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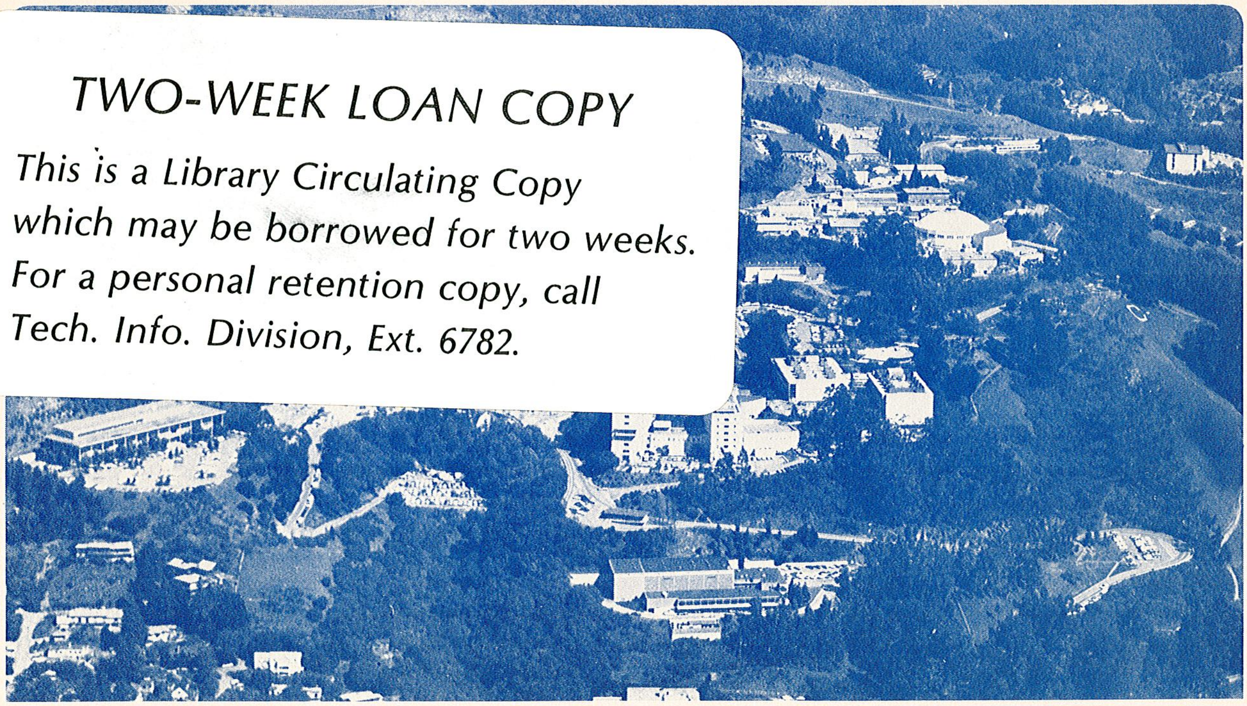

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Optimum Filter for Determination of the Position of an Arbitrary Waveform in the Presence of Noise

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Summary

This paper presents a general approach to solving the problem of defining the time of arrival of a signal in the noise, or the similar one of defining the position of a peak in a histogram. In particular, it solves the problem of a signal in white noise, giving the well know result that the optimum filter before a zero crossing discriminator should have an impulse response function equal to the derivative of the waveform to be observed. It then solves the problem of estimating the position of a peak in a histogram in which the number of counts in each bin has a variance equal to the number of counts in that same bin. The resulting optimum filter is the derivative divided by the function itself. Examples of applying the results are given and the uniqueness of the solution to the set of non-linear simultaneous equations resulting from the problem is demonstrated.

Introduction

The problem of defining the time of arrival of a signal is well known in nuclear electronics. A variety of schemes are in use at present to determine the arrival time of a particle for the definition of a nuclear event, for example. The available methods use some form of linear processing as a step previous to the observation of the crossing of a threshold, at which time a timing signal is derived. In the nuclear electronics timing problem, the signal to be observed has an additive white noise component (high frequency series noise in the first amplifying element) and it is well known that the optimum linear filter to precede a zero crossing discriminator is one with an impulse response function equal to the derivative of the noise-free waveform to be observed.[1]

A closely related problem is the one of defining the position of a peak in a histogram with a limited number of counts. In that case, the histogram can be convolved with a suitable waveform (presumably antisymmetric) and the results can be scanned for a zero crossing. The bin number of that crossing would be the best estimate of the position of the peak. In the case of a histogram obtained from a number of independent events (as for example in pulse height analysis of nuclear events), the noise in the histogram bins is not simply additive to the histogram, but the variance in the number of counts in a bin is equal to the expected number of counts in that particular bin. Under those conditions the optimum filter waveform to use in the convolution is not known. (see Addendum)

This paper presents a general approach to solving the position determination problem for an arbitrary waveform under arbitrary noise conditions and solves the specific cases of

a) Additive white noise, giving the well known results indicated above, and

b) Poisson distributed noise in each bin for the histogram case.

The optimum filter in the latter case is shown to be the derivative of the noise-free function to be observed divided by the function itself. For the simple case of a Gaussian distributed function, the filter function is a ramp.

Tests by computer simulation of the characteristics of the derived filter function are shown and the uniqueness of the solution is demonstrated.

General formulation

Consider a general waveform $f(r_q)$, with values at discrete points r_q, whose position along the r-axis is to be estimated. Let $w(r_q)$ be a weighting function such that the convolution

$$g(r_p) = \sum_q w(r_q-r_p)\, f(r_q) \tag{1}$$

results in a function $g(r_p)$ whose zero crossing defines the best estimate of the position of $f(r_q)$.

In the case with no noise, $f(r_q) = f_o(r_q)$ and the result of the convolution will be $g_o(r_q)$. We then define the position r_o' of $f_o(r_q)$ by the equation

$$g_o(r_o) = 0 \tag{2}$$

In the case with noise, $g(r_o)$ will not be zero, in general, and we can calculate the variance of the error as

$$v_\varepsilon = E\left\{\left[g(r_o) - g_o(r_o)\right]^2\right\} = \sum_q w^2(r_q)\; E\left\{\left[f(r_q) - f_o(r_q)\right]^2\right\} \tag{3}$$

for the case where noise in uncorrelated at different points of the waveform $f(r_q)$.

We are interested in minimizing the error in the zero crossing of the noisy waveform, which to first order can be approximated by dividing v_ε by the $(\text{slope})^2$ of g_o at $r = 0$. Then, we can state the problem as: Find the values $w(r_q)$ for all q such that

$$F = v_\varepsilon / \left[d/dr(g_o)\big|_{r=0}\right]^2 = \text{minimum} \tag{4}$$

The numerator is defined by Eq. (3) and the denominator is found from Eq. (1) by

$$d/dr(g_o)\big|_{r=o} = \sum_q w(r_q)\ d/dr\ f_o(r_q-r)\big|_{r=o} \tag{5}$$

To simplify notation, we can define the function f'_{oq}, the derivative at r=o of the noiseless waveform displaced by r_q (or the derivative of f at $r = r_q$, which is the same) by:

$$f'_{oq} = d/dr\ f_o\ (r_q-r)\big|_{r=o} \tag{6}$$

Then, the minimum of the functional F of Eq. (4) can be found by setting:

$$\partial F/\partial w_q = 0 \tag{7}$$

for all q simultaneously and attempt to solve the set of resulting equations. The variables w_q are defined identically as $w(r_q)$.

Solution for Gaussian White Noise

As a check on the method, we calculate first the optimal filter for determining the position of a waveform when additive white Gaussian noise with variance a is present on the signal. We first find the variance of Eq. (3):

$$v_\epsilon = \sum_q w_q^2\ E\left\{\left[f(r_q)-f_o(r_q)\right]^2\right\} = a\ \sum_q w_q^2 \tag{8}$$

since the expected value in brackets is the variance of the additive Gaussian noise.

The functional to minimize is then, from Eqs. (4), (5), (6) and (8),

$$F = a\ \sum_q w_q^2 \Big/ \left[\sum_q w_q\ f'_{oq}\right]^2 \tag{9}$$

Taking the partial derivatives indicated by Eq. (7) and simplifying we get:

$$w_q\ \sum_p w_p\ f'_{op} - f'_{oq}\ \sum_p w_p^2 = 0 \text{ for any } q \tag{10}$$

Indices p and q run over the whole extension of the sampling region, and there are, therefore, as many equations to solve as sampling points. Since the equations are quadratic we can expect more than one set of solutions w_q. However, it will be shown that all possible sets of solutions are simply proportional to each other, and that, therefore, we only have to determine one set of w_q solutions to yield the optimum filter function.

Let's consider a set of solutions:

$$w_q = w_{q1} \quad \text{for} \quad 1 \leq q \leq m \tag{11}$$

Then, the summations $\sum_{p1} w_{p1}f'_{op1}$ and $\sum_{p1} w_{p1}^2$ in Eq. (10) are two specific numbers S_1 and R_1, respectively. From Eq. (10) we then have:

$$w_{q1} = \frac{R_1}{S_1}\ f'_{oq1} \tag{12}$$

or the optimum filter function is proportional to the derivative of the waveform function f. If we consider now another set of solutions w_{q2}, with the summations taking values S_2 and R_2, we find, from Eq. (9):

$$w_{q2} = \frac{R_2}{S_2}\ f'_{oq2} \tag{13}$$

and we see that the second solution is also proportional to f'_{oq}. It follows, therefore, that all possible solution sets are proportional to each other and we only need to consider one.

Solution for Poisson Noise in a Histogram

In that case where the number of counts in a bin follows Poisson statistics, as is the case in a histogram generated from independent events, we start from:

$$E\left\{\left[f(r_q)-f_o(r_q)\right]^2\right\} = f_{oq} \tag{14}$$

(i.e., the variance in the number of counts is equal to the expected number of counts).

The functional to minimize is then:

$$F = \sum_q w_q^2\ f_{oq} \Big/ \left[\sum_q w_q\ f'_{oq}\right]^2 \tag{15}$$

and the resulting set of equations is given by

$$f_{oq}w_q\ \sum_p w_p\ f'_{op} - f'_{oq}\ \sum_p w_p{}^2 f_{op} = 0 \text{ for any } q \tag{16}$$

Here, again, we can let $\sum_p w_p f'_{op} = S$ and $\sum_p w_p^2 f_{op} = R$ for a set of solutions, and we find that:

$$w_q\ f_{oq} = \frac{R}{S}\ f'_{oq} \tag{17}$$

or

$$w_q \propto \frac{f'_{oq}}{f_{oq}} \tag{18}$$

with all solution sets being again proportional to each other. Thus, the optimal waveform for filtering is the derivative of the waveform divided by the waveform itself.

Gaussian Waveform Histogram

A simple and useful application of the above development is found in determining the best estimate of the location of a Gaussian-shaped peak in a histogram with a limited number of counts.

From Eq. (18) we find the optimum convolution filter to be a ramp,

$$w_q \propto r_q \tag{19}$$

The width of the ramp should be truncated, in practice, at a point where the waveform is not expected to have any substantial number of counts, or where other histogram features would cause distortion in the results. The effects of this truncation will be investigated below.

For Gaussian waveform, it is possible to calculate the rms error expected in zero crossing of the filtered signal by finding $F^{1/2}$ from a continuous integral form of Eq. (15). For a Gaussian of standard deviation σ and a total number of counts N within the waveform, the rms error ε is obtained from:

$$\varepsilon = \left[\sqrt{2\pi}\ \ \sigma^5/N\right]^{1/2} \left\{ \left[\int_{-\infty}^{\infty} w^2(r)\ \exp(-r^2/2\sigma^2)dr\right]^{1/2} \Big/ \int_{-\infty}^{\infty} r\ w(r)\ \exp(-r^2/2\sigma^2)\ dr \right\} \quad (20)$$

The results of calculating Eq. (20) for filters consisting of a step, a ramp, a parabola and the derivative of the Gaussian are shown in Table 1. The rms error values are given in units of σ.

A check on the results has been made by computer simulation. The output of a Gaussian distributed random number generator has been histogrammed with four bins per standard deviation, for 100, 200, 1000 and 2000 total counts in one histogram. Convolutions have been carried out with the filter waveforms, with half-widths of 1, 2, 4 and 8 standard deviations. The zero crossing of the results have been obtained by linear interpolations between adjacent bins. One hundred histograms have been procesed for each entry in Table 1, where the expected error (in standard deviations) has been entered for each case.

The results of a Least Squares fitting to the log of the histogram values, taking points from the central 4σ are also shown at top right.

The effect of reducing the number of waveform samples from 4 per standard deviation to 2, 1, and 0.5 per σ has also been calculated for the optimum filter case, with a filter half-width of 4 σ. It is found that the calculated expected errors of Table 1 have to be increased by approximately 1% when 2 samples per σ are used, by 4% for 1 sample per σ, and by 23% for 0.5 samples per σ. The increase in expected error is somewhat steeper when the number of samples is lowered than would be calculated from the use of Eq. (15) for the Gaussian waveform case. The first order theoretical results are 7% increase at 0.5 samples per σ, 35% at 0.4, and 104% at 0.33 points per σ.

Discussion of Results

It is interesting to consider the intuitive aspect of the two particular solutions investigated in this paper given by Eqs. (12) and (18). With additive white noise, it is quite reasonable to find that the filter (weighting) function emphasizes the sections of maximum slope of the initial waveform in attempting to locate its position. For the case with Poisson distributed noise, Eq. (18) indicates that we want to emphasize the points with maximum slope provided that the function itself is not very high there, as noise increases in absolute terms with signal. The result that a ramp is the optimum filter for the Gaussian case can only be understood in that manner. Although not exhaustibly, the results of Table 1 prove the correctness of the mathematical treatment.

From the computer simulations of Table 1, for a Gaussian waveform, it appears that the theoretical expected error in position determination can best be approached when the filter half-width h is near 4 standard deviations or more (in the absence of other disturbing histogram features). It is also clear that the ramp gives consistently better results in all situations, except for very narrow h, when at low N a zero crossing may not be uniquely defined.

Conclusion

The method presented here provides one straight forward approach to solving the problem of position determination in the presence of noise. It can be considered quite general and it should be useful whenever the variance of Eq. (3) can be calculated or measured well and the final set of equations can be solved without excessive pain. The specific result obtained for the Poisson case can be useful in finding best estimates of energy for nuclear or x-ray spectral peaks, specially when few counts are obtainable. Improvements over Least Squares methods of better than 40% in expected error are achievable, for example.

Acknowledgments

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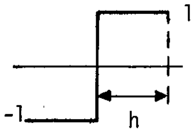
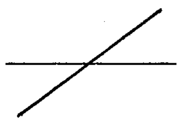
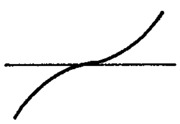
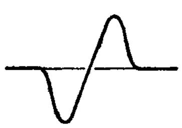
References

1. V. Radeka, IEEE Trans. Nucl. Sci., NS-21, No. 1, pp 51-64 (1979).

Addendum

After the completion of the manuscript, it was found that the result of Eq. (18) has been obtained by E. Gatti and V. Svelto (Nuclear Instruments & Methods, Vol. 39 (1966) pp. 309-312) for optimum filtering of timing with scintillation pulses.

Table 1. Calculated vs experimentally determined zero crossing errors with three kinds of of filters on a Gaussian waveform of standard deviation σ and a total of N counts. Results in units of σ.

	Calculated from Eq. (19)			Computer Simulation (4 bins per σ)					
					ε				
		N	ε	N	h=σ	h=2σ	h=4σ	h=8σ	LSq
Step Filter  (filter half width)	$\varepsilon = 1.26 \frac{\sigma}{N^{1/2}}$	100 200 1000 2000	0.126 0.0891 0.0398 0.0282	100 200 1000 2000	0.186 0.171 0.0682 0.0498	0.132 0.103 0.0439 0.0343	0.1226 0.0939 0.0392 0.0304	0.1226 0.0939 0.0392 0.0304	0.156 0.0935 0.0442 0.0286
					ε				
		N	ε	N	h=σ	h=2σ	h=4σ	h=8σ	
Linear Ramp  (Optimum)	$\varepsilon = \frac{\sigma}{N^{1/2}}$	100 200 1000 2000	0.1 0.0707 0.0316 0.0224	100 200 1000 2000	(*) (*) 0.0640 0.0477	0.123 0.0843 0.0347 0.0268	0.108 0.0735 0.0296 0.0226	0.108 0.0735 0.0296 0.0226	
					ε				
		N	ε	N	h=σ	h=2σ	h=4σ	h=8σ	
Quadratic Ramp  $\lvert w_q \rvert = r_q^2$	$\varepsilon = 1.085 \frac{\sigma}{N^{1/2}}$	100 200 1000 2000	0.1085 0.0767 0.0343 0.0243	100 200 1000 2000	(*) (*) 0.0660 0.0481	0.1315 0.0834 0.0356 0.0267	0.112 0.0736 0.0311 0.0231	0.1122 0.0732 0.0309 0.0231	
					ε				
				N	h=σ	h=2σ	h=4σ	h=8σ	
Derivative of Gaussian  $\lvert w_q \rvert = r_q \exp(-r_q^2/2\sigma^2)$				100 200 1000 2000	(*) (*) (*) 0.0485	0.1285 0.0989 0.0408 0.0315	0.1257 0.0964 0.0395 0.0306	0.1257 0.0964 0.0395 0.0306	

*Gross-errors in the procedure