Amplify-and-Forward Relay Networks with Variable-Length Limited Feedback

Xiaoyi (Leo) Liu, *Student Member, IEEE*, Hamid Jafarkhani, *Fellow, IEEE*, and Erdem Koyuncu *Member, IEEE*

Abstract

We study the channel quantization problem for amplify-and-forward (AF) relay networks and our target is to design a quantizer to minimize the outage probability. It is priorly known that any fixed-length quantizer with a finite-cardinality codebook cannot attain the same minimum outage probability as the case where all nodes in the AF relay networks have access to perfect channel state information (CSI). We propose variable-length quantizers with random infinite-cardinality codebooks for the sum and individual power constraints. We provide theoretical proofs and numerical simulations to validate that the proposed quantizers can achieve the full-CSI outage probabilities with finite average feedback rates.

Index Terms

Amplify-and-forward, variable-length quantizer, outage probability, feedback rate

I. Introduction

Cooperative diversity techniques have received significant attention since they can greatly enhance the spectral efficiency and extend the network coverage [1], [2]. In a wireless relay

This work was presented in part at the IEEE Global Communications Conference (GLOBECOM), December 2015. This work was supported in part by NSF Award CCF–1218771.

X. Liu, H. Jafarkhani and E. Koyuncu are with Center for Pervasive Communications & Computing, University of California, Irvine, Irvine, CA 92697–2625 USA (Email: {xiaoyil3, hamidj, ekoyuncu}@uci.edu).

network, the destination node receives signals from the source node with the help of relay nodes in the form of "distributed antennas." Several cooperation strategies, such as amplify-and-forward (AF), decode-and-forward, and compress-and-forward have been proposed in the literature. Among these, AF is an attractive solution with very low complexity that requires no decoding at relay nodes.

In the case of point-to-point wireless communication, the performance of the system depends on the availability of channel state information (CSI) at the transmitter and the design of the corresponding finite-rate feedback [3]–[5]. Similarly, the performance of wireless relay networks depends on the availability of CSI at the relay nodes and the destination node [6]–[8]. The destination node can acquire the entire CSI through training sequences from the source node and relay nodes. Meanwhile, although each relay node can have the knowledge of its own receiving channel via training sequences from the source node, it does not have a direct access to the channel from itself to the destination node. Thus, the relay nodes rely on the feedback information from the destination node [9]. Perfect CSI at the relay nodes requires an "infinite" number of feedback bits from the destination node, which is unrealistic due to the limitations of the feedback links. Hence, in practice, it is desired to design efficient transmission schemes based on quantized CSI for wireless relay networks.

There has been a lot of work on quantized channel feedback in wireless relay networks. In a cooperative network with a single AF relay in [6], power control methods have been analyzed to minimize the outage probability with limited feedback available at the transmitter. When the cooperative network has multiple relays, it is shown in [8], [9] that using relay beamforming achieves the full-CSI performance. Relay selection is possible to achieve the maximum diversity. However, it incurs a performance loss in array gain inevitably compared to relay beamforming in the full-CSI systems [10]. Moreover, relay beamforming based on quantized feedback from the receiver can be implemented in a distributed manner without complex coordination between relays. With the index fed back from the receiver, each relay can select the corresponding relay beamforming vector from the pre-defined codebook. Therefore, we only consider the channel

quantizers using relay beamforming in this paper. In a cooperative network with multiple AF relays, the capacity loss and bit error probability with quantized feedback have been studied in [7], when each relay node is subject to an individual power constraint. Also, [8] has investigated the optimal beamforming vector for relay nodes in the full-CSI scenario and the outage probability in the limited feedback scenario when the sum power constraint is imposed on the relay nodes. Compared to the full-CSI scenario where all relay nodes know the perfect CSI, the schemes in [7] and [8] always suffer from performance loss.

All of these previous schemes have relied on fixed-length quantizers (FLQs), in which the receiver feeds back the same number of bits for every channel state. In general, the receiver can send a different number of feedback bits for different channel states, resulting in a variable-length quantizer (VLQ). Recently, a VLQ has been proposed to achieve the full-CSI outage probability with a finite feedback rate for the non-cooperative setting of a multiple-input single-output (MISO) system [11]. One can thus expect that a VLQ structure will similarly offer high performance gains in cooperative networks. On the other hand, the results of [11] for MISO systems are not directly applicable to the VLQ design problem in AF relay networks due to the following reasons: (i) In such AF relay networks, the relay nodes are geographically apart from each other, which, unlike the co-located transmit antennas in a MISO system, prevents direct access to the CSI of others. (ii) The amplification of both signal and noise from the first hop brings in a highly-nonlinear dependence on the relay beamforming vector and the channel values to the instantaneous signal-to-noise ratio (SNR). However, in a MISO system, the SNR is simply given by the inner product of the beamforming and channel vectors. (iii) Both the sum and individual power constraints are considered for the AF relay networks. As shown in [9], the individual power constraint causes severe non-convexity to the SNR optimization, which further hampers the limited feedback design. Therefore, the distributed nature of the AF relay networks and the highly complicated SNR expressions result in great difficulties in the design and performance analysis of VLQs.

We overcome these difficulties by considering random quantizer codebooks instead of the struc-

tured codebooks presented in [11]. We also provide a framework for analyzing the performance of random codebooks using limited feedback in AF relay networks, and the derivations can be applied to many other scenarios with AF relays. We first prove that the outage probabilities of our proposed VLQs are the same as those of the full-CSI scenarios in the sum and individual power constraints, respectively. Then, for the average feedback rate of the proposed VLQ under the sum power constraint, we derive its upper bound to show it is finite. For the average feedback rate of the proposed VLQ under the individual power constraint, we are unable to theoretically prove it is finite due to the complicated SNR expression. Instead, we perform numerical simulations to verify it is finite and small.

The rest of this paper is organized as follows. The system model and problem formulation are described in Section II. In Section III, we first propose a VLQ with an infinite-cardinality random codebook for the sum power constraint, then, we prove the proposed VLQ achieves the same minimum outage probability as the full-CSI scenario does, and provide an upper bound on the average feedback rate. We deal with the VLQ for the individual power constraint in Section IV. Conclusions are drawn in Section V. We also provide some technical proofs in the appendices.

Notation: Bold-face letters refer to vectors or matrices. For a vector or matrix $\boldsymbol{x}$, $\boldsymbol{x}^{\top}$ represents its transpose, $\boldsymbol{x}^{\dagger}$ represents its conjugate transpose, $\|\boldsymbol{x}\|$ is the l^2-norm, and $[\boldsymbol{x}]_i$ denotes its i-th element. The sets of complex, real, and natural numbers are denoted by C, R, and N, respectively. The probability and expectation are represented by $\Pr\{\cdot\}$ and $\mathtt{E}[\cdot]$, respectively. We use the notation $\mathbb{CN}(\boldsymbol{a},\boldsymbol{b})$ to stand for a circularly-symmetric complex Gaussian random vector with mean of $\boldsymbol{a}$ and variance of $\boldsymbol{b}$. Similarly, $\mathbb{N}(\boldsymbol{a},\boldsymbol{b})$ is for a real Gaussian random vector. For any $x \in \mathsf{R}$, $\lfloor x \rfloor$ is the largest integer that is less than or equal to x and $\lceil x \rceil$ is the smallest integer that is larger than or equal to x. For any $x \in \mathsf{C}$, x^* is the conjugate, $\mathrm{Real}(x)$ is the real part, $\mathrm{Imag}(x)$ is the imaginary part, $|x| = \sqrt{[\mathrm{Real}(x)]^2 + [\mathrm{Imag}(x)]^2}$ is the absolute value and $\arg(x) = \arctan\left(\frac{\mathrm{Imag}(x)}{\mathrm{Real}(x)}\right)$ is the argument. For a logical statement ST, we let $\mathbf{1}\{\mathrm{ST}\} = 1$ when ST is true, and $\mathbf{1}\{\mathrm{ST}\} = 0$ otherwise. The column vector formed by stacking two column vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ together is

denoted as $[\boldsymbol{x}_1; \boldsymbol{x}_2]$. Finally, `rand()` returns a single uniformly distributed random number in the interval $(0, 1]$.

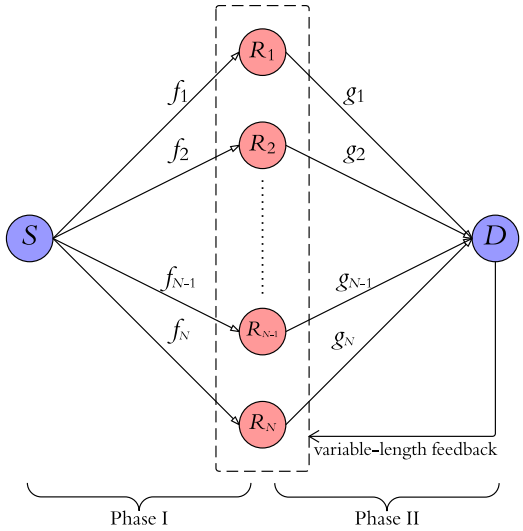

Fig. 1: System block diagram.

II. System Model and Problem Formulation

In the AF relay network depicted in Fig. 1, a source node S transmits to a destination node D with the aid of N AF relay nodes $R_1, \ldots, R_N$, where $N \geq 2$. Each node is equipped with only a single antenna. Assume that there is no direct link between S and D. Denote the channels from S to R_n and R_n to D by $f_n \sim \mathbb{CN}\left(0, \sigma_{f_n}^2\right)$ and $g_n \sim \mathbb{CN}\left(0, \sigma_{g_n}^2\right)$, respectively. Without loss of generality, we assume $\sigma_{g_1}^2 \leq \sigma_{g_2}^2 \leq \ldots \leq \sigma_{g_N}^2$. The entire channel state is represented by $\boldsymbol{H} = [f_1, \ldots, f_N, g_1, \ldots, g_N]^\top \in \mathbb{C}^{2N \times 1}$. We assume a quasi-static channel model, in which the channels vary independently from one block to another, while remain constant within each block.

In Phase I, the received signal at the n-th relay node R_n is

$$y_{R_n} = \sqrt{P_S}\, f_n x + v_{R_n},$$

where x is the information bearing symbol sent by S with $\mathbb{E}\left[|x|^2\right] = 1$ for each channel state

(the expectation is over all transmitted symbols), and P_S is the average transmit power at S.[1] The background noise v_{R_n} for $n = 1, \ldots, N$ is independent and modeled as $\mathbb{CN}(0, 1)$.

In Phase II, each relay node normalizes and retransmits its received signal y_{R_n}. The normalized signal to be re-transmitted with unit power at R_n is

$$x_{R_n} = \frac{y_{R_n}}{\sqrt{\mathrm{E}_{x, v_{R_n}}\left[\left|y_{R_n}\right|^2\right]}} = \frac{\sqrt{P_S}\, f_n x + v_{R_n}}{\sqrt{P_S\,|f_n|^2 + 1}}.$$

Thereafter, R_n sends $\sqrt{P_{R_n}} w_n^* x_{R_n}$, where P_{R_n} is the maximum transmit power at R_n and $P_{R_n}|w_n|^2$ is the actual-consumed transmit power. Without loss of generality, $P_S = P_{R_n} = P$ is assumed. Results for other values of P_S and P_{R_n} can be obtained similarly. The received signal at the destination node D is

$$\begin{aligned} y_D &= \sum_{n=1}^{N} g_n \sqrt{P} w_n^* x_{R_n} + v_D \\ &= \sum_{n=1}^{N} \frac{P w_n^* f_n g_n x}{\sqrt{P\,|f_n|^2 + 1}} + \sum_{n=1}^{N} \frac{\sqrt{P} w_n^* g_n v_{R_n}}{\sqrt{P\,|f_n|^2 + 1}} + v_D \\ &= \sqrt{P} \sum_{n=1}^{N} w_n^* \frac{f_n g_n}{\sqrt{|f_n|^2 + \frac{1}{P}}} x + \tilde{v}_D, \end{aligned} \tag{1}$$

where $\tilde{v}_D = \sum_{n=1}^{N} \frac{w_n^* g_n v_{R_n}}{\sqrt{|f_n|^2 + \frac{1}{P}}} + v_D$ and $v_D \sim \mathbb{CN}(0, 1)$ is the background noise at D. Given f_n and g_n, $\tilde{v}_D$ is distributed as $\tilde{v}_D \sim \mathbb{CN}\left(0, 1 + \sum_{n=1}^{N} |w_n|^2 \frac{|g_n|^2}{|f_n|^2 + \frac{1}{P}}\right)$. From Eq. (1), the signal-to-noise ratio (SNR) at D is given by

$$\Gamma(\boldsymbol{w}, \boldsymbol{H}) = P \frac{\left|\sum_{n=1}^{N} w_n^* \frac{f_n g_n}{\sqrt{|f_n|^2 + \frac{1}{P}}}\right|^2}{1 + \sum_{n=1}^{N} |w_n|^2 \frac{|g_n|^2}{|f_n|^2 + \frac{1}{P}}}, \tag{2}$$

where $\boldsymbol{w} = [w_1, \ldots, w_N]^\top$ is the relay beamforming vector.

[1]In the remainder of this paper, we refer to P as the transmit power instead of the average power over all transmitted symbols for conciseness.

A. Sum Power Constraint

Consider the sum power constraint for which the sum of the transmit power of all relay nodes is limited by P, i.e., w_n should satisfy $\sum_{n=1}^{N} |w_n|^2 = 1$, or $\|\boldsymbol{w}\|^2 = 1$ equivalently. The SNR expression of D in Eq. (2) can be reexpressed as

$$\Gamma(\boldsymbol{w}, \boldsymbol{H}) = P \frac{\boldsymbol{w}^\dagger \boldsymbol{h} \boldsymbol{h}^\dagger \boldsymbol{w}}{\boldsymbol{w}^\dagger (\boldsymbol{I} + \boldsymbol{D}) \boldsymbol{w}}, \tag{3}$$

where $\boldsymbol{h} = \left[\frac{f_1 g_1}{\sqrt{|f_1|^2 + \frac{1}{P}}}, \ldots, \frac{f_N g_N}{\sqrt{|f_N|^2 + \frac{1}{P}}} \right]^\top$, $\boldsymbol{I}$ is the $N \times N$ identity matrix and $\boldsymbol{D}$ is a $N \times N$ diagonal matrix with the n-th diagonal element being $\frac{|g_n|^2}{|f_n|^2 + \frac{1}{P}}$.

Consider outage probability as the performance measure throughout this paper. For a target data rate τ, outage occurs if $\frac{1}{2}\log_2(1 + \Gamma(\boldsymbol{w}, \boldsymbol{H})) < \tau$, or equivalently, $\Gamma(\boldsymbol{w}, \boldsymbol{H}) < 2^{2\tau} - 1 = \alpha$. In the rest of this paper, we refer to α as the outage threshold.

In the full-CSI scenario where all nodes are aware of a perfect knowledge of $\boldsymbol{H}$, the optimal beamforming vector $\boldsymbol{w}^\star_{\mathtt{SUM}}$ that maximizes $\Gamma(\boldsymbol{w}, \boldsymbol{H})$ is $\boldsymbol{w}^\star_{\mathtt{SUM}} = \frac{(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{h}}{\|(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{h}\|}$ [8], and the maximum SNR is

$$\Gamma(\boldsymbol{w}^\star_{\mathtt{SUM}}, \boldsymbol{H}) = P \sum_{n=1}^{N} \underbrace{\frac{|f_n|^2 |g_n|^2}{|f_n|^2 + |g_n|^2 + \frac{1}{P}}}_{=\Gamma_n}. \tag{4}$$

The minimum outage probability is then given as

$$\mathtt{Out}(\mathtt{Full}_{\mathtt{SUM}}) = \Pr\{\Gamma(\boldsymbol{w}^\star_{\mathtt{SUM}}, \boldsymbol{H}) < \alpha\} = \Pr\left\{\sum_{n=1}^{N} \Gamma_n < \frac{\alpha}{P}\right\} = \mathtt{E}_{\boldsymbol{H}}\mathbf{1}\left\{\sum_{n=1}^{N} \Gamma_n < \frac{\alpha}{P}\right\}. \tag{5}$$

In the limited-feedback scenario, assume the n-th relay node R_n only knows $|f_n|$ and the destination node D knows the entire channel state $\boldsymbol{H}$ [7], [8].[2] Define $\mathcal{W}_{\mathtt{SUM}} \triangleq \left\{\boldsymbol{w} : \boldsymbol{w} \in \mathsf{C}^{N\times 1}, \|\boldsymbol{w}\| = 1\right\}$. With an arbitrary quantizer $Q_{\mathtt{SUM}} : \mathsf{C}^{2N\times 1} \to \mathcal{W}_{\mathtt{SUM}}$, D maps $\boldsymbol{H}$ to some beamforming vector $Q_{\mathtt{SUM}}(\boldsymbol{H}) \in \mathcal{W}_{\mathtt{SUM}}$, then, feeds the index of $Q_{\mathtt{SUM}}(\boldsymbol{H})$ back to the relay nodes. The index of $Q_{\mathtt{SUM}}(\boldsymbol{H})$ is decoded at each relay node and $Q_{\mathtt{SUM}}(\boldsymbol{H})$ is recovered as the beamforming vector.

[2]One possible procedure of revealing the knowledge of $\boldsymbol{H}$ to the destination node D can be found in [7].

The resulting SNR is $\Gamma\left(Q_{\mathtt{SUM}}\left(\boldsymbol{H}\right),\boldsymbol{H}\right)$, and the corresponding outage probability is

$$\mathtt{Out}\left(Q_{\mathtt{SUM}}\right)=\Pr\left\{\Gamma\left(Q_{\mathtt{SUM}}\left(\boldsymbol{H}\right),\boldsymbol{H}\right)<\alpha\right\}.$$

B. Individual Power Constraint

Alternatively, we assume a maximum transmit power constraint P is imposed on each relay node. With the relay beamforming vector $\boldsymbol{\mu}=\left[\mu_1,\ldots,\mu_N\right]^{\top}$ (we use $\boldsymbol{\mu}$ to distinguish it from the notation $\boldsymbol{w}$ used for the sum power constraint), the power consumed at the n-th relay node R_n is $|\mu_n|P$, thus, $\boldsymbol{\mu}$ will be subject to $|\mu_n|\leq 1$ for $n=1,\ldots,N$. The optimal solution $\boldsymbol{\mu}^{\star}_{\mathtt{IND}}=\left[\mu_1^{\star},\ldots,\mu_N^{\star}\right]^{\top}$ that maximizes $\Gamma\left(\boldsymbol{\mu},\boldsymbol{H}\right)$ in Eq. (2) is given in [9, Theorem 1] as

$$\mu_n=\begin{cases}1, & n=\tau_1,\ldots,\tau_{i_0},\\ \lambda_{i_0}\phi_n, & n=\tau_{i_0+1},\ldots,\tau_N,\end{cases}\tag{6}$$

where $\phi_n=\frac{|f_n|}{|g_n|}\sqrt{|f_n|^2+\frac{1}{P}}$ for $n=1,\ldots,N$ and $\phi_{N+1}=0$; $(\tau_1,\ldots,\tau_N,\tau_{N+1})$ is an ordering of $(1,\ldots,N+1)$ satisfying $\phi_{\tau_1}\geq\phi_{\tau_2}\geq\cdots\geq\phi_{\tau_N}\geq\phi_{\tau_{N+1}}$ and $\tau_{N+1}=N+1$; $\lambda_i=\frac{1+\sum_{m=1}^{i}\frac{|g_{\tau_m}|^2}{|f_{\tau_m}|^2+\frac{1}{P}}}{\sum_{m=1}^{i}\frac{|f_{\tau_m}g_{\tau_m}|}{\sqrt{|f_{\tau_m}|^2+\frac{1}{P}}}}$; i_0 is the smallest i such that $\lambda_i<\phi^{-1}_{\tau_{i+1}}$. Thus, the minimum outage probability is

$$\mathtt{Out}\left(\mathtt{Full}_{\mathtt{IND}}\right)=\Pr\left\{\Gamma\left(\boldsymbol{\mu}^{\star}_{\mathtt{IND}},\boldsymbol{H}\right)<\alpha\right\}.\tag{7}$$

Define $\mathcal{U}_{\mathtt{IND}}\triangleq\left\{\boldsymbol{\mu}:\boldsymbol{\mu}\in\mathsf{C}^{N\times 1},|\mu_n|\leq 1,n=1,\ldots,N\right\}$. The relay beamforming vector selected by the quantizer $Q_{\mathtt{IND}}:\mathsf{C}^{2N\times 1}\rightarrow\mathcal{U}_{\mathtt{IND}}$ is $Q_{\mathtt{IND}}\left(\boldsymbol{H}\right)$, then, the achieved SNR is $\Gamma\left(Q_{\mathtt{IND}}\left(\boldsymbol{H}\right),\boldsymbol{H}\right)$ and the outage probability is $\mathtt{Out}\left(Q_{\mathtt{IND}}\right)=\Pr\left\{\Gamma\left(Q_{\mathtt{IND}}\left(\boldsymbol{H}\right),\boldsymbol{H}\right)<\alpha\right\}$.

In the subsequent sections, we will propose two VLQs respectively for the sum and individual power constraints, and show that the full-CSI outage probabilities $\mathtt{Out}\left(\mathtt{Full}_{\mathtt{SUM}}\right)$ in Eq. (5) and $\mathtt{Out}\left(\mathtt{Full}_{\mathtt{IND}}\right)$ in Eq. (7) can be achieved with finite average feedback rates.

III. Variable-Length Limited Feedback for the Sum Power Constraint

In this section, we first describe the proposed VLQ for the relay networks subject to the sum power constraint. Afterwards, we show the proposed VLQ can achieve the full-CSI outage

probaiblity $\mathtt{Out}(\mathtt{Full}_{\mathtt{SUM}})$ in Eq. (5) with a finite average feedback rate both theoretically and numerically.

A. Proposed VLQ

For any given $\boldsymbol{H}$, we propose a VLQ using the random codebook $\{\boldsymbol{w}_i\}_{\mathsf{N}}$, where $\boldsymbol{w}_i \in \mathcal{W}_{\mathtt{SUM}}$ is independent and identically distributed with a uniform distribution on $\mathcal{W}_{\mathtt{SUM}}$ for $i \in \mathsf{N}$ [12]. The random codebook provides a performance benchmark since if certain average performance is attained, one deterministic codebook can be found to surpass this average performance. Given $\{\boldsymbol{w}_i\}_{\mathsf{N}}$, the proposed VLQ is represented by

$$\mathrm{VLQ}_{\mathtt{SUM}} = \{\boldsymbol{w}_i, \mathcal{S}_i, \mathsf{b}_i\}, \tag{8}$$

where $\mathcal{S}_i$ denotes the channel partition region of $\boldsymbol{w}_i$ for $i \in \mathsf{N}$, $\boldsymbol{w}_i$ is the adopted relay beamforming vector when $\boldsymbol{H} \in \mathcal{S}_i$, and b_i is the binary feedback string representing the index of $\boldsymbol{w}_i$.

Different from the channel partition regions in FLQs which consist of channel states that achieve the best performance with the centroid codeword, the channel partition regions in $\mathrm{VLQ}_{\mathtt{SUM}}$ are set as

$$\mathcal{S}_i = \begin{cases} \{\boldsymbol{H} : \Gamma(\boldsymbol{w}_0, \boldsymbol{H}) \geq \alpha\} \cup \bigcap_{i \in \mathsf{N}} \{\boldsymbol{H} : \Gamma(\boldsymbol{w}_i, \boldsymbol{H}) < \alpha\}, & i = 0, \\ \{\boldsymbol{H} : \Gamma(\boldsymbol{w}_i, \boldsymbol{H}) \geq \alpha\} \cap \bigcap_{k=0}^{i-1} \{\boldsymbol{H} : \Gamma(\boldsymbol{w}_k, \boldsymbol{H}) < \alpha\}, & i \in \mathsf{N} - \{0\}. \end{cases} \tag{9}$$

For $i \in \mathsf{N}$, $\{\boldsymbol{H} : \Gamma(\boldsymbol{w}_i, \boldsymbol{H}) \geq \alpha\}$ is the set of channels that are in non-outage when $\boldsymbol{w}_i$ is the beamforming vector; $\{\boldsymbol{H} : \Gamma(\boldsymbol{w}_i, \boldsymbol{H}) < \alpha\}$ is its complementary set. For any $\boldsymbol{H}$ with $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) < \alpha$, all beamforming vectors lead to outage, then, $\mathrm{VLQ}_{\mathtt{SUM}}$ naively chooses $\boldsymbol{w}_0$ as the beamforming vector; for any $\boldsymbol{H}$ with $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) \geq \alpha$, $\mathrm{VLQ}_{\mathtt{SUM}}$ examines each beamforming vector in $\{\boldsymbol{w}_i\}_{\mathsf{N}}$ sequentially until it finds some $\boldsymbol{w}_i$ satisfying $\Gamma(\boldsymbol{w}_i, \boldsymbol{H}) \geq \alpha$. In terms of outage probability, the contribution of such $\boldsymbol{w}_i$ is identical to that of the optimal beamforming vector $\boldsymbol{w}^{\star}_{\mathtt{SUM}}$.

Variable-length coding is applied to encode the indices of $\boldsymbol{w}_i$ for $i \in \mathsf{N}$. Concretely, we let $\mathsf{b}_0 = \{0\}$, $\mathsf{b}_1 = \{1\}$, $\mathsf{b}_2 = \{00\}$, $\mathsf{b}_3 = \{01\}$ and so on for all binary strings in the set

$\{0, 1, 00, 01, 10, 11, \ldots\}$.[3] The length of $\mathtt{b}_i$ is $\lfloor \log_2(i+2) \rfloor$.

Based on the random codebook $\{\boldsymbol{w}_i\}_{\mathsf{N}}$, the outage probability and average feedback rate of the proposed quantizer $\mathrm{VLQ}_{\mathtt{SUM}}$ are

$$\mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right) = \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \Pr\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\} = \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right], \tag{10}$$

$$\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right) = \sum_{i=0}^{\infty} \lfloor \log_2(i+2) \rfloor \times \Pr\left\{\boldsymbol{H} \in \mathcal{S}_i\right\} = \sum_{i=0}^{\infty} \lfloor \log_2(i+2) \rfloor \times \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right]. \tag{11}$$

B. Outage Optimality

Theorem 1 states that the outage probability of our proposed quantizer $\mathrm{VLQ}_{\mathtt{SUM}}$ is the same as the full-CSI outage probability in Eq. (5). The proof of the theorem can be found in Appendix A.

Theorem 1. *For any $P > 0$, we have*

$$\mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right) = \mathtt{Out}\left(\mathtt{Full}_{\mathtt{SUM}}\right). \tag{12}$$

In the following, we provide an intuitive explanation of the result in Theorem 1. For a given $\boldsymbol{H}$ with $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) > \alpha$, to achieve the same non-outage performance as the optimal beamforming vector $\boldsymbol{w}^{\star}_{\mathtt{SUM}}$, one should use a unit-normal vector $\boldsymbol{w} \in \mathcal{W}_{\mathtt{SUM}}$ that is "close" enough to $\boldsymbol{w}^{\star}_{\mathtt{SUM}}$ such that $\Gamma\left(\boldsymbol{w}, \boldsymbol{H}\right) \geq \alpha$. We show that there exists a non-zero probability region in the unit sphere where all the unit-normal vectors result in non-outage. However, to "closely" represent $\boldsymbol{w}^{\star}_{\mathtt{SUM}}$ for any such $\boldsymbol{H}$, we need infinitely many beamforming vectors in the codebook $\{\boldsymbol{w}_i\}_{\mathsf{N}}$ to capture at least one in that non-outage region. Obviously, a FLQ with a finite feedback rate will not succeed. Whereas our VLQ proposed in Eq. (8) includes infinitely many beamforming vectors to achieve the full-CSI outage probability while perserves a finite average feedback rate.

[3]The proposed VLQ in Eq. (8) can be extended to the case of prefix-free codes. In other words, there is a prefix-free code for every quantizer designed in this paper. Suppose $\{\boldsymbol{w}_i\}_{\mathsf{N}}$ is a fixed-structured infinite-cardinality codebook whose performance is no worse than that of random codebooks. Therefore, it can achieve the full-CSI outage probability for the relay newtork. Let the codeword length of $\boldsymbol{w}_i$ be $l_i \triangleq \lceil 2\log_2(i+1)+1 \rceil$ for $i \in \mathsf{N}$ [13, Example 1]. It is straightforward to show that $\sum_{i \in \mathsf{N}} 2^{-l_i} \leq 1$. According to the Kraft's inequality, this code is prefix-free. Moreover, since $l_i = \lceil 2\log_2(i+1)+1 \rceil \leq 2\log_2(i+1)+2$, the average feedback rate of this code is also finite following the same derivations in the proof of Theorem 2.

C. Average Feedback Rate

Theorem 2 provides an upper bound on the average feedback rate of VLQ_{SUM}, the proof of which is presented in Appendix B.

Theorem 2. *For any $P > 0$, we have*

$$\text{FR}\left(\text{VLQ}_{\text{SUM}}\right) \leq C_0 + C_1 e^{-\frac{\alpha}{P \times \sigma_{g_N}}} \left[\frac{1}{P} + \frac{1}{P^N}\right]\left[1 + \left(\frac{\alpha}{P}\right)^N\right], \tag{13}$$

where $C_0, C_1 > 0$ are constants that are independent of α and P.

Since $e^{-\frac{\alpha}{P \times \sigma_{g_N}}} \left[\frac{1}{P} + \frac{1}{P^N}\right]\left[1 + \left(\frac{\alpha}{P}\right)^N\right]$ in Eq. (13) is bounded for any outage threshold $\alpha > 0$ and any transmit power $P > 0$, the average feedback rate of VLQ_{SUM} is finite. As shown in the numerical simulations, the average feedback rate can actually be very small.

D. Numerical Simulations

We provide numerical simulations of the outage probability and the average feedback rate of VLQ_{SUM}. We let $\alpha = 1$, and $\left(\sigma_{f_1}^2, \sigma_{f_2}^2\right) = (1, 0.8), \left(\sigma_{g_1}^2, \sigma_{g_2}^2\right) = (0.7, 0.9)$ for two relays; $\left(\sigma_{f_1}^2, \sigma_{f_2}^2, \sigma_{f_3}^2\right) = (1, 0.8, 0.6), \left(\sigma_{g_1}^2, \sigma_{g_2}^2, \sigma_{g_3}^2\right) = (0.5, 0.7, 0.9)$ for three relays; and $\left(\sigma_{f_1}^2, \sigma_{f_2}^2, \sigma_{f_3}^2, \sigma_{f_4}^2\right) = (1, 0.8, 0.6, 0.4), \left(\sigma_{g_1}^2, \sigma_{g_2}^2, \sigma_{g_3}^2, \sigma_{g_4}^2\right) = (0.3, 0.5, 0.7, 0.9)$ for four relays. Other values of α and channel variances will show similar simulation results. For each value of the transmit power P, a sufficiently large number of channel realizations are generated such that at least $1,000$ outage events can be observed. For each channel state realization with non-outage in the full-CSI case, a random relay beamforming vector $\boldsymbol{w} \in \mathcal{W}_{\text{SUM}}$ is generated repeatedly until one that makes for non-outage is found. With such simulation settings, the average feedback rate is computed as the average number of feedback bits, and the simulated outage probability is the number of outage incidents divided by the number of all channel state realizations. No endless iteration has occurred when $\boldsymbol{w}$ is generated in any channel state realization.

In Fig. 2, when $N = 2, 3$ or 4, the simulated average feedback rate is no larger than 3 bits per channel state for any P. In Fig. 3, we compare the outage probabilities of VLQ_{SUM} and the FLQ

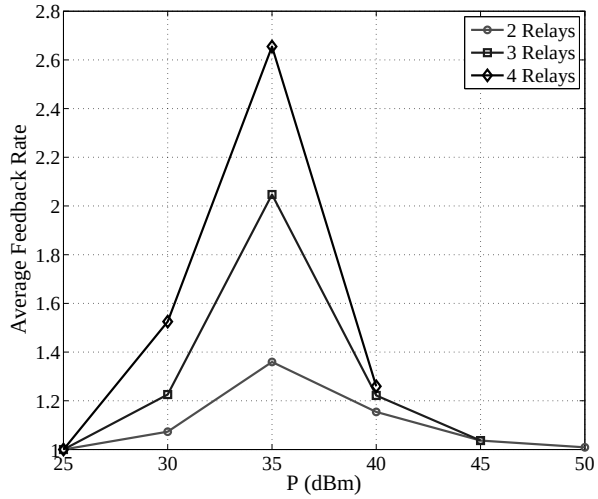

Fig. 2: Simulated average feedback rates of $\mathrm{VLQ}_{\mathtt{SUM}}$.

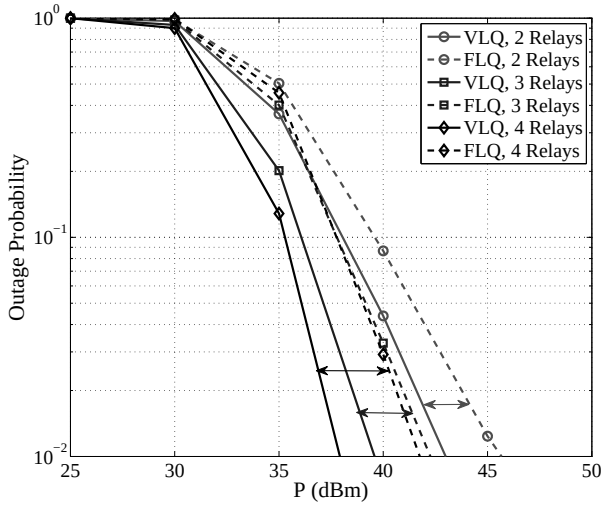

Fig. 3: Simulated outage probabilities of $\mathrm{VLQ}_{\mathtt{SUM}}$ and $\mathrm{FLQ}_{\mathtt{SUM}}$.

in [8] denoted by $\mathrm{FLQ}_{\mathtt{SUM}}$.[4] Given $\boldsymbol{H}$ and the random codebook $\{\boldsymbol{w}_i\}_{i=0,\ldots,2^B-1}$ where $\boldsymbol{w}_i \in \mathcal{W}_{\mathtt{SUM}}$, $\mathrm{FLQ}_{\mathtt{SUM}}$ chooses the relay beamforming vector as $\mathrm{FLQ}_{\mathtt{SUM}}(\boldsymbol{H}) = \mathrm{argmax}_{\boldsymbol{w}\in\{\boldsymbol{w}_i\}_{i=0,\ldots,2^B-1}} \Gamma(\boldsymbol{w}, \boldsymbol{H})$, thus, the feedback rate of $\mathrm{FLQ}_{\mathtt{SUM}}$ is B bits per channel state. We let $B = 2, 3, 3$ for $N = 2, 3, 4$,

[4]Theorem 1 has shown $\mathrm{VLQ}_{\mathtt{SUM}}$ achieves the full-CSI outage probability in Eq. (5). Hence, the simulated outage probability of $\mathrm{VLQ}_{\mathtt{SUM}}$ in Fig. 3 is also the simulated full-CSI outage probability.

respectively. These values of B are close to (but still larger than) the average feedback rates of $\text{VLQ}_{\texttt{SUM}}$ with the same relay network configurations in Fig. 2. Therefore, $\text{VLQ}_{\texttt{SUM}}$ shows great improvement in outage probability as compared to $\text{FLQ}_{\texttt{SUM}}$.

IV. Variable-Length Limited Feedback for the Individual Power Constraint

In this section, we propose a VLQ design for the relay network subject to the individual power constraint and prove it can attain the optimal outage probability in Eq. (7). Due to the intractable theoretical analysis on the average feedback rate of the proposed VLQ, numerical simulations are presented to show it is finite.

A. Proposed VLQ

For any given $\boldsymbol{H}$, the relay beamforming vector $\boldsymbol{\mu}_i = [\mu_{i,1}, \ldots, \mu_{i,N}]^\top \in \mathcal{U}_{\texttt{IND}}$ in the random codebook $\{\boldsymbol{\mu}_i\}_{\mathsf{N}}$ is constructed by

$$\begin{aligned} \mu_{i,n} &= \left|\mu_{i,n}\right| e^{j \arg(\mu_{i,n})}, \\ \left|\mu_{i,n}\right| &= \texttt{rand}(), \\ \arg(\mu_{i,n}) &= 2\pi \times \texttt{rand}(). \end{aligned} \tag{14}$$

The proposed VLQ for the individual power constraint is represented by

$$\text{VLQ}_{\texttt{IND}} \quad = \{\boldsymbol{\mu}_i, \mathcal{P}_i, \mathsf{d}_i\}, \tag{15}$$

where $\boldsymbol{\mu}_i$ is the assigned relay beamforming vector when $\boldsymbol{H}$ falls in the channel partition region $\mathcal{P}_i$, and d_i is the binary representation for the index of $\boldsymbol{\mu}_i$. Similar to Eq. (9), the channel partition region $\mathcal{P}_i$ is given by

$$\mathcal{P}_i = \begin{cases} \{\boldsymbol{H} : \Gamma(\boldsymbol{\mu}_0, \boldsymbol{H}) \geq \alpha\} \cup \bigcap_{i \in \mathsf{N}} \{\boldsymbol{H} : \Gamma(\boldsymbol{\mu}_i, \boldsymbol{H}) < \alpha\}, & i = 0, \\ \{\boldsymbol{H} : \Gamma(\boldsymbol{\mu}_i, \boldsymbol{H}) \geq \alpha\} \cap \bigcap_{k=0}^{i-1} \{\boldsymbol{H} : \Gamma(\boldsymbol{\mu}_k, \boldsymbol{H}) < \alpha\}, & i \in \mathsf{N} - \{0\}. \end{cases} \tag{16}$$

The design for $\mathtt{d}_i$ can also be inherited from that for $\mathtt{b}_i$ in $\mathrm{VLQ}_{\mathtt{SUM}}$, thus, the length of $\mathtt{d}_i$ is $\lfloor \log_2(i+2) \rfloor$. The key difference between $\mathrm{VLQ}_{\mathtt{IND}}$ and $\mathrm{VLQ}_{\mathtt{SUM}}$ lies in the construction of the beamforming vectors in the random codebook.

With $\{\boldsymbol{\mu}_i\}_{\mathtt{N}}$, the outage probability and average feedback rate of $\mathrm{VLQ}_{\mathtt{IND}}$ are

$$\mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{IND}}\right) = \mathtt{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}} \Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathtt{N}\right\} = \mathtt{E}_{\boldsymbol{H}} \mathtt{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}} \left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathtt{N}\right\}\right], \tag{17}$$

$$\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{IND}}\right) = \sum_{i=0}^{\infty} \lfloor \log_2(i+2) \rfloor \times \Pr\left\{\boldsymbol{H} \in \mathcal{P}_i\right\} = \sum_{i=0}^{\infty} \lfloor \log_2(i+2) \rfloor \times \mathtt{E}_{\boldsymbol{H}} \mathtt{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{P}_i\right\}\right]. \tag{18}$$

B. Outage Optimality and Average Feedback Rate

The following theorem shows that in the relay network with the individual power constraint, our proposed VLQ achieves the full-CSI outage probability in Eq. (7). The proof of the theorem is provided in Appendix C.

Theorem 3. *For any $P > 0$, we have*

$$\mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{IND}}\right) = \mathtt{Out}\left(\mathtt{Full}_{\mathtt{IND}}\right). \tag{19}$$

Due to the highly complicated expression of $\boldsymbol{\mu}^{\star}_{\mathtt{IND}}$ in Eq. (6) which hinders from further tractable analysis, we are unable to provide a closed-form upper bound on the average feedback rate $\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{IND}}\right)$ to theoretically prove its finity. However, we can still perform numerical simulations to verify this, i.e., Fig. 4 shows the average feedback rate will be finite under different simulation parameters and network configurations.[5]

In Fig. 5, we also compare the outage probabilities of $\mathrm{VLQ}_{\mathtt{IND}}$ and the FLQ in [7, Section V] denoted by $\mathrm{FLQ}_{\mathtt{IND}}$. The feedback rates of $\mathrm{FLQ}_{\mathtt{IND}}$ are chosen as $B = 2, 3, 4$ bits per channel state for $N = 2, 3, 4$, respectively. Although the average feedback rate of $\mathrm{VLQ}_{\mathtt{IND}}$ is smaller than that of $\mathrm{FLQ}_{\mathtt{IND}}$ with the same network configuration, $\mathrm{VLQ}_{\mathtt{IND}}$ has obtained much smaller outage probability compared to $\mathrm{FLQ}_{\mathtt{IND}}$.

[5]We use the same parameters for channel variances and $\alpha = 1$ here as in Section III-D.

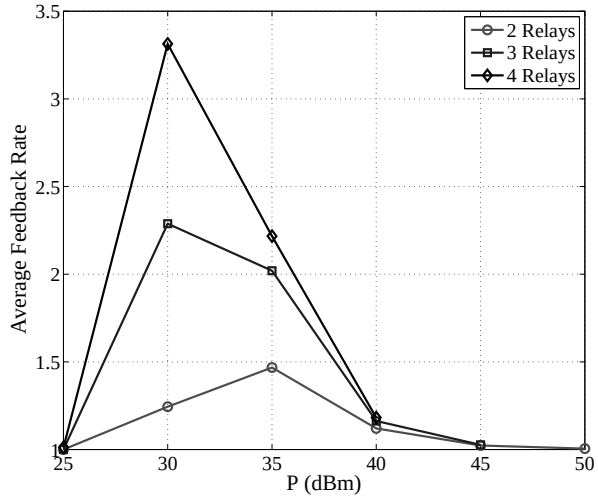

Fig. 4: Simulated average feedback rates of $\mathrm{VLQ}_{\mathtt{IND}}$.

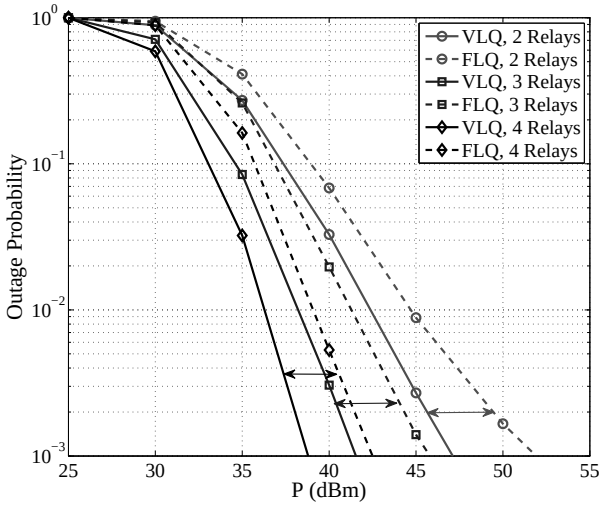

Fig. 5: Simulated outage probabilities of $\mathrm{VLQ}_{\mathtt{IND}}$ and $\mathrm{FLQ}_{\mathtt{IND}}$.

V. Conclusions

In this paper, we have proposed VLQs for the AF relay networks respectively subject to the sum and individual power constraints, and showed the proposed VLQs can achieve the full-CSI outage probabilities with finite average feedback rates. In the future, we intend to work on the

VLQ design for the multi-user relay networks with the sum or individual power constraint, and the goal is still to approach the full-CSI outage probability with a finite average feedback rate.

Appendix A: Proof of Theorem 1

Before presenting the detailed proof, let us summarize the main idea behind the proof first. Based on Eqs. (5) and (10), to prove $\mathtt{Out}\left(\mathtt{Full}_{\mathtt{SUM}}\right) = \mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right)$, it is equivalent to show:

1) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha$,

$$\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} = \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] = 1; \tag{20}$$

2) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) = \alpha$,

$$\begin{aligned}&\int_{\boldsymbol{H}\in\left\{\boldsymbol{H}\in\mathsf{C}^{2N\times 1}:\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star},\boldsymbol{H}\right)=\alpha\right\}} \mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\ &= \int_{\boldsymbol{H}\in\left\{\boldsymbol{H}\in\mathsf{C}^{2N\times 1}:\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star},\boldsymbol{H}\right)=\alpha\right\}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = 0,\end{aligned} \tag{21}$$

where $f_{\boldsymbol{H}}(\boldsymbol{H})$ is the probability density function (pdf) of $\boldsymbol{H}$;

3) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) > \alpha$,

$$\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} = \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] = 0. \tag{22}$$

For convenience, we define

$$\begin{aligned}&\mathcal{H} = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha\right\}, \quad \mathcal{H}'\left(\{\boldsymbol{w}_i\}_{\mathsf{N}}\right) = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}, \\ &\mathcal{H}'' = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) = \alpha\right\}, \quad \mathcal{H}''' = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) > \alpha\right\}.\end{aligned} \tag{23}$$

We omit the dependency of $\mathcal{H}'\left(\{\boldsymbol{w}_i\}_{\mathsf{N}}\right)$ on the realization of the random codebook $\{\boldsymbol{w}_i\}_{\mathsf{N}}$ and use $\mathcal{H}'$ for brevity.

Firstly, to prove Eq. (20), it is sufficient to show that $\mathbf{1}\{\boldsymbol{H} \in \mathcal{H}\} = \mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\} = 1$ for any $\boldsymbol{H} \in \mathcal{H}$ and $\{\boldsymbol{w}_i\}_{\mathsf{N}}$. Since $\boldsymbol{H} \in \mathcal{H}$, we have $\mathbf{1}\{\boldsymbol{H} \in \mathcal{H}\} = 1$ and $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha$. By the optimality of $\boldsymbol{w}_{\mathtt{SUM}}^{\star}$, $\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) \leq \Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) < \alpha$ for $i \in \mathsf{N}$, then, $\boldsymbol{H} \in \mathcal{H}'$ and $\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\} = 1$. Thus, $\mathbf{1}\{\boldsymbol{H} \in \mathcal{H}\} = \mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\} = 1$ stands for any $\boldsymbol{H} \in \mathcal{H}$ and $\{\boldsymbol{w}_i\}_{\mathsf{N}}$.

Secondly, to prove Eq. (21), it is equivalent to show that

$$\int_{\boldsymbol{H}\in\mathcal{H}''} \mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}\right\} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = \int_{\boldsymbol{H}\in\mathcal{H}''} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}'\right\}\right] f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = 0. \tag{24}$$

Since $\mathcal{H}\cap\mathcal{H}'' = \emptyset$, $\int_{\boldsymbol{H}\in\mathcal{H}''} \mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}\right\} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = 0$. Besides, we have

$$0 \le \int_{\boldsymbol{H}\in\mathcal{H}''} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}'\right\}\right] f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \le \int_{\boldsymbol{H}\in\mathcal{H}''} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = \Pr\left\{\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) = \alpha\right\} = 0,$$

due to the fact that the probability of a continuous random variable assuming a specific value is zero. Thus, $\int_{\boldsymbol{H}\in\mathcal{H}''} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}'\right\}\right] f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = 0$, and Eq. (24) holds.

Lastly, to prove Eq. (22), we will show $\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}\right\} = \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}'\right\}\right] = 0$ for $\boldsymbol{H}\in\mathcal{H}'''$ and given $\{\boldsymbol{w}_i\}_{\mathsf{N}}$. Since $\mathcal{H}\cap\mathcal{H}''' = \emptyset$, $\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}\right\} = 0$ for $\boldsymbol{H}$ in $\mathcal{H}'''$. To prove $\mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H}\in\mathcal{H}'\right\}\right] = 0$ for $\boldsymbol{H}\in\mathcal{H}'''$, by contradiction, assume $\exists\, \tilde{\boldsymbol{H}}\in\mathcal{H}'''$, s.t. $\mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}}\in\mathcal{H}'\right\}\right] = \varepsilon > 0$, then,

$$\begin{aligned}\mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}}\in\mathcal{H}'\right\}\right] &= \Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) < \alpha, \forall i\in\mathsf{N}\right\} \\ &\le \Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) < \alpha, \forall 0\le i\le K-1\right\} \overset{(\sharp)}{=} \left[\Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) < \alpha\right\}\right]^{K}, \end{aligned} \tag{25}$$

where $K\ge 1$ is an arbitrary finite natural number. The equality $(\sharp)$ holds because $\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right)$ for $i = 1,\ldots,K$ are mutually independent due to the independence of $\boldsymbol{w}_i$ for $i = 0,\ldots,K-1$ and given $\tilde{\boldsymbol{H}}$. To proceed, we need the following two lemmas, the proofs of which are in Appendix D.

Lemma 1. *If* $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) > \alpha$, *there exists* $\Pi\in(0,1)$ *such that for any* $\boldsymbol{w}\in\mathcal{W}_{\mathtt{SUM}}$ *with* $\left\|\boldsymbol{w}-\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right\|\le\Pi$, $\Gamma\left(\boldsymbol{w},\boldsymbol{H}\right)\ge\alpha$ *holds. The value of* Π *can be*

$$\Pi = \frac{\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) - \alpha}{2\sqrt{N}P\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)}. \tag{26}$$

Lemma 2. *Let* $\mathcal{W}_{\mathtt{R}} \triangleq \left\{\boldsymbol{w}_{\mathtt{R}} : \boldsymbol{w}_{\mathtt{R}}\in\mathsf{R}^{2N\times 1}, \|\boldsymbol{w}_{\mathtt{R}}\| = 1\right\}$. *For a fixed real vector* $\boldsymbol{u}\in\mathcal{W}_{\mathtt{R}}$, *a real number* $0\le t\le 1$, *and a random real vector* $\boldsymbol{v}$ *which is uniformly distributed on the real unit sphere*

$\mathcal{W}_{\mathrm{R}}$, we have

$$Pr\{\boldsymbol{u}^{\top}\boldsymbol{v} \geq t\} = \frac{1}{2}I_{1-t^2}\left(\frac{2N-1}{2}, \frac{1}{2}\right),$$

where $I_z(a,b) = \frac{1}{\beta(a,b)}\int_0^z x^{a-1}(1-x)^{b-1}\mathrm{d}x$ is the regularized incomplete beta function, $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$ is the beta function [14].

Using Lemma 1, for any given $\tilde{\boldsymbol{H}}$, we obtain

$$\Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) \geq \alpha\right\} \geq \Pr\left\{\left\|\boldsymbol{w} - \boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\| \leq \Pi\right\} = \Pr\left\{\mathrm{Real}\left\{\boldsymbol{w}^{\dagger}\boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\} \geq 1 - \frac{\Pi^2}{2}\right\}. \tag{27}$$

Note that $\mathrm{Real}\left\{\boldsymbol{w}^{\dagger}\boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\} = \left[\mathrm{Real}\left\{\boldsymbol{w}\right\}; \mathrm{Imag}\left\{\boldsymbol{w}\right\}\right]^{\top}\left[\mathrm{Real}\left\{\boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\}; \mathrm{Imag}\left\{\boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\}\right]$. Since $\boldsymbol{w}$ is uniformly distributed on the complex unit sphere $\mathcal{W}_{\mathtt{SUM}}$, $\left[\mathrm{Real}\left\{\boldsymbol{w}\right\}; \mathrm{Imag}\left\{\boldsymbol{w}\right\}\right] \in \mathsf{R}^{2N\times 1}$ is uniformly distributed on the unit real sphere $\mathcal{W}_{\mathrm{R}}$.[6] Using Lemma 2, it follows from Eq. (27) that

$$\begin{aligned}
\Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) < \alpha\right\} &= 1 - \Pr\left\{\Gamma\left(\boldsymbol{w}_i, \tilde{\boldsymbol{H}}\right) \geq \alpha\right\} \leq 1 - \Pr\left\{\mathrm{Real}\left\{\boldsymbol{w}^{\dagger}\boldsymbol{w}_{\mathtt{SUM}}^{\star}\right\} \geq 1 - \frac{\Pi^2}{2}\right\} \\
&= 1 - \frac{1}{2}I_{1-\left(1-\frac{\Pi^2}{2}\right)^2}\left(\frac{2N-1}{2}, \frac{1}{2}\right) = 1 - \frac{\int_0^{1-\left(1-\frac{\Pi^2}{2}\right)^2} x^{\frac{2N-3}{2}}\overbrace{(1-x)^{-\frac{1}{2}}}^{\geq 1}\mathrm{d}x}{2\times\beta\left(\frac{2N-1}{2}, \frac{1}{2}\right)} \\
&\leq 1 - \frac{\int_0^{1-\left(1-\frac{\Pi^2}{2}\right)^2} x^{\frac{2N-3}{2}}\mathrm{d}x}{2\times\beta\left(\frac{2N-1}{2}, \frac{1}{2}\right)} \leq 1 - \frac{\left(1-\left(1-\frac{\Pi^2}{2}\right)^2\right)^{\frac{2N-1}{2}}}{(2N-1)\times\beta\left(\frac{2N-1}{2}, \frac{1}{2}\right)} = \Phi < 1.
\end{aligned} \tag{28}$$

Letting $K = \lceil \log_{\Phi}\varepsilon\rceil + 1$ in Eq. (25), $\mathtt{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] \leq \Phi^K = \Phi^{\lceil\log_{\Phi}\varepsilon\rceil+1} < \Phi^{\log_{\Phi}\varepsilon} = \varepsilon$, which contradicts the assumption that $\mathtt{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] = \varepsilon$. Thus, $\mathtt{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\}\right] = 0$ for any $\boldsymbol{H} \in \mathcal{H}'''$, which completes the proof. ■

[6]It is known from [15] that $\boldsymbol{w}$ is generated by $\boldsymbol{w} = \frac{\boldsymbol{w}_{\mathsf{R}} + j*\boldsymbol{w}_{\mathsf{I}}}{\sqrt{\|\boldsymbol{w}_{\mathsf{R}}\|^2 + \|\boldsymbol{w}_{\mathsf{I}}\|^2}}$, where $\boldsymbol{w}_{\mathsf{R}}, \boldsymbol{w}_{\mathsf{I}} \in \mathbb{N}\left(\boldsymbol{0}_{N\times 1}, \frac{1}{2}\boldsymbol{I}_{N\times N}\right)$, and $\boldsymbol{w}_{\mathsf{R}}$ and $\boldsymbol{w}_{\mathsf{I}}$ are mutually independent. Thus, $\left[\mathrm{Real}\left\{\boldsymbol{w}\right\}; \mathrm{Imag}\left\{\boldsymbol{w}\right\}\right] = \frac{\left[\boldsymbol{w}_{\mathsf{R}}; \boldsymbol{w}_{\mathsf{I}}\right]}{\sqrt{\|\boldsymbol{w}_{\mathsf{R}}\|^2 + \|\boldsymbol{w}_{\mathsf{I}}\|^2}}$ is uniformly distributed on $\mathcal{W}_{\mathrm{R}}$.

Appendix B: Proof of Theorem 2

In this appendix, we will prove the upper bound on the average feedback rate in Theorem 2. Recall from Eq. (11) that $\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right) = \sum_{i=0}^{\infty} \lfloor \log_2(i+2) \rfloor \times \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right]$. Let $p = \Pr\left\{\Gamma\left(\boldsymbol{w}_i, \boldsymbol{H}\right) < \alpha\right\}$ for any given $\boldsymbol{H}$ and $\{\boldsymbol{w}_i\}_{\mathsf{N}}$. Based on Eq. (23) and the encoding rule in Eq. (9), for $i \geq 1$, we have $\mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right] = p^i(1-p)$ for $\boldsymbol{H} \in \mathcal{H}'' \cup \mathcal{H}'''$, and 0 for $\boldsymbol{H} \in \mathcal{H}$. Since $\lfloor \log_2(i+2) \rfloor \leq \log_2(i+2) \leq \log_2(2i+2) = 1 + \log_2(i+1)$, $\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right)$ is upper-bounded by

$$
\begin{aligned}
\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right) &\leq \sum_{i=0}^{\infty} \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right] + \sum_{i=0}^{\infty} \log_2(i+1) \times \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right] \\
&= 1 + \sum_{i=1}^{\infty} \log_2(i+1) \times \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{w}_i\}_{\mathsf{N}}} \left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{S}_i\right\}\right] \\
&= 1 + \int_{\hat{\mathcal{H}}} \underbrace{\left[\sum_{i=1}^{\infty} p^i(1-p) \times \log_2(i+1)\right]}_{=\Psi} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\
&\overset{(\spadesuit)}{\leq} 1 + \int_{\hat{\mathcal{H}}} \left(C_2 + C_3 \log \frac{1}{1-p}\right) f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\
&= 1 + C_2 \underbrace{\int_{\hat{\mathcal{H}}} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H}}_{\leq 1} + C_3 \int_{\hat{\mathcal{H}}} \log \frac{1}{1-p} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\
&\leq C_4 + C_3 \underbrace{\int_{\hat{\mathcal{H}}} \log \frac{1}{1-p} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H}}_{=\mathtt{FR}},
\end{aligned} \tag{29}
$$

where $\hat{\mathcal{H}} = \mathcal{H}'' \cup \mathcal{H}''' = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N \times 1}, \Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) = P \sum_{n=1}^{N} \Gamma_n \geq \alpha\right\}$ (Γ_n is given in (4)), $C_2 = \frac{6}{\log 2} + 3$, $C_3 = \frac{2}{\log 2}$ and $C_4 = 1 + C_2$. The inequality ($\spadesuit$) is from [16, Lemma 1]: $\Psi \leq p(1-p) + \left(\frac{6}{\log 2} + 2\right) p^2 + \frac{2}{\log 2} p^2 \log \frac{1}{1-p} \leq \left(\frac{6}{\log 2} + 3\right) + \frac{2}{\log 2} \log \frac{1}{1-p}$. Next, we will establish an upper bound on p first, then, substitute it into Eq. (29) and derive the upper bound on $\mathtt{FR}\left(\mathrm{VLQ}_{\mathtt{SUM}}\right)$.

The following lemma provides an upper bound on p, which originates from Eq. (28) in the proof of Theorem 1 in Appendix A.

Lemma 3. *We have* $p \leq 1 - \frac{\left(1-\left(1-\frac{\Pi^2}{2}\right)^2\right)^{\frac{2N-1}{2}}}{(2N-1) \times \beta\left(\frac{2N-1}{2}, \frac{1}{2}\right)}$, *where* $\Pi = \frac{\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}}, \boldsymbol{H}\right) - \alpha}{2\sqrt{N} P \left(\sum_{n=1}^{N} |g_n|\right)^2 \left(1 + \sum_{n=1}^{N} \frac{|g_n|^2}{|f_n|^2 + \frac{1}{P}}\right)}$ *is given in Eq.* (26), *and* $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} \mathrm{d}x$ *is the beta function.*

Substituting the upper bound on p in Lemma 3 into Eq. (29) yields an upper bound on FR as

$$\begin{aligned}\mathrm{FR} &\le \int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log \frac{(2N-1)\times\beta\left(\frac{2N-1}{2},\frac{1}{2}\right)}{\left(1-\left(1-\frac{\Pi^2}{2}\right)^2\right)^{\frac{2N-1}{2}}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &= \int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\left((2N-1)\times\beta\left(\frac{2N-1}{2},\frac{1}{2}\right)\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &\quad + \frac{2N-1}{2}\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{1}{1-\left(1-\frac{\Pi^2}{2}\right)^2} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}.\end{aligned}$$

Since $\beta\left(\frac{2N-1}{2},\frac{1}{2}\right) \le \frac{4}{2N-1}+1$ [17], $1-\left(1-\frac{\Pi^2}{2}\right)^2 = \frac{\Pi^2}{2}\left(2-\frac{\Pi^2}{2}\right) \ge \frac{\Pi^2}{2}\left(2-\frac{1}{2}\right) \ge \frac{\Pi^2}{2}$ and $\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \le 1$, the upper bound on FR is further derived as

$$\begin{aligned}\mathrm{FR} &\le \log(2N+3) + \frac{2N-1}{2}\int_{\sum_{i=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{2}{\Pi^2} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \le C_5 + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \left(\log\frac{1}{\Pi}\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &\le C_5 + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{2N^{\frac{1}{2}}\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)}{\sum_{n=1}^{N}\Gamma_n - \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &\le C_5 + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{2N^{\frac{3}{2}}\left(\sum_{n=1}^{N}|g_n|^2\right)\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)}{\sum_{n=1}^{N}\Gamma_n - \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &= C_5 + C_6\log\left(2N^{\frac{3}{2}}\right)\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\left(\sum_{n=1}^{N}|g_n|^2\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &\quad + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} + C_6\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{1}{\sum_{n=1}^{N}\Gamma_n - \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &\overset{(\ddagger)}{\le} C_7 + C_6\underbrace{\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \left(\sum_{n=1}^{N}|g_n|^2\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}}_{=\mathrm{FR}_1} + C_6\underbrace{\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\left(1+N\max_{n=1,\ldots,N}\frac{|g_n|^2}{|f_n|^2}\right) f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}}_{=\mathrm{FR}_2} \\ &\quad + C_6\underbrace{\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} \log\frac{1}{\sum_{n=1}^{N}\Gamma_n - \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}}_{=\mathrm{FR}_3}, \end{aligned} \tag{30}$$

where $C_5 = \log(2N+3) + (2N-1)\frac{\log 2}{2}$, $C_6 = 2N-1$, $C_7 = C_5 + C_6\log\left(2N^{\frac{3}{2}}\right)$. The inequality $(\ddagger)$ arises from the fact that $\int_{\sum_{n=1}^{N}\Gamma_n \ge \frac{\alpha}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \le 1$, $\log\left(\sum_{n=1}^{N}|g_n|^2\right) \le \left(\sum_{n=1}^{N}|g_n|^2\right)$ and $\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}} \le N\max_{n=1,\ldots,N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}} \le N\max_{n=1,\ldots,N}\frac{|g_n|^2}{|f_n|^2}$. Next, let us derive upper bounds on FR_k for $k = 1,\ldots,3$.

An upper bound on $\mathtt{FR}_1$ can be

$$\mathtt{FR}_1 \le \sum_{n=1}^{N} \int_0^{\infty} x f_{|g_n|^2}(x)\,\mathrm{d}x = \sum_{n=1}^{N} \int_0^{\infty} \frac{x}{\sigma_{g_n}} e^{-\frac{x}{\sigma_{g_n}}}\,\mathrm{d}x = \sum_{n=1}^{N} \sigma_{g_n} = C_8. \tag{31}$$

In $\mathtt{FR}_2$, the cumulative density function (cdf) of $\frac{|g_n|^2}{|f_n|^2}$ is $\Pr\left\{\frac{|g_n|^2}{|f_n|^2} < x\right\} = \frac{x}{x+\frac{\sigma_{g_n}}{\sigma_{f_n}}}$, then, the cdf of $\Upsilon = \max_{n=1,\ldots,N} \frac{|g_n|^2}{|f_n|^2}$ is $\Pr\{\Upsilon < x\} = \prod_{n=1}^{N} \frac{x}{x+\frac{\sigma_{g_n}}{\sigma_{f_n}}}$, and its pdf is

$$f_{\Upsilon}(x) = \sum_{n=1}^{N} \frac{\frac{\sigma_{g_n}}{\sigma_{f_n}}}{\left(x+\frac{\sigma_{g_n}}{\sigma_{f_n}}\right)^2} \prod_{n_1=1, n_1\neq n}^{N} \frac{x}{x+\frac{\sigma_{g_{n_1}}}{\sigma_{f_{n_1}}}} \le \sum_{n=1}^{N} \frac{\frac{\sigma_{g_n}}{\sigma_{f_n}}}{\left(x+\frac{\sigma_{g_n}}{\sigma_{f_n}}\right)^2} \le \left(\sum_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}}\right) \frac{1}{\left(x+\min\limits_{n=1,\ldots,N} \frac{\sigma_{g_n}}{\sigma_{f_n}}\right)^2}.$$

Thus, we obtain an upper bound on $\mathtt{FR}_2$ as

$$\begin{aligned}
\mathtt{FR}_2 &\le \int_{\boldsymbol{H}\in\mathbb{C}^{2N\times 1}} \log(1+N\Upsilon)\, f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\
&= \int_{0\le N\Upsilon\le 1} \underbrace{\log(1+N\Upsilon)}_{\le \log 2} f_{\Upsilon}(\Upsilon)\,\mathrm{d}\Upsilon + \int_{N\Upsilon>1} \underbrace{\log(1+N\Upsilon)}_{\le \log(2N\Upsilon)=\log(2N)+\log\Upsilon} f_{\Upsilon}(\Upsilon)\,\mathrm{d}\Upsilon \\
&\le \log 2 + \log(2N) + \sum_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}} \int_{\frac{1}{N}}^{\infty} \frac{\log\Upsilon}{\left(\Upsilon+\min\limits_{n=1,\ldots,N} \frac{\sigma_{g_n}}{\sigma_{f_n}}\right)^2}\mathrm{d}\Upsilon = C_9,
\end{aligned} \tag{32}$$

where $C_9 = \log 2 + \log(2N) + \sum_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}} \left(\frac{\log N}{\frac{1}{N}+\min_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}}} + \frac{\log\left(N \min_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}}+1\right)}{\min_{n=1}^{N} \frac{\sigma_{g_n}}{\sigma_{f_n}}} \right)$.

The derivation for the upper bound on $\mathtt{FR}_3$ relies on the following Lemma, a proof sketch of which is given in Appendix E.

Lemma 4. *For $N \ge 2$, the pdf of $\frac{\Gamma(\boldsymbol{w}^\star,\boldsymbol{H})}{P} = \sum_{n=1}^{N} \Gamma_n = \sum_{n=1}^{N} \frac{|f_n|^2|g_n|^2}{|f_n|^2+|g_n|^2+\frac{1}{P}}$ is upper-bounded by*

$$f_{\sum_{n=1}^{N}\Gamma_n}(x) \le e^{-\frac{x}{\sigma_{g_N}}} \left[D_0 x^{N-1} + D_1 \left(\frac{1}{P^{N-1}} + \frac{1}{P^N} \right) + \mathbf{1}\{N \ge 3\} \times D_2 \sum_{m=1}^{N-2} \left(\frac{x^m}{P^{N-m-1}} + \frac{x^m}{P^{N-m}} \right) \right], \tag{33}$$

where $D_0, D_1, D_2 > 0$ are constants that are independent of P. ■

Substituting Eq. (33) into Eq. (30), the upper bound on $\mathtt{FR}_3$ can be

$$\mathtt{FR}_3 = \int_{\frac{\alpha}{P}}^{\infty} \left(\log\frac{1}{x-\frac{\alpha}{P}}\right) f_{\sum_{n=1}^{N}\Gamma_n}(x)\mathrm{d}x = \int_0^{\infty} \left(\log\frac{1}{y}\right) f_{\sum_{i=1}^{N}\Gamma_i}\left(y+\frac{\alpha}{P}\right)\mathrm{d}y$$

$$\leq D_0 e^{-\frac{\alpha}{P\times\sigma g_N}} \int_0^\infty e^{-\frac{y}{\sigma g_N}} \left(\log\frac{1}{y}\right)\left(y+\frac{\alpha}{P}\right)^{N-1} \mathrm{d}x + D_1 e^{-\frac{\alpha}{P\times\sigma g_N}} \left(\frac{1}{P^{N-1}}+\frac{1}{P^N}\right)\int_0^\infty e^{-\frac{y}{\sigma g_N}} \left(\log\frac{1}{y}\right)\mathrm{d}y$$
$$+ \mathbf{1}\{N\geq 3\}\times D_2 e^{-\frac{\alpha}{P\times\sigma g_N}} \sum_{m=1}^{N-2}\left(\frac{1}{P^{N-m-1}}+\frac{1}{P^{N-m}}\right)\int_0^\infty e^{-\frac{y}{\sigma g_N}} \left(\log\frac{1}{y}\right)\left(y+\frac{\alpha}{P}\right)^{m}\mathrm{d}x. \quad (34)$$

The integral $\int_0^\infty e^{-\frac{y}{\sigma g_N}}\left(\log\frac{1}{y}\right)\mathrm{d}y$ in Eq. (34) is computed as

$$\int_0^\infty e^{-\frac{y}{\sigma g_N}}\left(\log\frac{1}{y}\right)\mathrm{d}y \overset{z=\log\frac{1}{y}}{=} \int_{-\infty}^\infty e^{-\frac{e^{-z}}{\sigma g_N}} z e^{-z}\mathrm{d}z \leq \int_0^\infty e^{-\frac{e^{-z}}{\sigma g_N}} z e^{-z}\mathrm{d}z \leq \int_0^\infty z e^{-z}\mathrm{d}z = 1. \quad (35)$$

Similarly, the integral $\int_0^\infty e^{-\frac{y}{\sigma g_N}}\left(\log\frac{1}{y}\right)\left(y+\frac{\alpha}{P}\right)^n \mathrm{d}y$ for $n\geq 1$ is bounded by

$$\int_0^\infty e^{-\frac{y}{\sigma g_N}}\left(\log\frac{1}{y}\right)\left(y+\frac{\alpha}{P}\right)^n \mathrm{d}y \leq 3\times 2^n n!\times\left(1+\sigma_{g_N}^n\right)\times\left(1+\left(\frac{\alpha}{P}\right)^n\right). \quad (36)$$

Applying Eqs. (35) and (36) to Eq. (34), we obtain

$$\mathtt{FR}_3 \leq 3\times 2^{N-1}(N-1)!\times\left(1+\sigma_{g_N}^{N-1}\right)D_0 e^{-\frac{\alpha}{P\times\sigma g_N}}\left[1+\left(\frac{\alpha}{P}\right)^{N-1}\right] + D_1 e^{-\frac{\alpha}{P\times\sigma g_N}}\underbrace{\left(\frac{1}{P^{N-1}}+\frac{1}{P^N}\right)}_{\leq 2\left[\frac{1}{P}+\frac{1}{P^N}\right]}$$
$$+\mathbf{1}\{N\geq 3\}\times D_2 e^{-\frac{\alpha}{P\times\sigma g_N}}\sum_{m=1}^{N-2}\underbrace{\left(\frac{1}{P^{N-m-1}}+\frac{1}{P^{N-m}}\right)}_{2\left[\frac{1}{P}+\frac{1}{P^N}\right]}\times 3\times\underbrace{\left(1+\sigma_{g_N}^m\right)}_{\leq 2\left(1+\sigma_{g_N}^N\right)}\times\underbrace{2^m m!}_{\leq 2^{N-2}(N-2)!}\underbrace{\left(1+\left(\frac{\alpha}{P}\right)^m\right)}_{\leq 2\left[1+\left(\frac{\alpha}{P}\right)^N\right]}$$
$$\leq C_{10}\underbrace{e^{-\frac{\alpha}{P\times\sigma g_N}}\left[1+\left(\frac{\alpha}{P}\right)^{N-1}\right]}_{\leq 1+e^{-(N-1)}(N-1)^{N-1}\sigma_{g_N}^{N-1}} + C_{11}e^{-\frac{\alpha}{P\times\sigma g_N}}\left[\frac{1}{P}+\frac{1}{P^N}\right] + C_{12}e^{-\frac{\alpha}{P\times\sigma g_N}}\left[\frac{1}{P}+\frac{1}{P^N}\right]\left[1+\left(\frac{\alpha}{P}\right)^N\right]$$
$$\leq C_{13}+C_{14}e^{-\frac{\alpha}{P\times\sigma g_N}}\left[\frac{1}{P}+\frac{1}{P^N}\right]\left[1+\left(\frac{\alpha}{P}\right)^N\right], \quad (37)$$

where $C_{10}=3\times 2^{N-1}(N-1)!\times\left(1+\sigma_{g_N}^{N-1}\right)D_0$, $C_{11}=2D_1$, $C_{12}=\mathbf{1}\{N\geq 3\}\times D_2\times 3\times(N-1)!2^{N+1}\left(1+\sigma_{g_N}^N\right)$, $C_{13}=C_{10}+C_{10}e^{-(N-1)}(N-1)^{N-1}\sigma_{g_N}^{N-1}$ and $C_{14}=C_{11}+C_{12}$. Substituting Eqs. (31), (32) and (37) into Eqs. (30) and (29) completes the proof of Theorem 2. ■

Appendix C: Proof of Theorem 3

Based on Eqs. (7) and (17), to prove $\mathtt{Out}\left(\mathtt{Full}_{\mathtt{IND}}\right)=\mathtt{Out}\left(\mathrm{VLQ}_{\mathtt{IND}}\right)$, it is equivalent to show:

1) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) < \alpha$,

$$\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} = \mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] = 1; \tag{38}$$

2) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) = \alpha$,

$$\begin{aligned}&\int_{\boldsymbol{H}\in\left\{\boldsymbol{H}\in\mathsf{C}^{2N\times 1}:\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star},\boldsymbol{H}\right)=\alpha\right\}} \mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\ &= \int_{\boldsymbol{H}\in\left\{\boldsymbol{H}\in\mathsf{C}^{2N\times 1}:\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star},\boldsymbol{H}\right)=\alpha\right\}} \mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] f_{\boldsymbol{H}}\left(\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = 0;\end{aligned} \tag{39}$$

3) For any $\boldsymbol{H}$ satisfying $\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) > \alpha$,

$$\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) < \alpha\right\} = \mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}\right] = 0. \tag{40}$$

We define

$$\begin{aligned}\mathcal{H} &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) < \alpha\right\}, \\ \mathcal{H}' &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) < \alpha, \forall i \in \mathsf{N}\right\}, \\ \mathcal{H}'' &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathsf{C}^{2N\times 1}, \Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) > \alpha\right\}\end{aligned}$$

The proofs of Eqs. (38) and (39) are similar to those of Eqs. (20) and (21) in Appendix A, thus omitted. To prove Eq. (40) is equivalent to show $\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}\right\} = \mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\}\right] = 0$ for $\boldsymbol{H}$ in $\mathcal{H}''$ and given $\{\boldsymbol{\mu}_i\}_{\mathsf{N}}$. Since $\mathcal{H} \cap \mathcal{H}'' = \emptyset$, $\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}\right\} = 0$ for $\boldsymbol{H}$ in $\mathcal{H}''$. To prove $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\}\right] = 0$ for any $\boldsymbol{H} \in \mathcal{H}''$, conversely, we assume $\exists\, \tilde{\boldsymbol{H}} \in \mathcal{H}''$, s.t. $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] = \varepsilon > 0$, then,

$$\begin{aligned}\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathsf{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] &= \Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \tilde{\boldsymbol{H}}\right) < \alpha, \forall i \in \mathsf{N}\right\} \\ &\leq \Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \tilde{\boldsymbol{H}}\right) < \alpha, \forall 0 \leq i \leq K-1\right\} = \left[\Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \tilde{\boldsymbol{H}}\right) < \alpha\right\}\right]^{K},\end{aligned} \tag{41}$$

where $K \geq 1$ is an arbitrary finite natural number. Using the upper bound derived in Eq. (45), for any $\boldsymbol{\mu}_i = \left[\mu_{i,1}, \ldots, \mu_{i,N}\right]^{\top}$ and $\boldsymbol{\mu}_{\mathtt{IND}}^{\star} = \left[\mu_{i,1}^{\star}, \ldots, \mu_{i,N}^{\star}\right]^{\top}$, we obtain

$$\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) - \Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) \leq \hat{\Xi} \times \sum_{k=1}^{N}\left|\mu_{i,k}^{\star} - \mu_{i,k}\right| = \hat{\Xi} \times \sum_{k=1}^{N}\left|\left|\mu_{i,k}^{\star}\right| - \left|\mu_{i,k}\right| \times e^{j\left[\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right]}\right|$$

$$= \hat{\Xi} \times \sum_{k=1}^{N} \sqrt{\left(\left|\mu_{i,k}^{\star}\right| - \left|\mu_{i,k}\right|\right)^2 + 2\underbrace{\left|\mu_{i,k}^{\star}\right| \cdot \left|\mu_{i,k}\right|}_{\leq 1} \cdot \underbrace{\left[1 - \cos\left(\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right)\right]}_{\leq \frac{1}{2}\left|\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right|^2}},$$

where $\hat{\Xi} = 2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1 + \sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)$. When $\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) > \alpha$, let $\delta = \frac{\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right)-\alpha}{\hat{\Xi}\sqrt{1+4\pi^2}N} > 0$, then, for any $\boldsymbol{\mu}_i$ satisfying $\left|\left|\mu_{i,k}^{\star}\right| - \left|\mu_{i,k}\right|\right| \leq \delta$ and $\left|\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right| \leq 2\pi \times \delta$, we have $\Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) - \Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) \leq \hat{\Xi}\sqrt{1+4\pi^2}N \times \delta = \Gamma\left(\boldsymbol{\mu}_{\mathtt{IND}}^{\star}, \boldsymbol{H}\right) - \alpha$, thus, $\Gamma\left(\boldsymbol{\mu}_i, \boldsymbol{H}\right) \geq \alpha$. For $\tilde{\boldsymbol{H}}$, it follows that

$$\begin{aligned}
&\Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \tilde{\boldsymbol{H}}\right) < \alpha\right\} = 1 - \Pr\left\{\Gamma\left(\boldsymbol{\mu}_i, \tilde{\boldsymbol{H}}\right) \geq \alpha\right\} \\
&\leq 1 - \Pr\left\{\left|\left|\mu_{i,k}^{\star}\right| - \left|\mu_{i,k}\right|\right| \leq \delta, \left|\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right| \leq 2\pi \times \delta, k = 1, \ldots, N\right\} \\
&= 1 - \prod_{k=1}^{N} \Pr\left\{\left|\left|\mu_{i,k}^{\star}\right| - \left|\mu_{i,k}\right|\right| \leq \delta\right\} \times \Pr\left\{\left|\arg\left(\mu_{i,k}\right) - \arg\left(\mu_{i,k}^{\star}\right)\right| \leq 2\pi \times \delta\right\} \\
&\overset{(\diamond)}{=} 1 - \prod_{k=1}^{N} \Delta_{1,k}\Delta_{2,k} = \Delta,
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
\Delta_{1,k} &= \left[\min\left(1, \left|\mu_{i,k}^{\star}\right| + \delta\right) - \max\left(0, \left|\mu_{i,k}^{\star}\right| - \delta\right)\right], \\
\Delta_{2,k} &= \left[\min\left(1, \tfrac{\arg\left(\mu_{i,k}^{\star}\right)}{2\pi} + \delta\right) - \max\left(0, \tfrac{\arg\left(\mu_{i,k}^{\star}\right)}{2\pi} - \delta\right)\right].
\end{aligned}$$

The equality $(\diamond)$ is derived from the fact that $\left|\mu_{i,k}\right|$ and $\arg\left(\mu_{i,k}\right)$ are uniformly distributed in $(0, 1]$ and $(0, 2\pi]$, respectively, as defined in Eq. (14). It can be readily observed that $0 < \Delta_{1,k}, \Delta_{2,k} \leq 1$, thus, $0 \leq \Delta < 1$. Substituting Eq. (42) into Eq. (41), we have $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] \leq \Delta^K$. When $\Delta = 0$, $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] = 0 < \varepsilon$; when $\Delta > 0$, letting $K = \lceil \log_{\Delta}^{\varepsilon} \rceil + 1$, we obtain $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] \leq \Delta^{\lceil \log_{\Delta}^{\varepsilon} \rceil + 1} < \Delta^{\log_{\Delta}^{\varepsilon}} = \varepsilon$. Both contradict the assumption that $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}}\left[\mathbf{1}\left\{\tilde{\boldsymbol{H}} \in \mathcal{H}'\right\}\right] = \varepsilon$. Hence, $\mathrm{E}_{\{\boldsymbol{\mu}_i\}_{\mathtt{N}}}\left[\mathbf{1}\left\{\boldsymbol{H} \in \mathcal{H}'\right\}\right] = 0$ for $\boldsymbol{H} \in \mathcal{H}''$, and the proof is complete. ■

Appendix D: Proof of Lemmas 1 and 2

A. Proof of Lemma 1

To prove Lemma 1, we first bound the gap between $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right)$ and $\Gamma\left(\boldsymbol{w}, \boldsymbol{H}\right)$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathtt{SUM}}$. Then, based on the upper bound on $\Gamma\left(\boldsymbol{w}_{\mathtt{SUM}}^{\star}, \boldsymbol{H}\right) - \Gamma\left(\boldsymbol{w}, \boldsymbol{H}\right)$, we find the conditions of $\boldsymbol{w}$ to satisfy

$\Gamma(\boldsymbol{w},\boldsymbol{H}) \geq \alpha$ for any $\boldsymbol{H}$ with $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}},\boldsymbol{H}\right) > \alpha$.

In order to upper-bound $\Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}},\boldsymbol{H}\right) - \Gamma(\boldsymbol{w},\boldsymbol{H})$, we successively alter each component of $\boldsymbol{w}^{\star}_{\mathtt{SUM}}$ until we reach $\boldsymbol{w}$, while keep track of the SNR variation at each step of the alteration [7, Appendix B]. Thus, $\overline{\Gamma(\boldsymbol{H})} = \Gamma\left(\boldsymbol{w}^{\star}_{\mathtt{SUM}},\boldsymbol{H}\right) - \Gamma(\boldsymbol{w},\boldsymbol{H})$ is decomposed as

$$\overline{\Gamma(\boldsymbol{H})} = \sum_{k=1}^{N} \overline{\Gamma_k(\boldsymbol{H})} = \sum_{k=1}^{N} \left[\Gamma\left(\boldsymbol{w}^{(k-1)},\boldsymbol{H}\right) - \Gamma\left(\boldsymbol{w}^{(k)},\boldsymbol{H}\right)\right], \tag{43}$$

where $\boldsymbol{w}^{(0)} = \boldsymbol{w}^{\star}_{\mathtt{SUM}}$, $\boldsymbol{w}^{(k)} = \left[[\boldsymbol{w}]_1, \ldots, [\boldsymbol{w}]_k, \left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_{k+1}, \ldots, \left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_N\right]^{\top}$ and $\boldsymbol{w}^{(N)} = \boldsymbol{w}$. Let

$$\begin{aligned}
&\tilde{f}_n = \tfrac{1}{|f_n|^2 + \frac{1}{P}}, \quad A_k = \textstyle\sum_{n=1}^{N} \left[\boldsymbol{w}^{(k-1)}\right]_n^{*} f_n g_n \sqrt{\tilde{f}_n}, \quad B_k = 1 + \textstyle\sum_{n=1}^{N} \left|\left[\boldsymbol{w}^{(k-1)}\right]_n\right|^2 |g_n|^2 \tilde{f}_n,\\
&\hat{A}_k = \textstyle\sum_{n=1}^{N} \left[\boldsymbol{w}^{(k)}\right]_n^{*} f_n g_n \sqrt{\tilde{f}_n} = A_k - \left(\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k^{*} - [\boldsymbol{w}]_k^{*}\right) f_k g_k \sqrt{\tilde{f}_k},\\
&\hat{B}_k = 1 + \textstyle\sum_{n=1}^{N} \left|\left[\boldsymbol{w}^{(k)}\right]_n\right|^2 |g_n|^2 \tilde{f}_n = B_k - \left(\left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k\right|^2 - |[\boldsymbol{w}]_k|^2\right) |g_k|^2 \tilde{f}_k.
\end{aligned}$$

From Eq. (2), $\overline{\Gamma_k(\boldsymbol{H})} = P\frac{|A_k|^2}{B_k} - P\frac{|\hat{A}_k|^2}{\hat{B}_k} = P\frac{|A_k|^2}{B_k} - P\frac{\left|A_k - \left(\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k^{*} - [\boldsymbol{w}]_k^{*}\right) f_k g_k \sqrt{\tilde{f}_k}\right|^2}{\hat{B}_k}$ is expanded as

$$\begin{aligned}
\overline{\Gamma_k(\boldsymbol{H})} &= P\tfrac{|A_k|^2}{B_k} - P\tfrac{|A_k|^2}{\hat{B}_k} - \underbrace{P\tfrac{\left|\left(\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k^{*} - [\boldsymbol{w}]_k^{*}\right) f_k g_k \sqrt{\tilde{f}_k}\right|^2}{\hat{B}_k}}_{\geq 0} + 2P\tfrac{\mathrm{Re}\left\{A_k^{*}\left(\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k^{*} - [\boldsymbol{w}]_k^{*}\right) f_k g_k \sqrt{\tilde{f}_k}\right\}}{\hat{B}_k}\\
&\leq P\tfrac{|A_k|^2\left(\hat{B}_k - B_k\right)}{B_k \hat{B}_k} + 2P\tfrac{|A_k| \cdot \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k - [\boldsymbol{w}]_k\right| \cdot |f_k| \cdot |g_k| \sqrt{\tilde{f}_k}}{\hat{B}_k}\\
&= P\tfrac{-|A_k|^2\left(\left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k\right|^2 - |[\boldsymbol{w}]_k|^2\right)|g_k|^2 \tilde{f}_k}{B_k \hat{B}_k} + 2P\tfrac{|A_k| \cdot \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k - [\boldsymbol{w}]_k\right| \cdot |f_k| \cdot |g_k| \sqrt{\tilde{f}_k}}{\hat{B}_k}\\
&\leq P\tfrac{|A_k|^2\left|\left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k\right|^2 - |[\boldsymbol{w}]_k|^2\right| \cdot |g_k|^2 \tilde{f}_k}{B_k \hat{B}_k} + 2P\tfrac{|A_k| \cdot \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k - [\boldsymbol{w}]_k\right| \cdot |f_k| \cdot |g_k| \sqrt{\tilde{f}_k}}{\hat{B}_k}\\
&\overset{(\heartsuit)}{\leq} P|A_k|^2 \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k - [\boldsymbol{w}]_k\right| \cdot \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k + [\boldsymbol{w}]_k\right| \cdot |g_k|^2 \tilde{f}_k + 2P|A_k| \cdot \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k - [\boldsymbol{w}]_k\right| \cdot |f_k| \cdot |g_k| \sqrt{\tilde{f}_k},
\end{aligned} \tag{44}$$

where the inequality ($\heartsuit$) is because of the inequality $\left||c_1|^2 - |c_2|^2\right| \leq |c_1 - c_2| \times |c_1 + c_2|$ for $c_1, c_2 \in \mathsf{C}$ (the proof is omitted), and $B_k, \hat{B}_k \geq 1$. Since $\left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k + [\boldsymbol{w}]_k\right| \leq \left|\left[\boldsymbol{w}^{\star}_{\mathtt{SUM}}\right]_k\right| + |[\boldsymbol{w}]_k| \leq 2$, $|f_k| \sqrt{\tilde{f}_k} \leq 1$ and $|A_k| \leq \sum_{n=1}^{N} \left|\left[\boldsymbol{w}^{(k-1)}\right]_n\right| \times |f_n| |g_n| \sqrt{\tilde{f}_n} \leq \sum_{n=1}^{N} |f_n| |g_n| \sqrt{\tilde{f}_n} \leq \sum_{n=1}^{N} |g_n|$, it follows from Eq. (44)

that

$$\begin{aligned}\overline{\Gamma_k(\boldsymbol{H})} &\le 2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\times|g_k|^2\tilde{f}_k+2P\left(\sum_{n=1}^{N}|g_n|\right)\times\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\times|g_k|\\ &\le 2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\times\sum_{n=1}^{N}|g_n|^2\tilde{f}_n\\ &+2P\left(\sum_{n=1}^{N}|g_n|\right)\times\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\times\left(\sum_{n=1}^{N}|g_n|\right)\\ &=2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right).\end{aligned}$$

Then, $\overline{\Gamma(\boldsymbol{H})}$ in Eq. (43) is upper-bounded by

$$\begin{aligned}\overline{\Gamma(\boldsymbol{H})} &\le 2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)\sum_{k=1}^{N}\left|[\boldsymbol{w}^\star_{\mathtt{SUM}}]_k-[\boldsymbol{w}]_k\right|\\ &\le 2P\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)\sqrt{N\sum_{k=1}^{N}\left|\left[\boldsymbol{w}^\star_{\mathtt{SUM}}\right]_k-[\boldsymbol{w}]_k\right|^2}\\ &=\underbrace{2\sqrt{N}P\left(\sum_{n=1}^{N}|g_n|\right)^2\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)}_{=\Xi}\times\left\|\boldsymbol{w}^\star_{\mathtt{SUM}}-\boldsymbol{w}\right\|.\end{aligned}\tag{45}$$

When $\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\alpha>0$, letting $\Pi=\frac{\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\alpha}{\Xi}$. When $\left\|\boldsymbol{w}^\star_{\mathtt{SUM}}-\boldsymbol{w}\right\|\le\Pi$, $\Gamma(\boldsymbol{w},\boldsymbol{H})=\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\overline{\Gamma(\boldsymbol{H})}\ge\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\Xi\times\left\|\boldsymbol{w}^\star_{\mathtt{SUM}}-\boldsymbol{w}\right\|\ge\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\Xi\times\Pi=\alpha$.

To complete, let us verify that $0<\Pi<1$: (i) since $\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)-\alpha>0$ and $\Xi>0$, we have $\Pi>0$; (ii) since $\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)=P\sum_{n=1}^{N}\frac{|f_n|^2|g_n|^2}{|f_n|^2+|g_n|^2+\frac{1}{P}}<P\sum_{n=1}^{N}|g_n|^2<P\left(\sum_{n=1}^{N}|g_n|\right)^2$, $\Pi<\frac{\Gamma\left(\boldsymbol{w}^\star_{\mathtt{SUM}},\boldsymbol{H}\right)}{\Xi}<\frac{1}{2\sqrt{N}\left(1+\sum_{n=1}^{N}\frac{|g_n|^2}{|f_n|^2+\frac{1}{P}}\right)}<1$. ■

B. Proof of Lemma 2

Similar to [15, Eqs.(23)-(24)], we have $\Pr\{\boldsymbol{u}^\top\boldsymbol{v}\ge t\}=\frac{S_{2N,t,\text{cap}}}{S_{2N}}$, where $S_{2N,t,\text{cap}}$ is the surface area of the spherical cap formed by the intersection of the subspace $\boldsymbol{u}^\top\boldsymbol{v}\ge t$ and the real unit hyper-sphere $\mathcal{W}_{\mathrm{R}}$. From [18], we obtain $S_{2N}=\frac{2\pi^N}{(N-1)!}$ and $S_{2N,t,\text{cap}}=\frac{\pi^N}{(N-1)!}I_{1-t^2}\left(\frac{2N-1}{2},\frac{1}{2}\right)$. Then, Lemma 2 is obtained by dividing $S_{2N,t,\text{cap}}$ by S_{2N}. ■

Appendix E: Proof of Lemma 4

Induction is applied to prove the upper bound on the pdf of $\sum_{n=1}^{N} \frac{|f_n|^2|g_n|^2}{|f_n|^2+|g_n|^2+\frac{1}{P}}$ in Eq. (33). We first derive an upper bound on the pdf of $\Gamma_n = \frac{|f_n|^2|g_n|^2}{|f_n|^2+|g_n|^2+\frac{1}{P}}$. Then, the base case where $N = 2$ and the inductive step are proved based on the upper bound on the pdf of Γ_n.

The cdf of Γ_n is calculated as $\Pr\{\Gamma_n < x\} = 1 - \frac{e^{-\frac{x}{\sigma_{g_n}}-\frac{x}{\sigma_{f_n}}}}{\sigma_{g_n}} \int_0^\infty e^{-\frac{y}{\sigma_{g_n}}-\frac{x^2+\frac{x}{P}}{y\times\sigma_{f_n}}} \mathrm{d}y$. By taking derivative of $\Pr\{\Gamma_n < x\}$ with respect to x, the pdf of Γ_n is

$$
\begin{aligned}
f_{\Gamma_n}(x) &= e^{-\frac{x}{\sigma_{g_n}}-\frac{x}{\sigma_{f_n}}} \frac{\frac{1}{\sigma_{g_n}}+\frac{1}{\sigma_{f_n}}}{\sigma_{g_n}} \int_0^\infty e^{-\frac{y}{\sigma_{g_n}}-\frac{x^2+\frac{x}{P}}{y\times\sigma_{f_n}}} \mathrm{d}y + \frac{e^{-\frac{x}{\sigma_{g_n}}-\frac{x}{\sigma_{f_n}}}}{\sigma_{g_n}\sigma_{f_n}} \left(2x+\frac{1}{P}\right) \int_0^\infty \frac{1}{y} e^{-\frac{y}{\sigma_{g_n}}-\frac{x^2+\frac{x}{P}}{y\times\sigma_{f_n}}} \mathrm{d}y \\
&\overset{(\S)}{=} 2e^{-\frac{x}{\sigma_{g_n}}-\frac{x}{\sigma_{f_n}}} \frac{\frac{1}{\sigma_{g_n}}+\frac{1}{\sigma_{f_n}}}{\sigma_{g_n}} \sqrt{\frac{\sigma_{g_n}\left(x^2+\frac{x}{P}\right)}{\sigma_{f_n}}} \boldsymbol{K}_1\left(2\sqrt{\frac{x^2+\frac{x}{P}}{\sigma_{g_n}\sigma_{f_n}}}\right) + 2\frac{e^{-\frac{x}{\sigma_{g_n}}-\frac{x}{\sigma_{f_n}}}}{\sigma_{g_n}\sigma_{f_n}} \left(2x+\frac{1}{P}\right) \boldsymbol{K}_0\left(2\sqrt{\frac{x^2+\frac{x}{P}}{\sigma_{g_n}\sigma_{f_n}}}\right) \\
&\overset{(\P)}{\leq} e^{-\frac{x}{\sigma_{g_n}}} \left(\frac{1}{\sigma_{g_n}}+\frac{1}{\sigma_{f_n}}\right) + 2\frac{e^{-\frac{x}{\sigma_{g_n}}}}{\sigma_{g_n}\sigma_{f_n}} \left(2x+\frac{1}{P}\right) \boldsymbol{K}_0\left(\frac{2x}{\sqrt{\sigma_{g_n}\sigma_{f_n}}}\right) \\
&\overset{(\maltese)}{\leq} e^{-\frac{x}{\sigma_{g_N}}} \left(\frac{1}{\sigma_{g_n}}+\frac{1}{\sigma_{f_n}}\right) + 2\frac{e^{-\frac{x}{\sigma_{g_N}}}}{\sigma_{g_n}\sigma_{f_n}} \left(2x+\frac{1}{P}\right) \boldsymbol{K}_0\left(\frac{2x}{\sqrt{\sigma_{g_n}\sigma_{f_n}}}\right),
\end{aligned} \tag{46}
$$

where the equality (§) is from $\int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} \mathrm{d}x = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} \boldsymbol{K}_\nu\left(2\sqrt{\beta\gamma}\right)$, and $\boldsymbol{K}_\nu(z)$ is the modified bessel function of the second kind [14, Eq.(3.471.9)]; the inequality (¶) is because $\boldsymbol{K}_1(x) \leq \frac{1}{x}$ [7, Eq.(25)] and $\boldsymbol{K}_0(\cdot)$ is a decreasing function [14, Eq.(3.471.9)]; the last inequality (✠) is based on our assumption that $\sigma_{g_1}^2 \leq \sigma_{g_2}^2 \leq \ldots \leq \sigma_{g_N}^2$.

In the base case where $N = 2$, the pdf of $\Gamma_1 + \Gamma_2$ is the convolution of $f_{\Gamma_1}(x)$ and $f_{\Gamma_2}(x)$, given by

$$
\begin{aligned}
f_{\Gamma_1+\Gamma_2}(x) &= \int_0^x f_{\Gamma_1}(r) f_{\Gamma_2}(x-r) \mathrm{d}r \\
&\leq \int_0^x \left[e^{-\frac{r}{\sigma_{g_N}}} \left(\frac{1}{\sigma_{g_1}}+\frac{1}{\sigma_{f_1}}\right) + 2\frac{e^{-\frac{r}{\sigma_{g_N}}}}{\sigma_{g_1}\sigma_{f_1}} \left(2r+\frac{1}{P}\right) \boldsymbol{K}_0\left(\frac{2r}{\sqrt{\sigma_{g_1}\sigma_{f_1}}}\right)\right] \\
&\quad \times \left[e^{-\frac{x-r}{\sigma_{g_N}}} \left(\frac{1}{\sigma_{g_2}}+\frac{1}{\sigma_{f_2}}\right) + 2\frac{e^{-\frac{x-r}{\sigma_{g_N}}}}{\sigma_{g_2}\sigma_{f_2}} \left(2x-2r+\frac{1}{P}\right) \boldsymbol{K}_0\left(\frac{2x-2r}{\sqrt{\sigma_{g_2}\sigma_{f_2}}}\right)\right] \mathrm{d}r.
\end{aligned}
$$

Using (46), $\int_0^\infty \boldsymbol{K}_0(ax)\mathrm{d}x = \frac{\pi}{2a}$, $\int_0^\infty \boldsymbol{K}_0^2(ax)\mathrm{d}x = \frac{\pi^2}{4a}$ for $a > 0$ [14, Eq.(6.511.12)-(6.511.13)],

$\boldsymbol{K}_0(x) \leq \frac{2}{x}$ for $x > 0$ [7, Eq.(27)] and after basic mathematical calculations, we obtain

$$f_{\Gamma_1+\Gamma_2}(x) \leq C_{15} x e^{-\frac{x}{\sigma_{g_N}}} + C_{16}\frac{e^{-\frac{x}{\sigma_{g_N}}}}{P} + C_{17}\frac{e^{-\frac{x}{\sigma_{g_N}}}}{P^2},$$

where

$$\begin{aligned}
C_{15} &= \prod_{n=1}^{2}\left(\frac{1}{\sigma_{g_n}}+\frac{1}{\sigma_{f_n}}\right) + \frac{\pi}{\sqrt{\sigma_{g_2}\sigma_{f_2}}}\left(\frac{1}{\sigma_{g_1}}+\frac{1}{\sigma_{f_1}}\right) + \frac{\pi}{\sqrt{\sigma_{g_1}\sigma_{f_1}}}\left(\frac{1}{\sigma_{g_2}}+\frac{1}{\sigma_{f_2}}\right) + \frac{16}{\prod_{n=1}^{2}\sqrt{\sigma_{g_n}\sigma_{f_n}}},\\
C_{16} &= \frac{\pi}{2\sqrt{\sigma_{g_2}\sigma_{f_2}}}\left(\frac{1}{\sigma_{g_1}}+\frac{1}{\sigma_{f_1}}\right) + \frac{\pi}{2\sqrt{\sigma_{g_1}\sigma_{f_1}}}\left(\frac{1}{\sigma_{g_2}}+\frac{1}{\sigma_{f_2}}\right) + \frac{4\pi}{\prod_{n=1}^{2}\sqrt{\sigma_{g_n}\sigma_{f_n}}},\\
C_{17} &= \frac{\pi^2\sqrt{\max_{n=1,\ldots,N}\sigma_{g_n}\sigma_{f_n}}}{\prod_{n=1}^{2}\sqrt{\sigma_{g_n}\sigma_{f_n}}}.
\end{aligned}$$

In the inductive step, given the upper bounds on $f_{\Gamma_{k+1}}(x)$ and $f_{\sum_{i=1}^{k}\Gamma_i}(x)$, we derive an upper bound on $f_{\sum_{i=1}^{k+1}\Gamma_i}(x) = \int_0^x f_{\sum_{i=1}^{k}\Gamma_i}(r) f_{\Gamma_{k+1}}(x-r)\mathrm{d}r$ as

$$\begin{aligned}
f_{\sum_{i=1}^{k+1}\Gamma_i}(x) &\leq \int_0^x e^{-\frac{r}{\sigma_{g_N}}}\left[C_{18}r^{k-1} + \mathbf{1}\{k\geq 3\}\times C_{19}\sum_{m=1}^{k-2}\left(\frac{r^m}{P^{k-m-1}}+\frac{r^m}{P^{k-m}}\right) + C_{20}\left(\frac{1}{P^{k-1}}+\frac{1}{P^k}\right)\right]\\
&\quad\times e^{-\frac{x-r}{\sigma_{g_N}}}\left[\frac{1}{\sigma_{g_{k+1}}}+\frac{1}{\sigma_{f_{k+1}}}+\frac{2}{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}\left(2x-2r+\frac{1}{P}\right)\boldsymbol{K}_0\left(\frac{2x-2r}{\sqrt{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}}\right)\right]\mathrm{d}r\\
&\leq e^{-\frac{x}{\sigma_{g_N}}}\left[C_{18}x^{k-1} + \underbrace{\mathbf{1}\{k\geq 3\}}_{\leq \mathbf{1}\{k+1\geq 3\}=1}\times C_{19}\sum_{m=1}^{k-2}\left(\frac{x^m}{P^{k-m-1}}+\frac{x^m}{P^{k-m}}\right) + C_{20}\left(\frac{1}{P^{k-1}}+\frac{1}{P^k}\right)\right]\\
&\quad\times\int_0^x\left[\frac{1}{\sigma_{g_{k+1}}}+\frac{1}{\sigma_{f_{k+1}}}+\frac{2}{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}\left(2x-2r+\frac{1}{P}\right)\boldsymbol{K}_0\left(\frac{2x-2r}{\sqrt{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}}\right)\right]\mathrm{d}r\\
&\leq e^{-\frac{x}{\sigma_{g_N}}}\left[C_{18}x^{k-1} + \mathbf{1}\{k+1\geq 3\}\times C_{19}\sum_{m=1}^{k-2}\left(\frac{x^m}{P^{k-m-1}}+\frac{x^m}{P^{k-m}}\right) + C_{20}\left(\frac{1}{P^{k-1}}+\frac{1}{P^k}\right)\right]\\
&\quad\times\underbrace{\int_0^x\left[\frac{1}{\sigma_{g_{k+1}}}+\frac{1}{\sigma_{f_{k+1}}}+\frac{2}{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}\left(2y+\frac{1}{P}\right)\boldsymbol{K}_0\left(\frac{2y}{\sqrt{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}}\right)\right]\mathrm{d}y}_{\leq x\left(\frac{1}{\sigma_{g_{k+1}}}+\frac{1}{\sigma_{f_{k+1}}}\right)+\frac{2\left(2x+\frac{1}{P}\right)}{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}\int_0^\infty \boldsymbol{K}_0\left(\frac{2y}{\sqrt{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}}\right)\mathrm{d}y = x\left(\frac{1}{\sigma_{g_{k+1}}}+\frac{1}{\sigma_{f_{k+1}}}\right)+\frac{\pi\left(2x+\frac{1}{P}\right)}{2\sqrt{\sigma_{g_{k+1}}\sigma_{f_{k+1}}}}}.
\end{aligned}$$

After trivial mathematical manipulations, we can obtain the upper bound on $f_{\sum_{i=1}^{k+1}\Gamma_i}(x)$ in Eq. (33) when $N = k+1$. By the law of induction, Lemma 4 stands for any $N \geq 2$. ■

References

[1] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3180, Dec. 2004.

[2] M. O. Hasna and M. S. Alouini, "End-to-end performance of transmission systems with relays over rayleigh-fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 6, pp. 1126–1131, Nov. 2003.

[3] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2735–2747, Oct. 2003.

[4] J. C. Roh and B. D. Rao, "Transmit beamforming in multiple-antenna systems with finite rate feedback: a VQ-based approach," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 1101–1112, Mar. 2006.

[5] S. Ekbatani and H. Jafarkhani, "Combining beamforming and space-time coding using quantized feedback," *IEEE Trans. Wireless Commun.*, vol. 7, no. 3, pp. 898–908, Mar. 2008.

[6] N. Ahmed, M. A. A. Khojastepour, A. Sabharwal, and B. Aazhang, "Outage minimization with limited feedback for the fading relay channel," *IEEE Trans. Commun.*, vol. 54, no. 4, pp. 659–669, Apr. 2006.

[7] E. Koyuncu, Y. Jing, and H. Jafarkhani, "Distributed beamforming in wireless relay networks with quantized feedback," *IEEE J. Select. Areas Commun.*, vol. 26, no. 8, pp. 1429–1439, Oct. 2008.

[8] Y. Zhao, R. Adve, and T. J. Lim, "Beamforming with limited feedback in amplify-and-forward cooperative networks," *IEEE Trans. Wireless Commun.*, vol. 7, no. 12, pp. 5145–5149, Dec. 2008.

[9] Y. Jing and H. Jafarkhani, "Network beamforming using relays with perfect channel information," *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2499–2517, Jun. 2009.

[10] ——, "Single and multiple relay selection schemes and their achievable diversity orders," *IEEE Trans. Wireless Commun.*, vol. 8, no. 3, pp. 1414–1423, Mar. 2009.

[11] E. Koyuncu and H. Jafarkhani, "Variable-length limited feedback beamforming in multiple-antenna fading channels," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 7140–7164, Nov. 2014.

[12] C. K. Au-Yeung and D. J. Love, "On the performance of random vector quantization limited feedback beamforming in a MISO system," *IEEE Trans. Wireless Commun.*, vol. 6, no. 2, pp. 458–462, Feb. 2007.

[13] E. Koyuncu and H. Jafarkhani, "Very low-rate variable-length channel quantization for minimum outage probability," in *IEEE Data Compression Conference (DCC)*, pp. 261–270, Mar. 2013.

[14] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., A. Jeffrey and D. Zwillinger, Eds. Academic Press, Mar. 2007.

[15] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finite rate feedback in multiple-antenna systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2562–2579, Oct. 2003.

[16] X. Liu, E. Koyuncu, and H. Jafarkhani, "Multicast networks with variable-length limited feedback," *IEEE Trans. Wireless Commun.*, vol. 14, no. 1, pp. 252–264, Jan. 2015.

[17] H. Alzer, "Sharp inequalities for the beta function," *Indagationes Mathematicae*, vol. 12, pp. 15–21, 2001.

[18] S. Li, "Concise formulas for the area and volume of a hyperspherical cap," *Asian J. Math. Stat.*, pp. 66 – 77, 2011.