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Author

Pellegrini, Claudio

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CROSSING OF AN INCOHERENT INTEGRAL RESONANCE IN THE ELECTRON RING ACCELERATOR*

Claudio Pellegrini† and Andrew M. Sessler

Lawrence Radiation Laboratory
University of California
Berkeley, California

January 26, 1970

ABSTRACT

In one mode of operation of an electron ring accelerator (ERA), at the end of compression rings are slowly moved through the radial integral betatron resonance $Q_r = 1$. Although the coherent radial oscillation frequency of the ring as a whole remains below unity, the oscillation frequencies of individual electron are (incoherently) caused to pass through the resonance because of the additional focusing from ions trapped in the ring. In this paper the effect of field errors on ring major and minor radii is evaluated--theoretically--for the cases in which the spread in the square of the electron oscillation frequency (Δ^2) is (a) much larger and (b) much smaller than the contribution to the square of the oscillation frequency from the ions (Λ^2). It is shown that for the ERA, where case (b) applies, the increase in ring minor dimensions, for given field errors and rate of resonance crossing, is less than in case (a) by a factor of $(\Delta/\Lambda)^2$. Numerical examples show that the degradation of ring quality in case (b) should, with suitable attention to the design and construction of the ERA apparatus, be acceptably small.

1. INTRODUCTION

In the electron ring accelerators (ERA) now being studied at Dubna, Berkeley, Karlsruhe, and Garching,[1] an electron ring is compressed in a magnetic field having field index $n \equiv -\frac{r}{B}\frac{\partial B}{\partial r}$ such that $0 < n < 1$. At the end of compression positive ions are captured in the ring, which is subsequently extracted from the compressor and brought into an accelerating column having a constant magnetic field and hence $n = 0$.

During the compression process the radial betatron frequency $\omega_r = Q\,\Omega$, where Ω is the revolution frequency and Q is approximately given by $(1 - n)^{1/2}$, stays below Ω or, equivalently, Q stays below unity. The capture of ions in the electron ring introduces an additional focusing force on the electron, which has the effect of increasing Q. During the extraction process n goes to zero, so that, in the absence of ions or other additional forces, Q would become equal to unity. As a result of both effects Q crosses the value $Q = 1$.

As is well known, when $Q = 1$ an integer resonance is excited. This can produce a large displacement of the electron orbits and hence a beam loss. Moreover, even if the beam is not lost it is possible that the crossing of the resonance could produce a large increase in beam dimension and a corresponding decrease in the electric field that keeps the ions inside the ring. As a consequence, the external electric field which is applied so as to accelerate the ring would have to be lowered to an uninterestingly small value.

The increase in oscillation amplitude of a single particle crossing an integral resonance at a rate $r = d\omega_r^2/dt$ is given approximately by

$$x_s = \left(\frac{\pi}{\Omega \dot{r}}\right)^{1/2} R\, \Omega^2 \left(\frac{\Delta B}{B}\right), \qquad \text{(I-1)}$$

where R is the beam radius, Ω the revolution frequency, and $(\Delta B/B)$ the magnetic field perturbation driving the resonance[2-5].

Formula (I-1) shows, using typical ERA parameters, that in order to maintain the increase in amplitude within reasonable limits, the requirements on the magnetic field are very strong; for instance, assuming $x_s = 0.1$ cm, $R = 3$ cm, and $\dot{Q} = 10^4\ \text{sec}^{-1}$, one has $(\Delta B/B) < 10^{-5}$. Various possibilities have been suggested for reducing $\dot{Q}$, so as to avoid crossing the resonance: The use of image forces obtained by surrounding the electron ring with a dielectric cylinder[6] or a slotted metallic cylinder[7], or keeping $Q > 1$ throughout compression and acceleration of the ring by using the azimuthal magnetic field generated by a current along the axis of the ring[8].

The use of image forces seems to provide a practical way to avoid the resonance crossing when there are only few ions in the ring, but not when the ring is charged with more ions than of the order of 1% of the electrons. The use of an azimuthal magnetic field to keep Q always above unity requires currents in the conductor on the axis of the order of 10^5 A --an inconvenient, but possible, design requirement.

It has, however, been pointed out by Van der Meer[9], on the basis of qualitative arguments, that the application to the ERA of the formula for the single-particle increase of amplitude during the resonance crossing may be incorrect. In this paper we study the effect of resonance crossing in detail. In particular we consider the case when Q would stay below unity in the absence of ions (i.e., the coherent integral resonance is <u>not</u> crossed), but is shifted above unity by the ion focusing force (i.e., the incoherent integral resonance <u>is</u> crossed). We find that in this case the formula (I-1) is not valid and that the behavior of the beam in crossing the incoherent resonance depends on the ratio of the spread in the square of the frequency in the electron ring, Δ^2, to the shift in the square of the frequency, Λ^2, induced by the ions.

The results described by (I-1) applies only when the condition

$$\frac{\Delta^2}{\Lambda^2} \gg 1 , \qquad \text{(I-2)}$$

since in this case each electron behaves as a single electron having a frequency $(\omega^2 + \Lambda^2)^{1/2}$, where ω is the frequency due to the external magnetic field and image forces, and Λ is the shift in frequency caused by the ions. Thus resonance crossing leads to an increase in beam minor dimensions, but no change in the beam center of mass.

On the contrary, in the case more often encountered in the ERA, when

$$\frac{\Delta^2}{\Lambda^2} \ll 1 , \qquad \text{(I-3)}$$

there is a (small) change in the local beam center of mass, but the beam minor dimension increase is smaller, by a factor of (Δ^2/Λ^2), than that expected on the basis of (I-1). Hence the limit on the tolerable magnetic field imperfections, $\Delta B/B$ (which is set by the strong requirement of small minor dimensions of the ring), is lowered and can more easily be satisfied. Thus our detailed analysis supports the general conclusions of Van der Meer and is in qualitative agreement with observation[10].

That the simple formula (I-1) does not apply to circular electron beams partially or totally neutralized by ions is of importance, also, for electron storage rings. In this case, too, due to the long beam lifetime, a large number of ions are captured by the beam, when clearing field electrodes are not used. Once again, the frequency shift introduced by the ions can cause a crossing of an integer resonance. Both the conditions that Q remain below the nearest integer during the ion loading process and condition (I-3) are well satisfied in storage rings. However, in this paper we have considered only azimuthally uniform beams, while the electron beam of a storage ring is bunched. Hence, we cannot directly apply our results to storage rings. Notwithstanding, we think that, at least to a first approximation, the results of this work indicate that also in the case of the storage ring the crossing of the resonance produces only a beam widening, and that this widening is not too dangerous because of the strong reduction introduced by the factor Δ^2/Λ^2 . This conclusion is in agreement with the experimental observations performed on electron storage rings.

2. FORMULATION OF THE PROBLEM

We assume that the electrons move on a circular orbit with a constant angular velocity Ω , and that they oscillate in a direction orthogonal to this orbit under the action of the focusing forces due to the external magnetic field and to the ions. The ions are assumed to have zero angular velocity and to oscillate in the same direction as the electrons under the action of the focusing force due to the electrostatic field of the electrons. We ignore ion-ion forces, since in practice the ion density is sufficiently low that these terms are negligible.

Let us call x_k, θ_k and ξ_j, ψ_j the transverse and the azimuthal coordinates of the kth electron and jth ion. The equations of motion can be written as

$$\ddot{x}_k(t) + \omega_k^2(t)x_k(t) + \Lambda_k^{(i)2}(t)[x_k(t) - \bar{\xi}(t,\theta_k)]$$

$$+ \Lambda_k^{(e)2}(t)[x_k(t) - \bar{x}(t,\theta_k)] = a\cos(\bar{n}\theta_k + \phi) ,$$

$$\theta_k = \Omega t + \alpha_k ,$$

$$\ddot{\xi}_j(t) + M_j^2[\xi_j(t) - \bar{x}(t,\psi_j)] = 0 ,$$

$$\psi_j = \text{const}, \qquad \text{(II-1)}$$

where $\omega_k^2 x_k$ is the focusing force due to the magnetic field, the $\Lambda_k^{(e)2}$ term describes the force of electron on electrons, $\Lambda_k^{(i)2}[x_k - \bar{\xi}(t,\theta_k)]$ and $M_j^2[\xi_j - \bar{x}(t,\psi_j)]$ are the forces between ions and electrons and $a\cos(\bar{n}\theta_k + \phi)$ is the perturbation in the guide magnetic field. Note that we consider only field bump errors and do not include gradient error terms as they are--in practice--negligible[4]. We consider only the $\bar{n}$-Fourier component in the magnetic field perturbation, where $n\Omega \simeq \omega_k$.

The electron-ion forces are written, in the linear approximation, as proportional to the distance of the kth particle from the local center of mass of the particles of the other species, $\bar{x}(t,\theta)$ and

$\bar{\xi}(t,\psi)$. The local center of mass can be defined, with the help of the step function $S(\theta)$, as

$$\bar{x}(t,\theta) = \frac{\sum_k x_k(t)\, S(\theta_k - \theta)\, S(\theta + d\theta - \theta_k)}{\sum_k S(\theta_k - \theta)\, S(\theta + d\theta - \theta_k)},$$

$$\bar{\xi}(t,\psi) = \frac{\sum_j \xi_j(t)\, S(\psi_j - \psi)\, S(\psi + d\psi - \psi_j)}{\sum_j S(\psi_j - \psi)\, S(\psi + d\psi - \psi_j)}. \qquad \text{(II-2)}$$

The nonlinearities of this force, as well as the nonlinearities in the external focusing force, are taken into account approximately by allowing a dependence of ω^2, M^2, $\Lambda^{(e)2}$, and $\Lambda^{(i)2}$ on some of the parameters of the particles such as oscillation amplitude or energy. Newton's third law implies a subsidiary condition amongst the $\Lambda_k^{(i)}$ and M_j. We need not invoke this relation, as well be seen below.

The quantities ω_k, M_j, $\Lambda_k^{(e)}$, and $\Lambda_k^{(i)}$ are functions of time, because of the changes in the external magnetic field and in the number of ions with time. Both these variations are assumed to be very slow compared with the electron and ion oscillation period.

We are only interested in studying the closed-orbit perturbations due to the magnetic field imperfections, i.e., the particular solution of the nonhomogenous (II-1).

We will first consider the case in which ω_k, $\Lambda_k^{(e)}$, M_k, and $\Lambda_k^{(i)}$ are constant in time. Since the driving force, $a\cos(\bar{n}\theta + \phi)$, is periodic with respect to θ , we look for a solution having the same periodicity. Let us assume

$$x_k(t) = A_k \cos(\bar{n}\,\theta_k + \phi) ,$$

$$\xi_j(t) = B_j \cos(\bar{n}\,\psi_j + \phi). \qquad \text{(II-3)}$$

The local centers of mass are then given by

$$\bar{x}(t,\theta) = \bar{A} \cos(\bar{n}\theta + \phi),$$

$$\bar{\xi}(t,\psi) = \bar{B} \cos(\bar{n}\psi + \phi) . \qquad \text{(II-4)}$$

The amplitudes $\bar{A}$, $\bar{B}$ are given, in the case of a beam containing N_e electron and N_i ions uniformly distributed along the circumference, and assuming that the distribution of the A_k, B_k is independent of the azimuthal position, by

$$\bar{A} = \left(\frac{1}{N_e}\right) \sum_{k=1}^{N_e} A_k ,$$

$$\bar{B} = \left(\frac{1}{N_i}\right) \sum_{j=1}^{N_i} B_j . \qquad \text{(II-5)}$$

Substituting (II-3) and (II-4) into (II-1), we obtain

$$B_j = \bar{A} ,$$

$$A_k \left\{\omega_k^2 + \Lambda_k^{(e)2} + \Lambda_k^{(i)2} - \bar{n}^2\Omega^2\right\} - \Lambda_k^{(e)2}\,\bar{A} - \Lambda_k^{(i)2}\,\bar{B} = a . \qquad \text{(II-6)}$$

By use of (II-5), the system of (II-6) can be reduced to

$$A_k \left\{ \omega_k^2 + \Lambda_k^{(e)2} + \Lambda_k^{(i)2} - \bar{n}^2\Omega^2 \right\} - \Lambda_k^{(i)2}\,\bar{A} - \Lambda_k^{(e)2}\,\bar{A} = a. \tag{II-7}$$

The first of (II-6), together with (II-5), shows simply that, under the action of the external perturbation, the local ion center of mass undergoes the same displacement as the local electron center of mass.

This result is also valid for slow changes of ω_k, M_k, $\Lambda_k^{(e)}$, and $\Lambda_k^{(i)}$, so that in general we can reduce the equations of (II-1) to an equation for the electrons only, namely

$$\ddot{x}_k + \omega_k^2(t)x_k + \Lambda_k^2(t)[x_k - \bar{x}] = a\cos(\bar{n}\theta_k + \phi), \tag{II-8}$$

where we have set $\Lambda_k^2 = \Lambda_k^{(e)2} + \Lambda_k^{(i)2}$. When ω_k and Λ_k are constant in time this clearly reduces to (II-7).

3. NORMAL MODE ANALYSIS

We have reduced the problem to solving (II-8), which task is accomplished in this and the next two sections. We can limit ourselves to the case in which the variation in time of ω_k and Λ_k is small compared with $\bar{n}\Omega$. It is then possible to perform a power-series expansion of these quantities, and to consider only terms up to first order, namely, to write

$$\omega_k^2 = \omega_k^2(t_0) + r(t - t_0)\,,$$

$$\Lambda_k^2 = \Lambda_k^2(t_0) + r'(t - t_0)\,. \tag{III-1}$$

We also assume that r and r' are different from zero only in a time interval $t_0 - t_1$ during which the resonance is crossed, and that the initial and final values of ω_k^2 and $\omega_k^2 + \Lambda_k^2$ are respectively well below and well above the resonant value $\bar{n}^2\Omega^2$. Notice, also, that we have assumed r and r' to be equal for all particles. This is a good approximation when the frequency spreads for both ω and Λ are small compared with $\bar{n}\Omega$.

We can now obtain a solution of (II-8), assuming x_k to be of the form

$$x_k(t) = \sum_{n=1}^{N_e} A_n(t)\, C_k^{(n)} \exp[i(\bar{n}\theta_k + \phi)] , \qquad \text{(III.2)}$$

where the $A_n(t)$ are unknown functions and the $C_k^{(n)}$ are a complete orthonormal set of vectors defined as the eigenvectors of the linear system of equations

$$[\omega_k^2(t_0) + \Lambda_k^2(t_0)]C_k^{(n)} - \Lambda_k^2(t_0)\, \bar{C}^{(n)} = \Gamma^2_{(n)}\, C_k^{(n)}, \quad n = 1, \cdots, N_e, \qquad \text{(III-3)}$$

where $\bar{C}^{(n)}$ is defined as

$$\bar{C}^{(n)} = \frac{1}{N_e} \sum_{k=1}^{N_e} C_k^{(n)} ,$$

as follows from (II-2) and (III-2); and $\Gamma^2_{(n)}$ is an eigenvalue. Substituting (III-2) into (II-8) and using (III-1) and (III-3), we get

$$\sum_{n=1}^{N_e} \left\{ \ddot{A}_n + 2i\bar{n}\Omega A_n + [\Gamma^2_{(n)} - \bar{n}^2\Omega^2 + (r + r')(t - t_0)]A_n \right\} C_k^{(n)}$$

$$- r'(t - t_0) \sum_n A_n \bar{C}^{(n)} = a . \qquad \text{(III-4)}$$

Using the orthonormality property of the $C_k^{(n)}$, we obtain

$$\ddot{A}_n + 2i\bar{n}\Omega \dot{A}_n + [\Gamma^2_{(n)} - \bar{n}^2\Omega^2 + (r + r')(t - t_0)]A_n$$

$$- r'(t - t_0)N_e \sum_{m=1}^{N_e} A_m \bar{C}^{(m)} \bar{C}^{(n)} = aN_e\bar{C}^{(n)}. \qquad \text{(III-5)}$$

We assume that $A_n(t)$ is a function varying slowly with respect to the characteristic oscillation periods, so that it is possible to neglect the second derivative of $A_n(t)$ in (III-5) and write it as

$$2i\bar{n}\Omega \dot{A}_n + [\Gamma^2_{(n)} - \bar{n}^2\Omega^2 + (r + r')(t - t_0)]A_n$$

$$- r'(t - t_0)N_e \sum_{m=1}^{N_e} A_m \bar{C}^{(m)} \bar{C}^{(n)} = a N_e \bar{C}^{(n)} . \qquad \text{(III-6)}$$

The problem is now reduced to finding the $C_k^{(n)}$ and $A_n(t)$, i.e., to solving (III-3) and (III-6).

The solution will depend on the ratio Δ^2/Λ_0^2, where Δ^2 is the width of the distribution of the frequencies ω_k^2, and Λ_0^2 is the average value of Λ_k^2. (We assume that the widths of the distribution of ω_k and Λ_k are small compared with the average values of ω_k and Λ_k.)

In the remainder of this paper we will study only the two cases

$$\text{(a)} \qquad \frac{\Delta^2}{\Lambda_0^2} \ll 1 \; ,$$

and

$$\text{(b)} \qquad \frac{\Delta^2}{\Lambda_0^2} \gg 1 \; ,$$

for both of which solutions of (III-3) and (III-6) can be obtained.

We also notice that we are interested in the determination of the two quantities

$$\bar{x} = \left| \frac{1}{N_e} \sum_k x_k \right| = \left| \sum_n A_n(t)\, \bar{C}^{(n)} \right| \tag{III-7}$$

and

$$\delta = \left\{ \frac{1}{N_e} \sum_k |x_k - \bar{x}|^2 \right\}^{1/2} = \left\{ \frac{1}{N_e} \sum_n |A_n(t)|^2 - \bar{x}^2 \right\}^{1/2} , \tag{III-8}$$

which are the local center-of-mass amplitude and the root-mean-square (rms) beam size. Both $\bar{x}$ and δ^2, as well as (III-6), depend on the $C_k^{(n)}$ only through the average values $\bar{C}^{(n)}$.

4. DETERMINATION OF THE EIGENVECTORS

In this section we determine the eigenvectors and eigenvalues of (III-3) in the two cases: (a) $\Delta^2/\Lambda_0^2 \ll 1$, and (b) $\Delta^2/\Lambda_0^2 \gg 1$. We consider case (a) first; case (b) is rather trivial and is discussed at the end of this section. It is convenient to start by solving (III-3) for the case of zero frequency spread. The eigenvectors $C_k^{(n)0}$ are given, for $\Delta = 0$, by

$$C_k^{(n)0} = \frac{1}{\sqrt{N}} e^{2\pi ink/N} , \qquad \text{(IV-1)}$$

$$\bar{C}^{(n)0} = \frac{1}{\sqrt{N}} \delta_{n,0} , \qquad \text{(IV-2)}$$

where we have employed N as a notation for N_e. The corresponding eigenvalues are

$$\Gamma^2_{(n)0} = \omega_0^2 + \Lambda_0^2[1 - \delta_{n,0}] . \qquad \text{(IV.3)}$$

Notice that all the $\Gamma^2_{(n)}$ are equal, with the exception of $\Gamma^2_{(0)}$.

For a small frequency spread, we can use perturbation theory to determine the $C_k^{(n)}$. Let us rewrite Eq. (III-3) as

$$\left(\underset{\sim}{H}^{(0)} + \underset{\sim}{H}^{(1)}\right) \underset{\sim}{C}^{(n)} = \Gamma^2_{(n)} \underset{\sim}{C}^{(n)} , \qquad \text{(IV-4)}$$

where $\underset{\sim}{C}^{(n)}$ is a vector of components $C_k^{(n)}$,

$$\underset{\sim}{H}^{(0)}_{k\ell} = (\omega_0^2 + \Lambda_0^2) \delta_{k\ell} - \frac{\Lambda_0^2}{N} , \qquad \text{(IV-5)}$$

$$\underset{\sim}{H}^{(1)}_{k\ell} = (\omega_k^2 + \Lambda_k^2 - \omega_0^2 - \Lambda_0^2) \delta_{k\ell} - \frac{\Lambda_k^2 - \Lambda_0^2}{N} , \qquad \text{(IV-6)}$$

and ω_0^2 and Λ_0^2 are the average values of ω_k^2, Λ_k^2. For $\underset{\sim}{H}^{(1)} = 0$, $\underset{\sim}{C}^{(n)}$ is equal to $\underset{\sim}{C}^{(n)0}$ as given by (IV-1), and $\Gamma^2_{(n)} = \Gamma^2_{(n)0}$ as given by (IV-3)

To apply perturbation theory when $\underset{\sim}{H}^{(1)} \neq 0$, one must remember that the unperturbed solution is degenerate (all eigenfunctions, except $\underset{\sim}{C}^{(0)0}$, belong to the same eigenvalue), and use instead of the $\underset{\sim}{C}^{(n)0}$'s , for $n \neq 0$, a linear combination of these vectors such that $\underset{\sim}{H}^{(1)}$ is diagonalized. Calling these new vectors $\underset{\sim}{\phi}^{(n)}$, one has

$$\underset{\sim}{\phi}^{(0)} = \underset{\sim}{C}^{(0)0} , \qquad \text{(IV-7)}$$

and

$$\underset{\sim}{\phi}^{(n)} = \sum_{t=1}^{N-1} B_t^{\,n}\, \underset{\sim}{C}^{(t)0} \quad \text{for } n \neq 0 , \qquad \text{(IV-8)}$$

where

$$B_t^{\,n} = \frac{1}{(N-1)^{1/2}}\, e^{2\pi int/(N-1)} . \qquad \text{(IV-9)}$$

It is easy to verify that

$$\left(\underset{\sim}{\phi}^{*(n)}\, \underset{\sim}{H}^{(1)}\, \underset{\sim}{\phi}^{(m)}\right) = 0;\ n,m \neq 0 \text{ and } n \neq m, \qquad \text{(IV-10)}$$

and

$$\left(\underset{\sim}{\phi}^{*(n)}\, \underset{\sim}{H}^{(1)}\, \underset{\sim}{\phi}^{(n)}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{t=1}^{N-1} (\omega_k^{\,2} + \Lambda_k^{\,2} - \omega_0^{\,2} - \Lambda_0^{\,2})$$

$$\times \left\{\exp 2\pi it \left[\frac{n}{N-1} + \frac{k}{N}\right]\right\}; \quad n \neq 0 , \qquad \text{(IV-11)}$$

and that

$$\left(\underset{\sim}{\phi}^{*(0)}\, \underset{\sim}{H}^{(1)}\, \underset{\sim}{\phi}^{(n)}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{t=1}^{N-1} (\omega_k^{\,2} - \omega_0^{\,2})$$

$$\times \frac{1}{(N-1)^{1/2}} \left\{\exp 2\pi it \left[\frac{n}{N-1} + \frac{k}{N}\right]\right\}. \qquad \text{(IV-12)}$$

The solution of (IV-4) is now given by

$$\underset{\sim}{C}^{(n)} = \underset{\sim}{\phi}^{(n)} + \sum_{m\neq n} A_m{}^n \underset{\sim}{\phi}^{(m)} , \qquad \text{(IV-13)}$$

and, to first order in the perturbation, one has

$$A_m{}^n = \frac{\left(\underset{\sim}{\phi}^{*(m)} \underset{\sim}{H}^{(1)} \underset{\sim}{\phi}^{(n)}\right)}{\Gamma^2_{(n)} - \Gamma^2_{(m)0}} , \qquad \text{(IV-14)}$$

$$\Gamma^2_{(n)} = \Gamma^2_{(n)0} + \left(\underset{\sim}{\phi}^{*(n)} \underset{\sim}{H}^{(1)} \underset{\sim}{\phi}^{(n)}\right) . \qquad \text{(IV-15)}$$

Notice that with our choice of ω_0^2, Λ_0^2 one has also

$$\left(\underset{\sim}{\phi}^{*(0)} \underset{\sim}{H}^{(1)} \underset{\sim}{\phi}^{(0)}\right) = 0 ,$$

so that there is no first-order correction to the coherent frequency $\Gamma^2_{(0)}$. The quantities $\bar{C}^{(n)}$ are now easily obtained, and, to first order, one has

$$\bar{C}^{(0)} = \frac{1}{\sqrt{N}} + \text{first order term}, \qquad \text{(IV-16)}$$

$$\bar{C}^{(n)} = \frac{1}{\sqrt{N}} \frac{\left(\underset{\sim}{\phi}^{*(0)} \underset{\sim}{H}^{(1)} \underset{\sim}{\phi}^{(n)}\right)}{\Lambda_0^2} \qquad \text{(IV-17)}$$

$$= \frac{1}{N\Lambda_0^2} \sum_{k=0}^{N-1} \sum_{t=1}^{N-1} (\omega_k^2 - \omega_0^2)$$

$$\times \frac{1}{\left(N(N-1)\right)^{1/2}} \left\{\exp 2\pi i t\left[\frac{k}{N} + \frac{n}{N-1}\right]\right\} .$$

We can now use these results to simplify (III-6). In the case $n = 0$ the equation contains zero-order terms and first-order terms in Δ^2/Λ_0^2. Neglecting the first-order terms, one has

$$2i\bar{n}\Omega \dot{A}_0 + [\omega_0^2 - \bar{n}^2\Omega^2 + r(t - t_0)] A_0 = a\sqrt{N} . \qquad \text{(IV-18)}$$

For $n \neq 0$ (III-6) contains first- and second-order terms in Δ^2/Λ_0^2. Keeping only lowest-order terms, one has

$$2i\bar{n}\Omega \dot{A}_n + [\omega_0^2 + \Lambda_0^2 - \bar{n}^2\Omega_0^2 + (r + r')(t - t_0)] A_n$$

$$- r'(t - t_0)\sqrt{N} \; \bar{C}^{(n)} A_0 = a N \bar{C}^{(n)} . \qquad \text{(IV-19)}$$

In case (b) the coupling between particles is negligible and the eigenvalues are almost equal to the single particle frequencies, i.e.,

$$\Gamma^2_{(n)} = \omega_n^2 + \Lambda_n^2 + O[(\Lambda_0^2/\Delta^2)^2] . \qquad \text{(IV-20)}$$

The corresponding eigenfunctions are

$$C_k^{(n)} = \delta_{n,k} + O[(\Lambda_0^2/\Delta^2)] , \qquad \text{(IV-21)}$$

and the $\bar{C}^{(n)}$ are given, to lowest order, by

$$\bar{C}^{(n)} = \frac{1}{N} . \qquad \text{(IV-22)}$$

Equation (III-6) now becomes, neglecting the coupling between particles,

$$2i\bar{n}\Omega \dot{A}_n + [\omega_n^2 + \Lambda_n^2 - \bar{n}^2\Omega^2 + (r + r')(t - t_0)]A_n = a . \qquad \text{(IV-23)}$$

5. DETERMINATION OF THE AMPLITUDE FUNCTIONS

In this section we solve (IV-18), (IV-19), (IV-23) for the functions $A_n(t)$.

5.1. Case b

We start from (IV-23), which we write in the form

$$\dot{A}_n(t) - i\, g_n(t)\, A_n(t) = -i\, \bar{a}\,, \tag{V-1}$$

where

$$g_n(t) = \frac{1}{2\bar{n}\Omega}\left[\omega_n^{\,2} + \Lambda_n^{\,2} - \bar{n}^2\Omega^2 + (r + r')(t - t_0)\right], \tag{V-2}$$

$$\bar{a} = \frac{a}{2\bar{n}\Omega}\,. \tag{V-3}$$

The solution of (V-1), with the initial condition $A(t_0) = 0$, is

$$A_n(t) = -i\bar{a}\left\{\exp\left[i\int_{t_0}^{t} g_n(t')dt'\right]\right\}\int_{t_0}^{t} dt' \exp\left[-i\int_{t_0}^{t'} g_n(t'')dt''\right]. \tag{V-4}$$

Evaluating the integrals, and using the notation

$$D_n = (\omega_n^{\,2} + \Lambda_n^{\,2} - \bar{n}^2\Omega^2)/2\bar{n}\Omega\,,$$

$$p = (r + r')/2\bar{n}\Omega\,, \tag{V-5}$$

one has:

$$\int_{t_0}^{t} \exp\left\{-i\left[D_n(t' - t_0) + \frac{1}{2}\,p(t' - t_0)^2\right]\right\} dt' = \sqrt{\frac{\pi}{p}}\;\exp[iD_n^{\,2}/(2p)]\left\{h\left(\frac{p(t - t_0) + D_n}{\sqrt{\pi p}}\right) - h\left(\frac{D_n}{\sqrt{\pi p}}\right)\right\}, \tag{V-6}$$

where

$$h(x) = C(x) - i\,S(x)\,, \qquad \text{(V-7)}$$

and $C(x)$, $S(x)$ are the Fresnel integrals.

It is usually possible, when p is small and the integral extends from well below to well above the resonance, to make the approximation

$$\frac{D_n}{\sqrt{\pi p}} \ll -1\,,$$

$$\frac{p(t - t_0) + D_n}{\sqrt{\pi p}} \gg 1\,. \qquad \text{(V-8)}$$

Since $C(\pm\infty) = S(\pm\infty) = \pm\frac{1}{2}$, one has in this case

$$\int_{t_0}^{t} \exp\left\{-i[D_n(t' - t_0) + \frac{1}{2}\,p(t' - t_0)^2]\right\} dt'$$

$$\approx \left(\frac{2\pi}{p}\right)^{1/2} \exp\,[+i(D_n^{\,2}/2p) - i\pi/4]. \qquad \text{(V-9)}$$

The value of A_n after crossing the resonance is then given by

$$A_n \approx -i\bar{a}\left(\frac{2\pi}{p}\right)^{1/2} \exp\left\{i[D_n(t - t_0) + \frac{p}{2}\,(t - t_0)^2]\right\}\exp[+i(D_n^{\,2}/2p) - i\pi/4]$$

$$= -i\bar{a}\left(\frac{2\pi}{p}\right)^{1/2} \exp\left\{i\left[\frac{p(t - t_0) + D_n}{\sqrt{2p}}\right]^2 - i\pi/4\right\}. \qquad \text{(V-10)}$$

The final amplitude after crossing the resonance is therefore

$$|A_n| = \bar{a}\,(2\pi/p)^{1/2} = a[\pi/\bar{n}\Omega(r + r')\,]^{1/2}, \qquad \text{(V-11)}$$

a well-known result.

5.2. Case a

In this case the frequency spread Δ^2 is small compared with the frequency shift Λ_0^2. The situation is described by (IV-18) and (IV-19), and is clearly more complicated than case (b). The procedure is to solve (IV-18) for A_0, substitute the result in (IV-19), and solve for A_n. The result will be different according to whether the coherent frequency, ω_0, does or does not cross the resonance. We will consider here only the case in which ω_0 does not cross the resonance (i.e., the coherent integral resonance is not crossed), since this is the situation which usually confronts us in practice. Under this assumption one can neglect the variation in time of the coherent frequency and of A_0, and obtain from (IV-18)

$$A_0 = \frac{a\sqrt{N}}{\omega_0^2 - (\bar{n}\,\Omega)^2} . \qquad \text{(V-12)}$$

Substituting this in (IV-19) one obtains

$$\dot{A}_n - i(D + p(t - t_0))A_n = -i\,\bar{a}\,N\,\bar{C}^{(n)}[1 + q(t - t_0)] , \qquad \text{(V-13)}$$

where

$$D = \frac{\omega_0^2 + \Lambda_0^2 - \bar{n}^2\Omega^2}{2\bar{n}\Omega} ,$$

$$p = \frac{r + r'}{2\bar{n}\Omega} ,$$

$$\bar{a} = \frac{a}{2\bar{n}\Omega} ,$$

$$q = \frac{r'}{\omega_0^2 - \bar{n}^2\Omega^2} . \qquad \text{(V-14)}$$

The solution can again be written, assuming $A_n(t_0) = 0$, as

$$A_n(t) = -i\,\bar{a}\,N\,\bar{C}^{(n)} \exp\left\{ i \int_{t_0}^{t} [D + p(t' - t_0)]dt' \right\}$$

$$\times \int_{t_0}^{t} dt'[1 + q(t' - t_0)]\exp\left\{ -i \int_{t_0}^{t'} [D + p(t'' - t_0)]dt'' \right\}.$$

(V-15)

The integrals of (V-15) can be evaluated by using (V-6) and

$$\int_{t_0}^{t} dt'(t' - t_0)\exp\left\{-i[D(t'-t_0) + \frac{p}{2}(t'-t_0)^2]\right\}$$

$$= \frac{2}{p}\, e^{i(D^2/2p)} \left\{ -\frac{D}{2\sqrt{\pi p}} \left[h\left(\frac{p(t - t_0) + D}{\sqrt{\pi p}}\right) - h\left(\frac{D}{\sqrt{\pi p}}\right) \right] \right.$$

$$\left. + \frac{1}{2}\left[\sin\left(\frac{p(t - t_0) + D}{\sqrt{2p}}\right)^2 + i \cos \frac{D^2}{2p} \right] \right\}. \qquad \text{(V-16)}$$

Assuming the conditions (V-8) to be satisfied, one obtains an amplitude, after crossing the resonance,

$$A_n(t) \approx -i\,\bar{a}\,N\,\bar{C}^{(n)} \left(\frac{2\pi}{p}\right)^{1/2} \left\{1 - \frac{qD}{\pi p}\right\} \exp\left\{ i \left[\frac{p(t-t_0) + D}{\sqrt{2p}}\right]^2 - i\pi/4 \right\},$$

(V-17)

where negligible contributions from the last term of (V-16) have been dropped. By use of (V-14), (V-17) can be written as

$$A_n(t) \approx -i \frac{a\sqrt{\pi}\ N\bar{C}(n)}{(\bar{n}\Omega(r + r'))^{1/2}} \left[1 - \frac{r'}{\pi(r + r')} \left(1 + \frac{\Lambda_0^2}{\omega_0^2 - \bar{n}^2\Omega^2} \right) \right]$$

$$\times \exp \left\{ i \frac{\left[\omega_0^2 + \Lambda_0^2 - \bar{n}^2\Omega^2 + (r + r')(t - t_0)\right]^2}{4n\Omega(r + r')} - i \frac{\pi}{4} \right\} . \tag{V-18}$$

6. EVALUATION OF BEAM POSITION AND SIZE

We are now in a position to evaluate the local center-of-mass displacement, $\bar{x}$, and the rms beam width, δ, which were defined in (III-7) and (III-8).

6.1. Case a: $\Delta^2/\Lambda_0^2 \ll 1$

Using (IV-16), (IV-17), (V-12), and (V-18), and introducing the quantities

$$\Delta^4 = \frac{1}{N} \sum_{k=0}^{N-1} (\omega_k^2 - \omega_0^2)^2 , \tag{VI-1}$$

$$\Delta_1^4 = \frac{1}{N} \sum_{k=0}^{N-1} (\omega_k^2 - \omega_0^2)^2 e^{-2\pi ik/N} , \tag{VI-2}$$

so that Δ^2 is the rms spread in the square of the frequency shift, one obtains

$$\bar{x} = \frac{a}{\omega_0^2 - (\bar{n}\Omega)^2} - ia\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2}\left[1 - \frac{r'}{\pi(r + r')}\left(1 + \frac{\Lambda_0^2}{\omega_0^2 - \bar{n}^2\Omega^2}\right)\right]$$

$$\times \exp\left\{i \frac{\left[\omega_0^2 + \Lambda_0^2 - \bar{n}^2\Omega^2 + (r+r')(t-t_0)\right]^2}{4\bar{n}\Omega(r + r')} - i\frac{\pi}{4}\right\} \frac{\Delta_1^4}{\Lambda_0^4}, \tag{VI-3}$$

$$\delta^2 = \left\{a\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2}\left[1 - \frac{r'}{\pi(r + r')}\left(1 + \frac{\Lambda_0^2}{\omega_0^2 - \bar{n}^2\Omega^2}\right)\right]\right\}^2 \times \left\{\frac{\Delta^4}{\Lambda_0^4} - \frac{|\Delta_1^4|^2}{\Lambda_0^8}\right\}$$

$$- \frac{a}{\omega_0^2 - \bar{n}^2\Omega^2} \quad a\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2}\left[1 - \frac{r'}{\pi(r + r')}\left(1 + \frac{\Lambda_0^2}{\omega_0^2 - \bar{n}^2\Omega^2}\right)\right]$$

$$\times 2\,\mathrm{Re}\left\{\frac{\Delta_1^4}{\Lambda_0^4} \exp\left[i \frac{\left[\omega_0^2 + \Lambda_0^2 - \bar{n}^2\Omega^2 + (r+r')(t-t_0)\right]^2}{4\bar{n}\Omega(r + r')} - i\frac{\pi}{4} - i\frac{\pi}{2}\right]\right\}. \tag{VI-4}$$

If

$$\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2} \gg \frac{1}{\omega_0^2 - \bar{n}^2\Omega^2}, \tag{VI-5}$$

and

$$\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2} \frac{\Delta_1^4}{\Lambda_0^4} \ll \frac{1}{\omega_0^2 - \bar{n}^2\Omega^2}, \qquad \text{(VI-6)}$$

then (VI-3) and (VI-4) become

$$\bar{x} \approx \frac{a}{\omega_0^2 - \bar{n}^2\Omega^2}, \qquad \text{(VI-7)}$$

$$\delta \approx a\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2} \frac{\Delta^2}{\Lambda_0^2}\left[1 - \frac{r}{\pi(r + r')}\left(1 + \frac{\Lambda_0^2}{\omega_0^2 - \bar{n}^2\Omega^2}\right)\right]. \qquad \text{(VI-8)}$$

Equation (VI-7) shows that, when the rate of change of ω and Λ and the frequency spread are such as to satisfy (VI-5) and (VI-6), the local beam center of mass is essentially not influenced by the resonance crossing (but only by the proximity of the coherent integral resonance). However, and under the same conditions, the crossing of the resonance can lead to an increase of beam size, as shown by (VI-8).

It is interesting to compare these results with the increase in amplitude of a single particle crossing the resonance. For a single particle the amplitude after crossing is given by

$$x_s = a\left(\frac{\pi}{\bar{n}\Omega r}\right)^{1/2}.$$

Taking, for the sake of comparison, $r' = 0$ the increase in beam size, δ, is seen to be equal to x_s multiplied by the factor Δ^2/Λ_0^2, i.e., the ratio of frequency spread to frequency shift.

As a numerical example consider the case of an ERA with parameters $\Omega = 10^{10}\ \text{sec}^{-1}$, R (ring radius) ≈ 3 cm, $\omega_0 - \Omega \approx 2 \times 10^{-2}\ \Omega$, $r = 2\omega_0\ (d\omega_0/dt) \approx 2 \times 10^{24}\ \text{sec}^{-3}$, $r' = 0$, $(\Delta/\Lambda) \approx 10^{-1}$, $a = -R\Omega^2(\Delta B/B) \approx 3 \times 10^{20}\ (\Delta B/B)\ \text{sec}^{-2}$, $\bar{n} = 1$. The quantity r corresponds to a case such that ω_0/Ω changes by 0.1 in 10 μsec, a value typical for the ERA. One sees that (VI-5) and (VI-6) are satisfied for these parameters. From (VI-7) and (VI-8) one has

$$\bar{x} \approx 30\ (\Delta B/B)\ \text{cm},$$

$$\delta \approx 37.5\ (\Delta B/B)\ \text{cm},$$

so that a value of $\Delta B/B$ less than 10^{-3} should suffice to keep the effect of the resonance crossing within tolerable limits.

6.2. Case b: $\Delta^2/\Lambda_0^2 \gg 1$.

From (V-10), (IV-21), and (IV-22) and from (III-7) and (III-8), we have

$$\bar{x} = a\left(\frac{\pi}{\bar{n}\Omega(r + r')}\right)^{1/2} \frac{1}{N} \sum_k \exp\left\{i\left[\left(D_k + p(t - t_0)\right)^2/2p\right]\right\} \tag{VI-9}$$

and

$$\delta^2 = \frac{a^2\pi}{\bar{n}\Omega(r + r')}\left\{1 - \frac{1}{N^2} \sum_{k,h} \exp\left[i\left(\frac{D_k + p(t - t_0)}{\sqrt{2p}}\right)^2 - i\left(\frac{D_h + p(t - t_0)}{\sqrt{2p}}\right)^2\right]\right\}. \tag{VI-10}$$

Assuming, again, that condition (V-8) is satisfied, (VI-9) and (VI-10) become, to a good approximation,

$$\bar{x} \approx 0 , \tag{VI-11}$$

$$\delta \approx a \left\{ \frac{\pi}{\bar{n}\Omega(r + r')} \right\}^{1/2} . \tag{VI-12}$$

These last results are equivalent to saying that each particle behaves as a single particle, so that, because of the large frequency difference between particles, their center of mass averages to zero and one gets essentially only a beam widening. But the width increase is larger, by a factor of Δ^2/Λ_0^2, than that obtained in case (a).

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† Permanent address: Laboratori Nazionali di Frascati, Frascati (Roma), Italy.

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