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Bar-Natan Skein Modules in Black and White

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Bar-Natan Skein Modules in Black and White

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

Mathematics

by

Lyla Fadali

Committee in charge:

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2016

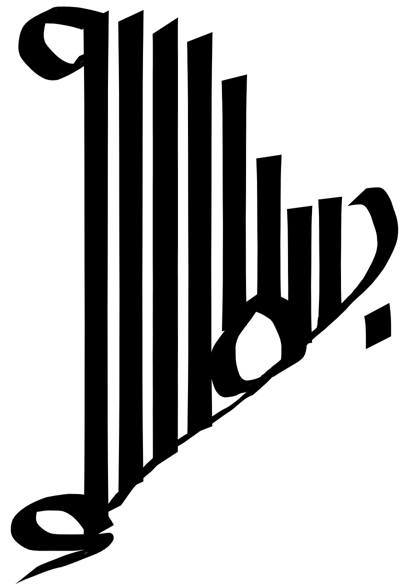
The dissertation of Lyla Fadali is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2016

DEDICATION



To Meru, light of my life; to my grandfather and my brother, mathematicians before me; to my father, who set me on this journey, and to my mother, as to mothers do we owe infinite debt for so many things.

EPIGRAPH

I have lived on the lip
of insanity, wanting to know reasons,
knocking on a door. It opens.
I've been knocking from the inside!
—Jalalludin ar-Rumi,
translated by Coleman Barks with John Moyne

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To Katie, my evil twin, my colleague and co-conspirator, my friend, my sister, my self. When friendship is so casually reliable and consistently present, it is easy to take for granted in the endless stream of day-to-day jokes, rants, mischief, and favours. But in reflecting on my years at UCSD, it is clear how very much I owe you, in mathematics and in life. I wish you all the success and happiness in the world—you completely deserve every bit of it.

To other friends, happily too many to be named, I might similarly mention the steady passing of days in easy togetherness. For many conversations, mathematical and otherwise, food, frisbee, D&D, much encouragement, silliness of all sorts, lots of distraction, and generally making grad school perhaps a bit *too* much fun, my deep and abiding thanks. More than anything, in you, for the first time in my life, I learned the meaning of community, and though we have shared years, still I wish I had found more time to savour the sweetness of your friendship. Here is to the relish of future years.

To my husband, Meru, who comforted, encouraged, chastised and sometimes wheedled as necessary, and above all, believed in me, even when I did not. At a loss to put in words the depths of such bonds, we can only resort to the tiredest of clichés. Dearest and best friend, I have so enjoyed our time together over the years, and so look forward to the rest of my life with you, whatever it holds.

Finally, it is to be expected that this dissertation owes a great deal to my advisor, Justin Roberts. But perhaps unexpected is the tremendous amount of

patience, wit, and charm that have accompanied your counsel. Bright curiousity about everything has put seemingly all of mathematics at your command (all the best parts, anyway), and for eight years, you have delighted and entertained me with a parade of beautiful ideas and discussions, not just about ‘maths,’ but music and sport, ancient literature, and I don’t know what else. You are a mathematician who also knows how to live—a true topologist—and there is no one I admire more. Thank you for the inspiration.

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ABSTRACT OF THE DISSERTATION

Bar-Natan Skein Modules in Black and White

by

Lyla Fadali

Doctor of Philosophy in Mathematics

University of California, San Diego, 2016

Professor Justin Roberts, Chair

Following the work of Asaeda and Frohman, we explore a variation of Bar-Natan skein modules which can be defined as a TQFT using Kevin Walker's fields and local relations. We prove analogous results to Asaeda and Frohman for this variation, and discuss the potential to compute skein modules of manifolds by decomposing the manifold into pieces and tensoring together the skein modules of the pieces. We give computations toward that end, and on the way give an application of the cyclic seiving phenomenon.

Chapter 1

Introduction

1.1 Context

In the beginning was the Jones polynomial, and the Jones polynomial led to Khovanov homology [Kho00], and Khovanov homology was great [BN02]. Indeed, its discovery generated a mass of new work in low-dimensional topology, and in wandering the wilds of knot homologies, I came across Bar-Natan skein modules.

Khovanov's knot invariant is a (1+1)-dimensional topological quantum field theory (TQFT), a functor sending circles to vector spaces and cobordisms between circles to linear maps between vector spaces. I have always found the interplay of topology and algebra in TQFT pleasing, but Bar-Natan made Khovanov homology particularly beautiful in his remarkably lovely paper [BN05]. While Khovanov shows invariance under Reidemeister moves on the algebraic side, after applying the TQFT, Bar-Natan gives a simple proof of invariance *before* applying the TQFT, just working topologically. The formal mechanism for this miracle is a category of (possibly dotted) cobordisms between circles (or tangles). To make the proof go through, he needed to impose some local relations on the surfaces (in addition to isotopy relations).

Using these relations, Asaeda and Frohman introduced a skein theory of surfaces in 3-manifolds, the Bar-Natan skein module. In their paper [AF07], they calculate a few examples and prove a result for Seifert-fibred spaces. Some additional work has followed, but relatively little is understood about Bar-Natan skein modules as of yet.

Russell calculated the skein module for the solid torus with longitudinal curves in its boundary, which she proved is isomorphic to the homology of the (n, n)-Springer variety. Consequently, she shows the skein has a well-defined comultiplication, but it cannot be a Frobenius extension with this structure [Rus09]. Kaiser discusses Bar-Natan skein modules in a more general framework. Several sets of relations have been studied in the search for Khovanov-like homology theories [BN05, BNM06], as well as extensions over unoriented surfaces [TT06], which are interesting to the community for their connexion with virtual knots and links. Kaiser's work encompasses these possibilities with a general 2-dimensional Frobenius algebra, giving miscellaneous results such as presentations with generators and relations ([Kai09], cf. [Kho06]). Paul Drube and Jeffrey Boerner studied Bar-Natan skein modules for $\mathfrak{sl}(n)$ specifically [BD12]. Kaiser has also discussed some aspects of TQFTs from fields (discussed below) in connection with Bar-Natan skein modules [Kai14].

Skein modules have been considered interesting objects in their own right for many years [Prz06, Prz], including incarnations inspired by Khovanov homology [APS04, APS06]. Most of these are concerned with skein modules of links in manifolds, rather than surfaces. Kaiser has worked on connecting these link skeins with Bar-Natan skein modules [Kai13].

We are particularly interested in the Bar-Natan skein module's role as a TQFT and its relation to higher categories as described in [Wal06, MW11]. According to Walker's formalism, a TQFT can be constructed as a system of fields modulo local relations. A topological field is essentially a collection of functors from i-manifolds to **Set** for $0 \leq i \leq n$, with good behavior required with respect to boundary, gluing, etc. As a first step along these lines, higher Hochschild homology is an interesting research topic. We can compute the Bar-Natan skein module of the solid torus, $D^2 \times S^1$, as HH_0 of the cylinder category associated to the disk D^2, which naturally leads us to ask what the higher Hochschild homology is (see [Lod92, Web07]). Taking this even farther, we can ask what the blob complex of the associated functor is for general manifolds.

There is a correspondence between Walker-style TQFTs and higher categories. In general, a system of surface fields in a 3 manifold should correspond to a 3-category: objects (0-morphisms) are the colours of the regions, 1-morphisms are a colouring of surfaces, 2-morphisms a colouring of seams, and 3-morphisms a colouring of vertices. Some of these could be trivial. In particular, a skein of 'true surfaces' (i.e. not foams) should give a some kind of 3-category with trivial 2- and 3-morphisms.

It is interesting to compare and contrast this with Reshetikhin-Turaev (RT) invariants and Turaev-Viro (TV) invariants. RT invariants corresponds to a 3-categories with trivial 0- and 1-morphisms (or equivalently, to pivotal braided monoidal categories), naturally dealing with knots and links. On the other hand, TV invariants correspond to non-braided 2-categories. We expect the Bar-Natan skein module to be more similar to TV invariants than RT. They seem more appropriate given the connexion Bar-Natan skeins have to planar algebras, which have no crossings. As there is a relationship between RT and TV invariants, we might expect a relationship between the blob functor for Khovanov homology and for Bar-Natan skeins.

We optimistically conjecture that Bar-Natan skein modules relate to Ko Honda's contact category in a meaningful way. In our variation (with black and white colouring), the objects are the same in both categories. Ritz showed a relation between the contact category and the Burau representation of the braid group [Rit10], which also seems promising.

This thesis

In a general context, there is really an omission in Bar-Natan's definition of a surface category, which is how to deal with a Möbius strip. (Note that having a Möbius strip is not a local condition.) His definition is perfectly functional for his purposes and for Asaeda/Frohman and Russell, but it becomes a problem for our work. There are four 'correct' definitions of a blob functor which give a variation of a Bar-Natan skein module.

- **unoriented:** We allow nonorientable surfaces, which makes the skein more complex, but perhaps does not really add anything.

- **oriented:** Everything is completely oriented. This creates a huge number of distinct objects with all possible orientations, which is annoying.

- **Blanchet foams:** Define a skein of foams as defined in [Bla10]. Given its role 'fixing' Khovanov homology, this might be in some sense be the 'most correct' way to orient things. This is also perhaps the most natural way to take TV invariants up a dimension categorically.

- **black/white:** The complement of a surface gets a checkerboard colouring, a simplified version of the oriented skein. This gives a 3-category with 2 objects.

We choose to study the last of the options because it seems like the simplest variation which is not trivial, and because of its apparent connexion to subfactors and contact geometry as mentioned above.

Organisation

We begin with some basics about Khovanov homology. While not all of this material is directly relevant, we include it because we imagine the landscape of Bar-Natan skein modules somewhat parallels that of Khovanov homology. It also played an important role shaping our thinking about a 'local model.' By this we mean that the skein of a larger, more complicated manifold can be computed by tensoring skeins of smaller, simpler pieces over the appropriate algebra. A more proximate inspiration for this approach is Walker's formalism of TQFTs [Wal06, MW11], which we discuss in Section 1.3, before we come to the meat of our work.

Both the Asaeda-Frohman and Russell papers use a 'hard-core topology' approach—basically thinking really hard about how surfaces sit inside a 3-manifold. We use this approach in chapter 2, where we prove results analogous to Asaeda and Frohman's for black-and-white skein modules. In chapter 3, our work is an attempt to reproduce and expand known calculations of Bar-Natan skein modules using a local model. We give a different proof of Heather Russell's result for the solid torus with longitudinal boundary curves, and extend this result to the torus with p/q curves on the boundary. Along the way, we establish the relationship between the twisted and untwisted tori using the cyclic seiving phenomenon. In chapter 4, we discuss further aspects of surface categories.

1.2 Khovanov homology

Khovanov homology is a 'categorification' of the Jones polynomial. Before we say what that means, we recall some essential facts about the Jones polynomial.

1.2.1 The Jones polynomial

In 1984, Vaughan Jones discovered a new knot polynomial, which came to be called the Jones polynomial. It is that holy grail of invariants, easy to compute yet still very powerful; for example, it is strong enough to distinguish the right- and left-handed trefoil knots.

One way to define the Jones polynomial—and the best way for the purposes of Khovanov homology—is to use the Kauffman bracket.

The Kauffman bracket

Definition. The Kauffman bracket of an unoriented link diagram is defined by the following three properties:

- $\langle \emptyset \rangle = 1$. The empty diagram is allowed, and its bracket is 1.
- $\left\langle \bigcirc \amalg D \right\rangle = (q+q^{-1}) \langle D \rangle$. An unknotted, unlinked component can be removed from a diagram at the cost of multiplying by $(q+q^{-1})$.
- $\left\langle \times \right\rangle = \left\langle)(\right\rangle - q \left\langle \asymp \right\rangle$. The bracket of a crossing is a linear combination of the 0-smoothing and the 1-smoothing. The 0-smoothing is the one you get by walking on the 'zero level' and turning right on the 'one level'

Remark 1.2.1. As we have defined it, the Kauffman bracket is not actually invariant under *any* of the Reidemeister moves without normalising.

Definition. The Jones polynomial (defined on an oriented diagram D of a link L) is given by

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle.$$

Theorem 1.2.2. *The Jones polynomial is a link invariant.*

Sadly, we topologists have completely failed to standardise our notation and normalisation for the Kauffman bracket and Jones polynomial. Thus, by tradition,

- The Kauffman bracket uses the variable A, where $q = -A^{-2}$. Tradition also defines the bracket of a crossing by $\langle \times \rangle = A \langle\,)(\, \rangle + A^{-1} \langle \asymp \rangle$. With this relation, the Kauffman bracket is invariant under R2 and R3, but not R1. Now the fix is obvious, since resolving $R1$ just multiplies by a factor of $A^{\pm 3}$ corresponding to the writhe $w(D) = n_+ - n_-$. This makes $J(A) = A^{-3w(D)} \langle D \rangle$. I find it easiest to see the relation between A and q by looking at the state sum, rather than using the skein relation.

- The Jones polynomial uses the variable $t = A^{-4}$, commonly written $V(t) = A^{-3w(D)} \langle D \rangle \mid_{t = A^{-4}}$.

- In the original normalisation the bracket of the unknot is 1, rather than $q + q^{-1}$.

- Sometimes q is used as I have used t, that is, with $q = A^{-4}$. There is also a variable a with $a = q^{-1}$. This may or may not be exactly right.

The moral here is: if you ever have a conversation with someone else about the Jones polynomial, make sure you're both using the same definition.

State sum formula

A critical fact for Khovanov homology is that the Kauffman bracket (and Jones polynomial) can be written as a sum over 'states,' or complete resolutions. You can arrange them nicely in a 'cube of resolutions,' as in Figure 1.1.

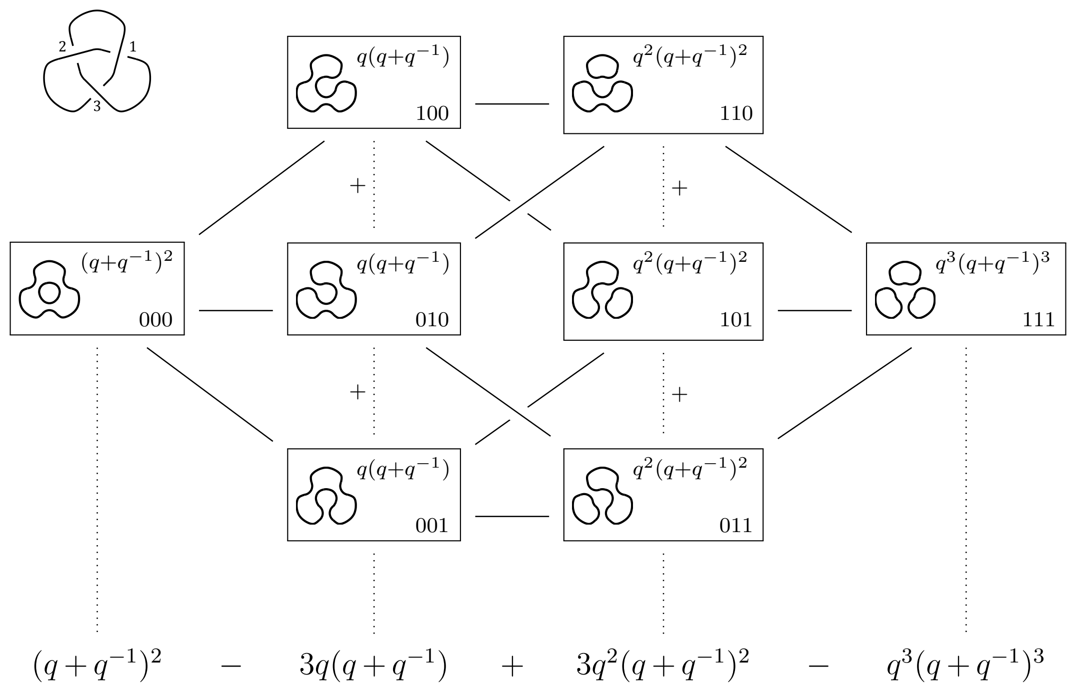


Figure 1.1: The cube of resolutions for the Kauffman bracket of the trefoil (Figure 1 in [BN02], used with permission).

Let r be the number of 1-smoothings in a state s.

Let k be the number of cycles in s.

If we arrange the cube with the all-zero smoothing on the left (as in the figure), r represents which column from the left the state s is in. To get the Kauffman bracket, it is convenient to sum down a column, and then take the alternating sum of the columns. Multiplying by a factor of q outside the sum gives the Jones polynomial. Thus, the state sum formula for the Jones polynomial is

$$J(K) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_s (-1)^r q^r (q+q^{-1})^k.$$

To see the relation between A and q, compare the formula above to

$$\sum (-A)^{-3w(D)} A^{\sum s_i} (-A^2 - A^{-2})^k.$$

Here $s_i = -1$ for a 1-smoothing, $+1$ for a 0-smoothing. The sum s_i is $-r + (n-r)$. The last factor obviously corresponds to $(q+q^{-1})^k$.

$$\begin{aligned} A^{-3w(D)}A^{\sum s_i} &= (-1)^w A^{-3n_+ + 3n_- - 2r + n_+ + n_-} \\ &= (-1)^w A^{-2n_+ + 4n_- - 2r} \\ &= (-A)^{-2(n_+ - 2n_- - r)} \cdot (-1)^{-2n_+ n_- + r} \\ &= (-1)^{n_-} q^{n_+ - 2n_-} (-q)^r. \end{aligned}$$

There is, in general, an annoying sign associated with the Kauffman bracket. We can understand it from the Witten-Reshetikhin-Turaev perspective of knot polynomials from quantum groups. Using unoriented diagrams for the Kauffman bracket is possible because $\mathfrak{sl}_2$ is self-dual, but it is not completely justified. The sign is a manifestation of the sloppiness. See Tingley's paper [Tin10], or on the Khovanov homology side, Clark/Morrison/Walker [CMW09], Blanchet [Bla10], or Caprau [Cap09].

1.2.2 Categorification

Consider the Euler characteristic of a space X. Here we see an integer as the 'shadow' of a larger structure, the homology of X:

$$\chi(X) = \sum_i (-1)^i \dim(H_i(X;\mathbb{Q})).$$

In this mindset, the paradigm of categorification, an integer is 'really' the alternating sum of dimensions of vector spaces, and a polynomial is the alternating sum of dimensions of *graded* vector spaces.

Thus, to categorify the Jones polynomial, we want

$$J(K) = \sum (-1)^r \mathrm{qdim}(V_r)$$

for some graded vector space V_r (up to some shift in grading). Recall $\mathrm{qdim}(W) = q^m \dim(W_m)$, where W_m is of homogeneous degree and $W = \oplus W_m$.

We can accomplish this if $\mathrm{qdim}(V_r) = q + q^{-1}$. Then a state in the cube of resolutions is associated to $q^r V^{\otimes k}$. The homological grading is given by r, which we normalise to go from $-n_-$ to n_+ We also have an overall shift in q-grading by $q^{n_+ - 2n_-}$.

Topologically, each state is related to the states following (connected by arrows) by a pair of pants, either upside down or right-side up. These put one in mind of multiplication and comultiplication, and, indeed, that is how we get the boundary maps for the chain complex (on the algebra side). In other words, we are applying a $(1+1)$-dimensional TQFT, and pants are sent to the multiplication and comultiplication in a unital Frobenius algebra. Sprinkling in signs according to a sort of exterior product rule (see [BN05]) makes $d^2 = 0$; hence, we have a chain complex.

Taking the (co)homology of this complex, we get

$$Kh(L) = \sum t^r \mathrm{qdim} H^r(L),$$

a bigraded homology theory.

Theorem 1.2.3. *$Kh(L)$ categorifies $J(L)$ (set $t = -1$), and is an invariant of L.*

Moreover, Khovanov homology is strictly stronger than the Jones polynomial. See [BN02].

Dror's category enables us to prove invariance in the world of topology, resulting in a very elegant proof [BN05]. Start with the category **Cob** whose objects are smoothings and morphisms are cobordisms between them. Make it pre-additive (linear) by taking formal linear combinations of morphisms and extending composition bilinearly. Impose the Bar-Natan relations S, T, 4-Tu (sphere=0, torus=2, and the 4 tubes relation—alternatively, we can use surfaces with dots and the relations in Figure 2.1—they're equivalent if 2 is invertible). These relations are a little mysterious, but they are needed to make the proof of invariance under Reidemeister moves go through. Call this $\mathbf{Cob}_{/l}$.

Then take $\mathbf{Mat}(\mathbf{Cob}_{/l})$, whose objects are formal direct sums of smoothings and matrices of linear combinations of cobordisms between them. Then take the category of formal chain complexes $\mathbf{Kom}(\mathbf{Mat}(\mathbf{Cob}_{/l}))$, and finally, quotient to get complexes of these up to homotopy equivalence. Call this $\mathbf{Kob}_{/h}$.

Computing complexes for the two pictures of each Reidemeister move, we get that they are equivalent in $\mathbf{Kob}_{/h}$, thus we have invariance for tangles. (I've glossed over the details for boundaries, but in Dror's proof, it's not very complicated - a major advantage of his method.)

1.2.3 Khovanov-like theories

Having categorified the Jones polynomial, we might ask if there was anything special about the Frobenius algebra we used, or if others will work too. The answer is yes, there is a whole family of them, but they are all isomorphic to Khovanov homology or one of two other variations, Lee homology and Bar-Natan homology [Kho06, BN05, Lee05].

Algebraic definitions of the maps for Khovanov homology (with $V = \mathbb{Q}[1, x]$, where 1 has q-degree $+1$ and x has q-degree -1) are

$$\begin{array}{ll} \iota(1) = 1 \text{ (unit)} & m(1 \otimes x) = m(x \otimes 1) = x \\ \epsilon(1) = 0 \text{ (counit)} & m(1 \otimes 1) = 1 \\ \epsilon(x) = 1 & m(x \otimes x) = 0 \\ \delta(1) = x \otimes 1 + 1 \otimes x & \delta(x) = x \otimes x. \end{array}$$

Modifying the multiplication $m(x \otimes x)$ and comultiplication $\delta(x)$ slightly gives Lee homology.

The q-grading of an element $\mathbf{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ of the chain complex is given by $\sum \mathrm{qdeg}(v_i) + r + n_+ - n_-$ (the latter part is the homological height). Grading is preserved by the differential in the Khovanov chain complex, but not in the Lee chain complex. However, the Lee complex can be filtered, and this filtration can be used to define the Rasmussen invariant [Ras04].

1.2.4 Gaussian elimination

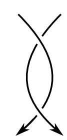
It is possible to simplify a Khovanov chain complex, using a procedure akin to Gaussian elimination for matrices.The idea is to use isomorphisms to simplify chain groups as in Figure 1.2, and then reduce the chain complex to a simpler, homotopically equivalent one. In effect, this amounts to pre-and post-composing differentials with lower triangular and upper triangular matrices (in such a way

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha_i"] \arrow[dr, "\varphi\circ\alpha_i"'] & A \arrow[r, "\alpha_{i+1}"] \arrow[d, "\varphi"] & \cdots \\
 & A' \arrow[ur, "\alpha_{i+1}\circ\varphi^{-1}"'] &
\end{tikzcd}$$

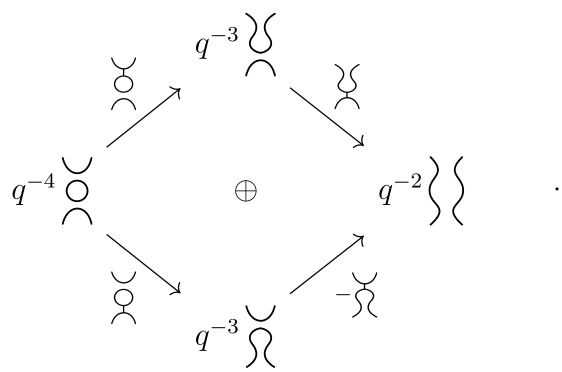
Figure 1.2: The key diagram, where φ is an isomorphism.

that they will cancel) to zero out the differential above and below the diagonal. [BN07] is excellent as usual.

Example 1.2.4. Consider the tangle with $(n_+, n_-) = (0, 2)$ below.



The Khovanov chain complex for this tangle is:



The first step in simplifying the complex, and the first application of our 'key diagram,' is to remove cycles using the isomorphism shown in Figure 1.3.

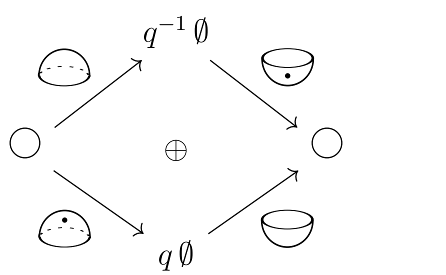
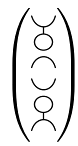
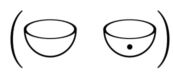


Figure 1.3: An isomorphism between a cycle and the empty picture with grading shifts.

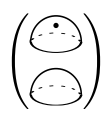
In our example, only the first smoothing has a circle, so we just need to precompose the differential from the first column to the second:



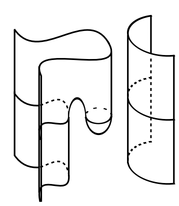
with the isomorphism:

 .

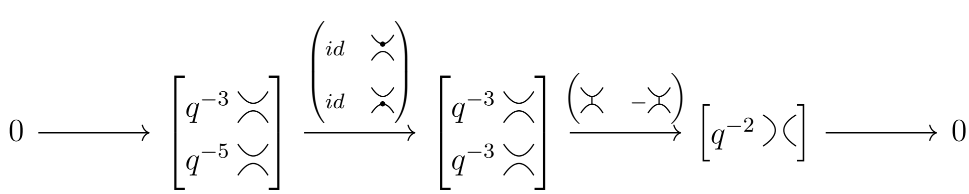
(If you, like me, started getting confused about the order of composition after you started learning about category theory, recall that precomposing by a matrix means multiplying on the right.) In general, we also need to postcompose the previous differential with the inverse isomorphism:



but the differential is zero in this case. The cups and caps are understood to be tensored with the identity cobordism as appropriate. Thus the first entry of the composition is:



which is isotopic to the identity between two arcs. That is good, because we need this entry to be an isomorphism for the Gaussian elimination part. We use planar pictures to represent this surface, with dots as necessary, so the complex becomes:



Call the first nonzero differential A and the second B.

Now we get to the Gaussian elimination. We will multiply A on the left by a lower triangular matrix L of morphisms that will cancel out the first column below the diagonal. This is postcomposition. To preserve the differential under the isomorphism, we precompose the next map, B, with L^{-1}. (Recall Figure 1.2. That's all we're doing.)

$$L = \begin{pmatrix} id & 0 \\ -id & id \end{pmatrix}$$

We will also precompose A with an upper triangular matrix U that cancels out the first row above the diagonal. (As with the step of eliminating the circle, we would normally need to postcompose the differential preceding A with U^{-1}, but the map is zero here.)

Doing the compositions, we get

$$LAU = \begin{pmatrix} id & \dot{\asymp} \\ 0 & \underset{\cdot}{\asymp} - \dot{\asymp} \end{pmatrix} \begin{pmatrix} id & -\dot{\asymp} \\ 0 & id \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & \underset{\cdot}{\asymp} - \dot{\asymp} \end{pmatrix}$$

and

$$BL^{-1} = \begin{pmatrix} \asymp & -\asymp \end{pmatrix} \begin{pmatrix} id & 0 \\ id & id \end{pmatrix} = \begin{pmatrix} 0 & -\asymp \end{pmatrix}$$

so we now have the complex

$$0 \longrightarrow \begin{bmatrix} q^{-3} \asymp \\ q^{-5} \asymp \end{bmatrix} \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & \underset{\cdot}{\asymp} - \dot{\asymp} \end{pmatrix}} \begin{bmatrix} q^{-3} \asymp \\ q^{-3} \asymp \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 & -\asymp \end{pmatrix}} \begin{bmatrix} q^{-2})(\end{bmatrix} \longrightarrow 0.$$

We now have split off an acyclic part, so this complex is homotopic to

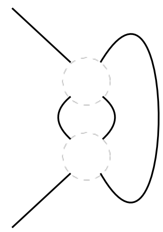
$$0 \longrightarrow \begin{bmatrix} q^{-5} \asymp \end{bmatrix} \xrightarrow{\underset{\cdot}{\asymp} - \dot{\asymp}} \begin{bmatrix} q^{-3} \asymp \end{bmatrix} \xrightarrow{-\asymp} \begin{bmatrix} q^{-2})(\end{bmatrix} \longrightarrow 0.$$

Once the complex is simplified, you can much more easily tensor it with another chain complex to get the chain complex for a bigger tangle.

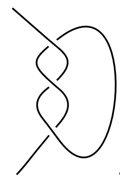
Example 1.2.5. Let's call the complex we just computed Φ, and call Ψ the complex of a single negative crossing:

$$0 \longrightarrow q^{-2} \asymp \xrightarrow{\asymp} q^{-1})(\longrightarrow 0\ .$$

When we tensor these, we will fit them together into the tangle:



which gives the complex for the 1-1 tangle of the trefoil:



To make sure the differential squares to zero, we need some extra negative signs when we tensor chain complexes together. We can accomplish the appropriate changes by negating every alternating row. (Alternating columns would also work.) This is pictured in Figure 1.4

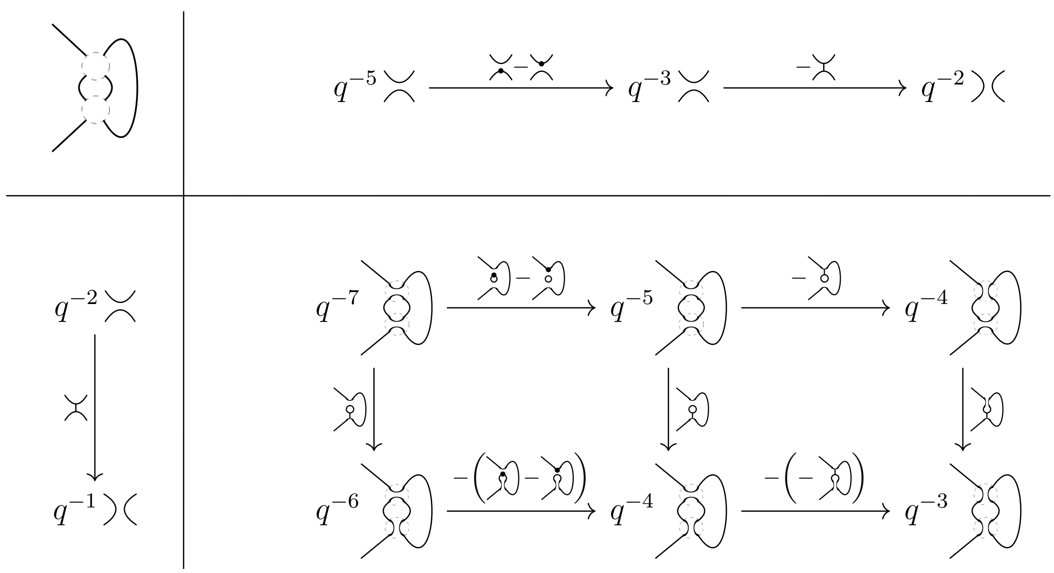


Figure 1.4: Tensoring two complexes to get the complex of a bigger tangle.

Now we can 'flatten' the complex by taking direct sums along the diagonals, which gives the complex in Figure 1.5. This complex can also be simplified.

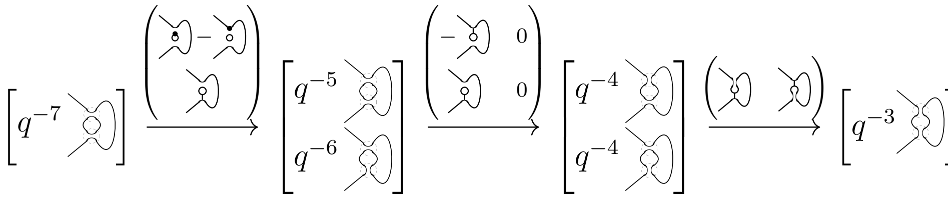


Figure 1.5: The flattened complex.

Related theorems

This section is from Scott Morrison's lecture at the MSRI workshop on link homologies [Mor10].

Theorem 1.2.6. *Every complex of a 1-1 tangle decomposes as a direct sum of complexes*

$$E = \ 0 \longrightarrow \Big) \longrightarrow 0$$

$$C_n = \ 0 \longrightarrow \Big) \xrightarrow[\text{with } n \text{ dots}]{\text{sheet}} \Big) \longrightarrow 0$$

with any overall q^k factor and in any homological height.

Theorem 1.2.7. *For a link with k components, exactly 2^{k-1} copies of E appear.*

The s-invariant of a knot K is the q-grading of the unique E for the knot's 1-1 tangle.

Conjecture 1.2.8. Only E, C_1, and C_2 appear in invariants of links.

Corollary 1.2.9. The s-invariant of a knot K is

$$s(K) = [Kh(K)]\,(q, t = -q^{-4})/(q + q^{-1})$$

where $[Kh(K)]\,(q, t)$ is Poincaré polynomial of the Khovanov homology.

1.3 TQFT from fields and local relations

Topological quantum field theory, better known as TQFT, originates in quantum mechanics, but was axiomatised mathematically by Atiyah (with inspiration from Segal) in 1988. According to Kevin Walker [Wal06], he was motivated to TQFTs using fields and local relations because he felt Atiyah's axioms strayed too far from the original physics formulation. His definition also fixes the 'anomoly' associated with Reshitikhin-Turaev invariants, thus giving a more unifield framework. However, the theory is not completely local for a non-semisimple TQFT, and the blob complex is a fix for that [MW11].

1.3.1 Fields

Let $\mathbf{M}^i$ be the category whose objects are compact oriented i-dimensional manifolds, and whose morphisms are orientation-preserving homeomorphisms.

Definition. An n-dimensional *system of topological fields* is a collection of functors

$$\mathcal{A}_i : \mathbf{M}^i \longrightarrow \mathbf{Set}$$

for $0 \leq i \leq n$. For a particular manifold M, we refer to $\mathcal{A}_i(M)$ as *i-fields* on M, or simply *fields* on M. The functors must behave well with respect to various properties of the category $\mathbf{M}^i$, as follows:

- Boundary maps. In each dimension $1 \leq i \leq n$, there must be maps

  $$\partial : \mathcal{A}_i(M) \longrightarrow \mathcal{A}_{i-1}(\partial M).$$

  If $\varphi \in Hom(M, N)$ is a homeomorphism, then the boundary map ∂ on fields commutes with φ and the boundary functors on M and N. In other words, we must have the following commutative diagram:

  $$\begin{tikzcd} \mathcal{A}_i(M) \arrow[r, "\mathcal{A}_i(\varphi)"] \arrow[d, "\partial"'] & \mathcal{A}_i(N) \arrow[d, "\partial"] \\ \mathcal{A}_{i-1}(\partial M) \arrow[r, "\mathcal{A}_{i-1}(\partial\varphi)"] & \mathcal{A}_{i-1}(\partial N) \end{tikzcd} .$$

For a field $c \in \mathcal{A}_{i-1}(\partial M)$, this enables us to define fields with a boundary condition as

$$\mathcal{A}_i(M, c) = \partial^{-1}(c).$$

- Orientation reversal. For each $c \in \mathcal{A}(\partial M)$, there must be a bijection

  $$\mathcal{A}(-M, \hat{c}) \longleftrightarrow \mathcal{A}(M, c).$$

  This also includes $\mathcal{A}(-M)$ when M is a closed manifold. This must commute with homeomorphisms and the boundary maps.

- Disjoint union. Fields on a disjoint union are identified with the product

  $$\mathcal{A}(M \amalg N) \cong \mathcal{A}(M) \times \mathcal{A}(N).$$

  This must commute with boundary maps, homeomorphisms, and orientation reversal.

- Gluing without corners. Suppose the boundary of M decomposes as

  $$\partial M = \Sigma \amalg -\Sigma \amalg T.$$

  By this, we mean that there is a homeomorphism between the two Σ components. Using the disjoint union map, fields on the boundary correspond to a product:

  $$\mathcal{A}(\partial M) \cong \mathcal{A}(\Sigma) \times \mathcal{A}(-\Sigma) \times \mathcal{A}(T).$$

  Composing the disjoint union map with the boundary map thus gives two maps to $\mathcal{A}(\Sigma)$, one using orientation reversal:

  $$\begin{tikzcd} \mathcal{A}_i(M) \arrow[r, "\partial"] & \mathcal{A}_{i-1}(\partial M) \arrow[rr, "\text{projection}"] \arrow[dr, "\text{projection}"'] & & \mathcal{A}(\Sigma) \\ & & \mathcal{A}(-\Sigma) \arrow[ur, "\text{orientation reversal}"'] & \end{tikzcd}.$$

  Let $\mathrm{Eq}_\Sigma \mathcal{A}_i(M)$ be the equaliser of the maps, i.e. the set of fields in $\mathcal{A}_i(M)$ on which the two maps agree. Note that $\mathrm{Eq}_\Sigma \mathcal{A}_i(M)$ injects into $\mathcal{A}_i(M)$ because we do not use isotopy classes of fields here. Isotopies will be incorporated later as local relations.

Let M_{gl} be the image of M with the two copies of Σ identified via the homeomorphism. Then we require that there is an injective gluing map

$$\mathrm{Eq}_\Sigma \mathcal{A}_i(M) \hookrightarrow \mathcal{A}_i(M_{gl}).$$

Like all the other maps, the gluing map must commute with boundary, etc. We also require that the map is surjective up to extended isotopy, which we will discuss below. The idea is that not every field can be cut along Σ because the field may not be transverse to Σ, but every field is equivalent to one that is. We intend to suggest the usual idea of transversality here; formally, we define a field F on M_{gl} to be *transverse* to Σ if F is in the image of the equaliser $\mathrm{Eq}_\Sigma(\mathcal{A}(M))$.

- Gluing with corners. Let M be a manifold with

$$\partial M = \Sigma \cup -\Sigma \cup T.$$

The two copies of Σ must be disjoint, but share boundary with T, i.e.

$$\partial(\Sigma \amalg -\Sigma) = \Gamma = \partial T.$$

Note that Γ has two connected components, $\Gamma = \partial\Sigma \amalg -\partial\Sigma$.

We beg forgiveness for spelling the next part out in painful detail, but we wish to escape our own confusion. Using gluing without corners, we get two maps to $\mathcal{A}(\Gamma)$:

$$\begin{tikzcd}
\mathcal{A}_{i-1}(\Sigma \amalg -\Sigma \amalg T) \arrow[r, "\partial"] & \mathcal{A}_{i-2}(\Gamma \amalg -\Gamma) \arrow[rr, "\text{projection}"] \arrow[dr, "\text{projection}"'] & & \mathcal{A}(\Gamma) \\
& & \mathcal{A}(-\Gamma) \arrow[ur, "\text{orientation reversal}"'] &
\end{tikzcd}.$$

Recall $\mathrm{Eq}_\Gamma \mathcal{A}(\Sigma \amalg -\Sigma \amalg T)$ is the set of fields that are equal under these two maps, i.e., the set of fields that agree on Γ. Let $\mathcal{A}_\Gamma(M) \subset \mathcal{A}_i(M)$ be the preimage of the equaliser $\mathrm{Eq}_\Gamma \mathcal{A}(\Sigma \amalg -\Sigma \amalg T)$ under the boundary map. We want the subset $\mathrm{Eq}_\Sigma \mathcal{A}_\Gamma(M)$ of fields in this preimage which are *also* equal under the two projection maps to $\mathcal{A}(\Sigma)$:

$$\begin{tikzcd}
\mathrm{Eq}_\Sigma \mathcal{A}_\Gamma(M) \arrow[r, hook] & \mathcal{A}_\Gamma(M) \arrow[r, "\partial"] & \mathrm{Eq}_\Gamma \mathcal{A}(\Sigma \amalg -\Sigma \amalg T) \arrow[r, "\text{proj.}"] \arrow[d, "\text{proj.}"'] & \mathcal{A}(\Sigma) \\
& & \mathcal{A}(-\Sigma) \arrow[ur, "\text{or. rev.}"'] &
\end{tikzcd}.$$

We can get ∂M from the disjoint union $\Sigma \amalg -\Sigma \amalg T$ by identifying the two copies of Γ. By our axiom for gluing without corners, this means there is an injective gluing map

$$\mathrm{Eq}_\Gamma(\mathcal{A}_{i-1}(\Sigma \amalg -\Sigma \amalg T)) \xhookrightarrow{\mathrm{gl}} \mathcal{A}_{i-1}(\partial M).$$

Recall that we define $\mathrm{Eq}_{\partial\Sigma}\mathcal{A}(T)$ as the equaliser of the two maps:

$$\begin{tikzcd}
\mathcal{A}_i(T) \arrow[r, "\partial"] & \mathcal{A}_{i-1}(\partial T = \partial\Sigma \amalg -\partial\Sigma) \arrow[rr, "\text{projection}"] \arrow[dr, "\text{projection}"'] & & \mathcal{A}(\partial\Sigma) \\
& & \mathcal{A}(-\partial\Sigma) \arrow[ur, "\text{orientation reversal}"'] &
\end{tikzcd}.$$

The gluing map ‘gl’ enables us to write an enormous sequence of maps from $\mathrm{Eq}_\Sigma \mathcal{A}_\Gamma(M)$ to $\mathrm{Eq}_{\partial\Sigma}\mathcal{A}(T)$, which we have hidden away in Figure 1.6. This long composition of maps induced by the boundary map $M \to \partial M$ is morally just a boundary map from fields that agree on Σ to fields that agree on $\partial\Sigma$.

Call T_{gl} the result of applying gluing without corners to T. Let M_{gl} denote the image of M with the two copies of Σ identified; then $\partial(M_{gl}) = T_{gl}$. We require that there is an injective map

$$\mathrm{glue} : \mathrm{Eq}_\Sigma \mathcal{A}_\Gamma(M) \longrightarrow \mathcal{A}(M_{gl})$$

which makes the following square commute:

$$\begin{tikzcd}
\mathrm{Eq}_\Sigma \mathcal{A}_\Gamma(M) \arrow[r, "\text{Fig. 1.6}"] \arrow[d, "\mathrm{glue}"'] & \mathrm{Eq}_{\partial\Sigma}\mathcal{A}(T) \arrow[d, "\mathrm{gl}"] \\
\mathcal{A}(M_{gl}) \arrow[r, "\partial"] & \mathcal{A}(T_{gl})
\end{tikzcd}.$$

$$
\begin{tikzcd}
\mathrm{Eq}_{\Sigma}\mathcal{A}_{\Gamma}(M) \arrow[d, hook] \\
\mathcal{A}_{\Gamma}(M) \arrow[d, hook] \\
\mathcal{A}(M) \arrow[d, "\partial"] \\
\mathcal{A}(\partial M) \arrow[d, "\text{restriction}"] \\
\mathrm{gl}(\mathrm{Eq}_{\Gamma}\mathcal{A}(\Sigma \amalg -\Sigma \amalg T)) \arrow[d, "\mathrm{gl}^{-1}"] \\
\mathrm{Eq}_{\Gamma}\mathcal{A}(\Sigma \amalg -\Sigma \amalg T) \arrow[d, hook] \\
\mathcal{A}(\Sigma) \times \mathcal{A}(-\Sigma) \times \mathcal{A}(T) \arrow[d, "\text{projection}"] \\
\mathcal{A}(T) \arrow[d, "\text{projection}"] \\
\mathrm{Eq}_{\partial\Sigma}\mathcal{A}(T)
\end{tikzcd}
$$

Figure 1.6: Enormous sequence of maps, morally a boundary map.

This map must also be compatible with boundary, etc. and must be surjective up to extended isotopy. In the future, we will refer to gluing maps both with and without corners as 'gl.'

Equivalently but with less detail, let c_{gl} be a field in $\mathcal{A}_{i-1}(T_{gl})$ which is transverse to Σ, and let c be the unglued version in $\mathcal{A}_{i-1}(T)$. Let $\mathcal{A}^c(M)$ be the preimage of c in $\mathcal{A}_i(M)$ under the boundary map (Figure 1.6). Using boundary maps and projection, we get two maps from $\mathcal{A}^c(M)$ to $\mathcal{A}(\Sigma)$, one using orientation reversal. Let $\mathrm{Eq}^c_\Sigma(\mathcal{A}(M))$ be the set of fields which are equal under the two maps. We require that there is an injective map

$$\mathrm{Eq}^c_\Sigma(\mathcal{A}(M)) \hookrightarrow \mathcal{A}(M_{gl}, c_{gl})$$

which is surjective up to extended isotopy, etc.

- Products. We require a map

$$\mathcal{A}_i(M) \longrightarrow \mathcal{A}_{i+1}(M \times I)$$

which sends $F \in \mathcal{A}(M)$ to $F \times I$, and which commutes with boundary, disjoint union, etc. If $\bar{\phi} : M \to M$ is a homeomorphism and $\phi : M \times I \to M \times I$ is a fibre-preserving homeomorphism which projects to $\bar{\phi}$, then

$$\phi(c \times I) = \bar{\phi}(c) \times I.$$

This enables us to define collar maps and extended isotopy.

Suppose Σ is an $(n-1)$-dimensional submanifold of ∂M, where M is an n-manifold. Let F be a field on M with ∂F transverse to $\partial\Sigma$. Define F_Σ to be the restriction of F to Σ. Glue $\Sigma \times I$ to M along Σ, and call the result $M \cup (\Sigma \times I)$. We get a field $F \cup (F_\Sigma \times I)$ on $M \cup (\Sigma \times I)$ by gluing the fields F and $F_\Sigma \times I$ together along Σ. Let φ be a homeomorphism $M \cup (\Sigma \times I) \to M$. Then the map

$$F \longmapsto \varphi(F \cup (F_\Sigma \times I))$$

is called a *collar map*, which we illustrate in Figure 1.7.

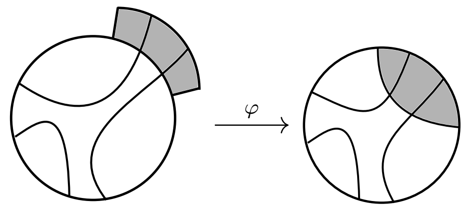


Figure 1.7: A collar map.

- We may enrich fields over a symmetric monoidal category such as **Vect**. If we do this, we must require that for the top dimension, $\mathcal{A}_n(M, c)$ is an object of this category, and maps between fields in $\mathcal{A}_n(M, c)$ must be morphisms in the category.

1.3.2 Local relations

Definition. A system of local relations is a collection of subsets of fields on balls which satisfy the following properties. Let B be an n-manifold homeomorphic to the standard n-ball, and let c be a field on ∂B. A collection of subsets $S(B, c) \subset \mathcal{A}(B, c)$ is a local relation if it satisfies the following:

- Functoriality. Relations are preserved under a homeomorphism $\varphi : B \to B'$ between n-balls, i.e. $\varphi(S(B, c)) = S(B', \varphi(c))$.

- Extended isotopy. An *extended isotopy* is an equivalence relation generated by collar maps and homeomorphisms isotopic to the identity. If two fields $F, G \in \mathcal{A}(B, c)$ are extended isotopic, then there is a relation between them in $S(B, c)$. This includes vanilla-flavoured isotopies.

- Ideal under gluing. If B splits into two sub-balls $B = B' \cup B''$ with $F \in S(B', c)$ a local relation on B' and $F'' \in \mathcal{A}(B'')$ a field on B'', then $F' \cup F'' \in S(B)$.

1.3.3 Getting a TQFT

Using an n-dimensional system of fields and local relations, we can define an $(n + \epsilon)$-dimensional TQFT. (Getting the $(n + 1)$-dimensional part of a TQFT requires extra conditions.)

Definition. Let M be an n-manifold. Let $S(M)$ be the subset of $\mathcal{A}(M)$ generated by fields of the form $F \cup G$, where $F \in S(B)$ is a local relation in some ball $B \subset M$ and $G \in \mathcal{A}(M - B)$ is a field on the complement of the ball. Then

$$\mathcal{C}(M) = \mathcal{A}(M)/S(M)$$

is a TQFT.

1.3.4 Higher categories

An n-category defines an n-dimensional system of fields if we have the following data:

- A cell decomposition of manifold M.
- Homeomorphisms around each k-cell to the standard sphere thought of as a bihedron, dividing $(k+1)$-cells into domain and range.
- Labelling each k-cell with an $(n-k)$-morphism according to the homeomorphism above.

Thus, given an n-category, we can define a TQFT from fields and local relations. There is also a dual construction in the reverse direction. According to this yoga, we should get assigments

$$\begin{aligned} n\text{-manifolds} &\rightsquigarrow \text{vector spaces} \\ (n-1)\text{-manifolds} &\rightsquigarrow \text{categories} \\ (n-2)\text{-manifolds} &\rightsquigarrow 2\text{-categories} \\ &\vdots \\ 0\text{-manifolds} &\rightsquigarrow n\text{-categories} \end{aligned}$$

In Chapter 3, we discuss the case of $(n-1)$-dimensional manifolds, which correspond to 1-categories, for the particular case of Bar-Natan skein modules.

Chapter 2

Black-and-white skein modules

2.1 Notation and basic definitions

Let M be a compact n-manifold, possibly with boundary. Let

$$\mathcal{A} : \mathbf{M}^n \longrightarrow \mathbf{Set}$$

be the functor which sends M to the set of oriented $(n-1)$-dimensional submanifolds F of M so that $M - F$ has a black-and-white checkerboard colouring with black on the side of the positive normal vector to F. The checkerboard colouring will allow us to gracefully handle orientations later on. We call $\mathcal{A}(M)$ *fields* on M. Note $\mathcal{A}(M) \neq \emptyset$ because F may be the empty submanifold.

More generally, let c be an $(n-2)$-submanifold of ∂M. Let $\mathcal{A}(M, c)$ be the set of oriented $(n-1)$-dimensional submanifolds F of M with $\partial F = c$, so that $M - F$ admits a checkerboard colouring as above. If there are no such submanifolds, we define $\mathcal{A}(M, c) = 0$.

Sometimes we will require some additional structure on our fields. We will use $\dot{\mathcal{A}}$ to refer to the functor which sends (M, c) to $(n-1)$-dimensional submanifolds F of M as above, where additionally, each connected component of F may have one or more dots away from ∂F. Dots are allowed to move freely on connected components of F, but not across components. We may use the notation $F^{\bullet}$ to refer to a connected submanifold with a single dot on it.

We will use the following conventions:

- $\mathbf{C}$ is a category and $\mathcal{F}$ is a functor.
- M is a 3-manifold; F, G are fields on M (2-dimensional submanifolds).
- Σ is a 2-manifold, such as ∂M; c, a, b are fields on Σ.
- Γ is a 1-manifold; p, q are fields on Γ.
- ξ is a 0-manifold; ϵ is a field on ξ.

Recall the Bar-Natan relations on surfaces, illustrated in Figure 2.1.

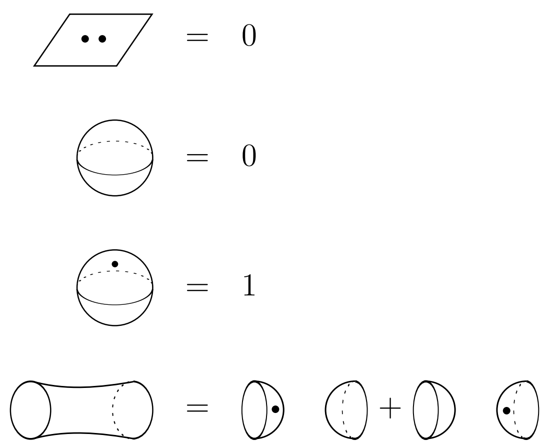


Figure 2.1: The Bar-Natan relations. In order from top to bottom, we refer to them as the 'two dots,' 'sphere,' 'dotted sphere,' and 'neck-cutting' relations.

The sphere relations hold only when the sphere bounds a ball. Strictly speaking, there are multiple versions of these relations, e.g. a sphere bounding a black ball, and a sphere bounding a white ball. The colouring of the region inside the neck must match the regions cut off by the disks in the neck-cutting relation.

Definition. Let R be a commutative, unital ring. Define $R\dot{\mathcal{A}}(M, c)$ to be R-linear combinations of elements of $\dot{\mathcal{A}}(M, c)$. Let $S(M, c; R)$ be the submodule of $R\dot{\mathcal{A}}(M, c)$ generated by the Bar-Natan relations and isotopy rel c.

The black-and-white skein module of M is then

$$\mathcal{B}(M, c; R) = R\dot{\mathcal{A}}(M, c)/S(M, c; R).$$

By [Wal06], $\mathcal{B}$ is a TQFT.

2.2 Simple examples

In the remainder of this chapter, we adapt the results in [AF07] on Bar-Natan skein modules to black-and-white skein modules. Many of the proofs remain largely unchanged.

Definition. A *pure state* is an element of the skein consisting of only one surface (which need not be connected). The surface may have any number of dots, and the complement may have either colouring.

Proposition 2.2.1 (AF Proposition 2.3)**.** The skein module of a manifold is generated by pure incompressible states.

Proof. Use the appropriately coloured neck-cuttings to reduce all compressible surfaces, then eliminate spheres bounding balls (black or white). After finitely many steps, we get a linear combination of pure incompressible states. □

Example 2.2.2. In S^3, all surfaces are compressible to spheres bounding balls, so the black-and-white skein is isomorphic to $R \oplus R$, generated by two copies of the empty surface, one with S^3 (the complement) coloured black, and one with S^3 coloured white.

Definition. We say surfaces F_0 and F_1 in M are *parallel* if there is an embedding of $F \times I$ into M with F_0 on one end and F_1 on the other.

Proposition 2.2.3 (AF Proposition 2.4)**.** Suppose a surface G has two parallel connected surfaces F_0 and F_1 bounding a handlebody with only one colour. (The colouring implies there are no other pieces of surface in between them.)

1. Let G_0 be the surface where F_0 has a dot and F_1 does not, and let G_1 be the surface with the dots the other way. Then $G_0 = -G_1$.

2. Let H be the surface where both F_0 and F_1 have dots. Then $H = 0$ unless G is a sphere, in which case H is equivalent to the surface with F_0 and F_1 removed.

Proof. 1. $F_0 \# F_1 = G_0 + G_1$ by neck-cutting.

If you remove a disk from a surface to get $F - D$, then you can view the rest of the surface as a 'mostly flat' disk-and-bands picture. The connect sum bounds a solid handlebody $(F - D) \times I$. If F is a sphere, the genus of the handlebody is 0; otherwise, it is at least 2. In either case, by neck-cutting and sphere relations, the connect sum is 0.

2. Similar. $(F_0 \# F_1)^\bullet = H + 0$. The connect sum is 0 unless F is a sphere, in which case it is 1.

□

In general, moving dots around can be difficult, but filtering by the number of connected components can help.

Definition. Let $\mathscr{F}_m$ be the span of surfaces in $\mathcal{B}(M; R)$ with representatives with fewer than m components.

It's easy to get this backward, so we'll spell out the details. We can always arbitrarily increase the number of components, e.g. by adding dotted spheres bounding balls. But we can only reduce the number (by 'regenerating' necks to connect components) if there are appropriate dots or some components are spheres. If you try filtering by submodules with greater than m components, the associated graded is trivial.

Remark 2.2.4. Other filtrations might be possible. The component filtration could be considered equivalent to filtering by Euler characteristic; we can also use a modified version called 'degree' (see Remark 4.2.1). Grading by degree obviously works, but the proof below doesn't go through. Filtering by the number of dots seems tempting, but it would need to be the number of dots on incompressible surfaces somehow. If we get rid of the dotted sphere relation, then the number of dots can only decrease, by regenerating necks.

Call the associated graded $\mathscr{G}_m = \mathscr{F}_m / \mathscr{F}_{m-1}$. If R is a direct product of simple rings, then $\mathcal{B}(M; R) = \bigoplus_m \mathscr{G}_m$.

Proposition 2.2.5 (AF Prop 2.5)**.** 1. With G_0 and G_1 surfaces as above, but not necessarily parallel, $G_0 = -G_1$ in $\mathscr{G}_m$.

2. A pure state of m components with more than one dot is zero in $\mathscr{G}_m$ (assuming M is a connected manifold).

Proof. 1. $F_0 \# F_1$ is in $\mathscr{F}_{m-1}$, so it is zero.

2. Use the first part to move dots around until one component has two dots.

□

Proposition 2.2.6 (AF Prop 2.6)**.** If M is a prime manifold, and $\{P_i^{\pm}\}$ are distinct pure incompressible states without dots and colouring on the complement indicated by $\pm$, then the $P_i^{\pm}$ are linearly independent.

Proof. For a surface G^+ consisting of P_i^+ union a bunch of dotted spheres bounding balls and k blank tori compressing to spheres bounding balls, define $f_i^+(G^+)$ to be 2^k, and otherwise 0. Similarly define $f_i^-(G^-)$. The $f_i^{\pm}$ vanish on relations and $f_i^{+/-}(P_j^{+/-}) = \delta_i^j$, so we get independence.

We need M to be prime because otherwise a torus can compress to two different things, meaning the functional is not well-defined. (Imagine a torus. Compressing it 'the obvious way' gives a sphere bounding a ball. But on the outside you could have a solid torus connect sum something nontrivial. The solid torus part means M is a lens space connect sum a nontrivial manifold.) □

For the remainder of this chapter, assume $\frac{1}{2} \in R$.

Example 2.2.7. The skein of $S^1 \times S^2$ is generated by pairs of spheres. More specifically, let Z^k be k parallel, nontrivial spheres (i.e. copies of $\{*\} \times S^2$). Let $\dot{Z}^k$ be the same as Z^k, but where one fixed sphere has a dot. Then

$$\mathcal{B}(S^1 \times S^2; R) = R[Z^2] \oplus R[\dot{Z}^2].$$

Proof. The incompressible surfaces in $S^1 \times S^2$ are nontrivial spheres. Since the spheres are parallel, we can use Proposition 2.2.3 to move dots between components and remove consecutive dotted spheres. Thus there is at most one dot, and we may assume without loss of generality that the dot is on the 'innermost' sphere, closest to $\{0\} \times S^2$. The colouring does not affect this simplification.

An element with a given colouring is isotopic to the element with the same surface and the opposite colouring, because a sphere can be isotoped around the back of the sphere, through $\{0\} \times S^2 \sim \{1\} \times S^2$.

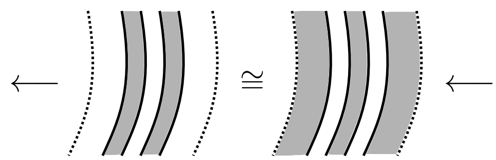


Figure 2.2: A diagrammatic view of isotopy 'around the back.'

Now we invoke Proposition 2.2.6 to show the independence of the blank spheres. $\square$

Remark 2.2.8. Note that while one of these surfaces is nulhomologous as a whole, the individual components are not.

Example 2.2.9 (A Similar Example, Ltd.)**.** The incompressible surfaces are in $T^3 = S^1 \times S^1 \times S^1$ are tori which are in one-to-one correspondence with triples of relatively prime integers (p, q, r) that are not all zero, with $(p, q, r) \sim (-p, -q, -r)$. (Think of the triple as giving a normal vector.) Let $F^k_{(p,q,r)}$ be the surface with k parallel blank (p, q, r) tori, and let $\dot{F}^k_{(p,q,r)}$ be the same but with one dotted torus. Then

$$\mathcal{B}(T^3; R) = \bigoplus_{(p,q,r)} R[F^2_{(p,q,r)}] \oplus R[\dot{F}^2_{(p,q,r)}].$$

The argument for $S^1 \times S^2$ can be applied here, too.

2.3 Seifert-fibred spaces

Before we discuss skeins of Seifert-fibred manifolds, first let us recall some basic facts. A Seifert-fibred space is a 3-manifold that can be written as a disjoint union of circles (fibres). Each fibre has a neighborhood that looks like a solid torus.

If the fibre is in the boundary of M, it has a neighborhood that looks like half a solid torus fibred by straight (vertical) lines. We will always assume M is compact.

Consider a solid cylinder $D^2 \times I$, where the top disk is rotated by $e^{2\pi i p/q}$, that is, p/q of a full twist. Identify x on the bottom with the image $\rho(x)$ on the top. This is a 'model Seifert fibring' of the solid torus. Every fibre in a Seifert-fibred manifold has a neighborhood diffeomorphic to a neighborhood of some fibre in a model Seifert fibring via a fibre-preserving diffeomorphism.

The *multiplicity* of a fibre Γ is the number of times nearby fibres intersect a small disk transverse to Γ. If the multiplicity is 1, the fibre is regular; if it is more than 1, the fibre is *singular* (or *exceptional* or *multiple*). Note p/q fibering is equivalent to p'/q' if $p/q = p'/q'$ mod 1, so assume $0 \leq p/q < 1$.

Example 2.3.1. In a model Seifert fibering the center fibre has multiplicity q, but any other fibre in the solid torus has multiplicity 1. (Consider a disk around it small enough to be away from the center.)

Singular fibres lie in the interior of M and are isolated. Identifying fibres to a single point gives a surface which is compact if M is compact, and away from the singular fibres, the projection map is a fibre bundle. Since there are no singular fibres in the boundary, the boundary consists of tori and Klein bottles (or just tori if M is orientable).

You can quotient M to get a set of fibres, equivalent to a surface F with some singular points, corresponding to the singular fibres. (This is also called an orbifold. A Seifert-fibred space is an S^1 bundle over a 2-dimensional orbifold.) Note there are actually two ways to do the quotienting:

- Quotient $D^2 \times I \to D^2$ by collapsing the interval, and then quotient by the action of ρ, getting a $q:1$ branched cover of the disk to itself away from the center point, which is the branch set.

- Map $D^2 \times I \to D^2 \times I$ by quotienting $(x,t) \sim (\rho(x), t)$, then collapse down to the disk. This way is fibre-preserving.

Example 2.3.2 (A few simple examples)**.**

- $S^2 \times S^1$, or any surface cross S^1, gives a trivial Seifert-fibred space, with no singular fibres. Note $S^1 \times S^2$ also has nontrivial fiberings (it does not have a unique Seifert fibering).

- Lens spaces.
- The Poincaré homology sphere has base a sphere with 3 singular fibres of multiplicity 2, 3, and 5.

This section heavily references [Bri07]. Seifert-fibred spaces are a basic topic in 3-manifold topology, so there are many references. For background material in this chapter, I have mostly relied on [Hat00]. I have mentioned other references in the text, where I have used them, except the Wikipedia article [Wik15f]. Hatcher recommends [Orl72].

2.3.1 Classification

Seifert-fibred spaces are classified by the base surface (its genus, orientability, number of boundary components), and the surgery slopes of the singular fibres. Mostly Seifert-fibred manifolds have a unique Seifert fibring, but there are a few exceptions, for example Lens spaces, S^3, and $S^1 \times S^2$. The reason is essentially the solid torus doesn't have a unique fibration, and gluing two solid tori together also does not have a unique fibration. There are a few other nonorientable Seifert-fibred spaces that also do not have unique fibrations.

If the base surface has no boundary, the sum of surgery slopes is called the *Euler number.* This is an invariant of the fibering.

2.3.2 Horizontal and vertical surfaces

Definition. A surface is *vertical* if it is a union of regular fibres. A surface is *horizontal* if it is transverse to all fibres.

Example 2.3.3. In $S^1 \times S^2$, a sphere $\{*\} \times S^2$ is horizontal; a tube S^1 cross a circle in S^2 is a vertical torus. Note that the horizontal spheres are incompressible, but there are no incompressible vertical surfaces in $S^1 \times S^2$.

Remark 2.3.4. Since vertical surfaces are disjoint from singular fibres, they are circle bundles, and thus must be annuli, tori, or Klein bottles. A connected vertical surface maps to a simple closed curve under the projection map to the base.

Remark 2.3.5. If a Seifert-fibred space has a horizontal (orientable) surface F, then the projection map is a branched cover, with multiplicity q branch points corresponding to multiplicity q fibres. Cutting M along F (to get $(M \mid F)$ in Hatcher's notation) gives an I-bundle over F, which is a mapping cylinder $F \amalg F \to G$ for some surface G.

If $(M \mid F)$ is disconnected, then F splits M into two I-bundles, each with only F as its boundary. An orientable I-bundle over an orientable surface has two boundary components, so the two I-bundles must be over non-orientable surfaces. Note M can still be orientable; e.g. the twisted I-bundle over $\mathbb{R}P^2$ is orientable. (It's analogous to the torus covering the Klein bottle.) In this case, the map $F \amalg F \to G_i$, $(i = 1, 2)$ is a nontrivial double cover. (Of course, there are some restrictions on which double coverings can happen based on Euler characteristic.) But this is only possible if the base is non-orientable.

A horizontal surface F must be orientable if the base is orientable, because you can coherently orient the fibres transverse to the base; this can be pushed along the fibres to give a coherent normal orientation along the horizontal surface. In this case $(M \mid F)$ is connected, $G = F$, and the map is a trivial double cover $F \amalg F \to F$. This shows M is an S^1-bundle over F, and implies M is homeomorphic to a mapping torus of some self-homeomorphism of S. (Gluing $(M \mid F)$ back together shows this.)

See [Hat00, Zul01] for more.

2.3.3 Incompressible surfaces

Theorem 2.3.6 (Waldhausen)**.** *In a connected, compact, irreducible Seifert-fibred space, any incompressible surface is isotopic to either a vertical surface, or a horizontal surface.*

The next two results give partial converses to Waldhausen's theorem.

Proposition 2.3.7. If M is an orientable Seifert-fibred space, and M has boundary, then there are horizontal surfaces in M. If M is orientable and has no boundary, horizontal surfaces exist if and only if the Euler number is zero.

Proposition 2.3.8. In a compact, irreducible Seifert-fibred space, every orientable horizontal surface is essential (incompressible and boundary-incompressible). This is true of connected, orientable vertical surfaces too, except a torus bounding a solid torus with a model Seifert fibering and at most one singular fibre, or an annulus cutting of a solid torus with the product fibering.

There are many such results; these are all mentioned in [Hat00].

2.3.4 A surface that compresses to a horizontal surface and a vertical surface

When we begin our investigation into chequered skein modules of Seifert-fibred manifolds, we will be interested in whether it is possible to have a compressible surface S with two distinct sequences of compressions, one leading to an incompressible horizontal surface H and the other leading to an incompressible vertical surface V. It turns out this *is* possible, as we'll see below.

Suppose M is a Seifert-fibred space, orientable, with closed, connected, orientable base Σ.

Proposition 2.3.9. If M has such a compressible surface S, then M fibres over the circle with fibre F, where F has genus at least 1.

Proof. Recall a connected vertical surface projects to a simple closed curve on the base. If there are 3 or fewer singular fibres, then on one side of a connected curve, there is only one singular point. There will thus be a compressing disk in V unless we have at least genus one base, or at least 4 singular fibres.

As discussed in Remark 2.3.5, M must fibre over the circle in order to have an orientable horizontal surface. Also, M must be a mapping torus of some homeomorphism h from F to itself. The homeomorphism h must be a map of finite order . To have a circle over a point $x \in F$, $h^k(x)$ must be equal to x for some k (otherwise it would be a line); thus h is a map of finite order [Hem76], p. 121.

The homeomorphism type of a surface bundle over the circle depends only on the isotopy class of the gluing homeomorphism. If h_1 and h_2 are isotopic homeomorphisms, then h_2^{-1} composed with the isotopy gives a map between $h_2^{-1}h_1$ and the identity. This induces a map on $F \times I$ (by taking the identity on I), which gives a well-defined map between the mapping tori M_{h_1} to M_{h_2}. (On the top, $(x, 0)$ maps to $h_2^{-1}h_1(x)$, and on the bottom, $(h_1(x), 1)$ gets mapped to $(h_1(x), 1)$. These points are identified in M_{h_2}.) In fact this gives a homeomorphism.

Homotopic homeomorphisms of a surface are in fact isotopic. [Eps66]

On the sphere, homotopy classes of maps are determined by degree. Since h is a homeomorphism, it must have either degree +1 or -1, for the identity or the antipodal map. (Another way of saying this is the mapping class group of the sphere is $\mathbb{Z}_2$.) Since we want an orientable manifold, we must have degree +1. These maps can have 2 fixed points (like a rotation), but since the maps are in the same isotopy class as the identity, the resulting spaces are all different Seifert fibrations of $S^1 \times S^2$. (Note lens spaces and S^3 also have fibrations with base S^2 and two singular points. These spaces - $S^1 \times S^2$, S^3, and lens spaces - do not have unique Seifert fibrations. In any case, these are not what we are looking for.) Thus if F has genus 0, then Σ has genus 0, but not enough singular points for M to have a vertical surface. $\square$

So let us consider maps of finite order on the torus. These relate to wallpaper groups, of which there are 17. Only 7 have a unique action (the other 10 are combinations of the 7, so they give different orbifolds, but no new maps). Two of the 7 have reflections and thus give nonorientable orbifolds.

The other maps, along with their orders and the orbifolds they produce, are given in Table 2.1.

Since they only have 3 singular points, the Seifert-fibred spaces corresponding to the last 3 maps in the table do not have vertical surfaces. The identity gives the Seifert-fibred space $S^1 \times S^1 \times S^1$, which we have already considered. That leaves

Table 2.1: Maps of finite order on the torus which give orientable orbifolds.

map	order	orbifold notation	orbifold description
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	o	torus
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	2	2222	'closed pillowcase' (sphere with 4 singular points of order 2)
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	442	'samosa' (sphere with 3 singular points of the given orders)
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	333	samosa
$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	6	632	samosa

the map of order 2 (the hyperelliptic involution) as the only possible example of a surface that can be compressed to both a horizontal incompressible surface and a vertical incompressible surface.

Much of this discussion is in [Hem76], p. 121-122; Wikipedia helped fill some gaps [Wik15d, Wik15g].

Example 2.3.10 (A specific example)**.** Thanks to Charlie Frohman for this example. Consider $T^2 \times I$ glued by the hyperelliptic involution. The base space is the sphere with 4 singular points. The vertical surfaces are tori separating 2 singular points. The horizontal surface is a torus. Projecting to the base is a branched cover with 4 branch points.

Take 2 horizontal tori, oriented oppositely, connected by 2 tubes. (The tubes are vertical, but not special otherwise, i.e. not around singular points or anything. This is a surface of genus 3. It is possible to compress it twice to get either 2 parallel horizontal tori, or 2 parallel vertical tori.

2.3.5 Black-and-white skeins of Seifert-fibred spaces

Let M be a compact, orientable Seifert-fibred manifold with orientable orbifold Σ and projection map $\psi : M \to \Sigma$. In addition, assume Σ is closed and connected; hence M is closed and connected.

Theorem 2.3.11. *The skein of M splits as a direct sum*

$$\mathcal{B}(M;R) = \mathcal{B}_V(M;R) \oplus \mathcal{B}_H(M;R),$$

where $\mathcal{B}_{V/H}$ is generated by checkerboard-coloured vertical/horizontal surfaces which may have dots.

Proof. Waldhausen's theorem implies that the right-hand side spans $\mathcal{B}(M;R)$. However, it is possible to have a compressible surface S with two distinct sequences of compressions, one leading to an incompressible horizontal surface H and the other leading to an incompressible vertical surface V.

We saw in Proposition 2.3.9 that if M has such a compressible surface S, then M fibres over the circle with fibre F, where F has genus at least 1.

Lemma 2.3.12. If M has a surface S as above, then both H and V are zero in the skein.

Proof. We can get S from H by attaching some 'mostly vertical' tubes to H.

If H has one component, $M - H$ cannot be checkerboard-coloured since $M \cong H \times S^1$. So H must have an even number of components with opposite orientations, and in particular, H has at least 2 components. These components must be parallel copies of the fibre F.

The surface S can't have more than n tubes connecting n components of H or else H is zero by the two dots relation. Further, because the components of H are parallel, with genus at least 1, S can't have *any* tubes, by Proposition 2.2.3. But since S must have at least one tube to be compressible to the two components, H is always zero. (In fact, S must have at least 2 tubes to be able to get a vertical surface by compressing.)

By Lemma 2.3.9, the components of H have at least genus 1, meaning S has at least genus 3 and $\chi(S) \leq -4$. Since we are only considering closed, orientable surfaces, any vertical surfaces are composed of tori [Hat00]. So we must perform at least two compressions; thus, there are at least two dots on each pure surface in V. This is zero, as discussed below. $\square$

Thus there are no relations between horizontal and vertical surfaces in M. $\square$

$\mathcal{B}_V$

Recall vertical surfaces are in one-to-one correspondence with elements of $\mathcal{A}(\Sigma)$, that is, 1-dimensional chequered submanifolds on the base Σ, and any connected vertical surface must be a torus. A vertical surface is separating if and only if the corresponding curve on Σ is separating. A connected sum of curves corresponds to 'annular sum' of tori along longitudes. (This can be extended to surfaces with boundary, but for the moment we restrict our attention to closed surfaces.)

Definition (Annular sum)**.** Let S_1 and S_2 be connected components of a 2-dimensional submanifold of a 3-manifold M. We allow S_1 and S_2 to be on the same component. Let A be an annulus properly embedded in M with each of its boundary components c_i properly embedded in S_i. The colouring on $M - S_1 \cup S_2$ must be such that it would be possible to connect sum S_1 and S_2 so the tube would coincide with A. Let N_i be a neighborhood of c_i with $\partial N_i = c_i' \cup c_i''$ and let A' and A'' be annuli parallel to A with $\partial A' = c_1' \cup c_2'$ and $\partial A'' = c_1'' \cup c_2''$.

Define annular sum $\#_A$ by

$$S_1 \#_A S_2 = (S_1 - N_1) \cup (S_2 - N_2) \cup A' \cup A''.$$

This depends on the choice of A.

Proposition 2.3.13 (AF Prop 5.2)**.** Let S_1' and S_2' be the components of $S_1 \#_A S_2$. (Either S_1 is the same component as S_2, or S_1' and S_2' are on the same component.) Then

$$\dot{S}_1 \cup S_2 + S_1 \cup \dot{S}_2 = \dot{S}_1' \cup S_2' + S_1' \cup \dot{S}_2'.$$

Proof. Annular sum is attaching a 1-handle (connect sum) followed by attaching a 2-handle (a neck-cutting, where the neck is like a taco shell on the connect sum neck). Thus either side is equal to $S_1 \# S_2$ by neck-cutting or reversing a neck-cutting. $\square$

Since vertical surfaces correspond to curves, we make the following definition.

Definition. For an orbifold Σ, define $af(\Sigma)$ as the submodule of $R\dot{\mathcal{A}}(\Sigma)$ generated by isotopy and the relations in Figure 2.3, which we call the Asaeda-Frohman relations. As with the Bar-Natan relations, each relation has two variations depending on the colourings. Define $\mathcal{B}(\Sigma; R) = R\dot{\mathcal{A}}(\Sigma)/af(\Sigma)$.

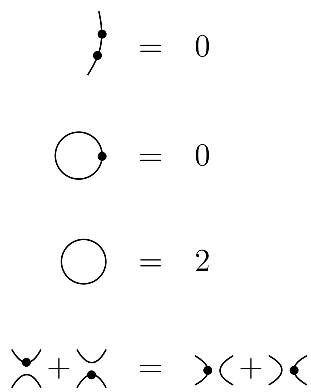


Figure 2.3: The Asaeda-Frohman relations on arcs: 'two dots,' 'dotted circle,' 'circle' and 'arc-sum' or 'neck-cutting.' The circles must bound a disk, possibly with one singular point (meaning they correspond to compressible tori).

Proposition 2.3.14 (AF Prop 5.5)**.** Twice a dotted separating curve is zero.

Sketch of proof. Using annular sum and Proposition 2.3.13, we can pull curves past genus. A curve that bounds singular points is zero by using the relations to isolate a singular point inside a curve, and then remove it. $\square$

Perhaps this corollary deserves emphasis:

Corollary 2.3.15 (AF Lemma 5.9)**.** If two circles cobound a surface $F \subseteq \Sigma$, a dot can be moved across F from one circle to the other with a negative sign.

Proof. The arc-sum relation implies the sum of the two pictures equals twice a dotted separating curve (since the two circles cobound a surface), which is zero by Proposition 2.3.14. □

Lemma 2.3.16 (AF Lemma 5.10)**.** A pure state with more than one dot is zero.

Proof. Let c and c' be the dotted circles, with adjacent circles $c_i, i = 1, 2 \ldots n$ in between. WLOG, the in-between circles are blank, and all the circles are distinct. The proof is by induction on n. We have $\dot{c} \cup c_1 \cup \dot{c}' = -c \cup \dot{c}_1 \cup \dot{c}' + 2(\dot{c \# c_1}) \cup \dot{c}'$, etc. We only know the connect sum is zero if it's separating, so we must keep doing this until we get two dots on c' in every term. Thus we get zero. For the induction step, moving the dot from c to c_1 reduces the number of in-between circles to $n-1$. □

Definition (Diagram, stack, weight, band)**.** Note that, even aside from checkerboard colouring, these definitions are slightly different from those in [AF07]. For a pure state α in $\mathcal{B}(\Sigma)$, we get the *unreduced diagram* $\widehat{\Gamma}(\alpha)$ by removing a small neighborhood of each curve. Each connected component of the resulting surface gives a vertex of $\widehat{\Gamma}$. Each circle gives an edge connecting vertices.

Two circles are in the same *stack* if they are connected by a bivalent vertex. If a circle is not connected to any bivalent vertices, it is in a stack by itself. Define the *weight* of a stack to be the number of circles in the stack. Checkerboard colouring eliminates some configurations, which we discuss further below.

Define the *diagram* $\Gamma(\alpha)$ by removing bivalent vertices and increasing the weight accordingly.

$\Gamma^{(0)}$ corresponds to the reduced set of connected components.

$\Gamma^{(1)}$ corresponds to the reduced set of edges e, i.e. to the stacks, along with each stack's weight and the $\mathbb{Z}_2$ homology class each circle in e determines.

We use e to refer to the stack, the edge, and the homology class simultaneously. Note that each circle in a stack has the same homology class, but different stacks may represent the same homology class.

For a stack e, define a *band* $N(e)$ as follows. If the weight ≥ 1, order the circles in the stack so that c_i and c_{i+1} are adjacent circles, and take $N(e)$ to be a connected surface in Σ containing the circles, with boundary $c_1 \amalg c_n$. The band may depend on the particular ordering. If the stack has weight 1, set $N(e)$ to be a regular neighborhood of the circle, disjoint from other bands. Then all bands are disjoint.

Note that dots can be moved through a stack easily (with a sign) by Corollary 2.3.15.

Lemma 2.3.17 (AF Lemma 5.12)**.** A dotted state α is zero if there is a connected component F in the complement of the bands $\Sigma - \cup_e N(e)^\circ$ with more than two distinct circles in α among its boundary components.

Proof. Let c_i, $i = 1, 2, 3$ be distinct circles in α among the boundary circles of F, and set α_i to be the state with the same circles as α but a dot on c_i. We can use Corollary 2.3.15 to move the dot across a surface with three boundary components, so we get $\alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_1$, so $\alpha = 0$. □

Call a graph that corresponds to a pure state *admissible*.

Lemma 2.3.18 (AF Lemma 5.15)**.** A graph is admissible if and only if for each $v \in \Gamma^{(0)}$,

$$\sum_{\substack{e \ni v \\ e \text{ not a loop}}} e = 0.$$

Proof. In a graph corresponding to a pure state, a vertex $v \in \Gamma^{(0)}$ corresponds to a surface with boundary corresponding to the edges connecting to v, thus the sum of the edges is zero in homology. (If an edge is not a loop, it is separating, hence zero in homology.) In the other direction, given a graph and data about the surface, the admissibility condition means the edges in the graph correspond to circles bounding a surface in Σ. Circles can be randomly assigned as necessary to get the weight of an edge. □

Lemma 2.3.19 (AF Lemmas 5.16-17)**.** By Lemma 2.3.17, any dotted state with a vertex with three or more distinct edges is zero. Recall that bivalent vertices can be reduced by adding the weights (admissibility requires that the two edges have the same homology class). An edge abutting a single vertex corresponds to a separating curve; we are assuming $\frac{1}{2} \in R$, so by Proposition 2.3.14 these are zero. Thus a reduced admissible graph corresponding to a dotted state consists of one of:

1. a single loop at a single vertex
2. two loops at a single vertex.
3. an edge between two vertices with a loop at each vertex.

(We assume the homology classes associated to the two loops in 2 and 3 are distinct.)

To simplify the description of $\mathcal{B}(\Sigma)$, recall the filtration introduced earlier, assuming R is a direct product of simple rings. Restrict it to $\mathcal{B}_V$ and project it to $\mathcal{B}(\Sigma)$, noting the number of components of a state in $\mathcal{B}_V$ is the same as for the corresponding state in $\mathcal{B}(\Sigma)$. Let $\mathscr{S}_i$ be the span of states with $\leq i$ components and with a dot on some component. Then $\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \dots$ and the limit of the $\mathscr{S}_i$'s, together with the undotted states, is $\mathcal{B}(\Sigma; R)$. Let $\mathscr{G}_i^\bullet = \mathscr{S}_i / \mathscr{S}_{i-1}$. Certainly in $\mathscr{G}_i^\bullet$ a state with more than one dot is zero.

If a loop has odd weight, the state does not admit a checkerboard colouring and thus is zero. For graphs of form 2 or 3, the state is also zero if a loop has even weight, by Lemma 2.3.17.

Proposition 2.3.20 (AF Prop 5.18)**.** A reduced admissible graph representing a state in $\mathscr{G}_i^\bullet$ is zero except when i is even; in that case $\mathscr{G}_n^\bullet$ is spanned by a single loop with weight 2 and some homology class in $H_1(\Sigma; \mathbb{Z}_2)$ on the loop.

Theorem 2.3.21 (AF Theorem 5.19, structure of $\mathcal{B}(\Sigma)$.)**.** *Let $H = H_1(\Sigma; \mathbb{Z}_2)$ and let A be the set of all reduced admissible graphs with no loops of odd weight.*

Then

$$\mathcal{B}(\Sigma) \cong \bigoplus_{b/w} RA \oplus \oplus_{n \geq 2 \ even} RH.$$

Proposition 2.3.22 (Prop 5.2)**.** $\mathcal{B}_V(M) \cong \mathcal{B}(\Sigma)$.

Proof. The Bar-Natan surface relations may not correspond to arc relations (because they are not necessarily vertical). However, the arc relations do all have obvious surface counterparts, so we get a well-defined map $\mathcal{B}(\Sigma) \to \mathcal{B}_V$ by taking the preimage of a curve under the projection. This map is clearly surjective. Next, we show $\mathcal{B}(\Sigma)$ injects into $\mathcal{B}_V$ to get equality.

We already know undotted incompressible surfaces are linearly independent, so we restrict our attention to dotted surfaces. Let $\dot{\mathscr{F}}'_n$ be the set of surfaces in $\mathcal{B}_V$ with representatives with n components or fewer and with a dot. We want to show the preimage of a basis $\alpha_i \in \mathscr{G}_n^\bullet$ is linearly independent in $\dot{\mathscr{G}}'_n = \dot{\mathscr{F}}'_n / \dot{\mathscr{F}}'_{n-1}$. By previous results, we know anything with more than one dot is zero. Assume the dot on each α_i is chosen arbitrarily and fixed on some component. Let A_i be the preimage $\psi^{-1}(\alpha_i)$, homeomorphic to $\alpha_i \times S^1$. We will define a linear functional to show these are linearly independent. Let $\varphi_i(B) = (-1)^l 2^k$ if B is a disjoint union of A_i with singly dotted spheres bounding balls and k blank tori compressible to spheres bounding balls. The dot may live on a different component, where l is the number of shifts to move the dot to the preferred component (this is all in terms of α_i, not A_i). We know l is well-defined mod 2 because α_i is not zero. Otherwise, set $\varphi_i(B) = 0$. We need to check that the φ_i's behave well on relations.

This is clear except for neck-cutting. Let A_∞ be the surface with the neck, A_+ the cut neck with the dot on the top disk, and A_- the cut neck with the dot on the bottom disk. The surface is the same outside of a ball containing these. We want to check $\varphi_i(A_\infty) = \varphi_i(A_+) + \varphi_i(A_-)$. We can assume $A_\pm$ do not have spheres bounding balls or tori bounding solid tori.

Since A_∞ has $n-1$ components, it is zero in $\dot{\mathscr{G}}'_n$.

If the surface underlying $A_\pm$ is incompressible but not isotopic to A_i, both sides of the equality are zero. This is also true if the surface is compressible, because it must again be not isotopic to A_i, which is incompressible with n components. Thus both sides of the equality are zero, and we have linear independence. $\square$

$\mathcal{B}_H$

As previously discussed, if M has a connected, orientable horizontal surface G, then M is homeomorphic to a mapping torus of a homeomorphism h from G to itself, and h is a map of finite order, obtained by following the (oriented) fibres in the positive direction.

We can decompose $H_1(M)$ using Mayer-Vietoris (with two cylinders $G \times I$). This shows $H_1(M) = \mathbb{Z} \oplus H_1(G)/(h_* - 1)$. The homeomorphism h gives a special class generating $\mathbb{Z}$, giving the circle part of M, and the rest of $H_1(M)$ is the invariant part of $H_1(G)$, which we denote $H_1(G)^h$.

Orient G, Σ, and M. If G' is a different (connected, oriented) horizontal surface, we get an ordered pair $(z(G'), d(G'))$ in $H_1(G)^h \oplus \mathbb{Z}$, where d is the degree of $\psi|_{G'}$. We get z by making G and G' transverse and taking the class of their oriented intersection in $H_1(G)^h$.

The image of this map is all indivisible elements. Two surfaces are isotopic if and only if they have the same image. Let $P(G)$ be indivisible elements with $d(G') > 0$. These are in one-to-one correspondence with isotopy classes of horizontal surfaces in M. Let F^k be k parallel copies of $F \in P(G)$, and let $\dot{F}^k$ be k parallel copies of F where one fixed copy has a dot.

Theorem 2.3.23 (AF 5.21)**.**

$$\mathcal{B}_H \cong \bigoplus_{F \in P(G)} \{F^{2k}\}_k \oplus \{\dot{F}^{2k}\}_k.$$

Two non-parallel horizontal surfaces cannot be disjoint. So an embedded surface only has parallel surfaces, and nonisotopic surfaces are necessarily not parallel (because the cylinder in between them would give an isotopy). From there, the proof is like $S^1 \times S^2$.

Putting it all together,

Theorem 2.3.24. *For M a Seifert-fibred space (orientable with orientable, closed, connected base), R a direct product of simple rings with $\frac{1}{2}$,*

$$\mathcal{B}(M; R) \cong \bigoplus_{b/w} RA \oplus \oplus_F \{F^{2k}\}_k \oplus \{\dot{F}^{2k}\}_k \oplus \oplus_{n \geq 2, even} RH.$$

Contrast this with Asaeda and Frohman's result for uncoloured Bar-Natan skein modules.

Theorem 2.3.25 (AF Theorem 5.22)**.** *Let $\Delta = \{(a,a)\} \in H \times H$. Let A' be the set of all reduced, admissible graphs (including those with loops of odd weights). Let $F^{\bullet}$ be a single copy of F with a dot.*

$$BN(M;R) \cong RA' \oplus \oplus_F(\{F^k\}_k \oplus F^{\bullet}) \oplus RH \oplus \oplus_{n\geq 2, even} \oplus \oplus_{n>1, odd} R(H \times H)/\Delta).$$

Chapter 3

Computations for a local model

It is pleasant to avoid thinking about manifolds with boundary, but we must face them if we want to compute $\mathcal{B}(M)$ by gluing together pieces of a manifold. Tori are fundamental building blocks for 3-manifolds (via link surgery, JSJ decompositions, Heegaard splittings) so this chapter is dedicated to computing the skein of the solid torus with some boundary curves of rational slope. In the process, we reprove some results of [Rus09].

Our first step is to compute the skein of the solid cylinder $D^2 \times I$. Understanding the category of a disk is a good starting point, since we can use it to understand other surface categories (see 4). We can think this as an algebra, but it is usually more convenient to think of it as a category.

3.1 Algebraic objects as categories

If G is a group, we can think of G as a category $\mathbf{G}$ with one object. All the elements of the group we view as homs from the object (which also gets called G) to itself. Composition is given by the group's multiplication.

If R is a ring, we need a *preadditive* category (also called a *linear* category), meaning the hom sets are abelian groups and composition is bilinear. So if $F, G \in Hom(x, y)$, then $F + G$ makes sense, and

$$E \cdot (F + G) = E \cdot F + E \cdot G,$$

etc., so all the ring structure makes sense, too.

Remark 3.1.1. Since a category has a unit for each object by definition, this is a unital ring.

Definition. A functor $\mathcal{M}$ between pre-additive categories is additive if for all objects x and y, the map $Hom(x, y) \to Hom(\mathcal{M}(x), \mathcal{M}(y))$ given by $F \mapsto \mathcal{M}(F)$ is a group homomorphism.

If $\mathbf{R}$ and $\mathbf{S}$ are rings thought of as categories, an additive functor between 'ring categories' is a ring homomorphism.

For the sake of comparison, let's recall some basic definitions.

Definition. A right-module A over a ring R is an abelian group $(A, +)$, with an operation (scalar multiplication) $A \times R \to A$ such that for all $r, s \in R$ and $a, b \in A$,

1. $(a + b)r = ar + br$
2. $a(r +_R s) = ar + as$
3. $a(rs) = (ar)s$
4. $1_R a = a$ if R has identity

A left-module is similar, but the action is on the left.

We can also think of a right $\mathbf{R}$-module as a (covariant) additive functor $\mathcal{M} : \mathbf{R} \to \mathbf{Ab}$. It sends the object R to a particular abelian group A. The literature often uses the convention that left modules are covariant and right modules are contravariant, but we use the opposite convention because it seems much more reasonable for composing morphisms and gluing cylinders together. Thus when we say module, we mean a right-module by default.

Let's check the properties:

1. $(a + b)r = ar + br$. This is true because r, a morphism in $Hom(R, R)$, gets sent to a morphism in $Hom(A, A)$ in the category $\mathbf{Ab}$, namely a group homomorphism from A to itself.

2. $a(r +_R s) = ar +_A as$. This holds because $\mathcal{M}$ is an additive functor, so the map $Hom(R, R) \to Hom(A, A)$ is a group homomorphism.

3. $a(rs) = (ar)s$

4. $1_R a = a$ if R has identity. The last two are true by functoriality, which says $\mathcal{M}(f \circ g) = \mathcal{M}(f) \circ \mathcal{M}(g)$ and $\mathcal{M}(id_R) = id_A$.

The moral is that the action of the category **R** on A behaves correctly as scalar multiplication in a module.

Example 3.1.2. For a preadditive category $\mathcal{C}$, the map $Hom_{\mathcal{C}}(A, -)$ is a covariant functor to **Ab**, giving a right-module.

A left **R**-module is a *contravariant* additive functor. Everything behaves the same way, except $\langle rs, a\rangle$, the action of rs on a, is the homomorphism $s \circ r$ evaluated on a instead of $r \circ s(a)$.

Example 3.1.3. For a preadditive category $\mathcal{C}$, the map $Hom_{\mathcal{C}}(-, B)$ is a contravariant functor to **Ab**, giving a left-module.

Over a commutative ring R, a right-module A can be given a left-module structure by defining $ra = ar$, [Hun74] p. 169. This is generally assumed to be true, so that A is a bimodule over R.

Definition. An R–R bimodule A (also called an R-bimodule) is both a left and right module over R.

Before we consider a categorical version of a bimodule, we will consider modules over an algebra.

Definition. An algebra A over a commutative ring R is an R-module A with multiplication $[\,\cdot\,,\cdot\,]$, satisfying bilinearity:

- $[ar + bs, c] = [a, c]r + [b, c]s$
- $[c, ar + bs] = [c, a]r + [c, b]s$.

An associative algebra has associativity and identity.

To have a category with the structure of an R-algebra, the category must be an R-linear category (with one object).

Definition. A category is called R-linear if the Hom sets $Hom(x, y)$ have an R-module structure (for all objects x and y), and composition is R-bilinear.

This is an algebra since $Hom(A, A)$ is an R-module, and composition being R-bilinear gives the R-bilinear multiplication we need for an algebra. Again, **A** has a unit, and composition is associative by definition of a category, so this is an associative algebra. (We assume that $Hom(A, A)$ in fact has the structure of an R-bimodule.)

Remark 3.1.4. Additivity is also sometimes included in the definition of an R-linear category, but all we need for an algebra is the definition above.

Definition. A right-module M over an algebra A (over a commutative ring R) is a unitary (meaning 1_R acts as the identity on M) right R-module such that M is also a right-module over the ring A, and

$$(ma)r = m(ar) = (mr)a$$

for all $r \in R, a \in A, m \in M$.

A homomorphism of algebra modules is both an R-module and A-module homomorphism.

In terms of categories, a module over an algebra is an additive functor from an R-linear category **A** (with one object) to **Ab**. Since **A** must have a unit (since it is a category), the R-module structure of M takes care of itself. The property $(ma)r = m(ar)$ is true by functoriality, and $(mr)a = m(ar)$ since we assume $ra = ar$.

Homomorphisms are natural transformations.

Now we return to the idea of an **A**–**A** bimodule M. It's not clear what it means to have a functor that is both covariant and contravariant (although it does still make sense to have both a left and right action on something), so we'll rephrase a bit. Let A^{op} be the opposite algebra of A, meaning it has the same elements as A, but multiplication acts on the left:

$$a \cdot b = ba.$$

Now we can think of an A–A bimodule as a right module over the enveloping algebra of A, defined as $A^e = A^{op} \otimes A$. The action of A^e on M is

$$m(a \otimes b) = amb.$$

Composition is

$$m(a \otimes b)(c \otimes d) = cambd.$$

This generalises nicely to categories, since we can think of the bimodule as a covariant functor $\mathbf{A}^{op} \otimes \mathbf{A} \to \mathbf{Ab}$. Note that a covariant functor on $\mathbf{A}^{op}$ is the same as a contravariant functor on $\mathbf{A}$, so in some sense this *is* a functor which is both co- and contravariant.

Example 3.1.5. For a preadditive category $\mathcal{C}$, the map $Hom_{\mathcal{C}}(-,-)$ is a covariant functor $\mathcal{C}^e \to \mathbf{Ab}$, giving a bimodule.

Proposition 3.1.6. Suppose categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are such that each object $c \in \mathcal{C}$ gives a functor $L_c : \mathcal{D} \to \mathcal{E}$ and each object $d \in \mathcal{D}$ gives a functor $R_d : \mathcal{C} \to \mathcal{E}$ so that the two functors agree, i.e. $L_c(d) = R_d(c)$.

We can combine the two functors into a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which sends a pair of objects (c, d) to $L_c(d) = R_d(c)$ if and only if for every pair of morphisms $F : c \to c'$ and $G : d \to d'$, the following square commutes:

$$\begin{tikzcd}
L_c(d) = R_d(c) \arrow[r, "R_d(F)"] \arrow[d, "L_c(G)"'] & R_d(c') = L_{c'}(d) \arrow[d, "L_{c'}(G)"] \\
L_c(d') = R_{d'}(c) \arrow[r, "R_{d'}(F)"'] & R_{d'}(c') = L_{c'}(d')
\end{tikzcd} .$$

It is easy to see this holds for $Hom_{\mathcal{C}}(A, -)$, $Hom_{\mathcal{C}}(-, B)$, and $Hom_{\mathcal{C}}(-,-)$. For more on bifunctors and product categories, see [ML78] pages 36-38.

Remark 3.1.7. A natural generalisation of modules over rings/algebras is just to allow any preadditive (linear) or R-linear category (so more than one object is possible).

The real advantage of using a category to describe the skein of the solid cylinder is being able to use different objects, so this is what we do below.

Other references for this section are [Hun74, Wik15c, Wik15e, Wik15a].

3.2 Surface categories

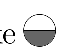
Now we begin our study of 'local' computations of chequered skein modules from Walker's formalism of TQFTs [Wal06, MW11].

Given an n-dimensional TQFT from fields and local relations, we can produce an k-category associated to an $n-k$-manifold. In the case of black-and-white skein modules, we can define a 'cylinder category' $\mathbf{C}(\Sigma)$ for a closed surface Σ using the skein $\mathcal{B}(\Sigma \times I)$. The objects are fields $a, b \in \mathcal{A}(\Sigma)$, and

$$Hom(a, b) \cong \mathcal{B}(\Sigma \times I, (a, \hat{b})).$$

The composition FG of $F \in Hom(a, b)$ and $G \in Hom(b, d)$ is the image under gluing in $\mathcal{B}(\Sigma \times I, (a, \hat{d}))$. Gluing is well-defined by a theorem of Walker, which we state below for black-and-white skeins.

We can generalise all of the above to get a category for a surface with boundary as well. Let Σ be a surface with $\Gamma = \partial\Sigma$, and let p, q be fields on Γ. Fix $c^\sharp \in \mathcal{A}(\Gamma \times I, (p, \hat{q}))$. The objects of the category are curve systems $a \in \mathcal{A}(\Sigma, p), b \in \mathcal{A}(\Sigma, q)$, and $\mathcal{B}(\Sigma \times I, (a, \hat{b}, c))$ is an algebra (or category) under gluing. In the case when $p = q$ and $c^\sharp = \{p\} \times I$, we will call this $\mathbf{C}(\Sigma, p)$.

Remark 3.2.1. Objects like  and , though isomorphic, are included as distinct objects. For convenience, we may refer to the equivalent category with only one object from each isomorphism class. This is a Morita equivalence; note in particular that Hochschild homology is preserved.

3.2.1 A Tale of Two Gluing Theorems

Gluing without corners

Let M be a 3-manifold with whose boundary has several components, two of which are homeomorphic. Thus we can write $\partial M = \Sigma \amalg -\Sigma \amalg T$. Let a be a field on Σ. We further require that the homeomorphism on Σ respects the checkerboard colouring on $\Sigma - a$. Define M_{gl} to be M with Σ and $-\Sigma$ identified via the homeomorphism. Fields on ∂M are isomorphic to the product of fields on the connected components, i.e.

$$\mathcal{A}(\partial M) \cong \mathcal{A}(\Sigma) \times \mathcal{A}(-\Sigma) \times \mathcal{A}(T).$$

Thus, if c is a field on T, then $(a, \hat{a}, c) \in \mathcal{A}(\partial M)$.

The gluing $M \to M_{gl}$ gives a map

$$\dot{\mathcal{A}}(M, (a, \hat{a}, c)) \longrightarrow \dot{\mathcal{A}}(M_{gl}, c).$$

Dots are assumed to be away from the boundary of surfaces in $\mathcal{A}(M, (a, \hat{a}, c)$, so they present no complications. This map induces a map

$$R\dot{\mathcal{A}}(M, (a, \hat{a}, c)) \longrightarrow R\dot{\mathcal{A}}(M_{gl}, c).$$

This in turn induces a map

$$\mathcal{B}(M, (a, \hat{a}, c)) \longrightarrow \mathcal{B}(M_{gl}, c),$$

since local relations in M still hold in M_{gl}.

Theorem 3.2.2. *Let L be the submodule of $\bigoplus_a \mathcal{B}(M, (a, \hat{a}, c))$ generated by elements of the form $FG - GF$, where $F \in \dot{\mathcal{A}}(\Sigma \times I, (a, \hat{b}))$ and $G \in \dot{\mathcal{A}}(\Sigma \times I, (b, \hat{a}))$. Then*

$$\mathcal{B}(M_{gl}) \cong \bigoplus_a \mathcal{B}(M, (a, \hat{a}, c))/L.$$

We could equally well view $\mathbf{C}(\Sigma)$ as an algebra with elements $F \in Hom(a, b)$ and $G \in Hom(c, d)$, and multiplication given by gluing. This is also a Morita equivalence, preserving Hochschild homology. We can then rephrase Theorem 3.2.2 as

$$\mathcal{B}(M \cup_\Sigma N) \cong \mathcal{B}(M) \otimes_{\mathbf{C}(\Sigma)} \mathcal{B}(N)$$

although Theorem 3.2.2 is slightly stronger in that it allows gluing a manifold to itself. See also [Kai09] for a special case.

Gluing with corners

Let M be a 3-manifold with $\partial M = \Sigma \cup (-\Sigma) \cup T$, and $\partial\Sigma = \Gamma, \partial T = -\Gamma \amalg \Gamma$. As before, define M_{gl} to be M with Σ and $-\Sigma$ identified, and define T_{gl} to be T with Γ and $-\Gamma$ identified so that $\partial M_{gl} = T_{gl}$.

Let $p \in \mathcal{A}(\Gamma)$ and let $c \in \mathcal{A}(T_{gl})$ be such that c is the image under gluing of $c^\sharp \in \mathcal{A}(T, (p, \hat{p}))$. The gluing theorem with corners is:

Theorem 3.2.3. *Let L be the submodule of $\bigoplus_{a \in \mathcal{A}(\Sigma, p)} \mathcal{B}(M, (a, \hat{a}, c^\sharp))$ generated by elements of the form $FG - GF$. Then*

$$\mathcal{B}(M_{gl}, c) \cong \bigoplus_{a \in \mathcal{A}(\Sigma, p)} \mathcal{B}(M, (a, \hat{a}, c^\sharp))/L.$$

3.3 The solid cylinder

Now we consider the particular case of the category of the disk. Let p be a collection of $2m$ distinct points in ∂D^2.

The category $\mathbf{C}(D^2, p)$

For convenience, we will call this $\mathbf{C}_{2m}$. The objects are checkerboard-coloured Temperley-Lieb pictures in D^2 with boundary p. The morphisms are surfaces in the 'can' between two such pictures with matching checkerboard colouring around the boundary of the disk.

An object a could include a closed component d (a disjoint circle away from the boundary of the disk). However, d is boring addition, because any surface in $Hom(a, -)$ must have a compressing disk just inside the manifold from d. Cutting this neck gives a disjoint disk D or $\dot{D}$ bounding d. As observed in [Kai09], algebraically, d's presence just means tensoring on two copies of the ground ring to the skein module, i.e.

$$\mathcal{B}(D^2 \times I, a \amalg d) \cong \mathcal{B}(D^2 \times I, a) \otimes R[D] \oplus R[\dot{D}].$$

When it comes to calculating the skein of the solid torus, a d is even less interesting, because the circle itself gives a compressing disk—it doesn't even create a boring generator. Thus we will neglect such possibilities. Note, however, a curve which cuts off a disk intersecting the boundary is still interesting.

Once we have chosen a pair of Temperley-Lieb pictures a and b in the disk, we know precisely what morphisms go between them.

Lemma 3.3.1. Up to local relations and isotopy, $Hom(a, b)$ has only one unmarked element F. If $a = b$, F consists of a bunch of 'curtains,' $a \times I$. If $a \neq b$, F is a composition of elementary saddles. The other morphisms in $Hom(a, b)$ are dotted versions of F, i.e. the underlying surface is the same, but each connected component may or may not have a dot.

Proof. Given a fixed set of boundary conditions a and b, $Hom(a, b)$ is the skein of the solid cylinder, $\mathcal{B}(D^2 \times I, (a, \hat{b}, p \times I))$. This cylinder is homeomorphic to the ball B^3 with some number of boundary circles, hence the incompressible surfaces are bunches of disks. Since the surface $a \times I$ is incompressible and there is only one such surface up to isotopy, we have the result for $a = b$.

If $a \neq b$, morsify the surface and consider the 'movie' from slicing the cobordism and progressing along the interval, perpendicular to the slices. Since we are assuming we have removed all closed components, any circle that appears must merge with an arc attaching to the boundary. Thus any minimum (or maximum by symmetry) must have a cancelling saddle, and the morsified surface is isotopic to one that only consists of saddles and curtains. □

For the remainder of this discussion, we continue to consider morsified surfaces with a fixed ordering of saddles. Note that there might be different orderings of saddles that give the same surface—there is only one surface up to *isotopy*.

Clearly, $Hom(b, a)$ is generated by the same marked surfaces as $Hom(a, b)$ in reverse. We denote the reverse surface of F by F^*.

Example 3.3.2 (The 12-morphism category $\mathbf{C}_4$)**.** The primordial example is the category of a disk with four points. There are two objects in the category, a '0-smoothing' and '1-smoothing.' There are twelve morphisms: identities from each object to itself (arcs cross I, blank curtains), and 3 more curtain morphisms for each object, one for each configuration of dots. The other morphisms are saddles, with and without a dot, and the same surfaces in reverse.

3.4 Hochschild Homology

To get the solid torus from the solid cylinder $D^2 \times I$, we glue together the ends os the interval to make a circle. The algebraic glue we need is Hochschild homology. See [MW11] for an explicit description of Hochschild homology's relationship to the circle.

The usual definition of Hochschild homology is for a bimodule M over an algebra A (over a commutative ring R). Tensor products are over R unless otherwise noted.

Definition. The nth chain group of the Hochschild chain complex of M is $C_n = M \otimes A^{\otimes n}$. From C_n to C_{n-1} we get a collection of maps d_i by multiplying successive pairs of elements, i.e.

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes \cdots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}.$$

The differential $d : C_n \to C_{n-1}$ is given by the alternating sum $d = \sum_{i=0}^{n} (-1)^i d_i$. The homology of this chain complex is the Hochschild homology $HH_*(A, M)$ of M. We write $HH_*(A)$ if $M = A$.

Example 3.4.1. If $M = A = R$, the complex is

$$\ldots \xrightarrow{0} R \xrightarrow{1} R \xrightarrow{0} R \longrightarrow 0.$$

So $HH_0(R) = R$ and $HH_{\geq 1} = 0$.

Example 3.4.2.
$$HH_0(A) = \frac{A}{ab - ba} = A/\left[A, A\right].$$

An algebra and a module walk into a bar complex.

Definition. The bar complex of an algebra A is the complex

$$\ldots \longrightarrow A^{\otimes n+1} \xrightarrow{d'} A^{\otimes n} \longrightarrow \cdots \longrightarrow A^{\otimes 2} \longrightarrow 0$$

with $d' = \sum_{i=0}^{n-1} (-1)^i d_i$, where d_i are as in the definition for the Hochschild complex.

Recall the enveloping algebra $A^e = A^{op} \otimes A$ allows us to define an A-bimodule as a right module over A^e by the action $m(a \otimes b) = amb$.

Proposition 3.4.3. The bar complex is a resolution of A as an A^e-module. (See [Lod92], 1.1.12.)

Proposition 3.4.4. If A is a projective module over R, then there is an isomorphism $HH_*(A, M) \cong Tor_n^{A^e}(A, M)$, so the bar complex gives a prescriptive method for computing Tor, and thus for computing Hochschild homology.

Categories

The Hochschild complex of an R-linear category $\mathcal{C}$ is

$$C_n = \bigoplus_{a_i \in Obj(\mathcal{C})} Hom(a_1, a_2) \otimes \cdots \otimes Hom(a_{n+1}, a_1),$$

with the obvious boundary map taking an alternate sum and composing pairs of morphisms.

For a bimodule over the category $\mathbf{C}$ (by which we mean a functor $M : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$), the Hochschild chain groups are

$$C_n = \bigoplus M(b_1, b_2) \otimes Hom(b_2, b_3) \cdots \otimes Hom(b_n, b_{n+1}).$$

If we have a functor $\mathcal{R} : \mathbf{C} \to \mathbf{C}$ we can also define twisted Hochschild homology with $d_i = m \otimes f_1 \otimes \ldots f_i f_{i+1} \ldots f_n$ for $1 < i < n$ as before, but with $d_1 = m\mathcal{R}(f_1) \otimes f_2 \otimes \cdots \otimes f_n$ and $d_n = f_n m \otimes f_1 \otimes \cdots \otimes f_{n-1}$. See [Sit05] pages 2-3 for the algebra version (referred to as 'natural twisted Hochschild homology') and variations. We use this with a literal twist in Section 3.6.

Other references used for Sections 3.4.1 and 3.4 are [Wei94] chapter 9, [Lod92] chapter 1, and [Wik15b].

3.4.1 Morita Equivalence

Definition. Two rings R and S are Morita equivalent if there is an R–S bimodule P and an S–R bimodule Q such that ${}_RP_S \otimes_S Q_R \cong R$ as an R–R bimodule and ${}_SQ_R \otimes_R P_S \cong S$ as an S–S bimodule.

Example 3.4.5 (The Standard Example)**.** R is Morita equivalent to $\mathcal{M}_n(R)$. P is the set of n-dimensional row vectors, and Q is the set of n-dimensional column vectors.

We can think of Morita equivalence as saying two rings are equivalent if they have the same modules. This is natural from the perspective of ring theory, because modules over a ring can be thought of as representations of the ring, and many properties of rings are characterised by their modules.

Lemma 3.4.6. In fact, R and S are Morita equivalent if and only if the categories of bimodules R–**mod**–R and S–**mod**–S are equivalent as categories.

In particular, this implies that thinking of a ring as a category is a Morita equivalence.

3.5 Hochschild homology of the disk category

When we glue the two ends of the cylinder together, we get a solid torus with boundary curves $c = p \times S^1$. Applying Theorem 3.2.1 to the skein modules, we have

$$\mathcal{B}(D^2 \times S^1, c) \cong \bigoplus_{a \in \mathcal{A}(D^2,p)} \mathcal{B}(D^2 \times I, (a, \hat{a}, p \times I)) / \langle FG - GF \rangle \,.$$

Rephasing this in terms of the category $\mathbf{C}_{2m}$, this says the skein

$$\mathcal{B}(D^2 \times S^1, c) \cong \bigoplus_{a,b \in Obj(\mathbf{C}_{2m})} Hom(a,b) \otimes Hom(b,a) / \langle FG - GF \rangle ,$$

which is precisely $HH_0(\mathbf{C}_{2m})$. Intuitively, we are saying the skein of the solid torus is the same as the skein of the solid cylinder, with additional relations that come from choosing where to cut the torus.

Let $F, G \in Hom(a,b)$, where F is the unmarked surface. By Lemma 3.3.1, every relation is of the form $FG^* = G^*F$. There is actually a simple presentation for the skein module of the solid torus with $2m$ longitudinal curves, namely, it suffices to consider F to be a single saddle along with $m-2$ curtains (as opposed to a composition of many saddles) [Rus09]. We will reprove Russell's result here in somewhat different language, more suited to our local framework, though it is very similar in spirit. (Also, see 2.3.)

Let HR_{2m} be the module spanned by $\dot{\mathcal{A}}(D^2, p)/af$, where af is the submodule generated by the Asaeda-Frohman relations (Definition 2.3.5).

Theorem 3.5.1. $\mathcal{B}(D^2 \times S^1, c) \cong HH_0(\boldsymbol{C}_{2m}) \cong HR_{2m}$.

Proof. The incompressible surfaces in $D^2 \times S^1$ are annuli, so the generating morphisms in $\mathbf{C}_{2m}$ are curtains, which obviously correspond to the generators of HR_{2m}. So to prove the theorem, we need to show that the relations from taking Hochschild homology are the same as the relations in HR_{2m}.

Like Russell, we induct, but we instead induct on the number of saddles in F. By Lemma 3.3.1, this is equivalent to inducting on the number of compressing disks in FG^*. The base case, $k = 1$, gives the most fundamental relation, which Russell calls a Type I relation. Her 'Type II' relation is in some sense less basic, because it can be obtained by sticking a dotted curtain onto the Type I surfaces.

Remark 3.5.2. A key observation for the proof is that dots can slide through saddles, $Id_a^{\bullet} F = F Id_b^{\bullet}$. Thus adding dots to an old relation gives us another relation.

Assume that if if $F \in Hom(a,b)$ has k saddles or fewer, then a relation $FG = GF$ can be obtained by a linear combination of Type I and II relations.

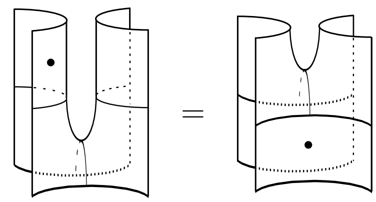


Figure 3.1: Moving a dot through a saddle

We want to show $sFF^*s^* = F^*s^*sF$ is also a linear combination of Type I and II relations, where s is an elementary saddle with curtains. (Note there can be no more than m saddles, by the two-dots relation.)

Consider sFF^*s^*. We can think of the FF^* as giving a bunch of dots on a single saddle with curtains (s or s^*), and use square brackets to indicate that we are thinking of FF^* as the surface with all of its necks cut - a defeated hydra. We have $s([FF^*]s^*) = ([FF^*]s^*)s$, because a relation with a single saddle is obtainable with Type I and II relations. Now reverse the neck cutting - bringing the FF^* hydra back to life - and think of this surface instead as $F(F^*[s^*s])$, which is $(F^*[s^*s])F$, using the induction hypothesis.

□

3.6 Now with a twist

Now we will rework the first part of the chapter for boundary curves that twist around the torus. As before, we start by understanding the cylinder.

3.6.1 A twisted cylinder

Consider the homeomorphism τ^j of the cylinder which acts by fixing one end of the cylinder and rotating the other end by $j\pi/m$ counter-clockwise. For convenience in our definitions, we also refer to the functor $\mathcal{R}^j : \mathbf{C}_{2m} \to \mathbf{C}_{2m}$ which simply rotates all objects and morphisms by $j\pi/m$.

Define $\mathcal{T}^j$ to be the bifunctor $\mathcal{T}^j : \mathbf{C}_{2m}^{op} \otimes \mathbf{C}_{2m} \to \mathbf{Ab}$ which sends a pair of objects (a, b) to the skein module

$$\mathcal{T}^j(a, b) = \mathcal{B}(\tau^j(D^2 \times I, (a, b, p \times I))).$$

On a pair of morphisms $F : a \to a'$ and $G : b \to b'$, we define

$$\mathcal{T}^j(F,G) = F^* \cup_{(D^2,a)} \mathcal{T}^j(a,b) \cup_{(D^2,\mathcal{R}(b))} \mathcal{R}^j(G).$$

There is also an inverse functor $\mathcal{T}^{-j}$ corresponding to rotating the end clockwise.

We abuse the notation and also refer to $\mathcal{T}^j\mathbf{C}_{2m} = \bigoplus_{a,b\in Obj(\mathbf{C}_{2m})} \mathcal{B}(\tau^j(D^2 \times I, (a, b, p \times I)))$. We can view $\mathcal{T}^j\mathbf{C}_{2m}$ as a bimodule over the surface algebra of the disk instead of thinking of the bifunctor $\mathcal{T}^j$.

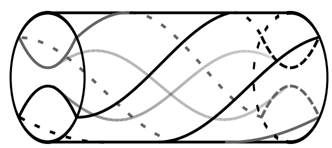


Figure 3.2: An element of $\mathcal{T}^{-2}\mathbf{C}_4$.

Clearly, $\mathcal{T}^j$ induces an isomorphism of R-modules

$$\mathcal{T}^j\mathbf{C}_{2m} \cong \mathbf{C}_{2m},$$

although they are not isomorphic as bimodules. Lemma 3.3.1 thus implies that for a given $a, b \in Obj(\mathbf{C}_{2m})$, there is still a unique blank morphism between them, either 'twisted curtains' or a collection of saddles with twisting boundary.

3.6.2 Come on torus, let's do the twist

Now let us consider the effect of gluing together the ends of $\mathcal{T}^j\mathbf{C}_{2m}$, giving a 'twisted torus.' As before, we accomplish this algebraically using Hochschild homology. We will not allow j to be odd as this would force black regions to be glued to white regions. If $k = j \mod 2m$, then τ^k gives a Dehn twist on the solid torus, and an isomorphism of skein modules:

$$HH_0(\mathcal{T}^k, \mathbf{C}_{2m}) \cong HH_0(\mathcal{T}^j, \mathbf{C}_{2m}).$$

Proposition 3.6.1. In fact, we will show that the skein only depends on the order of the rotation.

Blank generators

The incompressible surfaces in the twisted torus are still annuli, by the same standard facts as before. Each generator of $\mathcal{T}^{2k}\mathbf{C}_{2m}$ has $(m, k) = gcd(m, k)$ annuli. This is a result of elementary facts about cyclic groups, namely, the order of k in $\mathbb{Z}_m$ is $m/(m, k)$. See [Hun74], section 1.3.

Using the uniqueness of surfaces up to isotopy, we can use $a \in Obj(\mathbf{C}_{2m})$ to represent the blank surface in $\mathcal{T}^j(a, a) \otimes Hom_{\mathbf{C}_{2m}}(\mathcal{R}(a), a)$. Thus we can hope to reduce the skein of the twisted torus to a 2-dimensional presentation, as we did with $HH_0(\mathbf{C}_{2m})$. There is some ambiguity about the placement of dots, which we will address later.

If a picture is not symmetric, it is not a generator because it will need at least one saddle to get back to the starting picture (so that it can be closed up). The glued-up surface then has genus, i.e. a compressing disk. (We can see this using Euler characteristic. Assume that the twisted part has no saddles. If $a \neq \mathcal{R}^{2k}a$, then it takes at least one saddle to be able to close up the torus. But then we have 1 saddle and $m-2$ bands, i.e. $m-1$ disks. Gluing these along m arcs, we get $\chi = -1$, i.e. not annuli because $\chi \neq 0$. Actually, we can't have an odd number of saddles because compression increases χ by 2 (removing a circle and adding two disks) so starting from an odd χ would never give zero. But increasing the number of saddles just decreases the starting Euler characteristic.)

Example 3.6.2 ($\mathcal{T}^2\mathbf{C}_4$)**.** The curves connect in such a way that there are only two circles on the boundary, hence one annulus in the torus. However, there are in fact *two* non-boundary isotopic annuli bounding these curves. (The one that connects the 1st boundary point to the 2nd and the one that connects the 1st boundary point to the 4th, corresponding to the '0-smoothing' and '1-smoothing' objects.)

Cyclic sieving phenomenon

Since we can represent both the skein of the torus with product boundary $p \times I$ and the skein of the torus with twisted boundary curves with elements of $\dot{\mathcal{A}}(D^2, p)$, we might ask what the relationship is between the two sets of generators.

For simplicity, let us restrict our attention to blank generators for the moment. For the product boundary case, we know there are C_m blank generators, where

$$C_m = \frac{1}{2}\binom{2m}{m}$$

is the mth Catalan number.

When we apply a twist, we need to ask: how many fields in the disk are symmetric with respect to rotation by $2k\pi/m$? We can answer using known results about the cyclic sieving phenomenon.

Definition. Given a set X, an action of $C = \langle g \rangle$ of order N, and a polynomial $f(q)$, the triple $(X, C, f(q))$ is said to exhibit the *cyclic sieving phenomenon* if the number of elements fixed by g^d equals the evaluation $f(e^{2d\pi i/N})$ for all d.

In [RSW04] section 7, Reiner, Stanton, and White proved that the cyclic sieving phenomenon holds when X is the set of crossingless matchings on $2m$ points, g is rotation by π/m (order $2m$), and

$$f(q) = \frac{[2m]!}{[m+1]![m]!},$$

where

$$[n]! = [n][n-1]\ldots[2][1]$$

and

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Also see the survey by Sagan [Sag10] for a general introduction, and section 10 in particular.

In fact, Reiner, Stanton, and White proved something stronger, using noncrossing partitions. Consider m vertices of a convex m-gon ordered circularly $1, 2, \ldots, m$. A partition is noncrossing if the blocks of the partition correspond to subsets of the vertices whose convex hulls are pairwise disjoint. Noncrossing partitions are in one-to-one correspondence with Temperley-Lieb pictures on $2m$ points. We illustrate an example of the correspondence with $m = 4$ in Figures 3.3 and 3.4.

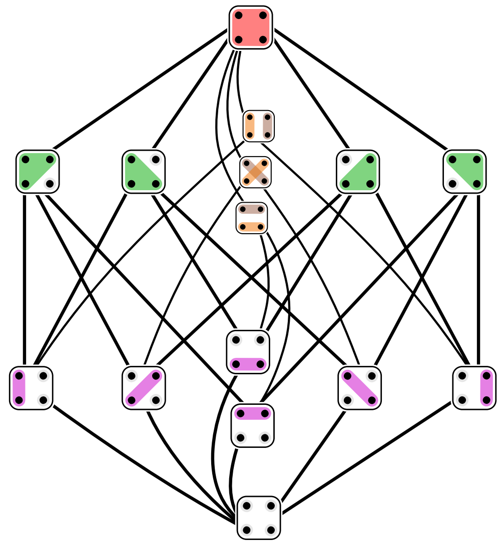


Figure 3.3: 'Set partitions 4; Hasse; circles' by Watchduck (a.k.a. Tilman Piesk). Licensed under CC BY 3.0 via Wikimedia Commons [Pie]. Points in a coloured region are in the same partition. The different colours indicate the rank of the partition.

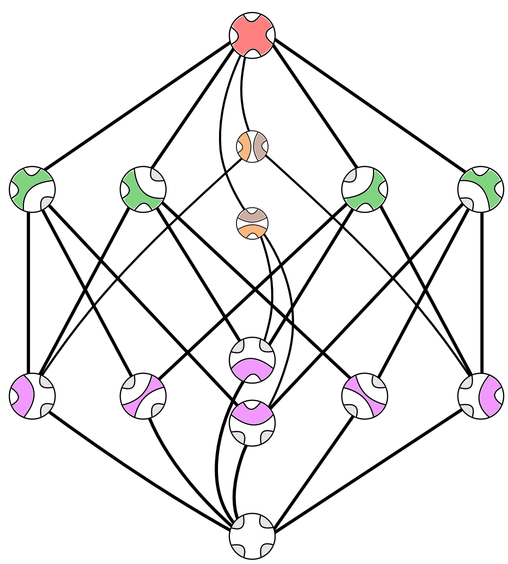


Figure 3.4: The corresponding Temperley-Lieb diagrams. Note the 'crossing partition' has no equivalent.

Theorem 3.6.3 (RSW)**.** *The number of pictures invariant under a rotation of order $d \geq 2$ is*

$$\sum_{j=0}^{j=m-1} \begin{cases} \dfrac{m-j}{m}\dbinom{\frac{m}{d}}{\frac{j}{d}}^2 & \text{if } d \mid j \\ \dfrac{j+1}{m}\dbinom{\frac{m}{d}}{\frac{j+1}{d}}^2 & \text{if } d \mid (j+1) \\ 0 & \text{otherwise.} \end{cases}$$

The sum is over the rank of the partition $j = m - b$, where b is the number of blocks in the partition.

As a corollary, this gives the number of blank generators of $\mathcal{T}^{2k}\mathbf{C}_{2m}$. Note that for $m \geq 1$, this number only depends on (m, k), the number of annuli that each generator has. We can invoke Proposition 2.2.6 to show the linear independence of the blank incompressible surfaces.

Example 3.6.4 ($k = 1$)**.** Taking $k = 1$ corresponds to rotation by $\frac{2\pi}{m}$, which has order $d = m$, meaning d only divides 0 and $(m-1)+1 = m$. The sum is then

$$1\binom{1}{0}^2 + 1\binom{1}{1}^2 = 2.$$

Example 3.6.5 ($m = 8, k = 2$)**.** The rotation here is by π, order 2. The sum is

$$1 + \frac{1}{4}\binom{4}{1}^2 + \frac{3}{4}\binom{4}{1}^2 + 2\frac{1}{2}\binom{4}{2}^2 + \frac{3}{4}\binom{4}{3}^2 + \frac{1}{4}\binom{4}{3}^2 + 1 = 70.$$

We give more examples in Table 3.1.

Dots and relations

We know one presentation for $HH_0(\mathcal{T}^{2k}, \mathbf{C}_{2m})$ is

$$\frac{\bigoplus_{a \in Obj(\mathbf{C}_{2m})} \mathcal{T}^{2k}(a, a)}{FT = T\mathcal{R}^{2k}(F)},$$

where $F \in Hom_{\mathbf{C}_{2m}}(a, b)$ and $T \in \mathcal{T}^{2k}(b, a)$. However, if $\mathcal{R}^{2k}a \neq a$ then

$$Fm = m\mathcal{R}^{2k}(F)$$

Table 3.1: Counts of blank generators of $\mathcal{T}^{2k}\mathbf{C}_{2m}$. This does not include dotted versions of generators or account for relations. C_m refers to the mth Catalan number and ${}_qC_m$ to the mth quantum Catalan number.

m	C_m	k	(m,k)	order	generators	${}_qC_m$
1	1	1	1	1	1	1
2	2	1	1	2	2	$1+q$
3	5	$1,2$	1	3	2	$1+q^2+q^3+q^4+q^6$
4	14	$1,3$	1	4	2	$(1+q^4)(q^2-q+1)(1+q+\cdots+q^6)$
4		2	2	2	6	
5	42	$1,2,3,4$	1	5	2	$(1+q^4)(q^2-q+1)(q^6+q^3+1)(q^4-q^3+q^2-q+1)(1+q+\cdots+q^6)$
6	132	$1,5$	1	6	2	
6		$2,4$	2	3	6	$(1+q^4)(q^2-q+1)(q^6+q^3+1)(q^4-q^3+q^2-q+1)(q^4-q^2+1)(1+q+\cdots+q^{10})$
6		3	3	2	20	
8	1430	4	4	2	70	–

only describes an unsymmetric element in terms of the symmetric generators. So the relation tells us no more than what we found in Section 3.6.2, namely, there is a smaller number of blank generators. We want to describe the skein in terms of this reduced set of generators, ideally using only planar pictures. Taking all possible configurations of dots gives us a spanning set of generators.

Thinking geometrically about the relation, it tells us that the surface does not have any necks in the cylinder composed in one order but it does have a neck in the other order. In other words, the relation merely says that this surface, which is incompressible in the cylinder, gives a compressible surface in the torus. As for the torus with product boundary curves, interesting new relations come from composing saddles in different orders because they give distinct necks in both orders of composition. (Cutting the necks in the two compositions amounts to compressing a surface in different ways in the torus.)

Note that the number of connected components limits the number of necks we need consider. For instance, if $(m, k) = 1$, there can be at most one neck (two saddles) or the element is zero in the skein. The tricky bit is that there is a lot of isotopy to go around; $(s_1 s_2)T = T\mathcal{R}^{2k}(s_1 s_2)$ is not an interesting relation, but $T(s_1 s_2)$ is isotopic rel boundary to $(Ts_1)s_2 = s_2(Ts_1)$, which might be an interesting relation. Relations in HR_{2m} between two symmetric pictures will still be relations in $\mathcal{T}HR_{2m}$, but are there any other relations?

Example 3.6.6 $(m = 6,\ k = 1)$**.** From Example 3.6.4, we know that there are two blank incompressible surfaces. Each is an annulus. The blank generators are linearly independent, but there may be a relation between the dotted versions. Using a (somewhat cumbersome) procedure of iterating Hochschild relations and isotopies rel boundary, we can in fact produce a relation. We start with a composition of saddles in $Hom(a, a)$ which is equivalent to a dotted annulus. We represent this with a row of pictures

$$a \to b \to \mathcal{R}^{2k}(a) = a.$$

Then we apply a Hochschild homology relation to show the above surface is equivalent to the surface represented by

$$b \to \mathcal{R}^{2k}(a) \to \mathcal{R}^{2k}(b).$$

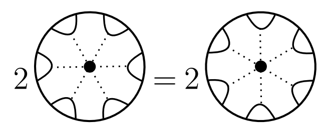
The isotopy changes the order of the two saddles, so we get

$$b \to d \to \mathcal{R}^{2k}(b).$$

Then we repeat until we get to a row with symmetric pictures on the end again.

We give two versions of this calculation. In Figure 3.5, we represent the cylinder with the objects on the ends drawn as disks, and in Figure 3.6 we use a bijection with Temperley-Lieb pictures on a line. The second version is less topologically accurate, but it is perhaps easier to see patterns.

This computation gives us the relation:



where the dotted lines indicate a 'quantum dot' that exists equally much on each arc.

In fact, not only are all the generators symmetric under the rotation, we can get all relations using only those which are symmetric under the rotation.

Theorem 3.6.7. *Let $V = \mathcal{A}(D^2, 2m)$. Let W be the subspace generated by type I and II relations, so $HR_{2m} = V/W$. Let $G = \mathbb{Z}_k$ and let V^G be the invariant part of V (the symmetric pictures under $\mathcal{R}^{2k}$). Then the kernel of the map $(V^G \hookrightarrow V \to V/W)$ gives the relations between symmetric pictures.*

Proof. Suppose $x, y \in V^G$ with $x - y = w \in W$. Let ϕ be the symmetriser $\frac{1}{k}\sum_0^k \phi^i(x)$. We have

$$\phi(x) - \phi(y) = \phi(w) = x - y.$$

So the kernel is $\phi(W)$, the symmetric relations. □

We can thus represent the skein with pictures in a 'cone' (a fraction of a disk) with fractional dots. (Perhaps, for the sake of the dots, it would be better

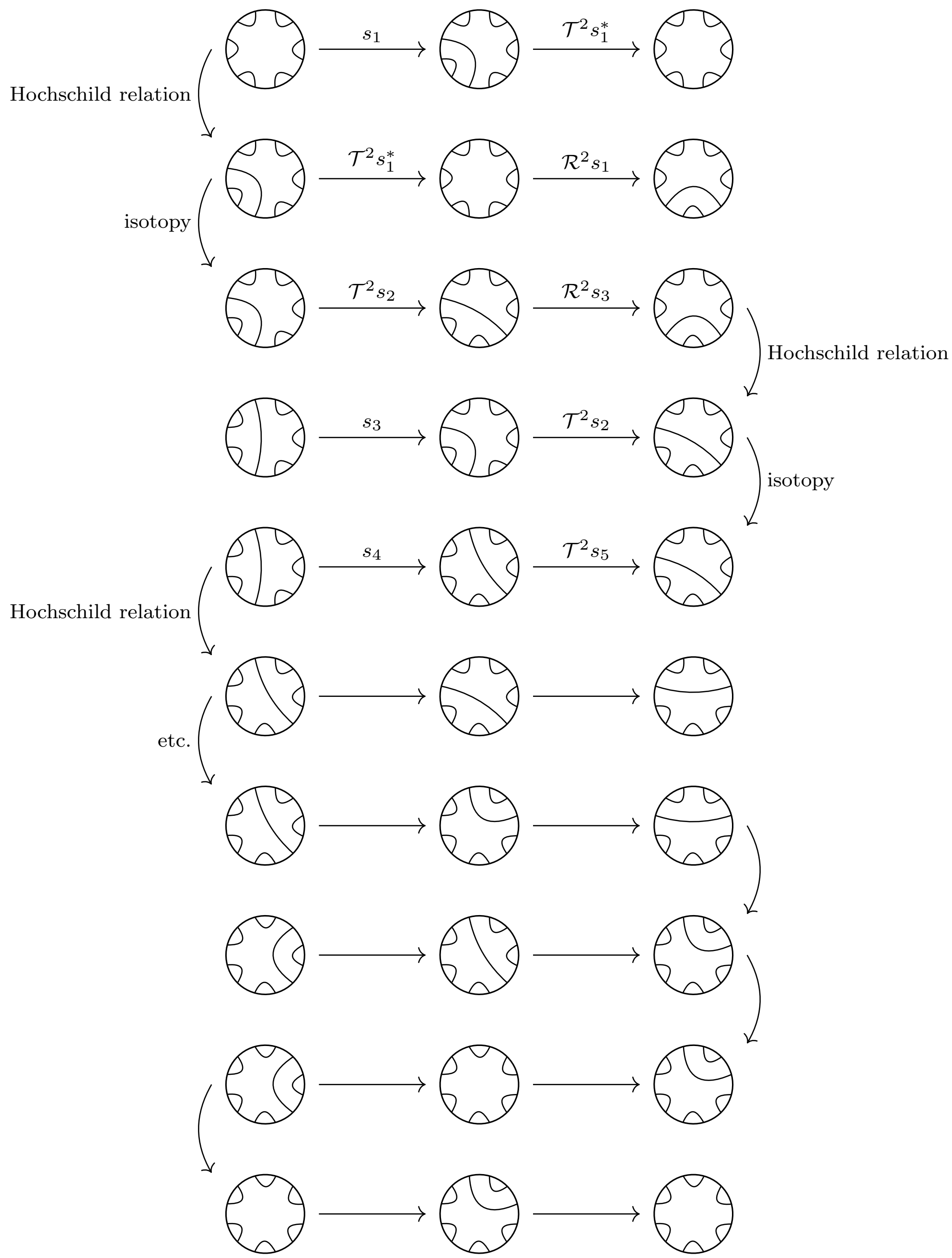


Figure 3.5: Disk form of calculation for Example 3.6.6.

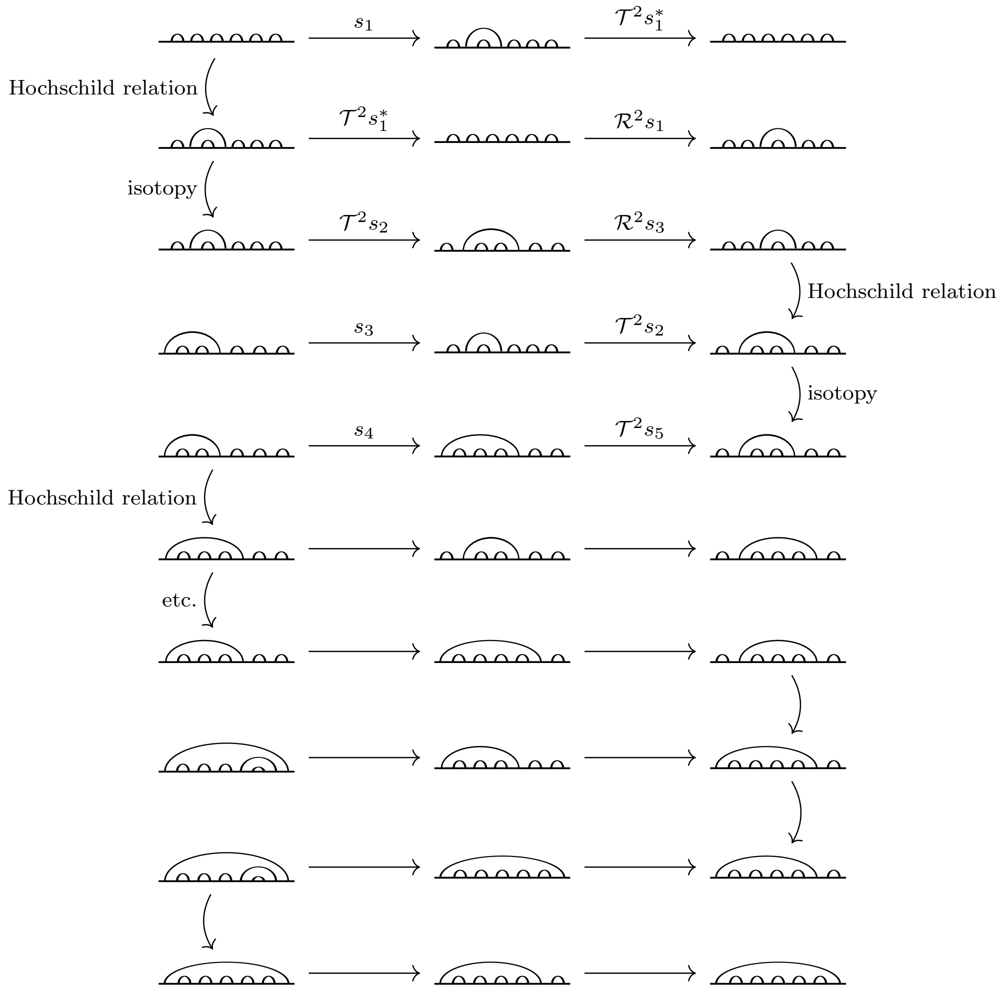


Figure 3.6: Flat version of Example 3.6.6.

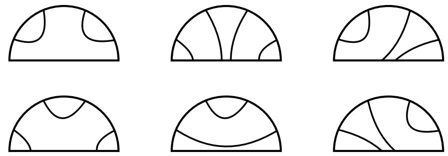


Figure 3.7: Blank cone pictures for $m = 4$ and $k = 2$.

to use a round version, but we have chosen the form given to distinguish it from the untwisted torus.) Then the Type I and II relations in the cone give all the relations.

Corollary 3.6.8. The skein depends only on the order of the rotation. In particular, any skein where m and k are relatively prime consists of the two non-boundary isotopic single annuli, with a relation saying the dotted versions of these annuli are equal.

3.7 A graphical calculus for relations

This is kind of fun.

Short form

The procedure in Example 3.6.6 is pretty repetitive; each isotopy has the same end pictures and Hochschild homology moves two pictures in the previous row to the next row. Eliminating these pictures and arranging each line in an L-shape with 4-periodicity, we get a condensed form which makes for faster calculations.

We can do the same thing with partitions of the vertices of a hexagon, pictured in Figure 3.9.

Best of all, we can do this with elements of $\mathbb{Z}_6$ corresponding to the vertices.

Proof of Corollary 3.6.8

Suppose $(m, k) = 1$. Then $HH_0(\mathcal{T}^{2k}\mathbf{C}_{2m})$ has three generators.

Proof. Our task is simply to prove there is a relation between the two dotted generators. We procede as for Example 3.6.6.

				$()$	
			$(m\,k\,2k)$	$(k\,2k)$	$()$
			$(m\,k\,2k\,3k)$	$(k\,2k\,3k)$	$(2k\,3k)$
		$\iddots$			
$(m\,k\,2k\ldots(m-1)k)$	$(k\,2k\ldots(m-1)k)$				
	$(m\,k\ldots(m-1)k)$				

This must happen because k has order m in $\mathbb{Z}_m$ $\square$

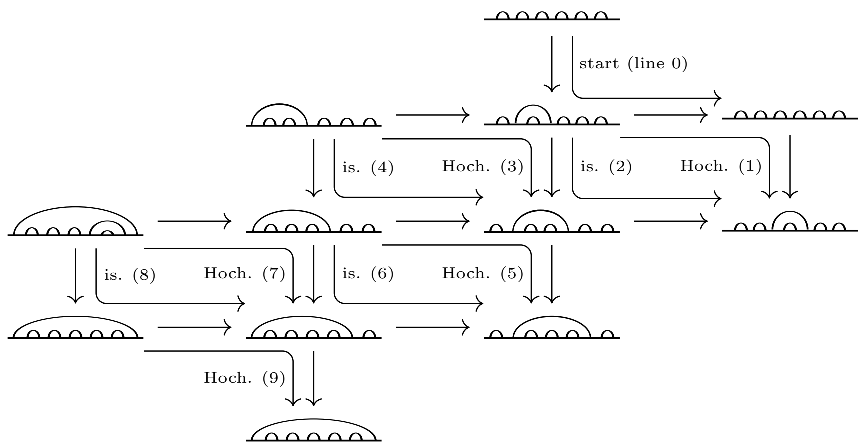


Figure 3.8: Short form of Example 3.6.6.

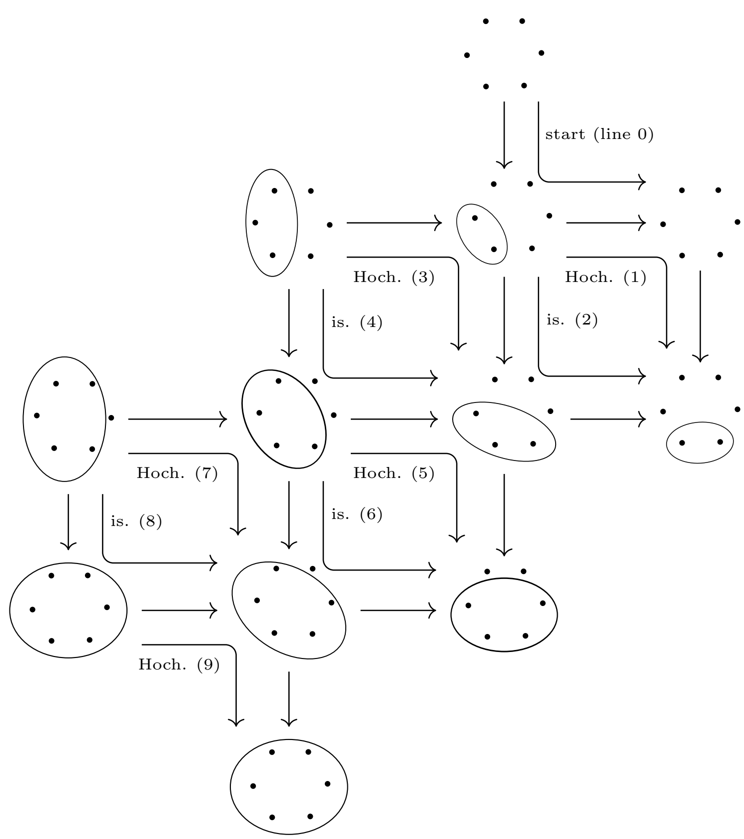


Figure 3.9: Lattice form of Figure 3.8.

		( )	
	(123)	(23)	( )
(61234)	(1234)	(234)	(34)
(612345)	(12345)	(2345)	
	(124356)		

Figure 3.10: Tabular form of 3.8.

Chapter 4

More surface categories

Previously, we discussed the Bar-Natan category of a disk. In this chapter, we give other computations and observations about surface categories. We rely on work from Chapter 3, since disks can be glued to themselves to make annuli and other planar surfaces. First we briefly describe planar algebras, which can be used to describe a 2-categorical structure of planar surface categories.

4.1 Planar algebras

Say you have disk with some marked points on its boundary, and perhaps missing some disks which also have points on their boundaries. A planar algebra describes the operad structure of putting such things inside other such things.

A little more carefully, say k is a natural number. A planar algebra $\mathscr{P}$ associates k to a vector space P_k which we think of as a disk with k boundary points. A planar tangle with some disks missing (in the words of [MPS10], a 'spaghetti and meatballs' diagram) is associated to a linear map from the tensor product $\otimes_i P_{k_i}$ of the inner disk vector spaces to the outer circle vector space P_{k_0}. (By a planar tangle, we mean a tangle embedded in the plane; in particular, there are no crossings.)

An annulus P_k to P_k with radial lines gives the identity. The composition must also satisfy relations so that putting things inside of other things in different orders gives a well-defined result.

Planar tangle pictures are not considered to be invariant under rotations, thus each boundary circle is marked with a dot so that rotated pictures can be distinguished. If there are no inner disks missing, the picture may be depicted as a rectangle, with the dot presumed to be on the left (this is the Temperley-Lieb algebra).

Further, for a standard subfactor planar algebra, the planar regions are given a checkerboard colouring. (This is a way of representing tensor products of R–S and S–R bimodules.) Obviously, this means that only even numbers of boundary points are allowed.

These planar pictures are objects of black-and-white cylinder categories; in this sense, the black-and-white skein module is a categorification of a planar algebra.

For a much better discussion of planar algebras than mine, see [MPS10]. For annular planar algebras in particular, see [Jon01].

4.2 A canopolis

In fact, planar surface categories form a canopolis, as defined in [BN05]. A canopolis is a collection of categories that behave like a 'city of cans,' which can be built up vertically or out horizontally.

Definition. Let $\mathscr{P}$ be a planar algebra which associates the vector space P_k to a natural number k. A canopolis over $\mathscr{P}$ is a collection of categories $\mathbf{C}_k$ with $\mathrm{Obj}(\mathbf{C}_k) = P_k$. In other words, the objects of categories in the canopolis *are* the planar algebra $\mathscr{P}$. The morphims between all objects also must form a planar algebra, and the two directions of composition must commute (vertical composition in a category $\mathbf{C}_k$ and horizontal composition in the morphism planar algebra).

Remark 4.2.1. The canopolis of planar surface categories is graded by 'degree,' a modified form of Euler characteristic, which counts a dot for -2 and a boundary component on the side of a can for $-1/2$.

The objects of the canopolis are generated generated by the two single arc objects in $\mathbf{C}_2$. The morphisms are generated by:

- the identities on the arcs
- a single dotted sheet D with $D^2 = 0$
- the saddles in $\mathbf{C}_4$
- cups and caps in $\mathbf{C}_0$.

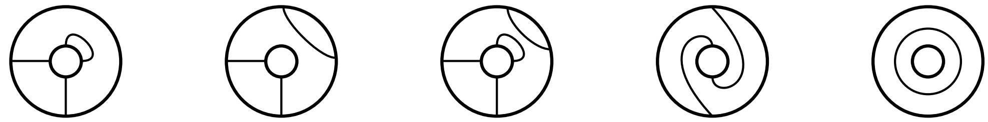
This is obviously true from standard facts of Morse theory; we have also discussed it in great detail for $\mathbf{C}_{2m}$. If A is an annulus, we know $A \times I$ is homeomorphic to $D^2 \times S^1$. Thus we know the morphisms are dictated by the boundary curves and are either disks (in the form of curtains or saddles) or annuli (possibly with dots) with Type I and II relations. Any other planar surface category can be obtained by tensoring annuli categories together over disk categories.

The Hochschild homology of the canopolis is also an algebra over the planar algebra operad. This means we can use the results of Chapter 3 to describe $HH_0(\mathbf{C}(\Sigma, p))$, since the relations occur in solid tori, between saddles composed in different orders. In particular, we can use arcs to describe the skein. (The proof of Proposition 2.3.5 actually works for surfaces with boundary as well, so we could use that argument if we preferred.) We are looking at the S^1-invariant part of the skein module here, which is not the same as $\mathcal{B}(\Sigma \times S^1, c)$ in general.

4.3 The annulus category

The objects of the annulus category are elements of annuluar planar algebras which (in the notation of [Jon01]) are generated by $\epsilon_i, \varepsilon_i, F_i, \sigma$, and ρ, pictured in Figure 4.1. We refer to the category of the annulus with $2m$ points on the inner circle and $2n$ points on the outer circle as $\mathbf{C}_{(2m,2n)}$. Jones uses $\pm$ rather than 0 to specify the colouring, but since we are working with a category and not an algebra, we are less concerned with the difference. We assume that colourings agree when we write an element as a product of generators.

Note that in the category, there is more than one unmarked incompressible surface in $Hom(a, b)$ for a given pair of objects a and b in $\mathcal{A}(A, (2m, 2n))$. However, if neither a nor b has a copy of σ, then there is a unique blank incompressible



$\epsilon_1 \in \mathrm{Obj}(\mathbf{C}_{(4,2)})$ $\varepsilon_1 \in \mathrm{Obj}(\mathbf{C}_{(2,4)})$ $F_1 \in \mathrm{Obj}(\mathbf{C}_{(4,4)})$ $\rho \in \mathrm{Obj}(\mathbf{C}_{(2,2)})$ $\sigma \in \mathrm{Obj}(\mathbf{C}_{(0,0)})$

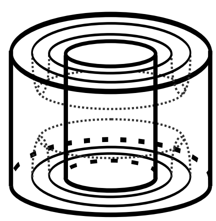
Figure 4.1: Examples of classes of generators of an annular planar algebra.

surface in $Hom(a, b)$. One way to see this is using 2-categorical structure; we can view the annulus as HH_0 of the skein associated to I^3. There is also a 3-categorical structure. (This requires embedding the cubes in $\mathbb{R}^3$.) This is perhaps worth considering further for the sake of higher categories, but its application in understanding annular planar algebras seems limited - using a canopolis seems to be a more useful way of thinking about 2-category structure.

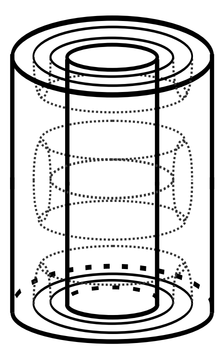
Remark 4.3.1. The category $\mathbf{C}_{(2m,2n)}$ is a $\mathbf{C}_{(2m,2m)} - \mathbf{C}_{(2n,2n)}$ bimodule. More generally, $\mathbf{C}(\Sigma, 2m)$ is a module over $\mathbf{C}_{(2m,2m)}$ for any surface Σ with $2m$ marked points in a boundary circle. In particular, $\mathbf{C}_{(2m,0)}$ is a module over $\mathbf{C}_{(0,0)}$.

Example 4.3.2 ($\mathbf{C}_{(0,0)}$)**.** The objects are generated by σ. Since $A \times I$ is homeomorphic to $D^2 \times S^1$, the morphisms are annuli with all combinations of dots, modulo Type I and II relations, as discussed in Chapter 3.

Remark 4.3.3. The morphism P below is an idempotent.

$P = \frac{1}{2}$ 

Composing P with itself gives

$P^2 = \frac{1}{4}$  $= P.$

Taking Hochschild homology, we get that a morphism a composition of trenches (like P) in $Hom(\sigma^2, \sigma^2)$ is equivalent to a compressible torus in $Hom(\emptyset, \emptyset)$. Let $\bar{\sigma} = (\sigma \times S^1)$, and let $\bar{\tau} = \bar{\sigma}^{\bullet}$. Using Type I and II relations in $A \times I$, we then get that a surface with two parallel copies of τ is zero in $\mathcal{B}(A \times S^1)$. Similarly, we can move dots between copies of $\bar{\tau}$ and $\bar{\sigma}$ with a negative sign.

Remark 4.3.4. Using the horizontal composition of the canopolis, we get that $HH_0(\mathbf{C}_{(0,0)})$ is a superalgebra.

Definition. Let R be a commutative ring. A *superalgebra* is an R-module with a direct sum decomposition

$$A = A_0 \oplus A_1$$

and a bilinear multiplication so that the image of $A_i \times A_j$ is a subset of A_{i+j}, where $i + j$ is interpreted mod 2.

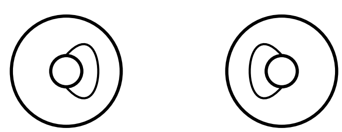
The odd part of $HH_0(\mathbf{C}_{(0,0)})$ is:

$$A_1 = \oplus_k R \left\langle \bar{\sigma}^{2k+1} \bar{\tau} \right\rangle \oplus \oplus_k R \left\langle \bar{\sigma}^{2k+1} \right\rangle$$

and the even part is:

$$A_0 = \oplus_k R \left\langle \bar{\sigma}^{2k} \bar{\tau} \right\rangle .$$

Example 4.3.5 ($\mathbf{C}_{(2,0)}$). The objects are ϵ_1 and ϵ_2 (in Figure 4.2) with some number of copies of σ.



ϵ_1 ϵ_2

Figure 4.2: Objects ϵ_1 and ϵ_2 of $\mathbf{C}_{(2,0)}$

The morphisms are curtains in $Hom(\epsilon_i, \epsilon_i)$, saddles in $Hom(\epsilon_i, \epsilon_j \sigma)$, and annuli for the copies of σ.

For HH_0, we still have the dot migration and consecutive dot relations from $\mathbf{C}_{(0,0)}$, so there is at most one $\bar{\tau}$, which we can assume is the innermost torus.

There is a saddle relation between $\bar{\epsilon}_i^{\bullet}\bar{\tau}$ and $\bar{\epsilon}_j^{\bullet\bullet}$, so there can be only one dot total on a generator. If we like, we can use the undotted saddle relation to move dots off tori, so we are left with the blank surfaces $\bar{\epsilon}_i\,\bar{\sigma}^k$, and the dotted surfaces are $\bar{\epsilon}_i^{\bullet}\,\bar{\sigma}^k$ as generators.

Example 4.3.6 ($\mathbf{C}_{(4,0)}$). The objects are products of the generators in Figure 4.1. Since it's easy to rotate pictures on a computer, we've put them all in Figure 4.3. Of course, we also ought to include σ^k.

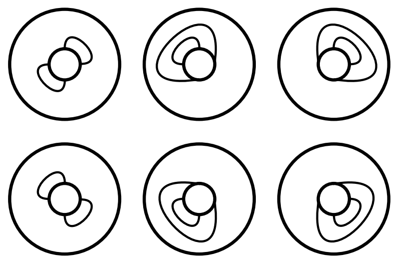
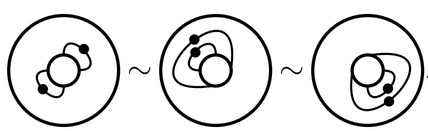


Figure 4.3: Objects of $\mathbf{C}_{(4,0)}$

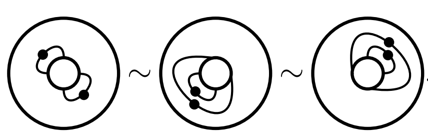
The morphisms are the usual mix of curtains, saddles, and annuli.

We again inherit relations for HH_0, so there can be at most one copy of $\bar{\tau}$. Furthermore, if an element includes $\bar{\tau}$ and has any other dotted surface, it is zero. However, surfaces with two dots (on different components) are not necessarily zero.

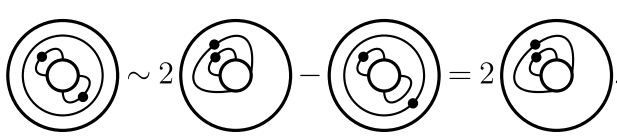
One equivalence class of surfaces with two dots corresponds to the following arc pictures:



Another corresponds to:



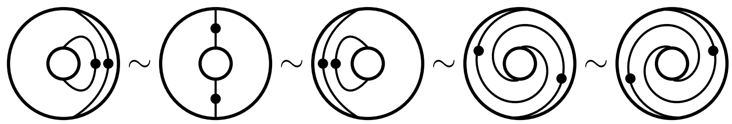
These are in fact the only two, as copies of sigma can be removed when there are two dots, as in the following example:



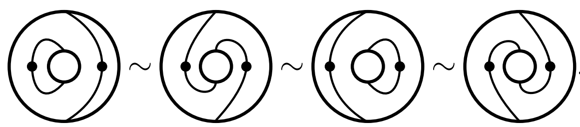
This procedure works inductively.

There are similar Type I relations, but the trick of removing circles only works with two dots. We can however move dots off of circles in that case.

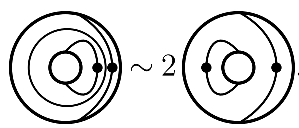
For $\mathbf{C}_{(2,2)}$, we see ρ make an appearance. The two classes of surfaces with two dots are (including higher powers of $\rho^{\pm}$ not pictured):



and



We can use the circle-removing trick to say:



It seems probable that other annulus and planar surface categories follow basically this pattern: dots can be moved around on tori; if a surface has two dots, one of which is one a torus, the surface is zero; when two dots are present, circles can be removed. There is potentially a better way to enumerate the generators of these skeins; perhaps it is worth noting that counting curves on surfaces has been surprisingly little studied [DKM15].

Bibliography

[AF07] Marta Asaeda and Charles Frohman. A note on the Bar-Natan skein module. *Internat. J. Math.*, 18(10):1225–1243, 2007.

[APS04] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. Categorification of the Kauffman bracket skein module of I–bundles over surfaces. *Algebraic & Geometric Topology*, 4(2):1177–1210, 2004.

[APS06] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. Categorification of the skein module of tangles. 2006.

[BD12] Jeffrey Boerner and Paul Drube. Generalized skein modules of surfaces. *J. Knot Theory Ramifications*, 21(1):1250006, 26, 2012.

[Bla10] Christian Blanchet. An oriented model for Khovanov homology. *Journal of Knot Theory and Its Ramifications*, 19(02):291–312, 2010.

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370 (electronic), 2002.

[BN05] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.

[BN07] Dror Bar-Natan. Fast Khovanov homology computations. *Journal of Knot Theory and Its Ramifications*, 16(03):243–255, 2007.

[BNM06] Dror Bar-Natan and Scott Morrison. The Karoubi envelope and Lee's degeneration of Khovanov homology. *Algebr. Geom. Topol.*, 6:1459–1469, 2006.

[Bri07] Matthew G. Brin. Seifert fibered spaces: Notes for a course given in the spring of 1993. *arXiv preprint arXiv:0711.1346*, 2007.

[Cap09] Carmen Caprau. The universal sl (2) cohomology via webs and foams. *Topology and its Applications*, 156(9):1684–1702, 2009.

[CMW09] David Clark, Scott Morrison, and Kevin Walker. Fixing the functoriality of Khovanov homology. *Geom. Topol.*, 13(3):1499–1582, 2009.

[DKM15] Norman Do, Musashi A. Koyama, and Daniel V. Mathews. Counting curves on surfaces. *arXiv preprint arXiv:1512.08853*, 2015.

[Eps66] D. B.A. Epstein. Curves on 2-manifolds and isotopies. *Acta Mathematica*, 115(1):83–107, 1966.

[Hat00] Allen Hatcher. Basic topology of 3-manifolds. *Unpublished notes available online at: http://www.math.cornell.edu/hatcher*, 2000.

[Hem76] John Hempel. *3-manifolds*, volume 349. American Mathematical Soc., 1976.

[Hun74] Thomas W. Hungerford. *Algebra*. Graduate Texts in Mathematics. Springer, 1974.

[Jon01] Vaughan F.R. Jones. The annular structure of subfactors, from: essays on geometry and related topics, vol. 1, 2, monogr. *Enseign. Math*, 38:401–463, 2001.

[Kai09] Uwe Kaiser. Frobenius algebras and skein modules of surfaces in 3-manifolds. In *Algebraic topology—old and new*, volume 85 of *Banach Center Publ.*, pages 59–81. Polish Acad. Sci. Inst. Math., Warsaw, 2009.

[Kai13] Uwe Kaiser. On extended Bar-Natan skein modules. Talk at AMS Western Fall Sectional Meeting, November 2013.

[Kai14] Uwe Kaiser. On constructions of generalized skein modules. *Banach Center Publications*, 2014.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[Kho06] Mikhail Khovanov. Link homology and Frobenius extensions. *Fund. Math.*, 190:179–190, 2006.

[Lee05] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Advances in Mathematics*, 197(2):554–586, 2005.

[Lod92] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[ML78] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 1978.

[Mor10] Scott Morrison. Khovanov homology II, functoriality, deformations and the s-invariant. Lecture at MSRI, January 2010. [Online at https://tqft.net/math/MSRI-Khovanov2.pdf; accessed 25-March-2016].

[MPS10] Scott Morrison, Emily Peters, and Noah Snyder. Skein theory for the D2n planar algebras. *Journal of Pure and Applied Algebra*, 214(2):117–139, 2010.

[MW11] Scott Morrison and Kevin Walker. Higher categories, colimits, and the blob complex. *Proc. Natl. Acad. Sci. USA*, 108(20):8139–8145, 2011.

[Orl72] Peter Orlik. *Seifert manifolds*. Springer-Verlag Berlin, 1972.

[Pie] Tilman a.k.a. Watchduck Piesk. Set partitions 4; Hasse; circles. Licensed under CC BY 3.0 via Wikimedia Commons.

[Prz] Józef H. Przytycki. Skein modules, chapter IX of the book Knots: from combinatorics of knot diagrams to the combinatorial topology based on knots. Accepted for publication, online at http://arxiv.org/abs/math/0602264.

[Prz06] Józef H. Przytycki. Skein modules of 3-manifolds. *arXiv preprint math/0611797*, 2006.

[Ras04] Jacob A. Rasmussen. Khovanov homology and the slice genus. *arXiv preprint math/0402131*, 2004.

[Rit10] Sandra Ritz. *A Categorification of the Burau Representation Via Contact Geometry*. PhD thesis, University of Southern California, 2010.

[RSW04] Victor Reiner, Dennis Stanton, and Dennis White. The cyclic sieving phenomenon. *Journal of Combinatorial Theory, Series A*, 108(1):17–50, 2004.

[Rus09] Heather M. Russell. The Bar-Natan skein module of the solid torus and the homology of (n, n) Springer varieties. *Geom. Dedicata*, 142:71–89, 2009.

[Sag10] Bruce E. Sagan. The cyclic sieving phenomenon: a survey. *arXiv preprint arXiv:1008.0790*, 2010.

[Sit05] Andrzej Sitarz. Twisted Hochschild homology of quantum hyperplanes. *K-theory*, 35(1):187–198, 2005.

[Tin10] Peter Tingley. A minus sign that used to annoy me but now I know why it is there. *arXiv preprint arXiv:1002.0555*, 2010.

[TT06] Vladimir Turaev and Paul Turner. Unoriented topological quantum field theory and link homology. *Algebr. Geom. Topol.*, 6:1069–1093, 2006.

[Wal06] Kevin Walker. TQFTs. [Online at http://canyon23.net/math/tc.pdf; accessed 30-April-2015], 2006.

[Web07] Ben Webster. Hochschild homology, 2007. [Online at http://sbseminar.wordpress.com/2007/07/22/hochschild-homology/; accessed 30-April-2015].

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wik15a] Wikipedia. Additive category, 2015. [Online at http://en.wikipedia.org/wiki/Additive_category; accessed 30-April-2015].

[Wik15b] Wikipedia. Hochschild homology, 2015. [Online at http://en.wikipedia.org/wiki/Hochschild_homology; accessed 30-April-2015].

[Wik15c] Wikipedia. Module (mathematics), 2015. [Online at http://en.wikipedia.org/wiki/Module_(mathematics); accessed 30-April-2015].

[Wik15d] Wikipedia. Orbifold, 2015. [Online at http://en.wikipedia.org/wiki/Orbifold; accessed 30-April-2015].

[Wik15e] Wikipedia. Preadditive category, 2015. [Online at http://en.wikipedia.org/wiki/Preadditive_category; accessed 30-April-2015].

[Wik15f] Wikipedia. Siefert fiber space, 2015. [Online at http://en.wikipedia.org/wiki/Seifert_fiber_space; accessed 30-April-2015].

[Wik15g] Wikipedia. Wallpaper group, 2015. [Online at http://en.wikipedia.org/wiki/Wallpaper_group; accessed 30-April-2015].

[Zul01] Louis Zulli. Seifert 3-manifolds that are bundles and semi-bundles. *Houston J. Math*, 27(3):533–540, 2001.