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ABSTRACT

An approximate quantum mechanical solution of the meson nucleon scattering problem for intermediate values of the coupling constant is presented. The particular case treated here is that of charged scalar mesons interacting with a static nucleon. In finding the cross section, attention is focused upon the matrix elements of the field and isotopic spin operators and on the equations of motion, no attempt being made to calculate explicitly the scattering state vector. It is shown that in both the weak and strong coupling limits the procedure described here gives the scattering correctly. For intermediate coupling the cross section must be found numerically. Computations have been carried out for several intermediate values of the coupling constant and the results are presented in the form of curves showing cross section vs. meson energy. Since certain information about the one nucleon problem (i.e., one real nucleon and no real mesons) is needed for these calculations, a detailed numerical solution of that problem has been carried out using the Tomonaga approximation. The relevant results of this work are presented in several graphs. Although it is not a part of the scattering problem, the calculation of the isobar separation in the strong coupling limit can be carried out so easily by the methods of this paper that a brief account of it is also given. In the appendix, a variational method of calculating the scattering state vector for intermediate coupling is described. It is shown that this, however, fails to give the correct strong coupling limit.

MESON NUCLEON SCATTERING. I

R. J. Riddell, Jr. and B. D. Fried

1. Introduction

One of the most fundamental problems of meson field theory is the task of giving a quantitatively satisfactory explanation of the increasingly abundant and detailed experimental data on the scattering of pions by nucleons. Since all qualitative estimates of the meson-nucleon coupling indicate that it is neither very large nor very small, it is imperative to develop a technique for calculating the scattering cross section which is valid for intermediate values of the coupling constant. Although it is clear that the problem must eventually be treated in a completely relativistic manner, using pseudoscalar meson theory, to date this has proved so formidable a task that a critical examination of the intermediate coupling region for the much simpler problem of an infinitely heavy nucleon seems worthwhile. Even with this restriction no exact solution for all values of the coupling has been found, and so we must look for suitable approximations.

In this paper we shall study the question of meson scattering from a fixed nucleon for intermediate values of the coupling constant, g. In order to avoid the algebraic complications involved in treating both spin and isotopic spin, we shall first discuss the simplest non-trivial case, the charged scalar field, although the same methods can be used in the more complicated cases--e.g. the pseudoscalar, charge symmetric field, which will be discussed in a subsequent paper.

The non-relativistic one-body problem (i.e., one real, fixed nucleon and no real mesons) can be treated by Tomonaga's variational procedure[1] which

[1] S. Tomonaga, Prog. of Theor. Phys. 2, 6 (1947).

uses as trial function a state vector in which only a few meson states are occupied (although no restriction is placed on occupation numbers). The crucial point in the justification of this method is its validity in both the weak and strong coupling limits. The ansatz that all mesons are in the same spatial state is certainly correct if, as in the weak coupling case, the probability of having more than one meson is very small. That this same approximation will also be successful for large values of the coupling constant is less obvious and a demonstration of the agreement between the strong coupling limit of the Tomonaga approximation and the conventional strong coupling (s.c.) theory[2] is essential in any attempt to make plausible the validity of the Tomonaga ansatz for the intermediate coupling region. In his original paper[1] on the subject, Tomonaga showed that his method does indeed give the correct s.c. value for the isobar separation.

Going on to the two-body problem of one real meson and one real nucleon, it seems natural to look for a consistent method of calculating the scattering cross section which shall satisfy the following three criteria:

1° For small values of g the cross-section is the same as that obtained from ordinary perturbation theory.

2° For large g the cross-section agrees with the result of the conventional strong coupling theory[2].

3° For intermediate values of g a numerical calculation of the scattering cross-section is feasible.

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[2] G. Wentzel, Helv. Phys. Acta 13, 269 (1940); W. Pauli and S. M. Dancoff, Phys. Rev. 62, 85 (1942); A. Kaufman, Phys. Rev. (to be published).

The first requirement is satisfied by almost any reasonable procedure, and it is quite easy to find some which also fulfil the third condition, since an approximate calculation--e.g., Tomonaga's--may be used to supply such information about the one-body problem as is needed in the scattering calculation. However, it is considerably more difficult to find a method which in addition gives the correct s.c. limit. It may well happen, of course, that a procedure which fulfils only two of the conditions will be closer to the exact answer in the intermediate coupling region than one which satisfies all three. In particular, our insistence that condition 2° be satisfied may seem unjustified since from qualitative indications, such as the failure to detect stable isobars, it appears that the actual value of g cannot be large enough to make s.c. theory applicable. However, it is equally certain that g is not small enough to justify the use of perturbation theory and in the absence of evidence to the contrary it seems reasonable to place more confidence in a method which is correct in <u>both</u> the weak and strong coupling limits.

In this paper we shall describe a procedure (i.e., a set of approximations) for solving the meson nucleon scattering problem which satisfies all three of these requirements. Subsequent papers will deal with the application of this same method to the scattering problem for pseudo-scalar mesons and with its extension to the case of non-static nucleons. In section II we briefly discuss the Tomonaga solution of the one-body problem including some features which, to our knowledge, have not been described before. The solution of the scattering problem is described in section III, and graphs showing the variation of $d\sigma/d\Omega$ with energy and coupling constant are presented. Section IV shows how the same techniques may be used to find the s.c. isobar separation. A general discussion of the method and results is given in section V.

In appendix I a variational approach to the scattering problem is discussed. This method satisfies conditions 1^{o} and 3^{o} but gives only order of magnitude agreement with the s.c. result. Still another way of finding the scattering has recently been proposed by Maki, Sato and Tomonaga[3]. They have shown that it satisfies requirements 1^{o} and 3^{o}, and state that with a suitable modification the correct s.c. scattering can also be obtained.

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3 Z. Maki, M. Sato and S. Tomonaga, Prog. of Theor. Phys. 9, 614 (1953). We are indebted to Dr. T. Kinoshita for informing us of this work prior to its publication.

2. The One Body Problem in the Tomonaga Approximation

In terms of momentum space annihilation operators $A(\underline{k})$, $B(\underline{k})$ for positive and negative mesons, respectively, and the usual Pauli isotopic spin matrices, $\tau_{\pm}$, the hamiltonian for the charged scalar field is[4]

$$H = \int d\underline{k} \left\{ A^*(\underline{k})\, A(\underline{k}) + B^*(\underline{k})\, B(\underline{k}) - gR(\underline{k}) \left[A(\underline{k})\, \tau_- + B(\underline{k})\, \tau_+ + \text{c.c.} \right] \right\}, \tag{1}$$

where

$$R(k) \equiv \rho(\underline{k})/\sqrt{2\omega},$$

$$\omega = \sqrt{k^2 + \mu^2},$$

and $\rho(\underline{k})$ is the Fourier transform of the nucleon source density $\rho(\underline{x})$ satisfying

$$\int d\underline{x}\ \rho(\underline{x}) = 1 .$$

As usual,

$$\left[A(\underline{k}), A^*(\underline{k}')\right] = \left[B(\underline{k}), B^*(\underline{k}')\right] = \delta(\underline{k} - \underline{k}') \tag{2}$$

and other commutators are zero. (We use units in which $\hbar$, c and μ (meson mass) have the value one.)

To solve the Schrodinger equation HF = EF for a state of total charge Q we introduce Tomonaga's ansatz

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4 G. Wentzel, Quantum Theory of Fields, (Interscience Publishers, Inc., New York, 1949).

$$F = \sum_n c_{n+Q,n} \frac{(a^*)^{n+Q} (b^*)^n}{\sqrt{n!\,(n+Q)!}} N + \sum_n c_{n+Q-1,n} \frac{(a^*)^{n+Q-1} (b^*)^n}{\sqrt{n!\,(n+Q-1)!}} P , \tag{3}$$

where

$$a = \int d\underline{k}\ \phi_+ (\underline{k})\ A(\underline{k}) \tag{4}$$

and b is defined similarly. The state vector N (or P) denotes the state: "bare neutron (or proton) plus meson vacuum". The $c_{n,m}$ and $\phi_\pm$ (which we take to be real functions) are to be determined by minimizing $\langle F | H | F \rangle$ subject to

$$\langle F | F \rangle = 1 \qquad \text{and} \qquad \int d\underline{k} \left[\phi_\pm (\underline{k}) \right]^2 = 1 ,$$

i.e.,

$$\delta \left\{ \langle F | H | F \rangle - \nu \langle F | F \rangle - \nu_+ \int \phi_+^2 - \nu_- \int \phi_-^2 \right\} = 0 . \tag{5}$$

Since

$$\langle F | H | F \rangle = \langle F | H^T | F \rangle , \tag{6}$$

with

$$H^T = a^* a\, \omega_+ + b^* b\, \omega_- - \left[\gamma_- (g_+ a + g_- b^*) + \text{c.c.} \right] \tag{7}$$

and

$$\omega_\pm = \int d\underline{k}\ \omega \phi_\pm^2 , \qquad g_\pm = g \int d\underline{k}\ R \phi_\pm ,$$

the variation with respect to the $c_{n,m}$ just leads to a simplified Schrodinger equation,

$$H^T F = \mathcal{E} F , \tag{8}$$

which can be solved anlytically in the weak and strong coupling limits and numerically in the intermediate coupling region. (The Lagrange multiplier ν, of course, is just $\mathcal{E}$.)

In carrying out the variation with respect to $\phi_\pm$ it is convenient to use the easily verified relation

$$\delta_{\phi_+} F = \left[\int d\underline{k} \ \delta\phi_+ (\underline{k}) \ A^*(\underline{k}) \ a \right] F \tag{9}$$

for the change in F produced by a small variation,

$$\phi_+ (\underline{k}) \longrightarrow \phi_+ (\underline{k}) + \delta\phi_+ (\underline{k}) .$$

We then have, from (5),

$$\langle F | H - \mathcal{E} \ | \ A^*(\underline{k}) \ aF \rangle + \text{c.c.} - \nu_+ \phi_+ (\underline{k}) = 0 .$$

Using the commutation relations (2) and noting that by (3) and (4)

$$A(\underline{k}) \ F = a \ \phi_+ (\underline{k}) \ F , \tag{10}$$

we obtain

$$\left[\langle F | a^*(H - \mathcal{E}) \ a | F \rangle \ \phi_+ + \langle F | \omega \ \phi_+ \ a^* - gR \Upsilon_- | aF \rangle + \text{c.c.} \right] - \nu_+ \phi_+ = 0 .$$

As in (6), H can now be replaced by H^T so we find

$$\left[N_+ (\omega - \omega_+) + M_+ \ g_+ - \nu_+ \right] \phi_+ - g \ M_+ \ R = 0 , \tag{11}$$

with

$$N_+ = \langle F | a^* a | F \rangle ,$$

$$M_+ = \tfrac{1}{2}\left[\langle F | a \, \Upsilon_- | F \rangle + \text{c.c.} \right] .$$

Since multiplying (11) by $\phi_+(\underline{k})$ and integrating gives $\nu_+ = 0$, we have finally

$$\phi_+(\underline{k}) = \frac{gM_+}{N_+} \cdot \frac{R(\underline{k})}{(\omega - \omega_+ + gM_+/N_+)}$$

or

$$\phi_+(\underline{k}) = (gM_+/N_+)\left[R(\underline{k})/(\omega - \lambda_+)\right] , \qquad (12)$$

where λ_+ is given in terms of gM_+/N_+ by the normalization requirement on ϕ_+ ,

$$(gM_+/N_+)^2 \int d\underline{k} \; \frac{R^2(\underline{k})}{(\omega - \lambda_+)^2} = 1 . \qquad (13)$$

A similar equation holds for ϕ_- .

Numerical calculations for $\phi_\pm$ and F have been carried out using the IBM Card Programmed Computer at UCRL, Livermore, California. In Fig. 1 through Fig. 4 we have presented some of the results of this work. Curves corresponding to the charge 2 isobar have been dotted for values of g below 8.66, where that state becomes unstable. Since it seems certain that the total hamiltonian H has no stable bound states of charge 2 or higher, for small g, we feel that the Tomonaga state vector F_2 has little physical meaning in that region. For the same reason, the entire $\mathcal{E}_3$ curve is dotted since $\mathcal{E}_3 - \mathcal{E}_1 > \mu$ for all values of g which we used.

Actually, the scattering calculations to be described in section III require very little information about the one-body problem. Besides the isobar energy separations, we need only the matrix elements $\langle 0 | \tau_+ | 1 \rangle$ and $\langle 1 | \tau_3 | 1 \rangle$ (for ordinary scattering) and $\langle 1 | \tau_+ | 2 \rangle$ and $\langle 2 | \tau_+ | 3 \rangle$ (for charge exchange scattering). Since the variational calculation of the Tomonaga state described above is to be carried out anew for each value of the total charge Q of (3), $\phi_\pm$ and all quantities related to them-- $\lambda_\pm$, a, b, H^T, ε, $N_\pm$, etc.--depend on Q and really ought to carry a label Q which we have omitted in this section for simplicity. Because of the charge symmetry of the hamiltonian it is sufficient to solve the one-body problem for $Q \geq 1$ since $\varepsilon_Q = \varepsilon_{1-Q}$; $\lambda_{Q\pm} = \lambda_{1-Q,\mp}$; $N_{Q\pm} = N_{1-Q,\mp}$; etc.

Since we have used a static approximation for the nucleon, it is necessary to introduce a cut-off in all of the calculations. We have chosen the one which seems most convenient for calculations:

$$\rho(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int d\underline{x}\, \rho(\underline{x})\, e^{i\,\underline{k}\cdot\underline{x}} = \begin{cases} (2\pi)^{-3/2} , & |\underline{k}| < M \\ 0 , & |\underline{k}| > M \end{cases} \tag{14}$$

where $M = 6.62\,\mu$ is the nucleon mass.

3. Meson Scattering.

Of the three requirements discussed in section I, the most difficult to satisfy seems to be that of agreement with s.c. theory. Thus, using a technique reminiscent of that theory, we shall work directly with the matrix elements of the field and isotopic spin operators in solving the scattering problem. In contrast to the approach customary in perturbation theory, we shall not attempt to calculate explicitly the relevant state vectors. The latter contain more information than we actually require whereas the matrix elements are closely related to observable quantities. For instance, suppose that $|Q\rangle$ is the state vector of a physical nucleon ($Q = 1$ for proton, $Q = 0$ for neutron) while $|Q+1, \underline{p}+\rangle$ represents a state of total charge $Q+1$ in which a positive meson of momentum $\underline{p}$ is incident upon the charge Q nucleon. (It is to be emphasized that we mean these to be exact eigenstates of the total hamiltonian:

$$H|Q\rangle = E_Q|Q\rangle \qquad H|Q\,\underline{p}+\rangle = E_{Q\,\underline{p}+}|Q\,\underline{p}+\rangle\ .) \tag{15}$$

Then $\langle Q|A(\underline{x})|Q+1, \underline{p}+\rangle$ is asymptotically equal to the wave function of the scattered positive meson, i.e.,

$$\langle Q|A(\underline{x})|Q+1, \underline{p}+\rangle \longrightarrow (2\pi)^{-3/2}\left[e^{i\,\underline{p}\cdot\underline{x}} + \alpha\, e^{i\,pr}/r\right] \tag{16}$$

for $r = |\underline{x}| \to \infty$, and $|\alpha|^2$ gives the scattering cross section. If the coupling is strong enough so that stable isobars of higher charge, e.g., $Q = 2$ and $Q = 3$, exist, then charge exchange scattering may also occur for

$\omega' = \omega_p - E_{Q+2} + E_Q > \mu$. In a similar way we can then find the cross section for that process from $\langle Q+2 \mid B(\underline{x}) \mid Q+1, \underline{p}+\rangle$, which is asymptotically the wave function of the outgoing negative meson,

$$\langle Q+2 \mid B(\underline{x}) \mid Q+1, \underline{p}+\rangle \longrightarrow (2\pi)^{-3/2} \alpha' e^{ip'r}/r \;, \tag{17}$$

where

$$p' = \sqrt{\omega'^2 - \mu^2} \;.$$

In the following we need consider only the case where the incident meson is positive since it follows from the charge symmetry of the hamiltonian that $\sigma(\pi^+ + P) = \sigma(\pi^- + N)$, etc.

In order to find these matrix elements, we turn to the equations of motion. From the hamiltonian (1) we obtain the operator equations

$$-i\,\mathring{A} \equiv [H, A] = -\omega A + gR\tau_+ \tag{18}$$

$$-i\,\mathring{B} = -\omega B + gR\tau_- \tag{19}$$

$$-i\,\mathring{\tau}_+ = g\int d\underline{k}\; R(A + B^*)\,\tau_3 \tag{20}$$

$$-i\,\mathring{\tau}_3 = 2g\int d\underline{k}\; R\left[(A^* + B)\tau_+ - (A + B^*)\tau_-\right] \tag{21}$$

together with corresponding relations for the Hermitian conjugate operators, A^*, B^*, and τ_- . In a matrix representation of these operators which uses as basis a complete set of energy eigenstates, these equations give rise to a set of coupled integral equations for the matrix elements. Of course, in addition to the matrix elements (16) and (17), an infinite number of others are included in this set.

Since we cannot solve these coupled equations exactly, we shall introduce the approximation of neglecting all but a few of the matrix elements involved. We shall demonstrate that this procedure can be carried out in such a way that <u>all</u> of the requirements set forth in section I are satisfied.

The basis of energy eigenstates will consist of the following states:

$$\begin{aligned}
&|Q\rangle &&\sim \text{ nucleon isobar of charge } Q; \\
&|Q+1, \underline{p}+\rangle &&\sim \text{ positive meson of momentum } \underline{p} \text{ incident on a nucleon isobar of charge } Q; \\
&|Q-1, \underline{p}-\rangle &&\sim \text{ negative meson of momentum } \underline{p} \text{ incident on a nucleon isobar of charge } Q;
\end{aligned} \tag{22}$$

together with states similar to these representing two, three, ... incident mesons. For small values of g, Q may assume only the values 0 and 1. However, when g is large, isobars of higher charge are stable and in that case Q must be allowed to take on all integral values from Q_m to $1 - Q_m$ where Q_m is the largest charge for which

$$E_{Q_m} - E_1 \leq \mu .$$

(In the extreme s.c. limit $Q_m \rightarrow \infty$ since $E_Q - E_{Q-1}$ is of order g^{-2}.)

The approximations which we shall make are the following, in which C represents any of the operators A, B, γ_+ , γ_3, etc.:

I. We neglect all matrix elements involving states with more than one incident meson.

II. We set

$$\langle Q\, \underline{p}\, j \,|\, C \,|\, Q'\, \underline{p}'\, j'\rangle = \delta_{jj'}\, \delta(\underline{p}-\underline{p}')\, \langle Q+1 \,|\, C \,|\, Q' \pm 1\rangle, \tag{23}$$

where j, j' are + or − and the upper sign is to be used for $j = -$, the lower for $j = +$.

III. We approximate one-body matrix elements by the values obtained from the Tomonaga solution of the one-body problem

$$\langle Q | C | Q' \rangle = \langle F_Q | C | F_{Q'} \rangle , \tag{24}$$

where F_Q is the lowest charge Q eigenstate of the Tomonaga hamiltonian,

$$H_Q^T F_Q = \mathcal{E}_Q F_Q . \tag{25}$$

In addition, the isobar energies,

$$E_Q = \langle Q | H | Q \rangle , \tag{26}$$

are approximated by the corresponding Tomonaga values $\mathcal{E}_Q$.

By actually computing the values of the matrix elements which have been neglected, it can be shown, a posteriori, that the approximations I and II are valid in both the weak and strong coupling limits. (That III is correct in both limits follows from Tomonaga's work[1].) In general, I and II correspond to the assumption that the physical nucleon is not affected very much by the incident meson. The quantitative justification of all three assumptions for intermediate values of g can be accomplished by an iterative procedure to be described in a subsequent paper.

To solve the scattering problem, we may begin with the equation for $\langle 0 | A | 1\, \underline{p} + \rangle$. From (18) we find

$$(\omega - \omega_p) \langle 0 | A | 1\, \underline{p} + \rangle = g\, R \langle 0 | \tau_+ | 1\, \underline{p} + \rangle \tag{27}$$

since

$$E_{Q\, \underline{p}\, +} = E_{Q \mp 1} + \omega_p .$$

Solving (27) with the boundary condition indicated by (16) we obtain

$$\langle 0 | A(\underline{k}) | 1\, \underline{p} + \rangle = \delta(\underline{k} - \underline{p}) + \frac{g\, R(\underline{k})}{\omega - \omega_p - i\varepsilon} \langle 0 | \tau_+ | 1\, \underline{p} + \rangle . \tag{28}$$

Thus, to compute the cross-section for ordinary π^+, neutron scattering,

$$d\sigma/d\Omega = \left| 4\pi^2 \omega_p\, g\, R(\underline{p}) \langle 0 | \tau_+ | 1\, \underline{p} + \rangle \right|^2 , \tag{29}$$

we need to know $\langle 0 | \tau_+ | 1\, \underline{p} + \rangle$. From (20), we have

$$-\omega_p \langle 0 | \tau_+ | 1\, \underline{p} + \rangle = g \int d\underline{k}\, R \sum_n \langle 0 | A + B^* | n \rangle \langle n | \tau_3 | 1\, \underline{p} + \rangle ,$$

where $| n \rangle$ denotes any eigenstate of H and the summation is over all values of n. Invoking assumption I, we can reduce this to the much simpler equation

$$-\omega_p \langle 0 | \tau_+ | 1\, \underline{p} + \rangle = g \int d\underline{k}\, R \Big\{ \langle 0 | A + B^* | 1 \rangle \langle 1 | \tau_3 | 1\, \underline{p} + \rangle$$

$$+ \sum_{j=\pm} d\underline{p}' \langle 0 | A + B^* | 1\, \underline{p}'\, j \rangle \langle 1\, \underline{p}'\, j | \tau_3 | 1\, \underline{p} + \rangle \Big\}$$

or, using assumption II,

$$-\omega_p \langle 0 | \tau_+ | 1\, \underline{p} + \rangle = g \int d\underline{k}\, R \Big\{ \langle 0 | A + B^* | 1 \rangle \langle 1 | \tau_3 | 1\, \underline{p} + \rangle$$

$$+ \langle 0 | A + B^* | 1\, \underline{p} + \rangle \langle 0 | \tau_3 | 0 \rangle \Big\} . \tag{30}$$

In addition to the one-body matrix elements, which by assumption III are to be considered as known, we have now introduced the new matrix elements $\langle 1 | \tau_3 | 1\, \underline{p} + \rangle$ and $\langle 0 | B^* | 1\, \underline{p} + \rangle$, so we continue writing equations of motion until we have a closed set (number of equations equal to the number of unknown matrix elements).

Before carrying out this procedure in general, we shall consider the two limiting cases (small g and large g) in order to illustrate the method for simple cases and also to demonstrate that conditions 1° and 2° are satisfied.

(a) Weak Coupling

In this case, the Tomonaga solution of the one-body problem (which here coincides with ordinary perturbation theory) tells us that $\langle 0 | A + B^* | 1 \rangle$ is of order g while $\langle 0 | \tau_3 | 0 \rangle = 1 + \mathcal{O}(g^2)$. Furthermore, we have from (19)

$$\langle 0 | B^* | 1\, \underline{p} + \rangle = \frac{gR}{\omega + \omega_p} \langle 0 | \tau_+ | 1\, \underline{p} + \rangle \tag{31}$$

so that if in (30) we retain only the terms of lowest order in g we get

$$-\omega_p \langle 0 | \tau_+ | 1\, \underline{p} + \rangle = g \int d\underline{k}\, R \langle 0 | A | 1\, \underline{p} + \rangle$$

or, using (28)

$$\langle 0 | \tau_+ | 1\,\underline{p}\,+\rangle = -g\,R(p)/\omega_p \quad , \tag{32}$$

where again terms of higher oder in g have been dropped. Substituting (32) into (29), we find for $\pi^+ + N \rightarrow \pi^+ + N$ the cross-section

$$d\sigma/d\Omega = g^4 (16\,\pi^2\,\omega^2)^{-1} \quad , \tag{33}$$

in agreement with perturbation theory[4]. In a similar way we can show that the cross-section for $\pi^+ + P \rightarrow \pi^+ + P$ is also given by (33).

(b) Strong Coupling

When g is very large, the first term on the right side of (30) is larger by two orders of g than the second term, in contrast to the situation for weak coupling. To see this we note first that according to the Tomonaga solution of the one-body problem (which is here the same as conventional s.c. theory to leading order in g^{-1}), $\langle 0 | \tau_3 | 0 \rangle$ is of order g^{-4}, while $\langle 0 | A + B^* | 1 \rangle$ is of order g [5]. In addition, we shall see[6] that $\langle 1 | \tau_3 | 1\,\underline{p}\,+\rangle$ is of order g^{-3} while $\langle 0 | A + B^* | 1\,\underline{p}\,+\rangle$ is of order g^0, so the first term within the curly brackets of (30) is of order g^{-2} while the second is of order g^{-4}. Then (30) becomes

$$\langle 0 | \tau_+ | 1\,\underline{p}\,+\rangle = \frac{-g}{\omega_p} \int d\underline{k}\; R \langle 0 | A + B^* | 1 \rangle \langle 1 | \tau_3 | 1\,\underline{p}\,+\rangle \quad . \tag{34}$$

From (18) and (19) we find

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5 These statements can also be obtained directly from the equations of motion without recourse to the Tomonaga approximation--cf. section IV.

6 The a posteriori justification of assumptions concerning the relative magnitude of various terms is characteristic of most s.c. calculations[2].

$$\langle 0 | A | 1 \rangle = \langle 0 | B^* | 1 \rangle = g\, R \langle 0 | \tau_+ | 1 \rangle / \omega$$
$$= g\, R/2\omega \qquad (35)$$

since the Tomonaga (or s.c.) solution of the one body problem gives[7]

$$\langle 0 | \tau_+ | 1 \rangle = \tfrac{1}{2} \qquad (36)$$

Then from (34),

$$\langle 0 | \tau_+ | 1\, \underline{p} + \rangle = -g^2 K_1 \langle 1 | \tau_3 | 1\, \underline{p} + \rangle / \omega_p , \qquad (37)$$

where

$$K_n \equiv \int d\underline{k}\, R^2/\omega^n . \qquad (38)$$

Evidently we now need an equation for $\langle 1 | \tau_3 | 1\, \underline{p} + \rangle$. From (21) we obtain

$$-\omega_p \langle 1 | \tau_3 | 1\, \underline{p} + \rangle = 2g \int d\underline{k}\, R \Big\{ \langle 1 | \tau_+ | 2 \rangle \langle 2 | A^* + B | 1\, \underline{p} + \rangle$$
$$+ \langle 1 | \tau_+ | 2\, \underline{p} + \rangle \langle 1 | A^* + B | 0 \rangle - \langle 1 | \tau_- | 0 \rangle \langle 0 | A + B^* | 1\, \underline{p} + \rangle$$
$$- \langle 1 | \tau_- | 0\, \underline{p} + \rangle \langle -1 | A + B^* | 0 \rangle \Big\} \qquad (39)$$

where, as in the derivation of (30), we have made use of assumptions I and II. Although (39) contains several new matrix elements it is quite easy to

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7 This corresponds to the fact that for large g the physical nucleon involves a large number of bound mesons and it is equally likely for the bare nucleon to be either a proton or a neutron.

express them all in terms of $\langle 1 | \tau_3 | 1\,\underline{p}+ \rangle$ and thus obtain the desired closed set. Thus, from (19) we have

$$\langle 2 | B | 1\,\underline{p}+ \rangle \left(\omega - \omega_p + E_2 - E_0\right) = g\,R \langle 2 | \tau_- | 1\,\underline{p}+ \rangle ,$$

which, with the boundary condition (17), gives

$$\langle 2 | B | 1\,\underline{p}+ \rangle = \frac{g\,R}{\omega - \omega_p + E_2 - E_0 - i\epsilon} \langle 2 | \tau_- | 1\,\underline{p}+ \rangle . \tag{40}$$

In the s.c. limit where the isobar separation is of order g^{-2} this reduces to

$$\langle 2 | B | 1\,\underline{p}+ \rangle = \frac{g\,R}{\omega - \omega_p - i\epsilon} \langle 2 | \tau_- | 1\,\underline{p}+ \rangle . \tag{41}$$

In a similar fashion we find

$$\langle 2 | A^* | 1\,\underline{p}+ \rangle = \frac{g\,R}{\omega + \omega_p} \langle 2 | \tau_- | 1\,\underline{p}+ \rangle , \tag{42}$$

$$\langle 0 | B^* | 1\,\underline{p}+ \rangle = \frac{g\,R}{\omega + \omega_p} \langle 0 | \tau_+ | 1\,\underline{p}+ \rangle \tag{43}$$

where we have again neglected the isobar separations.

In calculating $\langle 1 | \tau_+ | 2\,\underline{p}+ \rangle$ from (20) we follow a procedure slightly different from that which led to (30). Instead of writing

$$\langle 1 | (A + B^*)\,\tau_3 | 2\,\underline{p}+ \rangle = \sum_n \langle 1 | A + B^* | n \rangle \langle n | \tau_3 | 2\,\underline{p}+ \rangle , \tag{44}$$

we can equally well use

$$\langle 1 | (A + B^*) \mathcal{T}_3 | 2\ \underline{p}+ \rangle = \sum_n \langle 1 | \mathcal{T}_3 | n \rangle \langle n | A + B^* | 2\ \underline{p}+ \rangle, \tag{45}$$

since the isotopic spin operators, $\mathcal{T}$, certainly commute with the field operators A and B. The form (45) has the advantage that upon applying assumptions I and II we obtain on the right hand side no new matrix elements. Thus, we find

$$-\omega_p \langle 1 | \mathcal{T}_+ | 2\ \underline{p}+ \rangle = g \int d\underline{k}\ R \Big\{ \langle 1 | \mathcal{T}_3 | 1 \rangle \langle 1 | A + B^* | 2\ \underline{p}+ \rangle + \langle 1 | \mathcal{T}_3 | 1\ \underline{p}+ \rangle \langle 0 | A + B^* | 1 \rangle \Big\}$$

which, in consequence of the remarks preceeding (34)[8] reduces to

$$\langle 1 | \mathcal{T}_+ | 2\ \underline{p}+ \rangle = -\frac{g}{\omega_p} \int d\underline{k}\ R \langle 0 | A + B^* | 1 \rangle \langle 1 | \mathcal{T}_3 | 1\ \underline{p}+ \rangle,$$

or

$$\langle 1 | \mathcal{T}_+ | 2\ \underline{p}+ \rangle = -\frac{g^2 K_1}{\omega_p} \langle 1 | \mathcal{T}_3 | 1\ \underline{p}+ \rangle, \tag{46}$$

if we use (35). In an entirely similar fashion we find the remaining matrix elements

$$\langle 2 | \mathcal{T}_- | 1\ \underline{p}+ \rangle = \frac{g^2 K_1}{\omega_p} \langle 1 | \mathcal{T}_3 | 1\ \underline{p}+ \rangle, \tag{47}$$

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8 Because of the charge symmetry of the hamiltonian, $\langle 0 | \mathcal{T}_3 | 0 \rangle = -\langle 1 | \mathcal{T}_3 | 1 \rangle$.

and

$$\langle 1 | \tau_- | 0\ \underline{p}+\rangle = \frac{g^2 K_1}{\omega_p} \langle 1 | \tau_3 | 1\ \underline{p}+\rangle . \tag{48}$$

Substituting (28), (35), (37), (41), (42), (43), (46), (47), and (48) into (39) we find

$$- \omega_p \langle 1 | \tau_3 | 1\ \underline{p}+\rangle = 2g \int d\underline{k}\ R \left\{ \langle 1 | \tau_+ | 2\rangle \cdot \frac{2g^3 K_1}{\omega_p} \frac{R}{\omega^2 - \omega_p^2 - i\epsilon} \right.$$

$$\cdot \langle 1 | \tau_3 | 1\ \underline{p}+\rangle - \frac{g^3 K_1 R}{\omega_p\ \omega} \langle 1 | \tau_3 | 1\ \underline{p}+\rangle - \langle 1 | \tau_- | 0\rangle$$

$$\cdot \left[\delta(\underline{k} - \underline{p}) - \frac{2g^3}{\omega_p} \frac{R}{\omega^2 - \omega_p^2 - i\epsilon} \langle 1 | \tau_3 | 1\ \underline{p}+\rangle \right]$$

$$\left. - \frac{g^3 K_1 R}{\omega_p\ \omega} \langle 1 | \tau_3 | 1\ \underline{p}+\rangle \right\} . \tag{49}$$

Using the Tomonaga (or s.c.) one-body matrix elements[7],

$$\langle 1 | \tau_- | 0\rangle = \langle 0 | \tau_+ | 1\rangle^* = \langle 1 | \tau_+ | 2\rangle = \tfrac{1}{2} , \tag{50}$$

we find

$$\langle 1 | \tau_3 | 1\ \underline{p}+\rangle = \frac{g\ R(p)\ \omega_p}{\omega_p^2 - 4\ g^4\ K_1 \left[K_1 - \tfrac{1}{2}\ I(\omega_p) \right]} ,$$

or, in the s.c. limit,

$$\left\langle 1 \left| \tau_3 \right| 1\ \underline{p} + \right\rangle = \frac{R(p)\ \omega_p}{2\ g^3\ K_1 \left[I(\omega_p) - 2\ K_1 \right]} , \tag{51}$$

where

$$I(x) = \int d\underline{k}\ \frac{2\ R^2\ \omega}{\omega^2 - x^2 - i\epsilon} = \int d\underline{k}\ \frac{\rho^2(k)}{\omega^2 - x^2 - i\epsilon} . \tag{52}$$

For our choice (14) of ρ , this becomes

$$I(\omega_p) = \frac{1}{2\pi^2} \left[M + \frac{p}{2} \log \frac{M - p}{M + p} + \frac{\pi\ i\ p}{2} \right] . \tag{53}$$

Although $I(\omega_p)$ diverges for $M \longrightarrow \infty$, the denominator of (51) remains finite since

$$K_1 = \frac{1}{4\pi^2} \left[M - \tan^{-1} M \right] . \tag{54}$$

Thus, with

$$J(\omega_p) = \frac{I(\omega_p) - 2\ K_1}{\omega_p^2} = \int d\underline{k}\ \frac{\rho^2(k)}{\omega^2 (k^2 - p^2 - i\epsilon)} \tag{55}$$

we find

$$J(\omega_p) \xrightarrow[M \to \infty]{} \frac{\mu + i\ p}{4\pi\ \omega_p^2} . \tag{56}$$

For M finite,

$$\langle 1 | \tau_3 | 1\, \underline{p} + \rangle = R(\underline{p}) \left[2\, g^3\, K_1\, \omega_p\, J(\omega_p) \right]^{-1} , \tag{57}$$

and so

$$\langle 1 | \tau_3 | 1\, \underline{p} + \rangle \xrightarrow[M \to \infty]{} 2 \pi \omega_p\, R(\underline{p}) \left[g^3\, K_1\, (\mu + i\, p) \right]^{-1} . \tag{58}$$

Now we are in a position to calculate the desired scattering cross-sections. From (29) and (58) we find for the ordinary π^+, neutron cross-section:

$$d\sigma/d\Omega = (4\, \omega_p^2)^{-1} . \tag{59}$$

Moreover, comparing (37) and (47) we see that

$$\left| \langle 2 | \tau_- | 1\, \underline{p} + \rangle \right| = \left| \langle 0 | \tau_+ | 1\, \underline{p} + \rangle \right|$$

and so it follows from (41) that the cross-section for charge exchange π^+, neutron scattering is also given by (59). Both of these results are in agreement with s.c. theory, as are the other cross-sections (π^+ on proton) which can easily be found in the same way.

We have now demonstrated that our approximations I, II and III lead to results for meson scattering which agree with both the perturbation and s. c. theories in the appropriate limits. The general nature of our procedure should by now be fairly clear, so we shall simply outline the derivation of the formulas for the cross-sections in the general case (g arbitrary).

(c) General Case

In contrast to our procedure in (a) and (b), we must now retain in our equations all matrix elements which remain after assumptions I and II have been invoked. For instance, both of the terms on the right side of (30) must be retained for, as we have seen, one is important for small g, the other for large g, and we may expect them to be of comparable magnitude in the intermediate coupling region. In addition, assumption III now assumes its full importance, for while the necessary one-body matrix elements-- $\langle 0 | \tau_+ | 1 \rangle$, $\langle 1 | \tau_3 | 1 \rangle$, $\langle 1 | \tau_+ | 2 \rangle$, and $\langle 2 | \tau_+ | 3 \rangle$ -- can be found analytically in the weak and strong coupling limits, for intermediate values of g they must be computed numerically from some solution (e.g., Tomonaga's) of the one-body problem.

Except for these two points, the development proceeds in a manner very similar to that in (b). Since a few more matrix elements are involved, the algebra is somewhat more complicated, but the extra effort is not wasted. If we begin by looking for the matrix elements which describe the scattering of the π^+ on neutron, then we find that the closed set automatically includes the matrix elements needed for the ordinary scattering of π^+ on proton.

It appears expeditious to solve for the A and B matrix elements, as in (28) and (40), thereby eliminating them from the equations. We are then left with algebraic equations for the τ matrix elements. To find the cross-sections we need four of these: $\langle 0 | \tau_+ | 1\, \underline{p} + \rangle$ and $\langle 2 | \tau_- | 1\, \underline{p} + \rangle$ for the π^+, neutron ordinary and charge exchange scattering, $\langle 1 | \tau_+ | 2\, \underline{p} + \rangle$ and $\langle 3 | \tau_- | 2\, \underline{p} + \rangle$ for the π^+, proton scattering. By using equations like (30), these may be expressed in terms of $\langle 1 | \tau_3 | 1\, \underline{p} + \rangle$. (Actually, $\langle 3 | \tau_- | 2\, \underline{p} + \rangle$ involves $\langle 2 | \tau_3 | 2\, \underline{p} + \rangle$ but this, also, can be expressed in terms of $\langle 1 | \tau_3 | 1\, \underline{p} + \rangle$:

$$\langle 2 | \tau_3 | 2\, \underline{p} + \rangle = \frac{I(0)}{I(\Delta)} \frac{\langle 0 | \tau_+ | 1 \rangle}{\langle 1 | \tau_+ | 2 \rangle} \langle 1 | \tau_3 | 1\, \underline{p} + \rangle \tag{60}$$

where

$$\Delta = E_2 - E_1 .$$

We obtain (60) by using, for instance, an equation like (30) for $\langle 1 | \tau_+ | 2\, \underline{p} + \rangle$.) Substituting these into (39) we obtain

$$\langle 1 | \tau_3 | 1\, \underline{p} + \rangle = -2\, g\, R(p) \langle 0 | \tau_+ | 1 \rangle^* \Big\{ g^2 \langle 1 | \tau_3 | 1 \rangle$$

$$\cdot \left[\frac{I(\omega_p)}{D_-(\omega_p)} + \frac{I(0)}{D_+(\omega_p)} \right] + 1 \Big\} \times \Big\{ -\omega_p + 2\, g^4\, I(0) \left| \langle 0 | \tau_+ | 1 \rangle \right|^2$$

$$\cdot \left[\frac{I(0)}{D_+(\omega_p)} - \frac{I(\omega_p)}{D_-(\omega_p)} \right] + 2\, g^4 \alpha\, I(\Delta) \left| \langle 1 | \tau_+ | 2 \rangle \right|^2$$

$$\cdot \left[\frac{I(\Delta)}{D_-(\omega_{21})} - \frac{I(\omega_{12})}{D_+(\omega_{12})} \right] \Big\}^{-1} , \tag{61}$$

where

$$D_{\pm}(x) = x \pm g^2 I(x) \langle 1 | \tau_3 | 1 \rangle$$

$$\omega_{Q, Q'} = \omega_p + E_Q - E_{Q'} \qquad (62)$$

$$\alpha = \begin{cases} 0 & \text{for } g \text{ such that } \Delta > 1 \\ 1 & \text{for } g \text{ such that } \Delta < 1 . \end{cases}$$

Knowing (61) we can compute the $\tau_{\pm}$ matrix elements and from these find the scattering cross-sections using equations like (29).

In Figs. 5 through 8 the results of these calculations are presented in the form of curves showing $d\sigma/d\Omega$ (in units $(\hbar/\mu c)^2$) vs. ω for several values of g. A semi-log plot has been used so that all curves could be drawn to the same scale. In Figs. 5 and 7, the ordinary π^+, neutron and π^+, proton cross-sections are given for g = 1, 2, 3, and 5.25. The cross-sections for higher g values are given in Figs. 6 and 8. According to the Tomonaga solution of the one body problem, the charge 2 isobar is stable for $g > 8.66$ (cf. Fig. 2). Thus, the π^+, neutron charge exchange cross-sections are also shown for g = 10.5 and g = 15. (Since the charge 3 isobar is not stable for the range of coupling constants which we used, the π^+, proton charge exchange scattering is not considered.) For g = 5.25, 7.46, 10.5, and 15 the one body matrix elements needed in the scattering calculation ($\langle 0 | \tau_+ | 1 \rangle$, $\langle 0 | \tau_3 | 0 \rangle = - \langle 1 | \tau_3 | 1 \rangle$, and $\langle 1 | \tau_+ | 2 \rangle$) have been found from a numerical solution of the Tomonaga problem as described in section 2. These were also used to draw the curves of Fig. 1, the matrix elements for other choices of g then being obtained from Fig. 1. As explained in section 5, all of these cross-section curves should be taken seriously only below the meson production threshold, $\omega < 2$.

4. The S.C. Isobar Separation

It is of interest to note that once the matrix elements for the scattering cross-section have been found the isobar separation can be determined by similar techniques. Here we shall carry this through for the s.c. limit where an analytic solution is possible. From (20) we obtain

$$\langle Q - 1 \,|\, \tau_+ \,|\, Q \rangle (E_{Q-1} - E_Q) = g \int d\underline{k}\, R \Big\{ \langle Q - 1 \,|\, A + B^* \,|\, Q \rangle \cdot \langle Q \,|\, \tau_3 \,|\, Q \rangle + \sum_{j=\pm} \int d\underline{p}\, \langle Q - 1 \,|\, A + B^* \,|\, Q\, \underline{p}\, j \rangle \langle Q\, \underline{p}\, j \,|\, \tau_3 \,|\, Q \rangle \Big\} , \tag{63}$$

or, using the other order for $(A + B^*)\tau_3$:

$$\langle Q - 1 \,|\, \tau_+ \,|\, Q \rangle (E_{Q-1} - E_Q) = g \int d\underline{k}\, R \Big\{ \langle Q - 1 \,|\, \tau_3 \,|\, Q - 1 \rangle \cdot \langle Q - 1 \,|\, A + B^* \,|\, Q \rangle + \sum_{j=\pm} \int d\underline{p} \langle Q - 1 \,|\, \tau_3 \,|\, Q - 1, \underline{p}\, j \rangle \langle Q - 1, \underline{p}\, j \,|\, A + B^* \,|\, Q \rangle . \tag{64}$$

If the scattering problem has been solved, then the only unknowns are the isobar separation and the one-body matrix elements. Now[7]

$$\langle Q - 1 \,|\, \tau_+ \,|\, Q \rangle = \frac{1}{2} ,$$

so using (18) and (19)

$$\langle Q - 1 \,|\, A \,|\, Q \rangle = \langle Q - 1 \,|\, B^* \,|\, Q \rangle^* = \frac{g\, R}{2\, \omega} + \mathcal{O}(g^{-1}) .$$

However, the s.c. limit for $\langle Q | \Upsilon_3 | Q \rangle$ is of much higher order in (1/g) and cannot be obtained from such simple considerations. We calculate it along with the isobar separation in the following way:

(i) Set $Q = 1$ in (63). Since $E_1 = E_0$, the left side is zero and we can solve for $\langle 1 | \Upsilon_3 | 1 \rangle$.

(ii) Equate the right hand sides of (63) and (64) and solve for

$$\langle Q | \Upsilon_3 | Q \rangle - \langle Q-1 | \Upsilon_3 | Q-1 \rangle .$$

(iii) From (i) and (ii) compute $\langle Q | \Upsilon_3 | Q \rangle$.

(iv) From (63) and (iii) compute $E_Q - E_{Q-1}$.

In carrying this out one must be careful to use for the scattering matrix elements the result obtained before letting $M \to \infty$, since in (63) and (64) $\underline{p}$ ranges from 0 to M. For instance, we must use (57) for $\langle 1 | \Upsilon_3 | 1 \, \underline{p} + \rangle$ and not the $M \to \infty$ form (58). The limit $M \to \infty$ must be taken only *after* the integrations in (63) and (64) have been performed.

The result of step (i) is

$$\langle 1 | \Upsilon_3 | 1 \rangle = -(g^4 K_1)^{-1} \int d\underline{p} \, \rho^2(\underline{p}) \left[2\omega_p^4 \, | J(\omega_p) |^2 \right]^{-1} \tag{65}$$

where $J(\omega_p)$ is defined by (55). The integral (65) has been evaluated by A. Kaufman[2]. Using his result we have

$$\langle 1 | \Upsilon_3 | 1 \rangle = -(4 g^4 K_1)^{-1} \left[1/K_3 - 1/2 \, (K_1)^2 \right] \xrightarrow[M \to \infty]{} -(4 g^4 K_1 K_3)^{-1} \tag{66}$$

From (ii) and (iii) we then find

$$\langle Q \,|\, \tau_3 \,|\, Q \rangle = (1 - 2\,Q)(4\,g^4\,K_1\,K_3)^{-1} \tag{67}$$

so that

$$E_Q - E_{Q-1} = (Q - 1)/g^2\,K_3 \tag{68}$$

and

$$E_Q = (Q - \tfrac{1}{2})^2/2\,g^2\,K_3 + \text{const.}$$

which is in agreement with conventional s.c. theory[2].

5. Discussion of Method and Results

In the preceding sections we have seen that our approximate method of solving the matrix equations of motion is correct in both the weak and strong coupling limits. To establish its validity in the intermediate region, the approximations of section 3 could be investigated by an iterative procedure in which one-body matrix elements and those of the form $\langle Q | C | Q', \underline{p}j, \underline{p}'j' \rangle$ or $\langle Q, \underline{p}j | C | Q', \underline{p}'j' \rangle$ are calculated from the equations of motion, using for $\langle Q | C | Q', \underline{p}j \rangle$, etc. the values found previously. Since the latter are only known numerically in the intermediate region, this calculation is fairly complicated and will be discussed in a subsequent paper.

In some respects our procedure resembles the Tamm-Dancoff approximation, since states with more than one incident meson are neglected. However, the total number of mesons allowed is considerably larger than in most Tamm-Dancoff calculations for no limitation is placed on the number of mesons bound to the nucleon. Any such restriction would, in fact, preclude agreement with s.c. theory where the average number of bound mesons is approximately $g^2 K_2/2$.

As can be seen from Figs. 5 through 8, the shape of the cross-section vs. energy curves for intermediate values of g is quite different from the $d\sigma/d\Omega \propto 1/\omega^2$ which characterizes the two limiting cases. The ordinate in these curves is actually the $|\alpha|^2$ or $|\alpha'|^2$ of (16) or (17). These give the scattering cross-section when the energy is below the threshold for meson production, but when $\omega > 2$ the meson production must be taken into account. Although this will be left for a subsequent paper, we have here carried the calculations beyond $\omega = 2$ to facilitate a discussion of the variation with g. For $g = 2$ the π^+, proton curve has a small peak which moves to the right as g increases. When $g = 7.46$ this peak is centered at $\omega = 4.25$ and has become very sharp and extremely large. For higher g values the peak decreases in height and moves back towards $\omega = 1$. The π^+, neutron ordinary scattering curves behave in a similar fashion, while the charge exchange cross-section in the region of interest $(1 < \omega < 2)$ is characterized by a very sharp rise from the threshold.

The essential test of any theory--comparison with experiment--cannot be applied to our results since we have used scalar mesons. Nevertheless, it would perhaps be desirable to "explain" the qualitative nature of the curves, e.g., in terms of resonance with a virtual isobar level. Even this proves difficult since as g increases from 2 to 8.66 (where the charge 2 isobar becomes stable) the peak in the $d\sigma/d\Omega$ vs. ω curve moves to the right, while we would expect that the excitation energy of a virtual isobar should decrease as g increases. As yet we have not found a simple "explanation" for the variation of $(d\sigma/d\Omega)$ with ω and g in terms of properties of the single nucleon. Since the scattering problem for intermediate values of g has not been extensively studied heretofore, it is difficult to say whether our results are "reasonable". This question will be resolved, however, by a comparison of the experimental pion, nucleon scattering data with the results of the calculations for pseudoscalar mesons which are now in progress.

One of the most unpleasant consequences of our static approximation for the nucleon is the necessity for a cut-off. Since this cannot be avoided in a non-relativistic treatment, we can simply hope that it is not very important for the low energy mesons. To verify this, we intend to repeat some of the above calculations using a different choice for the cut-off. Moreover, a generalization of the methods described in this paper to include recoil and relativistic effects is now being investigated. If a renormalization procedure can be found, it will be possible to eliminate the cut-off altogether. In this connection, it may be noted that even in its present, static form this matrix formulation has the advantage that only energy differences, and not the energies themselves, appear in the equations.

We wish to express our very sincere appreciation to Mr. James Baker and Mrs. Joan Lafon for their excellent computational work on both the scattering and the one body problem, and to Messrs. Robert Oeder and Lawrence Lasnik who carried out the solution for the one body state vectors on the IBM Card Programmed Calculator at UCRL, Livermore. We are also indebted to Dr. J. Lepore for many interesting discussions of the Tomonaga approximation and the strong coupling theory.

Appendix I. Variational Calculation of Meson Scattering.

The problem of meson scattering can also be treated by a modification of the Hulthén variational principle[9, 10]. For the state vector describing the scattering of a positive meson of initial momentum $\underline{p}$ by a proton we make the ansatz

$$\psi = a_s^* \,|1\rangle + b_s^* \,|3\rangle + \eta \,|2\rangle, \tag{A.1}$$

where $|Q\rangle$ is again the state vector corresponding to the nucleon isobar of charge Q. The scattering operators a_s and b_s are defined by

$$a_s = \int d\underline{k}\; \chi_+^*(\underline{k})\, A(\underline{k}) ,$$

$$b_s = \int d\underline{k}\; \chi_-^*(\underline{k})\, B(\underline{k}) , \tag{A.2}$$

where $\chi_+(\underline{k})$ is the Fourier transform of a function which for large r has the form of an incident plane wave plus an outgoing scattered wave, while χ_- is asymptotically just a scattered wave; viz.

$$\chi_+(\underline{x}) \longrightarrow (2\pi)^{-3/2} \left[e^{i\,\underline{p}\cdot\underline{x}} + \alpha\, e^{i\,p\,r}/r \right],$$

$$\chi_-(\underline{x}) \longrightarrow \alpha'\, e^{i\,p\,r}/r(2\pi)^{3/2} . \tag{A.3}$$

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9 Hulthén, Kungl. Fysio. Sällskapets Lund Förhand. 14 (1944), 1.

10 This approach has been used by R. Christian and T. D. Lee in their work on the Tomonaga approximation (to be published).

The functions $\chi_{\pm}$ and the constant η are to be determined by making $\langle\psi \mid E - H \mid \psi\rangle$ stationary, with

$$E = E_1 + \omega_p = E_1 + \sqrt{p^2 + \mu^2} . \tag{A.4}$$

Before carrying out this variation, we shall make a few remarks concerning the choice of the ansatz (A.1). The first term in (A.1) corresponds to the ordinary π^+, proton scattering while the second term takes account of the charge exchange scattering which can occur when the coupling is sufficiently strong. (Since we are primarily interested in the s.c. limit for this method we shall consider first the case in which g is large enough to make the Q = 2 and Q = 3 isobars stable.) The role of the third term can be appreciated if the variation with respect to η is carried out. We obtain

$$\delta\eta^* \langle 2 \mid \psi\rangle = 0$$

which shows that the value of η simply makes ψ orthogonal to ψ_2,

$$\eta = -\langle 2 | a_s^* | 1 \rangle - \langle 2 | b_s^* | 3 \rangle . \tag{A.5}$$

The importance of including this term may be understood from the following:

1. The scattering problem for a neutral scalar field, which is closely related to the s.c. limit of charged theory, may be treated by an ansatz analogous to (A.1). (Of course, it can also be solved exactly.) If a term corresponding to the $\eta|2\rangle$ of (A.1) is included, the correct answer, $d\sigma/d\Omega = 0$, is obtained. However, if this term is omitted, the cross-section is no longer zero. Instead, we find

$$(d\sigma/d\Omega)_{\text{neut.}} = \left| \omega_p/\pi - (p/\pi)\log(p+\omega_p)/\mu + \mu/2 + i\,p \right|^{-2} . \qquad (A.6)$$

2. As shown below, the charged meson scattering may be calculated by approximating the $|Q\rangle$ in (A.1) by the corresponding Tomonaga state vector, F_Q. If we consider the s.c. limit of the H^T problem and retain only the leading order terms in g, then we obtain $d\sigma/d\Omega = 0$, indicating that higher order corrections to the Tomonaga state vector must be considered. If, however, the $\eta|2\rangle$ term is omitted, then the cross-section turns out to be just 1/4 of $(d\sigma/d\Omega)_{\text{neutral}}$ which does not agree with the correct s.c. result (59). This discrepancy has been pointed out by Christian and Lee[10]. In this case it is clear that higher order corrections to the Tomonaga approximation could not remove this lack of agreement.

A final point concerning the ansatz (A.1) is that we shall consider only the case $\mu \leq \omega_p \leq 2\mu$. At higher energies, additional terms would of course be needed in (A.1) to take account of meson production, etc.

Returning now to the variational problem

$$\delta \langle \psi | E - H | \psi \rangle = 0$$

we see that variation with respect to χ_+^* gives

$$\int d\underline{k}\ \delta \chi_+^*(\underline{k}) \langle 1 | A(\underline{k}) [E - H] | \psi \rangle = 0.$$

Commuting $A(\underline{k})$ with (E - H) and taking account of (15) and (A.4) we find

$$\omega_p \langle 1 | A(\underline{k}) | \psi \rangle - \langle 1 | \omega\, A(\underline{k}) - g\, \tau_+\, R(\underline{k}) | \psi \rangle = 0$$

which by manipulations similar to those used in section 2, can be reduced to

$$(\omega_p - \omega)\chi_+ + (\omega_p - \omega)\Big\{ \langle F_1 | a_1^* \; a_1 | F_1 \rangle \phi_{1+} \int \chi_+ \phi_{1+}$$

$$+ \langle F_1 | b_1^* \; a_3 | F_3 \rangle \phi_{3+} \int \chi_- \phi_{1-} + \eta \langle F_1 | a_2 | F_2 \rangle \phi_{2+} \Big\}$$

$$+ g R \Big\{ \langle F_1 | a_1^* \tau_+ | F_1 \rangle \int \chi_+ \phi_{1+} + \langle F_1 | b_1^* \tau_+ | F_3 \rangle \int \chi_- \phi_{1-}$$

$$+ \eta \langle F_1 | \tau_+ | F_2 \rangle \Big\} = 0 \,, \tag{A.7}$$

where, according to (A.5),

$$\eta = - \langle F_2 | a_2^* | F_1 \rangle \int \chi_+ \phi_{2+} - \langle F_2 | b_2^* | F_3 \rangle \int \chi_- \phi_{2-} \,. \tag{A.8}$$

Here we have made the approximation of replacing $|Q\rangle$ by the corresponding eigenstate, F_Q, of H_Q^T. The integral equation for χ_- can be obtained from (A.7) by the substitutions

$$a \longleftrightarrow b$$

$$1 \longleftrightarrow 3 \quad \text{(in all subscripts)}$$

$$\omega_p \longleftrightarrow \omega_p - \Delta_{13}$$

where $\Delta_{13} = E_3 - E_1$.

These coupled integral equations for χ_+ and χ_- can be solved once the Tomonaga matrix elements appearing in (A.7) and (A.8) are known. Using Tomonaga's s.c. solution[1] of (8), we have evaluated these matrix elements in the s.c. limit. Table I shows the leading term, and also the first (i.e., $1/g^2$) correction thereto, for all of the relevant matrix elements. (The $\lambda_\pm$ calculated from (13), which are also listed for convenience, agree with the expressions given by Tomonaga[1], who obtained them in a somewhat different way.)

From Table I, we see at once that the right side of (A.7) contains terms of order g^2, g^0, ... However, upon substituting the various matrix elements into (A.7) we find that <u>all terms of order g^2 cancel</u>. Without the g^{-2} corrections to the matrix elements there would be <u>no scattering</u>. (As pointed out above, this cancellation does not occur if the $\eta|2\rangle$ term is omitted from (A.1).) With the aid of equation (12) and Table I we obtain

$$(\omega - \omega_p)\chi_+(\underline{k}) = -(\omega - \omega_p + \Delta_{13})\chi_-(\underline{k}) = J_1(\underline{k})$$

$$\equiv \frac{R(\underline{k})}{4\omega}\left\{\left[\frac{1}{K_3} + \frac{\omega_p K_4}{2(K_3)^2} - \frac{\omega_p}{2K_2}\right]\int \frac{R\chi}{\omega} - \frac{\omega_p}{K_3}\int\frac{R\chi}{\omega^2}\right\} -$$

$$- \frac{R(\underline{k})}{4\omega^2}\frac{\omega_p}{K_3}\int\frac{R\chi}{\omega} + \frac{R(\underline{k})}{8K_2}\int\frac{R\chi}{\omega} \qquad (A.9)$$

where

$$\chi \equiv \chi_+ - \chi_- .$$

Since we are concerned with terms of order g^{o}, we may drop Δ_{13} which is of order g^{-2}. Taking the sum and difference of the χ_+ and χ_- equations we find two elementary, uncoupled equations

$$(\omega - \omega_p)(\chi_+ + \chi_-) = 0 \qquad (A.10)$$

$$(\omega - \omega_p)\chi = 2 J_1 \qquad (A.11)$$

According to (A.10)

$$\chi_+(\underline{k}) + \chi_-(\underline{k}) = \delta(\underline{k} - \underline{p})$$

i.e., χ_+ and $(-\chi_-)$ can differ only by a plane wave so that the direct and charge exchange scattering cross-sections are equal, in agreement with s.c. theory. From (A.11) it is easy to obtain a solution of the form

$$\chi(\underline{k}) = \delta(\underline{k} - \underline{p}) - \frac{f(\omega)}{\omega - \omega_p - i\epsilon} , \qquad (A.12)$$

where $f(\omega)$ can be found explicitly by solving a pair of algebraic equations. In the limit of a point source, $\rho(\underline{x}) \longrightarrow \delta(\underline{x})$, the scattering amplitude $f(\omega_p)$ has the value

$$f(\omega_p) = \left[R(\underline{p})\right]^2 \omega_p^{-2} \left[K_3 + C(\omega_p)\right]^{-1} ,$$

where

$$C(\omega_p) \equiv \int d\underline{k} \; \frac{R^2(\underline{k})}{\omega^2(\omega - \omega_p - i\epsilon)} =$$

$$= (4\pi^2 \omega_p^2)^{-1}\left[\omega_p - p \log\left[(p + \omega_p)/\mu\right] + \pi\mu/2 + \pi i p\right] . \qquad (A.13)$$

Consequently, both the ordinary and charge exchange cross-sections have the value

$$d\sigma/d\Omega = \frac{1}{4}\left|\frac{\omega_p^2}{4\mu} + \frac{\omega_p}{\pi} - \frac{p}{\pi}\log\frac{p+\omega_p}{\mu} + \frac{\mu}{2} + i\,p\right|^{-2} . \tag{A.14}$$

In Fig. we have plotted the s.c. limt of $d\sigma/d\Omega$ as a function of incident meson energy. The solid curve is the correct s.c. result, (59). The one below it represents equation (A.14) which is seen to be correct only in order of magnitude. Lest we be tempted to accept this as a partial fulfillment of condition 2° of the introduction, we have plotted in Fig. also $\frac{1}{4}(d\sigma/d\Omega)_{neutral}$ as given by (A.6), i.e., the result obtained by omitting the $\eta|2\rangle$ term from the ansatz (A.1). It seems to us that this spurious neutral scalar cross-section has very little to do with the actual charged scalar problem and yet it, too, shows what might be called order of magnitude agreement with the correct charged scalar s.c. result. We conclude from this that nothing less than exact agreement with the s.c. answer can be accepted as fulfillment of condition 2°.

Finally, in the weak coupling limit the isobars of charge 2 and 3 do not exist so that no question concerning the $\eta|2\rangle$ term arises. The ansatz (A.1) then simplifies to

$$\psi = a_s^* \,|1\rangle \tag{A.15}$$

and an analysis like that which led to (21) gives

$$(\omega_p - \omega)\left[\chi_+ + N_{1+}\,\phi_{1+}\int\chi_+\,\phi_{1+}\right] + g\,R\,M_{1+}\int\chi_+\,\phi_{1+} = 0 . \tag{A.16}$$

As Christian and Lee[10] have shown, the cross section obtained from (A.16) for small g agrees with that given by ordinary perturbation theory, so that condition 1° is satisfied. For intermediate values of g the cross-section can be found numerically by using for the various Tomonaga matrix elements required in (A.7) or (A.16) the values computed from a numerical solution of the one-body problem, and so this method also satisfies condition 3°.

So far we have discussed only the π^+, proton scattering. However, the problem of π^+, neutron can be treated in an exactly analogous fashion. The appropriate ansatz there is

$$\psi = a_s^* |0\rangle + b_s^* |2\rangle + \eta' |1\rangle$$

for large g or

$$\psi = a_s^* |0\rangle + \eta' |1\rangle$$

for small g. In this case, the $\eta' |1\rangle$ term should be included for all values of g, since $|1\rangle$ (real proton) and $|0\rangle$ (real neutron) are always stable. Again, conditions 1° and 3° are satisfied but not 2°.

TABLE I

$$N_{Q+} \equiv \langle F_Q \mid a_Q^* \; a_Q \mid F_Q \rangle = \frac{1}{4} v^2 \left[1 + 2(Q-1) v^{-2} \right]$$

$$N_{Q-} \equiv \langle F_Q \mid b_Q^* \; b_Q \mid F_Q \rangle = \frac{1}{4} v^2 \left[1 - 2Q\, v^{-2} \right]$$

$$M_{Q+} \equiv \langle F_Q \mid a_Q \; \tau_- \mid F_Q \rangle = \frac{1}{4} v \left[1 + (Q - \tfrac{1}{2}) v^{-2} \right]$$

$$M_{Q-} \equiv \langle F_Q \mid b_Q \; \tau_+ \mid F_Q \rangle = \frac{1}{4} v \left[1 - (Q - \tfrac{1}{2}) v^{-2} \right]$$

$$\langle F_1 \mid b_1^* \; a_3 \mid F_3 \rangle = \frac{1}{4} v^2 (1 + v^{-2}) \exp(L\, v^{-2})$$

$$\langle F_1 \mid a_2 \mid F_2 \rangle = \frac{1}{2} v (1 + v^{-2}) \exp(\tfrac{1}{4} L\, v^{-2})$$

$$\langle F_3 \mid b_2 \mid F_2 \rangle = \frac{1}{2} v (1 + 2\, v^{-2}) \exp(\tfrac{1}{4} L\, v^{-2})$$

$$\langle F_1 \mid b_1^* \; \tau_+ \mid F_3 \rangle = \frac{1}{4} v (1 - \tfrac{3}{2} v^{-2}) \exp(L\, v^{-2})$$

$$\langle F_1 \mid \tau_+ \mid F_2 \rangle = \langle F_2 \mid \tau_+ \mid F_3 \rangle = \frac{1}{2} \exp(\tfrac{1}{4} L\, v^{-2})$$

$$\langle F_3 \mid a_3^* \; \tau_- \mid F_1 \rangle = \frac{1}{4} v (1 + \tfrac{3}{2} v^{-2}) \exp(L\, v^{-2})$$

$$\lambda_{Q\pm} = \left[\pm (Q - \tfrac{1}{2}) - K_1 K_3 (K_2)^{-2} \right] / g^2 K_3$$

where

$$v = g (K_2)^{\frac{1}{2}} \left\{ 1 + g^{-2} \left[1 - K_1 K_3 (K_2)^{-2} \right] (K_2)^{-1} \right\}$$

$$L = 1 - K_2 K_4 (K_3)^{-2}$$

$$K_n = \int d\underline{k} \; R^2(\underline{k}) \; \omega^{-n}$$

FIGURE CAPTION

Figure 1. Matrix elements of the one-body problem computed from the Tomonaga approximation.

Figure 2. Nucleon isobar energies computed from the Tomonaga approximation. The charge 2 isobar is stable for $g > 8.66$.

Figure 3. $\lambda_{\pm}$ vs. g for the isobars of charge 1 and charge 2 in the Tomonaga approximation as computed from (13).

Figure 4. Average number of positive and negative bound mesons in the meson cloud surrounding the charge 1 and 2 nucleon isobars.

Figure 5. Cross section vs. total energy for π^+, neutron scattering with $g = 1, 2, 3,$ and 5.25. The result of lowest order perturbation theory for $g = 1$ is also shown for comparison.

Figure 6. Cross section vs. total energy for π^+, neutron scattering with $g = 7.46$, 10.5 and 15. The charge exchange cross section for the latter two coupling constants is denoted by (ex.). The s.c. theory result is also included.

Figure 7. Cross section vs. total energy for π^+, proton scattering with $g = 1, 2, 3,$ and 5.75. The result of lowest order perturbation theory for $g = 1$ is also shown for comparison.

Figure 8. Cross section vs. total energy for π^+, proton scattering with $g = 7.46$, 10.5 and 15. The s.c. theory result is also included.

Figure 9. S.c. limit of the variational scattering calculation described in Appendix I. The lower curve is a plot of (A.14) while the upper curve shows the result obtained if the $\eta|2\rangle$ term is omitted from the ansatz (A.1). The s.c. theory[2] cross section is also shown for comparison.

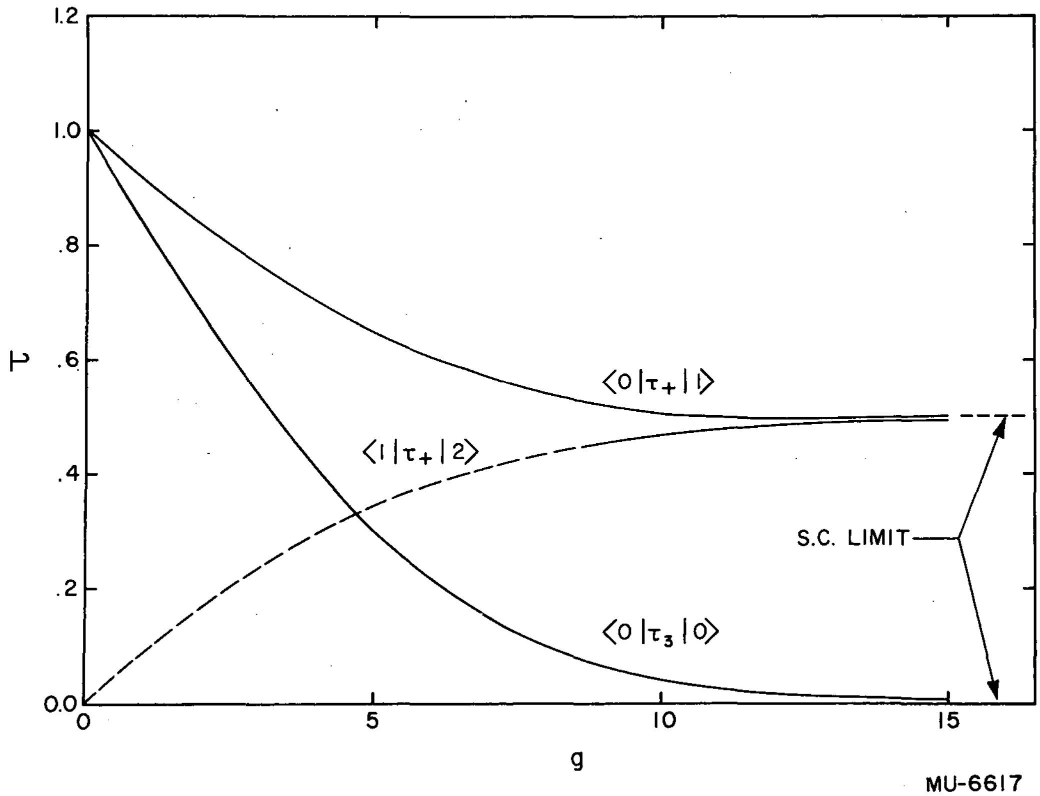


Fig. 1

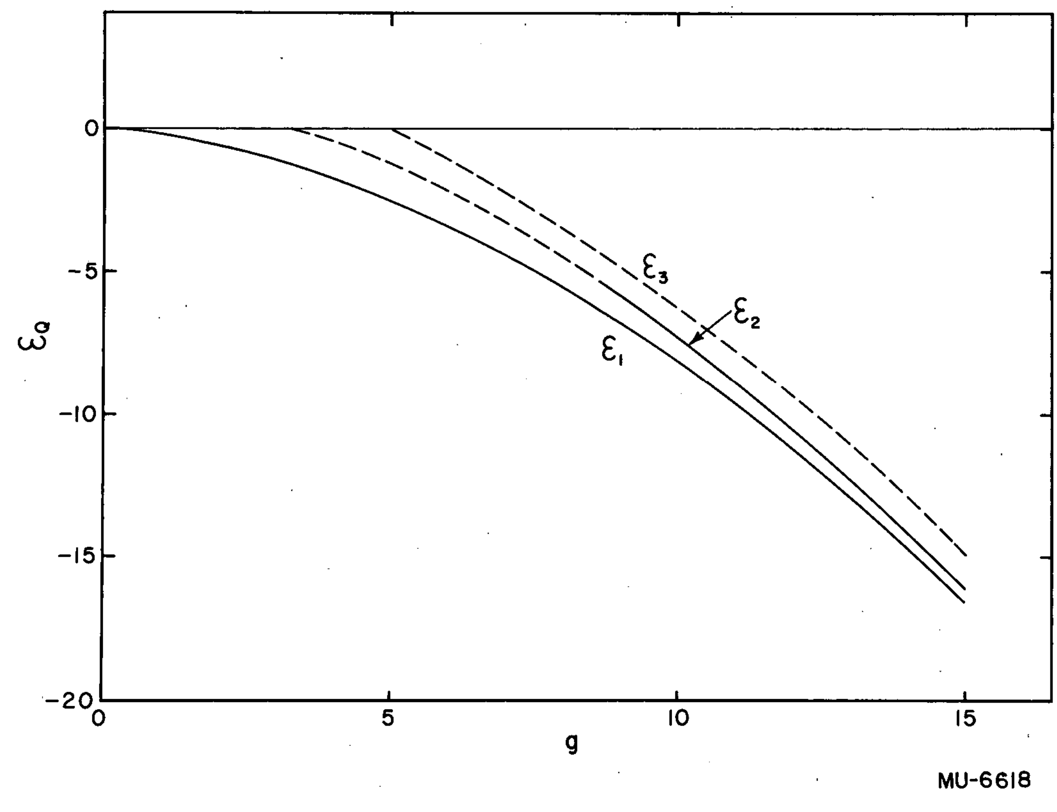
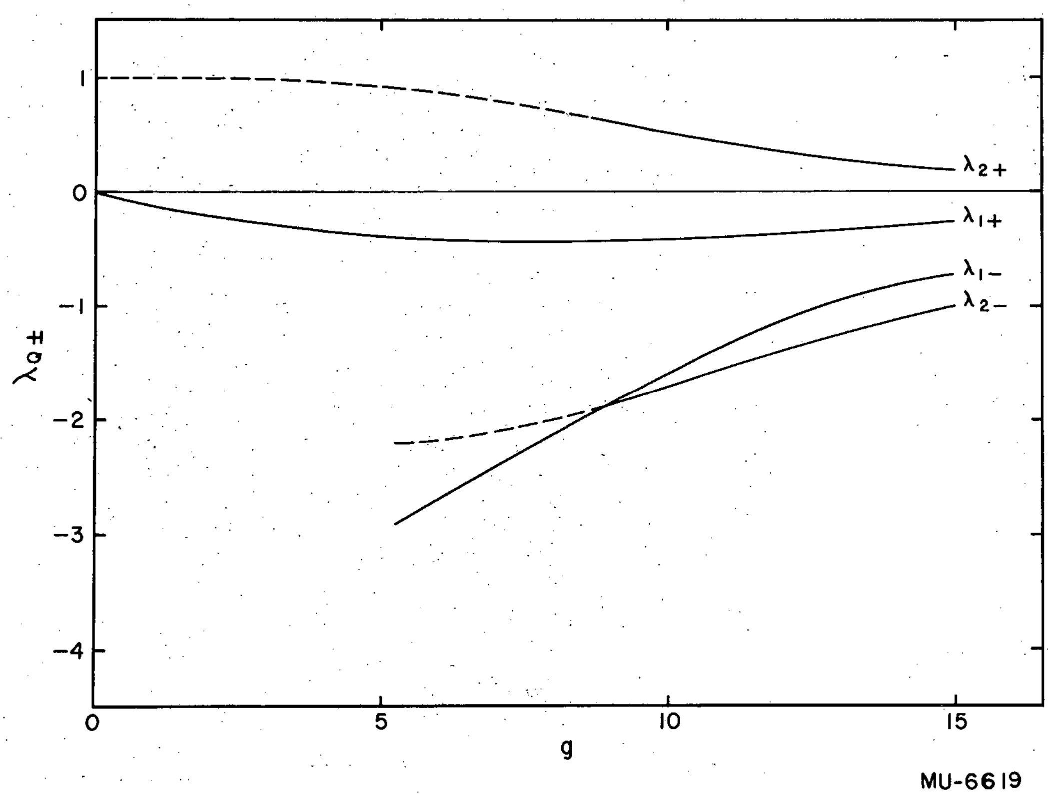

0
-5
-10
-15
-20
$\mathcal{E}_q$
0
5
10
15
g
$\mathcal{E}_3$
$\mathcal{E}_2$
$\mathcal{E}_1$
MU-6618

Fig. 2



1
0
−1
−2
−3
−4
$\lambda_{a\pm}$
0
5
10
15
g
λ_{2+}
λ_{1+}
λ_{1-}
λ_{2-}
MU-6619

Fig. 3

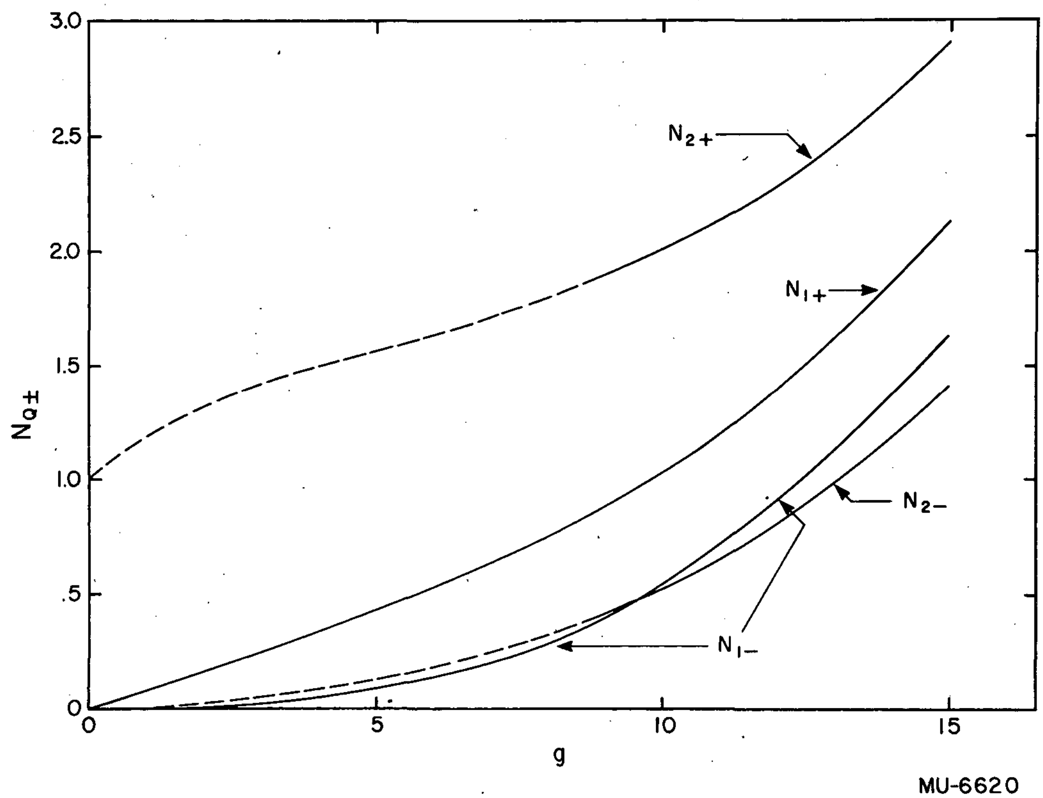
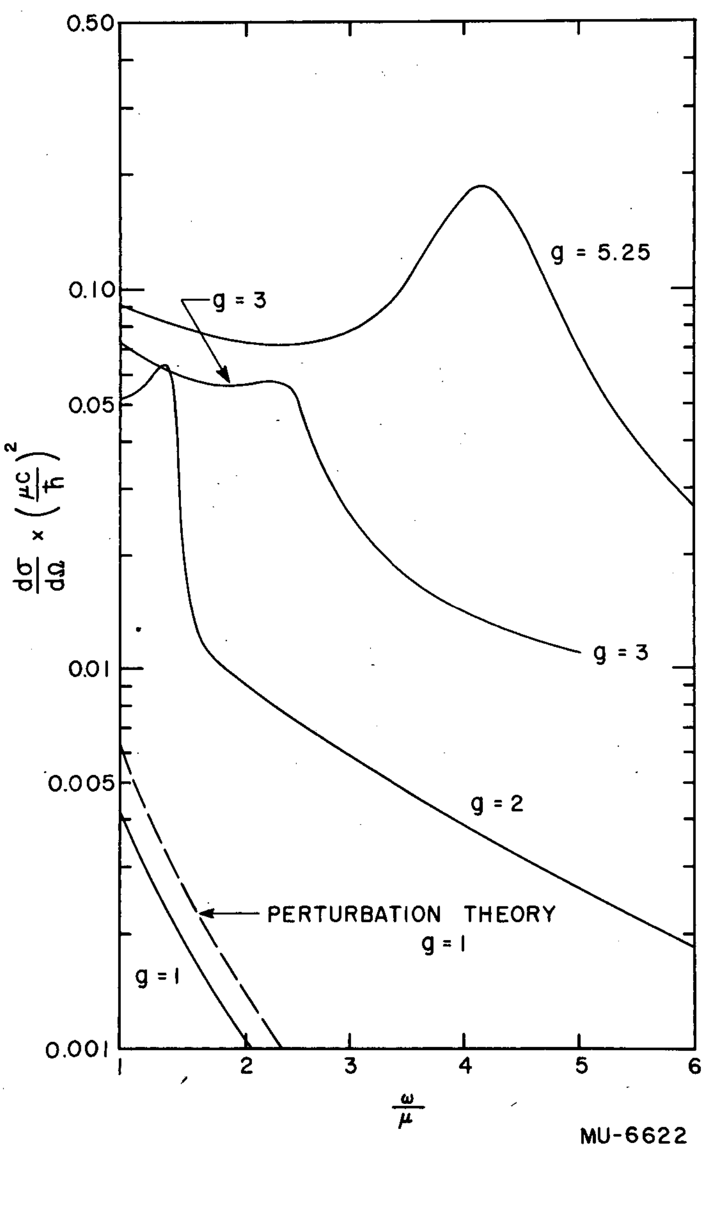


Fig. 4



0.50
0.10
0.05
0.01
0.005
0.001
$\frac{d\sigma}{d\Omega} \times \left(\frac{\mu c}{\hbar}\right)^2$
g = 5.25
g = 3
g = 3
g = 2
PERTURBATION THEORY
g = 1
g = 1
1
2
3
4
5
6
$\frac{\omega}{\mu}$
MU-6622

Fig. 5

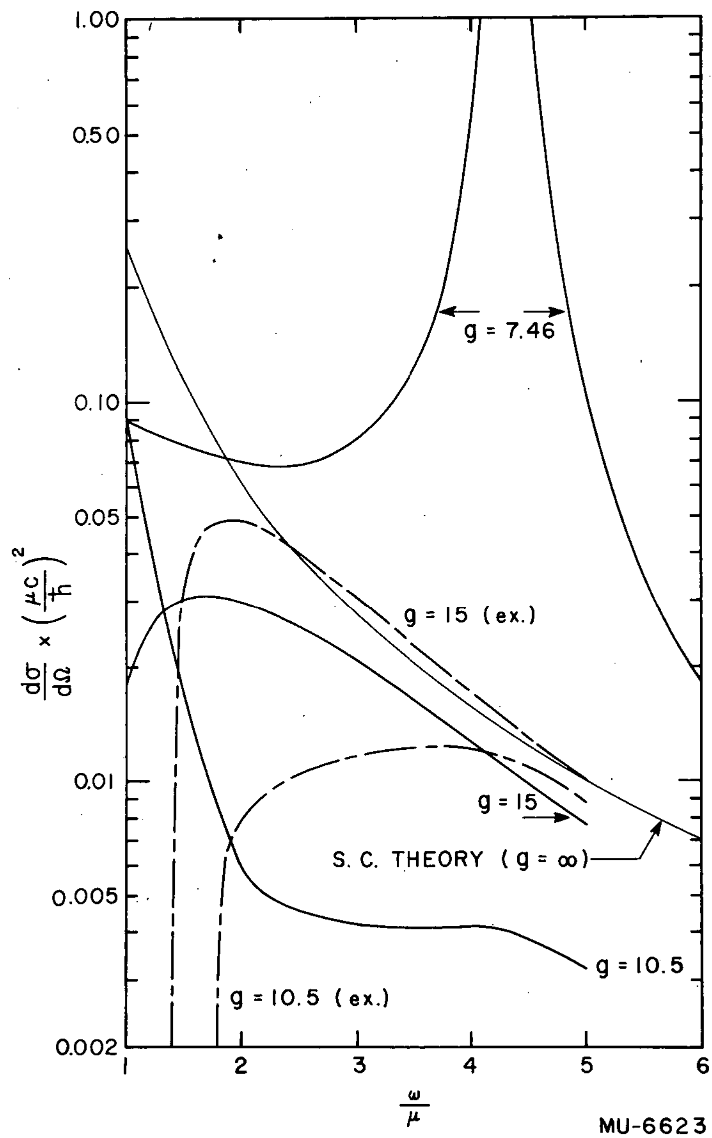

1.00
0.50
0.10
0.05
0.01
0.005
0.002
$\frac{d\sigma}{d\Omega} \times \left(\frac{\mu c}{\hbar}\right)^2$
g = 7.46
g = 15 (ex.)
g = 15
S. C. THEORY (g = ∞)
g = 10.5
g = 10.5 (ex.)
1
2
3
4
5
6
$\frac{\omega}{\mu}$
MU-6623

Fig. 6

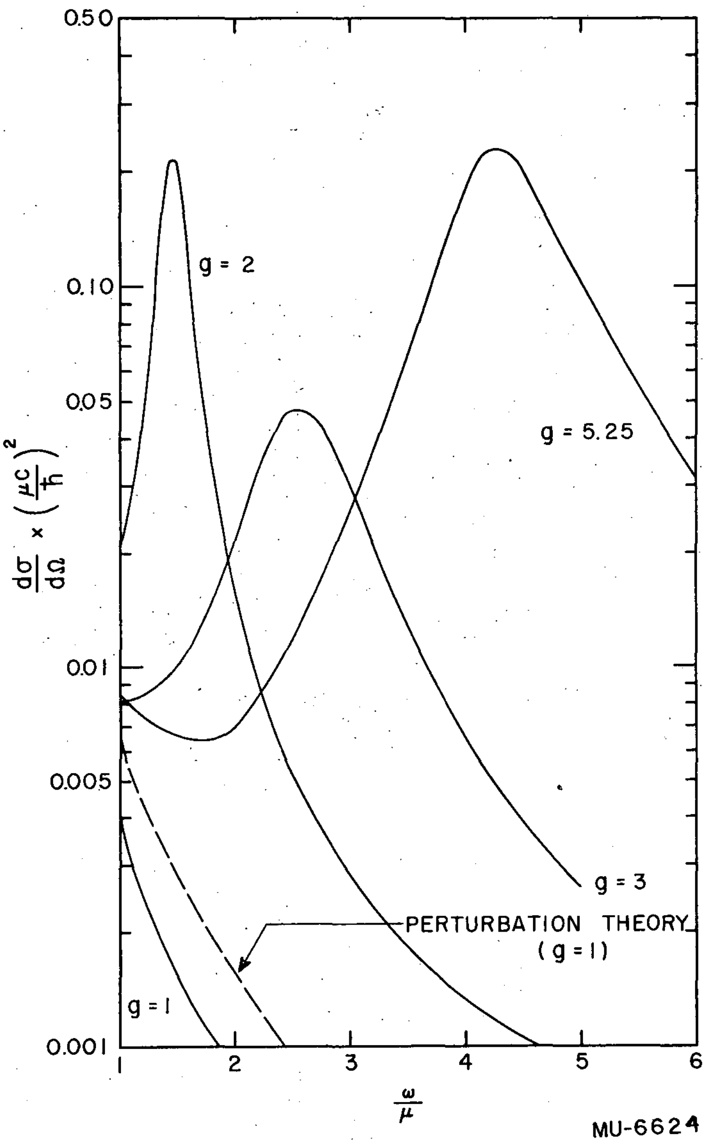

0.50
0.10
0.05
0.01
0.005
0.001
$\frac{d\sigma}{d\Omega} \times \left(\frac{\mu c}{\hbar}\right)^2$
g = 2
g = 5.25
g = 3
PERTURBATION THEORY
(g = 1)
g = 1
1
2
3
4
5
6
$\frac{\omega}{\mu}$
MU-6624

Fig. 7

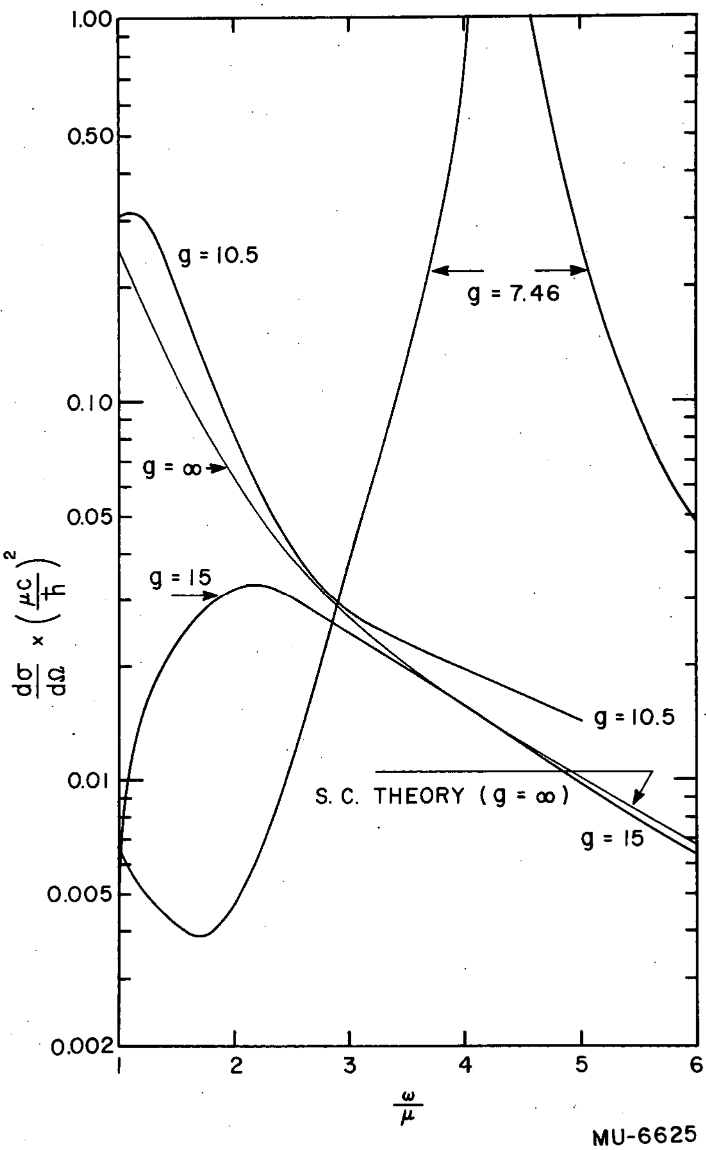

1.00
0.50
0.10
0.05
0.01
0.005
0.002
g = 10.5
g = 7.46
g = ∞
g = 15
g = 10.5
S. C. THEORY (g = ∞)
g = 15
1
2
3
4
5
6
$\frac{d\sigma}{d\Omega} \times \left(\frac{\mu c}{\hbar}\right)^2$
$\frac{\omega}{\mu}$
MU-6625

Fig. 8

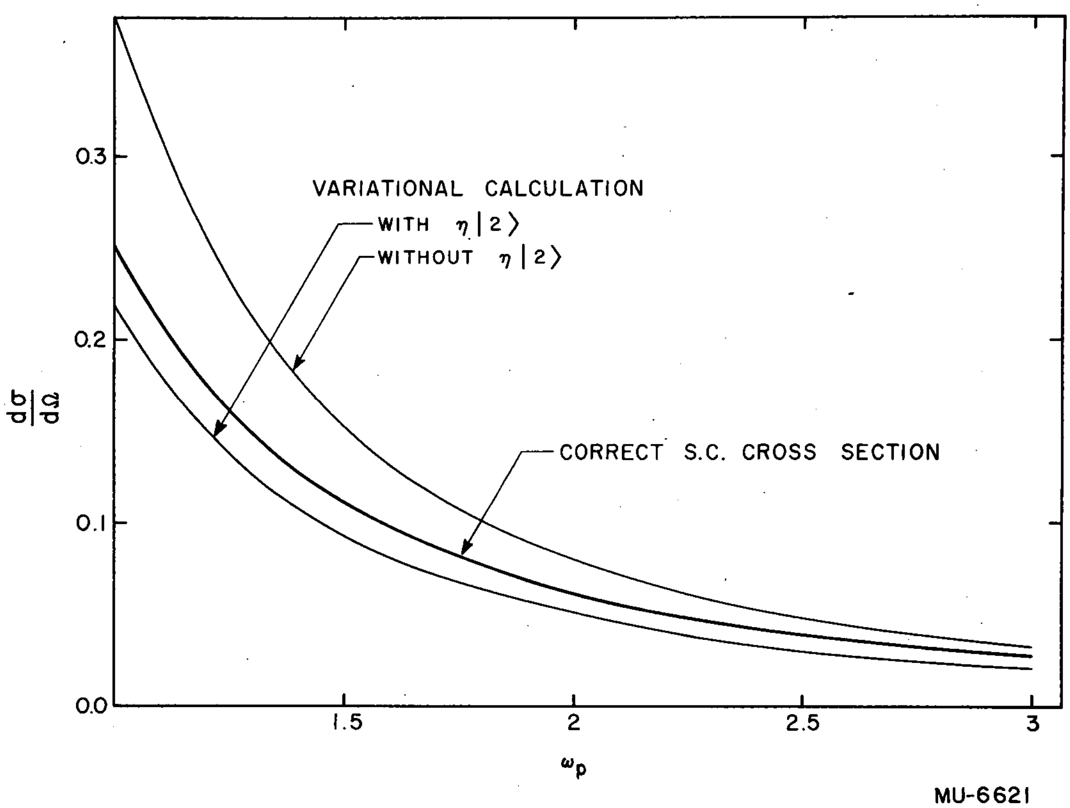

0.3
0.2
0.1
0.0
1.5
2
2.5
3
VARIATIONAL CALCULATION
WITH η | 2⟩
WITHOUT η | 2⟩
CORRECT S.C. CROSS SECTION
dσ/dΩ
ω_p
MU-6621

Fig. 9