry proposes a pop-up topic. The proposition and subsequent diagramming of the scenario constitute an instance of mathematical play. Through this instance of mathematical play, two of Jerry's group members, Fiona and Candace engage in mathematical argumentation to try to convince Jerry his proposed scenario does not constitute a valid mathematical perspective. Had Jerry not initiated the consideration of this scenario, an instance of mathematical play, the discussion of whether a perspective can consist of only points on a single line would likely not have occurred. Thus, the instance of mathematical play led to this particular occurrence of argumentation. Further instances of mathematical play leading to considering pop-up topics, and argumentation and justification will be presented in Chapter 6.

There is some evidence suggesting mathematical play with the mathematical tools used in the course led students to developing (a) an understanding of the coordination between the different representations of mathematical situations, such as the two versions of Alberti's Window, and (b) assisted students in developing and understanding of the mathematical ideas at hand. For example, the first instance in which the class was asked to determine the projection of a square on the tabletop that resides between the window and the eyepiece, there were multiple instances of mathematical play with the Alberti's Window within the groups. In particular, some groups either moved their eyepiece to the edge of the table and imagined the window, or they moved the window to the floor. For one group, this moving of the eyepiece to the edge of the table, an episode I will discuss in subsection 4.4.3, convinced one student that her initial prediction of what the projection would look like was incorrect. In Chapter 6, I address instances in which students developed an understanding of the coordination between the two representations of the Alberti's Window through mathematical play.

4.4 Providing the Opportunity for Mathematical Play

In this section, I discuss the instructional and course aspects that contributed to the opportunity for mathematical play. In particular, the instructor influence in the course, the nature of the task at hand, and affordances and limitations of mathematical tools, provided the opportunity to engage in mathematical play. I first discuss instructor influence, followed by the nature of the task, and end with affordances and limitations of mathematical tools.

4.4.1 Instructor Influence

Dr. R. structured the course such that students worked in groups on activities that could help them develop their understanding of ideas in projective geometry, and he worked hard to cultivate productive classroom social norms (Cobb & Yackel, 1996), which are the patterns of behaviors considered acceptable by the classroom culture. Dr. R. negotiated classroom social norms that supported students sharing their thinking. For example, he frequently brought the class back together during activities so that students could share their findings, and he seemed to carefully listen to and acknowledge students' contributions, often revoicing (O'Connor & Michaels, 1993) how he interpreted their ideas. In one instance, Alejo presented his group's idea about how a square would project when located between the window and the viewer, and where the location of the vanishing point would be as a result. Alejo expressed his group was thinking there would be a vanishing point below the tabletop, to which the sides of the square would converge toward. Dr. R. first clarified with Alejo that his idea meant the vanishing point would be located lower than the height of the eyepiece. Dr. R. reserved judgment of the group's idea and then looked to other groups for their ideas.

Dr. R: So here we have a hypothesis that the vanishing point doesn't match the straight eyepiece (extending a line with his finger straight our in front of him from his eye). Right? Because if it were, it would be above [the projected square], but here you get it below.
Alejo: Yeah, those are our ideas.
Dr. R: Ideas, yes. Ideas are important. Yes? (indicating to a student with his had raised.

Here, Dr. R. acknowledged Alejo's group's idea, reserving judgment, and then looked to other groups for their ideas. This demonstrates the ways in which Dr. R. cultivated classroom social norms that supported students sharing their thinking. In addition, when particular groups of students had been quiet for periods of time, Dr. R. called them out to share, saying such things as, "Francis, what do you think?"

Typically, the way in which Dr. R. began class discussions in which students shared their thinking allowed for a variety of responses. During the first day on which the class used the Alberti's Window to project parallel lines, for example, Dr. R. called the class back together by saying, "Alright, so let's share your main observations or questions or conjectures that you made about how um, parallel lines get projected from some plane to the Alberti's Window." This request for students to share is in contrast to a more narrow question, such as, "How do parallel lines project?," which might be asked by a different instructor. This open prompt for students to share made room for students to present their findings, as well as introduce questions they had about unclear aspects of the activity. This appeared to make students feel comfortable sharing the various observations they made during the activity, as exhibited by students' willingness and enthusiasm to share.

On multiple occasions, Dr. R. suggested to the class they should be playing with the mathematical ideas at hand. For example, both when students began working with the physical and the synthetic aspects of projective geometry, Dr. R. indicated to the class they would be "playing with parallel lines" and "playing with projections," respectively. Similarly, when the class began proving theorems in projective geometry, Dr. R. suggested that often the beginning of proving theorems in projective geometry is “playing with the axioms.” While the instructor did not suggest to students that they carry out mathematical exploration through autonomous and freeform activity, the suggestion of *playing* with mathematical entities has the potential to encourage students to explore more freely than they otherwise might, as it does not limit them to particular actions. The class session in which Dr. R. suggested the class play with parallel lines had the greatest number of instances of mathematical play of all the class sessions. Specifically, there were 11 instances of mathematical play across the five participants during this activity.

At the end of a whole-class conversation about the first chapter of A Mathematician’s Lament, as well as the emergence of contemporary art, Dr. R. further encouraged mathematical play by explicitly stating, while mentioning the art project assignment, “[T]he intention of this art project is for you to play with mathematics, and for you to feel the freedom to do so, and that nobody will tell you not to double fold.” Here, Dr. R. is referencing a story a student conveyed about her son getting into trouble at school for folding a paper in half twice, while his class had been instructed to fold a paper in half. Here, Dr. R. communicated his support of students engaging in mathematical play and indicates they should not fear being judged for exploring and using mathematical ideas in the way they choose.

In addition to explicitly suggesting students should play with mathematical ideas, such as parallel lines, projections, and axioms, Dr. R. demonstrated his approval of mathematical play through his actions during whole-class discussions – particularly the ways in which he allowed conversations to progress and entertained students' questions, and modeled mathematical play. During the first discussion regarding how parallel lines project onto the Alberti's Window, the major focus of the discussion was how circles project, as several groups had chosen to play with projecting circles after they had projected parallel lines. Dr. R. did not try to focus the conversation solely on parallel lines; rather, he allowed the class conversation to progress until there was an opportunity to return to discussing parallel lines. In this instance, Alejo questioned another student about whether the compression of a projection occurs faster vertically than it does horizontally.

> Alejo: But shouldn't it squish faster [horizontally], than [vertically]? 'Cause I mean look at your railroad tracks. If you take these tracks to be tangents on the side of the circle, like imagine the circle is now in a square, then you would see it shrink [horiz.] faster than [vertically].

Alejo cited the look of railroad tracks to support his claim that projections will compress faster vertically. The student to whom Alejo directed the question was unsure about the answer, and Dr. R. capitalized on this by suggesting the class consider two parallel lines of string on the floor to be railroad tracks. He requested Alejo come to the front of the class to trace the projection of the string onto the Alberti's Window. Alejo went to the front of the room and traced the projection of the parallel strings onto the window (*Figure 34*).

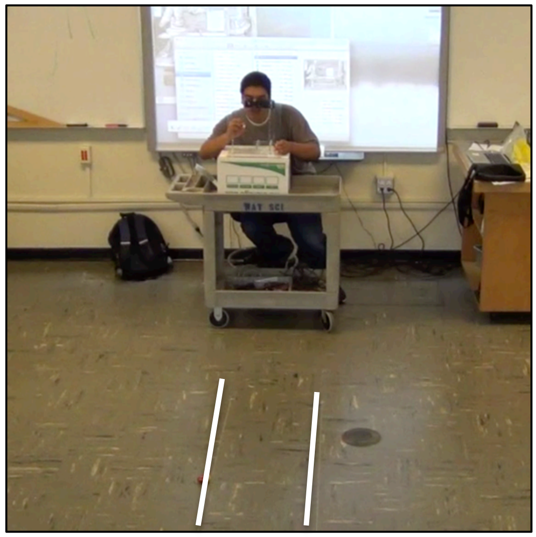


Figure 34. Alejo traces the parallel strings onto the window at the front of the classroom.

When Alejo finished, Dr. R. noted how the projection of the strings was converging, and the class conversation moved toward how not being centered with the image being projected affects the projection. Jerry interrupted the conversation with a question.

> Jerry: I just have a question. What happens if you're laying on the ground and looking at it? Do they still converge, or just at a slower rate?
> Dr. R: We'll do that in a moment.

Here, Jerry proposes as "what if" question that could result in mathematical play. Dr. R. suggested they return to the question momentarily. Dr. R. had Alejo perform another projection of the strings with equal distances marked off on the strings. After Alejo traced this new projection, and the class discussed how as you get farther away from the window, the shorter the projected lengths, Dr. R. returned to Jerry's inquiry.

Dr. R: But Jerry also had another question. What was your question? If you go on the floor?
Jerry: Yeah, if you just look at it on the same level, as where it is.
Dr. R: (places Alberti's Window on the floor) So Jerry, it's your turn to get on the floor.
(class laughs)

In this instance, Dr. R. could have replied that the projected lines would converge faster when you lay on the ground. Instead, however, he asked Jerry to carry out the action he had proposed (*Figure 35*). That is, Dr. R. had Jerry engage in mathematical play at the front of the class. Dr. R. encouraged mathematical play behavior by having Jerry follow through with his "What if" question. When Jerry finished tracing the projection, Dr. R. noted, "So you see how much more dramatic they get closer to each other," pointing out what the class should notice from the instance of mathematical play and validating the usefulness of Jerry's exploration.

Figure 35. Jerry lies on the floor and traces the parallel strings onto the window.

Dr. R. further validated student engagement in mathematical play through drawing attention to the activity of groups engaged in play. During an activity in which the class was attempting to determine where the vanishing point, or horizon line, is located on the window, Dr. R. called the class back together for a conversation and directed attention to a group engaged in mathematical play.

> Alright, let's do some uh, sharing of this, vanishing point. Let's share what you are trying to do, what you found, or what you didn't find. What are you doing? (indicating to a particular group with their Alberti's Window on a rolling cart). You seem to be very engaged into something.

The group explained they were using the Alberti's Window on the cart to try to determine at what point, as they increased the distance between the window and the paper with the parallel lines, would the two projected lines meet on the window. After they finished their explanation, two other groups presented their ideas. Dr. R. then directed attention to another group that had also been engaged in mathematical play during the task. This situation was described in the vignette at the beginning of section 4.2, in which one group member held the parallel lines and rolled back in a chair and another group member traced the projection. Dr. R. asked this group to explain what they were doing to try to find the vanishing point. The way in which Dr. R. drew attention to the two groups that had been engaged in mathematical play demonstrated how he valued mathematical play.

Instructor influence may also hinder mathematical play. For instance, consider the episode presented as the illustration of mathematical play, in which Jerry and Alejo's group was engaged in play by using the mirror to try to determine the projection of parallel lines from behind the eyepiece. Recall that I intervened during this episode, asking the group what they were doing with the mirror and redirecting their activity to using a string to determine the projection of the parallel lines. In this case, my intervention – while done with the best of intentions – halted the mathematical play of the group, which may or may not have helped the group to develop their understandings of projective geometry.

4.4.2 Nature of Task

As I indicated previously, the nature of a task influences the determination of whether a scenario constitutes mathematical play. Consequently, the nature of a task can influence the opportunity for mathematical play. The degree to which a task is open – in the sense of how rigid and explicit the instructional steps are for the task – affects the opportunity for students to engage in mathematical play. Tasks that are more open allow more room for students to explore mathematical ideas in flexible ways.

In the beginning of the work with the physical Alberti's Window, Dr. R. oriented the class to the task by focusing attention on the particular components of the mathematical tool, rather than on the particular actions the group should carry out with the tool.

> And so here we have the eyepiece. The eyepiece consists of a little hole where you put your eye, so that you keep your eye in the same spot. And so you can, the eyepiece, you can move it up and down. So you go like this (looking through the eyepiece). And, here we have, this is all made, made here in the lab. So this is the window. So we call it Alberti's Window, in honor of Battista Alberti. And so the idea

> is you put this on your tabletop, and you put your eye somewhere, and on the other side here, with a piece of paper, you draw something. And then with a marker… you trace the object on the window. And so we are going to study the projection of whatever you have on the tabletop, with respect to this window. And this (indicating to the hole in the eyepiece) being the center of projection. And so in each table you will set up one of these. And I suggest that we start working with parallels. So you will have just parallel lines, and see where the parallel lines get projected on the window. So you will play with parallel lines in different directions, and then we will have a discussion of what happens with parallel lines. And then we'll work with other shapes. Yes? Okay.

In these minimal instructions, the focus is on the components of Alberti's Window, rather than on the particulars of what groups should do with the window. Dr. R. notes the height of the eyepiece can be adjusted, yet does not suggest this is something the groups should experiment with. Similarly, in his description of how to use the window, he mentions they should "put your eye somewhere," and does not tell them specifically where they should set the eyepiece, nor that they might consider changing the location of the eyepiece. Finally, Dr. R. gives the instructions that they should set up the window on their table and "play" with parallel lines in different directions. Note he does not suggest they can have their paper with parallel lines at different distances from the window. Of the multiple aspects of the Alberti's Window that can be varied – the height of the eyepiece, the location of the eyepiece, the location of the image to be projected, and the image to be projected – Dr. R. specified only the image to be projected, and that they should vary the direction of that image. This left multiple aspects of the Alberti's Window the groups could choose to vary, if they chose to engage in mathematical play. It's interesting to note that not all groups remained working with parallel lines for the duration of the task. Several groups moved on very quickly to working with shapes such as circles and stars.

During this task, many groups engaged in playing with the multiple variable aspects of the Alberti's Window, such as the height of the eyepiece, the distance of the eyepiece from the window, and the distance of their lines from the window. And, while the instructor indicated to the class that they should "play with parallel lines in different directions," which most groups seemed to do by turning the paper on which their lines were drawn, so that the lines sat at different angles to the window, some groups went a step further and lifted the paper from the tabletop, holding the paper perpendicular to the tabletop.

In contrast to the previous task instructions, the subsequent task instructions, in which Dr. R. focused the class on determining where a set of projected parallel lines would meet on the Alberti's Window, were more guiding. In this next task, he directed the class in the following way,

> You see that in general, like we have here, we have here um, parallel lines, they tend to converge (showing one group's results from the previous task). So where do they think that they will meet? So this is what we are going to work on. The point where they meet is called the vanishing point. So where is that vanishing point? Yes?

A student interrupts, asking, "Wouldn't it be on the horizon?" Dr. R. acknowledges this and continues,

> Okay so we need to, to experiment with that. What is the horizon then? So uh, we cannot, whatever lines we have on the floor will end, will be finite. But you can extrapolate. Like you can um, keep going, and they will meet somewhere. This is the vanishing point. Here, you say the vanishing point is on the horizon, let's figure out

what the horizon is or how to locate the horizon. So you will work on parallel lines, but always, in this case, on the tabletop. So the parallel lines are on the tabletop. And you try different heights of your eyepiece, uh and try to see, but keep record, like you see that there is here like a scale (picking up eyepiece and showing the increments marked on the stand). So you say if I have it at nine, this, this is a vanishing point. Then have it at five, this is my vanishing point, and so forth. We will keep track of what the horizon line is and what the vanishing point is.

In this scenario, Dr. R. gives more detailed and rigid directions for how the groups should carry out the task of determining the horizon line and vanishing points. Specifically, he instructs the class to project the parallel lines with varying heights of the eyepiece and to keep record of where the projected lines meet. In this task, he limits the degree of openness by providing more specified instructions.

While instances of mathematical play still occurred during this more constrained task, in which Dr. R. directed the class to vary the height of the eyepiece and determine the vanishing point, there were fewer instances of mathematical play than during the task in which Dr. R. focused the task instructions on the components of the Alberti's Window, rather than on the particular activity students should carry out with the window. In the less constrained case, between the five chosen participants, there were nine instances of mathematical play. In the more constrained case, with the less open-ended task, there were four instances of mathematical play between the five participants.

During the synthetic projective geometry component of the course, in which the course was focused on the axioms, definitions, and particular proofs in projective geometry, instances of mathematical play were less frequent than when the class considered the physical aspects of projective geometry, which may be a result of more guided tasks. For example, during the class introduction to Desargues' Theorem, which states that two triangles that are perspective from a point are also perspective from a line, Dr. R. guided the class though constructing the necessary diagrams to prove Desargues' Theorem. First, Dr. R. guided the class through drawing a diagram of two triangles that are perspective from a point (*Figure 36*). Specifically, Dr. R. first drew a pencil with three lines on the smart board, and then had the groups do the same on their individual whiteboards. Next, Dr. R. drew a triangle with each of its vertices on the pencil, then a second triangle. The groups of students followed the same steps on their whiteboards. Dr. R. followed a similar demonstrate-follow procedure for having the groups construct two triangles that are perspective from a line (*Figure 37*). In this task, between the five chosen participants, there were no instances of mathematical play. This is not particularly surprising, as the groups were reproducing the actions of Dr. R. Subsequently, Dr. R. guided the class through the proof of Desargues' Theorem, which also resulted in no instances of mathematical play for any of the five chosen participants.

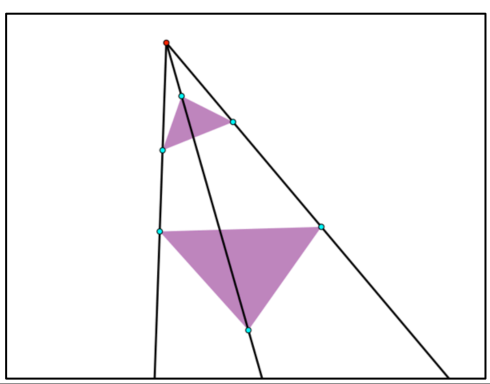


Figure 36. Two triangles that are perspective from a point.

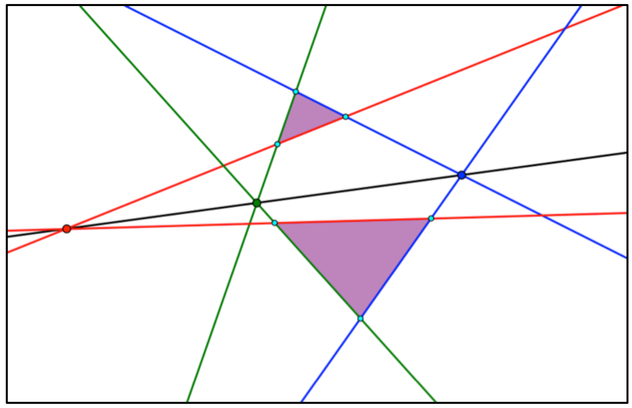


Figure 37. Two triangles that are perspective from a line.

4.4.3 Affordances and Limitations of Tools

All mathematical tools have affordances and limitations, and these can have an effect on mathematical play. In this course, students used several different mathematical tools. During the first component of the course, in which the class considered the physical aspects of and problems that give rise to projective geometry, students worked with both the physical and the GSP versions of the Alberti's Window. During the second component of the course, in which the class considered the axioms, definitions, and theorems of projective geometry, students primarily worked with white boards and markers, as well as Geometer's Sketchpad, starting with blank sketches.

In general, the affordances and limitations of the two versions of the Alberti's Window are related to the static versus dynamic nature of the two tools. Working with GSP provided the opportunity to watch aspects of a GSP sketch change while other aspects were being manipulated, which cannot be done with the physical window. For example, with the GSP Alberti's Window, students could watch the projection of an image change while the original image is dragged from one location to another. Similarly, the projection of an image changes when the eye height or eye distance lines are moved up or down. If a student wanted to see how a projection changed with the physical Alberti's Window, she would have had to trace the projection of the image multiple times, changing one of the variable aspects each time.

Another difference between the physical and GSP versions of the Alberti's Window is that the physical window requires students to use their imaginations to project images that reside between the window and the viewer, as well as behind the viewer. This is a result of the design of the physical window, where only images that are on the opposite side of the window from the viewer can be traced onto the window. The GSP version of the Alberti's Window, on the other hand, allowed students to instantly see the projection of an image that was located between the window and the viewer, or behind the viewer.

This static versus dynamic character also arises between using a whiteboard or GSP to construct diagrams of perspectives and projections. When using a whiteboard, any changes to a diagram must be made through erasing and drawing new aspects of the diagram. Geometer's Sketchpad, on the other hand, allows students to drag points and lines to different locations, which immediately shows students how their diagram changes based on the new locations.

If we consider the dynamic nature of GSP to be an affordance of a mathematical tool, while the static nature of the physical Alberti's Window, as well as the whiteboard, to be limitation, then both affordances and limitations of the mathematical tools gave rise to mathematical play. I first discuss two affordances that gave rise to mathematical play, followed by two limitations that gave rise to mathematical play.

During an instance in which the class was working with the GSP version of the Alberti's Window, Jerry and Alejo engaged in mathematical play using the affordance of the dynamic nature of GSP. The group had been discussing a homework assignment in which they were asked if they could create a piece of contemporary art using projective geometry. Carla, one of the group's members, mentioned that a projected grid is projective geometry and so it could constitute an art piece. Alejo began to construct a grid in GSP and Jerry got excited to think about how an already projected grid, as they had created in a homework assignment (*Figure 38*), would react when projected again.

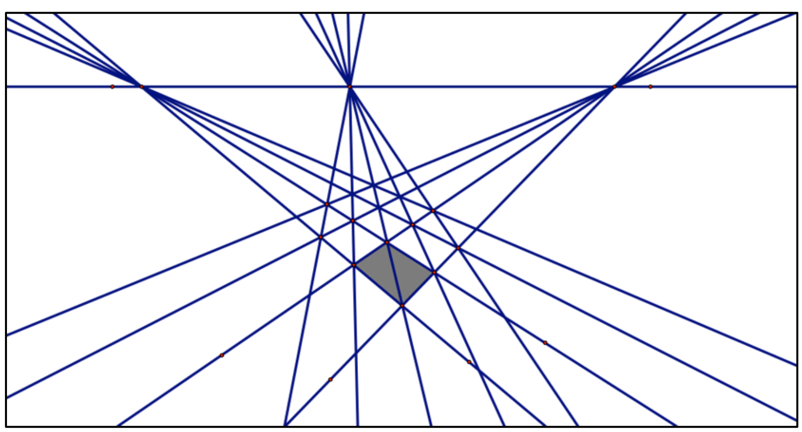


Figure 38. A perspective grid students created for a homework assignment

Alejo: Oh! What if I, what if I do draw a checkerboard? Okay, so let's say I took a still line.
Jerry: Oh, if you transform the checkerboard!?
Alejo: Like I drew the checkerboard and then transform it.
Jerry: That'd be nuts!
Alejo: I don't know, I don't know.
Jerry: I really want to see what it looks like. Let's do it.
Alejo: Okay, let's try it.

Jerry became intrigued by the idea of projecting the already projected grid and expressed his interest in exploring the situation. Since the class was working on a different task, and the class had not discussed projecting an image that, in a sense, had already been projected, Jerry was initiating mathematical play. As Jerry was expressing his interest in and confusion about projecting a perspective grid, Carla was resistant, saying, "I'm not going to imagine that." Jerry replied, "We won't have to imagine it, we'll see it." Jerry's response highlights the affordance of the GSP Alberti's Window, being able to instantly see the projection of an image.

Alejo, with Jerry helping, proceeded to construct a grid in GSP and then project it using the built-in transformation (*Figure 39*). The grid Alejo constructed and projected was not a perspective grid, but a simple rectangular grid, which was not what Jerry was particularly interested in seeing. After the grid was projected, Alejo proceeded to drag the grid to various locations on the screen, which resulted in the projection changing accordingly, based on where the original image was with respect to the horizon line, the baseline, and the eye distance line.

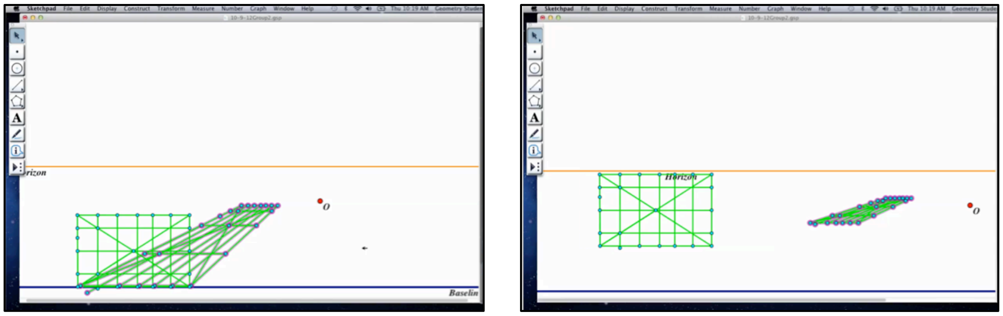


Figure 39. Alejo's projected grid on Geometer's Sketchpad, in two different locations.

Alejo relinquished control of the computer to Jerry, and Jerry proceeded to construct a perspective grid, thus already appearing to be projected, which is what he had been interested in seeing the projection of (*Figure 40*). Jerry projected his perspective grid and moved the grid to various locations on the screen, including locations in which part of the grid was above the baseline, below the baseline, and below the eye distance line.

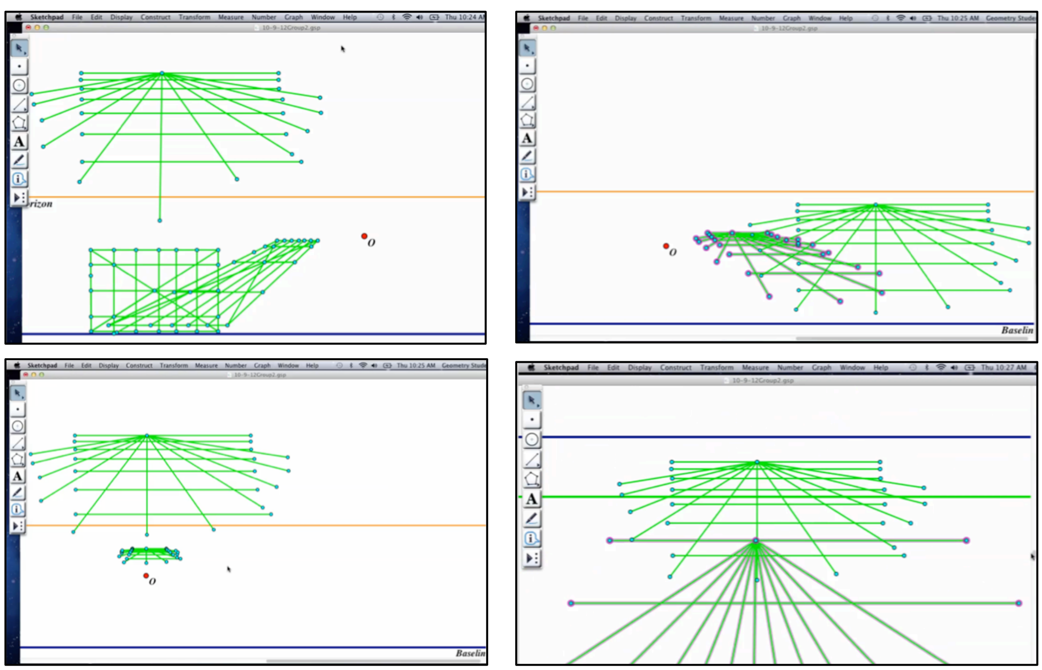


Figure 40. Jerry's perspective grid and its projection at various locations on the screen.

At one point, Jerry had part of his perspective grid below the eye distance line, and he scrolled multiples times up and down the screen, tracking the projection, which appeared both above the horizon line and below the baseline (*Figure 41*). Jerry also changed the color of the perspective grid and its projection, perhaps to distinguish between the two (*Figure 42*). From start to finish, Alejo and Jerry's mathematical play with the grid lasted, intermittently, for approximately 30 minutes. The affordance of the dynamic nature of GSP, as well as the built-in projection transformation, is what allowed Alejo and Jerry to play with the projection of the rectangular grid and perspective grid. It is reasonable to assume that without the GSP Alberti's Window sketch, Jerry and Alejo would not have explored the projection of the perspective grid, at least to the same extent, since they would have had to trace the projection each time they moved the grid had they been using the physical Alberti's Window. And, in cases where the grid was on the same side of the window as the viewer, they would have had to imagine what the projection looked like, rather than being able to see it – as Jerry indicated.

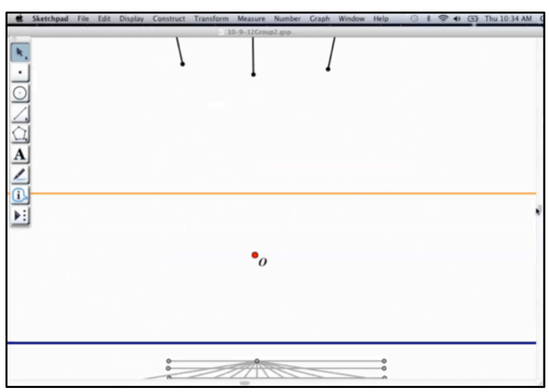


Figure 41. Jerry's projected perspective grid, in which part of the projection appears above the horizon line and part appears below the baseline.

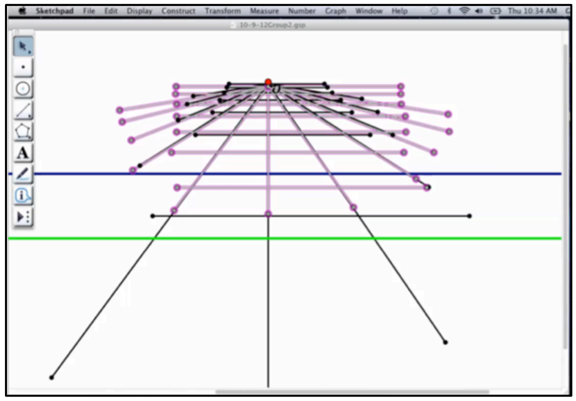
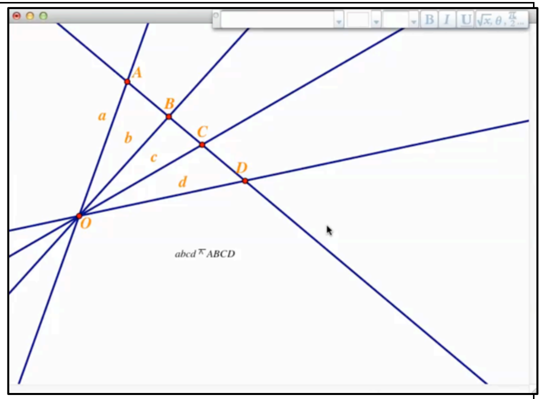
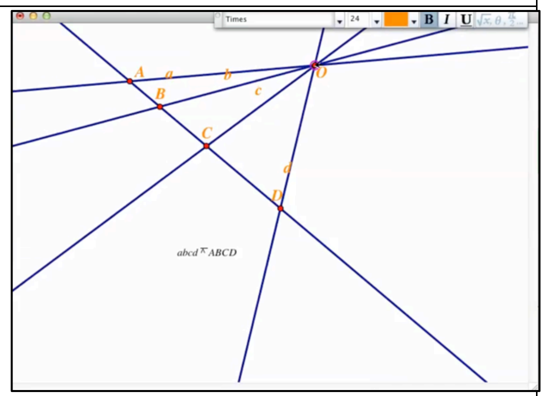
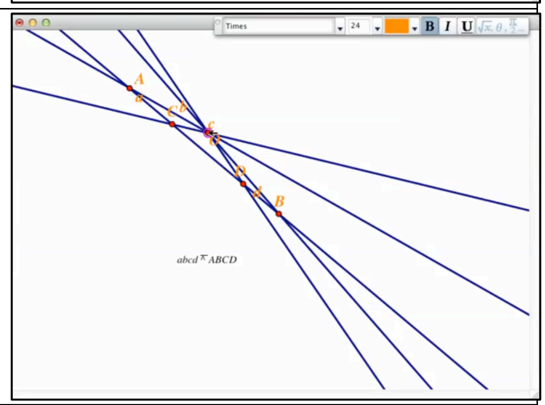
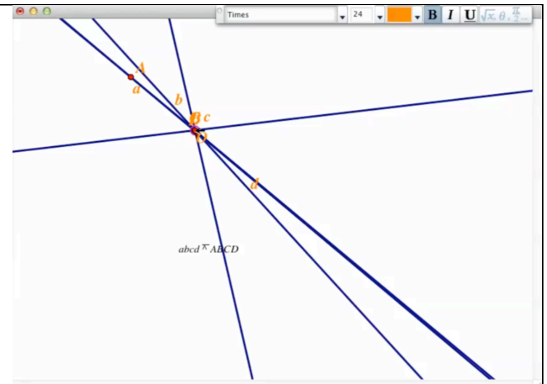


Figure 42. Jerry's projected perspective grid, in which the color of the grid and its projection differ.

Mathematical play as a result of the dynamic nature of GSP also occurred during the second component of the course, in which the class considered the axioms, definitions, and theorems of projective geometry. The nature of the mathematical play was similar to the example with the GSP Alberti's Window, where students in a group would manipulate their constructed perspectives by dragging components to various locations on the screen. For example, during the first day on which the class was working with definitions in projective geometry, Fiona and Jerry began to play with the location of a center of projection. Specifically, the group constructed a projection in which a pencil with four lines projected to a range of four points, through the center *O*. Dr. R. comes to talk with the group and mentions that regardless of where the point *O* is located, the diagram they have will still constitute a projection. The dynamic nature of GSP, as well as Dr. R.'s statement about having a valid projection even if the center of projection is moved, sparked mathematical play in the group.

184	Dr. R: So uh if, if you move the center *O*, it's still, this statement (indicating to the statement of correspondence between the pencil and the range that Fiona has written) is still valid. What happens is that the points and lines move, but it's still a projection between these lines and these points. (*Dr. R. leaves the table*)	
185 186	Jerry: Try flipping it to the other side. Fiona: So like, (moves *O* to the opposite side of the line) that?	
187	Jerry: Now make it really close.	

188	Jerry: Make it on the same line that makes the range.	
189	Fiona: (laughs)	
190	Jerry: Wow that's confusing.	
	(Fiona continues moving the center of projection)	
191	Jerry: I don't really understand what's happening there.	
192	Candace: But if you put it, if you put the point *O* on the line, then it's no longer a range, because that line becomes a pencil as well, because it goes through the point *O*.	
193	Jerry: Okay that, that's confusing. Oh I see what you're saying, the line.	
194	Candace: Because that's what makes it a pencil is that it goes through the point *O*.	
195	Jerry: Or is it, that's part of a pencil. That's still range. Is it both or is it just a pencil then?	
196	Candace: It would just become a pencil.	

Here, the affordance of the dynamic aspect of GSP gave rise to mathematical play with the center of projection. Specifically, the group moved the center of projection to the opposite side of the range line, moved it close to the range line, and then placed it on the range line. During the process, the group could see how the change in the location of the center of projection changed the look of their projection. Note this is also an example of mathematical play leading to mathematical justification and argumentation, as Candace is proposing if they put the center of projection on the range line then they no longer have a range and a pencil. She provides the justification that since, if the center of projection were on the range line, then the range line would go through point O, resulting in the range line becoming part of the original pencil.

The limitations of mathematical tools that gave rise to mathematical play were generally related to the physical Alberti's Window. One example of this is during the episode highlighted in the illustration of mathematical play construct in section 4.2.3, in which Jerry and Alejo's group used a mirror to try to determine how parallel lines behind the eyepiece would project onto the window. In the episode, the group engages in mathematical play with the mirror, since the physical Alberti's Window does not directly lend itself to determining the projection of images behind the eyepiece. That is, students cannot place an image behind the eyepiece and simply trace the projection onto the window. Rather students must imagine the projection of such images.

Another instance in which the physical limitations of the Alberti's Window led to mathematical play came while Trisha's group was trying to determine the projection of a square located between the eyepiece and the window (*Figure 43*). The resulting projection in this scenario is a trapezoid-like shape in which the top of the trapezoid has a shorter length than the bottom. In this scenario, with the square between the window and the eyepiece, similar to when an image is behind the eyepiece, the projection cannot simply be traced onto the window, and therefore must be imagined.

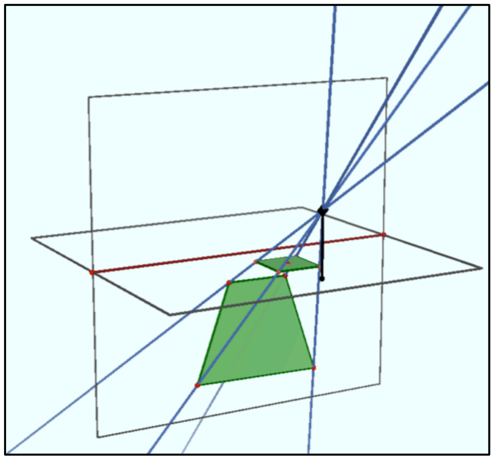
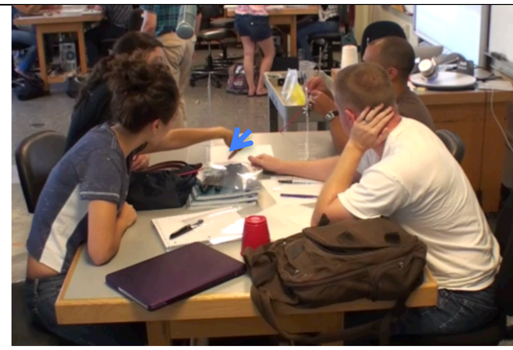


Figure 43. The projection of a square that rests between the window and the viewer.

At the beginning of the episode, Trisha explains to Candace, one of her group members, that the lines of projection will go from the eyepiece, to the square, and then pass through the table to project below the table.

197	Trisha: You want to imagine this is going down,	 Mike and Trisha each put the backs of their fingers to the window and move their hands downward toward the table.

198	Trisha: so the string would continue to go down.	 Trisha moves her hand to the opposite side of the window and points at an angle down toward the table in front of the window.
199 200 201	Trisha: And it would hit at a point underneath, like by your feet. Candace: Then how do you draw that? Trisha: You have to imagine like, what it would look like.	 Trisha places her fingertips at the base of the window.

Candace appears confused, and Trisha walks around to the end of the table where the Alberti's Window is located to explain further. Trisha and Mike, another group member, try to explain and discuss how to determine what the projection of the square will look like. There is a string going through the eyepiece, and Trisha holds the string to the vertices of the square, alternating vertices. She appears to be trying to imagine how the string would extend to create the projection. She picks up a transparency that was sitting on the table and holds it next to the edge of the table, as if it were an extension of the window. As she does this, she says, "I want it to be like, there," (*Figure 44*) highlighting the limitation of the physical Alberti's Window not extending below the tabletop. Mike then tries to explain to the group where the four vertices will project onto the window, indicating the projection would be a trapezoid with the top side shorter than the bottom side. Trisha returns to her seat. The two other group members, Candace and Francis, did not seem to follow Mike's explanation, with Candace saying, "I'm not explaining it." After a moment, Trisha has an idea about moving the eyepiece to the edge of the table.

Figure 44. Trisha holds a transparency next to the Alberti's Window, as if extending the window below the tabletop.

202	Trisha: Well you could actually do it if you came to the end of my table. So like, say you were here.	 Trisha pulls the eyepiece to the edge of the table closest to her.

203	Trisha: 'Cause it extends.	 Trisha moves her right hand in a straight line downward next to the table.
204	Trisha: Wait, where's the square? This is totally cheating.	 Trisha places the paper with the square on it under the eyepiece to where the square is off the table and the eyepiece is holding the paper to the table.

Trisha holds the string to the vertices of the square on the paper, then moves her hand underneath the paper and draws her hand away from the paper, as if extending the line of the string (*Figure 45*). She continues this for just over one minute, then she tells Mike she still thinks the trapezoid will be larger on the top. Mike asks Trisha where he window is located with respect to the table. They determine Trisha is thinking about the projection incorrectly. Where, with the eyepiece at the edge of the table and the square sticking out in front of the eyepiece, she was imagining the window at the edge of the table, putting the square in front of the window, rather than between the window and the eyepiece. Trisha notes that this is why she thought the trapezoid would be bigger on the top.

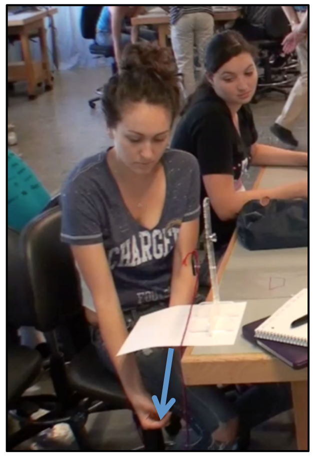


Figure 45. Trisha draws her hand down away from the paper, as if extending the string down below the paper.

Eventually the group places the eyepiece on the edge of the table and the window on the floor. The group does not have the opportunity to come to a conclusion based on their mathematical play, as Dr. R. calls the class back together and asks a particular group to share their findings.

Trisha's activity in this episode constitutes mathematical play, as her idea to move the eyepiece to the edge of the table was not a suggestion made by the instructor. Rather, her idea appeared to stem from her desire to extend the window below the tabletop. The mathematical play in this episode resulted from the limitation of the physical Alberti's Window where students are unable to simply trace the projection of images on the same side of the window as the viewer. If the group had been using software such as Cabri 3D (refer to *Figure 43*), which Dr. R. later used to demonstrate projections, then the mathematical play likely would not have occurred in the same way.

Since both affordances and limitations of mathematical tools, and in particular the dynamic versus static nature of the tools, can result in mathematical play. This suggests in tool design, or while determining which tools to use in a course, the instructor should consider the affordances and limitations of tools. And, specifically, how the affordances and limitations might result in mathematical play.

4.4.4 Section Summary

In this section, I illustrated three aspects in the projective geometry course that provided the opportunity for mathematical play. In particular, I discussed the influence of the instructor, the nature of the task, and the affordances and limitations of mathematical tools. In this course, Dr. R. provided the opportunity for and encouraged mathematical play through multiple actions. Dr. R. cultivated classroom social norms that supported students sharing their thinking. He responded to students sharing their thinking in ways that acknowledged their ideas, yet reserved judgment, which allowed students to feel comfortable sharing their ideas. The way in which Dr. R. requested for groups to share their thinking with the class provided an open floor for presenting a wide variety of things, including observations, questions, and modes of discovery. In whole class discussions, when students inquired about particular aspects of mathematical ideas or situations, such as wondering how the projection of parallel lines would change if the viewer were very low to the plane on which the lines are located, Dr. R. capitalized on the opportunity to model mathematical play to the class by having the student follow through on the actions associated with the inquiry, such as asking Jerry to lay on the floor and project the parallel lines onto the window.

In Dr. R.'s explanation of tasks to the class, he referred to "playing" with mathematical entities on multiple occasions, including in his description of the artistic projects, in which he informed the class the purpose of the project was for students to play with mathematics, and feel the freedom to do so. Furthermore, Dr. R. encouraged mathematical play through drawing attention to groups engaged in mathematical play. I contrasted the ways in which Dr. R. encouraged and supported mathematical play with an instance in which a group was engaged in mathematical play and I redirected their activity.

The nature of the task contributed to providing students the opportunity to engage in mathematical play, in the sense that tasks in which students had more freedom to interpret the instructions and explore avenues for finding solutions allowed students to engage in autonomous and freeform activity. Tasks in which the instructions were more rigid gave less opportunity for play, although occasionally when task instructions were rather rigid, some students found ways to engage in mathematical play by exploring ways to solve the problem different from the way in which Dr. R. proposed. Dr. R. supported this play through not interfering with groups engaged in play and providing an open floor for the students to discuss their process and findings. Tasks in which students followed more closely the actions of the instructor, such as following the steps to construct a diagram for a particular proof, provided fewer opportunities for students to engage in mathematical play.

Affordances and limitations of mathematical tools also contributed to providing students the opportunity to engage in mathematical play. In particular, the affordances of dynamic geometry software allowed students to engage in mathematical play through exploring different constructions of mathematical situations, such as a chain of perspectives, by dragging the mathematical entities in the sketch, such as points and lines, to various locations on the screen. The limitations of mathematical tools that provided students the opportunity to engage in mathematical play arose out of students attempting to find way to use the mathematical tool to carry out a task, when the tool did not naturally lend itself to carrying out the task.

4.5 Chapter Summary

In this chapter I discussed the mathematical play construct, a mathematical practice in which students in the course engaged while working on problems in projective geometry. I defined mathematical play as the exploration of mathematical ideas through individual or group actions that are both autonomous and freeform – where by autonomous I mean the actors have minimal concern with what others around them are doing, or with what others think about what they are doing, and by freeform I mean the details of the actions are not scripted or prescribed. I noted that mathematical play can include engagement with physical devices, computer programs, acts of imagination, and social interactions, as well as inscriptions, which was illustrated throughout the episodes I highlighted in the chapter.

I illustrated the mathematical play construct through an episode from the data in which a group of students engaged in mathematical play with the Alberti's Window and a mirror while trying to determine how a set of parallel lines behind the eyepiece would project onto the window. I elaborated on why this was a case of mathematical play, noting the ways in which the activity was autonomous and freeform.

Next, I illustrated two benefits of mathematical play that arose in the course, students considering pop-up topics, and students engaging in argumentation and justification. I defined pop-up topic as a mathematical situation that had not yet been addressed in the course, and that arose organically from student activity. I illustrated these two benefits from engaging in mathematical play though an episode in which Jerry engages in mathematical play by proposing a pop-up topic. Jerry proposes a particular scenario in which all of the points in a perspective, the center of projection and range points, all lie on the same line. Considering the pop-up topic led two group members to engage in mathematical argumentation and provide justification for why they believed Jerry's proposed scenario would not constitute a perspective.

Finally, I discussed the elements of the learning environment that provided the opportunity for students to engage in mathematical play. Specifically, the instructor influence, the nature of the task at hand, and the affordances of limitations of tools contributed to the opportunity for students to engage in mathematical play. Since mathematical play is characterized by the autonomous and freeform nature of exploring mathematical ideas, mathematical play can be encouraged by instructors and fostered through task and tool design, yet individuals cannot be made to play.

Chapter 5

Acts of Imagination

The moving power of mathematical invention is not reasoning but imagination.
-Augustus de Morgan (1806-1871)

In the previous chapter, I discussed the first of two results of my first research question. In this chapter, I discuss the second result of my first research question:

> In the context of an activity-based projective geometry course, in what mathematical practices do students engage while working on problems in projective geometry?

In Chapter 4, I discussed mathematical play as a mathematical practice. The focus of this chapter is *acts of imagination* as a mathematical practice.

This chapter contains five sections. In the first section, I provide a very brief background into mathematics and the imagination in the literature. In the second section, I provide a description and illustration of my definition of acts of imagination as a mathematical practice. In the third section I address the way in which students engaged in acts of imagination in explaining and justifying mathematical situations. In the fourth section I discuss aspects of the learning situations that created the opportunity for students to engage in acts of imagination in the projective geometry course. In the final section, I provide a summary of the chapter.

5.1 Brief Background

There are myriad ways in which imagination might be conceived of. Stevenson (2003) reflected on the ways in which imagination has been considered, and described twelve different conceptions, some of which apply to fields such as mathematics, and others, he suggests, solely apply to fields such as art. For example, those conceptions of imagination that might apply to mathematics include, "The ability to think of something that is not presently perceived, but is, was, or will be spatio-temporally real" (p. 239) and "The ability to entertain mental images" (p. 243). Conceptions of imagination that apply to the artistic realm include, "The sensuous component in the appreciation of works of art or objects of natural beauty without classifying them under concepts or thinking of them as practically useful" (p. 253). Stevenson goes as far as to provide the most general of definitions for imagination, "The ability to think of (conceive of, or represent) anything at all." (p. 245)

Certainly, imagination can be considered a fundamental aspect of mathematics learning, particularly as many aspects of mathematics only exist symbolically and in the imagination (Mazur, 2004; Moschkovich, 2003). For example, consider two lines intersecting at a point at infinity. Symbolically, we can represent the infinite in mathematics by the symbol ∞, yet the infinite cannot physically be reached. At the same time, an individual can imagine and understand the infinite as a mathematical entity. Moschkovich (2003) briefly identifies imagining as a mathematical practice, in the sense of having to communicate about imaginary mathematical entities, such as infinity, zero, and lines that never meet, as well as in terms of "visualizing shapes, objects, and relationships that may not exist in front of our eyes" (p. 327).

At times, in mathematics, imagination has been conceived of as synonymous with visualization (Kotsopoulos & Cordy, 2009), bringing a picture into the minds eye, or "seeing the unseen" (Arcavi, 1999;). Descartes (as cited in Stevenson, 2003), in the Sixth *Meditation*, compared his ability to envision in his mind a shape such as a triangle and consider it as if it were present before him, yet an object such as a thousand-sided figure escaped his mind's ability to consider it as if it were in front of him. At the same time, he noted, he was able to conceive of a thousand-sided object mathematically and be able to reason with it and about it. Descartes used this comparison to distinguish between *imagining* and *pure understanding*, in which pure understanding refers to the latter situation.

In the field of mathematics, imagination has been conceived of in many ways, not just as related to visualization. At times, researchers have conceived of imagination in mathematics as inventiveness, or the ability to generate new mathematical ideas (Lovitt, 1924; Perkins, 1985; Saiber & Turner, 2009). For example, Perkins (1985) discussed imagination in terms of inventiveness (and at times, mental imagery or visualization), such as imagining a new theorem in geometry or being able to conceive of a problem in a different way, such that the problem becomes transformed to where the solution becomes clear. At other times, researchers have conceived of imagination as the notion of mathematical intuition, or being able to immediately comprehend mathematical ideas or situations without "conscious reasoning" (Saiber & Turner, 2009). More recently, researchers have begun to conceive of mathematical imagination as a phenomenon rooted in bodily engagement (Nemirovsky & Ferrara, 2009; Nemirovsky, Kelton, & Rhodehamel, 2012).

In the shift from considering imagination as specifically related to visualization, mental images, inventiveness, or intuition, Nemirovsky and Ferrara (2009) and Nemirovsky, Kelton, and Rhodehamel (2012), have begun to explore the role of the body and social interaction in imagining mathematics. Nemirovsky et al. (2012), explore *collective imagining* as social communication in mathematics. Drawing upon Casey's (1979) quasi-perceptual and Sartre's (2004) quasi-observation constructs, Nemirovsky et al. (2012) define collective imagining as "the social-interactive experience of bringing into (quasi-) presence something which is absent in the current surroundings of the participants" (p.131). They describe bringing an entity, such as an object, body, or sign, into quasi-presence as a phenomenon in which a group of actors is aware that the entity is not, in fact, physically present in their surroundings, but they act *as if* the entity is present. It is from this definition of collective imagining that I draw upon considerably in defining acts of imagination as a mathematical practice.

5.2 Acts of Imagination As A Mathematical Practice

"Let's do a little exercise. Let's stand up, please," requests Dr. R. Each of the students stands up from their chair. "So, we'll imagine that in front of each of us, you have to imagine that there is a window, that it's a plane, an infinite plane. It goes up, down, to the side, everything." Dr. R. indicates up, down, and to the sides with his hands, almost as if conducting an orchestra. "And so you imagine that, say for each, a railroad track," gesturing toward the floor in front of him. "So say, uh, with your left hand, so you follow slowly the railroad track." Dr. R. explains how they will trace the imaginary railroad track in front of them with the imaginary line that extends from their left hands, and then imagine where that line would intersect with the imaginary window. At the point at which the imaginary railroad tracks passes from being in front of Dr. R. to going behind him, he notes they will begin to use both hands as the line of projection. "And then as you pass it, we'll use the right hand. So imagine that the two arms are a line." He points at the railroad tracks behind him with his left hand, and stretches his right hand up toward the ceiling out in front of him (*Figure 46*). He begins to trace the railroad track behind him. As he traces away from himself behind his back, he lowers his right arm, keeping his two arms in a straight line. "As I go like that I, this line is going, down like this, projecting on the window."

Figure 46. Dr. R. points to the imaginary railroad track behind him with his left, continuing the straight line by extending his right arm and hand.

After his first explanation of the activity, he repeats what he wants the students to imagine, performing each action as he wants the students to do.

> So we'll, we'll do it um, each of us will do it silently for a moment. So we start with the left hand. The window is there. And then we slowly, trace, or sort of follow the track. So imagine that there is a line here [from your hand] that is leaving a trace on the window. So I go like that. At a certain point I hit the baseline, which means I hit the line where the floor meets the window. Right? So as I continue tracing the track, now the projection is under the floor. Yes? Under the floor… and it goes down down down down, farther and farther and farther and farther. Something very dramatic happens when you are looking exactly down. Because then in that case you project, it's a, it's a line parallel to the window, so in a way it doesn't meet the window, or it meets uh in the infinite. But then as you go behind myself, so the track keeps going, then I use my other arm, and as I, I go behind, then the, my two arms are a straight line. Right? One extreme touches the track, behind me. The other one touches the window. So I start from very very very high, and I start to come down on the window. Like this.

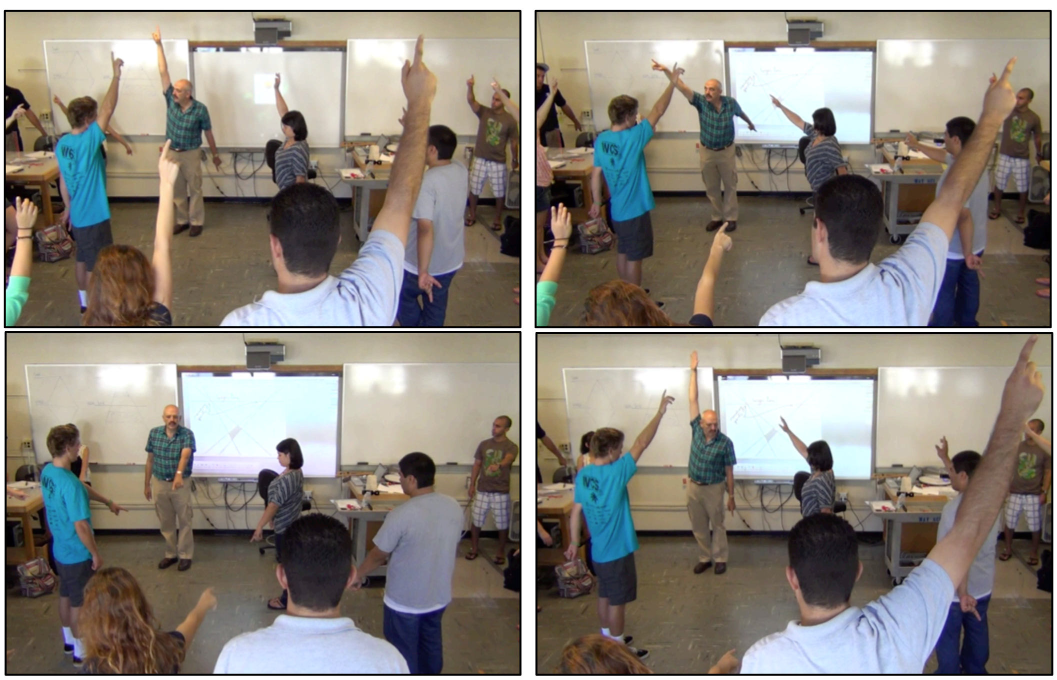


Figure 47. Dr. R. guides the class through an exercise in how to imagine the projection of a set of infinite railroad tracks.

As Dr. R. explains, the students follow his lead, tracing their imaginary railroad tracks with their left hands. As they trace the imaginary tracks behind themselves, they point at the track with their left hand and continue the straight line with their right arm and hand, pointing in front of them, up toward the ceiling, just as Dr. R. does. “So do it once more on your own and then we’ll talk about that. Try not to get distracted by others. Just focus on this exercise” The classroom is quiet, and the students trace their imaginary railroad tracks.

Dr. R.’s request in this vignette is an unusual one, in my experience, for a professor in a mathematics course, that the students should stand up and use their imaginations to consider a mathematical idea. Not only should they use their imaginations, they should engage their bodies in the process of imagining. Their bodies were the only aspect in the exercise not being imagined, yet their bodies determined the locations of the imaginary elements. For example, each stood in a set of imaginary railroad tracks that extended both in front of and behind them. Each had an imaginary window in front of them on which to trace the projection, and each used their left hand to guide the line of projection as it traced the imaginary track and intersected with the window. While the request was unusual in a mathematics course, the exercise became a way of reasoning for many students in imagining how the projections of certain images looked.

As the co-instructor of the course, when the class first began working with the physical Alberti’s Window to consider projections of images that reside on the same side of the window as the viewer, I interacted with different groups, checking to ensure they understood the directions of the task at hand, and checking to see how they were thinking about the mathematical ideas involved in the task. In the beginning, many groups looked for clarification about how they should create their predicted projections. Occasionally, I would check in with a group to see how they were thinking about a projection and they would explain they were using their imaginations to come up with their prediction, as if this might not be an acceptable solution method. For example, when Trisha's group was working on projecting a circle from different locations on the tabletop, I asked how things were coming along for them. One of her group members said to me, "Now, we're just basically, just kind of imagining," gesturing with his hand from his head to the paper on which their predicted projections were drawn. The questioning tone in the group member's voice and the look to me for approval indicated the group was unsure about imagination as justification and needed some assurance their approach was acceptable.

Imagining aspects of mathematical situations became commonplace in the course – in particular while students worked with the Alberti's Windows. Students provided mathematical reasoning through the use of gesture to indicate mathematical entities or situations, such as using their arms as a line of projection to justify why the projection of a certain image would appear below the tabletop. Sometimes the justification was through tracing with their fingertips an imaginary line from the eyepiece to the image, then explaining where this imaginary line of projection would intersect the Alberti's Window, as is the case in an episode I discuss in section 5.3. At other times, students placed their whole bodies in the mathematical situation to justify their thinking, using their arms as the line of projection to point at an imaginary image on the ground, and then following that line of projection to the imaginary window – such as Dr. R. had requested students do in the vignette at the beginning of this section. For example, in one instance, Candace, one of Trisha's group members, explained to me the projection of railroad tracks behind the viewer would appear above the horizon line by saying, "That's the whole this thing," pointing to the ground behind her back with one hand and pointing up toward the ceiling in front of her with the other hand, making a straight line with her two arms and lowering the arm in front of her slightly as she raised the arm behind her slightly, as if tracing the railroad track behind her (*Figure 48*). In this case, she was indicating if there were an image behind the eyepiece, then the line of projection would extend from the image (through the eyepiece) and then up toward the window, until the line of projection intersected the window.

Figure 48. Candace imagines how railroad tracks behind the viewer get projected onto the window above the horizon line. Candace points at the floor behind her body with her left hand, while her right hand and arm extend in a straight line from her left hand and arm. As she raises her left hand behind her, she lowers her right hand in front of her.

Students also seemed to use their imaginations to explain mathematical situations during the individual interviews I conducted. With certain questions, such as when I asked why a particular aspect of their artistic design looked a certain way, or when I asked them how an aspect of the GSP Alberti's Window corresponded with the physical Alberti's Window, students turned to creating the mathematical situation with mathematical entities that were not physically present, but instead were brought into being through imagination and bodily engagement. For example, as I describe in detail in the forthcoming illustration, Willow utilized imagination to explain why a certain projection in her artistic design became a larger projection than its original image, while the three other projections were smaller than their original images. In this instance, Willow used the computer screen, her hands, her body positioning, and her gaze to bring the mathematical situation into being. Similarly, Trisha imagined her hand as the physical Alberti's Window to explain what it meant, in terms of the physical Alberti's Window, to project the projection of an image in the GSP version of the window – an episode I discuss in more detail in Chapter 6.

This form of student activity, in which students imagined aspects of a mathematical situation, or imagined themselves as part of a mathematical situation, to bring the situation into being, is what led to the way in which I defined acts of imagination. In particular, I wanted to capture in my definition for acts of imagination, as a mathematical practice, the activity where students seemed to engage with aspects of a mathematical situation that were not physically present during the activity (or were physically present but not used), but that they utilized anyhow through imagination.

5.2.1 Definition of Act of Imagination

Through a grounded analysis of the classroom video data, considering my experiences engaging with groups of students in the course, and exploring the literature regarding mathematics and imagination, I arrived at a definition for an act of imagination as a mathematical practice that has been adapted from the definition for *collective imagining* put forth by Nemirovsky et al. (2012). Recall from section 5.1, Nemirovsky et al. (2012) defined collective imagining as "the social-interactive experience of bringing into (quasi-) presence something which is absent in the current surroundings of the participants." They describe bringing *something*, such as an object, body, or sign, into quasi-presence as a phenomenon in which a group of actors is aware that the *something* is not, in fact, physically present in their surroundings, but they act *as if* it is present.

During the analysis of my data, in particular my individual interviews with participants, it became apparent that students had likely engaged in bringing mathematical objects and situations into quasi-presence to reason about mathematical ideas while they were alone, and not solely while they were with their peers in the classroom. For these students who engaged in bringing mathematical objects and situations into quasi-presence while they were by themselves, the collective imagining then did not necessarily occur until they brought the reasoning to others for discussion, such as classmates during class time or myself during an interview. This led me to adapt the collective imagining definition to include *an individual* bringing mathematical entities into quasi-presence, without the company of others. This led to the following definition of an *act of imagination* as a mathematical practice:

> An *act of imagination* is a mathematical practice characterized by one or more individuals acting *as if* a mathematical situation or entity were present, despite the entity not being physically present in the current surroundings.

These acts of imagination could incorporate gesture, body positioning, eye gaze, verbal utterances, components of mathematical tools, as well as inscriptions. This definition slightly broadens the definition for collective imagining put forth by Nemirovsky et al. (2012), as it includes space for an individual to engage in an act of imagination while unaccompanied by others.

5.2.2 Identifying acts of imagination

In this subsection, I detail the aspects of student activity that indicated students engaging in acts of imagination. In particular, I discuss the actions of students that indicated bringing mathematical entities or situations into quasi-presence – that is, acting *as if* a mathematical entity or situation were present despite the entity or situation being physically absent from current surroundings. The details I describe in this section indicate the ways in which I identified acts of imagination in the data.

There were two general scales on which students brought mathematical entities and situations into quasi-presence. First, students brought mathematical entities into quasi-presence on a smaller physical scale, such as acting as if certain elements of a situation with a mathematical tool were present. For example, a student might use his hand as if it were the image to be projected, or as if it were the Alberti's Window itself, saying, for example, "Okay, the window is here." In one instance, which I discuss in detail later in this chapter, a student was working with the physical Alberti's Window to determine how an image between the window and the eyepiece gets projected. The student used her fingers to trace lines of projection from the eyepiece to an imaginary image hovering above the tabletop. She used this act of imagination to explain why the shape of the projection of the image would appear below the tabletop and be longer vertically than horizontally.

A second way in which students brought mathematics into quasi-presence was on a larger physical scale. In these instances, students appeared to insert their entire body, rather than just a hand or finger, into a mathematical situation. The student's body becomes part of the mathematical situation and the student orients herself to the other mathematical entities in the situations, some or all of which may also be imaginary. For example, a student might sweep her hands around her body to indicate that she is standing inside of a circle. Then, acting as if the circle were present, she might point toward the imaginary circle with one arm and finger extended, and follow the line of projection with her other arm and finger pointing in the exact opposite direction. In this case, the student is acting as if her arms constitute the line of projection, touching the circle behind her at one end, passing through a center of projection at the top of her sternum, and extending up in front of her toward the ceiling to intersect with the window. This is similar to the exercise in which Dr. R. engaged the students in the classroom in the vignette at the beginning of section 5.2.

As students engaged in acts of imagination, they utilized nearby components of mathematical tools, gesture, body positioning, eye gaze, and speech to express the mathematical situations about which they were imagining. I return here to the vignette in which Dr. R. requests the entire class to participate in imagining how a set of railroad tracks would project onto the window to illustrate they way in which I identified acts of imagination in the data.

At the beginning of his request, Dr. R. states that each of the students should imagine a window in front of them. As he says "you have to imagine that there is a window", he raises his hands high up in the air in front him, and then draws his hands downward, as if running the backs of his hands along a window (*Figure 49*). Here, Dr. R.'s speech and gesture bring the window into quasi-presence; while we can't actually see the window, we can imagine it where his hands traversed the space in front of him. In the activity that follows, Dr. R. provides explanation *as if* the window were still present in front of him, as he indicated.

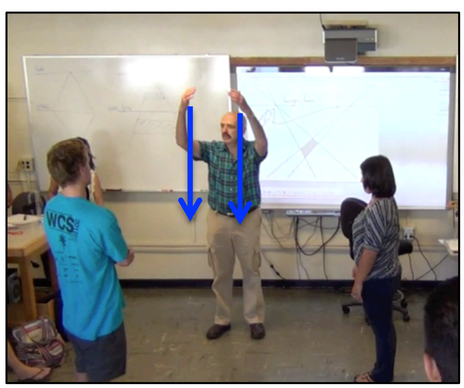


Figure 49. Dr. R. raises his arms in front of him, then draws his hands downward, as if his hand were sliding down a window.

As the activity continues, and Dr. R. begins to explain how they will use both of their arms, he tells the students to, "imagine that the two arms are a line." He begins to trace the imaginary railroad track going behind him, pointing to the imaginary track with his left hand and extending the line of his left arm with his right arm and hand. As he moves his left hand away from his body behind him, Dr. R. states, "As I go like that I, this line is going, down like this, projecting on the window." As he says "this line is going down" he shifts his gaze up toward his hand, as if looking past his hand (*Figure 50*). His gaze lowers accordingly with his right hand as his left hand moves further behind him.

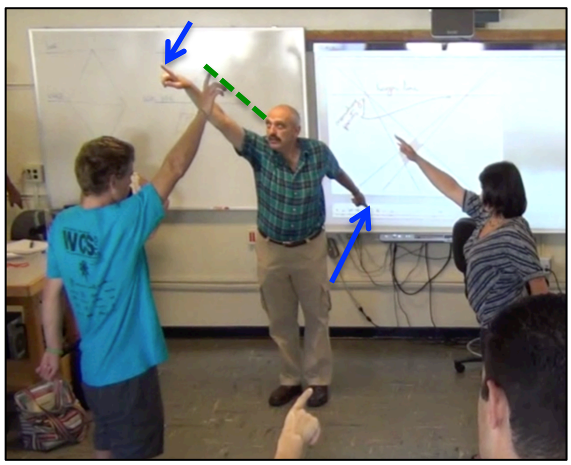


Figure 50. Dr. R. points to the imaginary railroad track behind him and follows the line of his arm to extend in front of him. He lowers his right hand as he lifts his left hand behind him, gazing past his right hand in the process.

Here, Dr. R. is acting as if his arms are a segment of the line of projection, and we can imagine the line extending beyond his fingertips. His gaze up toward his right hand assists in bringing the line of projection into quasi-presence, as he appears to be looking past his right hand, as if following the line of projection with his gaze up to where the line intersects with the window. Dr. R. is also acting as if he were standing in the center of a set of railroad tracks, as he points at one of the rails of the track with is left hand and traces it behind him.

These two brief illustrations of Dr. R.'s acts of imagination are examples of the way in which I identified acts of imagination in the data by attending to gesture, body positioning, eye gaze, and speech. In addition, I used aspects of mathematical tools to identify acts of imagination when those tools were incorporated into the act of imagination, for example when a student used a string extending from the Alberti's Window eyepiece to an image behind the eyepiece, then proceeded to reason about why the line would extend to reach the window above the horizon line somewhere. In general then, to identify an act of imagination, I looked for one or more students engaging in activity that suggested they were operating *as if* a mathematical entity or situation were present. This included the way in which the students used gesture, body positioning, eye gaze, speech, and aspects of mathematical tools to indicate aspects of mathematical situations or ideas that were not physically present (or were only partially present) in their surroundings.

5.2.3 Illustration

In this subsection I illustrate the act of imagination construct through an episode from the data. Since the background of the episode is pertinent to making sense of the episode, I first describe the context in which this episode occurred. I follow this with the illustration and an examination of why this episode constitutes an act of imagination.

5.2.3.1 Episode background. In this episode, which I selected for its clarity on what is being brought into quasi-presence, Willow is discussing an artistic design she created during the course. Recall that students created two artistic pieces during the course, using the GSP Alberti's Window (see Chapter 3 for a detailed description of the projects). I discuss these artistic pieces in more detail in Chapter 6. In this episode, I ask Willow why a certain projection in her design is larger than its original image, while the other three projections in her design are smaller than their original images. Her response is related to where the original image is located on the tabletop, as well as where the viewer's eye is located with respect to the window and the tabletop.

With the Alberti's Window, the projection of an image that is located on the tabletop on the opposite side of the window as the viewer appears above the baseline (where the window intersects the tabletop) and below the horizon line (which is the height of the eyepiece above the table). In addition, the projection of the image will be smaller in size than the original image (*Figure 51*). When an image is located between the window and the viewer, its projection is located below the baseline, and, depending on the height of the viewer and the viewer's distance from the window, the projection may be larger than the original image (*Figure 52*), both horizontally and vertically, or perhaps just larger horizontally. In particular, if the viewer is close to the image and relatively tall, then the projection of the image will appear larger than the original image (Note, in the case where the original image is located between the viewer and the window, the projection of the image will always be wider than the original image, but the height of the projected image may be less than the height of the original image).

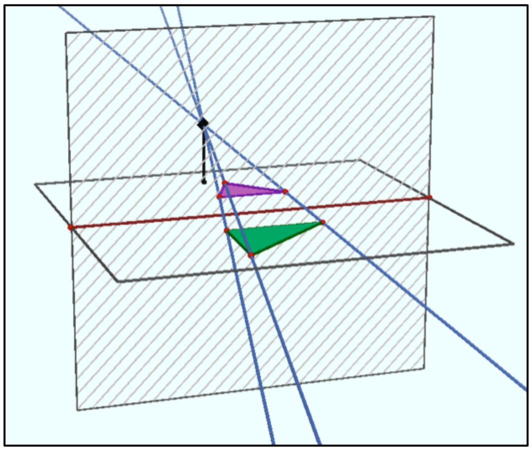


Figure 51. A green triangle on the opposite side of the window from the viewer being projected onto the window, resulting in the purple triangle.

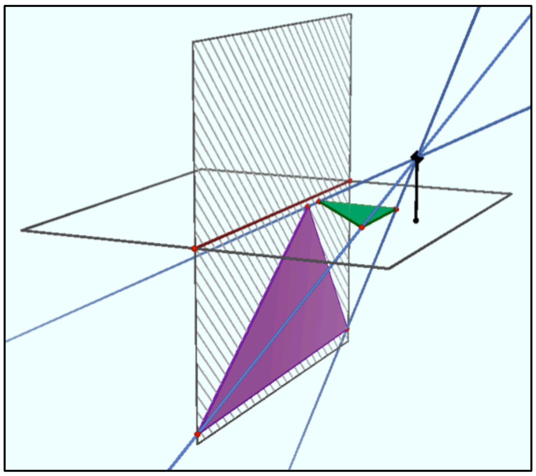


Figure 52. A green triangle between the window and the viewer being projected onto the window, resulting in the purple triangle.

Willow used the GSP version of the Alberti's window to create her artistic design, as was required in the project assignment. The GSP version of the Alberti's Window is a two dimensional version of the physical Alberti's Window, where the tabletop plane and the window plane are superimposed by rotating the tabletop plane up to the window plane about the intersection of the two planes, which is called the baseline (*Figure 53*).

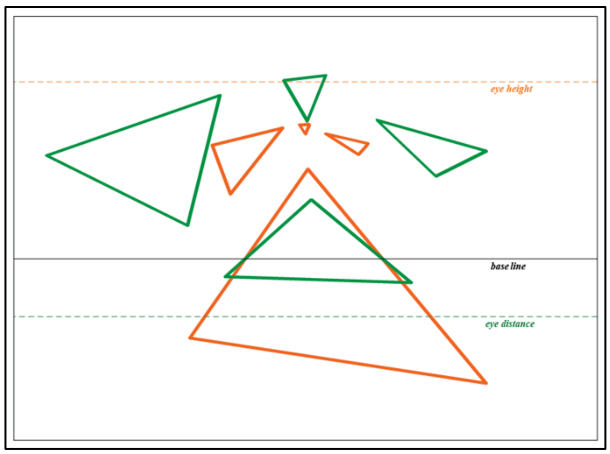


Figure 53. Willow's design construction in GSP. The green triangles are the original triangles. The orange triangles are the projections of the green triangles. The green line is the eye distance line, the orange line is the eye height line, and the black line is the baseline.

Willow's design, highlighted in this episode, consists of eight triangles, four of which are original images and four of which are the projections of those original images. Three of the original triangles are located on the opposite side of the window from the viewer, and the fourth original triangle is located partially on the opposite side of the window from the viewer and partially on the same side of the window as the viewer (*Figures 53 & 54*).

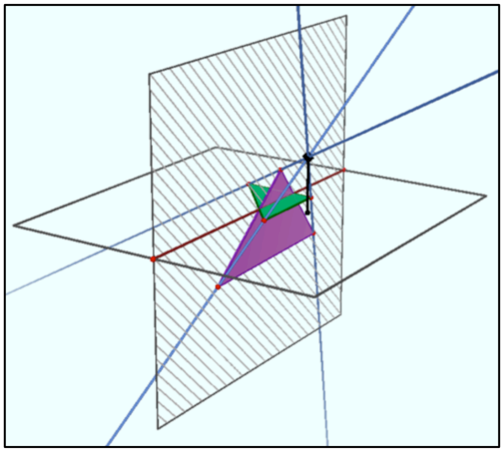
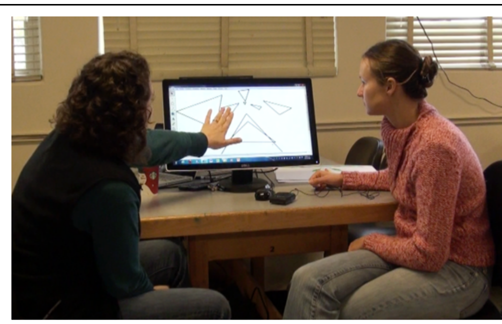
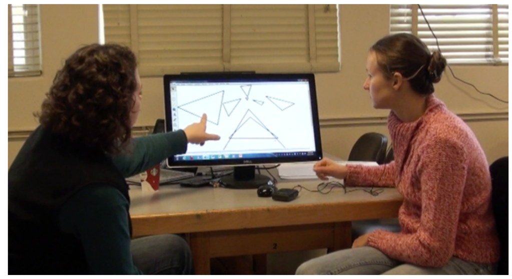
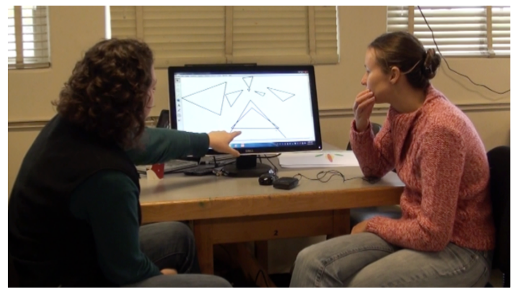
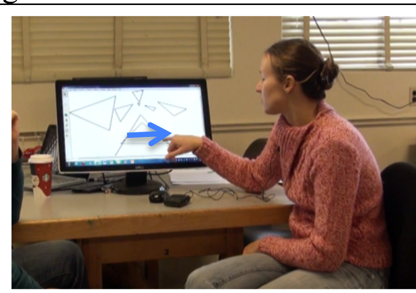
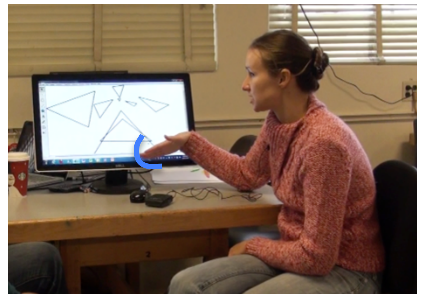
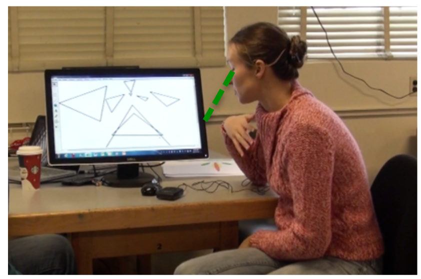


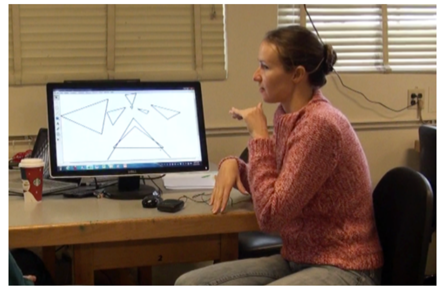
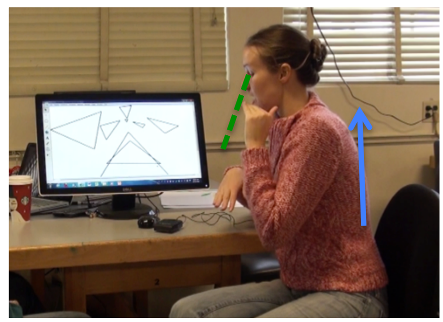
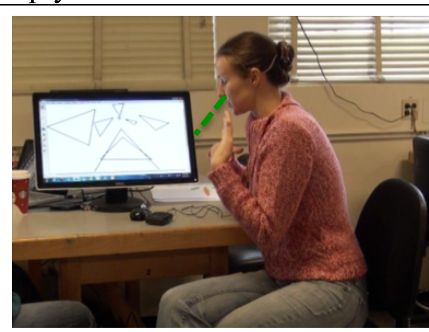
Figure 54. A depiction of a green triangle, partially on the opposite side of the window as the viewer and partially between the window and the viewer, being projected onto the window, resulting in the purple triangle.

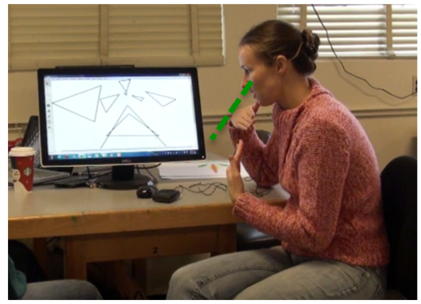
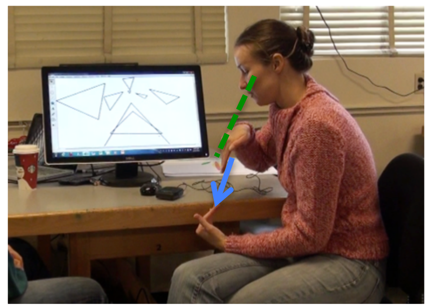
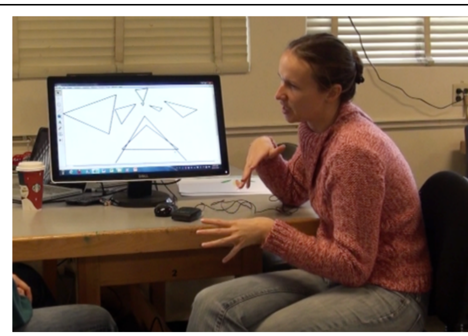
5.2.3.2 The episode. In this episode, Willow and I are discussing her artistic design she created in GSP. In particular, I ask Willow about a specific component of her design, why three of the four projected triangles in her piece were smaller than their original images, while the fourth projected triangle was much larger than its original image (refer to *Figure 53*). In *Figure 53*, the green triangles are the original triangles, while the orange triangles are the projections of the original green triangles.

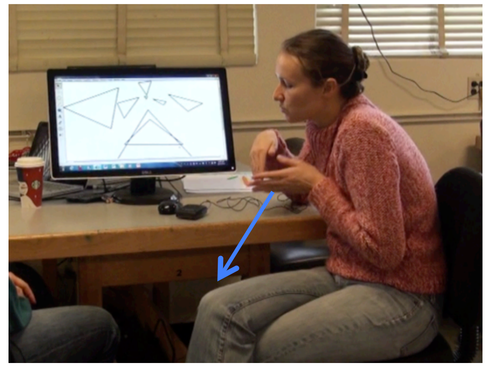
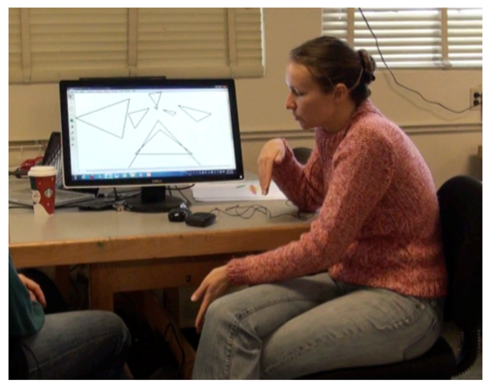
223	Int: So then, like these projections are all, smaller	 Interviewer indicates to the three smallest projections of the triangles.

224 225	Int: than their originals, Willow: Mm hm	 Interviewer indicates to the original images of the three smaller projections.
226	Int: but this one's so much bigger.	 Interviewer indicates to the largest of the projections.
227	Willow: Mm hm. This one's bigger because the,	 Willow points to the inside of the largest projection with her right pinky finger.
228	Willow: the bottom of it is	 Willow draws her right pinky finger across the screen at the location where the original triangle and its projection intersect.

229	Willow: below the baseline.	 Willow sweeps her right hand in an arc from the location at which the original triangle and its projection intersect to a lower place on the screen.
230	Willow: And because, I'm so much closer, the, the um,	 Willow draws her right hand toward her chest and leans slightly toward forward, and shifts her gaze down toward the table.
231	Willow: I'm so much closer to the window um,	 Willow moves her right hand closer to the computer screen, and leans her body slightly closer to the screen.

232	Willow: than I am high.	 Willow turns back to the interviewer, places one hand on top of the other, then separates her hands by lifting her left hand closer to her chin, while moving the right hand down slightly toward the table.
233	Willow: Like I'm, I'm higher up	 Willow sits up straighter in her chair, raising her right hand with her chin as she sits up. Willow shifts her gaze sharply downward toward the table.
234	Willow: and closer,	 Willow brings both her hands closer to her face, palms facing her chest, and leans forward.

235	Willow: so that the angle that I'm,	 Willow holds her hands with her fingertips lining up just below her gaze.
236	Willow: so I'm looking down, at it at like,	 Willow draws her left hand down toward her leg, orients her fingers up toward her eyes. She turns her right hand so her fingers face the fingers on her left hand. She moves her right hand down in a straight line to meet her left hand.
237	Willow: big angle.	 Willow keeps her hands along the same line she made with her fingers previously. She does two short pulses with her hands.

238	Willow: Um, so it's really projecting it	 Willow draws her left hand close to her body, fingers pointing away from her body. She quickly moves her left hand away from her body in a straight line.
239	Willow: far down, down the window.	

Here, Willow explains her understanding of why the projection of one of her triangles is larger than its original image, while the projections of the other three triangles are smaller than their original images. She states the projection is larger than the original image as a result of two things, the location of the original triangle and the positioning of the eyepiece. Specifically, the original triangle is partly "below the baseline," which refers to its location in the GSP sketch. An original image below the baseline in the GSP Alberti's Window corresponds to the original image being between the window and the viewer in the physical Alberti's Window scenario (refer to *Figures 53* & *54*). Willow also states she is "higher up and closer" to the window, which refers to the location and height of the eyepiece, or center of projection.

5.2.3.3 Acts of Imagination. In this episode, Willow acts as if many aspects of the mathematical situation are present in order to justify why the projected triangle is larger than its original image. She brings into quasi-presence the baseline of the Alberti's Window, the window component of the physical Alberti's Window, the image to be projected, and lines of projection.

As Willow begins to explain why the projected triangle is larger than the original triangle, she traces a line with her right-hand pinky finger across the computer screen where the original triangle and its projection intersect (line 228). She then drops her outstretched hand down to a lower location on the screen (line 229). As she does this, she states that the bottom of the original triangle is located "below the baseline." On the computer screen is the GSP sketch of her design, which does not include the baseline, as the baseline had been hidden (for the cutting of the stencil). Here, Willow acts as if the baseline were present in her explanation.

Next, Willow begins to explain why having part of the original triangle located below the baseline in GSP will result in the enlargement of the projected triangle. In her explanation, she coordinates between the physical and GSP versions of the Alberti's Window, perhaps as "below the baseline" refers to the GSP window, while most of her actions in her explanation refer to the physical window. When an original image is located below the baseline in the GSP version of the window, it corresponds to the image being located between the baseline and the viewer with the physical window (refer to *Figures 53 & 54*).

After Willow points out the relationship between the baseline and the original image, Willow inserts herself into the mathematical situation as she continues her explanation. She draws her outstretched hand close to her chest, turns her head and shoulders toward the computer screen, and leans in toward the screen (line 231). As she does this, she refers to how close she is to the window. Here, Willow is inserting herself into the mathematical situation and acting as if her hand, and then the computer screen, were the Alberti's Window.

Willow turns away from the computer screen and explains, with respect to the horizontal plane, where her eye is located with respect to the window. At first, Willow states she is much closer to the window than she is high up (from the tabletop). As she says this, she first leans in toward the computer screen, then turns back to me, as the interviewer, and indicates where the center of projection is in relation to the horizontal plane. She starts with her two hands one of top of the other and raises the top hand away from the bottom hand (line 232), up toward her chin, as if raising the height of an eyepiece. She directs her gaze toward the table in front of her, suggesting the direction of the line of projection going from her eye to the image being projected.

Willow restates her thinking, saying she is "higher up and closer," meaning her distance from the window is a shorter distance than the distance from her eye to the tabletop. As she says this, she sits up straighter in her chair, raising her right hand up with her chin (line 233), then moves both her hands up in front of her face, palms facing her, and leans her whole body toward her hands (line 234), again bringing the window into quasi-presence. She again directs her gaze down toward the table in front of her, suggesting the line of projection. Here, Willow is acting as if she is present in the physical Alberti's Window context, and as if her eye is the center of projection.

In the next part of her explanation, Willow notes that being higher up and closer to the window means the angle at which she is viewing the triangle image is sharper (lines 235-237). She looks down at an angle in front of herself, as if the image being projected were between her and the imaginary window she brought into quasi-presence. She explains this "big angle" results in the points on the image that are between the eyepiece and the window being projected "far down the window," indicating with a quick movement of her outstretched hand in a straight line from close to her face down toward the imaginary image that rests between her and the imaginary window (lines 238 & 239). This movement of her hand in the straight line is as if she were moving her hand along the line of projection, going from a center of projection, to the image, and extending to the window. As she moves her hand in this straight line, she is bringing into quasi-presence the line of projection that gives her the projected image "far down" on the window.

In this illustration, Willow engages in an act of imagination to explain why the projection of her triangle was larger than its original image. Through her gesture, body positioning, and gaze, Willow brings into quasi-presence the physical Alberti's Window situation, inserting herself into the mathematical situation. Specifically, she acts as if the window, the original image, and the lines of projection were present. I revisit this episode in Chapter 6.

5.2.4 Frequency of Acts of Imagination

Over the course of the 22 class sessions included in my analyses, there were a total of 126 instances of acts of imagination across the five participants. During the first component of the course, there were 104 instances of acts of imagination, and 22 during the second component of the course (recalling that each component consisted of 11 class sessions). The greatest number of instances of acts of imagination in any one class session was 30, which was across four of the five participants, as one participant was not in attendance. In only four of the 22 class sessions were there no instances of acts of imagination. The median number of instances of acts of imagination was 4.

Alejo and Willow each engaged in 19 instances of acts of imagination across the 22 class sessions. Trisha and Jerry engaged in 23 and 26 acts of imagination, respectively, across the 22 class sessions. Fiona engaged in the greatest number of acts of imagination across the 22 class sessions, with a total of 39 instances of acts of imagination.

5.2.5 Section Summary

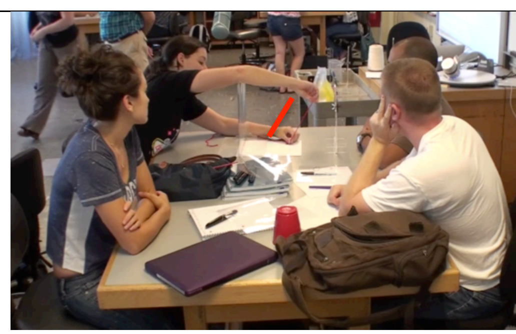
In this section, I provided a brief background into the literature involving imagining in mathematics. I recounted a vignette from an episode that occurred during a classroom session, and pointed to aspects of classroom activity that led me to considering acts of imagination as a mathematical practice. I then defined the act of imagination construct, which I stated is characterized by one or more students acting *as if* a mathematical situation or entity were present, despite not being physically present. I discussed the ways in which I identified acts of imagination in the data, and then illustrated the act of imagination construct through an episode from an interview with Willow, in which she acted as if the physical Alberti's Window situation were present to justify why a particular aspect of her artistic design occurred. In the next section, I provide further examples of students using acts of imagination to explain and justify mathematical ideas and situations. By explain, I mean the student described the details of a mathematical situation to others. For example, I students might how to determine the projection of an image at a particular location on the tabletop. By justify, I mean the mathematical situation being considered had already been discussed and some form of conclusion had been arrived at, or someone asked why a student believed something to be true. That is, the act of imagination is being used to refute an already formed conclusion or to provide support for an argument being presented.

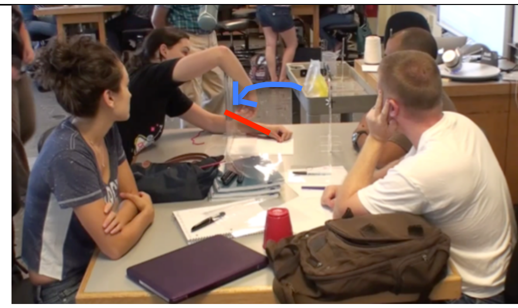
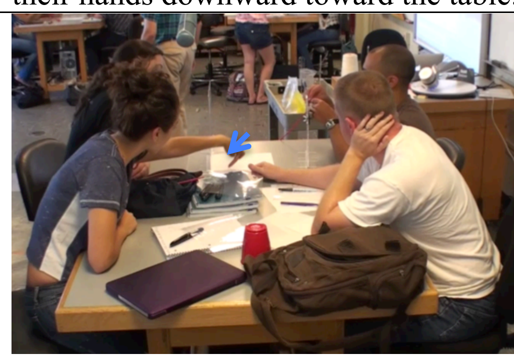
5.3 Acts of Imagination in Explaining and Justifying Mathematical Situations

In this section, I describe and illustrate ways in which acts of imagination played a role in students' explanation and justification of mathematical ideas and situations. During the component of the course in which groups explored situations with the physical Alberti's Window, particularly the situations in which the image to be projected rested on the same side of the window as the viewer, students frequently engaged in acts of imagination to explain and to justify why they predicted projections would look a particular way. In this section, I discuss the ways in which students used acts of imagination in explanation and in justification.

5.3.1 "It would hit at a point underneath, like by your feet."

During the first class session in which the class worked with projections of images located between the window and the viewer, Trisha engaged in an act of imagination to explain to a group member where the projection of the image would be located on the window. As the episode begins, Candace explains her theory about how they will project the square, sitting on the table between the window and the eyepiece, onto the window. Candace stretches the string from the eyepiece to a location on a square sitting on the table between their eyepiece and their window. She says her idea is that they will stretch the string form the image to the eyepiece, then rotate the string over to the window, holding the string to a point on the image (lines 240 & 241). Trisha and Mike both disagree with Candace, and Trisha engages in an act of imagination to explain to Candace how the projection of an image between the eyepiece and window will work.

240	Candace: So what I'm guessing is, you go through the eyepiece to a point,	 Candace stretches the string from the eypiece to a point on the square sitting between the eyepice and the window.

241 242 243 244	Candace: and then transfer it like this. Trisha: No Mike: No Candace: No?	 She then rotates the string from the eyepiece to the window, keeping the string touching the square.
245 246	Mike: You go straight down. Imagine this is projecting. Trisha: You want to imagine this is going down,	 Mike and Trisha each put the backs of their fingers to the window and move their hands downward toward the table.
247	Trisha: so the string would continue to go down.	 Trisha moves her hand to the opposite side of the window and points at an angle down toward the table in front of the window.

248	Trisha: And it would hit at a point underneath,	 Trisha places her fingertips at the base of the window.
249 250 251	Trisha: like by your feet. Candace: Then how do you draw that? Trisha: You have to imagine like, what it would look like.	 As she says, "like by your feet," Trisha moves her hand underneath the table.

In response to Candace's suggestion of how the projection would happen in this scenario, where the image to be projected rests between the window and the viewer, Trisha and Mike both begin to explain how they understand the projection will work in this scenario. Trisha and Mike each put their hand near the top of the window and move them straight down toward the base of the window (line 245 and 246), telling Candace they need to imagine the window extends below the tabletop. Here, Trisha and Mark set the stage for bringing the part of the window below the tabletop into quasi-presence by indicating they are imagining the extended window.

After introducing the idea of extending the window down below the tabletop, Trisha brings the line of projection, extending from the eyepiece to the image, into quasi-presence. She explains the string that originates at the eyepiece will "continue to go down," indicating the string would continue through the table. As she says this, she traces the imaginary string with her index finger, at an angle down toward the tabletop, in front of the window (line 247). As Trisha states that the imaginary string, which is the line of projection, will "hit at a point underneath, like by your feet," she indicates with her fingers in a downward motion toward the location where the window and the table come together (line 248), then sweeps her hand to underneath the table (line 249). Here, Trisha acting as if the line of projection passes through the tabletop and intersects the part of the window below the tabletop.

In this episode, Trisha acts as if several aspects of the mathematical situation were present. She first brought the extended window into quasi-presence by indicating with her fingers toward the bottom of the window and saying, "You want to imagine this is going down" (line 246). With this action and speech she is indicating the window extends beyond the small square of acrylic that sits atop their table. She follows this up with saying, "so the string would continue to go down," moving her hand to the opposite side of the window and pointing down toward the base of the window, brining the line of projection into quasi-presence. Here, Trisha acts as if there is a string extending from the eyepiece, down toward the table. Her suggestion that "the string would go down," indicates she is imagining the string passing through the table, "and it would hit at a point underneath, like by your feet," indicating where the line of projection would hit the window that extends below the tabletop.

Engaging in acts of imagination to explain mathematical situations to others was common while students were working with the Alberti's Window – in particular when the image was located on the same side of the window as the viewer. Students in the course also used acts of imagination to justify mathematical ideas and situations. By justify, I mean the situation being considered had already been discussed and some form of conclusion had been arrived at, or someone asked why a student believed something to be true. That is, the act of imagination is being used to refute an already formed conclusion or to provide support for an argument being presented. Over half of the instances of students engaging in acts of imagination were cases in which the students were explaining or justifying a mathematical idea or situation.

In the next section I present an episode in which Willow justifies her thinking to her group members, about a particular projection, through an act of imagination. I then present an episode in which Fiona explains her thinking to Dr. R. about what happens when a viewer tries to project a point directly next to her.

5.3.2 "The one that's right here, is gonna be oblong."

In this next episode, Willow uses an act of imagination to explain how she has changed her thinking about the projection of a circle that rests on the table between the eyepiece and the viewer, and to justify why the projection would look like a vertical ellipse. In this scenario, with the circle being projected resting between the window and the viewer, the projection of the circle will appear below the tabletop plane (*Figure 55*).

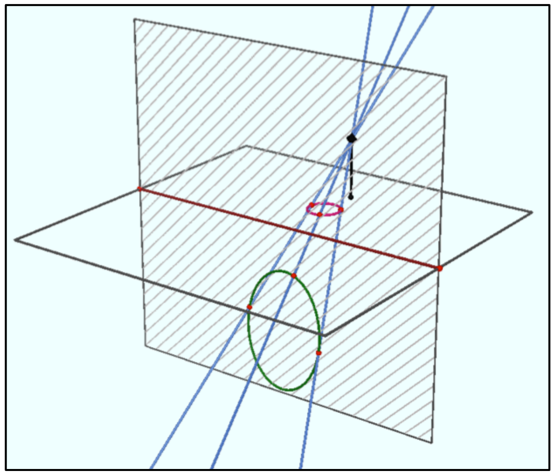


Figure 55. The pink circle resting between the window and the viewer is projected to the green ellipse.

Prior to this episode, Willow's group had predicted the projection of a circle between the eyepiece and the viewer should appear on the window below the tabletop, and would be "rounder" in shape than when the circle was on the opposite side of the window from the viewer. Specifically, Fiona, by using the idea that two parallel lines that are tangent to the circle being projected would meet at a vanishing point on the horizon line when projected, reasoned that the projection of the circle must be projected within the projection of those tangent lines (*Figure 56*). Observing that when a circle on the opposite side of the window from the viewer moved closer to the viewer, the elliptical projection of the circle remained a horizontal ellipse, but became taller, Fiona predicted the projection of the circle between the viewer and the window would remain a horizontal ellipse, but would become "rounder." Willow agreed with Fiona, saying, "Oh you're right, it's gonna be rounder."

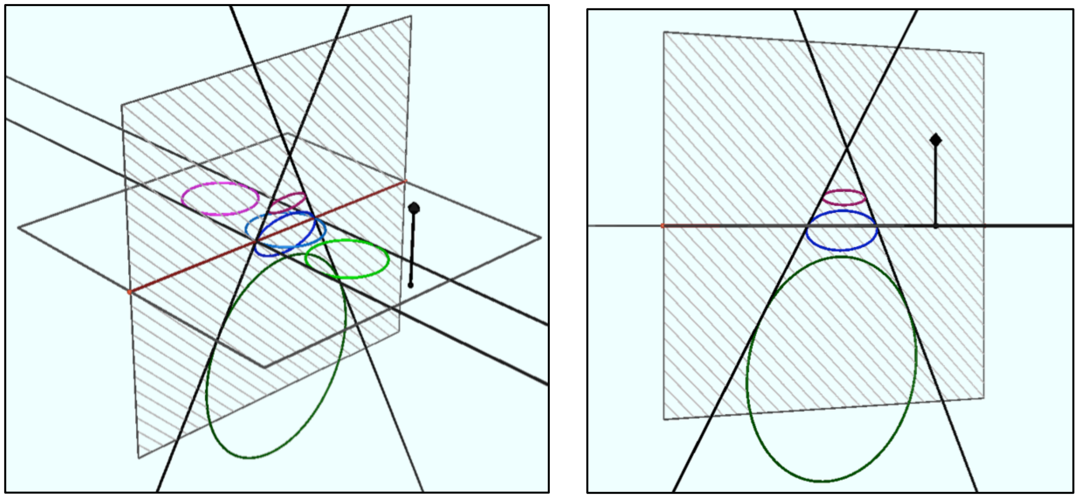
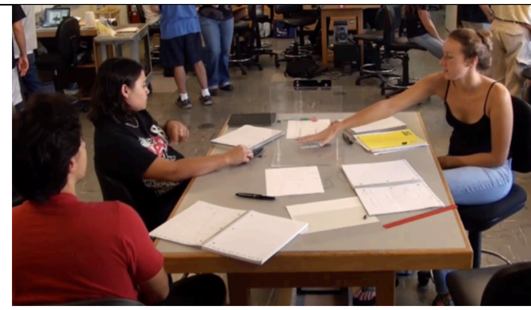
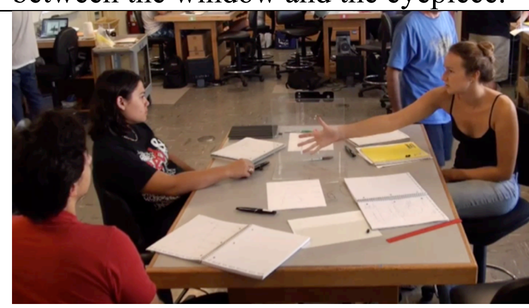
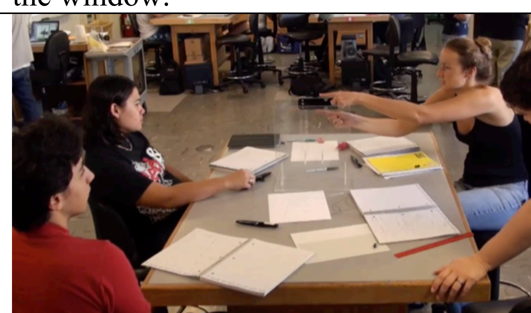
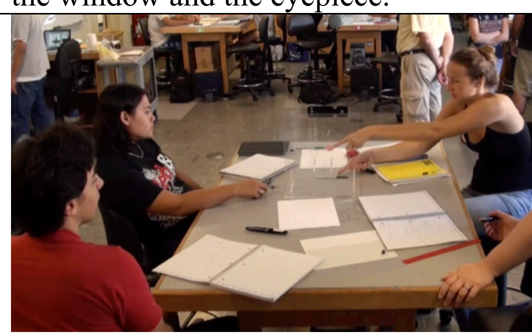
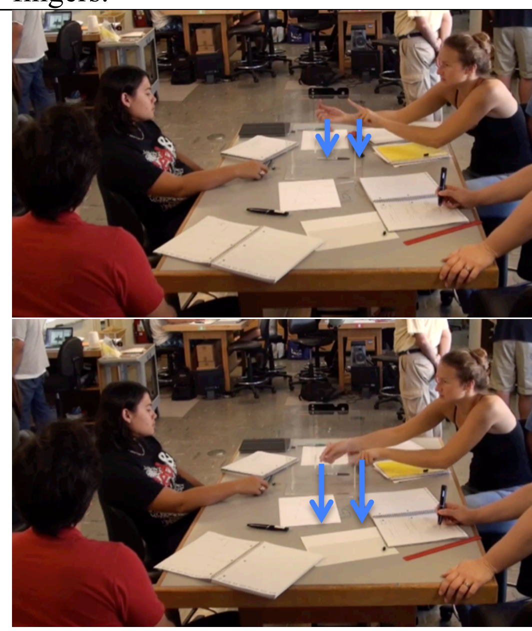


Figure 56. (*Left*): Three congruent circles tangent to a set of parallel lines being projected onto the Alberti's Window. (*Right*) A different view of the projected images of the three circles.

After predicting the projection of the circle resting between the window and the eyepiece would appear "rounder," yet still a horizontal ellipse, the group discussed how a circle would get projected if the eyepiece were in the center of the circle (For the curious reader, in the case where the eyepiece is in the center of the circle, the projection will appear as a hyperbola, in which one branch appears above the horizon line and one branch appears below the horizon line.) The group then discussed a feature they discovered on the SmartPens they were using to take notes, specifically, that the pens play back the recorded audio if the correct icon is tapped. The group went silent for just over one minute, during which time Willow looked down at her notes. Suddenly, Willow lifted her eyes from her notes and suggested their prediction for the projection of the circle between the window and the eyepiece will look different than they first thought. (Twelve minutes had passed since Fiona had first suggested the projection in question would look rounder.)

252	Willow: You know what, the one that's right here, is gonna be oblong.	 Willow taps the table with her palm, between the window and the eyepiece.
253 254	Jason: Oblong vertical? Willow: Yeah, vertical.	 Willow places her palm over the front of the window.
255	Willow: 'Cause, when it goes to the window, like	  Willow traces two lines through the air with her index fingers, originating at the

		eyepiece, moving toward the window, separating her fingers they get closer to the window.
256	Willow: here's the edges of the circle,	  Willow places her index fingers next to the eyepiece then slowly moves her fingers away from the eyepiece, while separating her fingers. She pauses and hovers her two index fingers, separated by a few inches, above the table, between the window and the eyepiece.
257	Willow: then it like, really separates	 Willow moves her two index fingers in straight lines toward the window, separating her fingers as she does so.

258	Willow: the top and bottom of the circle.	 Willow holds her hands flat and traces the same linear path toward the window that she had traced with her index fingers.
259	Willow: But it doesn't separate the sides as much.	 Willow holds her index fingers high up in front of the window, then traces lines straight down through the air.

In this episode, Willow engages in an act of imagination to justify why she believes the projection of a circle resting between the window and the eyepiece will be a vertical ellipse, rather than a "rounder" horizontal ellipse. At the beginning of the episode, Willow taps her palm on the table, between the window and the eyepiece, saying, "the one that's right here." Willow is referring to the case in which they are projecting a circle that rests on the table between the window and the eyepiece, which they had been discussing approximately twelve minutes earlier. Here, Willow sets the stage for bringing the circle into quasi-presence, through tapping her palm on the table, indicating to which case she is referring.

After clarifying for Jason that she thinks the ellipse will be oblong vertically, Willow begins to explain her thinking about the projection through acting as if certain mathematical entities, such as lines of projection and the circle to be projected, are present. Willow places her two index fingers near the hole in the eyepiece and traces two lines through the air toward the window, separating her two fingers vertically as she gets closer to the window (line 255). Here, Willow is bringing into quasi-presence the lines of projection. She returns her two fingers to the hole in the eyepiece and traces the lines of projection again, pausing halfway through to indicate points on the circle she is imagining projecting (line 256). She pauses with her fingers hovering above the table, separated slightly, and indicates, "here's the edges of the circle," bringing the circle she is projecting into quasi-presence. She continues tracing the lines of projection as if the lines were extending from the "edges of the circle" to the window (line 257). Willow then flattens her hands and slides her hands along the imagined lines of projection, saying "it really separates the top and bottom of the circle," indicating, approximately, where the two points she highlighted on the imaginary circle get projected to (line 258). Willow also indicates the projection of the "sides" of the circle isn't as dramatic a separation by tracing two straight lines down toward the table with her index fingers, as if the projection of the circle remains contained in these lines (line 259).

In this episode, Willow engages in an act of imagination through acting as if the circle being projected and the lines of projection were present in her surroundings, despite not being physically present. She uses this act of imagination to justify why she believes the projection of the circle that rests between the window and the eyepiece will be in the shape of a vertical ellipse, rather than a "rounder" horizontal ellipse. In this case, Willow is engaging in an act of imagination on a smaller scale – that is, she is operating as if she is on the outside of the mathematical situation, say, as an observer, rather than as a part of the mathematical situation herself.

As a side note, technically, both Willow and Fiona are correct about how the projection would look if the circle is resting on the tabletop between the window and the eyepiece. In addition to the location of an image with respect to the viewer and the window determining how a projection will look, the height of the eyepiece and how close the eyepiece is to the window also factor in to what the projection will look like (*Figure 57*). If, for example, the eye of the viewer were rather high above the tabletop, then the projection would appear as a vertical ellipse. If, however, the eye of the viewer were quite close to the tabletop, the projection would appear as a horizontal ellipse.

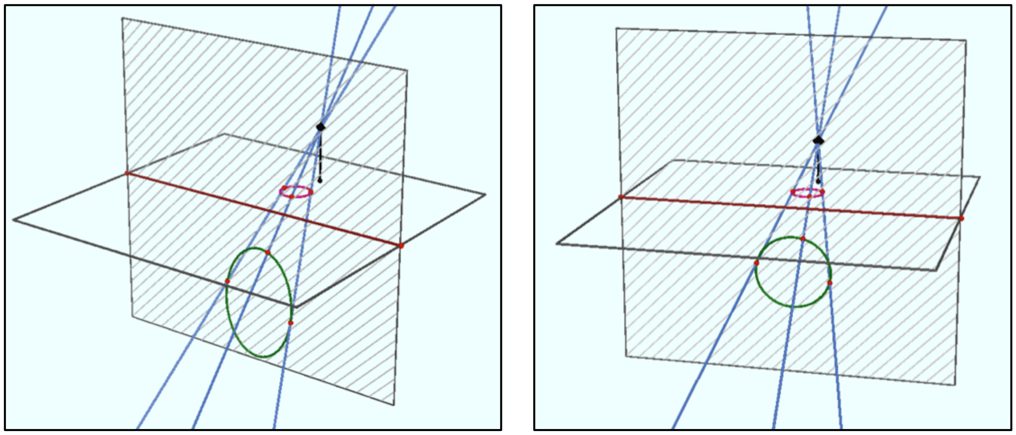


Figure 57. Two projections of a circle that are on at different locations on the tabletop, between the window and the viewer. (Left): The projection appears as a vertical ellipse. (Right): The projection appears as a horizontal ellipse.

5.3.3 "So that part just kind of misses."

In this next episode, Fiona uses an act of imagination to explain to Dr. R. why it is the case that when a point on an image is directly next to the viewer, or center of projection, the line of projection does not intersect the Alberti's Window. This episode takes place during an activity on a soccer field, in which the students were projecting one branch of a very large hyperbola on the field, and then imagining how the second branch would project. I explain the soccer field activity in further detail in a subsequent section. To understand this episode, we can imagine the viewer standing on a line that runs parallel to the window – which can be used to determine the distance of the viewer from the window. If we then create a line of projection from the viewer's eye down to any point on the line on which the viewer is standing, the line of projection will not intersect the window, as the line will be parallel to the window (*Figure 58*).

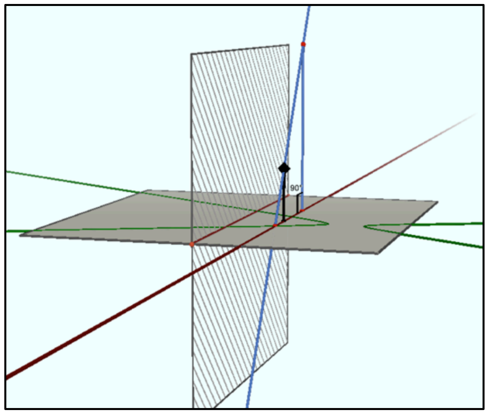
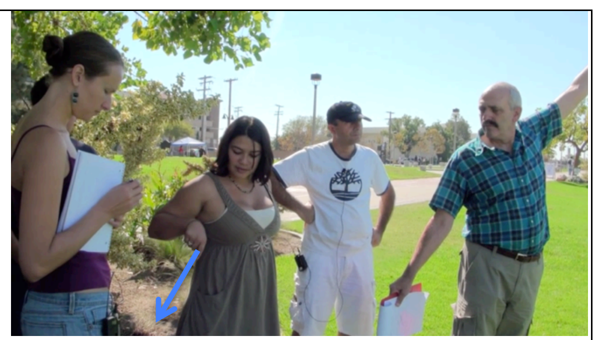


Figure 58. Projecting a point directly next to the viewer. The red line runs parallel to the window. The blue line is the line of projection. The line of projection runs parallel to the window.

In this episode, Fiona engages in an act of imagination in which she places herself into the mathematical situation and acts as if she is standing inside one branch of the hyperbola, and as if the window and the lines of projection were present. Again, in this episode, Fiona is providing an explanation to Dr. R. for why the projection of the points directly next to the viewer will not appear on the window.

260	Fiona: If it's right next to you	 Fiona draws her right hand close to her body, up toward her shoulder, extending her right index finger. She quickly pushes her hand away from her body, pointing toward the ground next to her, extending her right arm until it becomes straight.

261	Fiona: and straight across, it’s gonna be up here	  Fiona, watching her left hand, starts at her right shoulder, traces her finger across to her left shoulder, and extends her left arm so that it is in line with her right arm, pointing upward with her index finger.
262	Fiona: but then	 Fiona points in front of her with her nose, by moving her head slightly forward.

263	Fiona: there's no window	 Fiona places her two hands side-by-side, in front of her and higher than her face, palms facing away.
264	Fiona: the window doesn't curve,	  Fiona moves her hand away from each other in an arc, where the left hand curves over her head and the right hand curves down toward her side.

265	Fiona: so I can't draw on that.	 Fiona stands up straight and moves her arms so that her right hand is pointing back at the ground next to her and the left hand is pointing in the air, with her two arms in a line.
266 267 268 269	Fiona: So that part just kind of misses Dr. R: Yes. Fiona: Right? Dr. R: Yes.	 Fiona shakes her hands back and forth in front of herself as she says "kind of misses."

In this episode, Fiona inserts herself into a mathematical situation through acting as if several mathematical entities were present, and engages in an act of imagination to explain to Dr. R. why points directly next to the viewer do not appear on the window when projected. Fiona begins her explanation by pointing toward the ground next to her right side (line 260). She points here as if the part of the hyperbola they are discussing projecting is on the ground next to her. With her left hand, she points at her right shoulder, drawing her hand across from one shoulder to the other, then extending her left arm as if following the line of her right arm (line 261). Here, Fiona is acting as if there is a line of projection starting at the imaginary hyperbola on the ground and extending through her right arm to her left arm, and then continuing. Fiona follows this bringing into quasi-presence of the line of projection by pointing in front of her with her nose (line 262), and then drawing her hands in front of her, slightly above her head, flat palmed, as if the window were present directly in front of her (line 263). She accompanies this by saying, “there’s no window”, suggesting there is no window for the line of projection will intersect with. With her nose point, and her flat-palmed gesture in front of herself, Fiona is acting as if the window on which she is projecting is directly in front of her. She follows this by tracing an arc over he head with her two hands already in the air, and indicates, “the window doesn’t curve”, since the window is a flat plane (264). Here, Fiona is suggesting that since the window is a flat plane, and the line of projection runs parallel to the window, then line of projection will not intersect the window. She is also implicitly suggesting if the window were curved, then the line of projection could intersect with the window. She returns her arms to the position where her right arm is pointing to the ground beside her, and her left arm and hand follow that line, pointing up into the air to her left, once again, as if her arms were the line of projection (line 265). She states she “can’t draw on that,” indicating the line of projection will not intersect with the window, that it “just kind of misses” the window. She looks to Dr. R. for confirmation, to which Dr. R. replies, “yes.”

In this episode, Fiona brings several mathematical entities into quasi-presence. Specifically, Fiona imagines the hyperbola in which she stands, the line of projection extending from the point on the hyperbola directly next to her, the window on which she is considering projecting, and a window that curves over her head. First, she acts as if she is standing inside the hyperbola she is considering the projection of. Then, through following the line she creates with her right arm, she is acting as if the line of projection were present, using it to reason about why the line of projection will not intersect with the window. Fiona brings the window into the mathematical situation by pointing with her nose toward where the window would be located, as if the window were actually present. Finally, separate from the flat window on which Fiona is focused, she imagines a curved window, and gestures this curved window bending over her head. This curved window she imagines is such that the line of projection she is acting as if were present would intersect it at a point. She may have introduced this imaginary curved plane to highlight her point that her as if line of projection cannot intersect the flat plane window. Throughout the entire act of imagination, Fiona acts as if she is part the mathematical situation, where is standing in the hyperbola and her arms become the lines of projection.

5.3.4 Section Summary

In this section I discussed three instances of acts of imagination that illustrate the ways in which students utilized acts of imagination to explain and justify mathematical ideas and situations while working on problems in projective geometry. In general, students engaged in acts of imagination either on a smaller scale, in which they acted as if they were on the outside of a mathematical situation (for example, observing a mathematical situation), or on a larger scale, in which the students became part of the mathematical situation. In the case of Trisha explaining how they needed to think about the projection of a square resting between the eyepiece and the window, she engaged in an act of imagination as if she were on the outside of the situation, pointing out the mathematical entities of the situation, such as the lines of projection and the extended window. Similarly, in the case in which Willow justified why she thought the projection of the circle would be a vertical ellipse, rather than a horizontal ellipse, Willow engaged in an act of imagination in which she was on the outside of the mathematical situation, pointing out aspects of the situation. Finally, in the case of Fiona justifying to Dr. R. why the projection of a point directly next to the viewer would not appear on the window, Fiona engaged in an act of imagination as if she were part of the mathematical situation itself. For example, Fiona stood within the imaginary hyperbola branch, her arms became the line of projection, and she indicated the window directly in front of her.

Further instances of students explaining and justifying mathematical situations using acts of imagination, that arose during individual interviews with participants regarding their artistic pieces, are illustrated in Chapter 6.

5.4 Providing the Opportunity to Engage in Acts of Imagination

In this section, I discuss the instructional and course aspects that contributed to the opportunity for students to engage in acts of imagination. In particular, instructor influence, the nature of the task at hand, and *limitations* of mathematical tools provided the opportunity for students to engage in acts of imagination. I first discuss instructor influence, followed by the nature of the task, and end with limitations of mathematical tools.

5.4.1 Instructor Influence

Throughout the course, Dr. R. demonstrated his stance on the importance of students using their imaginations while developing an understanding of mathematical ideas. He demonstrated his stance on imagination in several ways. Dr. R. explicitly encouraged students to engage in acts of imagination, through task instructions, as well as through his choice of class activities and the mathematical tools employed in those activities, which I discuss later in this section. Dr. R. engaged the students in whole-class, and individual group, acts of imagination, and he accepted and encouraged mathematical justifications where acts of imagination were utilized. In addition, one day during class, he explicitly made clear his views on the role of imagination in mathematics.

The first instance in which Dr. R. explicitly encouraged the use of imagination occurred during the third day on which the class worked with the physical Alberti's Window. On this day, Dr. R. had the groups determine the way a square, located on the tabletop between the window and the viewer, projects onto the window. As I discussed in Chapter 4, the projection of a square in this scenario will appear on the window below the tabletop, and will be trapezoidal in shape (*Figure 59*), depending on the orientation of the square.

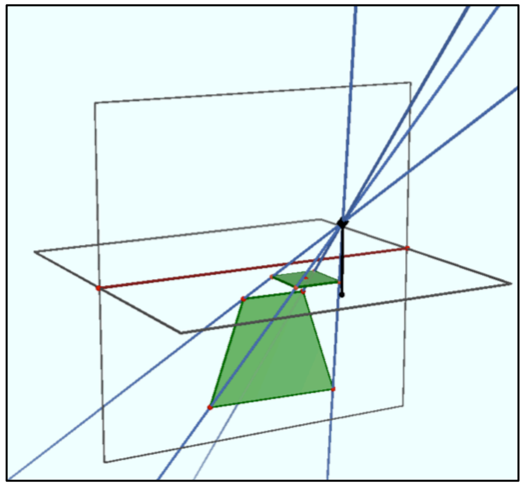


Figure 59. The projection of a square that sits on the tabletop between the window and the viewer.

Considering the projection of a square that sits on the tabletop and rests between the window and the eyepiece was the students' first introduction to extending the window and tabletop planes into infinite planes. While explaining the task, Dr. R. directly referenced how they would need to *imagine* how the projection would look.

> We now know how to project if you have something on the tabletop, but on the other side of the window. Now what happens if…, what happens if you have your square here, in between the eye and the window?

One or two students quietly say that you cannot see the square in that location. After a long pause, Dr. R. tells the groups to put their drawings of a square, which they used in the previous task, in between the window and the eyepiece. Dr. R. continues,

> So we say that we are extending the plane. In this case we are putting the piece ahead of, in front of the window. But we are also extending the plane of the tabletop and the window, both of them. And so if you look at, say, suppose that there is, let me use a piece of string (goes to get a piece of string). Suppose that uh, I will have a line of sight. So this end of the string is attached to my eye and it

will go like that (extending the string away from his eye, *Figure 60*). And so I go, I look through the eyepiece (goes to the table of a nearby group and threads the string through the hole in the eyepiece), and uh, here is my eye connected to the hole. So if I go look at this vertex (extending string from eyepiece to a vertex on the square, *Figure 61*), where does this line hit the window?

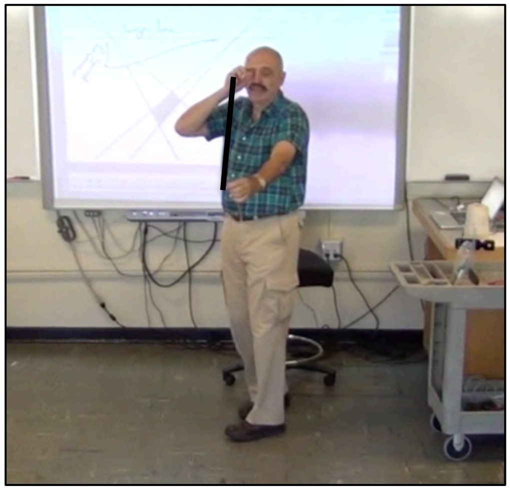


Figure 60. Dr. R. holds a piece of string to his eye with one hand and extend the string out in front of him with the other hand.

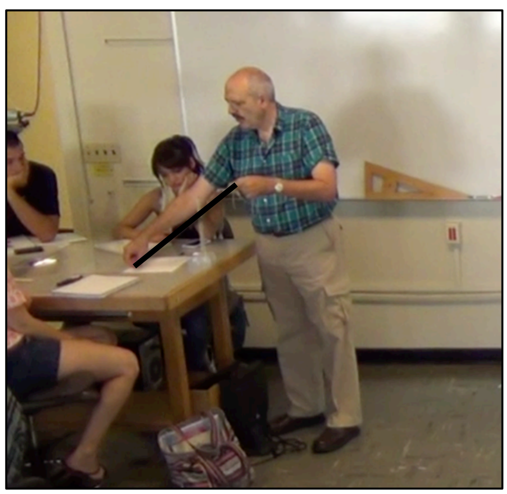


Figure 61. Dr. R. holds a piece of string that passes through the eyepiece, to a vertex on a square on the table.

One student replies that the line will hit the window “under the table.” Dr. R. reiterates this,

> It’s under the table. So you have to imagine that the window goes down. Well it goes in all directions. It’s a plane, an infinite plane. So work with a string and try to imagine, if you continue this [line of sight], and say you have the window under there (indicating under the table). So this is imaginary… How it will get projected, this square, underneath? So you draw it by hand, but this piece of string can help you to imagine it.

Here, Dr. R. explicitly tells the class they will need to imagine how the square gets projected onto the window. He highlights that the students will need to act as if the window and the tabletop planes are infinite, and then consider the projection of the square. He demonstrates with the string how they can think about the lines of projection going from the eyepiece to the square, and then to the window. Holding the string from the hole in the eyepiece to a vertex of the square on the table, Dr. R. says, “So if I go look at this vertex, where does this line hit the window?” The string Dr. R is holding forms a line segment on the line of projection that connects the eyepiece with the vertex on the square, and will determine where the vertex gets projected onto the infinite window. In this episode, Dr. R. invites the students to engage in an act of imagination with him, imagining the infinite window that extends below the tabletop, as well as the line of projection the string creates. At least one student, engaging in the act of imagination with Dr. R., suggests the imaginary line will hit the window “under the table.” During this activity in which the groups tried to determine the projection of the square resting between the window and the viewer, each of Jerry, Alejo, Fiona, Willow, and Trisha engaged in acts of imagination.

At times, Dr. R. specifically had students engage in acts of imagination. For example, during the first day on which students considered extending the window and tabletop planes, and projecting images on the same side of the window as the viewer, Dr. R. engaged the entire class in an act of imagination, which I illustrated in the vignette at the beginning of this chapter. Recall, in this vignette, Dr. R. requested the entire class stand up and engage in an act of imagination in which they used their arms as a line of projection to determine how a set of infinite railroad tracks project onto the window. In this exercise, Dr. R. asked the students to bring several things into quasi-presence, including the infinite window, a set of infinite railroad tracks, and a line of projection.

In addition to having the entire class engage in an act of imagination, at times Dr. R. encouraged particular groups to engage in acts of imagination, explaining to the group how to do so, or modeling how to do so. For example, during the activity in the soccer field, which I discuss in more detail later in this section, Fiona's group was trying to determine how a hyperbola would project onto the window if the viewer was standing between the two branches of the hyperbola, and the window was directly in front of the viewer *(Figure 62*). The group was able to trace onto the window the projection of the hyperbola branch that had been constructed on the soccer field, as it was in front of the viewer, and on the opposite side of the window. The group was then trying to determine how the branch of the hyperbola behind the viewer would project onto the window. In this episode, Dr. R. had checked in with the group to see how they were thinking about the projection of the hyperbola. As the episode begins, Fiona explains they are unsure whether the branch of the hyperbola behind the viewer will converge to a vanishing point on the horizon line, or if it will "stay wide" and not converge (*Figure 62 & 63*). Dr. R. leads the group through an act of imagination to help them imagine how the projection will look and determine an answer to Fiona's question.

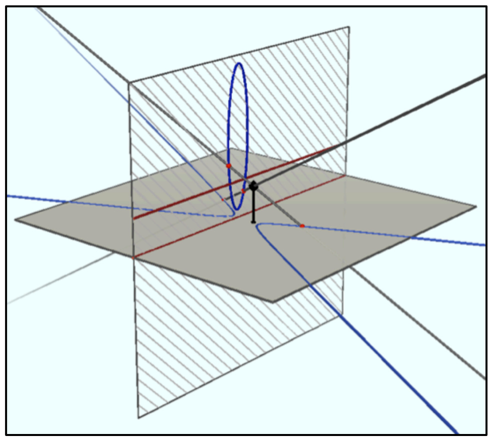


Figure 62. The projection of a hyperbola onto the Alberti's Window, where the viewer is standing between the branches of the hyperbola.

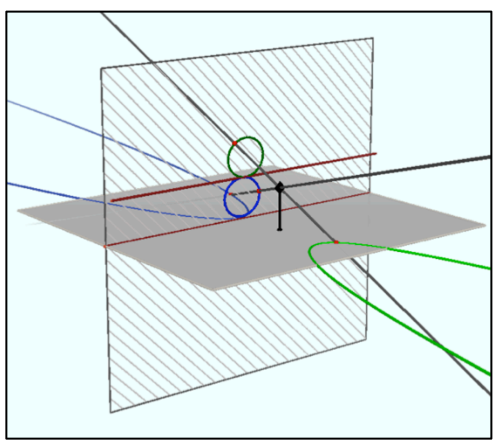


Figure 63. The projection of the hyperbola that Fiona suggests. This image is a projection of two parabolas, in which the projections converge at the horizon line.

292	Fiona: So then everything that's behind us, would be reflected upward.	  Fiona points behind herself with her right hand, then draws her hand in an arc over her head and points up in front of herself.
293	Fiona: But the problem is, does it, do we ever? Do, when it's behind us and it's reflected upward, do they connect or do they just stay wide? Or?	  Fiona makes a circular shape with her hand, then spreads her fingers so her hand makes a curved shape.

294 295	Dr. R: Well let's do something, uh, can, uh come here. *-Jason walks over to Dr. R -* Dr. R: So you imagine that there is that branch there, no?	 Dr. R. traces the imaginary hyperbola branch on the ground with his hand in the air.
296	Dr. R: And, I will borrow your hand. Someone, like Willow or someone else will just imagine that this line hits that branch,	 Dr. R. holds Jason's right arm out straight with his right hand. With his left hand, Dr. R. traces a straight line above Jason's arm.
297	Dr. R: and slowly, you know, will go like. Okay?	 Dr. R. traces the imaginary hyperbola branch with Jason's right hand.

298	Dr. R: And your job is, as Willow moves your hand down, you keep this straight, so you will go like this.	 Dr. R. holds Jason's left arm in a straight line with Jason's right arm as he traces the imaginary hyperbola branch with Jason's right hand.
299 300 301 302	Dr. R: Okay, let's do it. Willow: Okay. Dr. R: Slowly, Willow. Willow: Okay.	 Willow takes Jason's right arm.

Here, rather than directly provide Fiona with an answer to her question, by saying the projection of the hyperbola would not converge at the horizon line, Dr. R. instead models how the group should engage in an act of imagination to determine how the projection would look. Specifically, he positions Jason to engage in an act of imagination in which Jason's arms become the indicator for the line of projection, and another individual traces the branch of the hyperbola behind Jason with Jason's hand. The branch of the hyperbola is not physically present, and as such, the person guiding Jason's hand must act is if the hyperbola branch were present. Jason's role is to keep his arms in a straight line while another person traces the imaginary branch of the hyperbola with his hand. In this act of imagination, the lines of projection and the branch of the hyperbola are being brought into quasi-presence through the actors engaging in the act of imagination.

After Dr. R. models how the group should engage in the act of imagination, he guides the group through the act of imagination, where Willow moves Jason's hand, while Jason keeps his arms in a straight line. The group determines the projection of the branch of the hyperbola behind the viewer will not converge to a vanishing point on the horizon line. By modeling for the group how they can engage in an act of imagination to arrive at an answer to Fiona's question about whether the projection of the hyperbola will converge, Dr. R. is demonstrating his desire for students to engage in acts of imagination to predict the way projections look.

Dr. R. had a strong commitment to students using their imaginations to explore mathematical ideas and situations. On the fourth day of working with the physical Alberti's Window, Dr. R. explicitly expressed his approval of and commitment to imagining in learning in mathematics. On this day, Dr. R. requested the groups act as if they had a set of infinite railroad tracks on their tabletop, running from in front of the window to all the way behind the viewer (*Figure 64*). He asked the class to determine how the set of infinite railroad tracks projects onto the window. Dr. R. ended his explanation of the task by emphasizing that students should use their imaginations to find ways to justify their thinking about the projection of the railroad tracks, saying, "But then also think of how, how you imagine it. How, how, what do you do, to feel certain that the projection will go in some way in this plane."

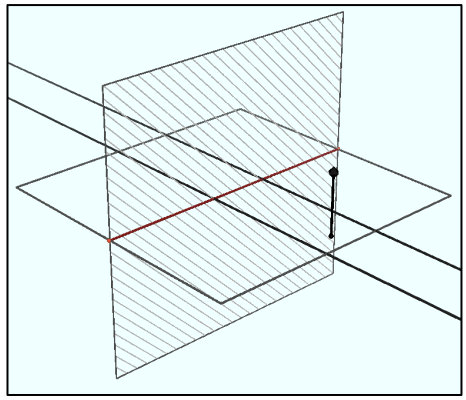


Figure 64. A viewer standing in between a set of railroad tracks.

While groups were working on determining how the set of infinite railroad tracks would project, Dr. R. interacted with groups to see how they were thinking about the task. During his interactions, multiple groups asked Dr. R. if their predictions of how the railroad tracks project were correct. In one particular instance, when a student asked, “Is that right?,” Dr. R. turned the question back on the student by asking, “What do you think, Francis?” Francis responded without confidence, “I think that’s right?” Dr. R. was quiet for a moment and then said, “I mean the, the point is not whether you get it right or not. It’s how you imagine it.”

Dr. R.’s interactions with groups in which students asked whether their predictions for the projections were correct seemed have concerned Dr. R., as he decided to address the issue with the entire class. When Dr. R. brought the class back together after this activity of projecting infinite railroad tracks, he addressed the issue of getting things right or wrong when predicting projections.

So I want to make a couple of general comments. Um, one, that this is not, eh, in several groups, you asked, "Well is this right, or "I'm getting it wrong, or?" This is not, it doesn't matter even if you get it

> right. It's how you think about it. And, and, whether you get it right because the two lines cross a certain way, is irrelevant unless you can have a way of imagining how these lines get generated. So this is a general comment. That's why, given that question of, of, "Do they go like this or like that?," well that, that's not the question. It's a question of how you see the whole scene and yourself in that.

Here, Dr. R. highlights the way in which he values explanation over having a correct answer to a mathematical question. He makes explicit his view that for predicting the projections of images, *how* students came to imagine the projection in such a way is more important to him than whether their prediction is accurate. Dr. R. continued his address to the class, expressing his beliefs about the role of imagination in mathematics.

> And the second general comment is that for, for some of us, mathematics… is a form of imagination. And learning mathematics is developing your imagination. So, much more important than saying, "I get the right answer", is how you develop our imaginations. And you don't develop imagination out of nowhere… There is all this work with the string and with the pieces of acrylic, and positioning yourself. So imagining is something you do with your body and with things. It's not something that happens in, in some uh, intangible way. So what you are doing now is, some of us think, is the essence of mathematics. You are, you are trying to figure out how to imagine.

In this second part of his address to the class, Dr. R. expresses his belief about the role of imagination in mathematics. In particular, he suggests that imagination "is the essence of mathematics," and that learning mathematics is a form of developing one's imagination. Furthermore, Dr. R. indicates to the class his desire for them to develop their imaginations through bodily engagement and with the mathematical tools. Here, Dr. R. provides further support for his commitment to students engaging in acts of imagination to make sense of mathematical situations.

Certainly, in addition to encouraging students' engagement with acts of imagination, Dr. R. also chose particular tasks and a mathematical tool with which students could engage to encourage them to use their imaginations to understand the mathematical ideas. In the next two subsections, I discuss some of these tasks and describe the aspects of the Alberti's Window tool that encouraged acts of imagination.

5.4.2 Nature of the task

Similar to the case with mathematical play, the nature of a mathematical task contributes to students' opportunities to engage in acts of imagination. In addition, the mathematical tools used in a task contribute to students' opportunities to engage in acts of imagination. Tasks that *require* students to use their imaginations, such as tasks in which students must imagine certain components of a mathematical situation, resulted in more acts of engagement than those tasks not requiring students to use their imaginations. For example, during the tasks in which students used the physical Alberti's Window to explore projections of images from one plane to another, the tasks in which students projected images on the opposite side of the window from the viewer resulted in fewer acts of imagination than those tasks in which students projected images on the same side of the window from the viewer. Specifically, during the first two days of using the Alberti's Window, where students projected images on the opposite side of the window from the viewer – meaning, the viewer was able to simply trace the projection onto the window with a marker – there were only three instances of acts of imagination across the five participants, and, specifically, only one participant engaged in acts of imagination. During the very first activity in which students considered the projection of an image on the same side of the window as the viewer, there were eight instances of acts of imagination across the five participants. This is not particularly surprising, as in the first case, where the image was on the opposite side of the window from the viewer, students were able to simply trace the projection of the image onto the window with a marker. In the second case, however, students were not able to simply trace the projection onto the window with a marker. Instead, students had to find a way to predict how the projection would look, which, generally, required students to use their imaginations.

There were many instances in this projective geometry course in which students participated in activities that *required* them to engage in acts of imagination to arrive at a solution. These activities included projecting infinite railroad tracks, projecting circles at different locations on the tabletop (Figure 65), and, perhaps most notably, projecting large-scale parabolas and hyperbolas. Each of these tasks resulted in acts of imagination for all five participants.

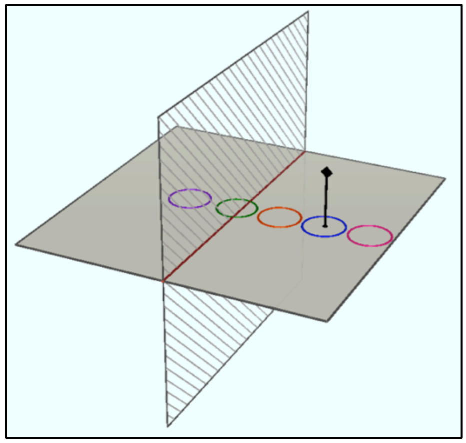


Figure 65. Five circles at different locations on the tabletop.

The activities in which students projected large-scale parabolas and hyperbolas occurred on a soccer field. On two consecutive class sessions, the class met at a soccer field to participate in an activity that required students to engage in acts of imagination in ways that projecting images that fit on a piece of paper does not. Specifically, on the soccer field, students were not able to extend a string from the image to the window to try to reason about the projection of the image. Instead, students had to find other ways to determine how the projection would look, and so the activities on the field required students to participate in imagining to a greater degree.

The tasks at the soccer field were to construct a very large parabola on the first day, and one branch of a hyperbola on the second day. At the soccer field, the groups of students were given the x-and y- coordinates of a set of points on the parabola (and hyperbola on the second day). With a line at the edge of the soccer field representing the x-axis on a coordinate plane, and a very long measuring tape representing the yaxis, groups used additional measuring tapes to plot their assigned set of points. Bright soccer cones were used to mark the points, and a very long rope – approximately 500 feet long – was used to make the shape of the parabola (or one branch of the hyperbola).

Once the parabola (or hyperbola) was constructed, students used their physical Alberti's Windows to determine the projection of the parabola from different locations. The first location from which the groups determined the projection was the case in which both the viewer and the window were behind the vertex, facing the vertex of the parabola (*Figure 66*). The second case was where the viewer and the window were in the center of the parabola, with the viewer facing away from the vertex (*Figure 67*). The third case was where the viewer was standing on the vertex, however the window was in the center of the parabola, and the viewer was facing the vertex (*Figure 68*). The cases were similar for when the groups determined the projection of the hyperbola, however in each case, the groups also had to imagine the second branch of the hyperbola behind the viewer (*Figure 69*).

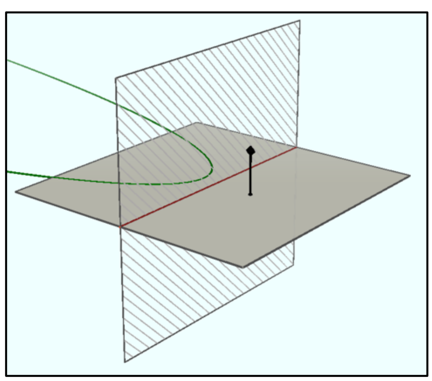


Figure 66. A viewer standing on the opposite side of the window from a parabola.

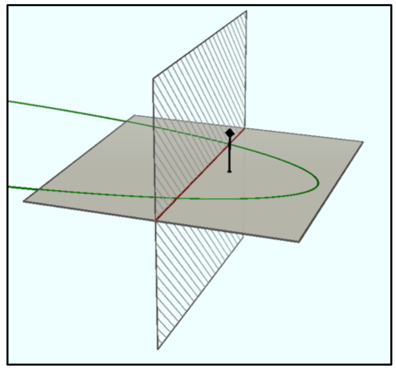


Figure 67. A viewer standing in the center of a parabola.

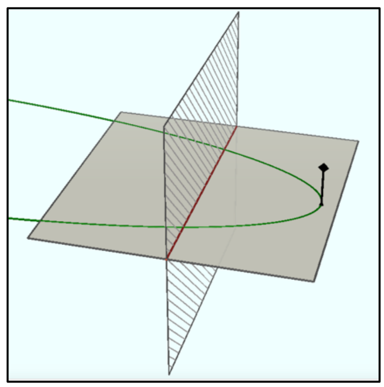


Figure 68. A viewer standing on the vertex of a parabola.

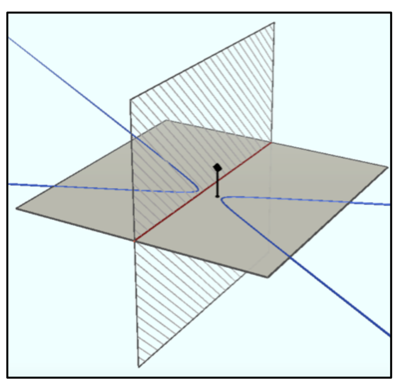


Figure 69. A viewer standing in between the two branches of a hyperbola.

In the case where both the viewer and the window were behind the vertex, students were able to simply trace the projection of the parabola onto their window. In the case where the viewer and the window were in the center of the parabola, students were able to trace the projection of the part of the parabola that was located in front of them, but they had to imagine the projection of the part of the parabola behind them. In the case where the viewer was standing on the vertex and the window was located in the center of the parabola, it was not possible for students to reach the window to trace the projection onto the window. Since the viewer could not reach the window, students had to imagine the entire projection of the parabola. In each of the hyperbola cases, students had to imagine the projection of the branch of the hyperbola located behind the viewer.

During these two soccer field activities, each of the five participants engaged in acts of imagination (with the exception of Trisha on the hyperbola day, as she was not in attendance). Two instances of acts of imagination during these soccer field activities have been presented in this chapter. For example, in section 5.3.3 the instance in which Fiona using an act of imagination to explain to Dr. R. why it was the case that the projection of the points on the hyperbola directly next to the viewer will not appear on the window.

The tasks in which students participated during the second component of the course, in which students considered the axioms, definitions, and theorems in projective geometry, unlike the tasks in the first component of the course, generally did not require students to engage in acts of imagination in the same way as tasks during the first component of the course, in which students considered the physical aspects of and problems that give rise to projective geometry. As such, there were fewer instances of mathematical play during the second component of the course than the first component of the course.

5.4.3 Limitations of Tools

Recall from Chapter 5, affordances and limitations of mathematical tools provided students the opportunity to engage in mathematical play. Unlike with mathematical play, in this course, it appeared to be only the *limitations* of mathematical tools that provided students the opportunity to engage in acts of imagination. Since very few instances of acts of imagination occurred during the second component of the course, I focus this section on the limitations of the Alberti's Window that provided the opportunity for students engage in acts of imagination. I contrast the limitations of the physical Alberti's Window with a hypothetical situation in which students have access to software with affordances the physical Alberti's Window does not.

As I mentioned previously, in this projective geometry course, students worked with the physical and GSP versions of the Alberti's Window to develop an understanding of how an image on one plane gets projected onto another plane through a center of projection. During the first component of the course, which focused on the physical aspects of and the problems that give rise to projective geometry, students consistently worked with the physical Alberti's Window. The physical Alberti's Window had certain affordances that allowed students to easily project images from certain locations on the tabletop, specifically those images on the opposite side of the window from the viewer, as the viewer could simply trace the projection of the image onto the window. In instances in which the image to be projected was on the same side of the window as the viewer, for example behind the viewer, students could no longer trace the projection onto the window. This meant students needed to determine the projection in a different way, such as through *imagining* how the image would get projected onto the window.

Initially, the limitation of only being able to trace the projection of images located on the opposite side of the window from the viewer was not an issue, as Dr. R. started the class using the Alberti's Window with images on the far side of the window. However, when Dr. R. introduced the idea of infinitely extending the window and tabletop planes, students had to determine a different way to think of projecting images from the tabletop onto the window, specifically through acts of imagination.

This limitation of the physical Alberti's Window was a consequence of the window design. Other mathematical tools, without this limitation, could have been used in the course. For example, the computer program, Cabri 3D (Bainville & Laborde, 2004), from which the reader has already seen images, would allow for students to quickly see the projection of images located on the tabletop plane on the same side of the window as the viewer. Cabri 3D allows the user to create mathematical objects, such as planes, line, points, and conics. This affordance would allow the user to construct an Alberti's Window situation in which imagining a projection would be unnecessary, since students could simply create the desired scenario with the Cabri 3D software (*Figure 70*). Had Dr. R. chosen for students to use Cabri 3D for exploring projections from one plane to another plane through a center of projection, students would have had fewer opportunities to engage in acts of imagination, as they would not necessarily have encountered situations that required acts of imagination to determine projections. Thus, had the students used Cabri 3D, the tool without the limitations of the physical Alberti's Window, there may have been fewer instances of acts of imagination as students learned about projection.

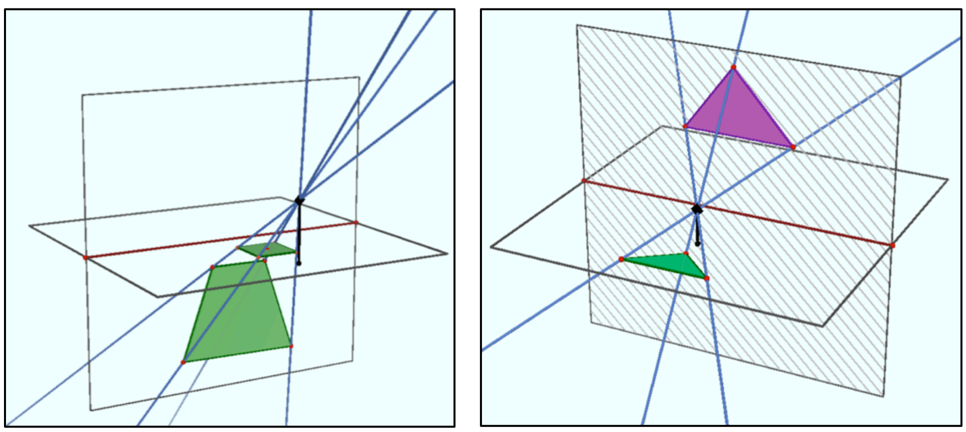


Figure 70. Two Cabri 3D images depicting projections of images from the tabletop to the window.

Similar to Cabri 3D, the GSP version of the Alberti's Window had affordances the physical Alberti's Window did not. Specifically, the GSP Alberti's window had the capability of instantly showing the projection of images at various locations on the tabletop. Had the students been introduced to the GSP version of the Alberti's Window earlier in the course, say instead of working with the physical window, then students may have missed the opportunity to engage in acts of imagination to reason about mathematical situations, since they could quickly and easily see the projection of images from any location on the tabletop.

During the second component of the course, in which students considered the axioms, definitions, and theorems in projective geometry, students typically used either a whiteboard with markers or the GSP software to construct diagrams of projections and perspectives. This highlights a second difference in the limitations and affordance of mathematical tools and their ability to proved the opportunity for student to engage in acts of imagination. While acts of imagination occurred much less frequently during the second component of the course, it was the limitations of mathematical tools, specifically the whiteboard and markers, that resulted in students engaging in acts of imagination.

5.4.4 Section Summary

In this section, I illustrated three aspects in the learning environment that provided the opportunity for students to engage in acts of imagination. In particular, I discussed the influence of the instructor, the nature of the task, and the limitations of mathematical tools. In this course, the instructor, Dr. R., provided students the opportunity to engage in acts of imagination through encouraging students to engage in acts of imagination while determining the projection of images from the tabletop to the Alberti's Window. He explicitly engaged the whole class, as well as groups of students, in act of imagination, and he made explicit his views that learning and doing mathematics is an imaginative activity. In addition, he made explicit that, to him, getting an answer correct was less important than being able to describe how they arrived at the answer, and how they came to image the mathematical relationships needed to arrive at the answer.

The nature of mathematical tasks and the limitations of mathematical tools in the course both contributed to students' engagement in acts of imagination. Specifically, some tasks in the course required students to engage in an act of imagination to arrive at a solution. In general, these tasks that required students engage in an act of imagination to arrive at a solution were tasks in which the limitations of the physical Alberti's Window played a role. Since the physical Alberti's Window did not allow students to simply trace the projection of images that were on the same side of the window as the viewer, students needed to find other ways to determine these projections, and in general, this included engaging in acts of imagination.

5.5 Imagination as the Essence of Mathematics

As I noted in section 5.4.1, in an address to the class, Dr. R. drew focus to his view on the importance of students developing their mathematical imaginations. He stated,

> [F]or some of us, mathematics… is a form of imagination. And learning mathematics is developing your imagination. So, much more important than saying, "[Did] I get the right answer?," is how you develop our imaginations. And you don't develop imagination out of nowhere… There is all this work with the string and with the pieces of acrylic, and positioning yourself. So imagining is something you do with your body and with things. It's not something that happens in, in some uh, intangible way. So what you are doing now is, some of us

> think, is the essence of mathematics. You are, you are trying to figure out how to imagine.

Here, Dr. R. suggested that imagination is "the essence of mathematics," and that learning mathematics is about developing one's mathematical imagination. If we consider imagination as the essence of mathematics, as Dr. R. suggested, then using one's imagination to come to understand mathematics is essential. It is reasonable to consider imagination as central to mathematics learning, as many aspects of mathematics exist only symbolically and in the imagination (Mazur, 2004; Moschkovich, 2003). Dr. R. takes the notion of imagination as *central* to mathematics learning one step further by indicating that mathematics learning is *constituted* by developing a mathematical imagination – specifically, developing ones ability to imagine mathematical ideas and situations.

From this perspective, developing proficiency in mathematics is akin to gaining an increasingly sophisticated mathematical imagination, and to be able to use one's mathematical imagination to explain and justify mathematical situations and arguments. The work of students in this course, in which they engaged in acts of imagination to explain and justify mathematical ideas and situations is then evidence of students' learning and developing proficiency in projective geometry ideas. To develop competence in mathematics then, students in the mathematics classroom should be using acts of imagination to explain and justify mathematical ideas and situations – a notion Dr. R. emphasized the importance of when he said,

> [I]t doesn't matter even if you get it right. It's how you think about it. And, and, whether you get it right because the two lines cross a certain way, is irrelevant unless you can have a way of imagining how these

> lines get generated… That's why, given that question of, of, "Do they go like this or like that?," well that, that's not the question. It's a question of how you see the whole scene and yourself in that.

If we consider imagination as the essence of mathematics, and assume Dr. R.'s statement that "imagining is something you do with your body and with things," then it is imperative that students learn how to imagine mathematical ideas and situations through participating in activities in the classroom that include the use of mathematical tools and bodily engagement, thus providing students the opportunity to engage in acts of imagination that foster the development of their mathematical imaginations.

5.6 Chapter Summary

In this chapter I discussed the act of imagination construct, a mathematical practice in which students in the course engaged while working on problems in projective geometry. Drawing upon the definition of collective imagining proposed by Nemirovsky et al. (2012), I defined an act of imagination as a mathematical practice characterized by one or more individuals acting *as if* a mathematical situation or entity were present, despite the entity not being physically present in the current surroundings. These acts of imagination could incorporate gesture, body positioning, eye gaze, verbal utterances, components of mathematical tools, as well as inscriptions, which was illustrated in the episodes I highlighted in the chapter.

I illustrated the act of imagination construct through an episode from the data in which a student engaged in an act of imagination to explain her understanding of why certain projections are larger than their original images, while other projections are smaller than their original images. I elaborated on why this was a case of an act of imagination, noting the ways in which the student had acted *as if* certain mathematical elements were present, and inserted herself into the situation.

Next, I illustrated the ways in which students engaged in acts of imagination while explaining or justifying the way in which they think about mathematical situations. In particular, students used acts of imagination to explain or justify their understanding in situations in which imagination was necessary to convey their understanding to others. Students engaged in acts of imagination to explain or justify acts of imagination to their peers as well as to Dr. R. and myself. I illustrated the ways in which students engaged in acts of imagination on a small-scale or large-scale – by this I mean the actor appeared to imagine herself or himself as an observer of a mathematical situation, or the actor appeared to take on the role of mathematical entities and imagine herself or himself as part of the mathematical situation, respectively.

Finally, I discussed the elements of the learning environment that provided students the opportunity to engage in acts of imagination. Specifically, the instructor influence, the nature of the task at hand, and the limitations of mathematical tools contributed to the opportunity for students to engage in acts of imagination. Dr. R. encouraged students to engage in acts of imagination through explicitly engaging students in acts of imagination, explaining task instructions by telling the class they would need to imagine mathematical situations, and by emphasizing his views on the importance of imagination in mathematics. Dr. R. developed tasks and selected a mathematical tool with certain limitations, the physical Alberti's Window, that required students to engage in acts of imagination in certain instances to find solutions to tasks.

Chapter 6

Artistic Engagement

In this chapter, I discuss the results of my second research question:

> In what ways can various means of artistic engagement enrich students' learning experiences and opportunities in an activity-based projective geometry course?

By *enrich*, I mean there is a form of value added to the mathematics course itself – something the students gained from engaging in the artistic components. This might include ways in which the artistic engagement supplemented or augmented the learning experiences and opportunities of the students in ways that would likely not have occurred without participating in the artistic engagement.

Recall, this particular activity-based projective geometry course included multiple forms of artistic engagement. First, students in the course participated in creating two artistic pieces using ideas from projective geometry (see Chapter 3, section 3.1.5.1 for a full description). For these artistic pieces, students created an artistic design, such as a visual pattern or scene, using the GSP version of the Alberti's Window. The designs were required to fit within a 13-inch by 10-inch frame, and projective geometry had to play a fundamental role in the design. Students were not required to demonstrate their understanding of projective geometry through their design; rather, they were to find inspiration in whatever piqued their interest and then use properties of projection to create their designs. As such, the final designs did not necessarily look like canonical images of projective geometry or linear perspective. Once the students finished creating their designs in GSP, stencils of the designs were cut. Students then used their stencils in conjunction with an airbrush to paint their design in any way they desired. Finally, students composed a written reflection regarding their experiences creating the artistic pieces.

A second from of artistic engagement in the course stemmed from of the roots of projective geometry. The first component of the course consisted of problems that gave rise to projective geometry, as well as the physical and spatial aspects of projective geometry. The history of projective geometry is rooted in Renaissance art (Andersen, 2007; Field, 1997; Kline, 1957), and as such, the first component of the course was also rooted in art and linear perspective. During this time, students analyzed multiple sketches and paintings, and engaged in considering the properties of linear perspective.

A third form of artistic engagement came in the form of readings, written reflections, and class discussions. During the course, students read two art-related writings, one related to considering mathematics as an art (Lockhart, 2009), and the other regarding Marcel Duchamp and the emergence of contemporary art (Gompertz, 2012). In addition, students paid a visit to the Museum of Contemporary Art. The students wrote reflective essays for each of the written pieces, and participated in a whole-class discussion regarding each of the written pieces, as well as their experiences visiting the museum.

This chapter is structured such that each section addresses the ways in which learning experiences or opportunities were enriched for one of the five participants. After addressing each of the five participants, I provide a summary in which I identify the overarching themes for the ways in which the participants' learning experiences and opportunities were enriched.

6.1 Trisha

When asked about her artistic abilities, Trisha juxtaposed her mother's artistic abilities with her own, saying, "My mom's a great artist…She has it. I didn't get it. I try really hard. I'm just, just not good at it." Although Trisha did not identify herself as an artist, she had a lifelong background in dance – learning to dance at a young age and then moving on to become a dance teacher for children and adults alike. Her relationship with dance found its way into her experiences in this course, both in her final artistic piece and in classroom discussions. In addition, her experiences in the course found a way into her dance instruction and choreography ideas, which I discuss later in this section. Furthermore, during Trisha's artistic engagement, she engaged in mathematical play and encountered pop-up topics. Recall, a pop-up topic is a mathematical idea or situation that had not previously been encountered in the classroom.

For her artistic pieces, Trisha found inspiration in personal aspects of her life – specifically in her life-long passion for dance – merging her personal experience with a mathematics assignment, an uncommon experience in traditional mathematics courses. Her inspiration for her second design was three-fold. First, she found herself intrigued by the three-dimensional chalk art discussed in class one day and decided she wanted to make a piece that had dimension to it. Second, she was inspired to create a spiral, relating the spiral to a pirouette – a move in dance in which the dancer does a complete turn or spin on one foot while the other leg is bent such that the foot is near the opposite knee (*Figure 71*). Trisha described her inspiration in the following way, "In dance my favorite thing to do is turn. I love doing pirouettes. The spiral makes me think about dance and the movements of dance." Finally, Trisha decided she would create the three-dimensional effect in her piece through the use of color (*Figure 72*). She chose to only use orange to paint her piece and noted she chose the color orange because "I also love nature and thought about a rising sunshine."

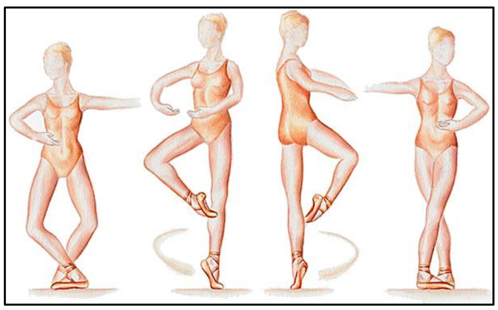


Figure 71. A visual representation of a pirouette. Retrieved on April 28, 2015 from: http://www.larousse.fr/encyclopedie/images/Pirouette/1005817

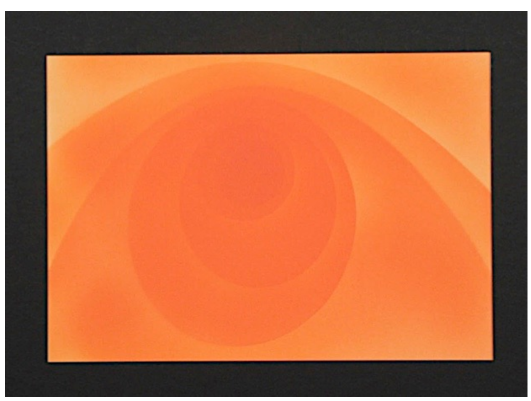


Figure 72. Trisha's second artistic piece, *Sunrise and Swirls.*

Trisha mentioned that she knew she wanted to make something spiral-like, and figured if she made a set of circles within circles, then through the painting process, with layers of the orange paint, she could make it look somewhat like a spiral with dimension. Trisha decided she wanted to use a circle to create her spiral design, but was not entirely sure how to execute the creation of her design. While determining how to construct her design, Trisha engaged in mathematical play with the projection of a circle in the GSP Alberti's Window. In an interview, Trisha described her process in the following way,

> So I literally just sat there and, had a circle and then just played with the circle, to see what happens with just one circle, and what happens when you move it, and when you do a projection. And then what happens when you move the eye [distance], everything. And, I noticed, ah well I remember from, um we were talking about what happens when you move the eye [distance], like closer, to the circle. And I was noticing that, originally you get a circle and then when you project it, the circle's smaller and it's about here (indicates lower on screen). And I noticed that when you would move the eye [distance], it would get closer and closer to your circle and it would get bigger. And then I, I finally got it to where the point where the circle was inside of the circle. And after I got that I was like, okay I totally got this project. Like I'll just keep doing that. Like, if I can get a circle inside of a circle then I'll get my image that I'm wanting. So I felt bad cause it was actually really easy to create. Like, this on-, really didn't take me that long. I was playing with the projections more than I, it took me to create my actual stencil. Um, so I spent most of the time just playing with the projections and really trying to understand, the, what happens when you do the projections and what happens when you go beyo-, past the baseline.

Trisha expressed that, during her design process, she allowed herself to freely explore how the variable aspects of the Alberti's Window – such as the location of the original image, the height of the eyepiece, and the distance of the eyepiece from the window – affect the projection of a circle. Since the directions for the artistic projects did not specify how students should create their pieces, the nature of her exploratory activity was freeform and autonomous, and so constituted mathematical play. This play with projections of circles led Trisha to discover that by moving the eye distance line above the baseline, she could obtain her original circle within its projection, which became the basis for her final design.

In creating her design, Trisha utilized two aspects, or processes, that were not mathematical situations that had been discussed in the classroom. The first of these two was moving the eye distance line in the GSP sketch to a location above the baseline, which Trisha expressed emerged from her playing with the GSP sketch and trying to understand what happens to projections of circles when “you go…past the baseline,” referring to moving the eye distance line above the baseline in the GSP Alberti’s Window (lines 340-343). This location change of the eye distance line, from below the baseline to above the baseline, is equivalent to moving the eyepiece onto the opposite side of the window from the usual position (*Figures 73 & 74*) – a context that was not discussed, nor likely would have arisen, in class. As such, Trisha was considering a pop-up topic.

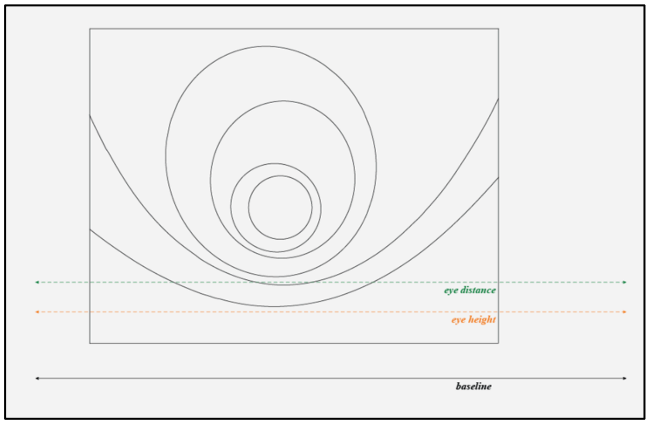


Figure 73. The GSP sketch of Trisha's design. The green line is the eye distance line, the orange line is the eye height line, and the black line is the baseline.

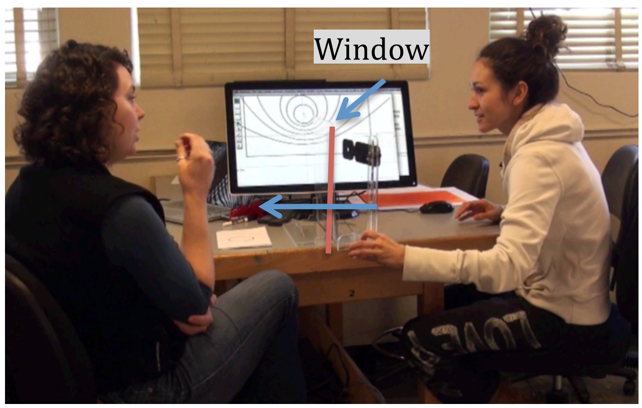
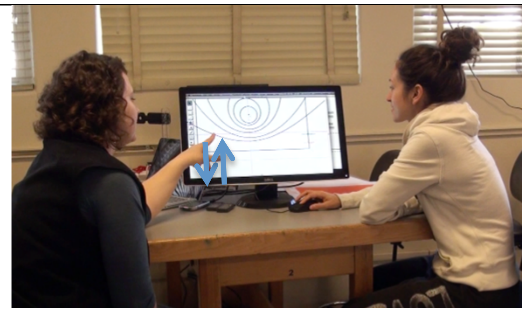
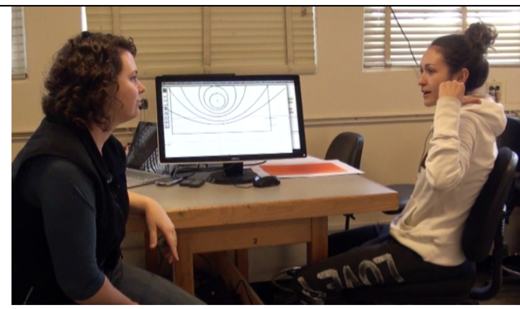
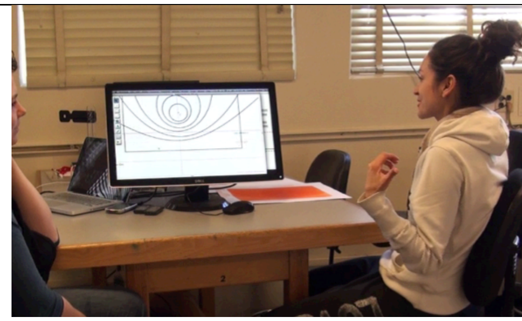
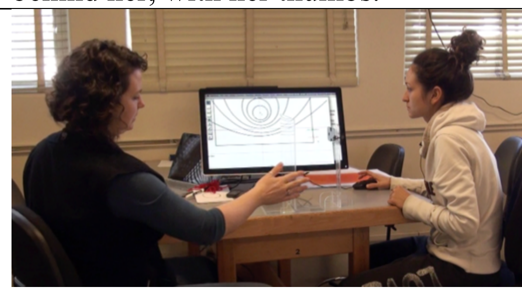
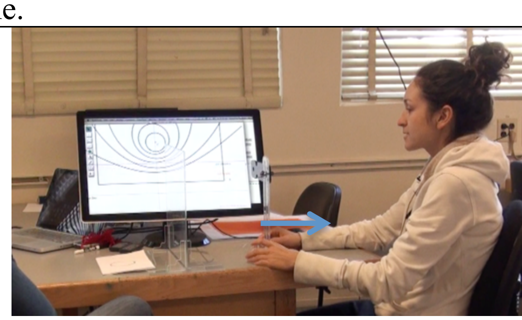
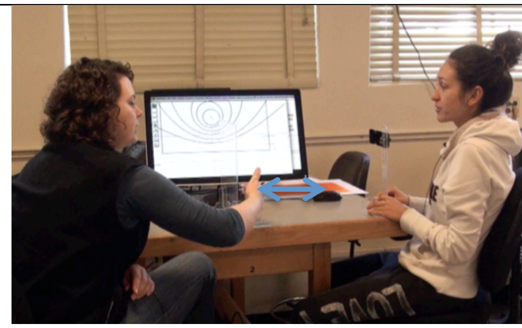
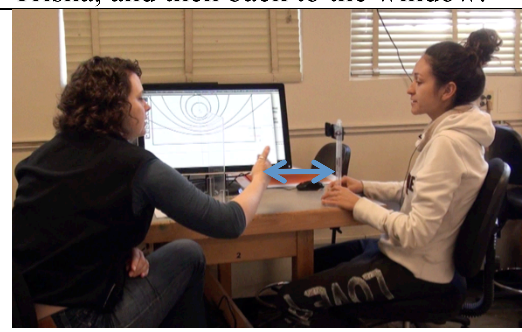
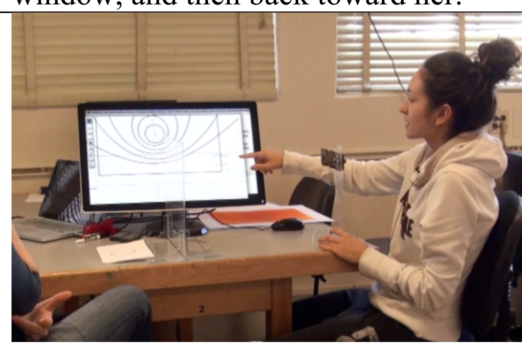


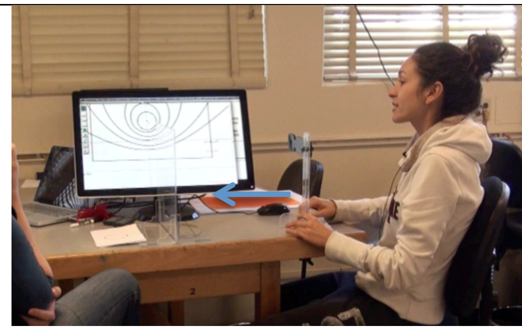
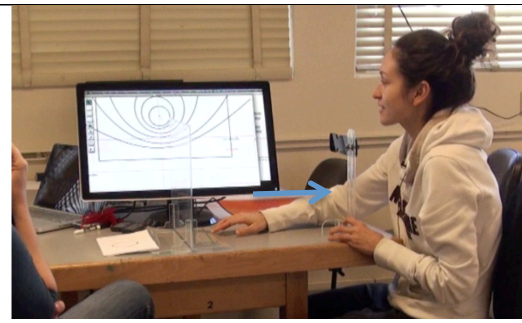
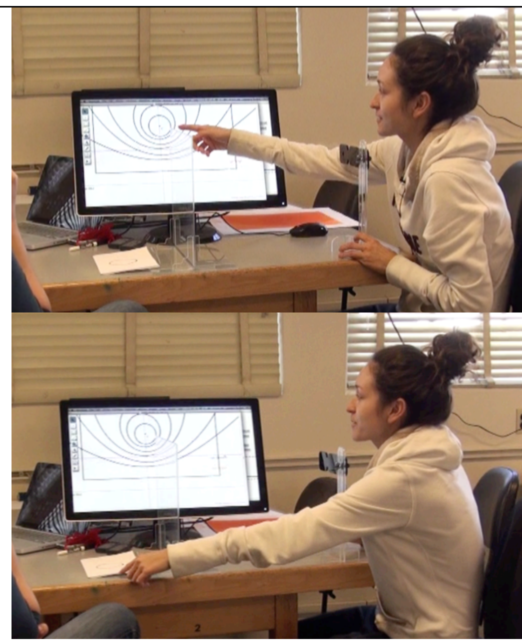
Figure 74. Trisha moving the eyepiece to the opposite side of the window

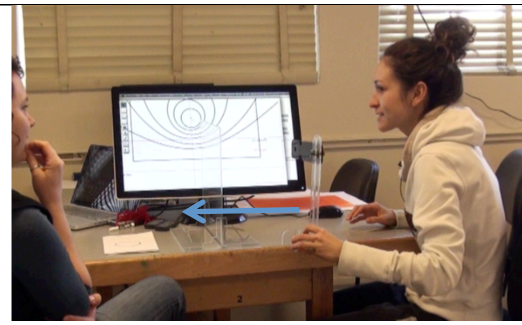
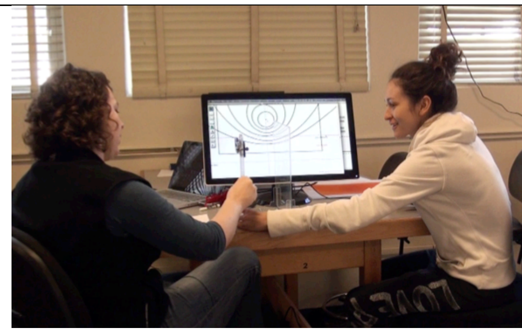
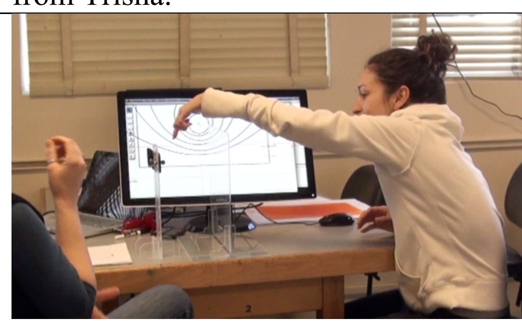
At the time of her design creation, she may not have realized she had created this scenario, however, in an interview, she worked through what the scenario meant in terms of the physical window, as well as why it had the effect it did on her design.

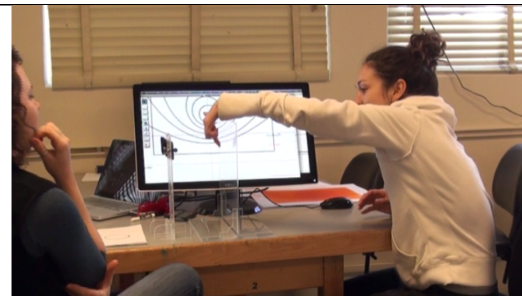
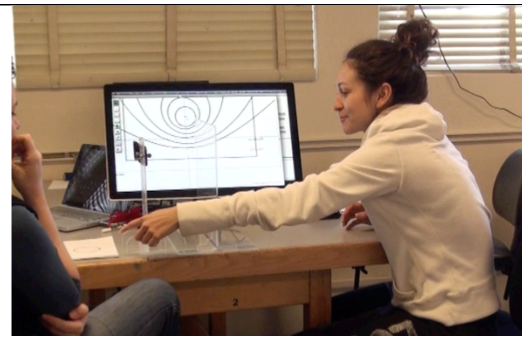
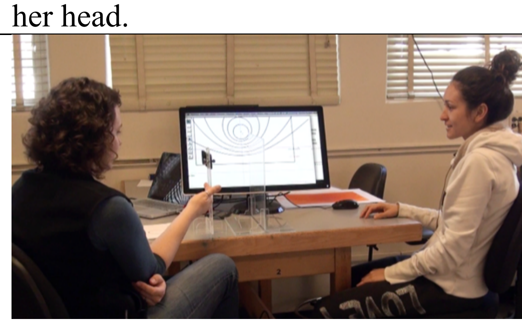
344 345 346	Int: What does that mean to move the eye distance, up, Trisha: That means like um, Int: above the baseline and the eye height?	 Interviewer points to eye distance line in GSP sketch, then points to where the eye distance line is typically located in the Alberti's Window sketch. She then traces a line with her finger from where the eye distance line is typically to where it is located in Trisha's sketch.
347	Trisha: means you're going behind.	 Trisha points over her shoulders, behind herself, with both her thumbs.
348 349	Trisha: Well we're obviously behind 'cause the, once you pass the baseline, then that means you're starting to do projections behind you. Int: So "we're obviously behind." What do you mean, "we're obviously behind"?	 Trisha moves her hand to the mouse. When she says "behind you," she slightly points over her shoulder with her left thumb.

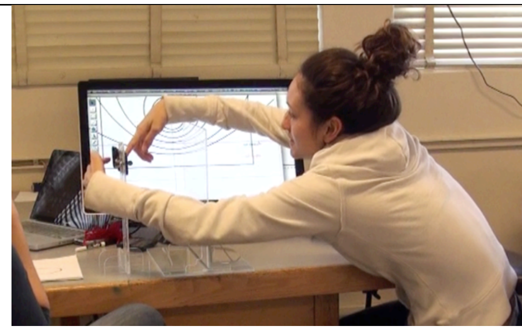
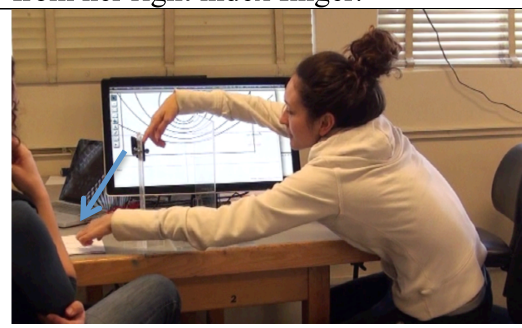
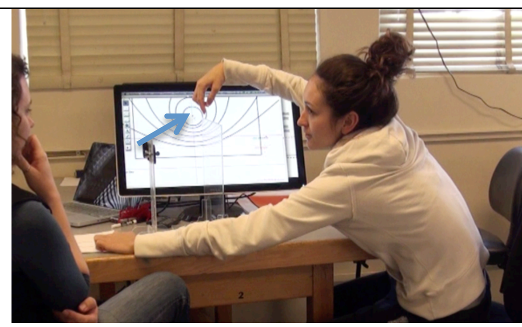
350 351 352	Trisha: Like, no, the, the object is now, we're now doing projections behind us. Int: The object is behind you? Trisha: Yes.	 Trisha again points over her shoulder, behind her, with her thumbs.
353	Int: So if you, ah. (reaches for physical Alberti's Window) If I gave you this, what's this situation (indicates to computer screen), uh as far as the eye distance, eye height, and baseline? As it relates to this, window.	 Interviewer gets the Alberti's Window and sets it up for Trisha.
	Trisha points out the eye height and baseline as they pertain to her design. She then turns to the eye distance line.	
354	Trisha: The eye distance would be like, this. Like moving it, away, from the baseline.	 Trisha pulls the eyepiece toward her.
355 356	Int: Okay so this is the eye height. Trisha: Because I don't know-,	 Interviewer places flat hand above the table. Raises and then lowers her hand from the table twice, to approximate the height of the eyepiece.

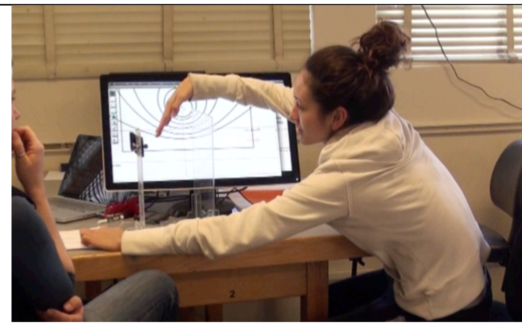
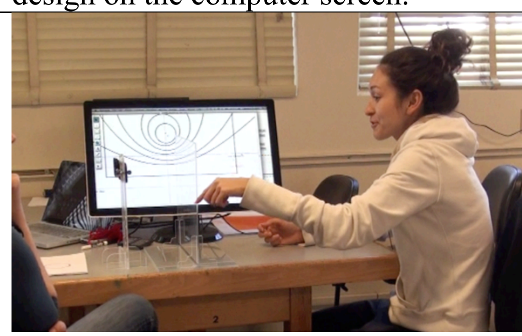
357	Int: And then this is the eye distance?	 Interviewer places flat hand near the window, parallel to window. Moves hand away from the window toward Trisha, and then back to the window.
358	Trisha: Yeah this is the eye distance but, Int: Okay	 Trisha slides the eyepiece toward the window, and then back toward her.
359	Trisha: I'm having a hard time, figuring out, ‘cause you're moving,	 Trisha points to the eye distance line on the GSP sketch.

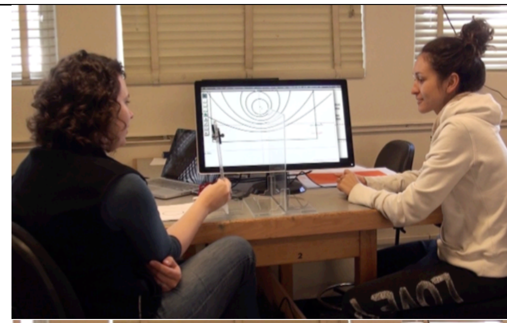
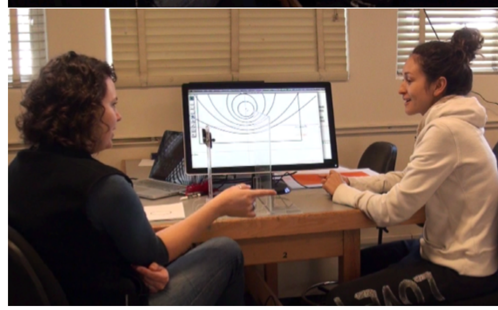
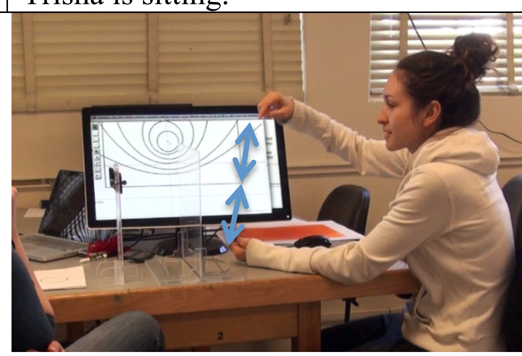
360 361	Trisha: you're moving it closer to the object. Int: Okay	 Trisha slides the eyepiece toward the window.
362	Trisha: So this is why I was saying it's like behind, 'cause in my head, I see the baseline	 Trisha draws the eyepiece back toward herself.
363 364	Trisha: and then I see the object in front of the baseline Int: Okay	 Trisha points to the circle in her design with her right hand. She then points to the paper with a circle on it on the opposite side of the window from her.

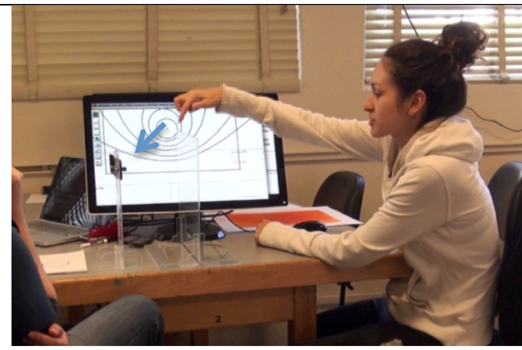
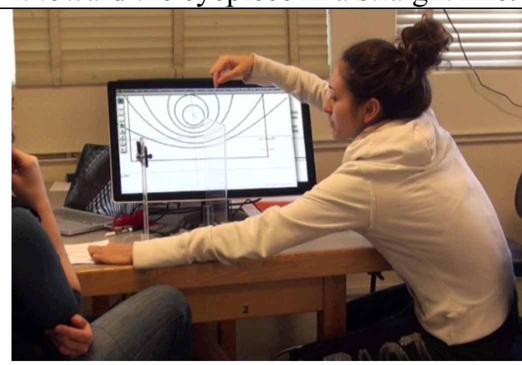
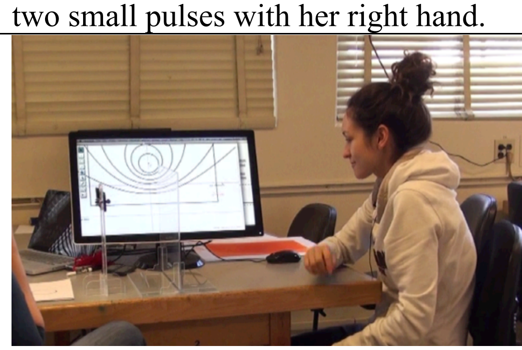
365	Trisha: And then I see the eye distance passing the baseline	 Trisha slides the eyepiece toward the window, then moves the eyepiece to the opposite side of the window from her.
366 367 368	Int: Okay. So it's over here? (moving eyepiece to other side) Trisha: So now it's like over there. Int: Okay.	 Interviewer helps Trisha place eyepiece on the opposite side of the window from Trisha.
369	Trisha: So now it's being, projected	 Trisha reaches over the window with her left hand and points toward the hole in the eyepiece.

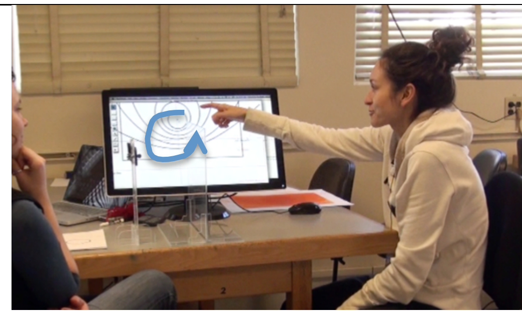
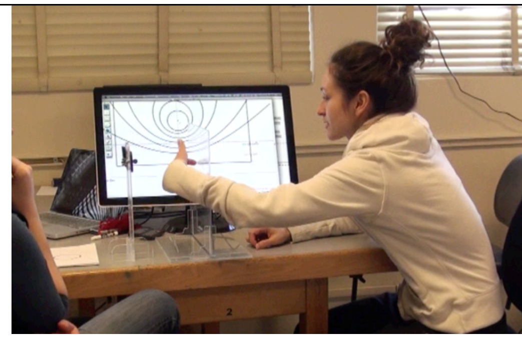
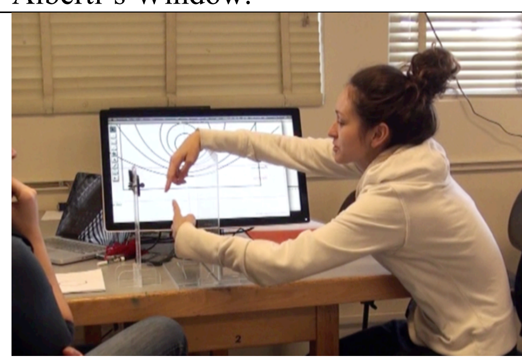
370	Trisha: and then like reflected	 Trisha points toward a spot on the window with her left hand.
371	Trisha: and bounced.	 Trisha points to the table on the opposite side of the window from her.
372	Trisha: I don't know. (laughs)	 Trisha sits back in her chair and shakes her head.
373	Int: (laughs) Well let's, let's talk about this So if you, if this was your situation.	 Interviewer indicates to the paper with the circle on the table, and then to the window.

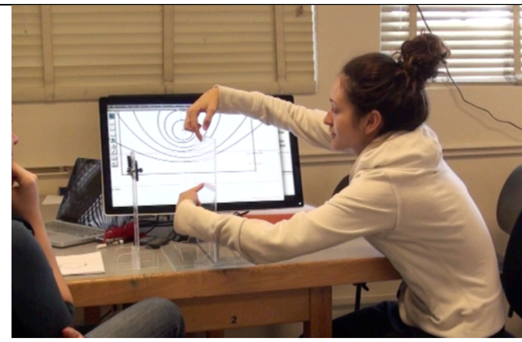
374 375	Trisha: You'd do the string thing. Like Int: Uh huh	 Trisha places her right index finger at the hole of the eyepiece. She than places her left index finger at the hole of the eyepiece, on the opposite side from her right index finger.
376 377	Trisha: so you'd put the string down right here Int: Uh huh	 Trisha traces a line with her left index finger from the hole in the eyepiece to the image of the circle on the table.
378 379	Trisha: and then it would hit and it would hit like, like here. Like that. Int: Uh huh	 Trisha traces a line with her right index finger from the hole in the eyepiece to a location above the window, but in line with the window. Her left index finger remains pointing at the circle.

380	Trisha: So it would be the string thingy.	 Trisha flattens her hand and traces a line with her hand from the point above the window she indicated toward the hole in the eyepiece.
381 382 383	Int: And is that what you have? Trisha: Yes. Int: Uh huh.	 Interviewer indicates toward Trisha's design on the computer screen.
384	Trisha: It's, I, I, I know this is what I have. I, I don't know how to explain it.	 Trisha points toward the circle image on the tabletop.

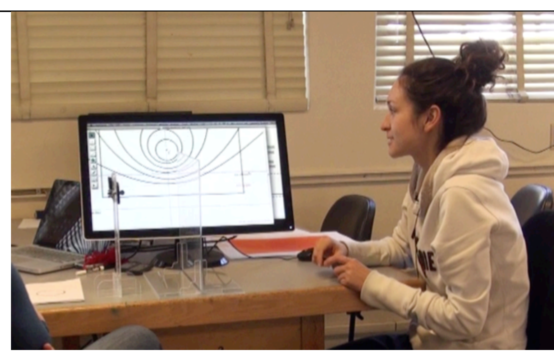
385 386 387	Int: What do you think about the fact that the, you are now on this side? Like the eye is now on this side of the window instead of that side of the window. Trisha: I obviously like that because I did that for my midterm too. Int: Uh huh	  Interviewer lifts the eyepiece up, shakes it slightly and puts it down. Interviewer then indicates by pointing to the side of the window on which Trisha is sitting.
388	Trisha: I think ‘cause it, it, in a way, it um, it’s, it’s like flipping the, the image in, in my head.	 Trisha lifts her right hand above the table. She then draws her left and right index finger tips together and then apart, lifting her left hand from the table, and lowering her right hand, then returns to the starting position.

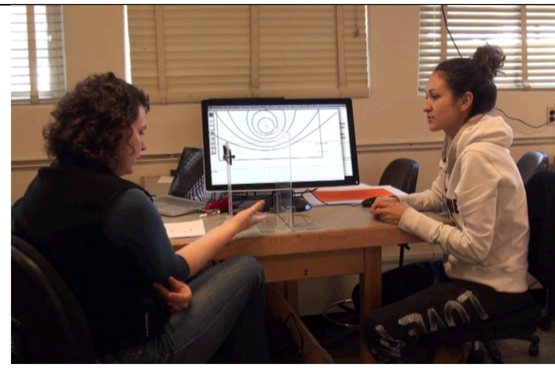
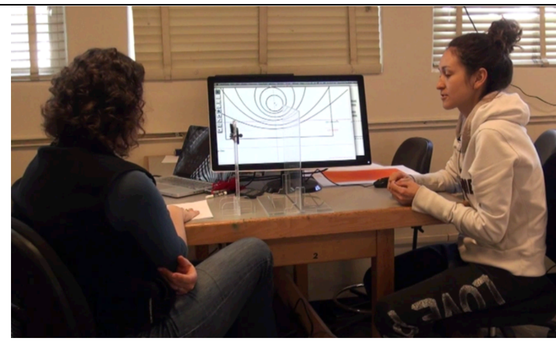
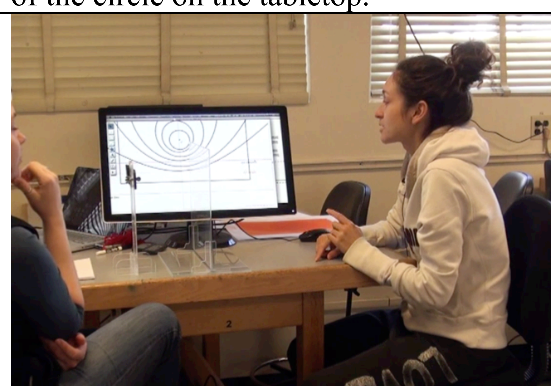
389	Trisha: 'Cause when you, when you do the string, you have to go down to hit it	 Trisha lifts her right hand, index finger extended, over the window and moves it toward the eyepiece in a straight line.
390 391	Trisha: and then it's going to be, like when you [project] it like the, this top of the circle will now be up here. Int: Uh huh	 Trisha points to the image of the circle on the tabletop with her left hand, while holding her right hand above and in line with the window. She then does two small pulses with her right hand.
392	Trisha: And then.	 Trisha sits back in her chair and goes quiet for a moment.

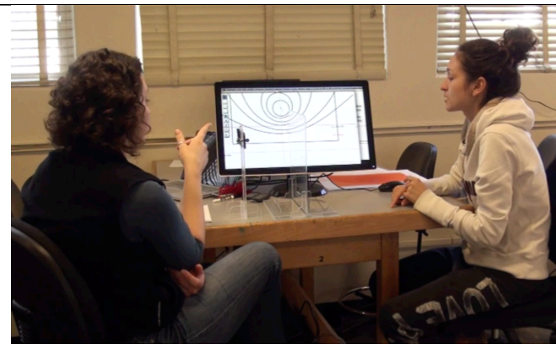
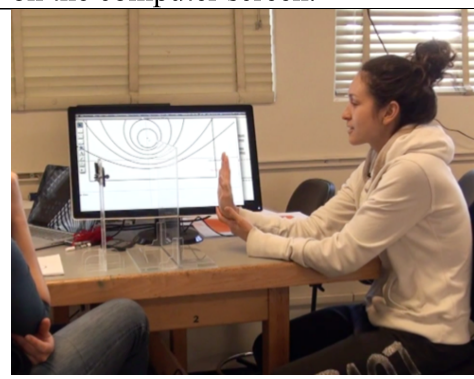
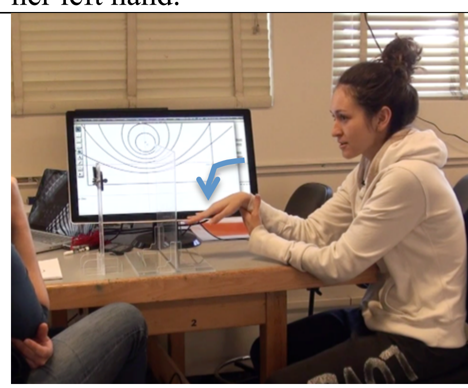
393 394	Trisha: Yeah, I'm still trying to figure out how it got on the outside. Int: Uh huh	 Trisha traces with her finger in a circular motion over the screen.
395	Trisha: I guess I'm trying to figure that out. But it would, make, a little bit of sense because, um when you're doing, your original circle, it'll be like right here.	 When Trisha says "right here," she points to a spot toward the center of the Alberti's Window.
396	Trisha: And then when you, when you do the string thing it'll,	 Trisha brings her two index fingers together in between the eyepiece and the window.

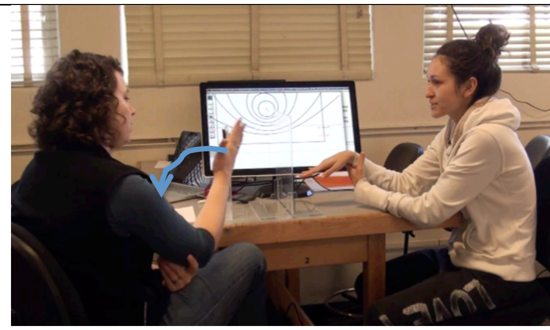
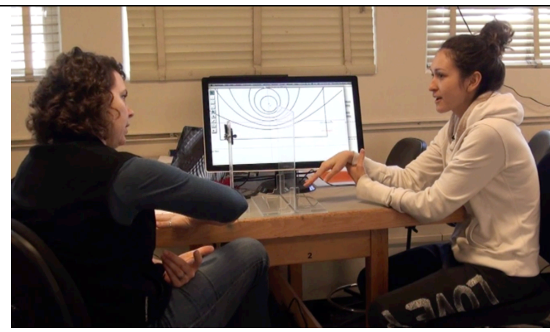
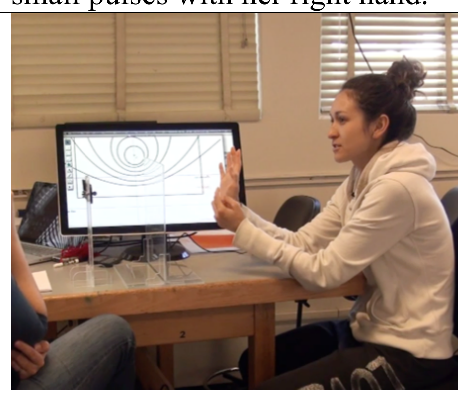
397 398	Trisha: it'll now be on the outer side of the circle. Int: Mm hm.	 Trisha moves her index fingers toward the window and separates them as she does so.

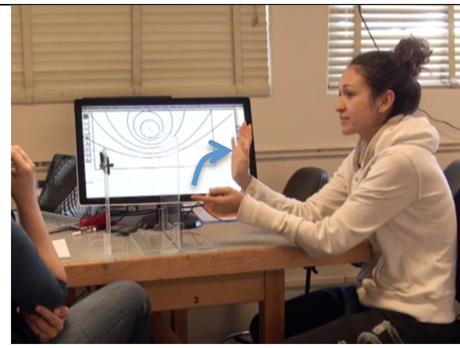
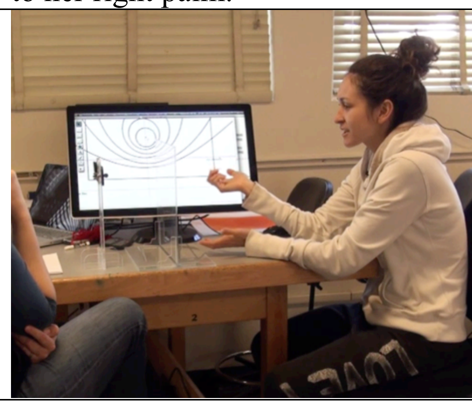
A second process Trisha used in her design creation, which had not been discussed in class, was projecting images that were the result of previous projections – taking projections of projections – in the GSP Alberti's Window. This was a process she had used in both her midterm design and her final design, and she noted in an interview that she found projections of projections intriguing. When asked why she thought that was, she replied, "I think it's 'cause I was trying to figure it out…like the whole projection of a projection." Trisha demonstrated the she had, in fact, tried to make sense of what it meant, in terms of the physical Alberti's Window, to project a projection in GSP.

399	Trisha: A projection of a projection is, you take, you take your object and then you project it. And then from there, you're going to, project that image.	 Trisha looks at the computer screen. Her hands in front of her move slightly as she explains

400	Int: Mm hm. So normally when we've been using this window (indicating the physical Alberti's Window),	 Interviewer indicates to the Alberti's Window.
401	Int: we project from the tabletop, onto the window. So what does it mean to do the first projection where you have this (indicates circle on table) onto the window, and then how do you get the second projection?	 Interviewer places her hand on the image of the circle on the tabletop.
402 403	Trisha: Mmm, yeah you're right, we've never, I was trying to figure that out actually. Int: Uh huh Trisha: That, I questioned that a lot when I was doing this	 As Trisha says, "when I was doing this," she points to her design on the computer screen.

404	Int: When you were doing this? (points to design on computer)	 Interviewer indicates to Trisha's design on the computer screen.
405	Trisha: Um, (nods) but I c-, I could imagine it um almost like this	 Trisha holds her right hand parallel to the window, and points at her right hand with her left hand.
406	Trisha: going back like down.	 Trisha drops her right hand to be close to parallel to the tabletop.

407	Int: Like onto the tabletop again?	 Interviewer pulls her hand up from the table and does a similar motion as Trisha, but with her entire arm, rather than just a hand.
408 409	Trisha: Yeah. And like printing an image on it. Int: Uh huh	 Trisha nods her head, and does a few small pulses with her right hand.
410	Trisha: And then pulling it back up	 Trisha draws her right hand back up to parallel with the window.

411	Trisha: And then projecting that image on there.	 Trisha indicates toward the table with the index finger on her left hand, then points to her right palm.
412	Trisha: That's the only thing I could come up with. Like, I seriously did think about that. And I questioned it, I go, “How come we never did that in class?”	

Here, Trisha indicates that she has, in fact, considered what a projection of a projection means in the context of the GSP Alberti’s Window. In her explanation, Trisha uses an act of imagination, where she utilizes her hand as if it were the physical window (lines 405-411), to illustrate and justify the way in which she has come to understand the projection of a projection. In her explanation, she indicates the way she understands the projection of a projection is that once you obtain the projection of your original image from the tabletop onto the window, the window will rotate down about the intersection of the window and tabletop and rest back on the tabletop (lines 405 & 406), printing the projected image on the tabletop (line 408). Finally, the projection of the original image, which now sits on the tabletop, is then projected onto the upright window (line 411).

Trisha's description of how she understands a projection of a projection is an accurate depiction of how this double projection works with the GSP version of Alberti's Window. The way in which the GSP version was designed was such that when you project an image, the program registers the image on the tabletop and projects it onto the window. Thus, despite the first projection technically being the projection onto the window, the GSP program registers the second projection as going from the tabletop onto the window again. This indicates that through the creation of and reflection on her artistic project, Trisha worked through making sense of the double projection in GSP. When asked about when she made sense of the situation, Trisha replied,

> It was more of an afterthought. Like when I was doing my reflection. It was more like, okay, I did a lot of reflections er, projections after projection, like of a projection. I was like, I did that in my midterm, I did that on my final, I obviously like it. I was like, I should probably figure out what it, means.

This double projection situation that arose from Trisha's mathematical play is one that likely would not have arisen in class, since the focus of the work in class with the Alberti's Window was single projections – both with respect to the physical and GSP versions of the window. This notion that a projection of a projection should be carried out in the way in which Trisha came to understand it was not trivial. It would be quite reasonable to think that a projection of a projection would result in the original image, as one might imagine a function and its inverse. Trisha described how this is, in fact, how she had originally thought about the double projection.

> [T]he thing I came up with, and I wasn't sure so I didn't write in on my final. It's ‘cause I wasn't sure if, when you had the projection, onto your paper, obviously if you take a projection of that projection it'll just be like the original image. Like you'll go back to what you had. So you had to have some sort of like, uh. Does that make sense? Like so say you take a projection right, and then you project that projection, just on the [physical window], you're gonna get back to where you started…It's just like, kind of going back and forth, back and forth. So I was like, there has to be some sort of, like, printing of the image going back down, and then redoing that to make it new, onto our new window.

Trisha expressed that her initial thinking about the relationship between the physical and GSP versions of Alberti’s Window, as it pertained to projections of projections, was incorrect, first saying she thought if you project a projection the result would “just be like the original image” (lines 420 & 421).

Through her artistic project – and perhaps in particular when she reflected on her artistic activity – she came to develop a better sense of the relationship between the two versions of the Alberti’s Window, where, after her projects, she was able to explain the coordination between the varying aspects of the two versions of the window, such as the location of the image on the table, the eye height line and the eye distance line. This highlights how Trisha was able to coordinate, and differentiate between, the similarities and dissimilarities of the two Alberti’s Window representations.

Trisha originally struggled with understanding the relationship between the two versions of Alberti’s Window. By the time the first art project was introduced as an assignment, the classwork with the physical Alberti’s Window had nearly come to a close. There were two occasions later in the course in which the physical window was used during tasks, but the main work with the window was over. Similarly, once the GSP version of the Alberti's Window was introduced, there were only three class sessions in which the GSP version was used. Therefore, the creation of her artistic piece, and her engagement in mathematical play while creating her piece, provided Trisha the opportunity to further develop her understanding of how the two representations of the Alberti's Window correspond. This indicates Trisha may not have developed the same understanding had she not engaged in creating her artistic pieces.

The ways in which Trisha's learning experiences were enriched through artistic engagement was not limited to her experiences creating artistic pieces using GSP. There were also ways in which her participation in the other components of artistic engagement, and in particular the art-related classroom discussions, proved significant for Trisha. During a classroom discussion regarding the first chapter of Paul Lockhart's, A Mathematician's Lament, the instructor of the course, Dr. R., discussed his view on the chapter saying he felt one of Lockhart's messages is that the playfulness in mathematics is missing from school mathematics. Dr. R. asked Trisha how she sees the differences and relationships between the practice of dance and the practice of mathematics. In her response, Trisha articulated how she sees the two activities as being similar, and expressed a fundamental difference she sees between the focus in her dance classes and the way in which Lockhart described school mathematics.

429 It's, it's almost exactly like doing math [as a mathematician]. I
430 mean you have, you have to figure out, when, when you're

> dancing, um, it's not like, "Am I doing this step right?" It's just like, "Am I having fun?" And, that's what I focus on when I'm teaching my classes is, not so much on the technique of it. Like, you know, "you're not doing it right." It's just, "Are we having fun today?"…

An implication of Trisha's response is that, unlike her dance classes, she sees the focus in the traditional mathematics classroom as one of accuracy of procedures, rather than student enjoyment. Her view that the practice of mathematics can be similar to the practice of dance in the way that both can focus on having fun is significant in that, in my experience, many students cannot fathom a way in which mathematics can be enjoyable and fun. She continued with connecting the practice of mathematics in the traditional classroom with an aspect of her dance classes she finds frustrating.

> I get frustrated in my dance classes because like um, I give them, we do like this creative circle and I'll just play music and I'm like, "Okay guys, you get to go and do whatever you want," and they'll just stand there. And they look at me like, they don't know what to do. I feel like that's how it is in math. Like they're so used to being told, "This is what you do." And then you give 'em a chance to be creative and they just stand there, completely just looking at me like with a blank stare, like, "I don't know what to do." And I'm like, "You can do whatever you want." And then they still, they just stand there. So, it's frustrating.

Here, Trisha makes an observation about the ways in which her dance students' inhibitions with respect to being creative is similar to the inhibitions mathematics students might experience when faced with exploring mathematics without direct instruction from a teacher. Trisha's description of the creative circle reflects the notion of the practice of mathematical play, in the sense that she is encouraging her students to engage in dance activity that is both autonomous and freeform. This connection indicates Trisha's acknowledgment of the creative nature of mathematics – an aspect of mathematical activity students in the traditional classroom rarely experience. These non-trivial connections between mathematics and dance that Trisha articulates, is an indication of her perspective on mathematics as a broader discipline – one that includes creativity, playfulness, and imagination.

On the final day of the course, during her video reflection about her artistic pieces, Trisha discussed the ways, over the course of the semester, in which mathematics, and specifically projective geometry, had influenced her creative intentions with respect to teaching and choreographing dance. Two aspects in particular arose. First, Trisha explained that she now saw more connections between mathematics and dance. She had previously mentioned that she liked to incorporate mathematical ideas into her dance classes, such as fractions (e.g., doing four quarter turns to make up a whole turn) and multiplication (e.g., tapping toes by multiples of two – "tap once, tap twice, tap four times, tap eight times"). By the end of the course, Trisha was considering other ways to incorporate mathematics into her dance activities. When asked about the ways in which the course changed her ideas about the connections between mathematics and art, Trisha focused on the connections between mathematics and dance.

> … I incorporate math more into my dance classes than I ever have. Little things to just the formations I make. I mean, I have to choreograph dances and I used to just be so simple, "Okay, four lines guys." Like, and now it's like, "No, I can do, I can do this different shape," and "What if I do this?" and then "I don't have to symmetric all the time." Like, "Let's be weird and let's not do four lines." And let's not just do like, I call it a bowling pin, like, one and then two and then three. I was like, "Let's be

> weird." So now I'm starting to play with uh different formations in my dance classes. And I'm starting to have ideas for my recital coming up, of doing a projection with a screen. And doing like a back light, and having a kid dance like behind it, and have it be big.

Here, Trisha indicates the ways in which her experiences in the course have begun to infuse her creativity in her dance classes. While it is possible Trisha may have considered the ways in which she could utilize the mathematics in the course into her dance classes without the artistic engagement included in the course, it is likely the connections came about as a result of the emphasis on connecting mathematics and art in the course.

The way in which Trisha's background in dance permeated her experiences in the projective geometry course is an unusual occurrence – and opportunity - for a course in mathematics. It is not uncommon to hear mathematics students grumble about how math has nothing to do with their own life – perhaps since traditional mathematics courses feel stripped of connections that are not, in my view, somehow contrived applications. This kind of weaving of personal experience through a course in mathematics could help the reluctant student to become more open to the myriad ways in which mathematics can relate to their own life.

In summary, Trisha's learning experiences and opportunities were enriched through artistic engagement in several ways. Through the creation of her artistic pieces, Trisha engaged in mathematical play and considered mathematical situations that likely would not have arisen during classwork. Through these two activities, Trisha developed a more sophisticated sense of the coordination between the two representations of the Alberti's Window. Both through her creation of artistic pieces and through her reflection on the ways in which mathematics and art are connected, Trisha was able to make non-trivial connections between her personal passion of dance and the field of mathematics.

6.2 Jerry

Jerry was one of the few students in the course who identified himself as having some artistic ability prior to the course. He mentioned his great grandmother was an artist, and this was passed down to his grandmother, to his mother, and then to him. When talking about creating artistic pieces when he was younger, he noted, "I've always liked things like that." He mentioned he had taken an art class in high school, saying, "It was a really good class for me." He mentioned he had even considered the idea of becoming an art teacher in the future, but had decided to pursue teaching mathematics instead.

In this projective geometry course, his mathematical curiosity permeated his experiences with artistic engagement. In his artistic pieces, Jerry found inspiration through mathematical play with a particular aspect of a homework assignment. By the end of the course, Jerry developed ways to connect mathematics and art in non-trivial ways, and had begun to consider the ways in which mathematics might influence his future artwork.

For his artistic pieces, Jerry found inspiration in a homework assignment in which students were required to create a projected grid in GSP. This projected grid construction was carried out by creating a horizon line, placing two vanishing points on the horizon line, creating two lines coming from each vanishing point to create the first tile of the grid, then filling in appropriate lines, including the diagonals of the tiles, to fill out the grid to at least a five-by-five (*Figure 75*).

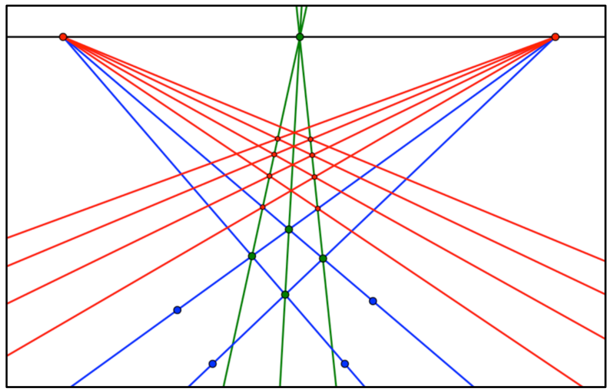


Figure 75. A perspective grid. The blue lines created the first tile. The green lines are the diagonals of the tiles. The red lines are the other necessary lines to fill out the grid.

Jerry's interest in creating an artistic piece inspired by the projected grid arose while working on a task with his group during class. At the time, the group was working with the GSP Alberti's Window. The group began to discuss a homework assignment in which they were asked if they could create a piece of contemporary art using projective geometry. Carla, one of Jerry's group members, mentioned that the projected grid is projective geometry and that it could constitute an art piece. Alejo, another group member, began to construct a grid in GSP, and Jerry got excited to think about how an already projected grid, such as the one they constructed for homework, would react when projected onto the window.

Alejo: Oh! What if I, what if I do draw a checkerboard? Okay, so let's say I took a still line.
Jerry: Oh, if you transform the checkerboard!?

Alejo: Like I drew the checkerboard and then transform it.
Jerry: That'd be nuts!
Alejo: I don't know, I don't know.
Jerry: I really want to see what it looks like. Let's do it.
Alejo: Okay, let's try it.

Jerry became excited and intrigued by the idea of projecting the already projected grid and expressed his interest in exploring the situation. Since the class was working on a different task, and the class had not discussed projecting an image that, in a sense, had already been projected, Jerry was initiating mathematical play – recalling from Chapter 4, I defined the practice of mathematical play as "the exploration of mathematical ideas through individual or group actions that are both autonomous and freeform."

As Alejo was constructing the projected grid, Jerry expressed his excitement about considering this situation for an artistic piece, exclaiming, "Dude, that's a good idea. That's a, *that's* the piece right there." Additionally, Jerry expressed confusion about the projection, "You're projecting something that already looks three dimensional, so what's gonna happen? I'm so confused right now. What is it gonna, is it just gonna look like if you projected it down-?" Jerry is likely suggesting, while expressing his confusion, that he imagines projecting an already projected image would equate to projecting the image back onto the tabletop to result in the original image. This arose in an interview in which Jerry explained how he thought about what a projection of a projection in the GSP Alberti's Window equated to when working with the physical window.

That's interesting 'cause I think like, what I would think is a projection of the projection, I think would like map back onto itself

> almost. But that wouldn't make sense. Like, like almost like a derivative and an antiderivative type thing. But it's not the same, because it's almost, it's more like a double derivative. ‘Cause you're not, ‘cause that's what I would originally think is if you started with a grid and then you projected it, and then you projected it back. But it's not like you're projecting it back, you're projecting it then you're projecting it again. So as it, almost as if there's like a second window that you place, after you have that. So there's one window here, your original image, and then you project it. And then there's a second, or would you just use the same window and then project it again.

Jerry explains the way he has come to understand the projection of a projection in the GSP Alberti’s Window in terms of the physical window. He compares the double projection to taking a double derivative in calculus, rather than a derivative and an antiderivative – suggesting one would carry out the same procedure with the projection as was done with the original image, rather than carrying out an inverse-type procedure with the projection. Again, this notion of taking a projection of a projection with the GSP Alberti’s Window was not a topic that had been intended as a discussion topic in class. As such, Jerry, through the reflection on his artistic piece, came to make sense of a mathematical situation, the coordination of these mathematical tools, outside of class.

With the projected perspective grid constructed in GSP, Jerry and Alejo, with Alejo relinquishing control of the computer to Jerry, continued to engage in mathematical play, manipulating the grid and the variable aspects of the GSP Alberti’s Window, intermittently, for nearly 30 minutes. After that time, Jerry tells Alejo that this projected grid idea is going to be his project. As a result of Carla’s indication that a checkerboard could constitute art, and through the mutual mathematical play by Jerry and Alejo, Jerry found inspiration for his artistic piece.

This mathematical play continued for Jerry, as he indicated, while he created his final design, which he used for his first artistic piece and then altered for his final piece. In the process, Jerry tried to make his design such that the original perspective grid and its projection wouldn't have overlapping tiles – in particular because this would create more pieces of the stencil, and hence become more difficult to airbrush. Jerry noted, "I just played with it until I got that. And I thought it was really cool that they overlapped each other, that they met on the baseline." (*Figure 76*)

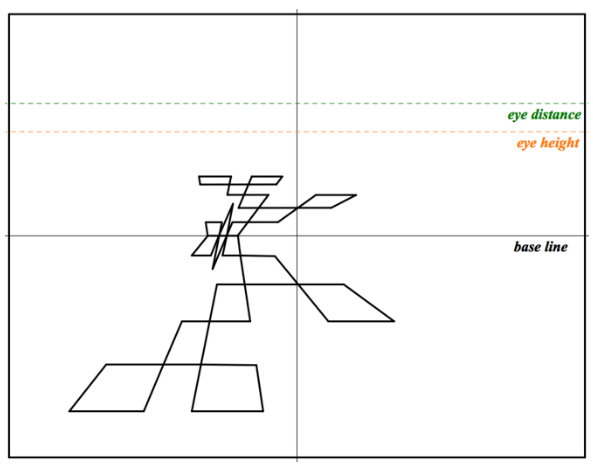
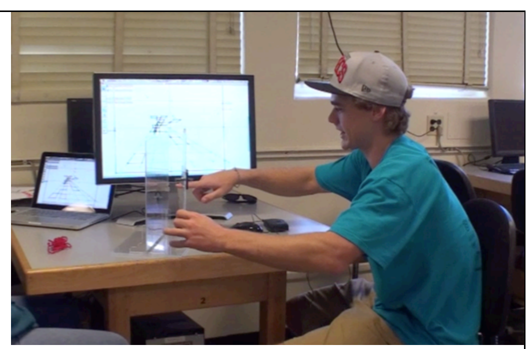


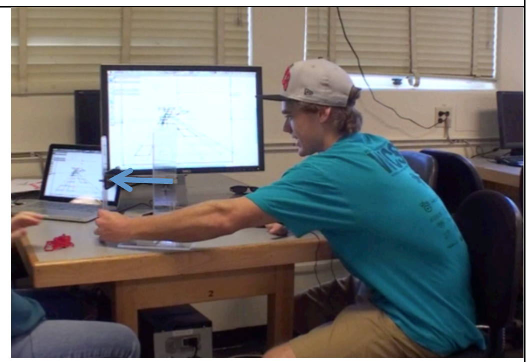
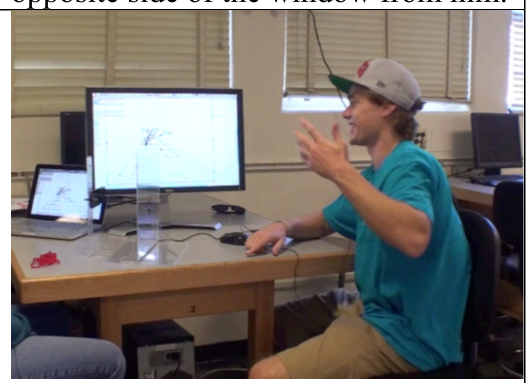
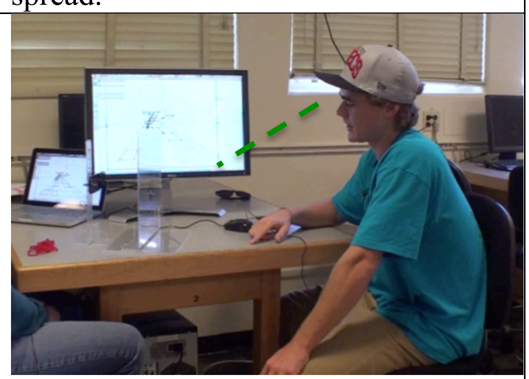
Figure 76. GSP sketch of Jerry's design. The green line is the eye distance line, the orange line is the eye height line, and the black horizontal line is the baseline.

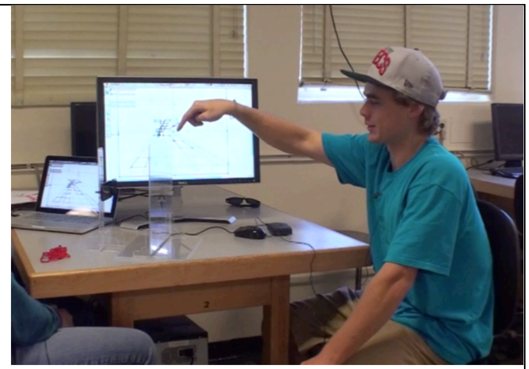
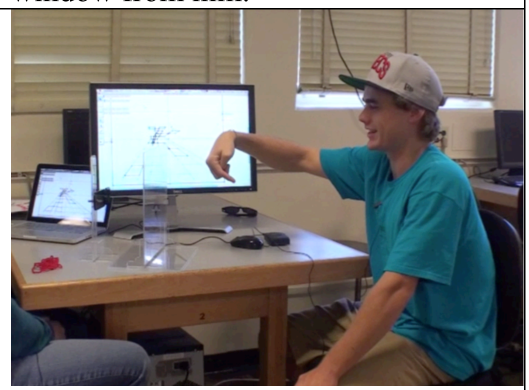
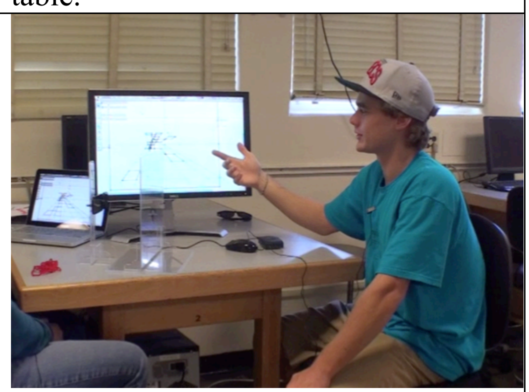
In the process of creating his design and trying to obtain the perspective grid and its projection such that the tiles would not overlap, Jerry created a mathematical situation that had not been discussed in class. Specifically, similar to Trisha, Jerry had moved the eye distance line above the baseline, corresponding to moving the eyepiece to the opposite side of the window from the usual classroom setup with the physical window. Jerry admitted that while he was creating his finalized design, through engaging in mathematical play, he did not consider the implications of moving the eye distance line above the baseline. Instead, he noted, “It was more aesthetic than, than actually thinking about it projective geometry-wise.” Jerry, like Trisha, in his effort to create a design he found aesthetically pleasing, created a situation that had not been considered in class. He said it was not until he wrote his reflection on his first artistic piece that he became aware of the mathematical situation he had created. Jerry explained,

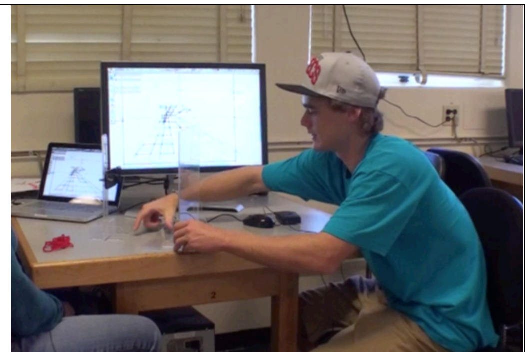
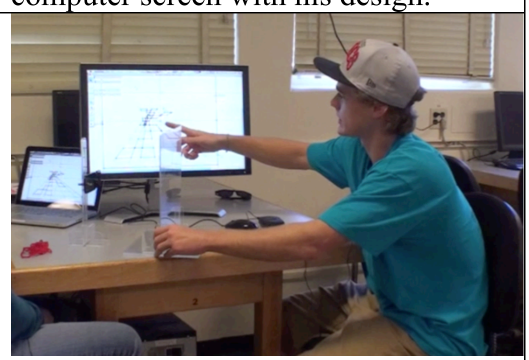
> And then so I was playing, I was just playing with it. I wasn't really thinking about the consequences of what I was doing and I realized that, I don't remember exactly what it was but I think I had like my eye distance above the horizon or some, like something really, weird. ‘Cause I remember I was playing with it, the other day. When I was writing the reflection, I was like playing around with it and I was like, “How did I make it look like this?”

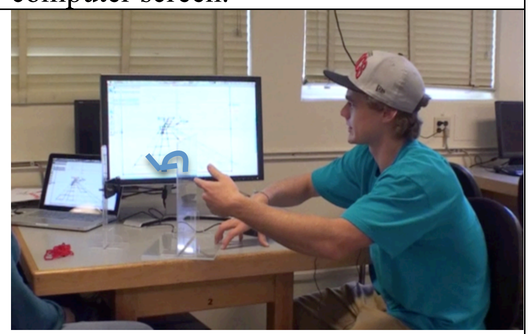
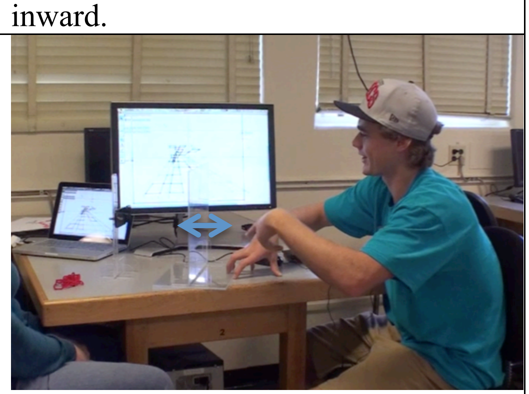
After realizing the arrangement of the variable aspects in the GSP Alberti’s Window he had created during his design construction, Jerry expressed confusion about the mathematical situation.

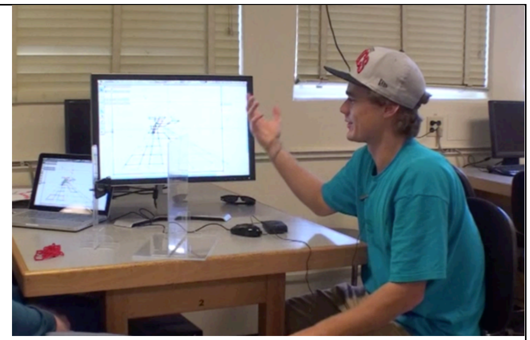
But then the eye distance is past the eye height	 Jerry points at the window with his

		right hand at the level of the height of the hole in the eyepiece. He holds the eyepiece next to the edge of the window.
489	so would it be, on the opposite side?	 Jerry moves the eyepiece to the opposite side of the window from him.
490	Like I (laughs), I was trying to think of it when I was writing the reflection and I was just like, this, I don't even know what I did. Like this doesn't make sense.	 Jerry lifts his left and in the air, fingers spread.
491	I don't understand how, that even projects.	 Jerry gazes toward the Alberti's Window.

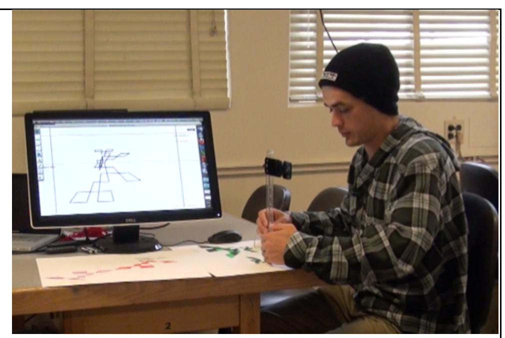
492	And then I, it got me confused like, oh then do you have to go to the other side [of the window]	 Jerry points with his right index finger toward the opposite side of the window from him.
493	and look at it from there?	 Jerry turns his pointing finger toward himself, slightly angled toward the table.
494	But then it's just reversed of, the original projection.	 Jerry indicates to his design on the computer screen.

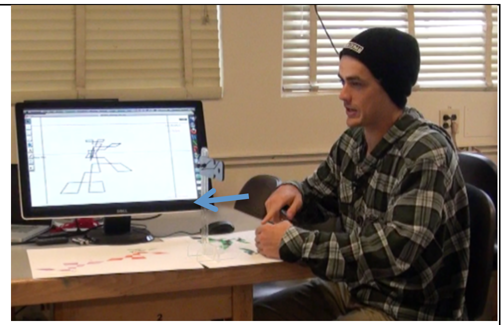
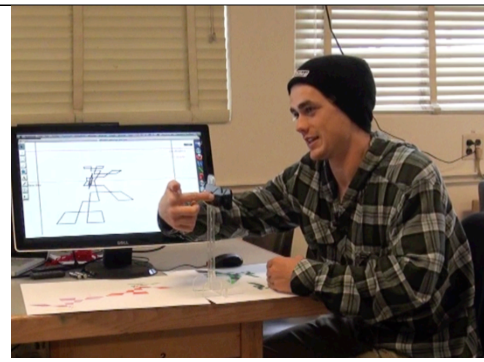
495	And then it's like, where is the, is the projection, is the image that I'm trying to, display right here?	  Jerry moves his right hand to the opposite side of the window from him, his fingers spread as if holding an object. He then gazes toward the computer screen with his design.
496	No ‘cause the baseline would be,	 Jerry points with his right index finger at the baseline on the computer screen.

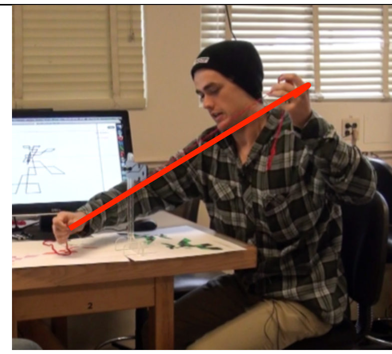
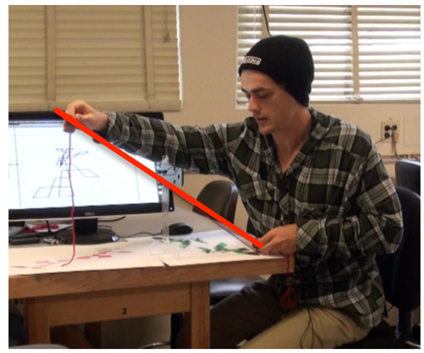
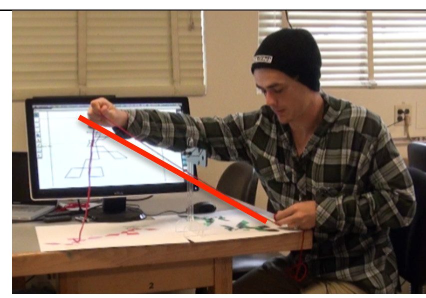
497	this is the original image so it'd be before the baseline.	 Jerry moves his right hand to the same side of the window as him, his fingers pointing toward the table and spread as if holding an object from the top. He continues to look toward the computer screen.
498	Yeah the eye distance is on the opposite side of the horizon.	 Jerry traces and upside down U shape with his left index finger, pointing inward.
499	So it's like how do you even,	 Jerry points back and forth two times between his right hand and the window.

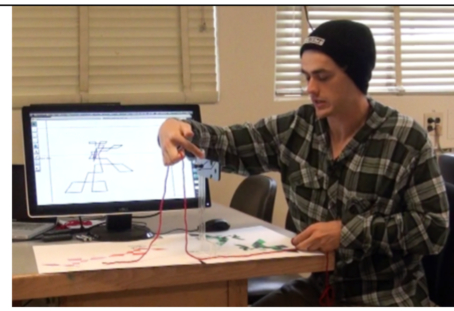
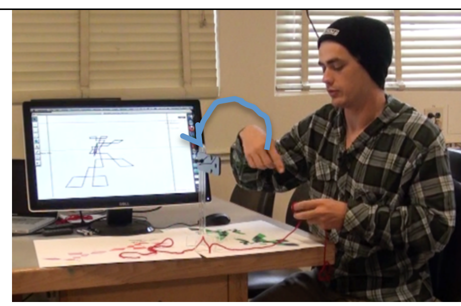
500	how did that even give me a shape? That's why it was really confusing.	 Jerry lifts his right hand into the air and drops it back to the table.

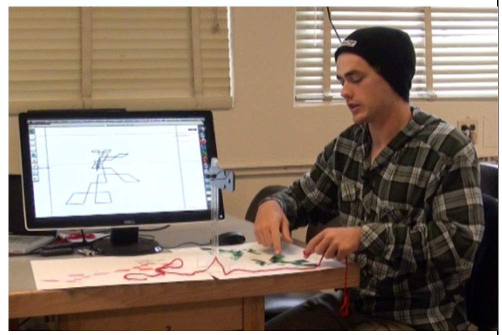
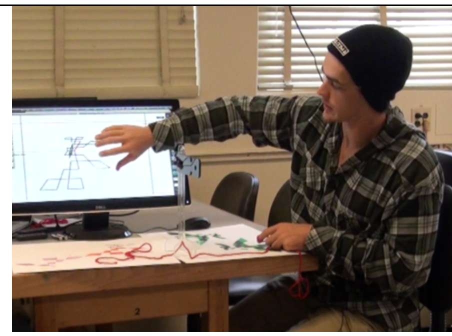
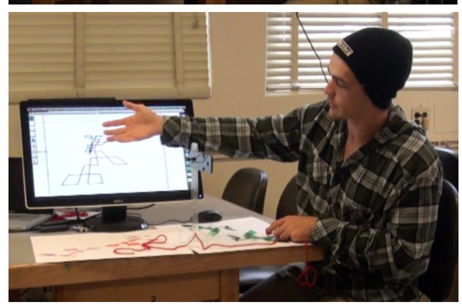
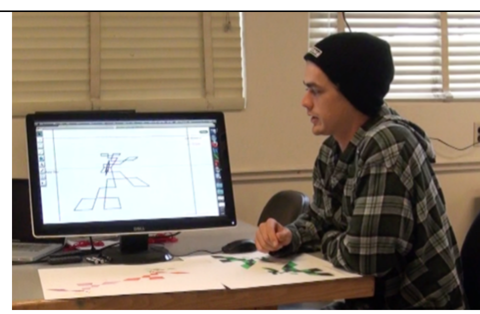
While Jerry did not make sense of this situation during the creation of or the reflection on his design, he did work through making sense of the situation during an interview. Subsequently, in the interview after creating his final artistic piece, Jerry expressed his understanding of the situation in which the eye distance line is placed above the baseline, in terms of the physical Alberti's Window.

501	So that would be the baseline.	 Jerry holds the eyepiece on the table with his right hand. Jerry places his left hand on the table in front of the eyepiece, fingers pointed inward.

502	So the eye distance was past the baseline.	 Jerry slides the eyepiece away from him with his right hand, and then points to his left hand with his right index finger.
503	So then I thought, if you were looking from the other side it would make sense to have a projection.	 Jerry points with his right index finger at the hold on the eyepiece, pointing in the direction he is sitting.
504	But if you were looking from this side everything would be behind you.	 Jerry leans forward and looks through the eyepiece.

505	And then using the string I realized that, the line of sight goes both ways.	  Jerry holds a string with both hands, right hand down toward the table further away, and right hand in the air closer to him. The string is passing close to the hole in the eyepiece. He then switches the vertical position of his hands, so that his left hand is close to the table and his right hand is higher in the air.
506	So that everything mapped on over here would end up, or wait, the projection, going back here	 Jerry taps the tabletop with the left hand holding the string.

507	would end up, or wait	 Jerry draws his right hand slightly down and toward himself, then returns his hand to where it started.
508	the projection going back here	 Jerry points at the hole in the eyepiece with his right index finger pointed toward himself.
509	would end up reversed for the projection	 Jerry draws his right index finger toward himself, then traces a counterclockwise arc away from himself.

510	of the original image.	 Jerry points to the table in front of him.
511	And so that's why everything gets like flipped, or inverted.	  Jerry places his hand over part of his design on the computer screen, then rotates his hand clockwise until the back of his hand faces the screen.
512	And so, that was really interesting and it actually like, it re-, it like, I guess it confirmed, what we had been learning.	 Jerry places the eyepiece and string behind the computer screen and continues to look at the screen.

Here, Jerry articulates how he has come to understand the way in which an image gets projected when the eyepiece is on the opposite side of the window from usual. He explains that he now understand that a string going through the eyepiece representing the line of sight will originate at the image, regardless of where the image is located on the tabletop, and extend until it intersects with the window (line 505). And, since Jerry's original image was on the opposite side of the window from the eyepiece, he explains how the projected image is similar to an inverted version of the original image.

Jerry utilized essentially the same design for both of his artistic projects. For his second project, Jerry included a second stencil that would allow him to fill out more of the grid. After the completion of his first artistic piece, Jerry expressed how certain aspects of his design – for example, where the horizon line is really located in his piece – could be considered rather confusing aspects, since the perspective grid has a horizon line and his final design has a different horizon line. He stated that he "realized how complex it can be," and intended to play with this confusion and complexity found in mathematics – in particular as it related to the horizon line of the projection of an already projected image – with the creation of his second artistic piece. For his second piece, Jerry explained that aside from filling out his grid more, he also experimented with the way in which his grids lined up.

513 I also put 'em on the same horizon line this time. Because I, also
514 had that realization that I thought that would, just be more,
515 confusing. And I know that sounds weird that I wanted it to be
516 confusing, but I wanted it to, bring about more thought. Despite it
517 being so simple I wanted it to like, I wanted people to see it and be

like, "Whoa, like what's going on there?" And what's, what's being projected, what's real, what's the original image.

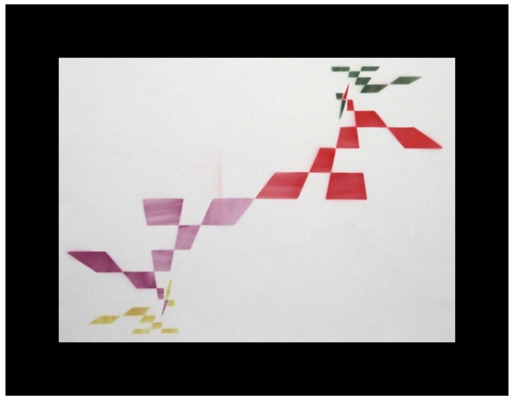


Figure 77. Jerry's midterm painting.

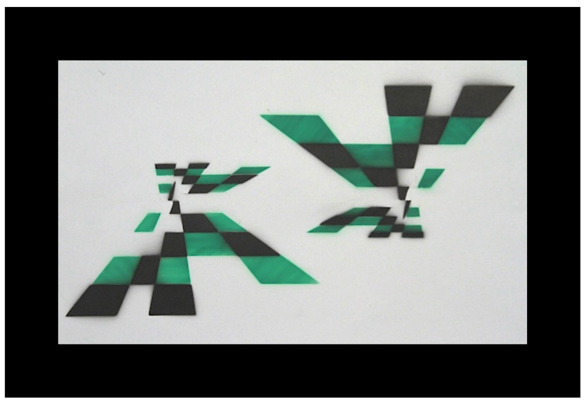


Figure 78. Jerry's final painting.

In general, expressing the confusion and complexity found in mathematics is not an opportunity students in the traditional mathematics course are confronted with. Jerry, however, was interested in, and succeeded with, highlighting these complexities through the playfulness with horizon lines in his design. Specifically, Jerry explained he lined up his two sets of projected grids on the line where it appears the projected grids would converge. He noted this line was not truly a horizon line, but rather the baseline in his GSP sketch, as the horizon line was higher in his sketch. Additionally, he explained, the way in which he had constructed his perspective grid resulted in a separate horizon line, where the original perspective grid converged. When asked whether he felt he was able to represent the confusion and complexities with his artistic piece, Jerry replied,

> I definitely do. Because, not, I don't feel like anyone would, would pick up on it. And I feel like that's another thing that carries over from when we visited the, the Museum of Contemporary Art is that to just anyone, if they looked at this they just be like, "Oh that's cool. It's like, it's like shapes like going together." But someone in this class who like understands projective geometry would, it might raise those questions of like what's converging where, what's the original image, what's projected, like how did it project that way.

Here, Jerry expresses he felt he was successful in creating an artistic piece that might bring about a certain kind of confusion or curiosity for students who had taken the projective geometry course. He notes the experience for someone who has not taken the course might be similar to the experience many students had on the visit to the Museum of Contemporary Art – specifically the acknowledgement of artistic pieces, but not necessarily understanding them. He goes on to draw similarities between his artistic piece and the experience he had while analyzing William Hogarth *Satire on False Perspective* (*Figure 79*).

Figure 79. William Hogarth's *Satire on False Perspective*. Retrieved from http://www.wikiart.org/en/william-hogarth/the-importance-of-knowing-perspective-absurd-perspectives

> …[W]ith uh, Hogarths's satire, how we, we noted all those things. Like I really like that, that piece of art because it, it was like this (indicating to his artistic piece). It was like… it brought about those questions. Like, "Wait how is the fishing pole in front of this, yet it's, like the lines don't match up?" And, so you just wonder, "Is this in front of this?" or is like - When really it's just two D, like none of it is behind or, in front of each other. So it's just like that, just get like perplexed, just lost in just, this simple image. That, that was a lot more complex, but.

The nature of the artistic project, in the sense that students had the freedom to explore inspiration from any source, provided Jerry an opportunity to explore visual representations of the complexity and confusion he observes in the field of mathematics, and to connect his mathematical and artistic activity with other artistic engagement in the course. In particular, Jerry connected his creation of artistic pieces with both the visit to the Museum of Contemporary art and the analysis of *Satire on a*

False Perspective.

Unlike most students in the course, Jerry identified as having some artistic aptitude prior to beginning the course. In an interview, he articulated how his typical artistic pieces differed from the pieces he created in this course. He explained that he felt his typical artistic works lacked direction from anything in particular, referring to it as “weird” and “random.” Comparing his previous artistic work to his artistic work in the projective geometry course, Jerry highlighted the aspect of mathematical play that arose for him in the creative process, stating “Whereas this was, really brought about by that mathematical thinking of like, ‘Oh, I wonder what happens if this happens.’" One aspect Jerry saw as similar between his typical artistic work and his artistic work in the course was that, despite having a mathematical influence, his artistic pieces were still open to interpretation, similar to his typical artwork.

Jerry expressed that his work creating mathematically inspired artistic pieces shifted his ideas about his relationship to art and influenced his ideas about what kinds of art he might create in the future. When asked about how he saw himself with respect to art prior to creating his artist pieces, Jerry replied,

> … I definitely didn't see that math, the mathematical like aspect of it. Because I just saw it more as, the expression. Like I was saying last time about how I would just create whatever I wanted. Like, the only guidelines would be from the art class, just like, oh there has to be this many shapes, you have to use this, this medium. But other than that it would be like, I'm just gonna throw random animals on there, whatever I want to do. Like it doesn't have to have like a structure or a, a plan even really. Whereas this has more, I had to like, solve the problem of, first how to create the image and then what I was gonna do with the image, how to fit the image on there, and then what I wanted it to appear as. I had to, I to more like play with it and plan it out. So,

> so it changed my, my sense of just being, of art just being, so now I almost see that there's two forms. Where there's this mathematical art that has like a real, a real point to it. Not to be punny, but. Like, like an actual point, like. And I'm not saying that art is pointless, but this one I, I feel like there are, more of the guidelines, like more of, you're trying to get, more of an actual goal, instead of just the pure determinate. And so just the pure um, expression, like creativity. There's also something like structuring it and, guiding what you do. So that's kind of my difference, is that now I feel like my artwork will include more, more aspects of like, one point perspective, things that carry throughout the piece instead of just being like… weird like dimensions, all different, like coming out of the walls and like stairs in places that don't make sense. Whereas now I feel like I'll try and actually get the realistic view that you can get with projective geometry.

Here, Jerry expressed how creating his mathematically inspired artistic pieces has changed his sense of what it and how he might create artistic pieces in the future. He notes he now recognizes the mathematical aspect of art and plans to use mathematical ideas to in his future artwork. Furthermore, in his explanation, Jerry touches on his view of the art projects as a type of problem solving activity, which he mentioned at a separate point in an interview he particularly enjoyed and gave him the sense of really doing mathematics.

In addition to the creation of his artistic pieces giving him the opportunity to express the confusion in and complexity of mathematics, and expanding his sense of art, Jerry expressed the project provided him an avenue to share mathematics with others. He explained that normally if you were to try to share mathematics, such as a proof, with someone who does not study mathematics, he expects the response would be one of disinterest or indifference. The artistic project, on the other hand, someone outside of mathematics may find more easily relatable.

> Jerry: I remember when I was making it, and… I showed my mom like, GSP and stuff. And she was just like, she thought it was the coolest thing. 'Cause it, it's, that's another thing that I really like about this is like. You can never share like, "Oh look at this cool proof I did in, in math" It's like, "S-sweet, I, I don't even know what those letters are. Like, what, alright thanks for showing me." But like, this program and like this project I was actually able to like explain what I was trying to do. And I was super excited about it…. And normally like when you explain like math to someone they're just like, "Whatever, just, just stop." But, but this is actually like, really interesting. And, you could actually like show people…
>
> Int: So this gave the opportunity to tell other people about, math.
>
> Jerry: Yeah. And show that math isn't just, like, proofs and numbers.
>
> Int: Is math, so some people do think that math is just numbers and symbols?
>
> Jerry: Mm hm
>
> Int: What is it to you?
>
> Jerry: I think it's that, that expression that, that just a new way of thinking like, like a problem solving mentality almost.

The creation of his artistic pieces, and the use of Geometer's Sketchpad to create his design, gave Jerry and avenue to share his experiences in mathematics with his mother. In addition, he noted he had the opportunity to share the experiences with students he was tutoring. He expresses, specifically, the artistic engagement provided him the opportunity to share the notion that mathematics is not merely a subject constituted by numbers, symbols, and proofs, but rather one that includes expression and a problem solving mentality.

On multiple occasions, Jerry explained he had a new way of thinking about mathematics in a way that correlated with his notion of artistic expression.

> Where I thought I was able to bring together the ideas of math and art was through self-expression. Because I feel like math involves a lot of, well math in the way I define it now, since reading like the Lockhart, the Lockhart article, because I feel like math isn't this procedural thing that we should be taught. I think it's, I think it's more, it should be more unique and like, personal, and you should actually figure things out for yourself, and like, not just take what people tell you, but actually try and play with the ideas, and work out your own reasoning for stuff. Because then it actually means more and it lasts for longer. And so, the same idea with this (indicating to his artistic piece), if you can self-express like, like you can write a proof your own way. Whereas, in art you can express, you can consider, whatever you want is considered art. It's your, it's up to you, it's your self-expression.

In general, in my experience, when students are asked to make connections between mathematics and art, they often provide vacuous statements such as, "I see now that math and art are one and the same," and fail to find real contact points between the subjects. Jerry, on the other hand, by suggesting one connection between mathematics and art is the way in which one can express oneself, described a non-trivial connection between mathematics and art.

For Jerry, the artistic engagement in the course provided him an opportunity to engage in mathematics play, and in doing so, consider situations that had not arisen in the classroom. In the creation of his artistic pieces, Jerry was able to experiment with visual representations of the complexity of and confusion in mathematics that he notices. Furthermore, his artistic pieces provided him an avenue to discuss mathematics with someone outside the field of mathematics, and to show how mathematics is a subject composed of more than just number, letters, and equations. Jerry's artistic engagement, and in particular his reading of and reflection on A

Mathematician's Lament, shaped his notions of what constitutes mathematical activity.

6.3 Willow

Similar to other students in the course, Willow did not consider herself an artist at the start of the course. Through her artistic engagement in the course, she came to think of herself as having more artistic ability than she previously thought. For one of her artistic pieces, Willow found inspiration in her mathematical activity, and engaged in mathematical play during the creation of her pieces.

For Willow's second artistic piece, she found inspiration in a homework assignment in which the class was asked to prove the duals of the axioms of projective geometry. The result of her effort to clarify the steps she took in one of her proofs was a series of lines of different colors (*Figure 80*). She explained her inspiration in the following way,

> I did that homework and… you know like you're doing a proof basically by drawing. And it's a step-by-step thing. But then I know when I turn it in …you're just going to see the final thing and… it's kind of h-hard to figure out what I started with and what each step was. So to try to, you know, tell you the difference, like this is step one, this is two, I changed the colors. So I'm like, okay all these lines are the first alignment, they are like blue. And then, okay here's the yellow point that I added, and then the lines coming from that are orange. And then, um some of the axioms took a lot of steps and I put a lot of connections and lines, so I ended up with like a whole rainbow, of lines and colors, and patterns that looked really cool. Um so I said, you know what, I can just take this. And I was almost going to just like take my exact GSP sketch from one of those axioms but there's so many lines, going all over the place that um, it would have been a mess, to stencil. Um so then I kind of simplified it and said, okay here, we'll just use this theorem. I'll sketch it out,

and um, and then use that.

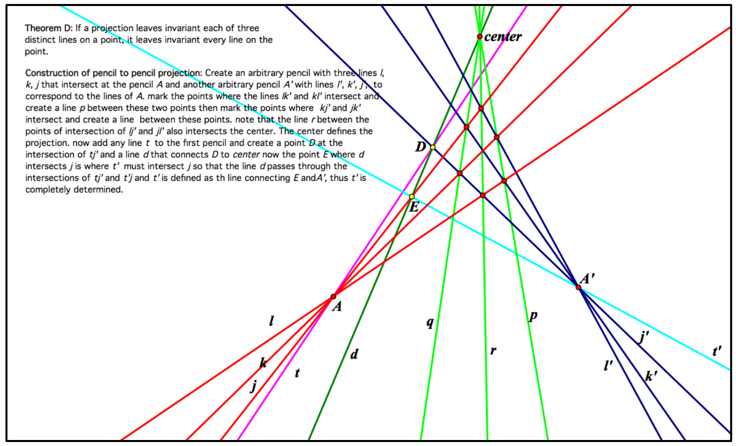


Figure 80. Willow's proof on a homework assignment in which she utilized colored lines to clarify the steps of the proof.

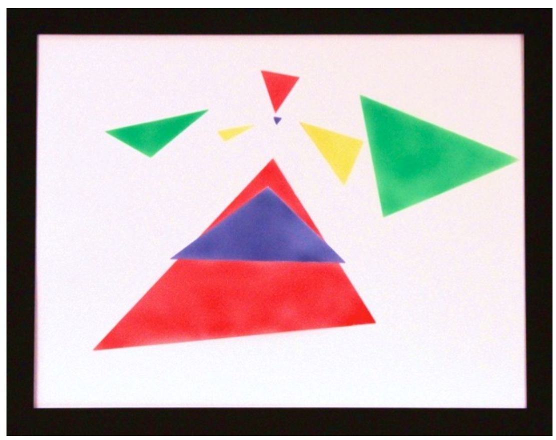


Figure 81. Willow's final artistic piece, *Perspective From A Point.*

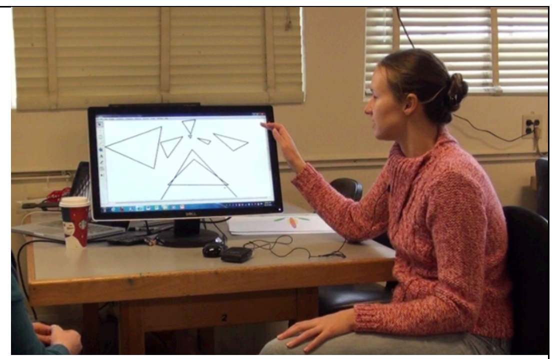
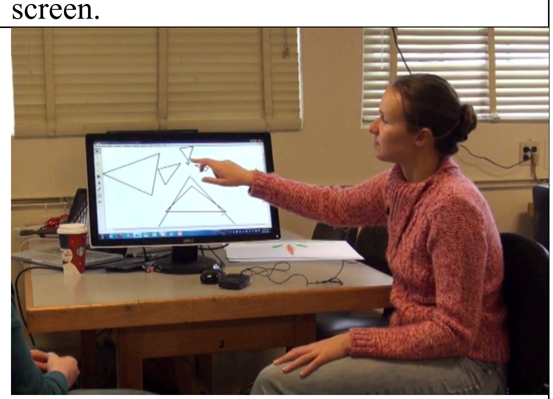
As Willow noted, she utilized many different colors of lines and points to create her proofs such that the reader could follow along with the many steps in the proof. Something about these colorful, visual patterns intrigued Willow, such that she chose to create an artistic piece related to them. She chose to create her artistic design based on Desargues' Theorem, which states that if two triangles are perspective from a point, then they are perspective from a line. Her piece consists of two sets of two triangles that are perspective from points, and the projections of those perspective triangles (*Figure 81*).

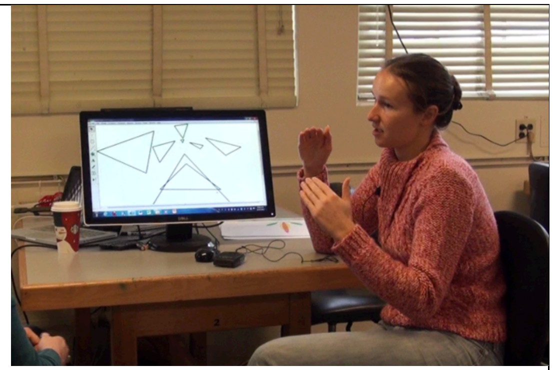
After creating her design, Willow became curious to know whether her projected sets of triangles would also be perspective from points. During the course, several days were spent discussing Desargues' Theorem, however, the class never had a discussion about what properties of perspective hold through projection. This means Willow became curious about a situation that was not discussed, nor was intended to be discussed, in the course. Since Willow chose to explore this idea by her own accord, and since the idea of considering which properties hold before and after projection had not been a topic of discussion in the course, her activity constitutes mathematical play. The way in which Willow chose to explore this idea was to connect the vertices of the projected triangles to see if each of the two projected sets were perspective from a point. She explained,

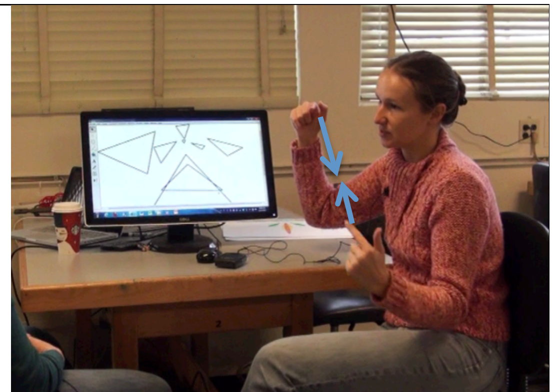
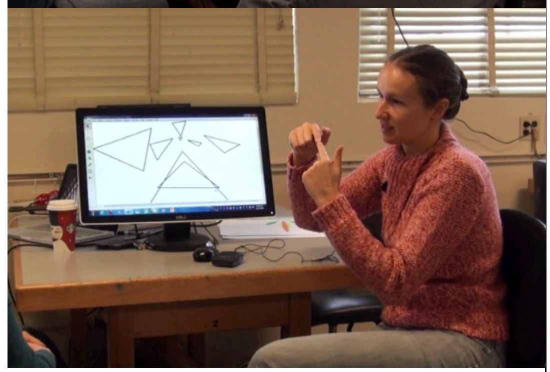
> And so I, after I did the projections of these triangles, I checked to see if they were perspective from a point by putting, um lines through each of their vertices. And indeed they did um all meet at one point. Um, I didn't check to see if that point, would have been a projection of the other point, but I assume it is. (laughs) It has to be, yeah.

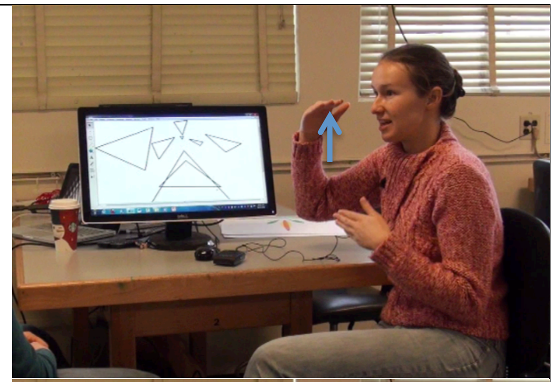
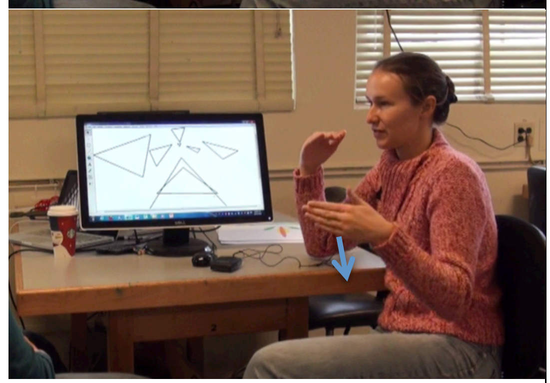
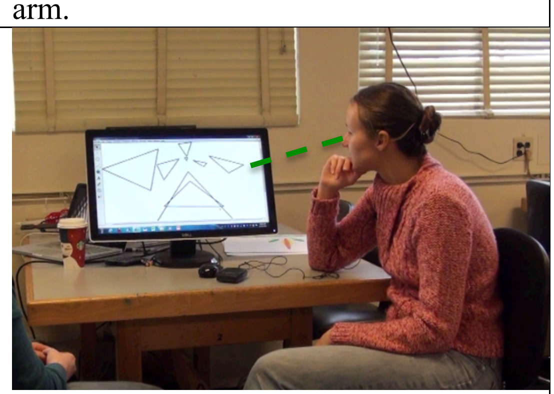
Willow's curiosity about her projected images and the properties of perspective though projection led her to engage in mathematical play by connecting the vertices of the projected triangles to confirm the triangles were in fact perspective from a point. This mathematical play corresponds to Willow exploring a mathematical situation not discussed in class. The result was that Willow confirmed for herself that Desargues' Theorem holds through projection.

Willow had a particularly developed sense of the relationship between the physical and GSP versions of Alberti's Window. Engaging in constructing her designs in GSP strengthened her sense of how the two mathematical tools relate. She explained after creating her first piece that keeping images and their projections from overlapping was a challenge that she faced. After creating her second piece, when asked what kinds of discoveries or realizations she made while creating her design, Willow responded,

626	[H]ow the eye height,	 Willow places her right hand near the top of her design on the computer screen.
627	you know to keep these things (an image and its projection) from overlapping	 Willow points back and forth between the two triangles (the original and the projection) in the top middle of her design.

628	I had to make the eye height not	 Willow draws her right hand in front of herself, fingers pointed in and palm facing down. She starts to raise her left hand, palm flat.
629	right on that piece.	 Willow places her outstretched pinky finger on her right hand over the original triangle image in the top middle of her design.

630	‘Cause that means it's going to project to exactly the same	  Willow raises her right hand to above shoulder height, while keeping her left hand closer to table height. She points her two index fingers at each other and draws her hands together until her fingertips meet.

631	height as it is far away.	  Willow draws her right hand, palm facing down up to her eye height. She takes her left hand, palm facing her right arm, and draws it away from her right arm.
632	Um, so I realized that.	 Willow looks back toward her design on the computer screen and nods her head.

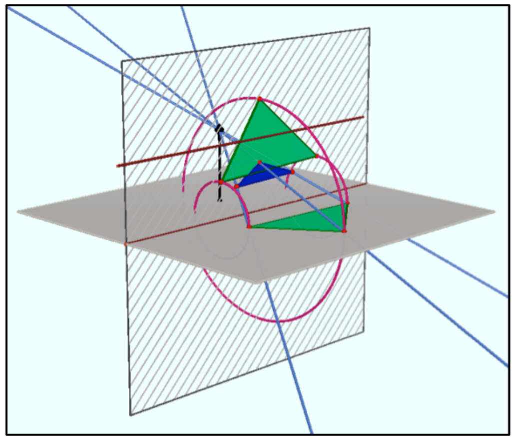


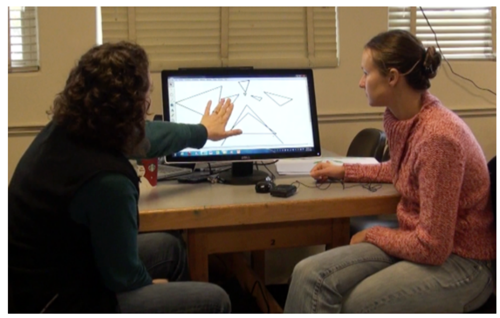
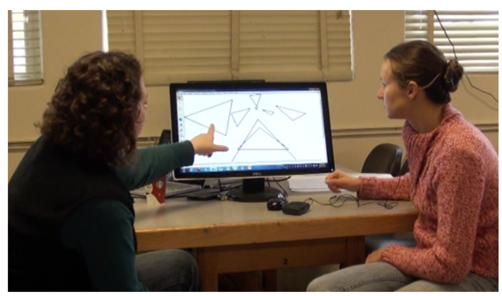
Figure 82. A depiction of a green triangle and its blue projection, in which the height of the horizon line and the distance of the green triangle from the window is approximately the same. The original green triangle is then rotated up onto the vertical plane. The green triangle and its blue projection overlap on the vertical plane

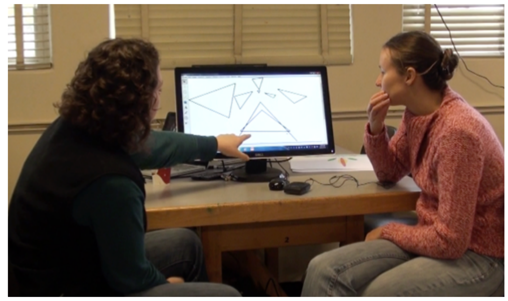
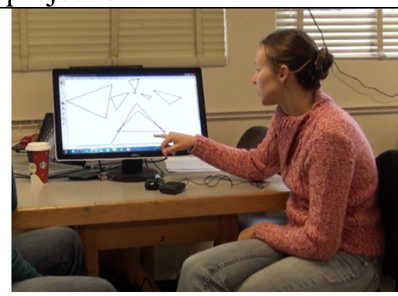
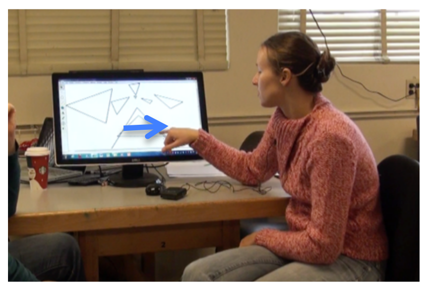
Here, Willow explains how if an original image and its projection in the GSP Alberti's window overlap, then it corresponds to the image being the same distance away from the window as the projection is high above the baseline (*Figure 82*). Willow determined that in terms of her design, this meant she had to raise the eye height line above or below the original image to a point at which the projection no longer overlapped. The understanding of this overlapping quality, and the ability to explain why the two images would overlap, is a rarity for students in the course, in my experience. Willow went on to explain a second aspect of the coordination of the physical and GSP versions of the window that she realized while creating her pieces.

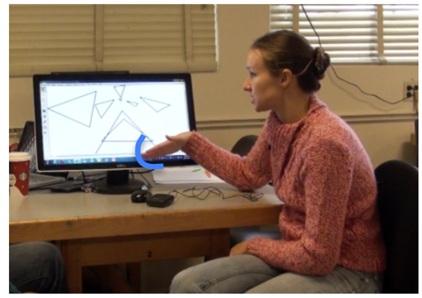
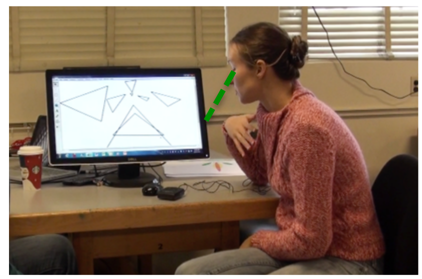
> [O]ne thing I didn't realize on my midterm was, I wasn't really thinking about how far I was away from the window, until afterwards. Um, but when I did this and I saw that this was elongating (icating to the projection of the largest triangle in her design) then I knew that I had to be, closer to the window. Um, and high up. 'Cause if I was, you know if I was like a foot away from the window but only six inches high, that still wouldn't

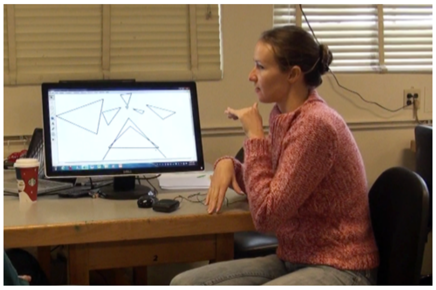
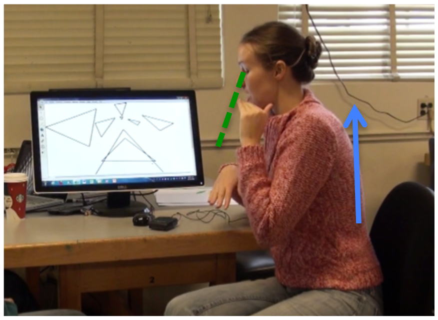
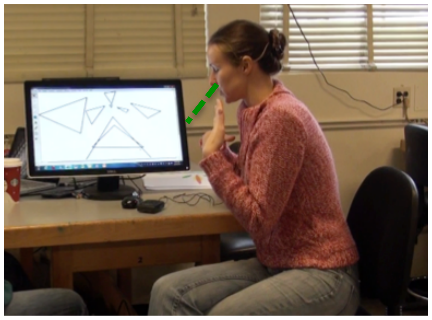
640 elongate even though it's coming in from the window (indicating
641 between the window and the eyepiece).

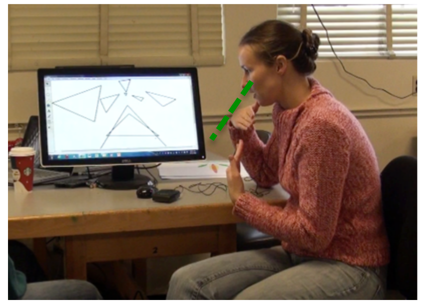
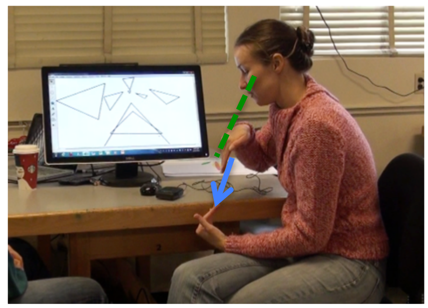
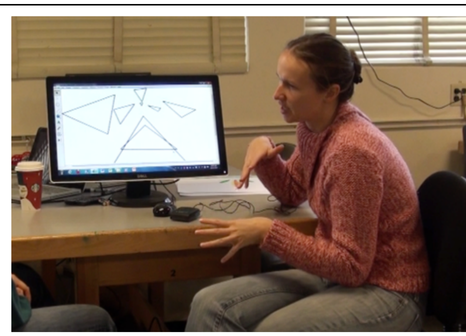
Willow further demonstrated her command of the coordination between the physical and GSP versions of the Alberti's Window when asked about a particular aspect of her second artistic piece. In particular, three of the four projected triangles in her piece were smaller than their original images, while the fourth projected triangle was much larger than the original triangle.

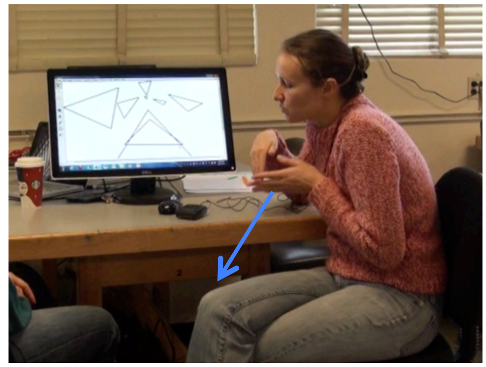
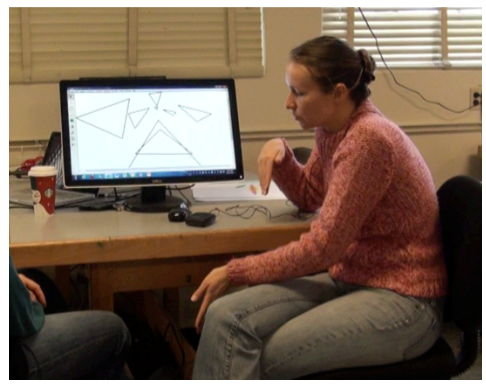
642	Int: So then, like these projections are all, smaller	 Interviewer indicates to the three smallest projections of the triangles.
643 644	Int: than their originals, Willow: Mm hm	 Interviewer indicates to the original images of the three smaller projections.

645	Int: but this one's so much bigger.	 Interviewer indicates to the largest of the projections.
646	Willow: Mm hm. This one's bigger because the,	 Willow points to the inside of the largest projection with her right pinky finger.
647	Willow: the bottom of it is	 Willow draws her right pinky finger across the screen at the location where the original triangle and its projection intersect.

648	Willow: below the baseline.	 Willow sweeps her right hand in an arc from the location at which the original triangle and its projection intersect to a lower place on the screen.
649	Willow: And because, I'm so much closer, the, the um,	 Willow draws her right hand toward her chest and leans slightly toward forward, and shifts her gaze down toward the table.
650	Willow: I'm so much closer to the window um,	 Willow moves her right hand closer to the computer screen, and leans her body slightly closer to the screen.

651	Willow: than I am high.	 Willow turns back to the interviewer, places one hand on top of the other, then separates her hands by lifting her left hand closer to her chin, while moving the right hand down slightly toward the table.
652	Willow: Like I'm, I'm higher up	 Willow sits up straighter in her chair, raising her right hand with her chin as she sits up. Willow shifts her gaze sharply downward toward the table.
653	Willow: and closer,	 Willow brings both her hands closer to her face, palms facing her chest, and leans forward.

654	Willow: so that the angle that I'm,	 Willow holds her hands with her fingertips lining up just below her gaze.
655	Willow: so I'm looking down, at it at like,	 Willow draws her left hand down toward her leg, orients her fingers up toward her eyes. She turns her right hand so her fingers face the fingers on her left hand. She moves her right hand down in a straight line to meet her left hand.
656	Willow: big angle.	 Willow keeps her hands along the same line she made with her fingers previously. She does two short pulses with her hands.

657	Willow: Um, so it's really projecting it	 Willow draws her left hand close to her body, fingers pointing away from her body. She quickly moves her left hand away from her body in a straight line.
658	Willow: far down, down the window.	

Here, Willow explains the coordination of the several variable aspects of the GSP Alberti's Window sketch that resulted in her projected image becoming larger than the original image. First, the coordination involved the location of part of the original image being below the baseline in the GSP window, which corresponds to the original triangle being partly in front of the physical window and partly between the window and the viewer (lines 647 & 648). Second, the coordination involved the relationship between the eye height and eye distance lines being such that the height of the eyepiece had a greater distance from the baseline, or horizontal plane, than the distance of the eyepiece from the window (lines 652 & 653). In her explanation, Willow supported her argument for why this situation with the GSP Alberti's Window results in a larger projection with an act of imagination (which was illustrated in Chapter 5). Her explanation of why an image and its projection might overlap, as well as her explanation for why and when the projection of an image placed in between the window and the viewer will elongate below the baseline, demonstrates her sophisticated understanding of how the mathematical tools coordinate.

Willow's view of herself as an artist shifted as result of her experiences creating her artistic piece. At the beginning of the course, Willow did not identify as an artist in any way. However by the end of the course, Willow expressed her views of herself in relation to art, and her views about art in general, had changed. When asked if the projects had changed her views about art, Willow responded,

> Willow: Yeah I think so. Um, I mean just the whole class has. Because art's not so much about having technical little detail skills. You know the class kind of showed like art is more how you think about things and what you see when you look at, at things. Um, I hung out with a lot of people that called themselves artists in college, and I was like, "Oh they're artists, but I'm not an artist" and now I'm like, "You know what, maybe I fit in more than I thought I did." Just ‘cause I liked the whole concept of, the art and math and, and thinking about things in that way.
>
> Int: So doing these types of things changed how you felt about,
>
> Willow: Yeah, about art. Yeah, in a good way. ‘Cause now I'm like, "Yes, I'm an artist."

Willow described how her notions of what art is about had changed, saying that art is more about your interpretations, rather than being about a very particular set of skills for art creation. Her response suggests that prior to the course, Willow considered art as a subject focused on technical skills, and a subject only for those who possess those skills. However, at the end of the course, Willow, expressed her perception of herself as an artist shifted, such that she came to identify herself as an artist. Willow continued,

> Willow: Um, I just like art more. Maybe in the same way that people are intimidated by math, I was intimidated by art. I was like, "Oh it's too hard. I just can't do it." Like "I don't get it, I'm never gonna get it." Like that exact same way I, that's how I felt about math. I mean that's how I felt about art!
> Int: Right
> Willow: And other people felt that way about math. And I'm like, "No", you know, "If you just look at it this way." And I felt like I, I don't know. I felt like everybody can do math. And I'm sure artists are like, "Everybody can do art." (laughs)
> Int: And what do you think now?
> Willow: Um, I guess everybody can do both. (laughs)

Here, Willow makes a comparison between those individuals who express a lack of aptitude for mathematics with her previous notion of herself as an artist. Specifically, prior to the course, she felt she lacked an aptitude for art, in the same way many feel they lack aptitude for mathematics. When asked about what she thinks the value is of creating art projects in a mathematics course, Willow responded,

> Um, I think a lot of, I guess this is still abstract art, but a lot of times we see math as something so abstract and not connected to the world. It's just something like we arbitrary made up these rules, and in this land of these rules this is, let's figure out what happens there, even though this land, doesn't really exist. But then when you do art, like you put it on paper, you can kind of see that, the land of those projective geometry axioms, can exist, like here in the real world. Um, so that's helpful. It makes it tangible. You know, ‘cause art's really tangible.

Willow was able to articulate one way in which she felt participating in creating art projects in a mathematics course might be useful for students. In particular, she suggested engaging in creating artistic pieces in a mathematics course may make mathematics more tangible for students and give them a way to connect mathematics to the real world.

Through artistic engagement in the course, Willow was able to find inspiration for the creation of her artistic pieces in her mathematical activity. During the creation of her second piece, Willow engaged in mathematical play, which resulted in her considering mathematical situations that had not been discussed in the classroom. Through artistic engagement, Willow shifted her thinking about art to a subject of perception, rather than technical skill, and came to identify herself as an artist. Finally, Willow was able to articulate a way in which she felt creating artistic projects in a mathematics course could be beneficial to students.

6.4 Fiona

Fiona identified as being mathematically inclined, but not artistically inclined – however, she noted she had a knack for writing poetry. By the end of the course, Fiona came to realize she had more artistic ability than she had previously recognized. For each of her artistic pieces, Fiona found inspiration in prior mathematical activity. During the creation of her pieces, Fiona engaged in mathematical play and came to a realization about a particular configuration of an image and its projection.

For her first artistic piece (*Figure 83*), Fiona found inspiration in observing a classroom of high school students creating constructions using only a compass to make circles. She explained,

692 While observing a classroom for a different course that I am taking,

I watched some freshman students in high school construct what they called “daisy wheels”. They were just playing with the construction of circles using compasses and using the same radius and having other circles have a point that is coincident with the radius. I thought, “that would be neat to see projected.” So I went home and constructed such a wheel and played with how the projection falls.

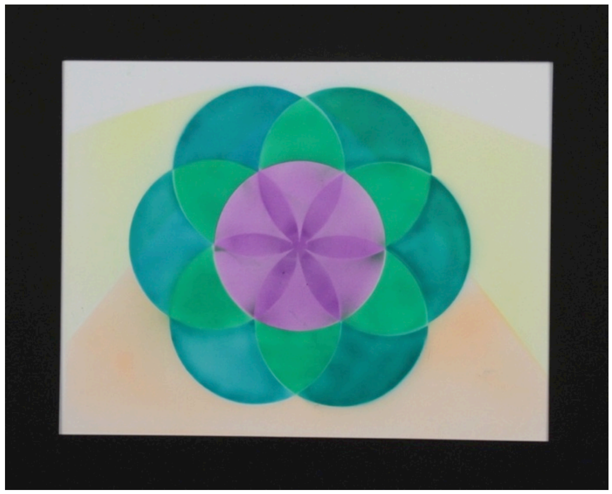


Figure 83. Fiona’s first artistic piece.

Fiona described her motivation coming from other students playing with mathematical ideas by using only a compass to create designs. Having projected circles in the projective geometry course, Fiona may have had a sense of how the “daisy wheel” might project and thought it would be interesting to see projected.

In describing the creation of her first piece, Fiona explained that she engaged in a great amount of play to obtain a design she found aesthetically pleasing. Fiona stated that while “playing” with the GSP Alberti’s Window before creating her design, she explored the way in which an image and its projection worked together, how small and how large she could make the projection, and how well she could match up the image and its projection. She stated, “I was just playing and watching the two work together.” She further explained,

> I wasn’t really thinking about the math. I was just playing. And like, I had, even before I, ‘cause I had the idea in my head that this [design] is what I wanted, but before I started doing it I was doing like, triangles, squares, and like, concave structures, and things like that. I was just playing with it. And then it’s like, “Oh, I gotta turn in my design.”

Fiona did not give much consideration to where aspects such as the baseline, horizon line, and eye distance line were located; rather, she was simply “trying to make it look fun” and make it look like the projections were “falling off the page.” In her final design, the way she was able to make the projection look like they were falling off the page was by lining up the top circles with the baseline. When an image touches the baseline, at the points at which it touches the baseline, the projection of those points will be identical. Fiona stated she engaged in “a lot of play” to get the projections to line up with the original images in the way she wanted, and she had not considered the property of the image and its projection being identical at the baseline until she began to explain it. This means that it was upon reflection that Fiona realized the mathematical situation she had created as a result of her play.

Fiona engaged in play again for her second artistic piece, in which she wanted to tie together a square with a circle. Fiona explained since she had used only circles and the projections of circles in her first piece, she wanted to use a square in her second piece, and somehow tie the two together. She decided to use the circle to construct the square, by creating two perpendicular radii of the circle and inserting the necessary lines to complete the square (*Figure 84*). Once she had those two elements, she began to play with the positioning of the square. She wrote in her reflection on her final piece,

> I tried several positions: my square above the base line, my square behind the base line, my square to left of the center, to the right of the center, my square horizontally angled, and so on and so forth. But I decided I wanted to have a subtle projection in my piece. Thus came about the pentagon shaped projection since the entire projection could not be included on the page. I lined the square even over the base line (the line created when the window meets the standing plane) so that the projection would be both above and below that line but the vertices would coincide. Once I had the square and its projection aligned, I revealed the construction of the square to add more elements to the painting.

Figure 84. Fiona’s second artistic piece, *Beyond the Square*.

In her piece, Fiona used the property that an image and its projection will meet at the baseline. She used this property to bring about a connection she was trying to make between the circle, the square, and its projection.

> [W]here the window meets the table, whenever you have your original piece there, the projection of that original piece will meet at the same points. So the effect of this (indicating to her GSP design), allowed them like, made the connection that they are the same. Or at least to me it made the connection that they’re the same. Because using the circle, you can see that the square is a part of the circle. It’s coming out of that. And then the projection

> itself is coming out of the square. So like, the connection has been made. Because if I had it anywhere else and it wasn't, conjoined at those points, it'd be like well, that's the projection. It doesn't necessarily mean anything. So I wanted them to connect, in a certain way.

Fiona explained that she had played with the configuration of the elements in her design, but that "playing with it and rotating [the square] around, this [particular design] made more sense." Fiona used the property of an image and its projection being identical at the baseline to "connect" the elements of her design. In her explanation of this aspect of her design, Fiona utilized an act of imagination to demonstrate her understanding of the property and the coordination of the physical and GSP versions of Alberti's Window.

Fiona did not feel she learned much mathematically from the creation of her second design – although she mentioned with her first design that she learned more about how images and their projections line up on the baseline. She did however describe how her perception of herself changed from her experiences creating the mathematically inspired artistic pieces. She explained,

> I learned more about myself, that I can be more creative than I give myself credit… This made me realize that I can be creative and I can do stuff, that's not just what I thought. Like I can be artsy. Like my brother was always artsy and he could draw, and he could create paintings, and do stuff like that. And we're both mathematically inclined. I was like, but I can't do the artsy stuff. I can't, like I couldn't draw. I couldn't be satisfied with anything that I drew. And so I guess, the fact that I'm satisfied with this art project that I had to do in a math class, I think, I can, I can be more artsy than I give myself credit. And I think that's something that I learned, and I'll use. … [T]his project and this process showed me that it, it's possible. I can do stuff like this. So. And I will use it, 'cause, now like I have my doodle book in my iPad. And I doodle a lot more.

Despite stating she now felt like she could do more creative things, Fiona said she still did not identify as an artist, stating, “I mean, I can do the artsy stuff, but I wouldn’t call myself an artist. Not even remotely close. Yeah, no.” She follows up with noting that she feels more capable of doing artsy things after having participated in these projects.

Artistic engagement provided an opportunity to engage in mathematical play and make the discovery that an image and its projection will meet on the baseline at same points. She utilized this property in her second artistic piece to connect the elements in her design in a way that made sense to her. Finally, as a result of creating the artistic pieces, Fiona felt as though she was more capable of doing artistic things than she one had believed.

6.5 Alejo

Alejo, similar to most of the students in the course, did not see himself as being artistic. He noted that, painting and drawing were “not my thing.” Yet, he was able to combine his art creation with his personal experiences for both of his artistic pieces. Additionally, Alejo engaged in some mathematical play while constructing his artistic pieces, and found ways to connect mathematics with art.

Alejo found inspiration for both his artistic pieces in personal hobbies. For his first artistic piece his inspiration came from his hobby of practicing a martial art. He mentioned the martial art he studies has a yin-yang concept to it, so he chose to create a yin yang design for his piece (*Figure 85*). Additionally, Alejo noted that he liked the way circles projected, and thought they would result in a nice symmetry.

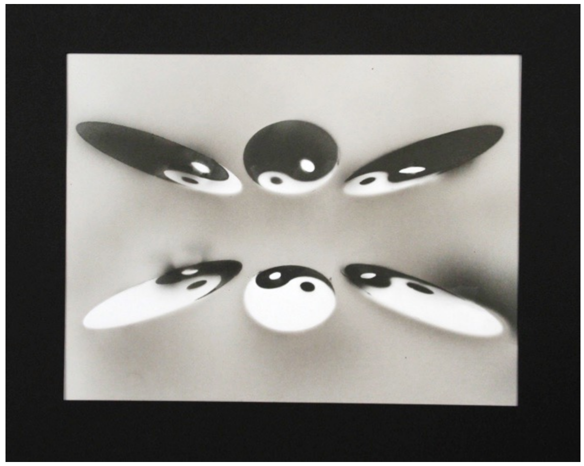
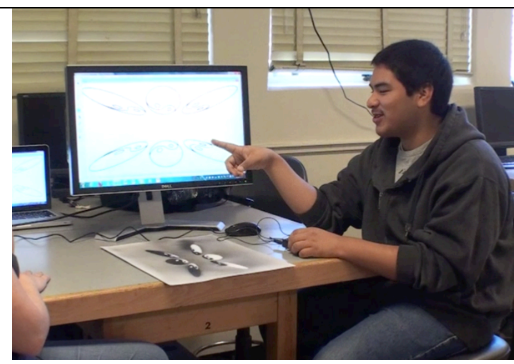
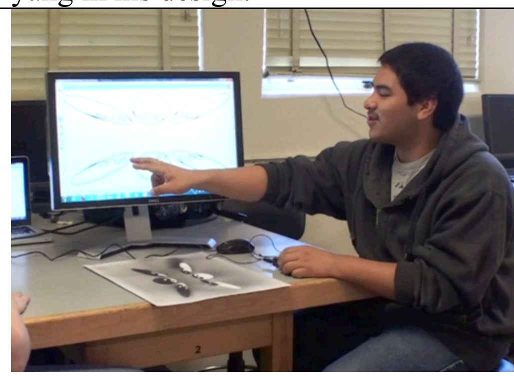
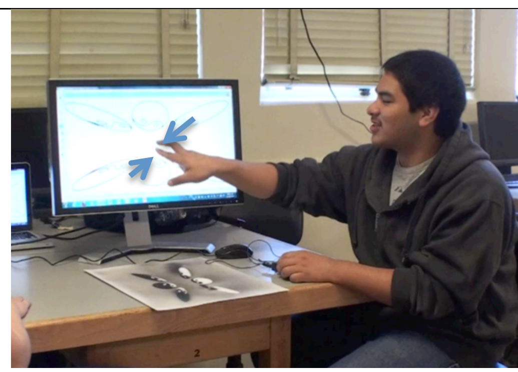
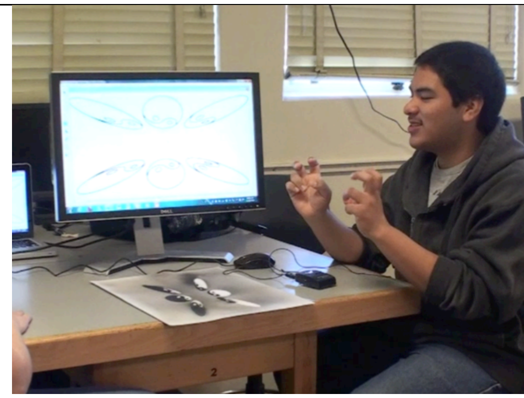


Figure 85. Alejo's first artistic piece.

743	I started practicing martial art, and so it's a yin yang concept, so I figured I'd do that…	 Alejo points to his design on the computer screen.
744	And I figured, I figured it'd always stay symmetrical. Like you know, there's just as much black here	 Alejo points to the upper-right-hand yin yang in his design.

745	but there's still just as much white here	 Alejo points to the lower-left-hand yin yang in his design.
746	So like if you, theoretically, you could take all these pieces	 Alejo points with two fingers at the lower-left-hand yin yang in his design.
747	and stick those together,	 Alejo holds his hand over his design on the computer screen. He stretches his middle finger toward the upper-right-hand yin yang and his index finger toward the lower-left-hand yin yang. He then draws his two fingers together in the center of the design.

748	and still have um the same amount of black and white.	 Alejo does quotes in the air when he says, “same amount.”

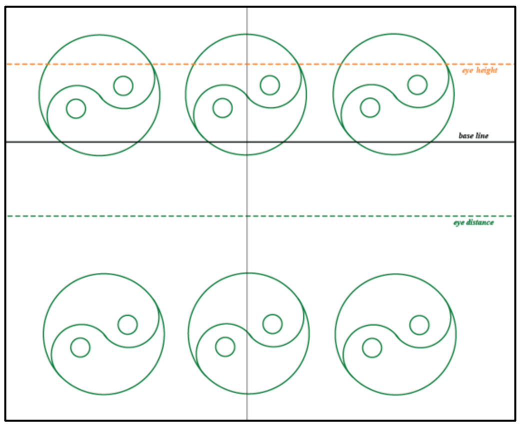


Figure 86. Alejo’s design before projection. The green line is the eye distance line. The black line is the baseline. The orange line is the eye height line.

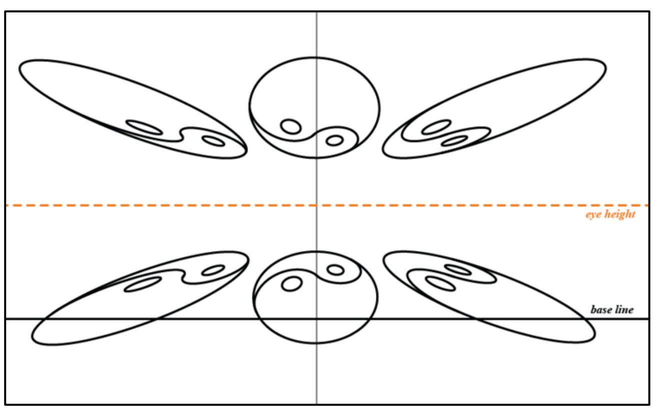
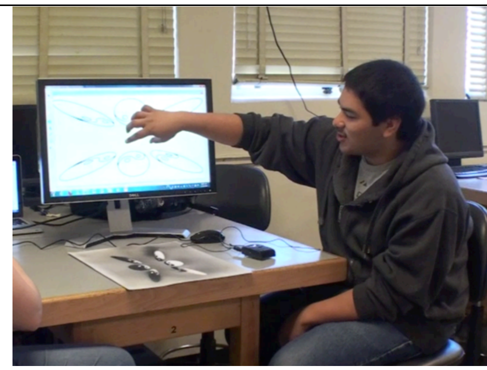
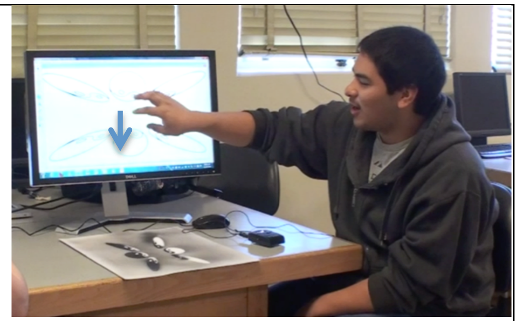
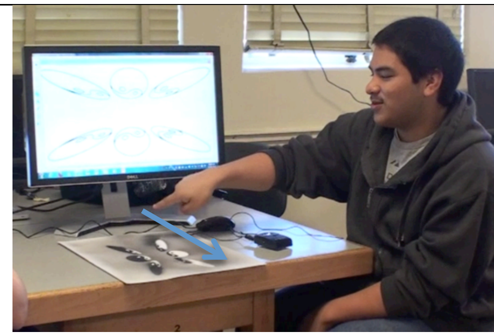


Figure 87. Alejo’s design after projection. The black line is the baseline. The orange line is the horizon line.

Alejo understood the projection of a yin yang symbol itself would not be symmetric, however he had an idea of how to create the symmetry he desired by projecting multiple copies of the symbol at different locations. When asked about why he chose to use the projection of circles and the yin yang symbol, Alejo explained,

749	[I]t stays very nice, symmetrical and stuff. Um I mean it, I mean if I project this circle, I mean this is what I, if I project a circle that was here,	 Alejo holds his hand over middle of his design. He then points to the middle of the design with his index finger as he says, “if I project a circle that was here.”
750	I’d get this, you know.	 Alejo points to the lower-middle yin yang symbol on the screen.

751	But um, but if I copy and paste that circle	 Alejo places his hand over the center of the screen, as if grasping an object. He then moves his hand to a lower position on the screen.
752	and just move it down,	 Alejo moves his hand down below the screen.
753	then I get its opposite, kind of uh, up here You know, the whole projection thing	 Alejo points to the upper-middle yin yang on the computer screen. He then points back and forth two times between the upper- and lower-middle yin yangs.

754	where you move it further below the eye [distance] and stuff, and get the other one.	 Aljeo points below the computer screen then draws his hand away from the screen and toward the edge of the table.

Here, Alejo explains that taking a projection of a yin yang symbol above the eye distance line will result in the projection being located on the screen below the horizon line. Then, by moving that same yin yang symbol down to a location below the eye distance line on the screen, you will obtain, a mirror image of the first projected yin yang symbol. The additional understanding needed is that to obtain a mirror image of the bottom right yin yang projection, a copy of the original yin yang would need to be projected from a location below the eye distance line and to the left of the center.

The sense of how to create a symmetrical image in the way in which Alejo constructed his piece is not trivial. This ability requires a sense of how images on the tabletop project from different locations on the tabletop with respect to the window (i.e., in front of the window, between the window and the viewer, and behind the viewer). Specifically, it requires the understanding that to obtain the opposite of a projection, the image must be moved to the opposite side of the eyepiece. In addition, it requires an ability to coordinate that sense of how images project with the way in which the physical and GSP versions of Alberti's Window relate to one another. Noticing and developing an understanding of the symmetric quality of projected images located in front of and behind the eyepiece, and then having the ability to transfer that understanding to the GSP version of the window demonstrates a sophisticated sense of the coordination between the two versions of the Alberti's Window. While Alejo may have already had a developed sense of this coordination before the first project, the project gave him the opportunity to test out and further develop his sense of the coordination.

For both his artistic pieces, Alejo engaged in mathematical play during the creative process. In particular, Alejo talked about playing with various shapes, such as circles and squares. He ultimately chose to work with circles rather than squares, since he found he didn't care for the way squares projected as trapezoids. After choosing to work with circles, and in particular the yin yang symbol, Alejo experimented multiple times with the location of his yin yang symbols in his sketch, as well as their projections.

> I would open it like, just like during breaks or something, just to look at it, and then just move it around, to see if I really wanted this [design] or not. I don't know. I tried a bunch of things. Like I just moved [the circle] down, and then um, this one would get really really big. And then um, the space would get really really close between these (the middle projection and the ones flanking it).

In addition to engaging in mathematical play by experimenting with the location of the yin yang symbols and their projections, Alejo experimented with the variable aspects of the GSP Alberti's Window, such as the location of the eye distance line and the eye height line. In an interview, Alejo explained he didn't care for the way in which moving the variable aspects in the GSP window eliminated the symmetry of his design, saying, "I did [move the eye height and eye distance], and then something would like squish here (*Figure 88*), and then this one would get really big (*Figure 89*)."

Figure 88. Alejo brings his fingertips together over the screen as he says, "something would squish here."

Figure 89. Alejo spreads his fingers apart over the screen as he says, "this one would get really big."

A final aspect to Alejo's mathematical play in creating his first design was similar to the mathematical play of Trisha and Willow as they created their own designs. In particular, Alejo experimented with projections of projections. He noted however, unlike Trisha and Willow, that didn't care for the way the second projection looked, and as a result remained with the single projection of his yin yang symbols.

In his video reflection for his second artistic piece, Alejo suggested that by creating his artistic pieces, and learning about projective geometry in such a way, he felt he was more likely to remember the ideas of projection. He stated, "I think the, doing the whole painting things, make the projective art um, or just projection in general, memorable. So um, because I learned it this way, um and did these kind of activities I, I definitely won't forget it as easily as other things."

Alejo began the semester stating that he neither cared for art, nor viewed himself as an artist. By the end of the semester, Alejo had made connections between mathematics and art that allowed him to think of himself as having more artistic ability than he previously considered. Alejo made two larger connections between mathematics and art, first with respect to what he considered as art, and second with respect to mathematics as an art.

At the end of the semester, Alejo explained, on more than one occasion, that he felt he previously had a misconception about what constituted art, and he mentioned he hated the idea of art related to his misconception. He explained his misconception regarding art was that art was required to be intricate and complex, providing da Vinci and Michaelangelo paintings as examples. Upon a brief reflection, Alejo realized a subtlety within his misconception that connected the practices of mathematics with the practices of art.

> So like those things (e.g. Renaissance art) that they're very um, intricate and have a lot of detail in them, and uh. So that to me is a misconception of art. Like, I don't know, it just has to be very complex. But then again we even saw that, The Last Supper painting and there was elements of projection in that. And, but projection itself isn't really that complex. So um, I don't know, it's just that misconception of uh, thinking that art has to be some kind of complex uh thing.

Here, Alejo acknowledges the misconception he felt he had about art prior to the course, that art was required to be complex and detailed. He then reexamines his notion of intricate and complex art by noting The Last Supper, while appearing complex, has simple projective elements in the underlying structure. Alejo drew a connection between this idea of a complex artistic piece having a simple underlying structure to mathematics, stating,

> Similar to math, um, now that I think about it, because uh, math is seen as this very difficult thing, there are so many equations or whatever, and it really isn't, it's really simple um, when you think about it.

The second connection Alejo drew between mathematics and art relates to regarding mathematics as an art. He explained,

> I really liked [A] Mathematician's Lament. That one's gonna stay with me for a while… 'Cause that's where, like I said um, that's where I kinda saw the math as art kind of thing. Like um, just, like the mathematician presupposes something that's true, and then he must work within that. Just like the artist, has something, and must work within those constraints.

Here, Alejo explains the way in which he thinks of mathematics as an art, and mathematicians as artists. He relates the two by suggesting that mathematicians choose a set of axioms, or define a particular object, and then must work within those constraints while exploring the mathematics of that realm. And similarly, the artist must choose a medium and then work within the constraints of the medium, noting later that artists are also limited to working with things that exist in the world, which he viewed as another constraint.

In an interview, when asked about whether these projects had changed his view of himself as an artist, Alejo replied, "Well, mathematicians are artists… I thought about it that way… The statement "mathematicians are artists" has more uh meaning to it than it did before this class and these projects." He followed this up later with,

> I don't know, well actually maybe, because of the fact, for the yin yang one, like I made, I purposely wanted those circles projected the way I wanted 'em to. And that is kinda, like an artist does exactly what he wants the piece to look out er, come out like. And sometimes even not. But I mean, I don't know. The artist's intention is to create something, right, so like when I was doing this, I was intentionally creating this, that way, and no other way. Or, you know, I wanted it that way and that's how I wanted it. So, I don't know, I guess I could say I'm more creative that way. I'm more artistic that way.

After creating his artistic pieces and reading the first chapter of A Mathematician's Lament, Alejo conceded to the idea he could consider himself an artist. However at another time, he presented a caveat, "I never really considered myself an artist but uh, until now. And even now I don't consider myself that great an artist."

Through artistic engagement in the course, Alejo was able to incorporate his personal hobbies into his mathematical activity, weaving his personal experiences with mathematics. While creating his artistic pieces, Alejo engaged in mathematical play and utilized his understanding of the ways in which the two versions of the Alberti's

Window correspond to obtain the symmetrical design he desired. Finally, Alejo made two significant connections between mathematics and art, which allowed him to think of himself as more of an artist than previously.

6.6 Summary

Each of the five participants experienced more than one way in which artistic engagement in the course enriched their learning experiences and opportunities. In this section I briefly summarize the overarching themes of the ways in which the chosen participants learning experiences and opportunities were enriched. The six themes I address are (a) fostering mathematical play, (b) making sense of pop-up topics, (c) coordinating mathematical tools, (d) weaving personal experiences with mathematics, (e) finding connections between mathematics and art, and (f) changing relationships with art.

6.6.1 Fostering Mathematical Play.

Each of the five chosen participants, in their individual interviews, indicated they engaged in mathematical play, to some extent, with the GSP version of Alberti's Window while creating their artistic pieces. This suggests creating the artistic pieces fostered mathematical play for students during the creation of their designs. The degree to which each of the participants engaged in mathematical play, and the particular actions they took while engaging in play, varied. Several of the participants noted the play in which they engaged was done for the sake of aesthetics – meaning, they manipulated the variable aspects in the GSP Alberti's Window until they found a design they found aesthetically pleasing. Others, like Fiona, discussed using play to watch an image and its projection work together, and to see how the variable aspects of the GSP window changed the projection of the image. For three of the chose participants, this mathematical play led to considering and making sense of mathematical situations that had not arisen in the classroom.

6.6.2 Making Sense of Pop-up Topics.

Three of the five chosen participants, Trisha, Jerry, and Willow, either while creating their artistic pieces, or while reflecting on the creative process, encountered and made sense of mathematical situations that were not intended as class discussion topics. This indicates artistic engagement, and in particular creating artistic pieces, in a mathematics course can lead students to mathematical inquiry.

Interestingly, Trisha and Jerry both made sense of what it means, in terms of the physical Alberti's Window, to place the eye distance line above the baseline in the GSP version of the window. Similarly, all three of Trisha, Jerry, and Willow made sense of what in means, again in terms of the physical Alberti's Window, to carry out a projection on an already projected image in the GSP version of the window.

6.6.3 Coordinating Mathematical Tools.

Three of the five chosen participants, Willow, Trisha, and Jerry, through the creation of or reflection on their artistic pieces, developed a more sophisticated understanding of the ways in which the two versions of the Alberti's Window coordinate. Specifically, they each developed a better sense of the ways in the variable aspects of the two representations of the Alberti's Window correspond, such as the location of the original image, the eye height, and the eye distance. This is particularly important, as, in my experience, students often struggle to make sense of how the two mathematical tools correspond. It is likely Alejo also developed a more sophisticated understanding through creating his artistic piece, however there was not enough evidence to draw that inference.

6.6.4 Weaving of Personal Experiences With Mathematics.

For Trisha and Alejo, creating the artistic pieces gave them an avenue to weave their personal experiences with mathematics. In particular, for his first artistic piece, Alejo created a design inspired by his experiences with studying a particular martial art, and the notion of yin yang. Similarly, for her second artistic piece, Trisha was inspired by her passion for dance to create a spiral-like design. Trisha's experiences weaving her personal experience with mathematics appeared deeper than Alejo's, as her dance experience seemed to permeate her mathematical experiences, and her mathematical experiences began to permeate her actions and intentions in her dance classes.

6.6.5 Connections Between Mathematics and Art.

Willow, Jerry, and Alejo each expressed particular ways in which they saw mathematics and art as connected. Willow drew parallels between the way in which she previously felt about art and the way in which people often say they do not understand mathematics. She alluded to the idea that by changing ones perspective, and trying to see something from a different point of view, then everyone could do both mathematics and art.

Jerry expressed a complex connection between mathematics and art, in that he came to view the similarity between the two subjects to be the role of self-expression. Jerry indicated he views the ability to create mathematical proofs in your own way, and figuring the proof out for yourself, is similar to the self-expression of creating artistic pieces.

Alejo made two significant connections between mathematics and art. First, he expressed the idea that a complex and intricate piece of artwork could have a simple underlying structure, and that this is similar to the way in which mathematics seems to be a complex subject, but the underlying structure is simple. Second, Alejo made the connection that both mathematicians and artist have to choose what they plan to work with, and then work within those constraints. For the mathematician, the constraints are such things as axioms and definitions, while for the artist, the constraints are the medium and the restrictions of the physical world.

6.6.6 Relationship with Art.

For each of Willow, Jerry and Fiona, the artistic engagement in the course altered their personal relationship with art. Fiona and Willow both experienced a change in their thinking about their own artistic abilities. Willow stated she came to consider herself as an artist, while Fiona remained unidentified as an artist, but felt she has more artistic skills than she previously had thought. Jerry, who identified as having artistic ability prior to taking the course, noted how, after creating his artistic pieces and engaging with projective geometry, he felt as if he would now incorporate more mathematics into his future artistic pieces.

Developing a more favorable relationship with art may encourage students to create future artistic pieces, in which they may incorporate mathematics. In creative future mathematically inspired artistic pieces, students may engage in mathematical play and explore mathematical ideas/situations previously unknown to them. Furthermore, it may develop students' confidence with respect to art and design, and encourage them to pursue other creative outlets.

6.7 Additional Notes

In this section I highlight two important aspects to keep in mind when considering the ways in which students learning experiences and opportunities in this course were enriched through artistic engagement. The two aspects I address are the nature of the artistic projects in the course and the reflective writing in which students engaged after creating each of their artistic pieces.

6.7.1 Nature of the Projects.

Recall the particular artistic projects the students engaged in twice during the course allowed students to explore their own inspiration for artistic creation. Students were required to create their designs using the Alberti's Window sketch in GSP, and they were given specific dimensions in which their design had to be contained. The students were allowed to include projected and non-projected elements in their pieces, and were told that projective geometry must play a fundamental role in their design. Other that those requirements, students had the freedom to create any type of design they liked. Furthermore, recall that the intention of the artistic project was not for students to demonstrate their understanding of projective geometry, but rather, students were to find inspiration in anything they liked, and then create a design based on that inspiration.

The nature of these artistic pieces, and the freedom students had, provided an opportunity for students to engage in mathematical play. Recall, in Chapter 4, I defined mathematical play as mathematical exploration that is both autonomous and freeform. Furthermore, recall the open-ended nature of a task influences mathematical play, in the sense that tasks with more flexibility lend themselves better to mathematical play. With its limited restrictions, and the requirement of using GSP, the design of these artistic projects allowed for students to engage in mathematical play to any extent they desired.

Features of the task that contributed to autonomous activity included that students were creating individual pieces, and thus were not likely to be reacting to criticism from others. In addition, students were given the freedom to create a design inspired by whatever piqued their interest, as long as projective geometry played a fundamental role in their design. The primary feature of the task that seemed to contribute to freeform activity was the use of the GSP Alberti's Window to create their designs. This allowed for easy manipulation of the variable aspects of the sketch, as well as the ability to move an image on the screen and watch the projection simultaneously change with it.

Given an art project in which they had to demonstrate their understanding of projective geometry, say, for example, creating an artistic piece that holds with the principals of linear perspective, may be less likely to foster play than the project assigned in this course. Students participating in a more constrained artistic project, such as suggested, may be less likely to engage in mathematical play, as they may be focused on the judgment to be placed on the piece. Similarly, focusing on creating a piece that hold with linear perspective could limit students' playfulness with shapes other than lines.

6.7.2 Importance of Reflective Writing.

Based on the interviews with participants, and in particular Trisha and Jerry, the reflective writing for the artistic projects played an important role in prompting students to think carefully about the mathematical situations represented in their artistic pieces. Trisha and Jerry noted they had not fully considered the mathematical situations in their pieces until they began to write their reflections. Without the requirement to write a reflective essay, in which they were explicitly asked to address the mathematics in their pieces, it is likely neither Jerry nor Trisha would have realized they had moved the eye distance line above the baseline in their designs. This suggests the importance of reflective writing at some time during the process of an artistic project in a mathematics course.

Ideas in Chapter 6, in part, have been published in the journal *Problems, Resources, and Issues in Mathematics Undergraduate Studies* (PRIMUS), 2016, in the article, Arguments for Integrating the Arts: Artistic Engagement in an Undergraduate Foundations of Geometry Course, Volume 26, Issue 4, pages 356-370.

Chapter 7
Summary and Discussion

In this chapter, I provide a summary of the results of my research questions, and implications that follow from the results. I then discuss the limitations of my study, which partially inform my interest in future directions of research. Using the results of my research questions and the limitations of my study, I describe some of my interests in future research. Finally, I discuss the significance of this study for the field of mathematics education, as well as for the STEAM (science, technology, engineering, arts, mathematics) movement.

7.1 Summary of Results

7.1.1 Mathematical Play

In chapter 4, I addressed the first of two results of my first research question. Specifically, I discussed mathematical play – the first of two mathematical practices in which students engaged in the activity-based projective geometry course, and on which I chose to focus my analyses. I defined mathematical play as the exploration of mathematical ideas through individual or group actions that are both autonomous and freeform – where by autonomous I mean the actors have minimal concern with what others around them are doing, or with what others think about what they are doing, and by freeform I mean the details of the actions are not scripted or prescribed. I noted that mathematical play can include engagement with physical devices, computer programs, acts of imagination, and social interactions, as well as inscriptions, which was illustrated throughout the episodes I highlighted in the chapter.

After illustrating the mathematical play construct, I discussed two benefits for students from engaging in the practice of mathematical play. First, engaging in mathematical play led students to consider pop-up topics, where by pop-up topic I mean mathematical situations that had not yet been addressed in the course, and that arose organically from student activity. Second, engaging in mathematical play led to the justification and argumentation of mathematical ideas. I illustrated these two benefits with an episode, which I split into two segments, in which a student engages in mathematical play through proposing a pop-up topic. The student proposes a particular scenario in which all of the points in a perspective, the center of projection and range points, all lie on the same line. Considering the pop-up topic led the group into an instance of mathematical argumentation in which two members of the group justified why they believed the scenario that arose from the pop-up topic would not constitute a perspective.

I then discussed what aspects of the learning environment provided the opportunity for students to engage in mathematical play. These aspects included instructor influence, the nature of the task at hand, and the affordances and limitations of mathematical tools. In this course, the instructor, Dr. R. created a learning environment that supported student engagement in mathematical play. He cultivated this learning environment through supporting and expecting students to share their thinking, drawing attention to groups engaged in mathematical play, modeling mathematical play by requesting students carry out their mathematical inquiries, and referring to "playing" while explaining tasks. The nature of mathematical tasks contributed students' opportunities to engage in mathematical play in that tasks with less rigid instructions gave students more freedom to explore mathematical ideas. The affordances and limitations of mathematical tools were related to the static versus dynamic nature between physical tools and computer software. The affordances of dynamic geometry software provided students the opportunity explore mathematical ideas by allowing students to easily drag components of a mathematical situation, such as lines and points, to various locations, instantly creating a new situation to consider. The limitations of physical tools, such as the physical Alberti's Window, provided students the opportunity to engage in mathematical play by prompting them to find creative ways to use the tools to determine solutions to tasks.

7.1.2 Acts of Imagination

In chapter 5, I discussed acts of imagination – the second of two mathematical practices in which students engaged, and on which I chose to focus my analyses. Drawing upon the definition of collective imagining proposed by Nemirovsky et al. (2012), I defined acts of imagination as a mathematical practice characterized by one or more individuals acting *as if* a mathematical situation or entity were present, despite the entity not being physically present in the current surroundings. These acts of imagination could incorporate gesture, body positioning, eye gaze, verbal utterances, components of mathematical tools, as well as inscriptions, which was illustrated in the episodes I highlighted in the chapter.

After illustrating the acts of imagination construct, I discussed the ways in which students engaged in acts of imagination during explanation and justification of mathematical ideas or situations. I demonstrated how students engaged in acts of imagination on a small-scale or large-scale, in which the actor appeared to imagine herself or himself as observer of a mathematical situation, or the actor appeared to imagine herself or himself as part of the mathematical situation, taking on the role of mathematical entities. Students engaged in acts of imagination to explain or justify acts of imagination to their peers as well as to Dr. R. and myself. In addition, I suspect, based on participants responses to questions in interviews, that students engaged in acts of imagination while by themselves and working on their artistic pieces.

Similar to mathematical practices, instructor influence and the nature of the task at hand provided students opportunity to engage in acts of imagination in this course. Unlike mathematical practices, however, it appeared to be solely the limitations of mathematical tools, rather than the affordances and limitations, that provided students opportunities to engage in acts of imagination. In this course, Dr. R. encouraged students to engage in acts of imagination through his explanations of tasks in which he informed students they needed to imagine projections, engaging the whole class or groups of students in acts of imagination, making explicit his view that imagination to the "essence of mathematics" and that having a correct answer is not as important as being able to explain and imagine the reasoning to arrive at an answer. In addition, Dr. R. selected tasks and mathematical tools that encouraged students to engage in acts of imagination. Dr. R. developed tasks that required students to engage in acts of imagination, including having students construct and determine the projections of a large-scale parabola and hyperbola, from different locations, on a soccer field. The limitations of the physical Alberti's Window tool provided the opportunity for students to engage in acts of imagination. Specifically, the Alberti's Window only allowed students to easily project, through tracing on the window with a marker, images located on the opposite side of the window from the viewer. This required students to find alternative ways to determine the projections of images on the same side of the window as the viewer, which prompted students to imagine how the images get projected.

7.1.3 Artistic Engagement

In chapter 6, I discussed the ways in which students learning experiences and opportunities were enriched through participating in various forms of artistic engagement in the course. I presented case studies of the ways in which five students' learning experiences were enriched through artistic engagement. After presenting each case, I discussed six ways in particular that students' learning experiences and opportunities were enriched. I took a broad approach to the term enrich, which included ways in which students learning experiences and opportunities were enriched that were not specific to the course, but were a result of the artistic engagement in the course. Artistic engagement in the course fostered mathematical play, led to students considering pop-up topics, helped students develop their coordination of mathematical tools, provided students the opportunity develop meaningful connections between mathematical an art, including weaving their personal experiences with their mathematical experiences, and initiated change in students' relationships with artistic engagement.

7.2 Implications

The results of this study suggests several implications for designing and teaching an activity-based course in mathematics in which the instructor is interested in students engaging in mathematical play and acts of imagination. In addition, this study suggests implications for incorporating artistic engagement into a mathematics course. In this section, I address these implications.

7.2.1 Mathematical Play

Engaging in mathematical play can benefit students by leading them to consider pop-up topics and participate in justification and argumentation of mathematical ideas and situations. As such, an instructor may be interested designing a course and cultivating an environment that fosters student engagement in mathematical play. The results of this study suggest providing students the opportunity to engage in mathematical play is related to instructor influence, the nature of the tasks in which students engage, and the affordances and limitations of mathematical tools.

The results of this study suggest that to create a course and an environment fostering mathematical play, the instructor needs to select mathematical tasks that avoid rigid instructions, and instead allow interpretation and freedom in solutions. If incorporating mathematical tools into the task, the instructor must consider the affordances and limitations of mathematical tools and envision and anticipate the ways in which students might explore mathematical ideas while using the tool, and then select or design tools that will give students the opportunity to explore. In this study, students engaged in mathematical play as a result of both the affordances of dynamic geometry computer software and the limitations of static physical devices provided different representations of similar mathematical situations. This suggests the possible benefit of having students utilize different mathematical tools – of both the dynamic and static nature – that are representations of the same mathematical situations.

Choosing tasks that avoid rigid instructions and mathematical tools that give students the opportunity to freely explore mathematical ideas is likely not enough to encourage students to consistently engage in mathematical play. In the course in which my study took place, Dr. R. contributed to students' opportunities to engage in mathematical play. This suggests that to cultivate an environment that fosters mathematical play, the instructor should emulate some of Dr. R's instructional practices. Specifically, Dr. R.'s instructional practices that cultivated an environment that fostered mathematical play were modeling mathematical play by requesting students carry out their mathematical inquiries, drawing attention to groups engaged in mathematical play, referring to "playing" while explaining tasks, and negotiating classroom social norms that create a space in which students feel comfortable, and feel they are allowed, to share any ideas or questions related to mathematical tasks.

7.2.2 Acts of Imagination

Similar to mathematical play, instructor influence, the nature of tasks, and the limitations of mathematical tools contribute to students' opportunities to engage in acts of imagination. This implies that to design a course in which students have the opportunity to engage in acts of imagination, the instructor needs to take these three aspects into consideration. Specifically, instructors need develop tasks and incorporate mathematical tools that require students to engage in acts of imagination in certain instances to find solutions to tasks. In considering which mathematical tools to incorporate into a course, the instructor needs to envision and anticipate the ways in which the limitations of the tools can ensure student will engage in acts of imagination. Aside from selecting tasks and mathematical tools that compel students to engage in acts of imagination, instructors can encourage students to engage in the practice through explicitly engage students in acts of imagination, emphasizing the importance of imagination in mathematics, and accepting acts of imagination as justification in mathematical arguments.

7.2.3 Incorporating Artistic Engagement

The results from Chapter 6 – the ways in which artistic engagement enriched students' learning experiences and opportunities – have implications for arts integration into mathematics courses. In this section I highlight two implications specifically for incorporating artistic projects into mathematics courses. The two implications I address are related to the nature of the artistic projects in the course and the reflective writing in which students engaged after creating each of their artistic pieces.

Recall that at the end of Chapter 6, I highlighted two significant considerations regarding the ways in which students learning experiences and opportunities were enriched as a result of the artistic engagement in this particular course. First, the nature of the artistic projects likely contributed to students learning experiences and opportunities being enriched. Specifically, since the students were not required to demonstrate their understanding of projective geometry through their artistic pieces, students had the freedom to create any artistic design, as long as projective geometry played a central role. This freedom to create any design likely contributed to such things as students connecting their personal experiences with their mathematical experiences, since they were able to find inspiration from any source. Furthermore, the freedom to create any design the liked likely contributed to students considering pop-up topics, since students did not solely need to focus on creating a piece true to linear perspective. This suggests that the design for an artistic project in a mathematics course should allow students the freedom to explore and use mathematical relationships to create an artistic piece that does not necessarily need to *display* their understanding of the mathematical relationships they used in the piece.

The second implication of the ways in which students learning experiences and opportunities were enriched as a result of the artistic engagement in the course is related to the reflective writing in which students engaged after the completion of creating their artistic pieces. This writing proved to be significant for students, since the need to explain the mathematics behind their piece required them to make sense of any unfamiliar mathematical situations that arose during the design process. This contributed to students considering and making sense of pop-up topics. Without engaging in reflective writing, it is possible students would not have explored and made sense of pop-up topics, since they would not have needed to explain the ways in which projective geometry played a role in their designs. This implies that when incorporating mathematically inspired artistic projects into a mathematics course, the instructor should consider assigning reflective writing related to the projects.

7.3 Limitations

In this section, I reflect on some of the limitation of this study. While the categories upon which I reflect are not mutually exclusive, I divide this section into three categories: data collection and analysis, mathematical play and acts of imagination, and artistic engagement. In the next section, I discuss some of my interests for future research, which, to some extent, are influenced by the limitations I highlight here.

7.3.1 Data collection and analysis

As I set out to collect the data for this study, I believe my eyes were bigger than my stomach, so to speak. Which is to say, I collected far more data than was realistic to analyze for a dissertation. Certainly, I had a sense of this going in to data collection, however I wanted to ensure a good selection of data to analyze. From my experiences with two previous semesters working with students in the same Foundations of Geometry course, I knew students differed significantly in their engagement in the course and the amount of time and effort they invested in their artistic pieces, which prompted the colossal data collection effort.

As a result of this large amount of data, I needed to limit the number of participants for which I analyzed the data. I It is possible that these participants were more apt to engage in mathematical play and acts of imagination than the participants I chose not to analyze, as they did appear to be more engaged in the course in general. This allows for the possibility that these mathematical practices were not as prevalent as they appeared. Similarly, it is possible the ways in which these participants' learning experiences were enriched through artistic engagement was different than students who were less engaged and invested in the course.

As a result of my interest in collecting data for four groups of four students, and as a result of the layout of the classroom, my options for camera placement were limited. This resulted in times during analysis that I could not see the activity of one or more of the participants in a group, for various reasons. For example, at times, students had their back turned to the camera, or their activity was partially blocked by other participants or furniture. In particular, there were times during which groups were working with the Alberti's Window and the group member looking through the eyepiece stood on the same side of the table as the camera was placed. This resulted in the student having their back to the camera and blocking the Alberti's Window from the view. While I attempted to remind groups to use the window on the opposite side of the table as the camera, I was not always successful. This means some instances of mathematical play or acts of imagination may have been missed. For example, *Figure 90* shows an instance in which a group's activity was blocked from the camera by a group member, as well as by the table.

Figure 90. An instance in which a group worked with the Alberti's Window, yet their activity was blocked from the camera.

During this time when the camera was blocked by one of the students in the group, the group was working on trying to determine the projection of a square sitting between the window and the eyepiece. The group had decided to try moving the window onto the floor while keeping the eyepiece on the table. One group member held the square in between the eyepiece and the window, while the entire group tried to imagine how the projection would look. This episode was an instance of both mathematical play and an act of imagination, however did not lend itself well to analysis, since it wasn't possible to see the actions of the group.

7.3.2 Mathematical Play and Acts of Imagination

As mentioned above, two limitations of my analyses for mathematical play and acts of imagination were data collection related. First, due to the vast quantity of data I collected, I had to limit my analyses to five participants in particular. The result of limiting my analyses to five participants could be that other participants engaged in mathematical play and acts of imagination more or less frequently, and could have experienced different benefits from engaging in the two mathematical practices. The second data-collection-related limitation of my analysis for mathematical play and acts of imagination was the placement of the cameras and the instances in which the view of participants was partially or fully blocked from the camera, as discussed above.

In addition to data-collection-related limitations, there are other limitations related to my analyses of mathematical play and acts of imagination. One particular limitation with identifying acts of imagination was that without access into students' thoughts, I was only able to identify acts of imagination from instances in which an actor or actors *overtly* engaged in bringing mathematical entities or situations into quasi-presence, through gesture, body positioning, eye gaze, and speech. It is likely that students engaged in acts of imagination, yet did not overtly act as if mathematical situations were present. For example, in the episode from section 5.3 in which Willow engaged in an act of imagination to justify why she believed the projection of a circle sitting between the eyepiece and the window would be ellipse that was "oblong vertical," I suspect Willow had been covertly engaging in an act of imagination prior to suggesting the ellipse would be "oblong vertical." Approximately twelve minutes prior to this episode, Willow's group had determined the projection of the circle would be a "rounder" horizontal ellipse. During the twelve minutes that passed, Willow did not engage in any overt acts of imagination, yet something changed her mind about how the projection of the circle would look. It is likely Willow engaged in a covert act of imagination – that is, engaging in an act of imagination without demonstrating an as if presence of mathematical entities through bodily engagement and speech. This is evidenced by Willow engaging in an act of imagination to justify her prediction about the projection being an "oblong vertical" ellipse. Assuming students did engage in covert acts of imagination that could not be identified during data analysis, this would mean the instances of acts of imagination in the course would be more frequent than my analyses indicate.

7.3.4 Artistic Engagement

There two categories of limitations I address here with respect to the ways in which artistic engagement can enrich students' learning experiences. The first category relates to the specific data I collected, and the second category deals with the generalizability of the results to other courses with artistic engagement components.

As I mentioned previously, due to the vast quantity of data I collected, I needed to limit my analysis to five participants. I chose five participants who I felt were articulate, thoughtful, and communicated their ideas well. Since I limited my analysis to these five participants, who seemed particularly engaged in the course, it may be my results of the ways in which student learning experiences were enriched through artistic engagement is not representative of all the students in the course. It is possible that less-engaged students may have had fewer instances of enrichments from artistic engagement.

A second challenge with determining how artistic engagement enriched students' learning experiences is that the majority of the data I was able to collect regarding students' artistic projects was self-report data. I had an interest in understanding more deeply the ways in which students went about creating their designs, as well as the struggles they faced and the discoveries they made. However, since the students created their artistic designs at home, I was not able to capture their design creation process as it happened. Instead, I attempted to draw out these experiences of design creation, including struggles and discoveries, during individual interviews, as well as through written reflections. The result is data pertaining to design construction, struggles, and discoveries was self-reported. This means I relied on students' memories about how they constructed their designs, what struggles they faced, and what discoveries they made, which suggests some of the information obtained from interviews and written reflections may not be precise.

The second category of limitations pertaining to the ways in which students learning experiences are enriched through artistic engagement is related to the generalizability of the results. The results about the ways in which artistic engagement enriched students learning experiences and opportunities in the projective geometry course cannot easily be generalized to other courses that include an artistic engagement component. Certainly, this particular projective geometry course with artistic engagement components lent itself well to artistic engagement, as projective geometry has its roots in art production (Andersen, 2007; Field, 1997). In addition, Dr. R. had a commitment to artistic engagement that may surpass that of other instructors.

Similarly, the particular artistic project in which students engaged, and the way in which students were encouraged to create what they desired, without being required to demonstrate their understanding of projective geometry though their artistic pieces, may have resulted in ways students' learning experiences and opportunities were enriched that may not have occurred had the purpose of the project been to demonstrate their understanding of the mathematics.

7.4 Reflections on Future Directions

In this section, I reflect on the ways in which this study has influenced my interests in future research topics. My future research interests have been influenced by multiple aspects of this study, including, certain results of the study, the limitations of the study, and aspects of the data that caught my interest. While I could fill numerous pages with emerging research interests and questions that arose from analyzing my data, here I keep it to six examples: the mathematics in which students engage in the activity-based projective geometry course, the relationship between mathematical play and acts of imagination, the role of communication and disposition in mathematical play and acts of imagination, the role of the aesthetic in mathematical play, arts integration in other courses at with different ages, and the relationship between practices in mathematics and practices in art.

7.4.1 The Mathematics in Which Students Engage

At the beginning of designing my study, one of my interests was the mathematics in which students engaged in this particular Foundations of Geometry course. This interest stemmed from examining prior collected data from two similar courses. For example, in Chapter 1, I provided two illustrations from previous data that indicated the course in which my study took place could serve as an interesting setting for investigating the ways in which students utilize while working on problems in projective geometry. In the first of the two illustrations, Ryan and Veronica discussed the projection of a circle onto the Alberti's Window when the viewer was standing in the middle of the circle. Ryan used the notion of a limit to determine the projection of the circle would be a hyperbola in which the both branches extend to infinity. In the second illustration, Veronica explained the horizon line on the Alberti's Window as being "like an infinity." As a consequence of instances such as these that I had observed in prior data, I had anticipated students in this course to engage in a rich variety of mathematical ideas from multiple areas of the field of mathematics. However, the data collected for this study provided fewer instances of students engaging with mathematical ideas from varying realms of mathematics than I had anticipated.

I remain interested in investigating the ways in which students draw upon mathematics from other courses and the way they bring these other mathematical topics to bear while working on problems in projective geometry, including the ways in which students struggle. In particular, I am interested in investigating the way in which students draw upon the notions of limits and infinity to reason about ideas in projective geometry – particularly as these are important topics with which students often struggle (Davis & Vinner, 1986, Monaghan, 2001; Singer & Voica, 2008; Tall & Tirosh, 2001; Tall & Vinner, 1981). I believe an activity-based projective geometry course such as the one in which my study took place could serve as a platform for students to develop their understanding of limits an infinity, as well as geometry, in general. I consider this particular semester of the Foundations of Geometry course as a single cycle in a developmental research cycle (Gravemeijer, 1994), in which following cycles, informed by the mathematics in which students engage in this course, could be developed to further engage students in topics such as limits and infinity. In addition, I would consider the ways in which other branches of mathematics, such as algebra, could be more explicitly drawn upon in the course.

7.4.2 Relationship Between Mathematical Play and Acts of Imagination

There was some evidence in the data of a relationship existing between students' engagement in mathematical play and their engagement in acts of imagination. For example, in the instance briefly mentioned above in section 7.3.1, a group of students was engaged in mathematical play trying to determine the projection of a square resting between the window and the viewer. In order to determine the projection, the group placed the Alberti's Window on the floor, kept the eyepiece on the tabletop, and held the square level with the table, between the window and the eyepiece. Once the group had arranged their mathematical tools, they engaged in an act of imagination to determine how the square would project. In particular, the group acted as if the lines of projection extending from the eyepiece to the square, and then extending to the window, were present. In this instance, it appeared the mathematical play and the act of imagination jointly emerged. In other instances, it appeared acts of imagination gave rise to mathematical play, and then contributed to the way in which the actor engaged in play. In future studies, I would look for relationships between the two mathematical practices, perhaps including as it pertains to mathematicians generating new mathematical ideas.

7.4.3 Communication & Disposition in Mathematical Play & Acts of Imagination

The reader may have noticed in Chapter 4, the chapter in which I addressed mathematical play, that one student in particular, Jerry, appeared in multiple illustrations of instances of mathematical play. This leads me to wonder what it was about Jerry's disposition that prompted him to frequently engage in mathematical play. Similarly, certain participants appeared to engage in acts of imagination more frequently. Some participants, who were not included in my analyses, appeared to only engage in a few acts of imagination throughout the duration of the course.

Similarly, it appeared that certain groups of participants engaged in mathematical play and acts of imagination more often than other groups – however this was difficult to determine definitively since the group compositions changed approximately the same time the nature of the tasks became more rigid. I suspect it is the case that group dynamics contribute to the frequency with which a group engages in mathematical play and acts of imagination. Furthermore, I suspect the frequency with which a group engages in the two mathematical practices is related to the disposition of the individual group members.

I would be interesting to investigate whether the disposition of students who engage in mathematical play or acts of imagination more frequently, as well as group dynamics, could be leveraged to cultivate an environment, through group composition, that fosters engagement in the two mathematical practices.

7.4.4 The Role of the Aesthetic in Mathematical Play

There was some evidence that, at times, aesthetic considerations played a role in students' mathematical play. In particular, some instances of mathematical play seemed to emerge from students finding mathematical situations interesting or uninteresting. For example, in one instance, during the first exploration with the physical Alberti's Window, in which groups were "playing with parallel lines in different directions," it appeared as if aesthetic consideration were contributing to Jerry's initiation of mathematical play. Specifically, Jerry would propose projecting the parallel lines from a particular location, and after the projection was traced, he might say, "That's boring. What if put them here," and place the lines in a different location. Similarly, as Willow's group was trying to determine how to proceed with the same activity of projecting parallel lines in different directions, Willow stated, "It will be boring if we all just do parallel lines. Let's do a house." One group member proceeded to sketch a very basic house, a rectangular shape with a triangular roof, and the group experimented with projecting the house.

This role of the aesthetic may be related to what Sinclair (2004) referred to as the evaluative and the generative roles of the aesthetic; the first of which deals with the judgment of aesthetic qualities of mathematical entities, and the second involves "nonpropositional, modes of reasoning used in the process of inquiry" (p. 264). Each of the evaluative an generative roles of the aesthetic (Sinclair, 2004) could serve a role in mathematical play – such as fostering or guiding play – and as such, would be interesting to investigate the relationship between the aesthetic and mathematical play.

7.4.5 Arts integration in other courses and with different ages

This projective geometry course, in which students participated in different forms of artistic engagement, was specifically an upper-division undergraduate level course. Recall that most of the students in the course were mathematics majors with a secondary teaching emphasis. Since artistic engagement in this course enriched students' learning experiences in many ways, I would like to investigate how artistic engagement might enrich students learning experiences in different classes, at different age levels, as well as for those who struggle with mathematics courses. I suspect artistic engagement could have positive effects for students at different age levels and at different levels of confidence with mathematics – in particular those who struggle with mathematical ideas or have difficulty connecting mathematics to their own lives.

7.4.6 Practices in Mathematics and Practices in Art

Through investigating the mathematical practices in which students engage, as well as the ways in which students learning experiences and opportunities are enriched through artistic engagement, an emergent interest of mine in future research is the points of contact between the practices in mathematics and the practices in art, and the way in which each may be leveraged to support the other. For example, the practices in art may leveraged to support students developing ways of engaging in mathematical practices and help students develop more productive beliefs and dispositions (Cobb & Yackel, 1996) in mathematics, and vice versa.

A beginning point for this future research, and one of the reasons I became intrigued by considering the points of contact between the practices in the two fields, is list of the Eight Studio Habits of Mind discussed by Hetland, Winner, Veenema, and Sheridan (2013) in the book Studio Thinking 2. These eight studio habits of mind include:

- understand art worlds: learning about the history of art and the current practices in art, as well as learning to communicate as an artist with other artists
- engage and persist: developing habits to embrace relevant problems within the art world and within ones own life, and developing focus to persevere at art-related tasks
- envision: learning to conceive of things that cannot directly be observed, developing the ability to play freely without a plan
- stretch and explore: learning to experiment beyond one’s perceived capacity, and learning to learn from, and appreciate what can be learned from, ones mistakes
- develop craft: learning to use (and care for) art production tools, as well as learning artistic convention
- reflect: learning to evaluate one’s own artistic works and the works of others, and communicate about aspects of artistic works and processes

- observe: learning to carefully attend to visual situations, to see what might not otherwise be seen
- express: learning to create artistic pieces that communicate an emotion, an idea, or a personal connection

These eight studio habits of mind can be connected to the practices in mathematics without much difficulty. For example, the *envision* habit might correspond to the mathematical practices presented in this study, acts of imagination and mathematical play. As a particular example, in developing a mathematical imagination, students must imagine entities that are unobservable, such as infinity. Similarly, the *stretch and explore* habit can be partially connected to mathematical play. As a particular example, during mathematical play, students may explore mathematical ideas by considering pop-up topics, which is similar to experimenting "beyond one's perceived capacity," as students have not yet learned about the topics that arise. As such, I find the notion of exploring the points of contact between practices in art and practices in mathematics, and the way each can may be leveraged to support the other, to be an intriguing research topic, and one that could potentially support students who struggle with mathematics but are interested in art.

7.5 Concluding Remarks

This dissertation contributes to the literature in the field regarding students' engagement in mathematical practices by illustrating the ways in which students engage in mathematical play and acts of imagination. In addition, it presses on the idealized versions of practices of mathematicians (Burton, 1999) included in K-12 the

NCTM and CCSS standards documents (NCTM, 2000; CCSS, 2010), which do not reflect these forms of practices that might generate new mathematical ideas. Furthermore, the consideration of imagination as the essence of mathematics proposes the notion of gaining mathematical competency as developing an increasingly more sophisticated mathematical imagination, and indicates the need for students to participate in classroom activities that foster engagement in acts of imagination.

This dissertation adds to the existing literature pertaining to arts integration, particularly as it pertains to mathematics courses. Generally, the focus in research on arts integration is developing students' creativity (Marshall, 2005; Wallace et al., 2010), promoting transfer between subjects (Catterall, 2002), and increasing students' performance on standardized exams, or achievement more broadly (Catterall et al. 1999; Deasey, 2002; Smithrim & Upitis, 2005). However, research on creativity and transfer presents challenges, as the conceptualizations of creativity and transfer vary significantly (Barnett & Ceci, 2002; Plucker et al., 2004), and some researchers question whether either can truly be measured. This study provides insight into the alternative ways in which arts integration might be beneficial for students, and, in doing so, supports the movement to integrate the arts into mathematics courses, and the STEM disciplines more broadly.

Furthermore, this study indicates the ways in which arts integration into mathematics courses might support mathematics students' development. In particular, artistic engagement, and, more specifically, creating artistic pieces using mathematical ideas, fostered mathematical play and led students to consider and make sense of pop-up up topics. Certainly, we would like students of mathematics to explore and make sense of mathematical ideas on their own, and integrating artistic projects such as the students in in this course created, engaged students doing just that.

For me personally, this dissertation helped me to develop many research skills, as I was primarily responsible for all data collection – planning, organizing, and orchestrating. Throughout data collection I conducted over sixty interviews with students, and while analyzing the data, I watched my interviewing techniques and skills develop. I learned about organizing large amounts of data, such that a snippet of data could be located quickly, and I developed my qualitative analysis skills. This dissertation has already begun to inform my own teaching in a Mathematics and Fine Arts course, in terms of attempting to cultivate an environment that fosters mathematical play and acts of imagination, as well as the way in which I design artistic projects in the course. Finally, this dissertation gave me a deeper passion for investigating arts integration into mathematics courses, and the ways in which it can benefit students' learning.

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