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Authors
Silverman, Dennis
Tan, Chung-I

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the vacuum state is a scalar under the ordinary $SU(3)$, the chimeral $SU(3)$ subgroup of $SW(3)$ will be realized as a Goldstone symmetry, in contrast to the ordinary $SU(3)$ subgroup, and our theory presented here would go through unchanged. These points will be discussed in greater detail elsewhere.

Lastly, we may make the following remark: We showed that b as a function of a is discontinuous at $a=-1$ and $a=2$. From this, we concluded that we may have essential singularities at these points, provided that b is an analytic function of a except for a few isolated points in the complex plane of the variable a. However, there is another possibility that b may have branch cuts, instead of the essential singularities, passing through points $a=-1$ and 2, since these will also give the desired discontinuity. An interesting possibility is the conjecture that the Kuo transformation $a\to(2-a)(1+4a)^{-1}$, $\epsilon_0\to-\frac{1}{3}(1+4a)\epsilon_0$ may transform physical quantities on the first Riemann sheet in this cut plane into those on the unphysical second sheet.

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Inclusion of Toller-Angle Dependence in the Multi-Regge Integral Equation*

DENNIS SILVERMAN AND CHUNG-I TAN
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
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The nonforward multiperipheral integral equation for the Reggeon-particle absorptive amplitude is generalized to include complete dependence on the Toller-angle variable.

I. INTRODUCTION

MUCH progress has been made in formulating the multiperipheral bootstrap equation using a multi-Regge production model.[1,2] In a recent publication,[3] Goldberger, Tan, and Wang have constructed a simplified integral equation for the Reggeon-particle absorptive amplitude $\mathcal{A}(p,p_0;Q)$ in a formulation of the multi-Regge model. Their construction seemed to depend on the assumption that the double Regge coupling is independent of the Toller angle ω, and an approximate justification for this assumption was suggested by Tan and Wang.[4] It is our purpose to show that an integral equation which includes the complete dependence on the Toller angle can be written for the absorptive amplitude $\mathcal{A}$. This establishes the full generality of the integral-equation approach through the $\mathcal{A}$ amplitude.

In the process of formulating this equation, we elaborate the relation between the ω angle and the other invariants. We then express the integration of the loop momentum in terms of a particular set of invariants which manifestly cover the entire phase space. These variables also allow us to explicitly continue the integral equation to the forward case $t=0$.

II. CHEW-GOLDBERGER-LOW EQUATION

Our starting point will be the Chew-Goldberger-Low (CGL) equation[1,5] for Regge multiperipheral dynamics where now arbitrary Toller angle dependence is assumed for the double Regge coupling $\beta(t',\omega',t'')$. The B amplitude introduced by CGL is related to the elastic two-body absorptive part $A(p,p_0;Q)$ for

$$(p+\tfrac{1}{2}Q)+(p_0-\tfrac{1}{2}Q)\to(p-\tfrac{1}{2}Q)+(p_0+\tfrac{1}{2}Q)$$

by

$$A(p,p_0;Q)=\int\frac{d^4p'}{(2\pi)^3}\delta^+((p-p')^2-\mu^2)\times g(t_+,t_+';t_-,t_-')B(p,p',p_0;Q).\quad(1)$$

We use the CGL equation for B with a double Regge coupling and propagator function $G(t_\pm',\omega_\pm',t_\pm'')=\beta^*(t_-',\omega_-',t_-'')\beta(t_+',\omega_+',t_+'')$, which is now assumed to depend on the Toller angles $\omega_\pm'$ [in contrast to Eq. (2) of Ref. 3]:

$$B(p,p',p_0;Q)=B_0(p,p',p_0;Q)+\int\frac{d^4p''}{(2\pi)^3}\delta^+((p-p')^2-\mu^2)G(t_\pm',\omega_\pm',t_\pm'')\times(\Sigma/\mu^2)^{\alpha(t_+')+\alpha(t_-')}B(p',p'',p_0;Q)\quad(2)$$

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[1] G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters **22**, 208 (1969).

[2] G. F. Chew and C. DeTar, Phys. Rev. **180**, 1577 (1969); I. G. Halliday and L. M. Saunders, Nuovo Cimento **60**, 494 (1969); A. H. Mueller and I. Muzinich, Ann. Phys. (N. Y.) (to be published); M. Ciafaloni, C. DeTar, and M. Misheloff, Phys. Rev. **188**, 2522 (1969); A. H. Mueller and I. J. Muzinich, Brookhaven Report No. BNL-13836 (unpublished).

[3] M. L. Goldberger, C.-I Tan, and J. M. Wang, Phys. Rev. **184**, 1920 (1969). We use a slightly different notation, $\mathcal{A}(p,p_0;Q)$, for the absorptive amplitude of the reaction Reggeon$(p+\frac{1}{2}Q)$+particle$(p_0-\frac{1}{2}Q)\to$ Reggeon$(p-\frac{1}{2}Q)$+particle$(p_0+\frac{1}{2}Q)$, while reserving $A(p,p_0;Q)$ for the physical on-shell absorptive amplitude.

[4] C.-I Tan and J. M. Wang, Phys. Rev. **185**, 1899 (1969).

[5] M. L. Goldberger, Erice Summer School, 1969 (unpublished), is a thorough and stimulating presentation of the integral-equation approach to multiperipheral dynamics.

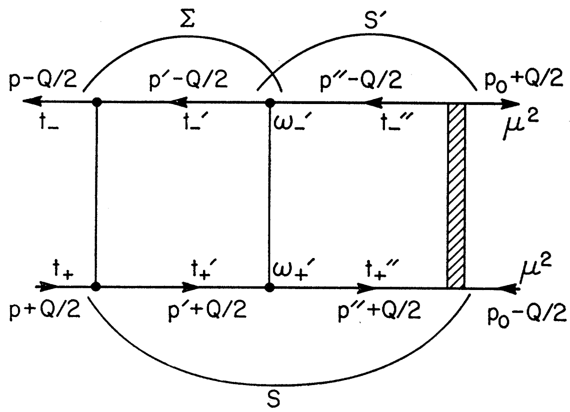

FIG. 1. Kinematic notation for nonforward multiperipheral diagram.

and

$$B_0(p,p',p_0;\,Q)=\pi g(t_+',\mu^2;\,t_-',\mu^2)\times(s/\mu^2)^{\alpha(t_+')+\alpha(t_-')}\delta^+((p'+p_0)^2-\mu^2). \quad (3)$$

We use the invariants (see Fig. 1) $t\equiv Q^2$, $t_\pm\equiv(p\pm\frac{1}{2}Q)^2$, $t_\pm'\equiv(p'\pm\frac{1}{2}Q)^2$, $t_\pm''\equiv(p''\pm\frac{1}{2}Q)^2$, $s\equiv(p+p_0)^2$, $s'\equiv(p'+p_0)^2$, $s''\equiv(p''+p_0)^2$, and $\Sigma\equiv(p-p'')^2$. Toller angles ω_+', ω_-' are spatial angles between planes formed by $(\mathbf{p}\pm\frac{1}{2}\mathbf{Q},\,\mathbf{p}-\mathbf{p}')$ and $(\mathbf{p}_0\mp\frac{1}{2}\mathbf{Q},\,\mathbf{p}''+\mathbf{p}_0)$ in the rest frame of the 4-vector $(p'-p'')$.

III. INTEGRAL EQUATION FOR NONFORWARD REGGEON-PARTICLE ABSORPTIVE AMPLITUDE

The basic approximations leading from the CGL equation to the simplified equation for the Reggeon-particle absorptive amplitude are the kinematic relations

$$\Sigma s'/s=f(t_+',\omega_+',t_+'')$$

and

$$\Sigma s'/s=f(t_-',\omega_-',t_-''),$$

where

$$f(t',\omega',t'')=\frac{\Delta(t',t'',\mu^2)}{\mu^2-t'-t''+2(t't'')^{1/2}\cos\omega'} \quad (4)$$

and

$$\Delta(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx.$$

Equation (4) is valid in the multi-Regge region of interest where s, s', and Σ are large compared to masses and t's, and where by strong ordering[6] $s',\Sigma\ll s$. Substituting these relations into the CGL equation, one finds

$$B(p,p',p_0;\,Q)=B_0(p,p',p_0;\,Q)+(s/s')^{\alpha(t_+')+\alpha(t_-')}\mathfrak{A}_0, \quad (5)$$

where

$$\mathfrak{A}_0=\int\frac{d^4p''}{(2\pi)^3}\delta^+((p'-p'')^2-\mu^2)G(t_\pm',\omega_\pm',t_\pm'')\times[f(t_-',\omega_-',t_-'')]^{\alpha(t_-')}[f(t_+',\omega_+',t_+'')]^{\alpha(t_+')}\times B(p',p'',p_0;\,Q). \quad (6)$$

[6] F. Zachariasen and G. Zweig, Phys. Rev. 160, 1322 (1967).

Because of the presence of $\omega_\pm'$, which are functions of p, p', p'', p_0, and Q, this quantity $\mathfrak{A}_0$ will in general depend on all vectors p, p', p_0, and Q. We will show in Sec. IV that under the same conditions leading to Eq. (4) the Toller angles $\omega_\pm'$ can be expressed as functions of only t_+', t_-', t_+'', t_-'', and t. Because of this, $\omega_\pm'$ will not depend on p and this will also be true of the functions $f(t_\pm',\omega_\pm',t_\pm'')$ and $G(t_\pm',\omega_\pm',t_\pm'')$. Hence we can rewrite Eq. (6) as

$$\mathfrak{A}_0(p',p_0;\,Q)=\int\frac{d^4p''}{(2\pi)^3}\delta^+((p'-p'')^2-\mu^2)\times\tilde{G}(t_\pm',t_\pm'',t)B(p',p'',p_0;\,Q), \quad (7)$$

where

$$\tilde{G}(t_\pm',t_\pm'',t)\equiv G(t_\pm',\omega_\pm',t_\pm'')\times[f(t_+',\omega_+',t_+'')]^{\alpha(t_+')}[f(t_-',\omega_-',t_-'')]^{\alpha(t_-')}. \quad (8)$$

This will now allow us to derive an integral equation for the nonforward Reggeon-particle absorptive amplitude with general double Regge coupling $\beta(t',\omega',t'')$. Defining the single-particle intermediate-state contribution, $\mathcal{A}_1(p,p_0;\,Q)=\pi g(t_+,\mu^2;\,t_-,\mu^2)\delta^+(s-\mu^2)$, the general integral equation for the $\mathcal{A}$ amplitude is

$$\mathcal{A}(p',p_0;\,Q)=\mathcal{A}_1(p',p_0;\,Q)+\int\frac{d^4p''}{(2\pi)^3}\delta^+((p'-p'')^2-\mu^2)\times G(t_\pm',t_\pm'',t)(s'/s'')^{\alpha(t_+'')+\alpha(t_-'')}\mathcal{A}(p'',p_0;\,Q), \quad (9)$$

where

$$\mathcal{A}(p',p_0;\,Q)\equiv\mathcal{A}_1(p',p_0;\,Q)+\mathfrak{A}_0(p',p_0;\,Q). \quad (10)$$

The loop integration will be expressed in terms of invariant variables after our discussion of the Toller angle.

IV. RELATION OF TOLLER ANGLES TO KINEMATIC INVARIANTS

We now show that the angles $\omega_\pm'$ depend only on $t_\pm'$, $t_\pm''$, and t, in the kinematic region in which Eq. (4) is valid. Because the angles $\omega_\pm'$ may be found from Eq. (4) in terms of $t_\pm'$, $t_\pm''$, and $\Sigma s'/s$, we need another relation between $\Sigma s'/s$ and $t_\pm'$, $t_\pm''$, t. The proof of our assertion depends crucially on the fact that $t=Q^2\neq0$. In this case, there are five momentum vectors in our problem: p_0, p, p', p'', and Q. Since they are 4-vectors, they cannot all be linearly independent. This may be expressed by the vanishing of the determinant of the 5×5 matrix of their scalar products. In the regions of interest where s, s', and Σ are large compared with masses and t's, the vanishing of the determinant to leading order gives us the relation for $\Sigma s'/s$:

$$(\Sigma s'/s)^2+2b(\Sigma s'/s)+c=0, \quad (11)$$

where coefficients b and c are functions of t_+', t_-', t_+'', t_-'', and t. Equation (11) yields two solutions:

$$\Sigma s'/s=\mu^2-\tfrac{1}{2}(t_+'+t_-'-\tfrac{1}{2}t)-\tfrac{1}{2}(t_+''+t_-''-\tfrac{1}{2}t)+(t_+'-t_-')(t_+''-t_-'')/t \pm[\Delta(t_+',t_-',t)\Delta(t_+'',t_-'',t)/4t^2]^{1/2}. \quad (12)$$

Equation (12), together with Eq. (4), allows us to solve for ω_+' and ω_-' in terms of $t_\pm'$, $t_\pm''$, and t and this completes our proof. The existence of two solutions indicates that special care has to be taken in transforming the loop-momentum integration in Eqs. (2) or (9) to invariant variables.[7]

V. TRANSFORMATION TO INVARIANT VARIABLES

The two roots of Eq. (12) indicate that a given set of values for the variables t_+'', t_-'', s'' corresponds to two values of p'' (p, p', Q being fixed). This results in two possible values of $\Sigma\equiv(p-p'')^2$. We will show how this nonuniqueness can be removed by making use of our knowledge that the physical ranges of ω_+' and ω_-' are $[0,2\pi]$. An alternative method is to exhibit the components of p'' in the center-of-mass frame in terms of the invariants; this is carried out in the Appendix.

We introduce the invariant variables,

$$\beta'\equiv\frac{p'\cdot Q}{(-t)^{1/2}}=\frac{t_+'-t_-'}{2(-t)^{1/2}},\quad \beta''\equiv\frac{p''\cdot Q}{(-t)^{1/2}}=\frac{t_+''-t_-''}{2(-t)^{1/2}},$$

$$\alpha'\equiv(-p'^2-\beta'^2)^{1/2}\equiv\left(\frac{\Delta(t,t_+',t_-')}{4t}\right)^{1/2}, \quad (13)$$

$$\alpha''\equiv(-p''^2-\beta''^2)^{1/2}=\left(\frac{\Delta(t,t_+'',t_-'')}{4t}\right)^{1/2},$$

where α' and α'' are taken positive. We note immediately that

$$t_\pm'=-\alpha'^2-[\beta'\mp\tfrac{1}{2}(-t)^{1/2}]^2,$$
$$t_\pm''=-\alpha''^2-[\beta''\mp\tfrac{1}{2}(-t)^{1/2}]^2, \quad (14)$$

and they are even functions of α', α''. We work in the region $\Delta(t,t_+,t_-)<0$, $t<0$, which guarantees in the integral equation that $\Delta(t,t_+',t_-')<0$ and $\Delta(t,t_+'',t_-'')<0$.[5] In terms of our new variables, Eq. (12) reads

$$\Sigma s'/s=\mu^2+(\alpha'\pm\alpha'')^2+(\beta'-\beta'')^2. \quad (15)$$

In order to cover the entire physical range in $\cos\omega_\pm'$ with α' and α'' positive, we must use both branches of Eq. (15). This is, however, equivalent to choosing, say, the branch with the minus sign, and allowing α' and α'' to take negative as well as positive values.

To verify these statements, we look, for example, in a two-dimensional plane of α'', $\beta''-\frac{1}{2}(-t)^{1/2}$. From Eq. (4) we find that the values of $|\cos\omega_+'|=1$ occur along the line through the origin $\alpha''/[\beta''-\frac{1}{2}(-t)^{1/2}]=\alpha'/[\beta'-\frac{1}{2}(-t)^{1/2}]$, with $\cos\omega_+'=+1$ for the segment with $\text{sgn}(\alpha'')=\text{sgn}(\alpha')$ and $\cos\omega_+'=-1$ when $\text{sgn}(\alpha'')=-\text{sgn}(\alpha')$. In this plane, fixed t_+'' corresponds to a circle about the origin, and to cover all values of ω_+' clearly requires α'' to be both positive and negative.

Now we rewrite Eq. (10) in terms of the newly defined variables:

$$\mathcal{A}(p',p_0;Q)=\mathcal{A}_1(p',p_0;Q)+\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d\alpha''\int_{-\infty}^{\infty}d\beta''\int_{\mu^2}^{s_-''}ds'' \times J(s',s'';\alpha'^2,\alpha''^2,\beta',\beta'')\tilde{G}(\alpha',\alpha'',\beta',\beta'',t) \times(s'/s'')^{\alpha(t_+'')+\alpha(t_-'')}\mathcal{A}(p'',p_0;Q), \quad (16)$$

where[5]

$$J(s',s'';\alpha'^2,\alpha''^2,\beta',\beta'')=\tfrac{1}{2}(\alpha''^2)^{1/2}(\alpha''^2s'^2-\rho s's''+\alpha'^2s''^2)^{-1/2},$$
$$\rho=(\beta'-\beta'')^2+(\alpha'^2+\alpha''^2)+\mu^2, \quad (17)$$

and

$$s_-''=\left(\frac{\rho-(\rho^2-4\alpha'^2\alpha''^2)^{1/2}}{2\alpha'^2}\right)s'.$$

Despite the double valuedness of the variables α'', β'' with respect to t_+'', t_-'', one can write the equation with variables t_+'', t_-'' because the solution $\mathcal{A}(p',p_0;Q)$ is a function of α'^2. This follows from the fact that $\mathcal{A}_1$ and J depend on α'^2 and the residue has the property

$$\tilde{G}(\alpha',\alpha'',\beta',\beta'',t)=\tilde{G}(-\alpha',-\alpha'',\beta',\beta'',t). \quad (18)$$

Using the symmetrized residue

$$G_s(\alpha'^2,\alpha''^2,\beta',\beta'',t)=\tfrac{1}{2}[\tilde{G}(\alpha',\alpha'',\beta',\beta'',t)+\tilde{G}(-\alpha',\alpha'',\beta',\beta'',t)], \quad (19)$$

the equation is written in the conventional form[3,5]:

$$\mathcal{A}(t_\pm';s',t)=\mathcal{A}_1(t_\pm';s',t)+\frac{1}{4(2\pi)^3}\int dt_+''dt_-''\theta(-\Delta(t,t_+'',t_-'')) \times\int_{\mu^2}^{s_-''}\frac{ds''}{s'}\frac{1}{L^{1/2}}G_s(t_\pm',t_\pm'',t) \times\left(\frac{s'}{s''}\right)^{\alpha(t_+'')+\alpha(t_-'')}\mathcal{A}(t_\pm'';s'',t), \quad (20)$$

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[7] We wish to thank Carlton DeTar for emphasizing this point. This question also arose in the group-theoretical treatment of the nonforward multi-Regge integral equation. (See the last two articles of Ref. 2.)

where

$$L^{1/2}=[(-t\alpha''^2)^{1/2}/2s']\times J^{-1}. \tag{21}$$

VI. FORWARD INTEGRAL EQUATION

The continuation of the integral equation to the forward direction, $t=0$, can be made directly in terms of the new variables. This is because the variables α'', β'' are just components of momenta which do not vanish at $t=0$ (see Appendix). In the forward direction the variables become

$$t_+''=t_-''=-\alpha''^2-\beta''^2\equiv t'',$$
$$t_+'=t_-'=-\alpha'^2-\beta'^2\equiv t',$$
$$\cos\omega_+'=\cos\omega_-'\equiv\cos\omega',$$

where $\cos\omega'$ is given by

$$\frac{\Sigma s'}{s}=\mu^2+(\alpha'-\alpha'')^2+(\beta'-\beta'')^2$$
$$=\frac{\Delta(t',t'',\mu^2)}{\mu^2-t'-t''+2(t't'')^{1/2}\cos\omega'}. \tag{22}$$

In terms of the α'', β'' variables, the Jacobian and integration limits of the integral equation (16) are not singular at $t=0$.

In applications of the forward integral equation it is convenient to integrate over s'', t'', and ω', so we now transform to these variables. This is facilitated by the transformation to polar coordinates in the α'', β'' plane with the ϕ' angle given by

$$\cos\phi'=\frac{\alpha'\alpha''+\beta'\beta''}{(-t')^{1/2}(-t'')^{1/2}}, \tag{23}$$

and

$$d\alpha''d\beta''=\tfrac{1}{2}d(-t'')d\phi'.$$

(The definition of ϕ' as a physical angle is given in the Appendix.) The forward Reggeon-particle absorptive part is given by the integral equation

$$\mathcal{A}(t';s')=\mathcal{A}_1(t';s')+\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dt''\int_0^{2\pi}d\phi'\int_{\mu^2}^{s'}\frac{ds''}{4s'}$$
$$\times\tilde{G}(t',t'';\cos\phi')(s'/s'')^{2\alpha(t'')}\mathcal{A}(t'';s''), \tag{24}$$

where the Jacobian has been approximated for $s''/s'\ll 1$. This form of the forward equation has been discussed by Low.[8] To transform from $d\phi'$ to $d\omega'$ we note the reciprocal relation of $\cos\phi'$ to $\cos\omega'$ from Eq. (22):

$$[\mu^2-t'-t''-2(t't'')^{1/2}\cos\phi'][\mu^2-t'-t''+2(t't'')^{1/2}\cos\omega']=\Delta(t',t'',\mu^2)$$

and

$$d\phi'=d\omega' f(t',\omega',t'')/\Delta^{1/2}(t',t'',\mu^2). \tag{25}$$

[8] F. Low, Brookhaven Report No. BNL 50162, 1969 (unpublished).

The forward integral equation is now rewritten as

$$\mathcal{A}(t';s')=\mathcal{A}_1(t';s')$$
$$+\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dt''\int_0^{2\pi}d\omega'\int_{\mu^2}^{s'}\frac{ds''}{4s'}\frac{f(t',\omega',t'')}{\Delta^{1/2}(t',t'',\mu^2)}$$
$$\times\tilde{G}(t',t'';\cos\omega')(s'/s'')^{2\alpha(t'')}\mathcal{A}(t'';s''). \tag{26}$$

This version of the forward integral equation has been formulated and its properties studied by Pinsky and Weisberger.[9]

VII. CONCLUSION

This integral equation for $\mathcal{A}(p',p_0;Q)$ may be regarded as a simplified multi-Regge model or as a large s approximation of the CGL equation. The treatment of the Toller-angle dependence in the $\mathcal{A}$ equation is now equally general as that of the CGL equation. The $\mathcal{A}$ equation seems to have all the essential physical content of the CGL equation and to have the additional advantage of being easier to work with. Following the analysis of Refs. 3 and 5, one finds Regge behavior for $\mathcal{A}(p',p_0;Q)$. Applications of this equation are discussed elsewhere.[10,11]

Note added in proof. After the completion of this work we were informed that the nonforward integral equation has also been formulated and studied by S. Pinsky and W. Weisberger, Princeton report (unpublished).

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We would like to thank M. L. Goldberger, J. M. Wang, and W. Weisberger for several valuable discussions.

APPENDIX

Although the negative values of α' and α'' have only been introduced formally, we shall show that they have well-defined meaning as components of the vectors p' and p'' at high energy.

We shall discuss the problem in the c.m. frame of $p+\frac{1}{2}Q$ and $p_0-\frac{1}{2}Q$. Let $p=-p_0$ be in the $\hat{z}$ direction, and let $\perp$ denote the projection of a vector on the x-y plane. It can be shown that

$$|Q_\perp^2|=-t+O(1/s^2),\quad Q_0=O(1/\sqrt{s}),\quad Q_z=O(1/\sqrt{s}),$$

and

$$|\mathbf{p}|\simeq|\mathbf{p}_0|\simeq p_0^0\simeq p^0=\tfrac{1}{2}\sqrt{s}+O(1/\sqrt{s}). \tag{A1}$$

It then follows that in the multi-Regge strong-ordering limit of Σ, $s'\ll s$; $s''<s'$:

$$t_\pm''=(p''\pm\tfrac{1}{2}Q)^2=-(p''\pm\tfrac{1}{2}Q)_\perp^2+O(\Sigma^2/s^2),$$
$$t_\pm'=(p'\pm\tfrac{1}{2}Q)^2=-(p'\pm\tfrac{1}{2}Q)_\perp^2+O(s'^2/s^2). \tag{A2}$$

[9] The treatment of the Toller angle and the integral equations at $t=0$ has been investigated by S. Pinsky and W. Weisberger, Phys. Rev. D (to be published).

[10] C.-I Tan and J. M. Wang, Phys. Rev. Letters **22**, 1152 (1969).

[11] D. Silverman and C.-I Tan, Phys. Rev. D (to be published).

We choose $Q_\perp$ to lie in the $\hat{x}$ direction so that α'', β'' are related to the components of p'' by

$$\beta''=\frac{p''\cdot Q}{(-t)^{1/2}}\cong-\frac{p_\perp''\cdot Q_\perp}{(-t)^{1/2}}=-p_x'' \tag{A3}$$

and, by using Eqs. (13) and (14),

$$\alpha''=p_y''.$$

Similarly,

$$\beta'=-p_x',\quad \alpha'=p_y'.$$

We see that it is necessary to take limits of $(-\infty,+\infty)$ for α'', β'' in order to cover the entire phase space.

Since the variables are related to components of vectors, it is interesting to introduce the angle ϕ' in the c.m. frame defined by

$$\cos\phi_+'=\frac{-\epsilon_{\mu\nu\rho\sigma}(p+\frac{1}{2}Q)^\nu(p_0-\frac{1}{2}Q)^\rho(p'+\frac{1}{2}Q)^\sigma\epsilon^\mu{}_{\lambda\tau\delta}(p+\frac{1}{2}Q)^\lambda(p_0-\frac{1}{2}Q)^\tau(p''+\frac{1}{2}Q)^\delta}{|\epsilon_{\mu\nu\rho\sigma}(p+\frac{1}{2}Q)^\nu(p_0-\frac{1}{2}Q)^\rho(p'+\frac{1}{2}Q)^\sigma|\,|\epsilon_{\mu\lambda\tau\delta}(p+\frac{1}{2}Q)^\lambda(p_0-\frac{1}{2}Q)^\tau(p''+\frac{1}{2}Q)^\delta|}. \tag{A4}$$

At large s, Σ, s' this becomes

$$\cos\phi_+'=\frac{1}{2(t_+'t_+'')^{1/2}}\left[\frac{s'\Sigma}{s}-(\mu^2-t_+'-t_+'')\right]$$
$$=\frac{\alpha'\alpha''+[\beta'-\frac{1}{2}(-t)^{1/2}][\beta''-\frac{1}{2}(-t)^{1/2}]}{(-t_+')^{1/2}(-t_+'')^{1/2}}. \tag{A5}$$

Using (A1) and (A3),

$$\cos\phi_+'=\frac{(p_\perp'+\frac{1}{2}Q_\perp)\cdot(p_\perp''+\frac{1}{2}Q_\perp)}{|p_\perp'+\frac{1}{2}Q_\perp|\,|p_\perp''+\frac{1}{2}Q_\perp|}. \tag{A6}$$

By changing $Q\rightarrow -Q$ in (A4), we also define $\cos\phi_-'$ and at large s, Σ, s',

$$\cos\phi_-'=\frac{1}{2(t_-'t_-'')^{1/2}}\left[\frac{s'\Sigma}{s}-(\mu^2-t_-'-t_-'')\right]$$
$$=\frac{\alpha'\alpha''+[\beta'+\frac{1}{2}(-t)^{1/2}][\beta''+\frac{1}{2}(-t)^{1/2}]}{(-t_-')^{1/2}(-t_-'')^{1/2}}$$
$$=\frac{(p_\perp'-\frac{1}{2}Q_\perp)\cdot(p_\perp''-\frac{1}{2}Q_\perp)}{|p_\perp'-\frac{1}{2}Q_\perp|\,|p_\perp''-\frac{1}{2}Q_\perp|}. \tag{A7}$$

We find that $\phi_\pm'$ are complementary angles to $\omega_\pm'$ by using (A5), (A7), and (4):

$$\mu^2-t_\pm'-t_\pm''-2(t_\pm't_\pm'')^{1/2}\cos\phi_\pm'=f(t_\pm',\omega_\pm',t_\pm'')$$
$$=\frac{\Delta(\mu^2,t_\pm',t_\pm'')}{\mu^2-t_\pm'-t_\pm''+2(t_\pm't_\pm'')^{1/2}\cos\omega_\pm'}. \tag{A8}$$