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FINAL REPORT TO
AEROJET-GENERAL CORPORATION
SOLID ROCKET PLANT
SACRAMENTO, CALIFORNIA

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FOREWORD

This report was prepared by the Institute of Engineering Research, University of California, Berkeley, for Aerojet-General Corporation, Solid Rocket Plant, Sacramento, California under Purchase Order S-605838-OP. The work forms part of a general study of mechanical properties of solid rocket propellants under Contract AF33(600)-40314 S.A. No. 1, under the direction of Dr. J. H. Wiegand, Head, Mechanical and Ballistic Properties Laboratory.

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ABSTRACT

Part I: Elastic and Viscoelastic Analysis of Orthotropic Cylinders

The equations for linear orthotropic elasticity are presented with applications to the thermoelastic behavior of thick-walled cylinders. Examples are given to illustrate (1) the quantitative effect of orthotropy and (2) a step by step technique for solving bilinear elastic cylinders. A solution method is given for orthotropic viscoelastic cylinders. This method does not depend on the anisotropic correspondence principle. Techniques for asymptotic solutions are discussed. A solution for the deformation of a nonlinear elastic orthotropic cylinder subjected to internal and external pressure is also presented.

Part II: Bilinear Elasticity with Applications to Thick-Walled Cylinders

A general bilinear elastic theory is developed. Representation of solid propellant mechanical properties by a bilinear model is considered. The ability of the theory to approximate the behavior of a typical propellant is investigated insofar as possible with limited experimental evidence. Analytical solutions to several problems are presented and numerical results are given.

Part III: Solution Method for Nonlinear Elasticity Problems with Applications to Thick-Walled Cylinders

A solution scheme is presented for second order elastic and thermoelastic problems. To illustrate the solution method three problems occurring in solid rocket motor design are extensively treated, i.e., pressurization, temperature effects and vertical slump of a thick-walled cylinder. To enable one to obtain material properties in a simple manner the uniaxial test is analyzed. The solutions are in terms of a perturbation series; for a second order theory, only the first two terms are needed. The homogeneous portion of the system of equations governing the behavior of each term in the series is identical to the classical elastic equations and the nonhomogeneous portion is a nonlinear function of the previous term. The method is extended to include those problems for which only an approximate solution to the classical problem is available.

Three classes of material behavior are considered; compressible, incompressible and near-incompressible. The latter class of material behavior is considered in order to refine the assumption of incompressibility and to be able to examine the validity of the incompressibility assumption.

In order to be able to consider thermal effects for a mechanically incompressible material a derivation of the constitutive equations for such a material including thermal effects is presented.

Part IV: Thermal Deformation of Viscoelastic Materials

The effect of temperature on linear viscoelastic stress analysis is investigated for materials which have a single time-temperature equivalence function. Such materials have been classified as thermorheologically simple by Schwarzl and Staverman.

Starting from the constitutive equations in integral form and modified for the effects of temperature, the displacement equations of equilibrium are derived in orthogonal curvilinear coordinates. The equations are then specialized for axisymmetric temperatures which are constant in the axial direction. Due to nonhomogeneity in the radial direction, solutions to this class of problems may not be readily obtained, consequently, subsequent analysis concerns the solution of problems with uniform temperatures varying only in time. The time dependence is shown to be obtained by solving a Volterra integral equation. The numerical technique of solution proposed by Lee and Rogers is introduced and numerical examples are presented for a hollow cylinder rigidly encased at the outer boundary and stress-free at the inner boundary. Finally, the solution of an infinite cylinder bonded to a thin elastic case is formulated in terms of a Volterra integral equation, again restricted to space independent temperatures.

TABLE OF CONTENTS

INTRODUCTION

The continuing increase in the size of solid propellant motors has placed a concomitant demand upon the analyst to develop suitable techniques for predicting grain structural integrity. In the absence, at present, of a general continuum theory of mechanical behavior which is both realistic and tractable, it has been necessary to attempt to classify grain structural integrity problems into categories and treat each class of problems with less general, yet reasonably satisfactory methods (in the engineering sense). To give examples we cite the following.

In recent years, an extensive propellant-centered literature in linear temperature-independent viscoelastic theory has arisen. Recently, work on linear thermoviscoelasticity has begun to appear. At the same time the importance of nonlinear behavior has been recognized, and Rivlin-type elastic theory has been applied to grain analysis. More recently recognition has been given to the necessity for taking into account the initiation of strain-induced anisotropy resulting from the dewetting phenomenon. Finally, the importance of introducing appropriate failure criteria, as a companion to stress and deformation analysis, has been noted.

This report, consisting of four parts, is but another contribution to the "categorized" treatment of solid propellant mechanics. It is hoped, however, that the ideas presented will stimulate the development of a suitable nonlinear, nonhomogeneous, anisotropic viscoelastic theory for propellant structural analysis.

PART I

ELASTIC AND VISCOELASTIC ANALYSIS OF ORTHOTROPIC CYLINDERS

by

S. B. DONG

INTRODUCTION

Stress analysis of cylindrical grains within the framework of linear isotropic elasticity and viscoelasticity has received considerable attention to date. An extensive body of information of this type has been reported by Williams, Blatz, and Schapery [1]. It is well-known, however, that filled propellants evince substantially different mechanical behavior in the presence of tensile stress fields than is found in compressive stress fields. This effect is a result of the presence of voids and the pullaway of the binder from the filler particles. Accordingly, a type of stress-induced anisotropy is developed in the propellant, necessitating consideration of anisotropic constitutive equations. The general problem involves the solution of boundary value problems for each subdomain of the body, defined by a particular state of stress, and the subsequent piecing-together of the solutions at common interfaces. In each instance the solution required will be that appropriate to an anisotropic body. The degree of stress-induced anisotropy encountered is at most orthotropy, thus the general discussion throughout this report will be restricted to this form of anisotropy. The analysis of bodies with a greater degree of anisotropy has been presented in [2]. Much of the present work has been drawn from Lekhnitskii [3] who, in addition to his own contributions, has summarized previous work in the field.

In view of the viscoelastic behavior of solid propellants under many circumstances, attention is drawn to the correspondence principle, first proposed by Alfrey for incompressible isotropic media [4] and generalized by Lee [5]. This principle was later extended for anisotropic bodies by Biot [6]. The forms of solution for many associated anisotropic elastic problems, however, are not readily invertible to recover the time-dependent viscoelastic response, although the inversion of the solution for an orthotropic thick-walled cylinder has been demonstrated by Spillers [7]. An alternative method of solution is to begin with the viscoelastic field equations and formulate a governing equation in both space and time. This governing partial differential equation may then be solved by a suitable technique. The investigation of an orthotropic cylinder following this approach is discussed.

Due to a characteristically low rigidity in solid propellants, large deformations may be sustained under loading and environmental conditions. Stress analysis of solids based on a non-linear theory must be adopted to account for these large deformations. A solution for the pressurization of an elastic orthotropic thick-walled cylinder is presented to illustrate the particular features of such an analysis.

1. Recapitulation of the Linear Thermoelastic Field Equations for Orthotropic Solids with Temperature-Independent Material Properties

Since many solid propellant configurations involve cylindrical geometries, the fundamental thermoelastic field equations will be summarized in cylindrical coordinates.

Stress Equations of Equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \Theta = 0 \qquad (1.1)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0$$

where R, Θ, and Z are body force components per unit of volume.

Strain-Displacement Relations

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \qquad (1.2)$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

where u, v, w are the components of the displacement vector in the r, θ, z directions.

Strain Compatibility Equations

$$\left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right]\epsilon_r + \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\epsilon_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r} + \frac{1}{r}\right]\gamma_{r\theta}$$

$$\frac{\partial^2\epsilon_r}{\partial z^2} + \frac{\partial^2\epsilon_z}{\partial r^2} = \frac{\partial^2\gamma_{rz}}{\partial r\partial z}$$

$$\frac{\partial^2\epsilon_\theta}{\partial z^2} + \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\epsilon_z = \frac{1}{r}\frac{\partial^2\gamma_{\theta z}}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial\gamma_{rz}}{\partial z}$$

$$\frac{2}{r}\frac{\partial^2\epsilon_r}{\partial\theta\partial z} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left[\frac{\partial\gamma_{r\theta}}{\partial z} + \frac{1}{r}\frac{\partial\gamma_{rz}}{\partial\theta} - \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\gamma_{\theta z}\right] \qquad (1.3)$$

$$+ \left[\frac{2}{r}\frac{\partial}{\partial r} + \frac{2}{r^2}\right]\gamma_{\theta z} - \frac{2}{r^2}\frac{\partial\gamma_{rz}}{\partial\theta}$$

$$2\left[\frac{\partial^2\epsilon_\theta}{\partial r\partial z} + \frac{1}{r}\frac{\partial}{\partial\theta}(\epsilon_\theta - \epsilon_r)\right] = \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial\gamma_{r\theta}}{\partial z} - \frac{1}{r}\frac{\partial\gamma_{rz}}{\partial\theta} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\gamma_{\theta z}\right]$$

$$\frac{2}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]\epsilon_z = \frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\gamma_{\theta z} + \frac{1}{r}\frac{\partial\gamma_{rz}}{\partial\theta} - \frac{\partial\gamma_{r\theta}}{\partial z}\right]$$

The constitutitive equation for a solid with cylindrical orthotropy written in matrix form* is

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* The individual matrices in Eq. (1.4) have been partitioned. The null submatrices in the off diagonal positions of the S_{ij} matrix indicate that extensional effects occur independently of shearing effects, i.e. they are uncoupled. For more general forms of anisotropy such coupling does occur, see, for example, [12].

$$\begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \hline \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{array}\right] \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \hline \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} + \begin{bmatrix} \alpha T \\ \alpha T \\ \alpha T \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.4}$$

The symmetric S_{ij} matrix defines the elastic compliances of the material. Thermal effects are accounted for by αT, where α is the coefficient of thermal expansion and T is the temperature change in the solid. Thermal isotropy has been assumed. The temperature function T must satisfy the heat conduction equation for a given problem.

It is sometimes convenient to deal with the inverse form of Eq. (1.4) given by

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \hline \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{array}\right] \begin{bmatrix} \epsilon_r - \alpha T \\ \epsilon_\theta - \alpha T \\ \epsilon_z - \alpha T \\ \hline \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \tag{1.5}$$

The C_{ij} in Eq. (1.5) are called the elastic moduli of the material and are related to the S_{ij} by a matrix inverse.

$$\left[C_{ij}\right] = \left[S_{ij}\right]^{-1} \tag{1.6}$$

Boundary conditions will be discussed in connection with specific problems.

2. Axisymmetrically Loaded Cylinders

Consider a finite hollow circular cylinder whose inner and outer radii are a and b, respectively. In the instance this cylinder is loaded by a system of forces and subjected to a temperature change, both of which are independent of the generatrix, the three dimensional problem reduces to one of plane strain. If, further, the cylinder is loaded axisymmetrically, the dependent variables become independent of θ and the governing equations become ordinary differential equations in the variable r. In the case of plane strain ϵ_z may be taken as a constant:

$$\epsilon_z = K \text{ (a constant)} \tag{2.1}$$

Therefore from the strain-stress relations (1.4) there results

$$\sigma_z = \frac{1}{S_{33}}\left[K - S_{13}\sigma_r - S_{23}\sigma_\theta - \alpha T\right] \tag{2.2}$$

The remaining normal components of strain become

$$\begin{aligned} \epsilon_r &= \beta_{11}\sigma_r + \beta_{12}\sigma_\theta + \left(1 - \frac{S_{13}}{S_{33}}\right)\alpha T + \frac{S_{13}}{S_{33}} K \\ \epsilon_\theta &= \beta_{12}\sigma_r + \beta_{22}\sigma_\theta + \left(1 - \frac{S_{23}}{S_{33}}\right)\alpha T + \frac{S_{23}}{S_{33}} K \end{aligned} \tag{2.3}$$

where

$$\beta_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}} \qquad (i,j = 1,2) \tag{2.4}$$

The relevant equilibrium and compatibility equations for torsionless axisymmetry take the following form

$$\sigma_\theta = \frac{d}{dr}(r\sigma_r) \tag{2.5}$$

$$\epsilon_r = \frac{d}{dr}(r\epsilon_\theta) \tag{2.6}$$

From Eqs. (2.5) and (2.6) with the help of Eq. (2.3), a differential equation in terms of σ_r can be formulated.

$$\beta_{22}\, r^2 \frac{d^2\sigma_r}{dr^2} + 3\beta_{22}\, r \frac{d\sigma_r}{dr} + (\beta_{22} - \beta_{11})\sigma_r$$

$$= \left[\frac{S_{13}}{S_{33}} - 1\right]\alpha r \frac{dT}{dr} + \left[\frac{S_{13} - S_{23}}{S_{33}}\right]\alpha T + \left[\frac{S_{13} - S_{23}}{S_{33}}\right] K \tag{2.7}$$

The boundary conditions for Eq. (2.7) are

$$\sigma_r(a) = h_1$$
$$\sigma_r(b) = h_2 \tag{2.8}$$

where h_1 and h_2 are prescribed constants.

By using Eq. (2.1) in the stress-strain relations (1.5) and the strain-displacement relations which, in the case of torsionless axisymmetry, are

$$\epsilon_r = \frac{du}{dr}$$
$$\epsilon_\theta = \frac{u}{r} \tag{2.9}$$

a governing equation in terms of the displacement u may be obtained:

$$C_{11}\frac{d^2u}{dr^2} + C_{11}\frac{1}{r}\frac{du}{dr} - C_{22}\frac{u}{r^2} = -\alpha(C_{11} + C_{12} + C_{13})\frac{dT}{dr}$$

$$- \alpha\left[C_{11} + C_{13} - C_{22} - C_{23}\right]\frac{1}{r}T + \left[C_{23} - C_{13}\right]K \qquad (2.10)$$

The boundary conditions are given by the displacement u and the derivative of the displacement u. The exact expression depends on whether a displacement or a stress boundary condition is specified.

When a cylinder is pressurized internally and externally and subjected to an extensional force at the ends, the governing equation (2.7) must be solved with the following boundary conditions: on the lateral surfaces

$$\sigma_r(a) = -p \qquad \sigma_r(b) = -q \qquad (2.11)$$

where p and q are the prescribed values of the applied pressures and on the ends

$$2\pi\int_a^b \sigma_r(r)\,r\,dr = P \qquad \text{(a prescribed value)} \qquad (2.12)$$

The solution to Eq. (2.7) is

$$\sigma_r(r) = C_1 r^{-1-k} + C_2 r^{-1+k} + f(r) + \frac{S_{13} - S_{23}}{S_{33}(\beta_{22} - \beta_{11})}K \qquad (2.13)$$

where

$$k = \sqrt{\frac{\beta_{11}}{\beta_{22}}} \qquad (2.14)$$

and f(r) is the particular solution for the non-homogeneous part of the differential equation for the terms involving T. C_1 and C_2 are constants of integration. For brevity, let

$$F(r) \equiv f(r) + \frac{S_{13} - S_{23}}{S_{33}(\beta_{22} - \beta_{11})} K \tag{2.15}$$

Evaluating C_1 and C_2 from the boundary conditions (2.11) there results

$$C_1 = \frac{[q + F(b)]\, ba^k - [p + F(a)]\, ab^k}{(\frac{b}{a})^k - (\frac{a}{b})^k}$$

(2.16)

$$C_2 = \frac{[p + F(a)]\, ab^{-k} - [q + F(b)]\, ba^{-k}}{(\frac{b}{a})^k - (\frac{a}{b})^k}$$

The remaining components of stress are

$$\sigma_\theta(r) = - C_1 k r^{-1-k} + C_2 k r^{-1+k} + \frac{d}{dr}\left(rF(r)\right) \tag{2.17}$$

$$\sigma_z(r) = \frac{1}{S_{33}}\Big[(kS_{23} - S_{13})C_1 r^{-1-k} - (S_{13} + kS_{23})C_2 r^{-1+k}$$

(2.18)

$$- S_{23} \frac{d}{dr}(rF) - S_{13}F + K - \alpha T\Big]$$

The value K may be found by substituting Eq. (2.18) into the end boundary condition (2.12). If an explicit expression for σ_z is known, the integral may be evaluated, giving an algebraic relationship between K and P. As the temperature field is not given explicitly, no attempt will be made here to obtain a general relationship, since this step of the solution is straight-forward for a given problem.

The displacement u is found from the stress-strain-displacement relations (2.3) and (2.9).

$$u(r) = C_1(\beta_{12} - k\beta_{22})r^{-k} + C_2(\beta_{12} + k\beta_{22})r^{k} + \frac{S_{23}K}{S_{33}} r$$

$$+ \beta_{22}\, r \frac{d}{dr}(rF) + \beta_{12}(rF) + (1 - \frac{S_{23}}{S_{33}}) r\alpha T \qquad (2.19)$$

The appearance of the independent variable r raised to non-integral powers containing the elastic coefficients is noteworthy, particularly with reference to the dependence of the displacement and stress distributions on the elastic coefficients. This, of course, has additional implications with respect to the solution of anisotropic viscoelasticity problems.

In the instance an orthotropic cylinder is subjected to torsion the problem of determining the stress and displacement distributions is exactly the same as that for the isotropic case. The reciprocal of the compliance S_{44} takes the place of the usual isotropic shear modulus. Since the solution of the torsion of cylinders may be found in any standard strength of materials text, no further consideration will be given here as the transition from isotropy to orthotropy is straightforward.

3. Examples

a. Internal Pressurization of a Cylinder of Hexagonal Material

Stress analysis of a thick-walled cylinder with hexagonal material properties was conducted to assess the effect of this particular kind of anisotropy which is characteristic of stress-induced anisotropy for a pressurized propellant cylinder in plane strain. The term hexagonal refers to a special form of orthotropy in which two of the three elastic compliances, S_{11}, S_{22}, and S_{33}, corresponding to radial, tangential, and axial directions, are identical. With this form of elastic symmetry, it is possible to reduce the number of independent elastic compliances from nine (for orthotropy) to five, viz.: S_{11}, S_{22}, S_{12}, S_{13}, S_{44}. The parameters S_{12}, S_{13} are associated with cross-effects while S_{44} is a shear compliance.

The values of the parameters adopted for the study are:

$$S_{11} = S_{33} = \frac{1}{540} \; ; \; S_{22} = \frac{1}{300} \; ; \; S_{13} = \frac{-1}{1080} \; ; \; S_{12} = S_{23}$$

$$\bar{k}_1 = \frac{S_{11}}{S_{22}} = 0.556 \; ; \; \bar{k}_2 = \frac{-S_{13}}{S_{22}} = 0.278 \; ; \; \bar{k}_3 = \frac{-S_{12}}{S_{22}} \tag{3.1}$$

The parameter $\bar{k}_3$ will be varied to obtain a family of curves. This parameter is essentially a measure of the cross-effect between the r and θ or the z and θ directions. No value was assigned to S_{44} since it does not appear in the expressions for the stresses given by Eqs. (2.13), (2.17), and (2.18). A plot of $\sigma_\theta(a)$ versus b/a for internal pressure only is shown in Fig. (3.1). The close proximity of the family of curves with $\bar{k}_3$ as a parameter discloses that the cross-effect has a negligible influence on the maximum stress for a particular value of $\bar{k}_1$. Thus, it is seen that the major factor in the difference between the maximum stress in the isotropic and hexagonal cases is the parameter $\bar{k}_1$, the ratio of the radial and tangential compliances. For solid propellants, which exhibit stress-induced orthotropy, $\bar{k}_1$ is less than unity. Consequently, the maximum stress lies below that for an isotropic cylinder for all values of b/a. The upper curve in the figure, corresponding to $\bar{k}_1 = 1.80$ and $\bar{k}_2 = \bar{k}_3 = 0.90$, is included to show the effect of interchanging the values of radial and tangential compliances.

b. Pressurization of a Cylinder with Stress-Induced Hexagonal Material Properties

When a cylinder is subjected to both internal and external pressure, the tangential stress σ_θ crosses over from tension to compression at some radius between the values of b and a. If the criterion for the change of values of the compliances is taken at the instance when one principal stress goes from tension to compression, then it is possible to solve this problem as two concentric cylinders, one of which is isotropic and one with hexagonal material properties. The theory of bilinear solids is presented in Part II of this report by Herrmann, who discussed a number of classes of bilinear materials with different cross-over criteria. The problem presented herein is a special case in one of the classes discussed, where the cross-over point is taken to be zero stress.

The following steps are taken for the solution of this bilinear elastic problem:

(1) Solve the following two boundary value problems:

<u>isotropy</u>

$$\frac{d^2\sigma_{ri}}{dr^2} + \frac{3}{r}\frac{d\sigma_{ri}}{dr} = 0 \tag{3.2}$$

with

$$\sigma_{ri}(b) = -q \qquad \sigma_{\theta i}(x) = 0$$

and <u>hexagonality</u>

$$\beta_{22} r^2 \frac{d^2\sigma_{rh}}{dr^2} + 3\beta_{22} r \frac{d\sigma_{rh}}{dr} + (\beta_{22} - \beta_{11})\, \sigma_{rh} = 0 \tag{3.3}$$

with

$$\sigma_{rh}(a) = -p \qquad \sigma_{\theta h}(x) = 0$$

where x is the radius at which σ_θ crosses over. In Eqs. (3.2) and (3.3), K and the thermal effects have been neglected. The solution for this step is

$$\sigma_{ri}(r) = - \frac{qb^2}{x^2 + b^2}\left[\left(\frac{x}{r}\right)^2 + 1\right] \tag{3.4a}$$

$$\sigma_{\theta i}(r) = \frac{qb^2}{x^2 + b^2}\left[\left(\frac{x}{r}\right)^2 - 1\right] \tag{3.4b}$$

$$\sigma_{rh}(r) = - \frac{pa}{\left(\frac{x}{a}\right)^k + \left(\frac{x}{a}\right)^{-k}} \frac{1}{r}\left[\left(\frac{x}{r}\right)^k + \left(\frac{x}{r}\right)^{-k}\right]$$

$$= - \frac{pa}{\cosh\left[k \ln \frac{x}{a}\right]} \frac{1}{r} \cosh k \ln \left(\frac{x}{r}\right) \tag{3.5a}$$

$$\sigma_{\theta h}(r) = \frac{pak}{\left(\frac{x}{a}\right)^k + \left(\frac{x}{a}\right)^{-k}} \frac{1}{r}\left[\left(\frac{x}{r}\right)^k - \left(\frac{x}{r}\right)^{-k}\right]$$

$$= \frac{pak}{\cosh\left[k \ln \frac{x}{a}\right]} \frac{1}{r} \sinh k \ln \frac{x}{r} \tag{3.5b}$$

(2) Determine the value of x by equating the radial stresses of both sub-domains at the interface $r = x$.

$$\sigma_{ri}(x) = \sigma_{rh}(x) \tag{3.6}$$

Substituting Eqs. (3.4a) and (3.5a) into Eq. (3.6) gives:

$$\frac{qb^2}{x^2 + b^2} = \frac{pa}{x\left[\left(\frac{x}{a}\right)^k + \left(\frac{x}{a}\right)^{-k}\right]} \tag{3.7a}$$

or

$$pa(x^2 + b^2) = 2qb^2 x \cosh\left[k \ln \left(\frac{x}{a}\right)\right] \tag{3.7b}$$

The solution of the transcendental equation (3.7b) gives the value of x.

In many problems of interest it is not the external pressure q which is prescribed, but rather a displacement boundary condition. Hence it is necessary to relate the external pressure to the displacement at the outer boundary. The displacements for the individual sub-domains may be found from Eqs. (2.3) and (2.9).

$$u_i(r) = \frac{qb^2 r}{x^2 + b^2}\left[(\beta_{22_i} - \beta_{12_i})\left(\frac{x}{r}\right)^2 - (\beta_{22_i} + \beta_{12_i})\right] \tag{3.8}$$

$$u_h(r) = \frac{pa}{2\cosh\left[k_h \ln\left(\frac{x}{a}\right)\right]}\left[(k_h\beta_{22_h} - \beta_{12_h})\left(\frac{x}{r}\right)^{k_h} - (\beta_{12_h} + k_h\beta_{22_h})\left(\frac{x}{r}\right)^{-k_h}\right] \tag{3.9}$$

where the subscripts h and i in the elastic coefficients distinguish them between hexagonal and isotropic properties. As an example consider the outer case to be rigid, i.e. $u_i(b) = 0$; then from Eq. (3.8) there results

$$\frac{x}{b} = \sqrt{\frac{\beta_{22_i} + \beta_{12_i}}{\beta_{22_i} - \beta_{12_i}}} \tag{3.10}$$

Eq. (3.10) shows that for a rigid outer case, the value of x/b is a constant, indicating that the value of x is independent of the ratio b/a and of the internal pressure (i.e., once the cylinder is pressurized, a cross-over point is immediately established and remains at that position as long as a pressure is maintained). To determine q in terms of p, substitute Eq. (3.10) into Eq. (3.7b).

$$q = \frac{pa}{b\cosh\left[k_h \ln \frac{b}{a}\sqrt{\frac{\beta_{22_i} + \beta_{12_i}}{\beta_{22_i} - \beta_{12_i}}}\right]}\left[\frac{\beta_{22_i}}{\beta_{22_i} - \beta_{12_i}}\right]\sqrt{\frac{\beta_{22_i} + \beta_{12_i}}{\beta_{22_i} - \beta_{12_i}}} \tag{3.11}$$

As a check, the displacements of both sub-domains must be the same at the interface, $u_i(x) = u_h(x)$. Equating Eqs. (3.8) and (3.9) for $r = x$ gives

$$q = \frac{pa}{b \cosh\left[k_h \ln \frac{b}{a} \sqrt{\dfrac{\beta_{22_i} + \beta_{12_i}}{\beta_{22_i} - \beta_{12_i}}}\right]} \left[\frac{\beta_{22_i}}{\beta_{22_i} - \beta_{12_i}}\right] \sqrt{\frac{\beta_{22_i} + \beta_{12_i}}{\beta_{22_i} - \beta_{12_i}}} \left[\frac{\beta_{12_h}}{\beta_{12_i}}\right] \tag{3.12}$$

Comparing Eqs. (3.11) and (3.12) shows that they are equal only if

$$\beta_{12_h} = \beta_{12_i} \tag{3.13}$$

Eq. (3.13) shows that there are only three relevant elastic constants instead of four in this type of stress-induced orthotropy. This relationship is a natural consequence of the cross-over criterion employed and may be seen more clearly in the direct formulation of the constitutive equations for a bilinear material as shown in Part II. Reflecting upon Eq. (3.10) it is reasonable to expect that x should depend on the elastic properties of both sub-domains.

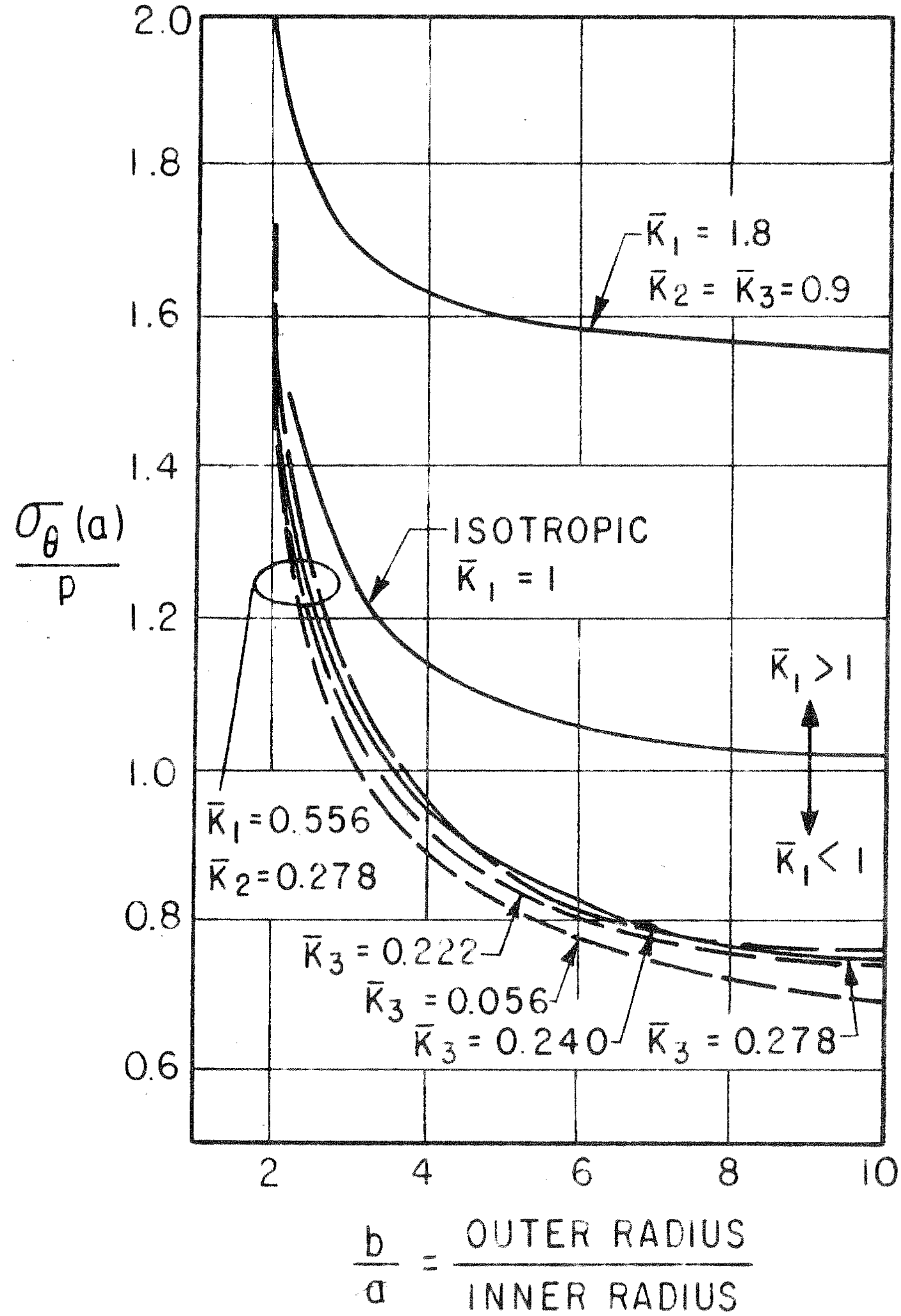

FIG.3-1 $\sigma_\theta(a)$ vs $\frac{b}{a}$ FOR INTERNAL PRESSURE

4. Correspondence Principle for Anisotropic Viscoelasticity

In linear isotropic viscoelasticity a useful analogy exists for the solution of boundary value problems. The correspondence principle has been repeatedly used for many isotropic problems. Following this principle an associated elastic problem with the proper boundary conditions in Laplace or Fourier transform space is inverted to obtain the viscoelastic response. This principle can be formally extended for anisotropic solids by replacing the elastic constitutive equations, (1.4) and (1.5), by viscoelastic constitutive equations

$$\sigma_{ij} = Q_{ij}^{kl} \, \epsilon_{kl} \quad \text{(summation on repeated indices)} \tag{4.1}$$

where Q_{ij}^{kl} is an operational tensor. Special forms of this tensor appear in [6] and [8].

The Laplace transform of Eq. (4.1) along with similar transforms of the equilibrium and compatibility equations, strain-displacement relations and boundary conditions are necessary to complete the formal analogy.

Using the correspondence principle and by inverting term by term the Mittag-Leffler expansion of the solution for the associated elastic orthotropic cylinder, Spillers has recovered the viscoelastic response [7]. In general the solution of viscoelastic problems may not be obtained in this straightforward manner. Recall from the solution of the cylinder problem in Section 2 that the independent variable r appears raised to a fractional power containing elastic coefficients. Although the viscoelastic response was obtained directly from the correspondence principle, the form of this solution seems to indicate that other orthotropic problems may not be readily invertible. An alternative method of solution is applied to the same orthotropic cylinder in the next section. The approach, however, is valid for other orthotropic viscoelastic problems.

5. Viscoelastic Solution for the Pressurization of a Cylinder

The problem of the pressurization of a cylinder is intended to illustrate a method of solution which does not involve the correspondence principle. The governing equation in stress σ_r, Eq. (2.7), is

$$\beta_{22} r^2 \frac{\partial^2 \sigma_r}{\partial r^2} + 3\beta_{22} r \frac{\partial \sigma_r}{\partial r} + (\beta_{22} - \beta_{11})\sigma_r$$

$$= \left[\frac{S_{13} - S_{33}}{S_{33}}\right]\alpha r \frac{\partial T}{\partial r} + \left[\frac{S_{13} - S_{23}}{S_{33}}\right]\alpha T + \left[\frac{S_{13} - S_{23}}{S_{33}}\right]K \qquad (5.1)$$

where $\beta_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}$ are now viscoelastic operators. The material properties are also assumed temperature independent. The theory of viscoelastic materials with temperature-dependent properties has been discussed by Muki and Sternberg [9] and the thermal deformation of viscoelastic cylinders with such materials properties is given in Part IV. The boundary and initial conditions for Eq. (5.1) are

$$\sigma_r(a,t) = -pH(t)$$
$$\sigma_r(b,t) = -qH(t) \qquad (5.2)$$

$$\sigma_r(r,0) = 0$$

$$\frac{\partial \sigma_r}{\partial t}(r,0) = 0 \qquad (5.3)$$

where H(t) is the Heaviside step function.

The solution of Eq. (5.1) is composed of two parts:

$$\sigma_r(r,t) = \sigma_{r1}(r,t) + \sigma_{r2}(r,t) \qquad (5.4)$$

where σ_{r2} is any arbitrary function chosen specifically to satisfy boundary conditions, but, in general, not satisfying the differential equation. Techniques of selecting this function for specific purposes will be discussed in the sequel. Eq. (5.1) is thus recast in the following form with homogeneous boundary conditions:

$$\beta_{22} r^2 \frac{\partial^2 \sigma_{r1}}{\partial r^2} + \beta_{22} r \frac{\partial \sigma_{r1}}{\partial r} + (\beta_{22} - \beta_{11}) \sigma r_1 = f(r,t) \qquad (5.5)$$

where

$$f(r,t) = \left[\frac{S_{13} - S_{33}}{S_{33}}\right] \alpha r \frac{\partial T}{\partial r} + \left[\frac{S_{13} - S_{23}}{S_{33}}\right] \alpha T + \left[\frac{S_{13} - S_{23}}{S_{33}}\right] K$$
$$- \left[\beta_{22} r^2 \frac{\partial^2 \sigma_{r2}}{\partial r^2} + \beta_{22} r \frac{\partial \sigma_{r2}}{\partial r} + (\beta_{22} - \beta_{11}) \sigma_{r2}\right] \qquad (5.6)$$

with

$$\sigma_{r1}(a,t) = 0$$
$$\sigma_{r1}(b,t) = 0 \qquad (5.7)$$

$$\sigma_{r1}(r,0) = \frac{\partial \sigma_{r1}(r,0)}{\partial t} = 0 \qquad (5.8)$$

A separation of variables technique on the homogeneous equation (5.5) is possible. Let

$$\sigma_{rl_h}(r,t) = R(r)\psi(t) \tag{5.9}$$

Substitution of Eq. (5.9) into Eq. (5.5) leads to the following two ordinary differential equations.

$$r^3 \frac{d^2R}{dr^2} + 3r^2 \frac{dR}{dr} - \lambda rR = 0 \tag{5.10}$$

with

$$R(a) = R(b) = 0 \tag{5.11}$$

and

$$\left[\beta_{11} - (1 + \lambda)\beta_{22}\right]\psi(t) = 0 \tag{5.12}$$

with

$$\psi(0) = \frac{d\psi(0)}{dt} = 0 \tag{5.13}$$

where λ is a separation constant.

Eq. (5.10) with homogeneous boundary condtions (5.11) is a Sturm-Liouville differential equation. From that system of equations a complete set of characteristic functions and the corresponding characteristic values may be generated. Any arbitrary function in the variable r may now be expanded in terms of this set of characteristic functions. The solution to Eq. (5.10) is

$$R(r) = A_1 r^{-1-\sqrt{1+\lambda}} + A_2 r^{-1+\sqrt{1+\lambda}} \tag{5.14}$$

Evaluating the constants A_1 and A_2 from the boundary conditions leads to the following characteristic equation, the roots of which are the characteristic values.

$$\left(\frac{a}{b}\right)^{2\sqrt{1+\lambda}} = 1 \quad (5.15)$$

The characteristic values are

$$\lambda_n = -\left(1 + \frac{\pi^2 n^2}{\gamma^2}\right) \quad (5.16)$$

where

$$\gamma = \ln \frac{a}{b} \text{ and } n \text{ is an integer.} \quad (5.17)$$

The corresponding characteristic functions are

$$R_n = \frac{1}{r} \sin\left(\frac{\pi n}{\gamma} \ln \frac{r}{a}\right) \quad (5.18)$$

The solution of Eq. (5.12) with homogeneous initial conditions (5.13) is identically zero.

However, with the complete set of characteristic functions at our disposal, the non-homogeneous term of Eq. (5.5) can be expanded in an infinite series.

$$f(r,t) = \sum_{n=1}^{\infty} C_n(t) R_n(r) \quad (5.19)$$

where

$$C_n(t) = \frac{1}{N^2} \int_a^b f(r,t) R_n(r)\, r dr \quad (5.20)$$

$$N^2 = \int_a^b R_n^2\, r dr \quad (5.21)$$

A particular solution may now be obtained using a mode superposition method. Let

$$\sigma_{rl_p} = \sum_{n=1}^{\infty} \psi_{n_p}(t) R_n(r) \quad (5.22)$$

where $\psi_{n_p}(t)$ are undetermined coefficients. Substitution of

Eqs. (5.22) and (5.19) into Eq. (5.5) gives

$$\sum_{n=1}^{\infty} \lambda_n \psi_{n_p} R_n + (1 - \frac{\beta_{22}}{\beta_{11}}) \psi_{n_p} R_n = \sum_{n=1}^{\infty} C_n R_n \tag{5.23}$$

Therefore the solutions to the typical equation in time gives the coefficients of the series solution.

$$\lambda_n \psi_{n_p}(t) + (1 - \frac{\beta_{22}}{\beta_{11}}) \psi_{n_p}(t) = C_n(t) \tag{5.24}$$

6. Asymptotic Solutions

The selection of σ_{r2} can be made judiciously to minimize the error in using only a finite number of terms of the series for σ_{r1_p}. For the purpose of discussion consider a Maxwell-type response. If a solution for early times is desired, σ_{r2} should be taken as the elastic response multiplied by the Heaviside unit function. Since the major contribution at early times is attributed to the elastic response, the series solution for σ_{r1_p} will represent only the deviation from this effect due to viscoelastic properties of the body. These viscoelastic effects will be small in comparison to σ_{r2}, hence early termination of the infinite series will incur no major error in the total solution.

On the other hand, if a long time solution is needed, σ_{r2} should be taken as the steady creep solution in the case of a Maxwell-type response. To obtain such an expression, it is necessary to substitute the viscosity coefficients in place of the elastic coefficients in the differential equation and seek a solution. This steady creep solution represents the bulk of the total response at very long times, and consequently the series σ_{r1_p} in this case is the deviation from this effect due to elasticity of the body. Again terminating the infinite series early will incur no major error in the total response. The roles of σ_{r2} and σ_{r1_p} for early and long times are thus interchanged.

The foregoing discussion on the selection of the function to satisfy boundary conditions need not be restricted to Maxwell-type responses. The qualitative approach used previously can be achieved systematically by examining the asymptotic forms of the constitutive equations. When a form of the constitutive equations is given, it is possible to predict the behavior of a solid at early or long times by reducing these equations to their asymptotic forms. A procedure of accomplishing this is to examine the Laplace transform of the constitutive equations. If t is the time variable and s is the transform variable, then as $s \to \infty$, $t \to 0$ and as $s \to 0$, $t \to \infty$. Performing the limiting process gives the asymptotic forms of these equations for early and long times. It is then possible to use this simplified form of the equations to obtain an asymptotic solution which will be used for σ_{r2}. Precaution should be taken in arriving at sensible relationships, i.e., for a limiting elastic response, a <u>stress-strain</u> relation should be obtained and for a limiting creep condition a <u>stress-strain rate</u> relation should be obtained.

A similar asymptotic process may be applied to the solution for σ_{r1_p}. A method for solving Eq. (5.24) is by Laplace transform. The ultimate form of a typical equation for the particular value of the parameter λ_n prior to the Laplace inversion is

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots\ldots + a_o}{s\left[b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots + b_o\right]} e^{st} ds \qquad (6.1)$$

It is possible to obtain an inversion for this expression by the method of partial fractions. However, for a high degree polynomial, the work involved is quite cumbersome. Since σ_{rl_p} represents only a small portion of the total response for a selected time interval, it is justifiable to use an approximate solution. Recall that $s \to 0$ and $s \to \infty$ is equivalent to $t \to \infty$ and $t \to 0$, respectively; the limiting process may again be applied. Consider the fraction

$$\frac{a_m s^m + a_{m-1} s^{m-1} + \ldots\ldots + a_o}{\left[s\ b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots + b_o\right]} \qquad (6.2)$$

As $s \to 0$, the lower order terms a_o, $a_1, \ldots, b_o$, $b_1, \ldots$ govern the value of the fraction. Hence the higher order terms in s may be neglected. The number of terms neglected depends on the accuracy of the desired solution. As an example, neglect all terms except a_o, a_1, b_o, b_1; then the asymptotic form is

$$\frac{a_1 s + a_o}{s\left[b_1 s + b_o\right]} \qquad (6.3)$$

Such an expression is easily invertible. Should more accuracy be needed, include a_2 and b_2. The expression in this case is

$$\frac{a_2 s^2 + a_1 s + a_o}{s\left[b_2 s^2 + b_1 s + b_o\right]} \qquad (6.4)$$

which is still a relatively easy inversion. Such an argument may be continued until the desired accuracy is attained.

A similar argument may be used when $s \to \infty$. The higher order coefficients predominate and it is justifiable to neglect the lower order terms. An example of this asymptotic form is

$$\frac{a_n s^n + a_{n-1} s^{n-1}}{s\left[b_n s^n + b_{n-1} s^{n-1}\right]} = \frac{a_n s + a_{n-1}}{s\left[b_n s + b_{n-1}\right]} \qquad (6.5)$$

This expression is equivalent to Eq. (6.3) which was noted to be readily invertible. For more accuracy additional terms must be included. By the foregoing techniques, a fairly accurate solution may be obtained with a minimum of computational effort.

Before eluding from this section attention is called to some other approximate methods of Laplace transform inversion. Schapery [10] has developed techniques which are applicable to stress analysis problems in quasi-static linear viscoelasticity. To use these methods it is only necessary to have knowledge of the associated elastic solution, either numerically or analytically. The principles underlying these techniques come from Irreversible Thermodynamics and a mathematical property of the Laplace Transform.

NON LINEAR ELASTIC ANALYSIS OF CYLINDERS
WITH CYLINDRICAL ORTHOTROPY

7. Fundamental Equations for Solids in Plane Strain Subjected to Axisymmetric Loads

Tensor notation will be used in this section; the reader is referred to [11] for more details. Adopting suitable convected coordinates

$$x^i = (r,\ \theta,\ z)\ ,\ i = 1,\ 2,\ 3 \tag{7.1}$$

the line elements in the undeformed and the deformed states are, respectively

$$\begin{aligned} (ds_o)^2 &= g_{ij}dx^i dx^j \\ (ds)^2 &= G_{ij}dx^i dx^j \end{aligned} \tag{7.2}$$

where the metric tensors g_{ij} and G_{ij} are

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G_{ij} = \begin{bmatrix} (1+\frac{du}{dr})^2 & 0 & 0 \\ 0 & r^2(1+\frac{u}{r})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.3}$$

The mixed form of the strain tensor is defined by

$$e^{i\cdot}_{\cdot j} = \frac{1}{2}\,(G_{ik} - g_{ik})g^{kj} \tag{7.4}$$

The strain-displacement relations are

$$e^i_j = \begin{bmatrix} \frac{du}{dr} + \frac{1}{2}(\frac{du}{dr})^2 & 0 & 0 \\ 0 & \frac{u}{r} + \frac{1}{2}(\frac{u}{r})^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

According to Green and Adkins [11], the strain energy density for curvilinear anisotropy must be expressed in terms of the physical components of strain. In anisotropy it is necessary to account for both the changes in geometry and in the preferred directions of curvilinear anisotropy. For a solid with cylindrical orthotropy, the strain energy density may be expressed as

$$W = W(e^1_1,\ e^2_2,\ e^3_3,\ e^1_2\ e^2_1,\ e^1_3\ e^3_1,\ e^2_3\ e^3_2,\ e^1_2\ e^2_3\ e^3_1) \quad (7.6)$$

For a second order theory, the explicit form of W is

$$\begin{aligned} W = {} & k_1(e^1_1)^2 + k_2\, e^1_1 e^2_2 + k_3 e^1_1 e^3_3 + k_4(e^2_2)^2 + k_5 e^2_2 e^3_3 \\ & + k_6(e^3_3)^2 + k_7 e^1_2 e^2_1 + k_8 e^1_3 e^3_1 + k_9 e^2_3 e^3_2 + k_{10}(e^1_1)^3 \\ & + k_{11}(e^1_1)^2 e^2_2 + k_{12}(e^1_1)^2 e^3_3 + k_{13}(e^2_2)^2 e^1_1 + k_{14}(e^2_2)^3 \\ & + k_{15}(e^2_2)^2 e^3_3 + k_{16}(e^3_3)^2 e^1_1 + k_{17}(e^3_3)^2 e^2_2 + k_{18}(e^3_3)^3 \\ & + k_{19} e^1_1 e^2_2 e^3_3 + k_{20} e^1_1 e^1_2 e^2_1 + k_{21} e^1_1 e^2_3 e^3_2 + k_{22} e^1_1 e^1_3 e^3_1 \\ & + k_{23} e^2_2 e^1_2 e^2_1 + k_{24} e^2_2 e^2_3 e^3_2 + k_{25} e^2_2 e^1_3 e^3_1 + k_{26} e^3_3 e^1_2 e^2_1 \\ & + k_{27} e^3_3 e^2_3 e^3_2 + k_{28} e^3_3 e^1_3 e^3_1 + k_{29} e^1_2 e^2_3 e^3_1 \end{aligned} \quad (7.7)$$

The stresses referred to the deformed space are given by

$$\sigma_{ii} = \frac{\overline{G}^i_i}{\sqrt{I_3}} \frac{\partial W}{\partial e^i_i} \quad \text{(no sum on i)} \tag{7.8}$$

where

$$(\sigma_{11}, \sigma_{22}, \sigma_{33}) \equiv (\sigma_r, \sigma_\theta, \sigma_z) \tag{7.9}$$

$$\overline{G}^i_i = g_{ii} G^{ii} = g^{ii} G_{ii} \tag{7.10}$$

$$I_3 = \frac{G}{g} = (1 + \frac{du}{dr})^2 (1 + \frac{u}{r})^2 \tag{7.11}$$

8. Solution for a Thick-Walled Cylinder Subjected to Internal and External Pressure

The problem of the pressurization of an orthotropic cylinder is governed by the following non-linear differential equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{u}{r}\frac{d\sigma_r}{dr} + \frac{du}{dr}\left(\frac{\sigma_r - \sigma_\theta}{r}\right) = 0 \tag{8.1}$$

with the following boundary conditions

$$\sigma_r(a) = -p_a \qquad \sigma_r(b) = -p_b \tag{8.2}$$

A method of solution is by a perturbation scheme. For details of this method, the reader is referred to Part III where a number of problems in non-linear isotropic elasticity are treated by this perturbation scheme. Only the results for the orthotropic cylinder will be stated here.

The radial displacement is given by

$$u(r) = ap_a\left[X_1\left(\frac{r}{a}\right)^{m_1+m_2} + X_2c^{m_1+m_2-1}\left(\frac{r}{a}\right)^{m_1-m_2}\right]$$

$$- ap_b\left[X_1c^{m_1-m_2-1}\left(\frac{r}{a}\right)^{m_1+m_2} + X_2\left(\frac{r}{a}\right)^{m_1-m_2}\right]$$

$$+ aX_1{}^2(p_a - p_bc^{m_1-m_2-1})^2\left[\frac{X_6}{X_{12}}\left(\frac{r}{a}\right)^{m_1+m_2} + \frac{X_9}{X_{12}}\left(\frac{r}{a}\right)^{m_1-m_2} - \frac{D_1}{X_3}\left(\frac{r}{a}\right)^{2m_1+2m_2-1}\right]$$

$$+ 2aX_1X_2(p_a - p_b{}^{m_1-m_2-1})(p_ac^{m_1+m_2-1} - p_b)\left[\frac{X_7}{X_{12}}\left(\frac{r}{a}\right)^{m_1+m_2} + \frac{X_{10}}{X_{12}}\left(\frac{r}{a}\right)^{m_1-m_2}\right.$$

$$\left. - \frac{D_2}{X_4}\left(\frac{r}{a}\right)^{2m_1-1}\right] + aX_2^2(p_ac^{m_1+m_2-1} - p_b)^2\left[\frac{X_8}{X_{12}}\left(\frac{r}{a}\right)^{m_1+m_2}\right. \qquad (8.3)$$

$$\left. + \frac{X_{11}}{X_{12}}\left(\frac{r}{a}\right)^{m_1-m_2} - \frac{D_3}{X_5}\left(\frac{r}{a}\right)^{2m_1-2m_2-1}\right]$$

The stresses are given by

$$\begin{pmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\end{pmatrix} = a^2p_a\left[X_1\begin{pmatrix}k_2 + 2k_1(m_1 + m_2)\\ 2k_4 + k_2(m_1 + m_2)\\ k_3 + k_5(m_1 + m_2)\end{pmatrix}\left(\frac{r}{a}\right)^{m_1+m_2-1}\right.$$

$$\left. + X_2c^{m_1+m_2-1}\begin{pmatrix}k_2 + 2k_1(m_1 - m_2)\\ 2k_4 + k_2(m_1 - m_2)\\ k_3 + k_5(m_1 + m_2)\end{pmatrix}\left(\frac{r}{a}\right)^{m_1-m_2-1}\right] \qquad (8.4)$$

$$- a^2 p_b \Bigg[X_1 c^{m_1 - m_2 - 1} \begin{pmatrix} k_2 + 2k_1(m_1 + m_2) \\ 2k_4 + k_2(m_1 + m_2) \\ k_3 + k_5(m_1 + m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 + m_2 - 1}$$

$$+ X_2 \begin{pmatrix} k_2 + 2k_1(m_1 - m_2) \\ 2k_4 + k_2(m_1 - m_2) \\ k_3 + k_5(m_1 - m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} \Bigg]$$

$$+ a^2 X_1{}^2 (p_a - p_b c^{m_1 - m_2 - 1})^2 \Bigg[\frac{X_6}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 + m_2) \\ 2k_4 + k_2(m_1 + m_2) \\ k_3 + k_5(m_1 + m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 + m_2 - 1}$$

$$+ \frac{X_9}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 - m_2) \\ 2k_4 + k_2(m_1 - m_2) \\ k_3 + k_5 \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} - \frac{D_1}{X_3} \begin{pmatrix} k_2 + 2k_1(2m_1 + 2m_2 - 1) \\ 2k_4 + k_2(2m_1 + 2m_2 - 1) \\ k_3 + k_5(2m_1 + 2m_2 - 1) \end{pmatrix} \left(\frac{r}{a}\right)^{2m_1 + 2m_2 - 2} \Bigg]$$

$$+ 2a^2 X_1 X_2 (p_a - p_b c^{m_1 - m_2 - 1})(p_a c^{m_1 + m_2 - 1} - p_b) \Bigg[\frac{X_7}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 + m_2) \\ 2k_4 + k_2(m_1 + m_2) \\ k_3 + k_5(m_1 + m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 + m_2 - 1}$$

$$+ \frac{X_{10}}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 - m_2) \\ 2k_4 + k_2(m_1 - m_2) \\ k_3 + k_5(m_1 - m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} - \frac{D_2}{X_4} \begin{pmatrix} k_2 + 2k_1(2m_1 - 1) \\ 2k_4 + k_2(2m_1 - 1) \\ k_3 + k_5(2m_1 - 1) \end{pmatrix} \left(\frac{r}{a}\right)^{2m_1 - 2} \Bigg]$$

(8.4)

Cont.

$$+ a^2 X_2^{\,2} (p_a c^{m_1+m_2-1} - p_b)^2 \Bigg[\frac{X_8}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 + m_2) \\ 2k_4 + k_2(m_1 + m_2) \\ k_3 + k_5(m_1 + m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 + m_2 - 1}$$

$$+ \frac{X_{11}}{X_{12}} \begin{pmatrix} k_2 + 2k_1(m_1 - m_2) \\ 2k_4 + k_2(m_1 - m_2) \\ k_3 + k_5(m_1 - m_2) \end{pmatrix} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} - \frac{D_3}{X_5} \begin{pmatrix} k_2 + 2k_1(2m_1 - 2m_2 - 1) \\ 2k_4 + k_2(2m_1 - 2m_2 - 1) \\ k_3 + k_5(2m_1 - 2m_2 - 1) \end{pmatrix} \left(\frac{r}{a}\right)^{2m_1 - 2m_2 - 2} \Bigg]$$

$$+ a^4 \begin{pmatrix} 3k_{10} + 3k_1 \\ k_{11} - \frac{k_2}{2} \\ k_{12} - \frac{k_3}{2} \end{pmatrix} \Bigg[p_a \left\{ X_1 (m_1 + m_2) \left(\frac{r}{a}\right)^{m_1 + m_2 - 1} + X_2 (m_1 - m_2) c^{m_1 + m_2 - 1} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} \right\}$$

$$- p_b \left\{ X_1 (m_1 + m_2) c^{m_1 - m_2 - 1} \left(\frac{r}{a}\right)^{m_1 + m_2 - 1} + X_2 (m_1 - m_2) \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} \right\} \Bigg]^2$$

$$+ a^4 \begin{pmatrix} k_{13} - \frac{k_2}{2} \\ 3k_4 + 3k_{14} \\ k_{15} - \frac{k_5}{2} \end{pmatrix} \Bigg[p_a \left\{ X_1 \left(\frac{r}{a}\right)^{m_1 + m_2} + X_2 c^{m_1 + m_2 - 1} \left(\frac{r}{a}\right)^{m_1 - m_2} \right\} \qquad (8.4)\ \text{cont.}$$

$$- p_b \left\{ X_1 c^{m_1 - m_2 - 1} \left(\frac{r}{a}\right)^{m_1 + m_2} + X_2 \left(\frac{r}{a}\right)^{m_1 - m_2} \right\} \Bigg]^2$$

$$+ a^4 \begin{pmatrix} 2k_{11} - 2k_1 + k_2 \\ 2k_{13} - 2k_4 + k_2 \\ k_{19} - k_3 - k_5 \end{pmatrix} \Bigg[p_a \left\{ X_1 (m_1 + m_2) \left(\frac{r}{a}\right)^{m_1 + m_2 - 1} + X_2 (m_1 - m_2) c^{m_1 + m_2 - 1} \left(\frac{r}{a}\right)^{m_1 - m_2 - 1} \right.$$

$$- p_b \left\{ X_1 (m_1+m_2) c^{m_1-m_2-1} (\tfrac{r}{a})^{m_1+m_2-1} + X_2 (m_1-m_2) (\tfrac{r}{a})^{m_1-m_2-1} \right\} \Big] \Big[p_a \Big\{ X_1 (\tfrac{r}{a})^{m_1+m_2}$$

$$+ X_2 c^{m_1+m_2-1} (\tfrac{r}{a})^{m_1-m_2} \Big\} - p_b \left\{ X_1 c^{m_1-m_2-1} (\tfrac{r}{a})^{m_1+m_2} + X_2 (\tfrac{r}{a})^{m_1-m_2} \right\} \Big] \tag{8.4}$$

where

$$c = \frac{b}{a} \tag{8.5}$$

$$m_1 = \frac{k_2 - 2k_1}{k_1} \qquad m_2 = \sqrt{(k_2 - 2k_1)^2 + 4k_1k_4} \tag{8.6}$$

$$X_1 = \frac{3k_2 - 4k_1 - 2\sqrt{4(k_2 - 2k_1)^2 + 4k_1k_2}}{(5k_2^2 - 8k_1k_2 - 16k_1k_4)(c^{m_1-m_2-1} - c^{m_1+m_2-1})} \tag{8.7}$$

$$X_2 = \frac{3k_2 - 4k_1 + 2\sqrt{4(k_2 - 2k_1)^2 + 4k_1k_4}}{(5k_2^2 - 8k_1k_2 - 16k_1k_4)(c^{m_1-m_2-1} - c^{m_1+m_2-1}}$$

$$X_3 = 2C_2\left[2(m_1+m_2)^2 - 3(m_1+m_2) + 1\right] + (C_1+C_7)\left[2m_1 + 2m_2 - 1\right] + (C_6 - C_1)$$

$$X_4 = 2C_2\left[2m_1^2 - 3m_1 + 1\right] + (C_1 + C_7)(2m_1 - 1) + (C_6 - C_1)$$

$$X_5 = 2C_2\left[2(m_1 - m_2)^2 - 3(m_1 - m_2) + 1\right] + (C_1 + C_7)\left[2m_1 - 2m_2 - 1\right] + (C_6 - C_1)$$

$$X_6 = K_1 \Big[\Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{2m_1 - 2m_2 - 2} - \Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{m_1 - m_2 - 1}$$

$$X_7 = K_2 \Big[\Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{2m_1 - 2} - \Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{m_1 - m_2 - 1}\Big]$$

$$X_8 = K_3 \Big[\Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{2m_1 - 2m_2 - 2} - \Big\{C_1 + (m_1 - m_2)C_2\Big\} c^{m_1 - m_2 - 1}\Big]$$

$$X_9 = K_1 \Big[C_1 + (m_1 + m_2)C_2\Big]\Big[c^{m_1 + m_2 - 1} - c^{2m_1 + 2m_2 - 2}\Big]$$

$$X_{10} = K_2 \Big[C_1 + (m_1 + m_2)C_2\Big]\Big[c^{m_1 + m_2 - 1} - c^{2m_1 - 2}\Big]$$

$$X_{11} = K_3 \Big[C_1 + (m_1 + m_2)C_2\Big]\Big[c^{m_1 + m_2 - 1} - c^{2m_1 - 2m_2 - 2}\Big]$$

$$X_{12} = (5k_2^2 - 8k_1k_2 - 16k_1k_4)(c^{m_1 - m_2 - 1} - c^{m_1 + m_2 - 1})$$

$$K_1 = C_3(m_1 + m_2)^2 + C_4 + C_5(m_1 + m_2) - \frac{D_1}{X_3}\Big\{C_1 + (2m_1 + 2m_2 - 1)C_2\Big\}$$

$$K_2 = C_3(m_1^2 - m_2^2) + C_4 + C_5 m_1 - \frac{D_2}{X_4}\Big\{C_1 + (2m_1 - 1)C_2\Big\} \qquad (8.8)$$

$$K_3 = C_3(m_1 - m_2)^2 + C_4 + C_5(m_1 - m_2) - \frac{D_3}{X_5}\Big\{C_1 + (2m_1 - 2m_2 - 1)C_2\Big\}$$

$$D_1 = (m_1 + m_2)^2 \left\{ C_2 + C_5 + C_7 + C_8 + 2C_2(m_1 + m_2 - 1) \right\}$$

$$+ (m_1 + m_2) \left\{ C_1 - C_2 + 2C_4 - C_5 + C_6 + C_{10} + C_5(m_1 + m_2 - 1) \right\}$$

$$+ (C_9 - C_1 - 2C_4)$$

$$D_2 = (m_1^2 + m_2^2)(C_2 + C_5) + a(C_1 - C_2 + 2C_4 - 2C_5 + C_6 + C_{10})$$

$$+ (m_1^2 - m_2^2) \left\{ C_5 + C_7 + C_8 + 2C_3(m_1 - 1) \right\} \qquad (8.9)$$

$$D_3 = (m_1 - m_2)^2 \left\{ C_2 + C_5 + C_7 + C_8 + 2C_2(m_1 - m_2 - 1) \right\}$$

$$+ (m_1 - m_2) \left\{ C_1 - C_2 + 2C_4 - C_5 + C_6 + C_{10} + C_5(m_1 - m_2 - 1) \right\}$$

$$+ (C_9 - 2C_4 - C_1)$$

$$C_1 = k_2 \qquad C_6 = k_2 - 2k_4$$

$$C_2 = 2k_1 \qquad C_7 = 2k_1 - k_2 \qquad (8.10)$$

$$C_3 = 3k_{10} + 3k_1 \qquad C_8 = 3k_{10} + k_1 - k_{11} + \frac{3k_2}{2}$$

$$C_4 = k_{13} - \frac{k_2}{2} \qquad C_9 = k_{13} - \frac{3k_2}{2} - k_4 - 3k_{14}$$

$$C_5 = 2k_{11} - 2k_1 + k_2 \qquad C_{10} = 2k_{11} - 2k_1 - 2k_{13} + 2k_4$$

The stresses and displacement when substituted back into the differential equation (8.1) will satisfy it within the degree of accuracy of the second order terms. Computation of numerical values of stress and displacement is contingent upon the knowledge of the strain energy function which can only be determined from experiments.

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PART II

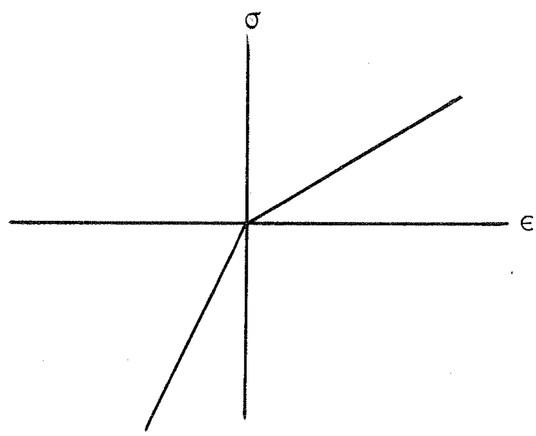
BILINEAR ELASTICITY WITH APPLICATIONS TO THICK-WALLED CYLINDERS

by

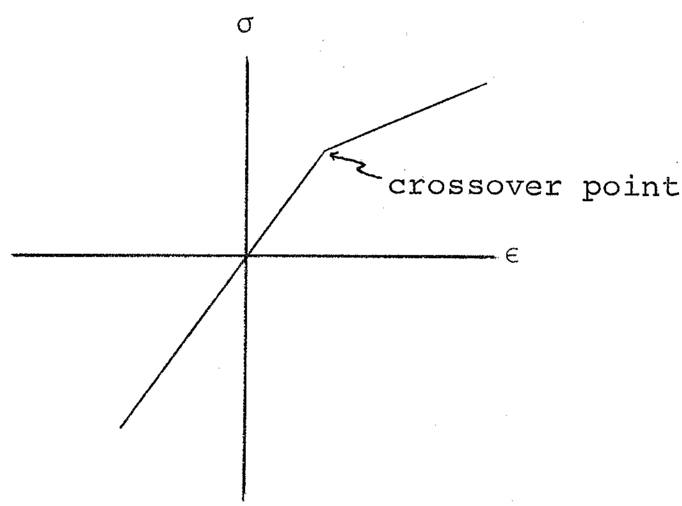
L. R. HERRMANN

INTRODUCTION

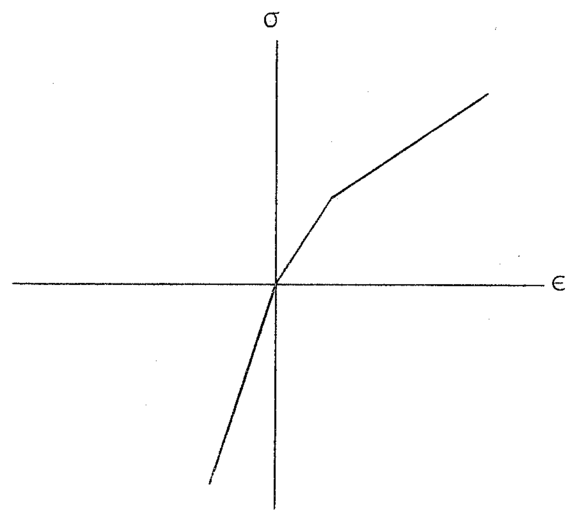
This investigation was initiated to attempt to approximate the behavior of propellants that exhibit different mechanical behavior in tension and compression. For example, one might experimentally obtain a uniaxial stress-strain curve of the form shown below:



We shall refer to the point of discontinuous rate of action as the "crossover point". In our study we have allowed this crossover point to occur at states other than the zero stress state; thus, we are able to accommodate a uniaxial stress-strain curve as shown below:



This generalization was made with the phenomenon of dewetting in mind, as it appears that dewetting usually occurs at some finite strain state. The following analysis may be easily extended to include trilinear materials without any conceptual difficulties, i.e., one might obtain a uniaxial stress-strain curve of the form:



It should be noted that the crossover point in reality need not be a sharply defined point as we have pictured it above but the phenomenon must be capable of being approximately represented by a series of linear steps. Thus we may have a bilinear fit of experimental data as is illustrated in Figure 2. In reality what we are doing is approximating a nonlinear material response by a series of zones of linear action; thus, more appropriately this analysis might be called "elastic zone analysis". The nonlinear aspects of the problem will in general be manifested in a nonlinear algebraic equation whose solution locates the zone boundaries, see for example Eq. (2.18). Therefore, although the resulting theories are for infinitesimal strains, superposition is in general no longer valid as the resulting equations are bilinear, not linear.

There are many ways to hypothesize a bilinear material depending on the criterion selected to define the crossover point; i.e., it is somewhat analogous to the yield condition in plasticity. We shall consider three simple classes of bilinear elastic materials corresponding to the following crossover criteria:

I) Principal strain criterion
II) Principal stress criterion
III) Mean stress criterion.

For the class I material we shall consider that the material passes from one linear phase of response to another as one of the principal strains passes through a critical value. We shall refer to this critical value as the "threshold" value. The first two criteria lead to the prediction of strain-or stress-induced anisotropy. This anisotropy is clearly apparent when one views the resulting constitutive equations, Eqs. (1.1) and (1.2). It will be noted that the preferred directions coincide with the principal stress directions and thus for that class of problems for which we do not know a priori the directions of the principal stresses, the governing equations for the class I and II materials become nearly intractable. The particular criterion that should be used to characterize a given material, of course, needs to be determined by experimental means; for example, one might analyze several different stress states as given by simple tests (see Sections 2 and 3).

It is to be noted that although we might have expected to be able to independently specify two or three elastic constants in each zone we have in totality only three independent elastic constants, as may be seen in Eqs. (1.1), (1.2) and (1.3). This restriction is not present in the proposed generalized theory of Section 7. The relationships relating the remaining constants, as found in a uniaxial test, are given by Eqs. (2.1), (2.2) and (2.3). They are different for each of the three classes of materials that we have considered and may be used as a guide in selecting the particular model to represent a given material, see Section 3.

NOTATION

(Additional notation will be explained as introduced.)

ϵ_i	Principal strain
ϵ^i_j	Strain
$e^i_j=\epsilon^i_j-\epsilon_m\delta^i_j$	Deviatoric strain
ϵ_r	Radial strain in cylindrical coordinates
ϵ_θ	Tangential strain in cylindrical coordinates
ϵ_z	Longitudinal strain in cylindrical coordinates
$\epsilon_m = \frac{1}{3}\epsilon^i_i$	Mean strain
$\theta = \epsilon^i_i$	First strain invariant
u	Radial displacement in cylindrical coordinates
σ_i	Principal stress
τ^i_j	Stress
$s^i_j=\tau^i_j-\sigma_m\delta^i_j$	Deviatoric stress
σ_r	Radial stress in cylindrical coordinates
σ_θ	Tangential stress in cylindrical coordinates
σ_z	Longitudinal stress in cylindrical coordinates

$\Theta = \tau_i^i$	First stress invariant
$\sigma_m = \frac{1}{3}\tau_i^i$	Mean stress
k_i	Elastic modulus
c_i	Elastic compliance
B	Bulk modulus
μ	Shear modulus
n	Class III elastic constant
E	Young's modulus
ν	Poisson's ratio
$\bar{\mu}$	Shear modulus of the motor case
$\bar{\nu}$	Poisson's ratio of the motor case
α	Thermal coefficient of linear expansion (assumed temperature independent)
e	Principal strain threshold value
s	Principal stress threshold value
h	Mean stress threshold value
T	Absolute temperature
T_o	Reference temperature

$T^*=T-T_o$	Relative temperature
T_a	Inner wall temperature
T_b	Outer wall temperature
a	Inner radius of thick-walled cylinder
b	Outer radius of thick-walled cylinder
$c = \frac{b}{a}$	Radii ratio
t	Thickness of motor case
P_i	Pressure on inner wall of thick-walled cylinder
P'	Interface pressure between thick-walled cylinder and motor case
x	Radial location of zone boundary
m_i	Exponent in anisotropic solution
β	Exponent in anisotropic solution
δ_j^i	Kronecker delta

1. Bilinear Constitutive Equations

The equilibrium and strain displacement equations remain unchanged from classical elasticity and therefore need not be considered here. The constitutive equations* for class I material become (see Section 5):

Zone 1 $\quad \epsilon_1 \le e \ , \ \epsilon_2 \le e \ , \ \epsilon_3 \le e$

$$\sigma_1 = c_1\epsilon_1 + c_2\epsilon_2 + c_2\epsilon_3$$

$$\sigma_2 = c_2\epsilon_1 + c_1\epsilon_2 + c_2\epsilon_3$$

$$\sigma_3 = c_2\epsilon_1 + c_2\epsilon_2 + c_1\epsilon_3$$

Zone 2 $\quad \epsilon_1 \ge e \ , \ \epsilon_2 \le e \ , \ \epsilon_3 \le e$

$$\sigma_1 = c_3\epsilon_1 + c_2\epsilon_2 + c_2\epsilon_3 + e(c_1 - c_3)$$

$$\sigma_2 = c_2\epsilon_1 + c_1\epsilon_2 + c_2\epsilon_3 \qquad (1.1)$$

$$\sigma_3 = c_2\epsilon_1 + c_2\epsilon_2 + c_1\epsilon_3$$

Zone 3 $\quad \epsilon_1 \ge e \ , \ \epsilon_2 \ge e, \ \epsilon_3 \le e$

$$\sigma_1 = c_3\epsilon_1 + c_2\epsilon_2 + c_2\epsilon_3 + e(c_1 - c_3)$$

$$\sigma_2 = c_2\epsilon_1 + c_3\epsilon_2 + c_2\epsilon_3 + e(c_1 - c_3)$$

$$\sigma_3 = c_2\epsilon_1 + c_2\epsilon_2 + c_1\epsilon_3$$

(continued)

*For simplicity of presentation the temperature terms have been omitted, but in subsequent sections we shall include them.

Zone 4 $\epsilon_1 \geq e$, $\epsilon_2 \geq e$, $\epsilon_3 \geq e$

$$\sigma_1 = c_3\epsilon_1 + c_2\epsilon_2 + c_2\epsilon_3 + e(c_1 - c_3)$$

$$\sigma_2 = c_2\epsilon_1 + c_3\epsilon_2 + c_2\epsilon_3 + e(c_1 - c_3) \qquad (1.1)$$

$$\sigma_3 = c_2\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 + e(c_1 - c_3)$$

The constitutive equations for class II material are (see Section 5):

Zone 1 $\sigma_1 \leq s$, $\sigma_2 \leq s$, $\sigma_3 \leq s$

$$\epsilon_1 = k_1\sigma_1 + k_2\sigma_2 + k_2\sigma_3$$

$$\epsilon_2 = k_2\sigma_1 + k_1\sigma_2 + k_2\sigma_3$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3$$

Zone 2 $\sigma_1 \geq s$, $\sigma_2 \leq s$, $\sigma_3 \leq s$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_2 = k_2\sigma_1 + k_1\sigma_2 + k_2\sigma_3 \qquad (1.2)$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3$$

Zone 3 $\sigma_1 \geq s$, $\sigma_2 \geq s$, $\sigma_3 \leq s$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

(continued)

$$\epsilon_2 = k_2\sigma_1 + k_3\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3 \qquad (1.2)$$

Zone 4 $\quad \sigma_1 \geq s\ ,\ \sigma_2 \geq s\ ,\ \sigma_3 \geq s$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_2 = k_2\sigma_1 + k_3\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_3\sigma_3 + s(k_1 - k_3)$$

For any given zone above we may write

$$\epsilon^{ij} = k^{ij}_{km}\,\tau^{km} + D^{ij},$$

then for an arbitrary set of orthogonal axes in that particular zone

$$\bar{\epsilon}^{ij} = \bar{k}^{ij}_{km}\,\bar{\tau}^{km} + \bar{D}^{ij}$$

where we may obtain $\bar{k}^{ij}_{km}$ and $\bar{D}^{ij}$ from k^{ij}_{km} and D^{ij} by means of fourth and second rank tensor transformations respectively; we set

$$k^{ij}_{km} = 0 \quad \text{for} \quad i \neq j \quad \text{or } k \neq m \quad \text{and } D^{ij} = 0 \text{ for } i \neq j.$$

The constitutive equations for class III materials are (see Section 5):

Zone 1 $\qquad \sigma_m \leq h$

$$\tau^i_j = (B - \tfrac{2}{3}\mu)\,\theta\delta^i_j + 2\mu\epsilon^i_j$$

or $$\epsilon^i_j = \frac{1}{3}\left(\frac{1}{3B} - \frac{1}{2\mu}\right)\Theta\delta^i_j + \frac{1}{2\mu}\tau^i_j$$

Zone 2 $$\sigma_m \geq h$$

$$\tau^i_j = (nB - \tfrac{2}{3}\mu)\theta\delta^i_j + 2\mu\epsilon^i_j + h(1 - n)\delta^i_j \qquad (1.3)$$

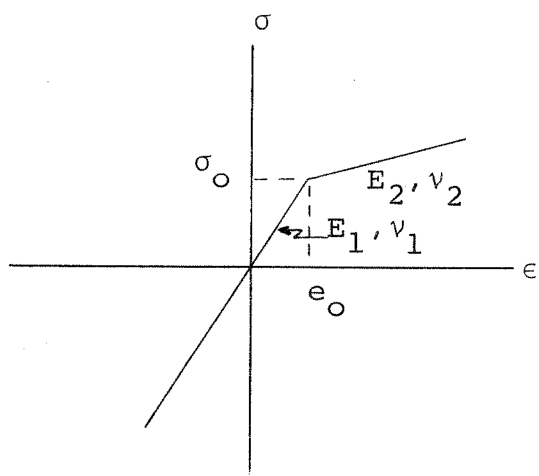
or $$\epsilon^i_j = \frac{1}{3}\left(\frac{1}{3nB} - \frac{1}{2\mu}\right)\Theta\delta^i_j + \frac{1}{2\mu}\tau^i_j - \frac{h(1 - n)}{3Bn}\delta^i_j$$

2. Elementary Solutions

We shall assume for simplicity that $s > 0$, $e > 0$ and $h > 0$.

We first consider the analysis of some simple tests that may be used to determine the proper threshold criterion and the appropriate elastic constants for a given material; for example, see Section 3.

Uniaxial Test



Class I

$$c_1 = \frac{(1 - \nu_1)E_1}{(1 - 2\nu_1)(1 + \nu_1)}$$

$$c_2 = \frac{\nu_1 E_1}{(1 - 2\nu_1)(1 + \nu_1)}$$

$$c_3 = \frac{E_2(1 - \nu_2)}{(1 - 2\nu_2)(1 + \nu_2)}$$

$$e = e_o$$

In addition for a Class I representation from continuity requirements the following relationship must be valid (see Section 5)

$$\frac{E_2 \nu_2}{(1 - 2\nu_2)(1 + \nu_2)} = \frac{E_1 \nu_1}{(1 - 2\nu_1)(1 + \nu_1)} . \qquad (2.1)$$

Class II

$$k_1 = \frac{1}{E_1}$$

$$k_3 = \frac{1}{E_2}$$

$$k_2 = - \frac{\nu_1}{E_1}$$

$$s = \sigma_o$$

In addition for a Class II representation as a result of continuity requirements the following relationship must be valid (see Section 5):

$$\frac{\nu_1}{E_1} = \frac{\nu_2}{E_2} \tag{2.2}$$

Class III

$$\mu_1 = \frac{E_1}{2(1 + \nu_1)}$$

$$B_1 = \frac{E_1}{3(1 - 2\nu_1)}$$

$$n = \frac{E_2}{3B_1(1 - 2\nu_2)}$$

$$h = \frac{\sigma_o}{3}$$

Additionally for a Class III representation from continuity the following relationship must be valid:

$$\frac{E_1}{2(1 + \nu_1)} = \frac{E_2}{2(1 + \nu_2)} \tag{2.3}$$

<u>Biaxial Test</u>

$$(\sigma_1 = \sigma_2 \ , \ \sigma_3 = 0)$$

Class I

Zone I $\quad \epsilon_1 \leq e$

$$\epsilon_3 = -\frac{2c_2}{c_1}\epsilon_1$$

$$\sigma_1 = (c_1 + c_2 - \frac{2c_2^2}{c_1})\epsilon_1$$

Zone 3 $\quad \epsilon_1 \geq e$

$$\epsilon_3 = -\frac{2c_2}{c_1}\epsilon_1$$

$$\sigma_1 = (c_3 + c_2 - \frac{2c_2^2}{c_1})\epsilon_1 + e(c_1 - c_3)$$

Class II

Zone 1 $\quad \sigma_1 \leq s$

$$\epsilon_1 = (k_1 + k_2)\sigma_1$$

$$\epsilon_3 = 2k_2\sigma_1$$

Zone 3 $\quad \sigma_1 \geq s$

$$\epsilon_1 = (k_3 + k_2)\sigma_1 + s(k_1 - k_3)$$

$$\epsilon_3 = 2k_2\sigma_1$$

Class III

Zone 1 $\sigma_1 \leq \frac{3h}{2}$

$$\epsilon_1 = \frac{1}{3}\left(\frac{2}{3B} + \frac{1}{2\mu}\right)\sigma_1$$

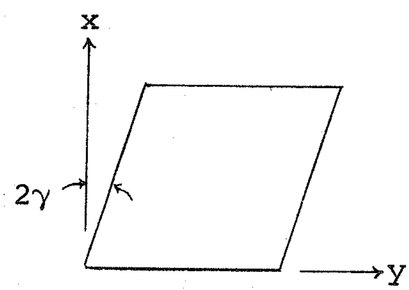
$$\epsilon_3 = \frac{2}{3}\left(\frac{1}{3B} - \frac{1}{2\mu}\right)\sigma_1$$

Zone 2 $\sigma_1 \geq \frac{3h}{2}$

$$\epsilon_1 = \frac{1}{3}\left(\frac{2}{3nB} + \frac{1}{2\mu}\right)\sigma_1 - \frac{h(1 - n)}{3nB}$$

$$\epsilon_3 = \frac{2}{3}\left(\frac{1}{3nB} - \frac{1}{2\mu}\right)\sigma_1 - \frac{h(1 - n)}{3nB}$$

<u>Pure Shear Strain Test</u> (see Section 6)



$\epsilon_{xx} = \epsilon_{yy} = 0, \epsilon_{xy} = \gamma$

Class II

Zone 1 $\sigma_1 \leq s$ or $\gamma \leq (k_1 - k_2)s$

$$\tau_{xx} = \tau_{yy} = 0$$

$$\tau_{xy} = \frac{1}{k_1 - k_2}\gamma$$

Zone 2 $\sigma_1 \geq s$ or $\gamma \geq (k_1 - k_2)s$

$$\tau_{xx} = \tau_{yy} = \frac{2(k_3-k_1)\left((k_1-k_2)s-\gamma\right)}{4k_1k_3 - k_2(k_1+k_3+2k_2)} \tag{2.4}$$

$$\tau_{xy} = \frac{s(k_3-k_1)(2k_1+k_2)+2\gamma(k_1+k_3+k_2)}{4k_1k_3-k_2(k_1+k_3+2k_2)}$$

Thus the threshold value of γ is

$$\gamma_c = (k_1 - k_2)s.$$

Class III Only one zone possible, $\sigma_m \leq s$ (as $\sigma_m \equiv 0$)

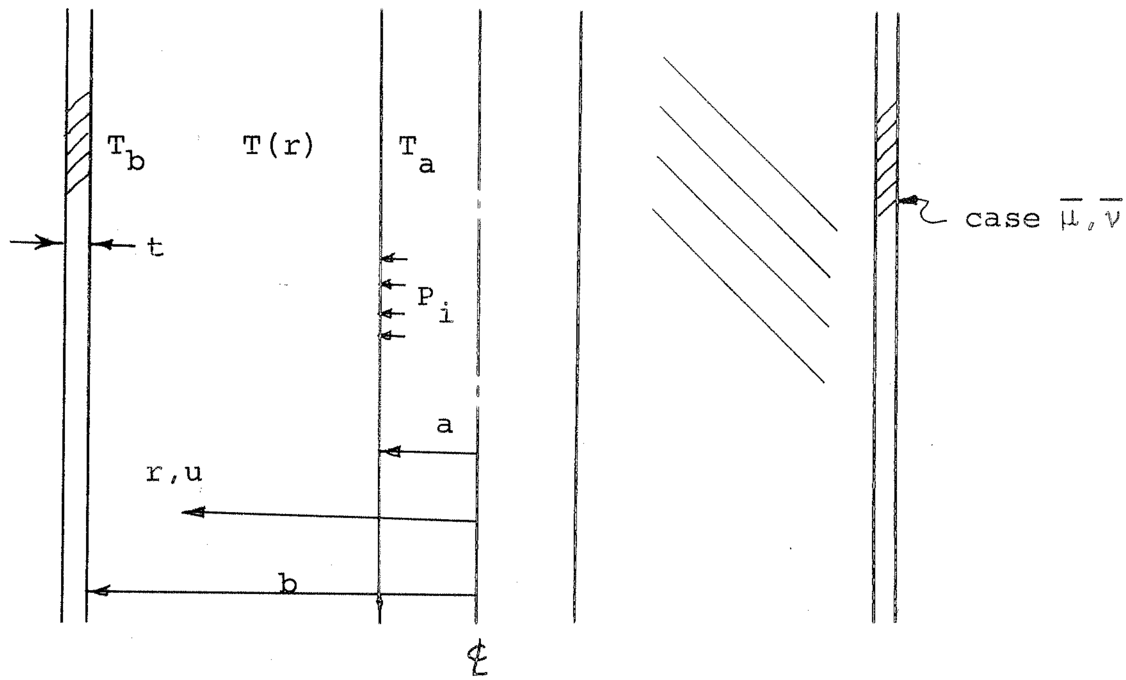
$$\tau_{xx} = \tau_{yy} = 0$$

$$\tau_{xy} = 2\mu\gamma \tag{2.5}$$

Inspecting Eqs. (2.4) and (2.5) we see that the results of a pure shear strain test would clearly show whether or not stress-induced anisotropy is present in a given material.

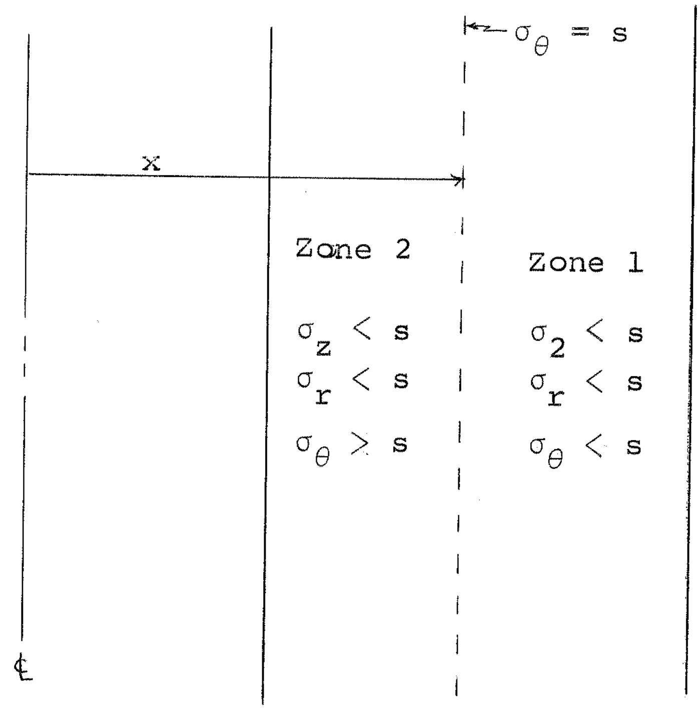
Axially Symmetric Plane Strain Thick-walled Cylinder Problems

A case-bonded thermoelastic cylinder in plane strain is shown below with identifying notation. As shall be shown later, Class I material appears to be a very poor model for actual propellants, therefore in the sequel we shall consider only Classes II and III. Depending upon the relative magnitudes of the various material parameters we may have an arbitrary distribution of zones; thus, one cannot anticipate all situations that may arise in a given physical problem. Therefore, we shall only be able to consider a few sample problems to illustrate the solution procedure that needs to be applied in a given problem.



Class II Material

Pressurization of a thick-walled cylinder, $\overline{\mu} = \overline{\nu} = 0$, with constant temperature, $T_a = T_b = T(r) = T_o$. For small internal pressure P_i, i.e., $P_i \leq P_c$, σ_r, σ_θ and σ_z are all less than the threshold value s; hence, the body behaves isotropically and the solution is well known [1]. At $P_1 = P_c$, $\sigma_\theta(a) = s$ and a zone boundary arises at r = a and propagates through the cylinder as P_i increases above P_c, i.e., we have



This state exists (assuming that $E_1 \leq E_2$, see Section 6) if

$$s\left(\frac{1}{2}c^{-m_2} + \frac{1}{2}c^{-m_1} - 1\right) \leq P_i \leq s\,\frac{c^2 - 1}{c^2 + 1}\,.$$

The resulting stress distributions will be as follows:

Zone 2: $a \leq r \leq x$

$$\sigma_r = -(P_i + s)\,\frac{\left(\frac{r}{x}\right)^{m_1} + \left(\frac{r}{x}\right)^{m_2}}{\left(\frac{a}{x}\right)^{m_1} + \left(\frac{a}{x}\right)^{m_2}} + s$$

$$\sigma_\theta = \sigma_r + r\,\frac{d\sigma_r}{dr}$$

$$u = r(k_2 - \frac{k_2^2}{k_1})\sigma_r + r(k_3 - \frac{k_2^2}{k_1})\sigma_\theta + s(k_1 - k_3)r$$

$$m_1 = -1 - \beta$$

$$m_2 = -1 + \beta$$

where

$$\beta = \sqrt{\frac{k_1^2 - k_2^2}{k_1 k_3 - k_2^2}}$$

Zone 1: $x \leq r \leq b$

$$\sigma_r = s \frac{1}{(\frac{b}{x})^2 + 1}\left[1 - (\frac{b}{r})^2\right]$$

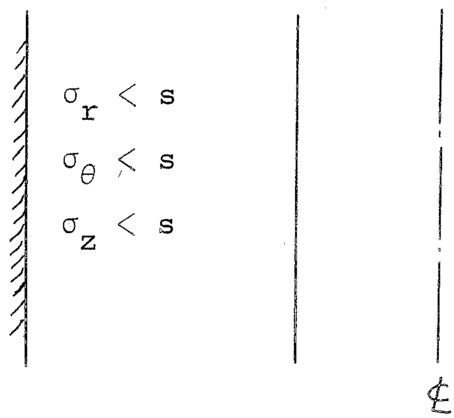
$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr}$$

$$u = r(k_2 - \frac{k_2^2}{k_1})\sigma_r + r(k_1 - \frac{k_2^2}{k_1})\sigma_\theta$$

where x is determined from the equation

$$\frac{c^2}{(\frac{x}{a})^2 + c^2} = \frac{\frac{P_i}{s} + 1}{(\frac{a}{x})^{m_1} + (\frac{a}{x})^{m_2}}.$$

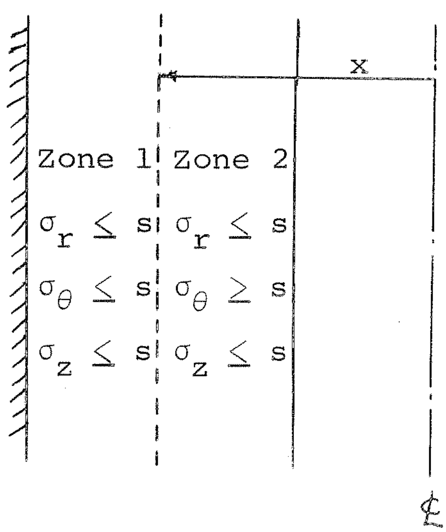
Uniform temperature drop of a thick-walled cylinder bonded to a rigid case, $\bar{\mu} = \infty$, $T_a = T_b = T(r) = T$. Let $T^* = T - T_o$. For $T^* \geq T_1$ we have



The material behaves isotropically and the solution is well known. When $T^* = T_1$, $\sigma_\theta(a) = s$ hence, a zone boundary arises (see Section 6) when

$$T^* = T_1 = -\frac{s}{2\alpha}\left[\frac{k_1}{c^2} + (k_1 + 2k_2)\right]$$

For $T_2 \leq T^* \leq T_1$, we have



In Zone 1

$$\sigma_\theta = A - \frac{D}{r^2} \tag{2.6}$$

$$\sigma_r = A + \frac{D}{r^2} \tag{2.7}$$

$$\sigma_z = -\frac{1}{k_1}(\alpha T^* + 2k_2 A) \tag{2.8}$$

$$u = (k_2 + k_1 - 2\frac{k_2^{\ 2}}{k_1})Ar + (k_2 - k_1)\frac{D}{r} + \alpha T^* r(1 - \frac{k_2}{k_1}) \tag{2.9}$$

$$D = x^2(A - s) \tag{2.10}$$

$$A = \frac{k_1 s(\frac{x}{b})^2 + \alpha T^*}{k_1(\frac{x}{b})^2 - (k_1 + 2k_2)} \tag{2.11}$$

In Zone 2

$$\sigma_r = \frac{F}{r^{\beta+1}} + Hr^{\beta-1} + s \tag{2.12}$$

$$\sigma_\theta = -\frac{\beta F}{r^{\beta+1}} + \beta H r^{\beta-1} + s \tag{2.13}$$

$$u = \left[k_2 - \beta k_3 + (\beta - 1)\frac{k_2^{\ 2}}{k_1}\right]\frac{F}{r^\beta} + \left[k_2 + \beta k_3 - (\beta+1)\frac{k_2^{\ 2}}{k_1}\right]H r^\beta$$
$$+ s\left[k_2 + k_1 - 2\frac{k_2^{\ 2}}{k_1}\right] r + \alpha T^*(1 - \frac{k_2}{k_1})r \tag{2.14}$$

$$F = Hx^{2\beta} \tag{2.15}$$

$$H = -\frac{as}{a^{\beta} + x^{2\beta}a^{-\beta}} \tag{2.16}$$

$$\beta = \sqrt{\frac{k_1^{\,2} - k_2^{\,2}}{k_1 k_3 - k_2^{\,2}}} \tag{2.17}$$

$$T_2 = s\left[\frac{2k_2}{c(c^{-\beta} + c^{\beta})} - (2k_2 + k_1)\right]$$

where one finds x from the equation

$$\frac{k_1 s(\frac{x}{b})^2 + \alpha T^*}{k_1(\frac{x}{b})^2 - (k_1 + 2k_2)} + \frac{(\frac{x}{b})^{\beta-1}\frac{s}{c}}{c^{-\beta} + (\frac{x}{b})^{2\beta}c^{\beta}} = s. \tag{2.18}$$

Class III Material

Pressurization of a case-bonded cylinder. When $\sigma_m \leq h$, (see Section 6) we have the usual classical solution, [1]. When $\sigma_m \geq h$, i.e., $P_i \geq P_c$, we have a second zone of action (for this particular problem the whole cylinder passes from one zone to the other at the same time), where

$$P_c = \frac{\dfrac{\mu(c^2-1)h}{c^2}\left[3x_1 - \dfrac{(1-n)}{3nB}\right] - \dfrac{h(1-n)}{\dfrac{c^2+3nBx_1}{(c^2-1)} + \dfrac{3(1-\bar{\nu})b\mu nBx_1}{\bar{\mu}t}}}{\dfrac{1}{c^2} - \dfrac{(\frac{1}{3nBx_1}+1)}{\mu(c^2-1)\left[\dfrac{c^2+3nBx_1}{3nBx_1\mu(c^2-1)} + \dfrac{(1-\bar{\nu})b}{\bar{\mu}t}\right]}}$$

$$x_1 = \frac{1}{3}\left(\frac{1}{3nB} + \frac{1}{\mu}\right)$$

$$\sigma_r = A + \frac{D}{r^2}$$

$$\sigma_\theta = A - \frac{D}{r^2}$$

$$u = \frac{r}{x_1}\left[\frac{1}{6\mu}\left(\frac{1}{3nB} - \frac{1}{2\mu}\right)\sigma_r + \frac{1}{6\mu}\left(\frac{2}{3nB} + \frac{1}{2\mu}\right)\sigma_\theta - \frac{h(1-n)}{6\mu nB}\right]$$

$$A = \frac{P_i - c^2 P'}{c^2 - 1}$$

$$D = \frac{b^2}{c^2 - 1}(P' - P_i)$$

$$P' = \frac{\dfrac{P_i}{(c^2-1)}\left[\dfrac{1}{3nBx_1} + 1\right] - \dfrac{h(1-n)}{3nBx_1}}{\dfrac{c^2 + 3nBx_1}{3nBx_1(c^2-1)} + \dfrac{\mu(1-\bar{\nu})b}{\bar{\mu}t}}.$$

Uniform temperature drop of a thick-walled cylinder bonded to a rigid case, $\bar{\mu} = \infty$. Again for $T^* \geq T_c$ we have the usual classical solution. As $T^* \rightarrow T_c$ the whole cylinder passes from one linear mode of action to another and we find

$$\sigma_r = A + \frac{D}{r^2} \tag{2.19}$$

$$\sigma_\theta = A - \frac{D}{r^2} \tag{2.20}$$

$$u = \frac{3}{2(\mu + 3nB)} Ar - \frac{1}{2\mu}\frac{D}{r} + \left(\alpha T^* - \frac{h(1-n)}{3nB}\right)\frac{9nBr}{2(\mu + 3nB)} \tag{2.21}$$

$$D = - a^2 A \tag{2.22}$$

$$A = \frac{h(1 - n) - 3\alpha T^* nB}{1 + \frac{1}{3c^2}\left(1 + \frac{3nB}{\mu}\right)} \tag{2.23}$$

$$T_c = - \frac{h}{\alpha} \frac{\frac{1}{\mu} + \frac{1}{3B}}{1 + \frac{3B}{\mu\left[1 + \frac{1}{3c^2}\left(1 + \frac{3B}{\mu}\right)\right]}} .$$

3. Model Fitting Considerations for a Particular Material

The plotted points in Figures 1 and 2 represent experimental data for a typical propellant that apparently experiences the dewetting phenomenon. The solid lines represent bilinear approximation of the uniaxial response. We find

B_1 = 74,600 psi		$k_1 = 1.243 \times 10^{-3}$ in^2/lb	
B_2 = 8,920 psi	which yields	$k_2 = -.621 \times 10^{-3}$ in^2/lb	
E_1 = 805 psi		$k_3 = 1.276 \times 10^{-3}$ in^2/lb.	
E_2 = 714 psi			

As the curves yielding B_1 , B_2 and E_1 are very easy to place accurately, we select these values as fundamental and calculate E_2 from the continuity relationships* (2.1), (2.2) and (2.3) respectively. We thus obtain for

Class I	E_2 =	6090 psi
Class II	E_2 =	784 psi
Class III	E_2 =	797 psi.

Comparing these values of E_2 with the experimental value of E_2 in Fig. 1, we see that the Class II and III models fit the experimental results much better than the Class I model. It is also apparent that the results from one simple test are not sufficient to select a crossover criterion for a given material. For example the results of the uniaxial test reported above would indicate that the above material may be represented by either a Class II or Class III model, but as we shall see in the next section, the two models give very different results when utilized in the solution of another problem. Thus it must be emphasized that one needs to examine the results of more than one stress state. If one were to perform a biaxial test ($\sigma_1 = \sigma_2 = \sigma$ and $\sigma_3 = 0$) on the above material and if it were to behave as a Class II material we would obtain Figures 3 and 4 whereas if it were to behave as a Class III material we would obtain Figures 5 and 6. The volume changes for the two postulated behaviors are compared in Figure 7. Although we do not have any biaxial test data taken from the same material as the uniaxial data presented

* We have called these equations continuity relationships since they arise from the fact that we have required a continuous action across the zone boundaries.

in Figures 1 and 2 we do have some biaxial test data taken from a somewhat similar propellant, as given in Figure 8. It will be noted that if we represent this data by a bilinear response, the crossover point would be much lower than predicted by the Class II representation (see Figure 4) for the first type of propellant. The crossover point would be slightly higher than that given by the Class III representation (see Figure 6), for the first propellant. If we make the crossover occur at the value as given by the Class III biaxial test (see Figure 6) the fit is still rather good (see Figure 9). From these very crude data, tentatively, it appears that the bilinear phenomenon may be represented by the Class III criterion. However, it must be emphasized that the above conclusion is a mere conjecture as it is based upon only two types of tests performed upon slightly different propellant formulations, tests which were not equilibrium tests and therefore are not a true indication of the elastic equilibrium action (also the accuracy of the tests is somewhat in question). It should be noted that the biaxial volume changes are about ten times greater than expected, see Figures 6 and 8; this may be due to one of the following causes: (1) different propellant formulations, (2) rate effects and/or (3) experimental inaccuracies. In light of these considerations it must be pointed out that the above analysis is merely indicative of the type of analysis and experiments that must be made to characterize the bilinear action of a given propellant.

4. Numerical Example

Making use of the elastic constants, obtained by fitting first a Class II then a Class III model to the uniaxial test data of Figures 1 and 2, we have evaluated some of the dependent variables that arise in the problem of the uniform temperature drop of a thick-walled cylinder bonded to a rigid case using the results from Section 2. The results are plotted in Figures 10 - 19. For the Class II representation the material behaves isotropically down to a temperature of 39.6° F at which time $\sigma_{\theta}(a) = s$, hence a zone boundary arises at $r = a$. As the material is further cooled this zone boundary moves across the cylinder and reaches the outer boundary at $T = 0^{\circ}$ F (see Figure 10), at this point $\sigma_z(b) = s$; thus a new zone boundary arises and proceeds toward $r = a$. Although the action within a given zone is linear, the fact that the zone boundary is moving results in a nonlinear response (see Figure 15).

One will note that the effects of the stress-induced anisotropy, for this problem, are very small and the results are nearly coincidental with the results obtained by assuming isotropic behavior throughout (see the dotted lines in Figures 11 - 15).

For the Class III representation of the same material the zone boundary arises throughout the cylinder at $T = 53^{\circ}$ F and thus the response is in two linear segments, see Figure 16. In this case the resulting response is considerably different from that obtained by ignoring bilinear effects (see the dotted lines). Comparing Figures 11 - 15 to Figures 16 - 19, respectively, one will see the considerable difference in results obtained by considering that the material behaves as a Class II or Class III model. As a uniaxial test tended to indicate that either model would suitably represent the material, the above results once again emphasize the need to consider several stress states in selecting an appropriate model for a given material. The data used in the previous calculations were as follows;

$c = 2,5,10$ (as indicated on the Figures)

$\overline{\mu} = \infty$

$\overline{\alpha} = 0$

$\alpha = 6.0 \times 10^{-5}$ in/in^{o}F

$B = 74,600$ psi

$n = .1195$

$\mu = 268$ psi

$h = 30$ psi

$k_1 = 1.243 \times 10^{-3}$ in^2/lb

$k_2 = -.621 \times 10^{-3}$ in^2/lb

$k_3 = 1.276 \times 10^{-3}$ in^2/lb

$s = 90$ psi

APPENDIX

5. Derivation of Bilinear Constitutive Equations

As the derivation of the constitutive equations for a Class I material is exactly paralleled by the derivation of the constitutive equations for a Class II material we shall only present the latter.

In the derivation of the governing field equations for a Class II bilinear material we shall assume that the material is at all time phenomenologically continuous and that within each zone of action it is linearly elastic and homogeneous. Thus the equilibrium equations and strain displacement relations are identical to those for classical elasticity.

Let the criterion for passage from one mode of action to another be the passage of the value of one of the principal stresses through the threshold value s. Thus we identify four zones, i.e.,

1. $\sigma_1 < s \ , \ \sigma_2 < s \ , \ \sigma_3 < s$

2. $\sigma_1 > s \ , \ \sigma_2 < s \ , \ \sigma_3 < s$

3. $\sigma_1 > s \ , \ \sigma_2 > s \ , \ \sigma_3 < s$

4. $\sigma_1 > s \ , \ \sigma_2 > s \ , \ \sigma_3 > s$.

Zones 1 and 4 will be isotropic; zones 2 and 3 will be anisotropic. Thus we may write

Zone 1 $\quad \sigma_1 < s \ , \ \sigma_2 < s \ , \ \sigma_3 < s$

$$\begin{aligned} \epsilon_1 &= k_1\sigma_1 + k_2\sigma_2 + k_2\sigma_3 \\ \epsilon_2 &= k_2\sigma_1 + k_1\sigma_2 + k_2\sigma_3 \\ \epsilon_3 &= k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3 \end{aligned} \tag{5.1}$$

Zone 2 $\sigma_1 > s \; , \; \sigma_2 < s \; , \; \sigma_3 < s$

$$\epsilon_1 = k_3\sigma_1 + k_4\sigma_2 + k_4\sigma_3 + D_1$$

$$\epsilon_2 = k_4\sigma_1 + k_5\sigma_2 + k_6\sigma_3 + D_2$$

$$\epsilon_3 = k_4\sigma_1 + k_6\sigma_2 + k_5\sigma_3 + D_3$$

Zone 3 $\sigma_1 > s \; , \; \sigma_2 > s \; , \; \sigma_3 < s$

$$\epsilon_1 = k_7\sigma_1 + k_8\sigma_2 + k_9\sigma_3 + D_3$$

$$\epsilon_2 = k_8\sigma_1 + k_7\sigma_2 + k_9\sigma_3 + D_3 \qquad (5.1)$$

$$\epsilon_3 = k_9\sigma_1 + k_9\sigma_2 + k_{10}\sigma_3 + D_4$$

Zone 4 $\sigma_1 > s \; , \; \sigma_2 > s \; , \; \sigma_3 > s$

$$\epsilon_1 = k_{11}\sigma_1 + k_{12}\sigma_2 + k_{12}\sigma_3 + D_5$$

$$\epsilon_2 = k_{12}\sigma_1 + k_{11}\sigma_2 + k_{12}\sigma_3 + D_5$$

$$\epsilon_3 = k_{12}\sigma_1 + k_{12}\sigma_2 + k_{11}\sigma_3 + D_5.$$

To insure a continuous* passage from one mode of action to another we must obtain relationships for the "interzone" (i.e. when $\sigma_i = s$) which are independent of the path traveled in reaching the interzone, i.e., the relations of Zone 1 as $\sigma_1 \rightarrow s^-$ must be identical to those of Zone 2 as $\sigma_1 \rightarrow s^+$, etc. Equating the constitutive equations of Zone 1 and Zone 2

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* A proposed generalization of the bilinear theory in which this condition is relaxed is outlined in Section 7.

as $\sigma_1 \rightarrow s$ we obtain

$$k_4 = k_2 \tag{5.2}$$

$$D_1 = s(k_1 - k_3) \tag{5.3}$$

$$k_5 = k_1 \tag{5.4}$$

$$k_6 = k_2 \tag{5.5}$$

$$D_2 = 0. \tag{5.6}$$

Equating the constitutive equations of Zone 2 and Zone 3 as $\sigma_2 \rightarrow s$ and using Equations (5.2) to (5.6), we obtain

$$k_7 = k_3 \tag{5.7}$$

$$k_9 = k_2 \tag{5.8}$$

$$k_8 = k_2 \tag{5.9}$$

$$D_3 = s(k_1 - k_3) \tag{5.10}$$

$$D_4 = 0 \tag{5.11}$$

$$k_{10} = k_1. \tag{5.12}$$

Equating the constitutive equations of Zone 3 and Zone 4 as $\sigma_3 \rightarrow s$ we obtain

$$\begin{aligned} k_{11} &= k_3 \\ k_{12} &= k_2 \\ D_5 &= s(k_1 - k_3). \end{aligned} \tag{5.13}$$

Using the above relationships we may write Equation (5.1) as

Zone 1

$$\sigma_1 \leq s \; , \; \sigma_2 \leq s \; , \; \sigma_3 \leq s$$

$$\epsilon_1 = k_1\sigma_1 + k_2\sigma_2 + k_2\sigma_3$$

$$\epsilon_2 = k_2\sigma_1 + k_1\sigma_2 + k_2\sigma_3$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3$$

Zone 2

$$\sigma_1 \geq s \; , \; \sigma_2 \leq s \; , \; \sigma_3 \leq s$$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_2 = k_2\sigma_1 + k_1\sigma_2 + k_2\sigma_3$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3$$

(5.14)

Zone 3

$$\sigma_1 \geq s \; , \; \sigma_2 \geq s \; , \; \sigma_3 \leq s$$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_2 = k_2\sigma_1 + k_3\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_1\sigma_3$$

Zone 4

$$\sigma_1 \geq s \; , \; \sigma_2 \geq s \; , \; \sigma_3 \geq s$$

$$\epsilon_1 = k_3\sigma_1 + k_2\sigma_2 + k_2\sigma_3 + s(k_1-k_3)$$

$$\epsilon_2 = k_2\sigma_1 + k_3\sigma_2 + k_2\sigma_3 + s(k_1 - k_3)$$

$$\epsilon_3 = k_2\sigma_1 + k_2\sigma_2 + k_3\sigma_3 + s(k_1 - k_3).$$

We shall now consider the stress-strain relations for arbitrary directions. Let y^i be principal axes (along which τ^{ii} acts) and x^i be arbitrary axes (along which $\bar{\tau}^{\alpha\beta}$ acts); both systems are assumed orthogonal.

Now
$$\bar{\epsilon}^{\alpha\beta} = \epsilon^{ij} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}$$

but
$$\epsilon^{ij} = 0 \quad \text{for} \quad i \neq j.$$

thus
$$\bar{\epsilon}^{\alpha\beta} = \epsilon^{ii} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i}$$

from (5.14) we may write

$$\epsilon^{ii} = k^{ii}_{jj} \tau^{jj} + D^{ii}$$

also
$$\tau^{jj} = \bar{\tau}^{\gamma\lambda} \frac{\partial y^j}{\partial x^\gamma} \frac{\partial y^i}{\partial x^\lambda}$$

thus
$$\bar{\epsilon}^{\alpha\beta} = \left(k^{ii}_{jj}\, \bar{\tau}^{\gamma\lambda} \frac{\partial y^j}{\partial x^\gamma} \frac{\partial y^j}{\partial x^\lambda} + D^{ii}\right) \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i}$$

$$\bar{\epsilon}^{\alpha\beta} = k^{ii}_{jj} \frac{\partial y^j}{\partial x^\gamma} \frac{\partial y^j}{\partial x^\lambda} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i} \bar{\tau}^{\gamma\lambda} + D^{ii} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i} \qquad (5.15)$$

therefore
$$\bar{k}^{\alpha\beta}_{\gamma\lambda} = k^{ii}_{jj} \frac{\partial y^j}{\partial x^\gamma} \frac{\partial y^j}{\partial x^\lambda} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i}, \quad \bar{D}^{\alpha\beta} = D^{ii} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^i}$$

or if we define $k^{ik}_{jm} = 0$ for $i \neq k$ or $j \neq m$ or both, then k^{ij}_{km} is a fourth rank tensor; similarly D^{ij} is a rank two tensor.

In the derivation of the constitutive equations for a Class III material we shall assume that the criterion for passage from one mode of action to another is governed by the passage of the mean stress σ_m through the threshold value h.

For the criterion of $\sigma_m = h$ we have two possible zones of action, i.e., Zone 1 ($\sigma_m \leq h$) and Zone 2 ($\sigma_m \geq h$). We shall let the quantities of Zone 1 be denoted by a subscript, e.g., B_1 etc.

Thus for Zone 1

$$\sigma_m \leq h$$

$$\sigma_m = 3B_1 \epsilon_m$$

$$s^i_j = 2\mu_1 e^i_j$$

Zone 2

$$\sigma_m \geq h$$

$$\sigma_m = 3B_2 \epsilon_m + D_m$$

$$s^i_j = 2\mu_2 e^i_j + D^i_j$$

Considering the interzone as in the previous derivation we have for

$$\sigma_m \rightarrow h^-$$

$$h = 3B_1 \epsilon_m$$

$$s^i_j = 2\mu_1 e^i_j$$

$$\sigma_m \to h^+$$

$$h = 3B_2\epsilon_m + D_m$$

$$s^i_j = 2\mu_2 e^i_j + D^i_j.$$

Thus we must have the following relations

$$\mu_1 = \mu_2$$

$$D^i_j = 0$$

$$D_m = h(1 - \frac{B_2}{B_1}).$$

Using the above equations we may write the constitutive equations as (let $n = \frac{B_2}{B_1}$ and $B = B_1$)

Zone 1 $\quad \sigma_m \leq h$

$$\tau^i_j = (B - \tfrac{2}{3}\mu)\theta\delta^i_j + 2\mu\epsilon^i_j$$

or $\quad \epsilon^i_j = \frac{1}{3}(\frac{1}{3B} - \frac{1}{2\mu})\Theta\delta^i_j + \frac{1}{2\mu}\tau^i_j$

Zone 2 $\quad \sigma_m \geq h$

$$\tau^i_j = (nB - \tfrac{2}{3}\mu)\theta\delta^i_j + 2\mu\epsilon^i_j + h(1 - n)\delta^i_j$$

or $\quad \epsilon^i_j = \frac{1}{3}(\frac{1}{3nB} - \frac{1}{2\mu})\Theta\delta^i_j + \frac{1}{2\mu}\tau^i_j - \frac{h(1 - n)}{3Bn}\delta^i_j.$

6. Derivation of Elementary Solutions

A. Analysis of a pure shear strain test ($\tau_{zz} = 0$) for a Class II material when $\sigma_1 \geq s$, $\epsilon_{xx} = \epsilon_{yy} = 0$, and $\epsilon_{xy} = \gamma$. For pure shear strain the principal axes will be at 45° to the x and y axis. Using Equation (5.15)

$$\epsilon_{xx} = 0 = (k_3 + 2k_2 + k_1)\tau_{xx} + 2(k_3 - k_1)\tau_{xy} + (k_3 + k_1)\tau_{yy} + 2s(k_1 - k_3) \tag{6.1}$$

$$\epsilon_{yy} = 0 = (k_3 + k_1)\tau_{xx} + 2(k_3 - k_1)\tau_{xy} + (k_3 + 2k_2 + k_1)\tau_{yy} + 2s(k_1 - k_3)$$

$$4\epsilon_{xy} = 4\gamma = (k_3 - k_1)\tau_{xx} + 2(k_1 + k_3 - 2k_2)\tau_{xy} + (k_3 - k_1)\tau_{yy} + 2s(k_1 - k_3). \tag{6.2}$$

From Equation (6.1)

$$\tau_{xx} = \tau_{yy} = \frac{2(k_3 - k_1)\left[(k_1 - k_2)s - \gamma\right]}{4k_1k_3 - k_2(k_1 + k_3 + 2k_2)}$$

and from Equation (6.2)

$$\tau_{xy} = \frac{s(k_3 - k_1)(2k_1 + k_2) + 2\gamma(k_1 + k_3 + k_2)}{4k_1k_3 - k_2(k_1 + k_3 + 2k_2)}.$$

Thus,

$$\sigma_1 = \frac{(k_3 - k_1)(4k_1 - k_2)s + 2(2k_1 + k_2)\gamma}{4k_1k_3 - k_2(k_1 + k_3 + 2k_2)}.$$

As we assumed that $\sigma_1 \geq s$, we find that when $\sigma_1 = s$,

$$\gamma_c = (k_1 - k_2)s.$$

B. Pressurization of a hollow cylinder* under plane strain for a Class II material.

The appropriate field equations are:

(1) equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6.3}$$

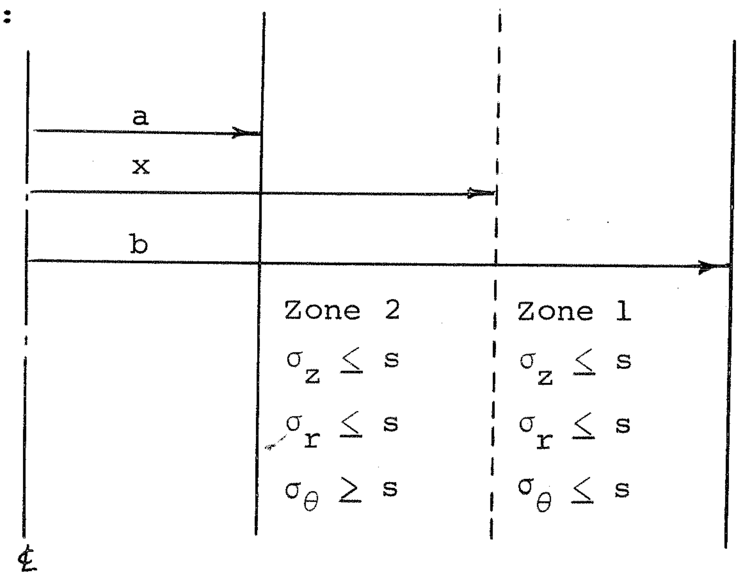
(2) strain-displacement equations

$$\epsilon_r = \frac{du_r}{dr}$$

$$\epsilon_\theta = \frac{u_r}{r} \tag{6.4}$$

$$\epsilon_z = 0$$

The constitutive equations are given by Equations (5.14). We shall consider loading conditions such that the following zones are present:



*The anisotropic thick-walled cylinder solutions are given in Part I and may be used to construct the bilinear solutions for a thick-walled cylinder where s = 0. However, we are here considering the situation where $s \neq 0$ and thus we shall construct the necessary anisotropic solutions as needed.

Thus in Zone 2, $a \leq r \leq x$

$$\epsilon_\theta = k_2\sigma_r + k_3\sigma_\theta + k_2\sigma_z + s(k_1 - k_3)$$

$$\epsilon_r = k_1\sigma_r + k_2\sigma_\theta + k_2\sigma_z \tag{6.5}$$

$$\epsilon_z = k_2\sigma_r + k_2\sigma_\theta + k_1\sigma_z$$

Combining Equations (6.3), (6.4) and (6.5) and solving the resulting differential equation we obtain

$$\sigma_r = Ar^{m_1} + Dr^{m_2} + s \tag{6.6}$$

$$\sigma_z = -\frac{k_2}{k_1}(\sigma_r + \sigma_\theta)$$

$$u = r(k_2 - \frac{k_2^2}{k_1})\sigma_r + r(k_3 - \frac{k_2^2}{k_1})\sigma_\theta + s(k_1 - k_3)r$$

$$\sigma_\theta = A(1 + m_1)r^{m_1} + D(1 + m_2)r^{m_2} + s \tag{6.7}$$

where $m_1 = -1 - \beta$

$$m_2 = -1 + \beta \tag{6.8}$$

and $$\beta = \sqrt{\frac{k_1^2 - k_2^2}{k_1k_3 - k_2^2}}\,. \tag{6.9}$$

The boundary conditions for this zone may be written as:

$$r = a \qquad \sigma_r = -P_i$$

$$r = x \qquad \sigma_\theta = s \tag{6.10}$$

Applying the boundary conditions to Equations (6.6) and (6.7) we obtain

$$A = \frac{-(P_i + s)(1 + m_2)x^{m_2}}{a^{m_1}(1 + m_2)x^{m_2} - a^{m_2}(1 + m_1)x^{m_1}}$$

$$D = \frac{(P_i + s)(1 + m_1)x^{m_1}}{a^{m_1}(1 + m_2)x^{m_2} - a^{m_2}(1 + m_1)x^{m_1}}.$$

Thus Equation (6.6) becomes

$$\sigma_r = -(P_i + s) \left[\frac{(\frac{r}{x})^{m_1} + (\frac{r}{x})^{m_2}}{(\frac{a}{x})^{m_1} + (\frac{a}{x})^{m_2}}\right] + s. \qquad (6.11)$$

In Zone 1

$$\epsilon_r = \frac{du_r}{dr} = k_1\sigma_r + k_2\sigma_\theta + k_2\sigma_z$$

$$\epsilon_\theta = \frac{u_r}{r} = k_2\sigma_r + k_1\sigma_\theta + k_2\sigma_z \qquad (6.12)$$

$$\epsilon_z = 0 = k_2\sigma_r + k_2\sigma_\theta + k_1\sigma_z$$

Thus

$$\sigma_z = -\frac{k_2}{k_1}(\sigma_r + \sigma_\theta)$$

Combining Equations (6.3), (6.4) and (6.12) and solving the resulting differential equation we obtain

$$\sigma_r = A + \frac{D}{r^2} \tag{6.13}$$

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} = A - \frac{D}{r^2}. \tag{6.14}$$

The boundary conditions may be written as

$$r = x \qquad \sigma_\theta = s$$

$$r = b \qquad \sigma_r = 0.$$

Applying the above boundary conditions to Equations (6.13) and (6.14) we find

$$D = -\frac{sb^2x^2}{b^2 + x^2}$$

$$A = \frac{sx^2}{b^2 + x^2}.$$

Then Equation (6.13) becomes

$$\sigma_r = \frac{sx^2}{b^2 + x^2}\left(1 - \frac{b^2}{r^2}\right). \tag{6.15}$$

To find x we set $\sigma_r(x^-) = \sigma_r(x^+)$. Using Equation (6.11) and (6.15) we obtain

$$\frac{s(x^2 - b^2)}{x^2 + b^2} = -(P_i + s)\left[\frac{2}{\left(\frac{a}{x}\right)^{m_1} + \left(\frac{a}{x}\right)^{m_2}}\right] + s$$

or

$$\frac{\left(\frac{b}{a}\right)^2}{\left(\frac{x}{a}\right)^2 + \left(\frac{b}{a}\right)^2} = \frac{\frac{P_i}{s} + 1}{\left(\frac{a}{x}\right)^{m_1} + \left(\frac{a}{x}\right)^{m_2}}. \tag{6.16}$$

We shall now investigate the pressure range for which the assumed zone configuration will exist. One limit will be when $x = b$; from Equation(6.16) we find

$$\frac{(\frac{b}{a})^2}{(\frac{b}{a})^2 + (\frac{b}{a})^2} = \frac{\frac{P_1}{s} + 1}{(\frac{a}{b})^{m_1} + (\frac{a}{b})^{m_2}}$$

or

$$P_1 = s\left[\frac{1}{2}(\frac{a}{b})^{m_2} + \frac{1}{2}(\frac{a}{b})^{m_1} - 1\right].$$

The other limit will be when $x = a$. To find this critical pressure we set $x = a$ in Equation (6.16)

$$\frac{s(a^2 - b^2)}{a^2 + b^2} = -P_2$$

or

$$P_2 = s\,\frac{(\frac{b}{a})^2 - 1}{(\frac{b}{a})^2 + 1}.$$

The assumption that $\sigma_r \leq s$ is obviously true. We also have assumed that $\sigma_z \leq s$; thus we must now impose this condition (it may be easily shown to be true in Zone 1) now (for $a \leq x \leq b$, $x \geq r \geq a$)

$$\sigma_z = -\frac{k_2}{k_1}(\sigma_r + \sigma_\theta) = -\frac{k_2}{k_1}(2\sigma_r + r\,\frac{d\sigma_r}{dr}).$$

Therefore we have

$$s \geq -\frac{k_2}{k_1}\left[-(P_i + s)\left[\frac{(2 + m_1)(\frac{r}{x})^{m_1} + (2 + m_2)(\frac{r}{x})^{m_2}}{(\frac{a}{x})^{m_1} + (\frac{a}{x})^{m_2}}\right] + 2s\right].$$

Using Equations (6.8) and (6.16) we have (note $-\frac{k_2}{k_1} = \nu_1$, see Section 2)

$$(1 - 2\nu_1) \geq \frac{\nu_1(\frac{b}{a})^2(\frac{x}{r})}{(\frac{x}{a})^2 + (\frac{b}{a})^2}\left[(\beta-1)(\frac{x}{r})^{\beta} - (\beta+1)(\frac{r}{x})^{\beta}\right]$$

but

$$(1 - 2\nu_1) \geq 0$$

$$\nu_1 > 0$$

$$\frac{(\frac{b}{a})^2(\frac{x}{r})}{(\frac{x}{a})^2 + (\frac{b}{a})^2} > 0.$$

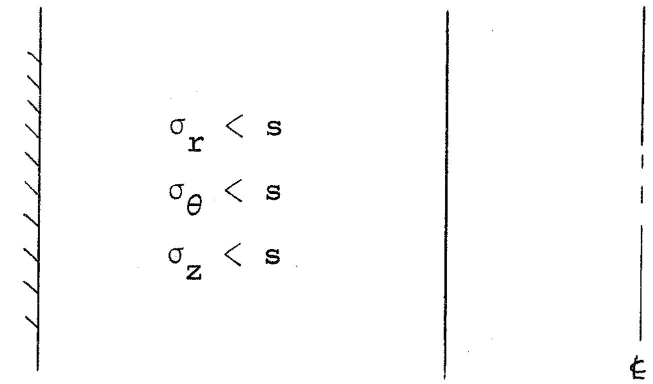
Also if $E_2 \leq E_1$, see Section 3, then $\beta \leq 1$ from Equation (6.9) and

$$\left[(\beta - 1)(\frac{x}{r})^{\beta} - (\beta + 1)(\frac{r}{x})^{\beta}\right] < 0$$

thus,

$$\sigma_z \leq s.$$

C. <u>Uniform temperature drop of a thick-walled cylinder bonded to a rigid case, Class II material</u>. Let $T^* = T - T_o$. For $0 \geq T^* \geq T_1$, we have



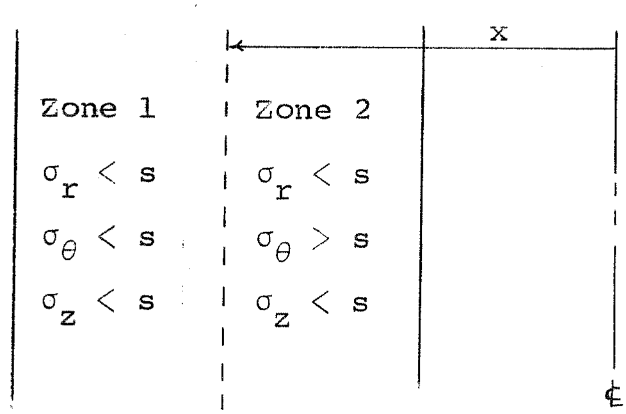
The material remains isotropic and the solution is well known, see Part III. A zone boundary will arise at $r = a$ when $\sigma_\theta = s$; from Part III we find

$$\sigma_\theta(a) = s = -\frac{6\alpha T_1 B}{1 + \frac{1}{3c^2}(1 + \frac{3B}{\mu})}$$

or

$$T_1 = \frac{-s\left[1 + \frac{1}{3c^2}(1 + \frac{3B}{\mu})\right]}{6\alpha B}. \tag{6.17}$$

For $T_2 \leq T^* \leq T_1$, we have



The equilibrium equation is given by Equation (6.3), the strain-displacement equation by Equation (6.4) and the constitutive equation* by Equation (5.14) where we add to each equation the term α T*. Combining the resulting equations and solving we obtain for Zone 1

$$\sigma_\theta = A - \frac{D}{r^2} \tag{6.18}$$

$$\sigma_r = A + \frac{D}{r^2} \tag{6.19}$$

$$\sigma_z = -\frac{1}{k_1}(\alpha T^* + 2k_2 A) \tag{6.20}$$

$$u = (k_2 + k_1 - 2\frac{k_2^2}{k_1})Ar + (k_2 - k_1)\frac{D}{r} + \alpha T^* r(1 - \frac{k_2}{k_1}), \tag{6.21}$$

The boundary conditions may be written as

$$r = b \qquad u = 0$$

$$r = x \qquad \sigma_\theta = s$$

hence

$$D = x^2(A - s) \tag{6.22}$$

$$A = \frac{k_1 s \frac{x^2}{b^2} + \alpha T^*}{k_1 \frac{x^2}{b^2} - (k_1 + 2k_2)}. \tag{6.23}$$

*At present we are not in possession of any experimental data which would indicate whether the linear coefficent of thermal expansion changes value as we pass from one zone of action to another or remains a constant; for simplicity we have assumed that it remains a constant.

For Zone 2

$$\sigma_r = \frac{F}{r^{\beta+1}} + Hr^{\beta-1} + s \tag{6.24}$$

$$\sigma_\theta = -\frac{\beta F}{r^{\beta+1}} + \beta Hr^{\beta-1} + s \tag{6.25}$$

$$\sigma_z = -\frac{\alpha T^*}{k_1} - \frac{k_2}{k_1}\left[2s + (\beta+1)Hr^{\beta-1} + (1-\beta)\frac{F}{r^{\beta+1}}\right] \tag{6.26}$$

$$u = \left[k_2 - \beta k_3 + (\beta-1)\frac{k_2^2}{k_1}\right]\frac{F}{r^\beta} + \left[k_2 + \beta k_3 - (\beta+1)\frac{k_2^2}{k_1}\right] Hr^\beta$$

$$+ s\left[k_2 + k_1 - 2\frac{k_2^2}{k_1}\right] r + \alpha T^*\left(1 - \frac{k_2}{k_1}\right) r \tag{6.27}$$

where

$$\beta = \sqrt{\frac{k_1^2 - k_2^2}{k_1 k_3 - k_2^2}} \tag{6.28}$$

The boundary conditions for Zone 2 may be written as

$$r = x, \qquad \sigma_\theta = s$$

$$r = a, \qquad \sigma_r = 0$$

which yields

$$F = Hx^{2\beta} \tag{6.29}$$

$$H = -\frac{as}{a^\beta + x^{2\beta}a^{-\beta}}\,. \tag{6.30}$$

The zone boundary x may be found from

$$\sigma_r(x^-) = \sigma_r(x^+)$$

Using Equations (6.18), (6.22), (6.23), (6.24), (6.29) and (6.30) we obtain

$$\frac{k_1 s \frac{x^2}{b^2} + \alpha T^*}{k_1 \frac{x^2}{b^2} - (k_1 + 2k_2)} + \frac{(\frac{x}{b})^{\beta-1} \frac{s}{c}}{c^{-\beta} + (\frac{x}{b})^{2\beta} c^{\beta}} = s. \qquad (6.31)$$

T_2 may be found by setting x = b in Equation (6.31):

$$\frac{k_1 s + \alpha T_2}{k_1 - (k_1 + 2k_2)} + \frac{\frac{s}{c}}{c^{-\beta} + c^{\beta}} = s$$

or

$$T_2 = \frac{s}{\alpha}\left[\frac{2k_2}{c(c^{-\beta} + c^{\beta})} - (2k_2 + k_1)\right]$$

It is now necessary to consider the assumption that $\sigma_z \leq s$ (as σ_z = constant in Zone 1 we see that the maximum σ_z in Zone 2 will at least have to be equal to σ_z in Zone 1 as, $\sigma_z(x^-) = \sigma_z(x^+)$; thus we need only consider Zone 2). Now in Zone 2

$$\sigma_z = -\frac{\alpha T^*}{k_1} - \frac{k_2}{k_1}\left[2s + \frac{Hx^{\beta}}{r}\left[(1+\beta)(\frac{r}{x})^{\beta} + (1-\beta)(\frac{x}{r})^{\beta}\right]\right].$$

We seek a maximum of this function in the interval $a \leq r \leq x$, thus setting

$$\frac{d\sigma_z}{dr} = 0 = \frac{-k_2}{k_1} H x^{\beta} \left[\frac{(1+\beta)}{x^{\beta}} (\beta-1) r^{\beta-2} - (1-\beta) x^{\beta} (\beta+1) r^{-\beta-2} \right]$$

or

$$r^{2\beta} = -x^{2\beta}.$$

Accordingly, no maximum occurs in this interval. It will be necessary to check the end points of the interval in each particular problem.

Now

$$\sigma_z(x) = -\frac{\alpha T^*}{k_1} - \frac{k_2}{k_1} (2s + 2H\, x^{\beta-1}) \tag{6.32}$$

$$\sigma_z(a) = - - \frac{\alpha T^*}{k_1} - \frac{k_2}{k_1} \left[2s + \frac{Hx^{\beta}}{a} \left[(1+\beta) (\tfrac{a}{x})^{\beta} + (1-\beta) (\tfrac{x}{a})^{\beta} \right] \right] \tag{6.33}$$

D. <u>Uniform temperature drop of a thick-walled cylinder bonded to a rigid case, Class III material</u>. From the classical elastic solution, see Part III, we see that the mean stress is a constant, thus the mean stress will reach its threshold value h throughout the cylinder at the same critical temperature T_1. Let us consider the case when $T^* \leq T_1$. Combining the equilibrium equation (6.3), the strain displacement equation (6.4) and the Zone 2 constitutive equation (1.3) and solving we obtain

$$\sigma_r = A + \frac{D}{r^2}$$

$$\sigma_\theta = A - \frac{D}{r^2}$$

$$\sigma_m = \frac{3nB\mu}{\mu+3nB}\left[\frac{A}{\mu} - \alpha T^* + \frac{h(1-n)}{3nB}\right]$$

$$u = \frac{3}{2(\mu+3nB)}Ar - \frac{1}{2\mu}\frac{D}{r} + (\alpha T^* - \frac{h(1-n)}{3nB})\frac{9nB}{2(\mu+3nB)}r.$$

The boundary conditions are

$$r = b,\ u = 0$$

$$r = a,\ \sigma_r = 0$$

which yield

$$D = -a^2A$$

$$A = \frac{h(1-n)-3\alpha T^*nB}{1 + \frac{1}{3c^2}(1+\frac{3nB}{\mu})}.$$

Note that the constitutive equations for Zone 1, see Equation (1.3), may be obtained from the constitutive equations for Zone 2, by letter h = 0 and n = 1 in the latter; thus we may obtain the solution for Zone 1 (i.e. $T^* \geq T_1$) by setting h = 0 and n = 1 in the above solution. For example, in Zone 1

$$u = \frac{3}{2}(\frac{1}{\mu + 3B})Ar + \frac{1}{2\mu}\frac{a^2A}{r} + \frac{a\alpha T^*B}{2(\mu + 3B)}r$$

$$A = \frac{-3\alpha T^*B}{1 + \frac{1}{3c^2}(1 + \frac{3B}{\mu})}$$

$$\sigma_m = \frac{3B\mu}{\mu + 3B}\left[\frac{A}{\mu} - \alpha T^*\right].$$

Now to find T_1 we set $\sigma_m = h$ and $T^* = T_c$ in the above equation i.e.

$$h = \frac{3B\mu}{\mu + 3B}\left[-\frac{3\alpha T_c B}{\mu\left[1 + \frac{1}{3c^2}(1 + \frac{3B}{\mu})\right]} - \alpha T_c\right]$$

$$\text{or} \quad T_c = \frac{-(\mu + 3B)h}{3B\mu\alpha} \; \frac{1}{1 + \frac{3B}{\mu\left[1 + \frac{1}{3c^2}\left(1 + \frac{3B}{\mu}\right)\right]}} .$$

E. <u>Pressurization of a thick-walled cylinder (Class III material) bonded to an elastic case</u>. As in the previous problem the classical solution, see [1], shows us that the complete cylinder passes from the first zone of action to the second at the instant that $P_i = P_c$. We find from Equations (6.3), (6.4) and (1.3) (for Zone 2)

$$\sigma_r = A + \frac{D}{r^2} \qquad (6.34)$$

$$\sigma_\theta = A - \frac{D}{r^2} \qquad (6.35)$$

$$\sigma_z = -2\frac{k_2}{k_1}A \qquad (6.36)$$

$$u_r = r(k_2 - \frac{k_2^2}{k_1})(A + \frac{D}{r^2}) + r(k_1 - \frac{k_2^2}{k_1})(A - \frac{D}{r^2}) - \frac{h(1-n)}{3Bn}(1 - \frac{k_2}{k_1})$$

$$3\sigma_m = \frac{\frac{A}{\mu} + \frac{h(1-n)}{3nB}}{k_1} \qquad (6.37)$$

where $$k_1 = \frac{1}{3}\left(\frac{1}{3nB} + \frac{1}{\mu}\right)$$

$$k_2 = \frac{1}{3}\left(\frac{1}{3nB} - \frac{1}{2\mu}\right).$$

We may write the boundary conditions as

$$r = a, \ \sigma_r = -P_i$$

$$r = b, \ \sigma_r = -P'$$

which yield

$$D = -a^2(A + P_i)$$

$$A = \frac{a^2P_i - b^2P'}{b^2 - a^2}.$$

Hence,

$$u_r = r\left[\left(k_2 + k_1 - \frac{2k_2^2}{k_1}\right)\frac{a^2P_i - b^2P'}{b^2 - a^2} - \left(1 - \frac{k_2}{k_1}\right)\frac{h(1-n)}{3nB}\right]$$

$$+ \frac{(k_2 - k_1)a^2b^2(P' - P_i)}{r(b^2 - a^2)}. \qquad (6.38)$$

The displacement of the motor case will be

$$\bar{u}_r = \frac{(1 - \bar{\nu}^2)}{\bar{E}}\frac{b^2P'}{t} \qquad (6.39)$$

and equating Equations (6.38) and (6.39), with $r = b$, we obtain

$$\frac{(1-\bar{\nu}^2)}{\bar{E}}\frac{b^2P'}{t} = b\left[\left(k_2 + k_1 - \frac{2k_2^2}{k_1}\right)\frac{a^2P_i - b^2P'}{b^2 - a^2} - \left(1 - \frac{k_2}{k_1}\right)\frac{h(1-n)}{3nB}\right]$$

$$+ \frac{(k_2 - k_1)a^2b^2(P' - P_i)}{b(b^2 - a^2)}$$

$$\text{or} \quad P' = \frac{2}{\dfrac{b^2 + 3nBk_1 a^2}{3nBk_1 \mu (b^2 - a^2)} + \dfrac{(1-\bar{\nu})b}{\bar{\mu} t}} \left[\frac{a^2 P_i}{2\mu (b^2 - a^2)} \left[\frac{1}{3nBk_1} + 1 \right] - \frac{h(1-n)}{6\mu n B k_1} \right]. \tag{6.40}$$

To determine the value of P_c we set $\sigma_m = h$. Note that as in the previous problem we obtain the Zone 1 solution by setting $n = 1$ and $h = 0$. Then

$$3hk_1 = \frac{a^2 P_c - b^2 P'}{\mu (b^2 - a^2)}$$

and using Equation (6.40)

$$3hk_1 = \frac{a^2 P_c}{\mu (b^2 - a^2)} - \frac{b^2}{\mu (b^2 - a^2)} \frac{2}{\dfrac{b^2 + 3Bk_1 a^2}{3Bk_1 \mu (b^2 - a)^2} + \dfrac{(1-\bar{\nu})b}{\bar{\mu} t}} \cdot$$

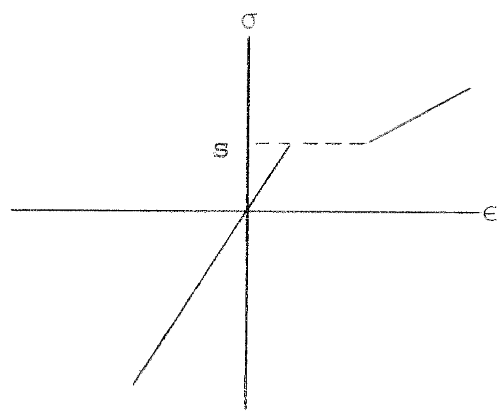
$$\frac{a^2 P_c}{2\mu (b^2 - a^2)} \left[\frac{1}{3Bk_1} + 1 \right]$$

or

$$P_c = \frac{3hk_1 \mu (b^2 - a^2)}{a^2 - \dfrac{a^2 b^2 (3Bk_1 + 1)}{\mu (b^2 - a^2) \left[\dfrac{b^2 + 3Bk_1 a^2}{\mu (b^2 - a)^2} + \dfrac{3(1-\bar{\nu}) bk_1 B}{\bar{\mu} t} \right]}} .$$

7. Proposed Generalization of the Bilinear Elastic Theory

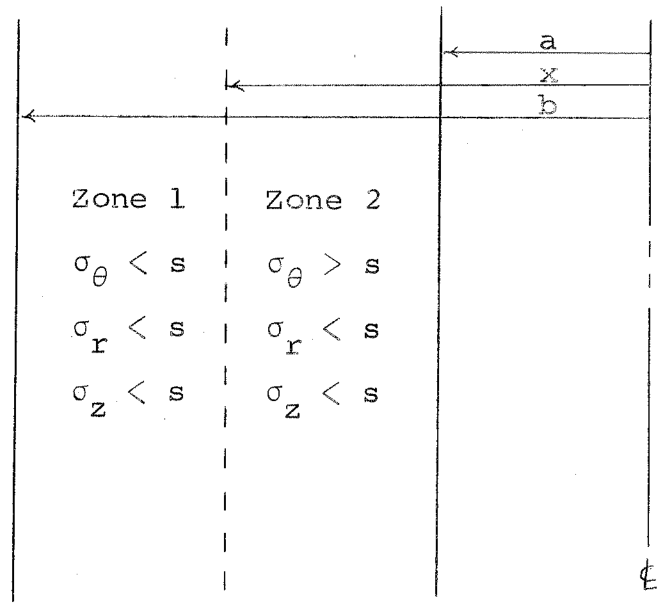
We shall now consider a generalization* of the bilinear elastic theory presented in Section 1. The generalization will be realized by allowing the constitutive equations to become discontinuous across the zone boundaries, whereas in the previous discussion we have only allowed the gradients of the constitutive equations to be discontinuous. Thus we would be able to accommodate a material which yields a uniaxial stress-strain curve of the form shown below, where the unloading curve is assumed to coincide with the loading curve.



Allowing the constitutive equations to be discontinuous at zone boundaries relieves us of the continuity conditions (2.1), (2.2) and (2.3) or their more general forms as given in Section 5 by Equations (5.2) through (5.13). Thus the general constitutive equations for a Class II material (where we allow discontinuous constitutive equations) are given by Equations (5.1). Similar considerations holds for Class I and Class III materials.

To illustrate the solution method that one would now employ we shall outline the method of solution for a simple problem, the pressurization of a thick-walled cylinder. Let us consider loading conditions compatible with the following:

*This generalization was suggested by Dr. Paul J. Blatz, California Institute of Technology.



As in Section 6 we would solve a separate problem for each zone, i.e., in Zone 1 we would use the Zone 1 constitutive equations, Equation (5.1) and the boundary conditions

$$r = b, \ \sigma_r^{(1)} = 0$$

$$r = x, \ \sigma_\theta^{(1)} = s.$$

The resulting solution will be in terms of an unknown parameter x. For Zone 2 we would use the Zone 2 constitutive equations and the results will be expressed in terms of three unknown constants, two constants of integration (A_1 and A_2) and the zone boundary x; let this solution carry superscripts (2). The three unknown parameters A_1, A_2 and x which appear in the two solutions are found from the following conditions

$$\sigma_r^{(2)}(a) = -P_i \quad \text{(Boundary condition)}$$

$$\sigma_r^{(2)}(x) = \sigma_r^{(1)}(x) \quad \text{(Equilibrium condition)}$$

$$u^{(2)}(x) = u^{(1)}(x) \quad \text{(Geometric condition).}$$

Note that in Section 6 we were able to use the condition $\sigma_\theta^{(2)}(x) = s$, see Equation (6.10), but as we have no longer required continuity of the constitutive equations across the zone boundaries this condition is no longer valid. Finally we must establish limits upon the loading conditions (in this problem upon P_i) such that the assumed zone distribution exists.

The above theory was not presented in detail, as yet no experimental evidence has been produced to indicate the necessity of such a theory.

REFERENCES

(1) Williams, M.L., Blatz, P.J. and Schapery, R.A., "Fundamental Studies Relating to Systems Analysis of Solid Propellants," GALCIT Report 101, California Institute of Technology, February 1961.

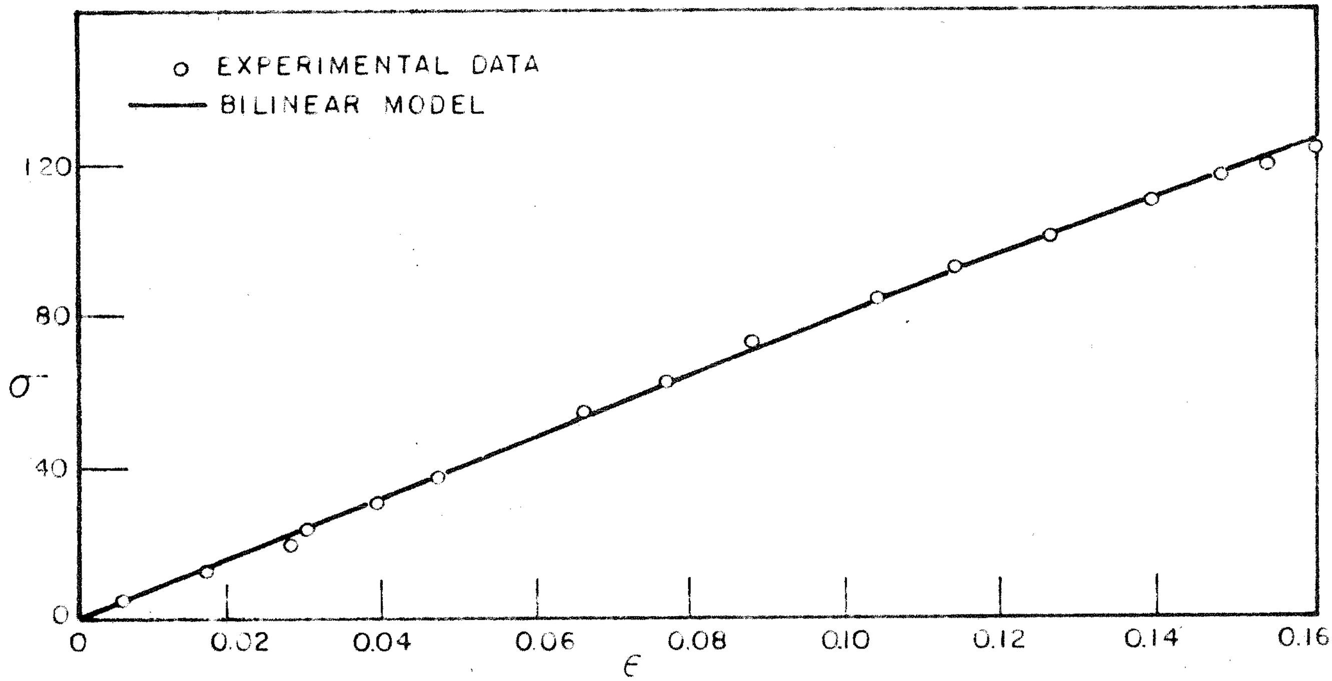


FIG. 1 UNIAXIAL TEST OF A TYPICAL PROPELLANT

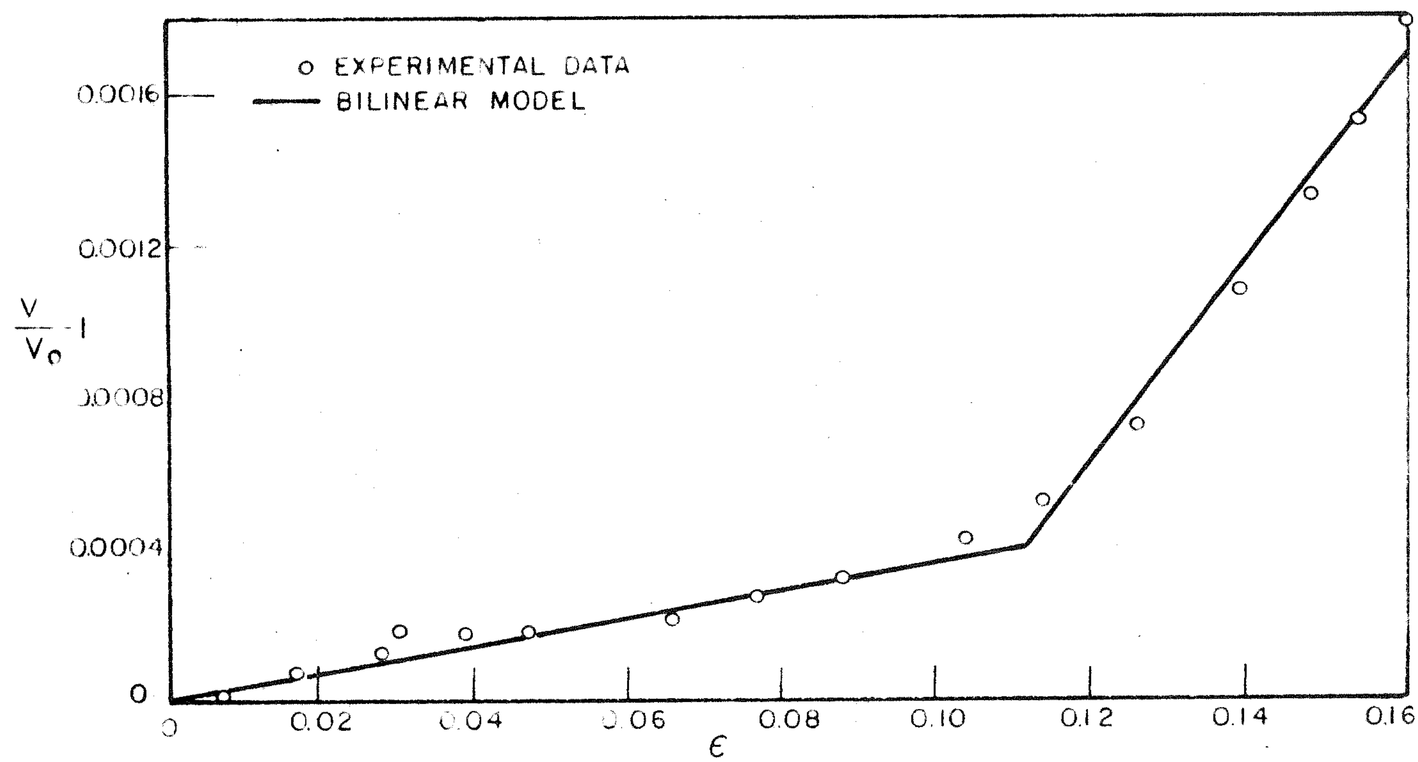


FIG. 2 UNIAXIAL TEST OF A TYPICAL PROPELLANT

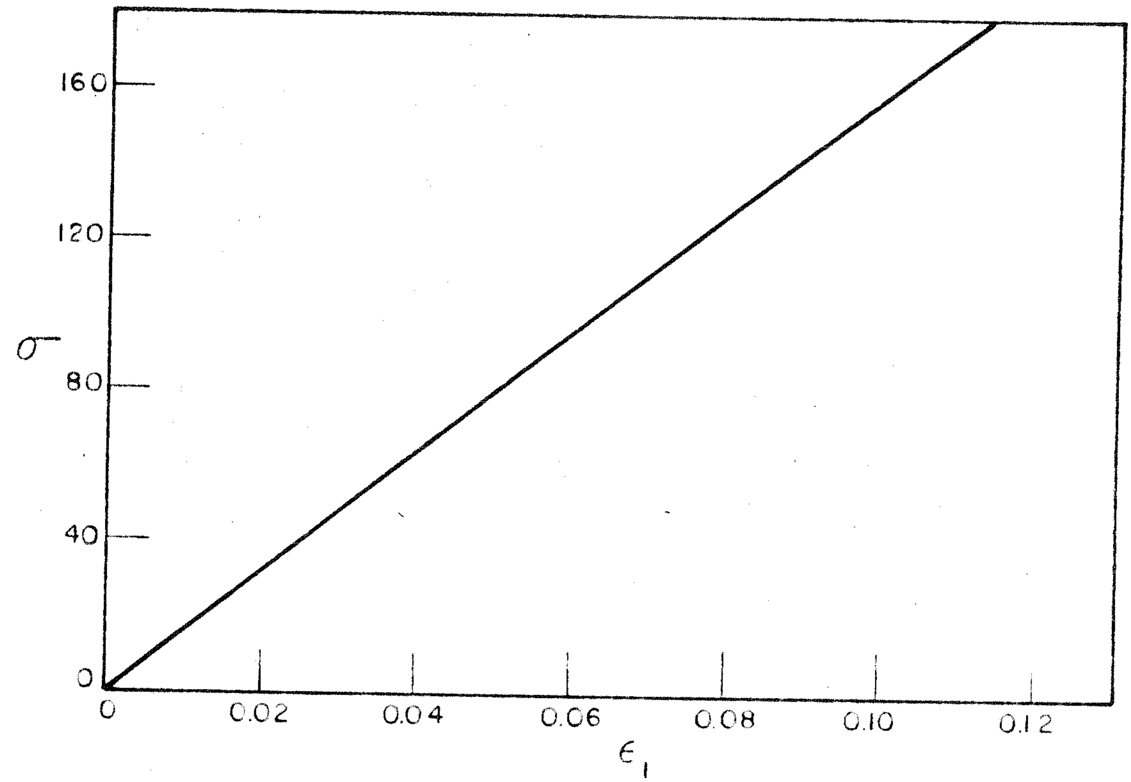


FIG.3 HYPOTHETICAL BIAXIAL TEST FOR A CLASS II MODEL

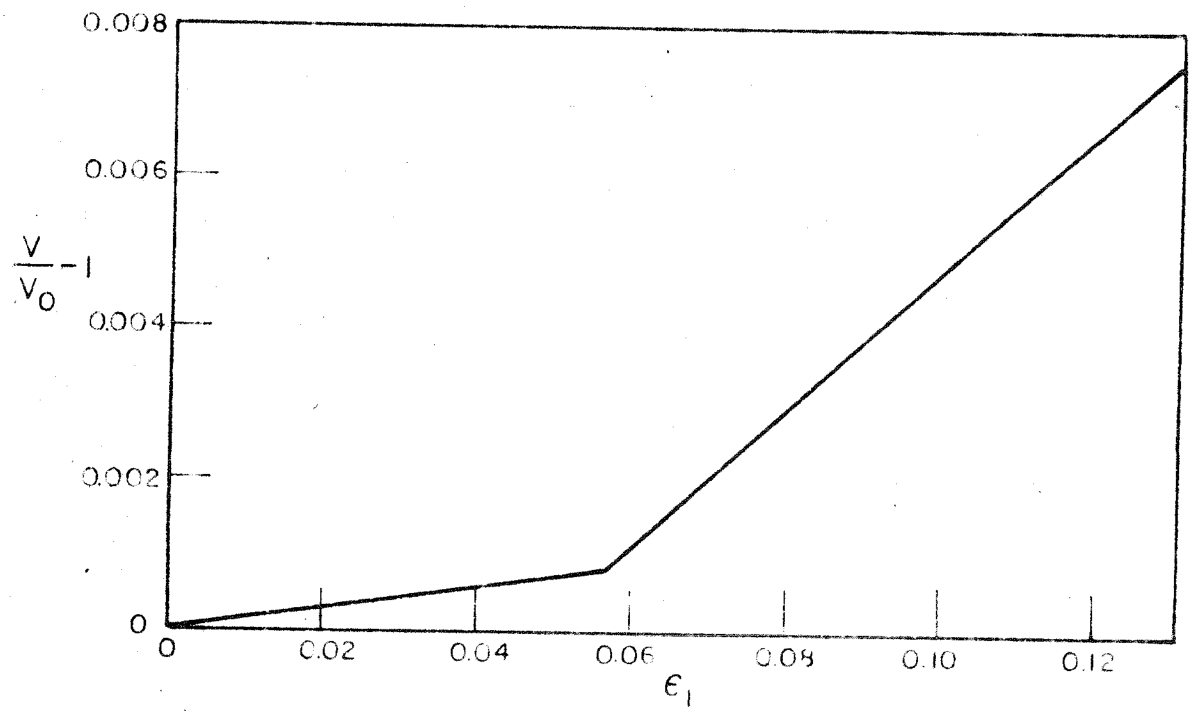


FIG.4 HYPOTHETICAL BIAXIAL TEST FOR A CLASS II MODEL

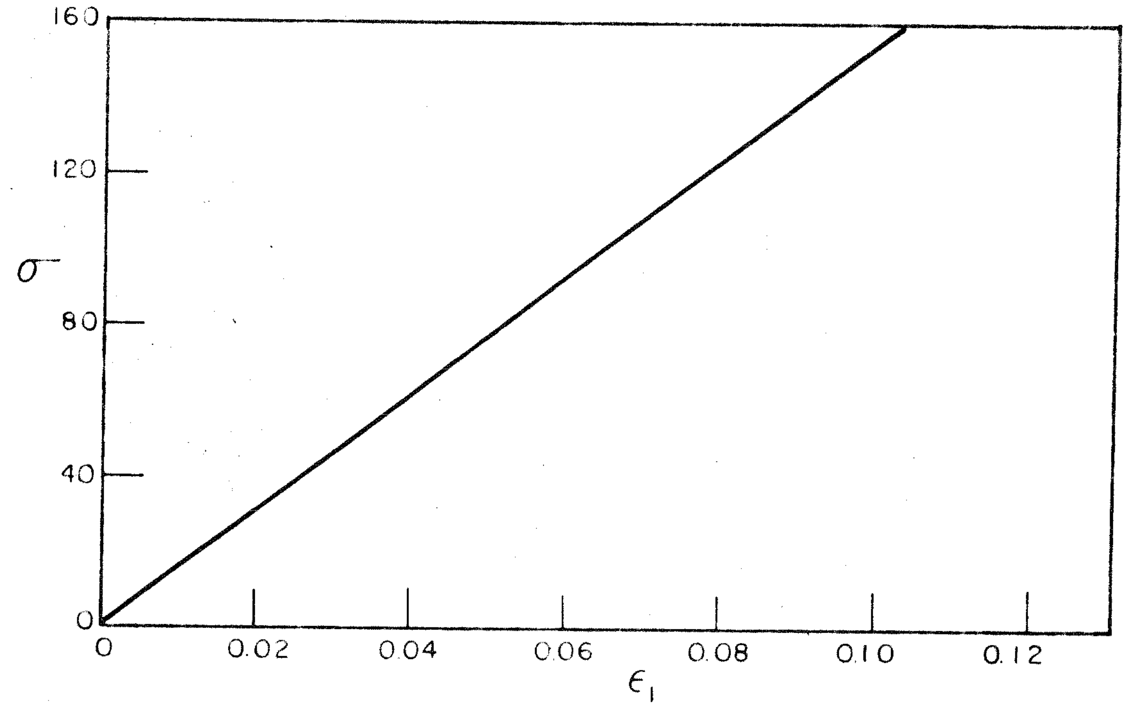


FIG. 5 HYPOTHETICAL BIAXIAL TEST FOR A CLASS III MODEL

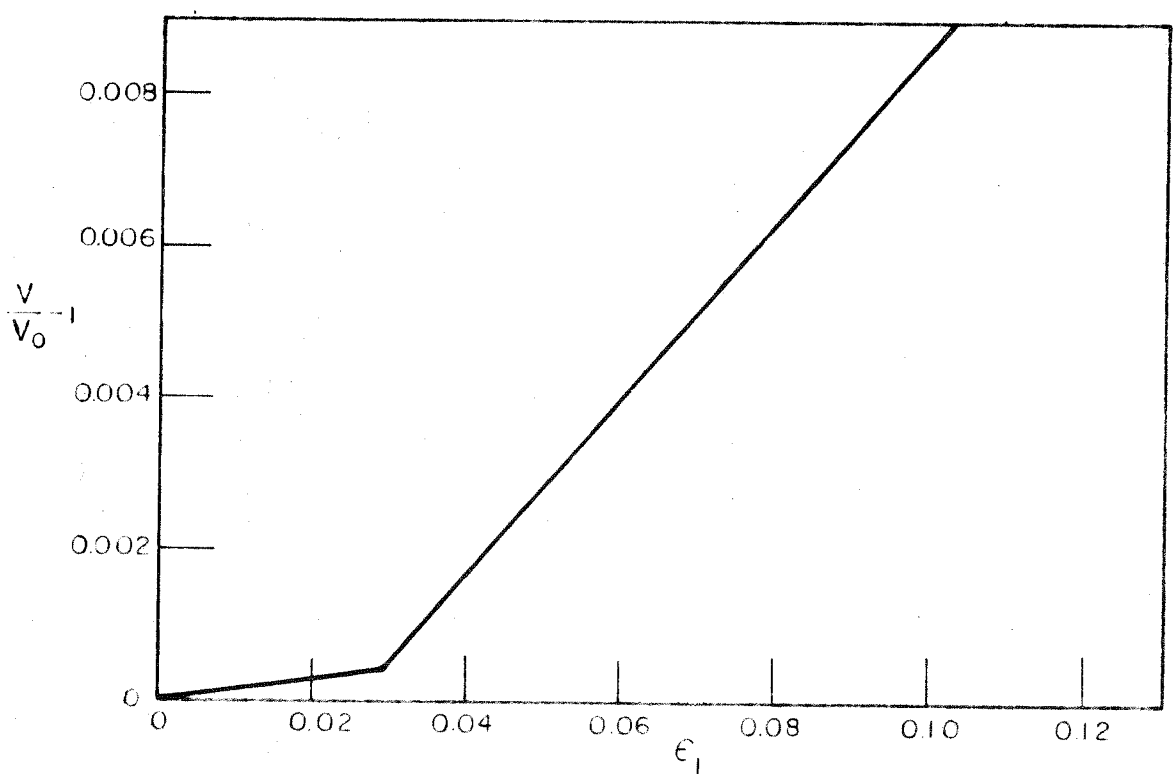


FIG. 6 HYPOTHETICAL BIAXIAL TEST FOR A CLASS III MODEL

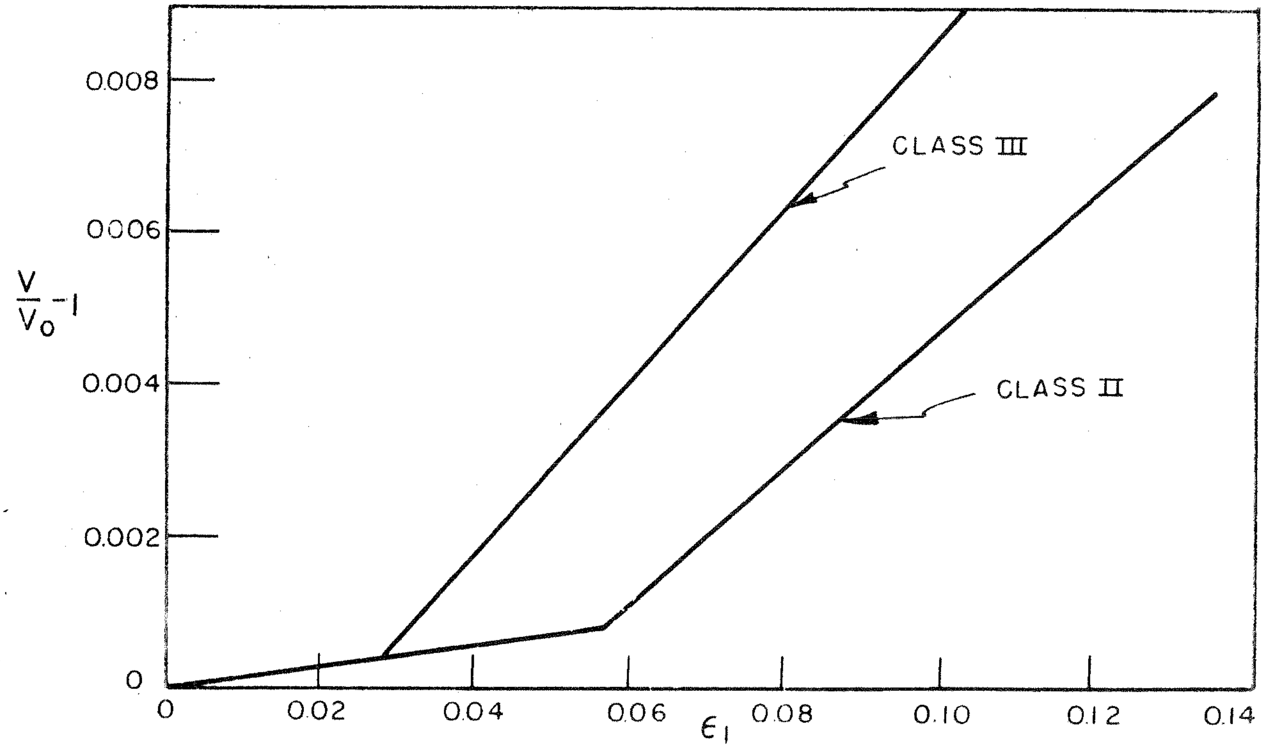


FIG. 7 COMPARISON OF HYPOTHETICAL BIAXIAL TEST FOR CLASS II AND III MODELS

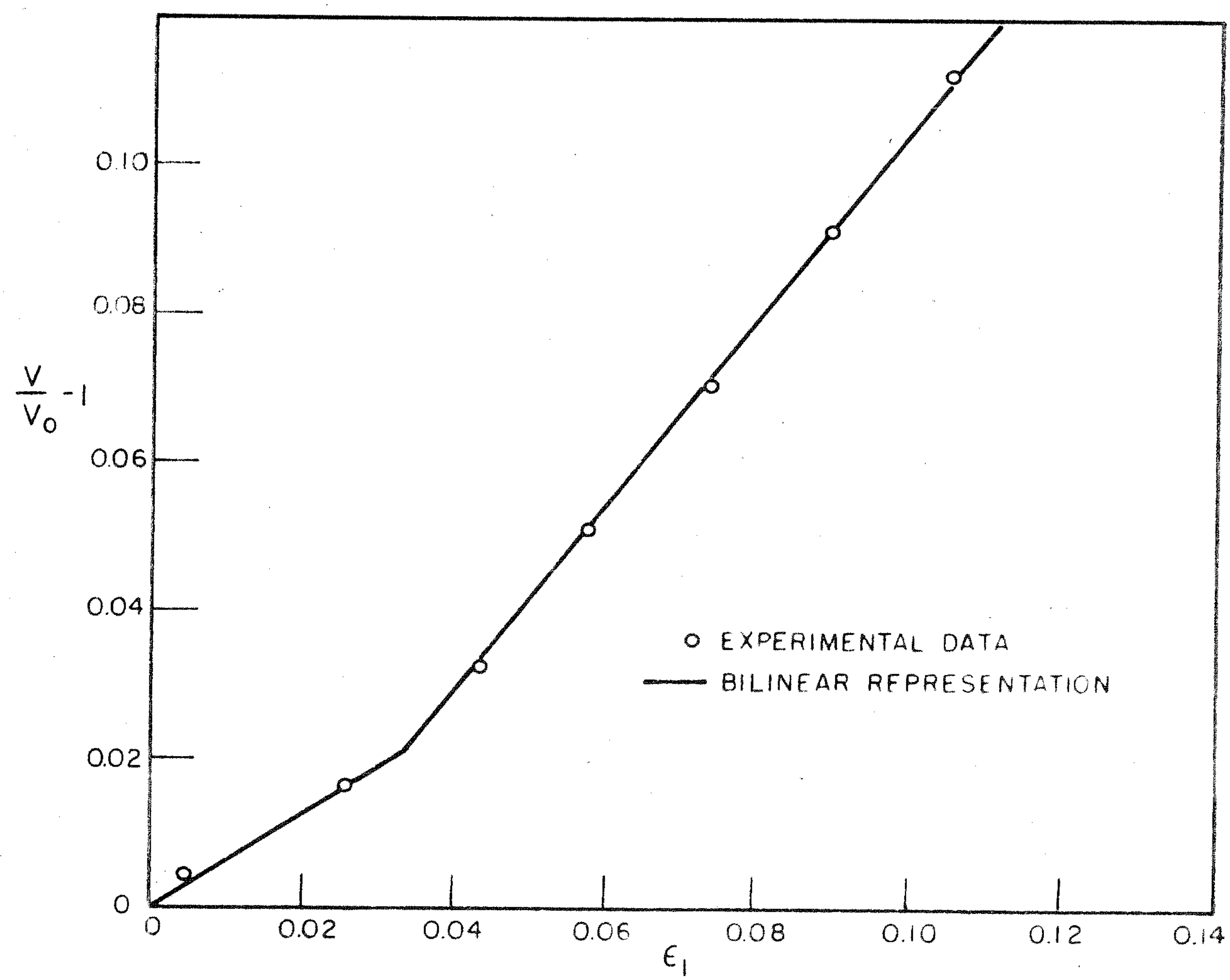


FIG. 8 BIAXIAL TEST OF A TYPICAL PROPELLANT

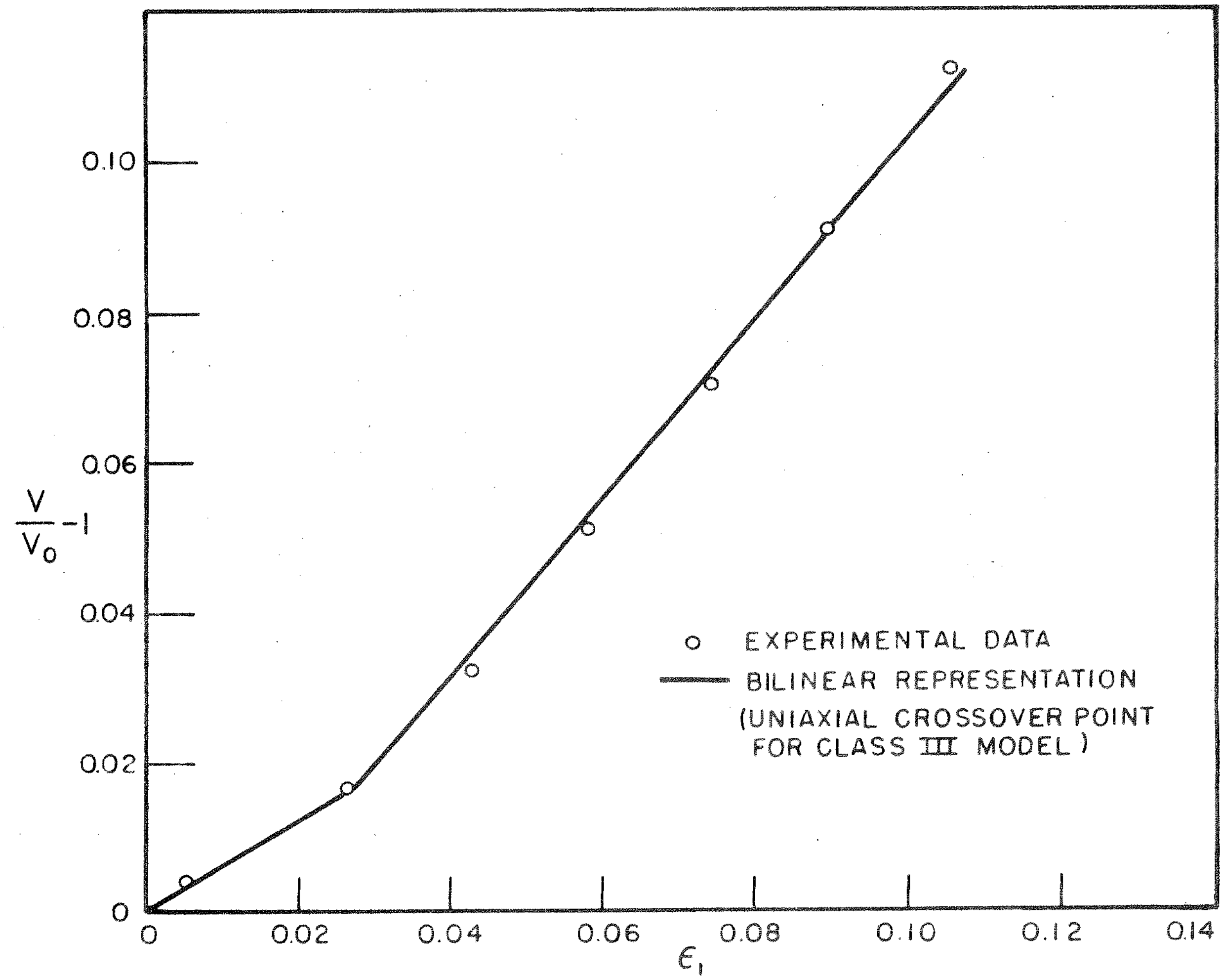


FIG. 9 BIAXIAL TEST OF A TYPICAL PROPELLANT

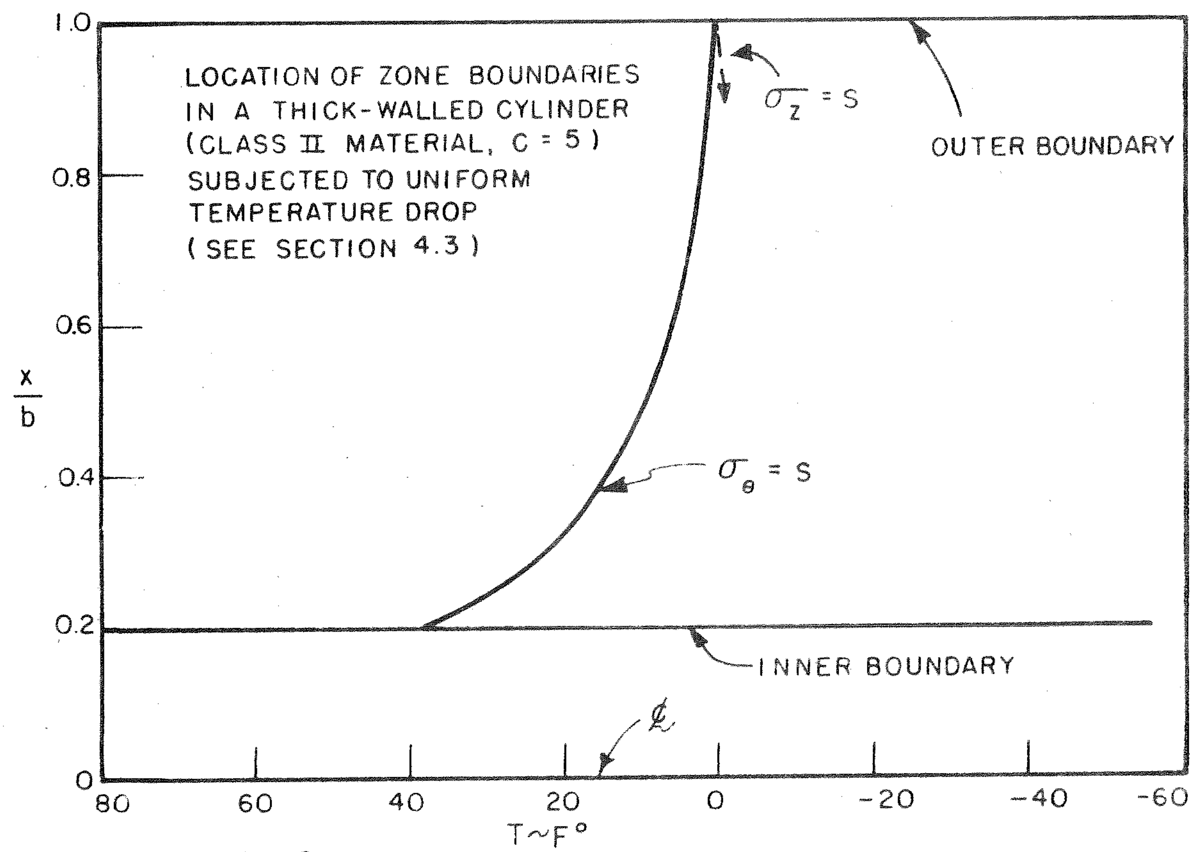


FIG. 10

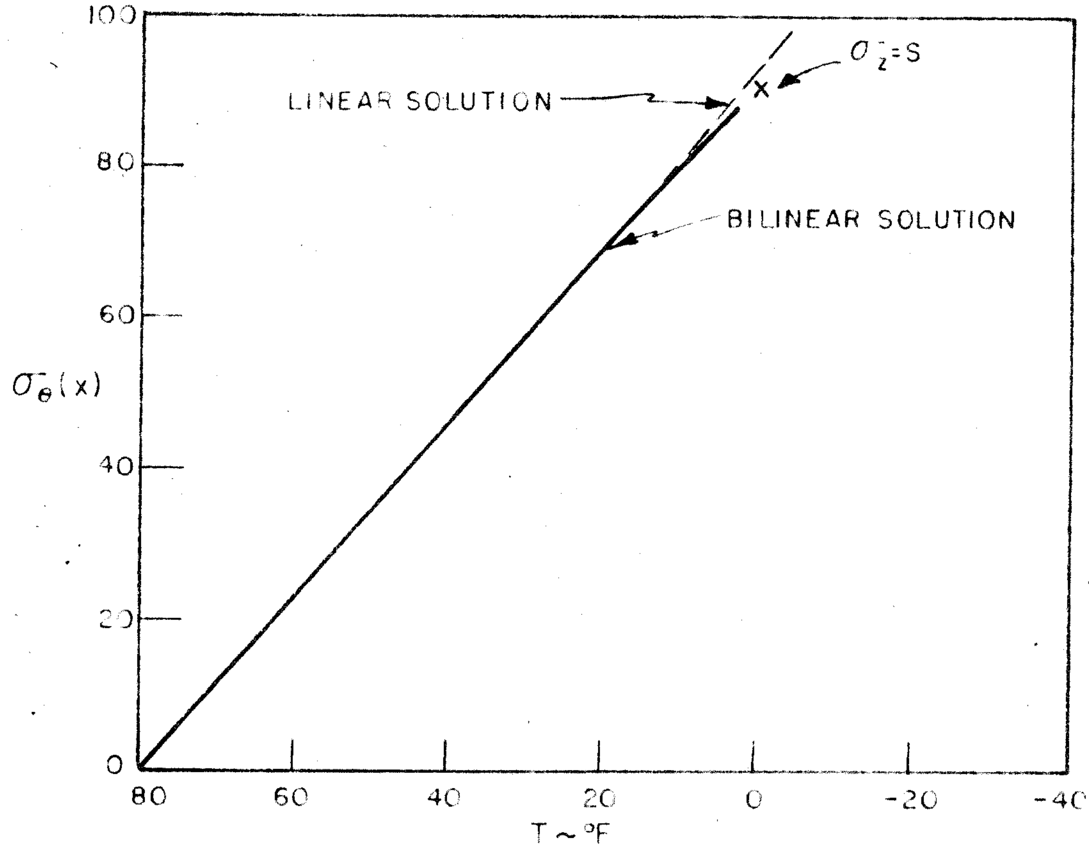


FIG.11 UNIFORM TEMPERATURE DROP OF A CLASS II THICK-WALLED CYLINDER, C = 5 (SEE SECTION 4.3)

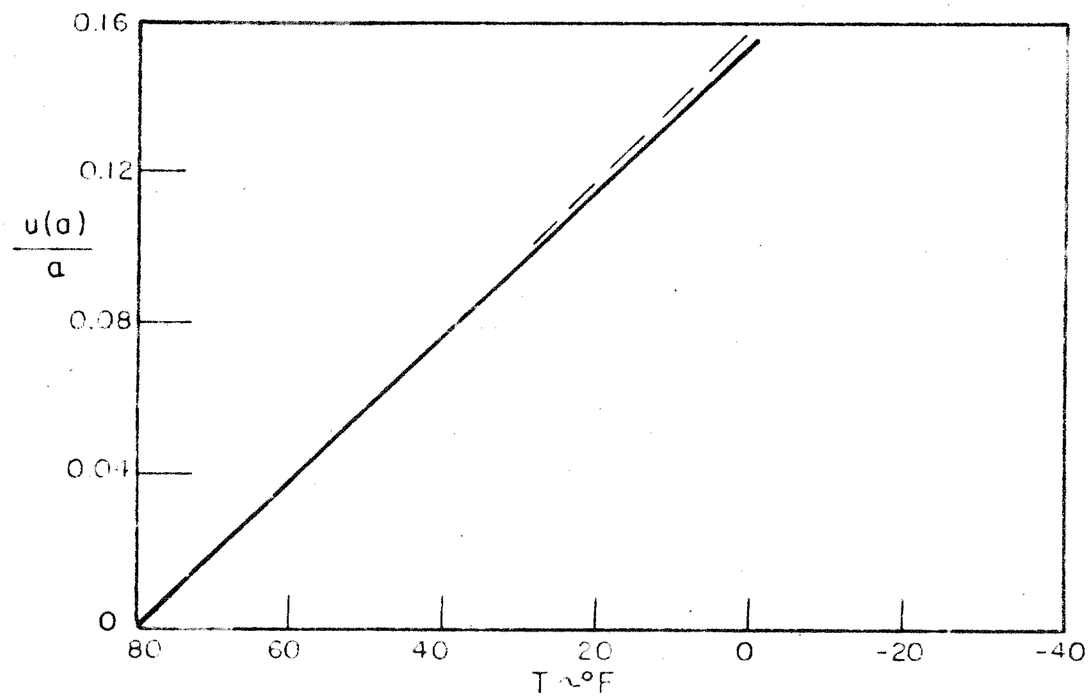


FIG. 12 UNIFORM TEMPERATURE DROP OF A CLASS II THICK-WALLED CYLINDER, C = 5 (SEE SECTION 4.3)

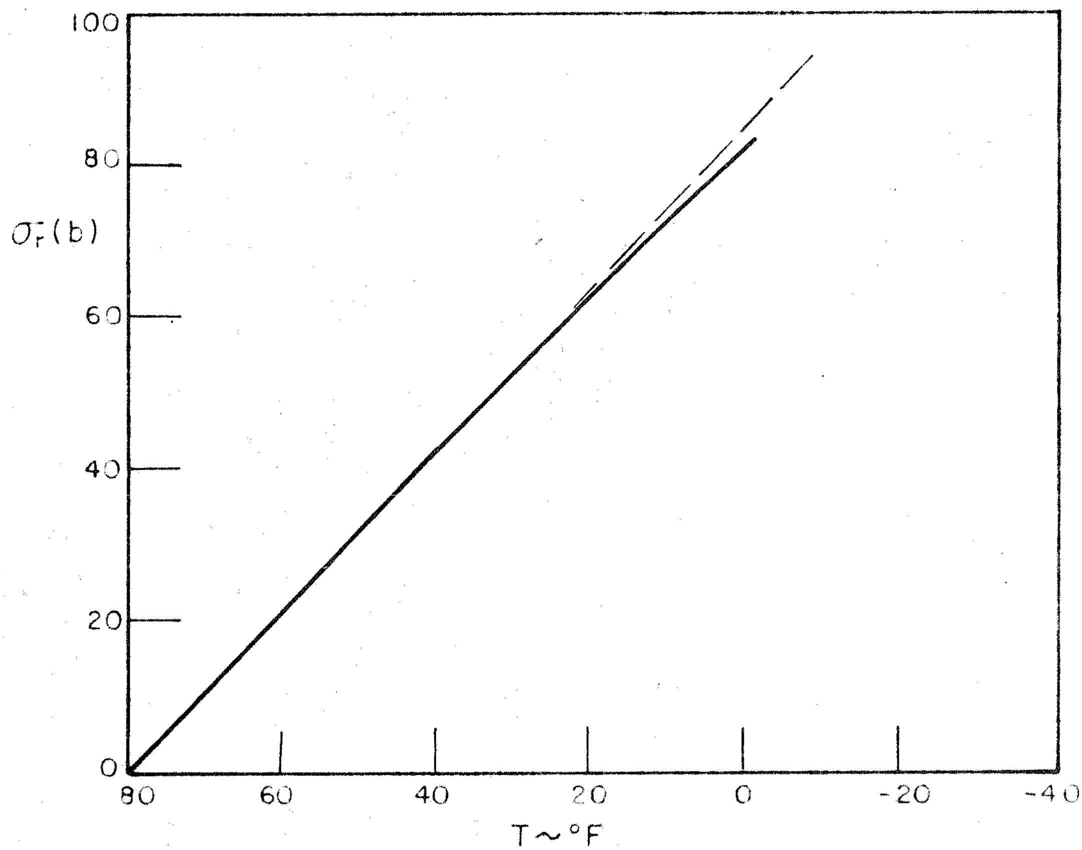


FIG. 13 UNIFORM TEMPERATURE DROP OF A
CLASS II THICK-WALLED CYLINDER
C = 5 (SEE SECTION 4.3)

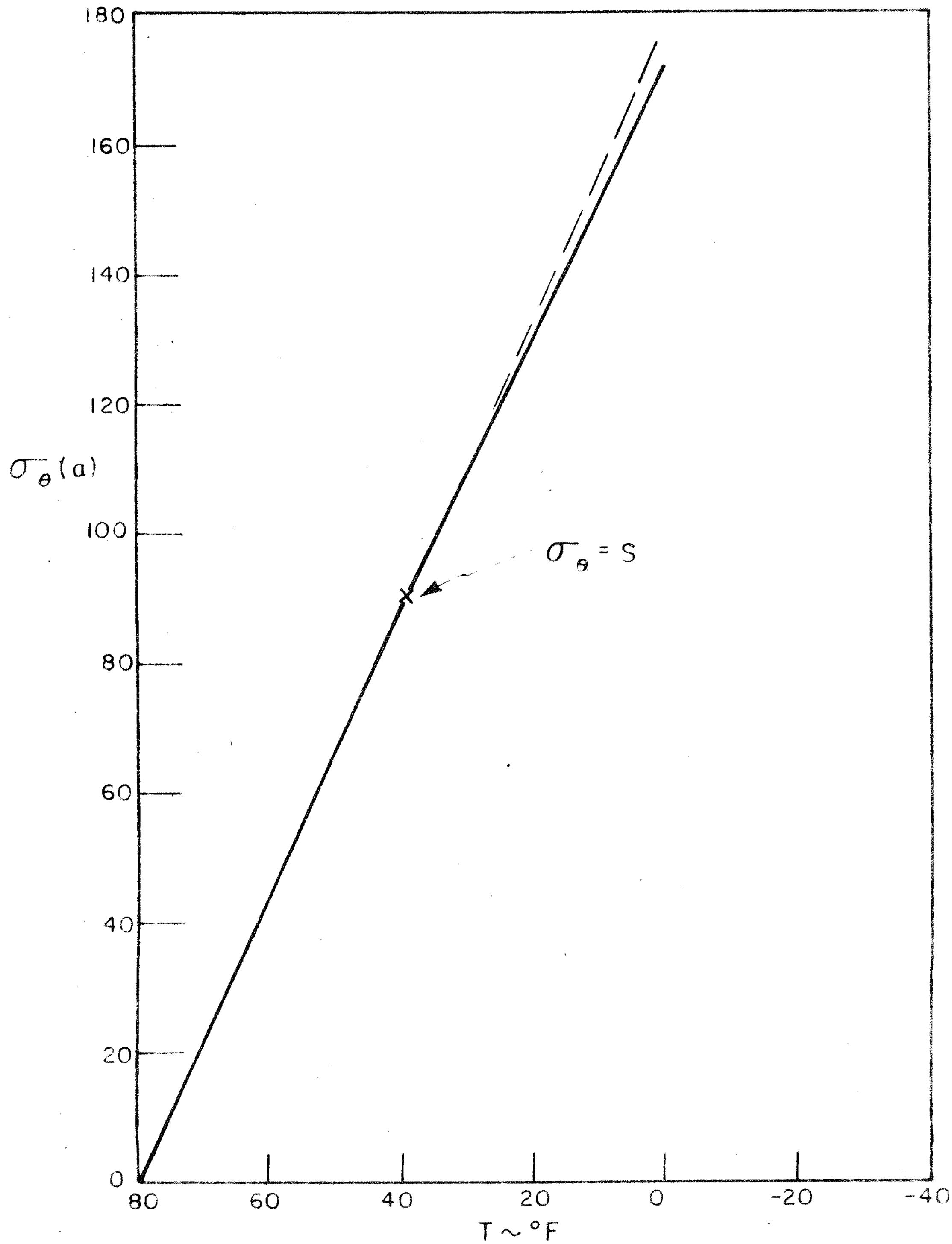


FIG. 14 UNIFORM TEMPERATURE DROP OF A CLASS II THICK-WALLED CYLINDER, C = 5 (SEE SECTION 4.3)

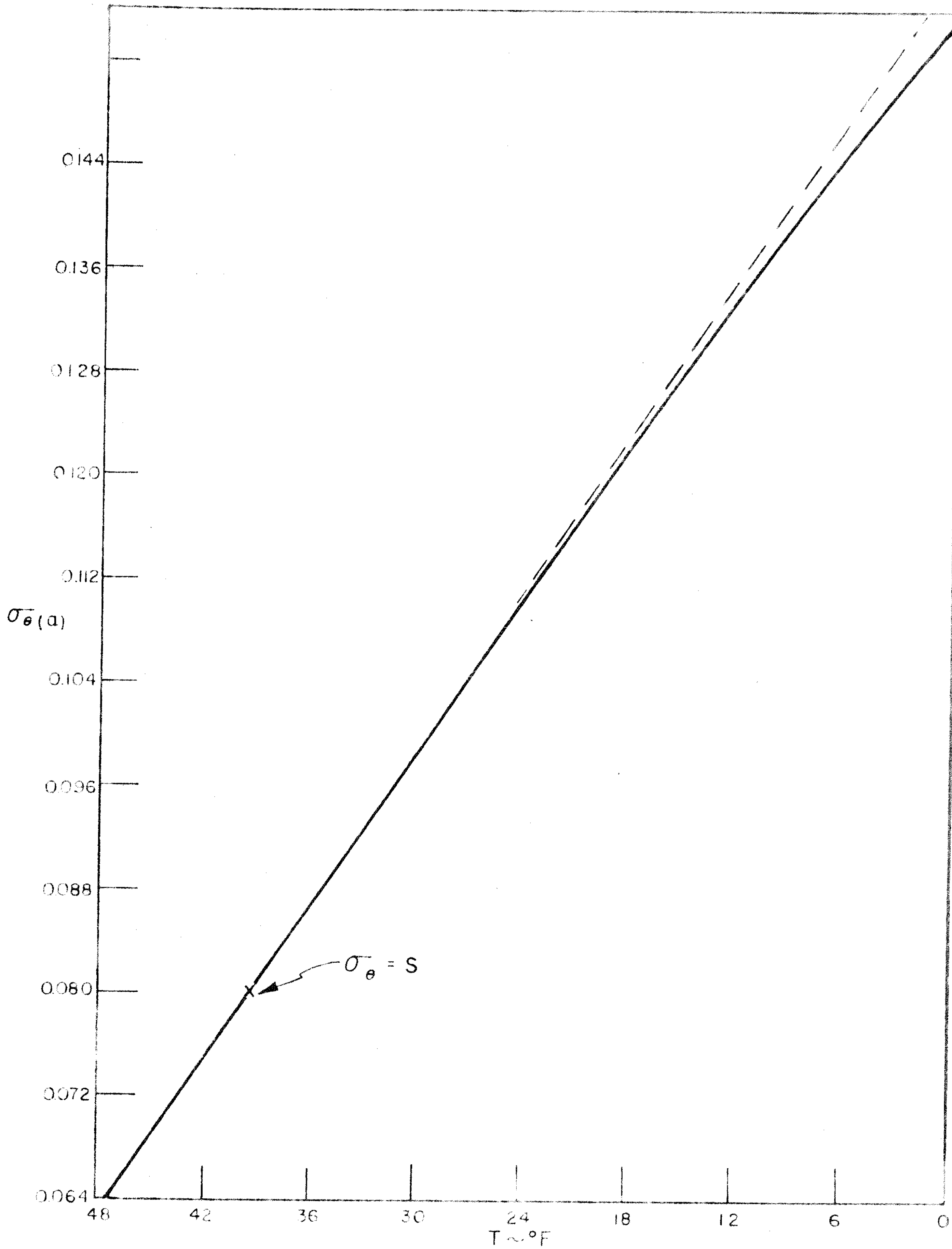


FIG 15 UNIFORM TEMPERATURE DROP OF A CLASS II THICK-WALLED CYLINDER, C = 5 ENLARGED PORTION OF FIG. 14 (SEE SECTION 4.3)

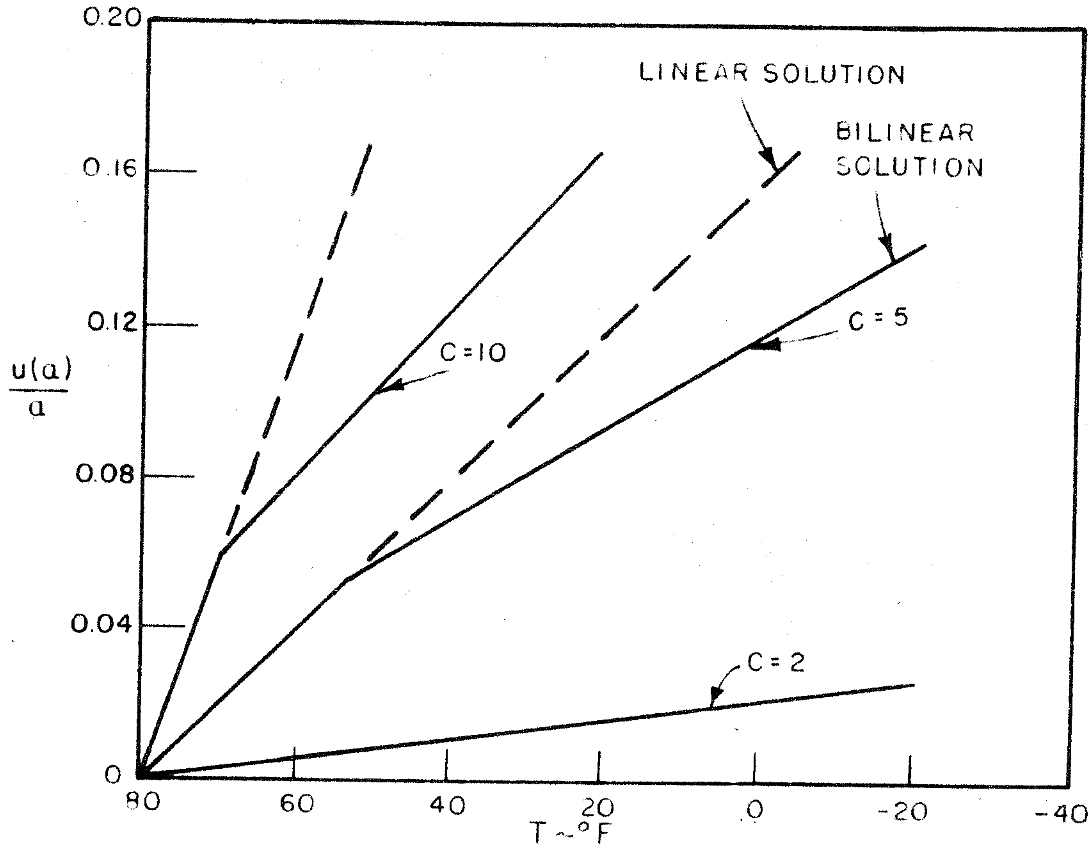


FIG. 16 UNIFORM TEMPERATURE DROP OF A CLASS III THICK-WALLED CYLINDER (SEE SECTION 4.3)

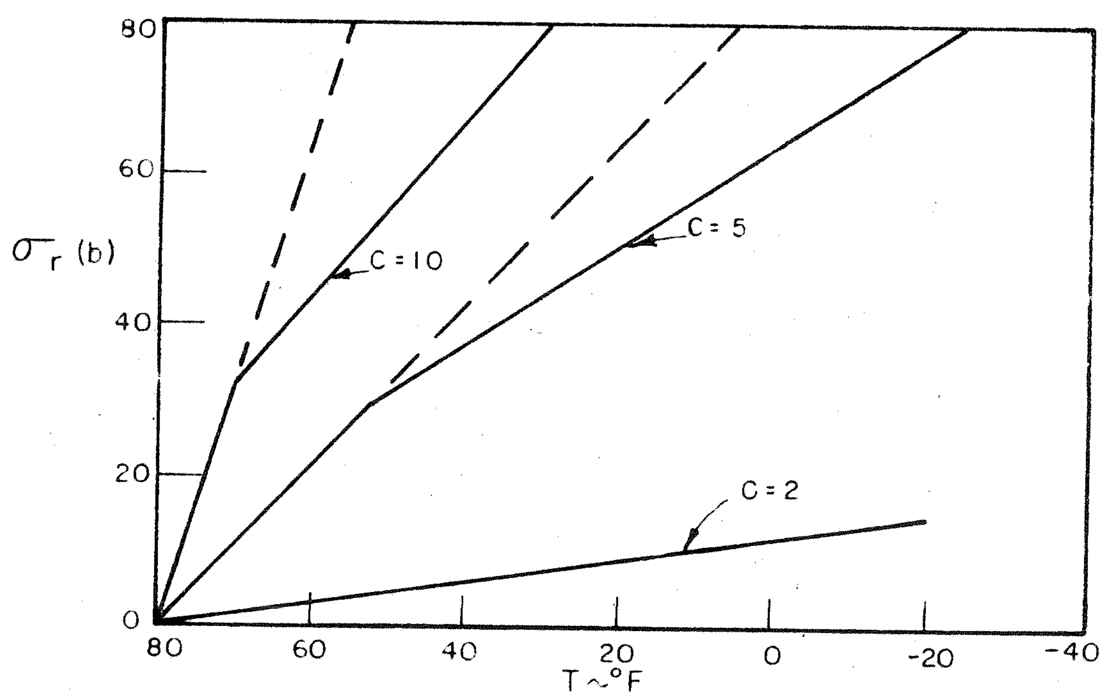


FIG. 17 UNIFORM TEMPERATURE DROP OF A CLASS III THICK-WALLED CYLINDER (SEE SECTION 4.3)

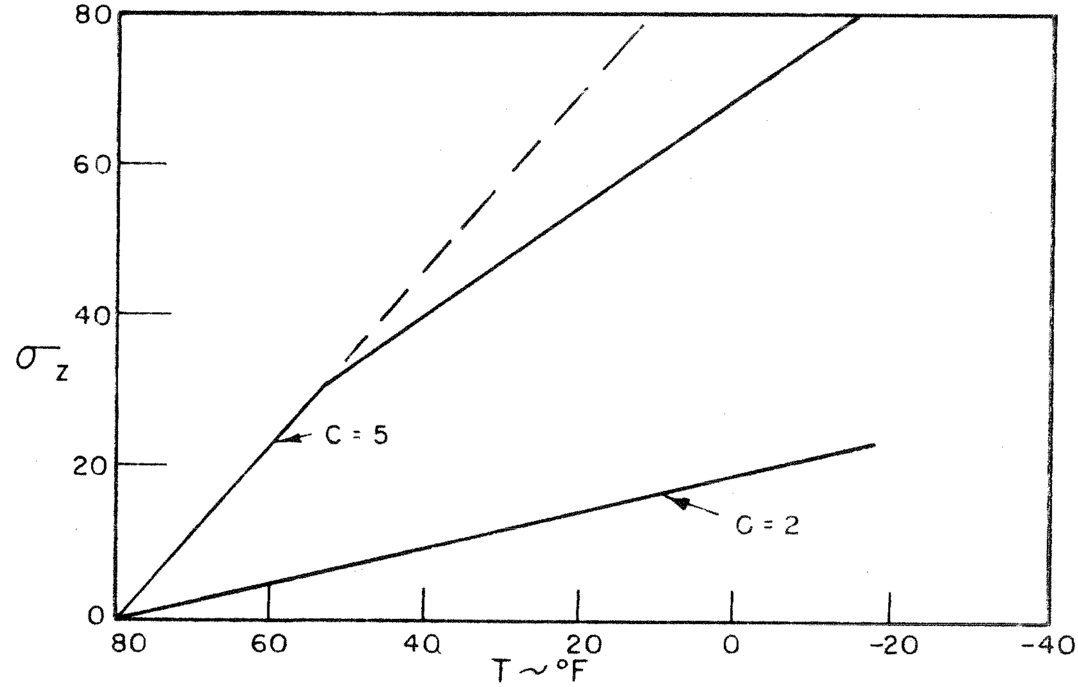


FIG.18 UNIFORM TEMPERATURE DROP OF A CLASS III THICK-WALLED CYLINDER (SEE SECTION 4.3)

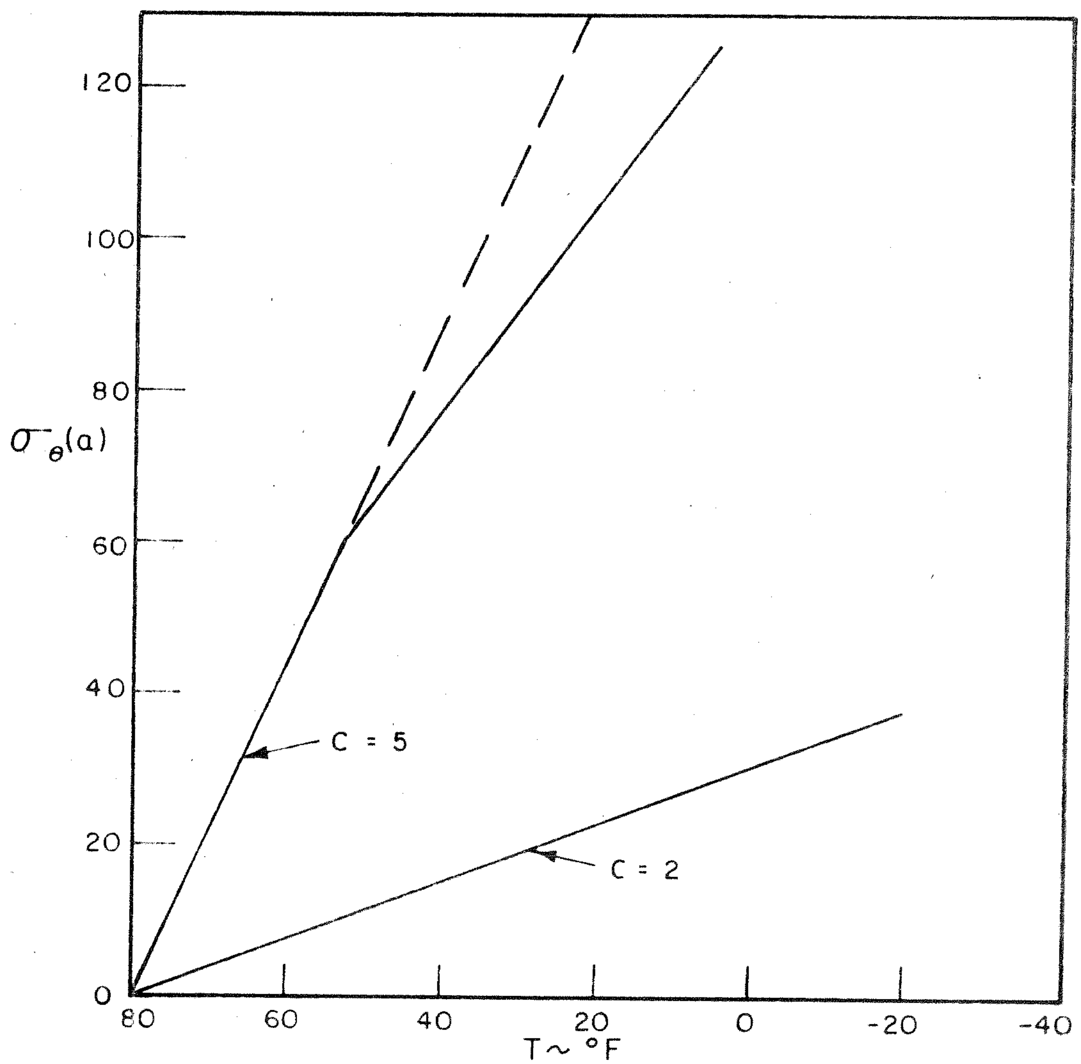


FIG.19 UNIFORM TEMPERATURE DROP OF A CLASS III THICK-WALLED CYLINDER (SEE SECTION 4.3)

PART III

SOLUTION METHOD FOR NONLINEAR ELASTIC PROBLEMS WITH APPLICATIONS TO THICK-WALLED CYLINDERS

by

L. R. HERRMANN

INTRODUCTION

The state of stress and deformation existing in an elastic body is governed by the finite elastic field equations [1] i.e., equilibrium equations, strain-displacement equations, and stress-strain laws (constitutive equations). These equations will in general be a set of non-linear partial differential equations. Classical elasticity (or first order elasticity) treats those boundary value problems for which this set of non-linear equations may be approximated by a linear set obtained from the finite elastic equations by (1) neglecting all powers of the displacement gradients in the equilibrium equations as compared to unity and neglecting second powers of the displacement gradients as compared to the first power in the strain-displacement relationships and (2) showing by experiment that for the range of strain of interest the constitutive equations are approximately linear. (This may be considered as one of the postulates for classical elasticity). The first assumption is justified if the strain is small. We shall consider a class of problems for which the above two assumptions are no longer justified but in which the strains are still relatively small; we shall base our theory on the following two conditions: (1) the displacement gradients are of such a magnitude that we may neglect their second powers as compared to unity and their first powers in the equilibrium equations and (2) the constitutive equations may be approximated by a truncated series of the second order (the extension to third and higher order theories will be obvious and offers no conceptual difficulties). We shall consider a theory in which the strains are still relatively small, since most problems susceptible to elastic analysis that occur in rocket motor analysis fall into this category.

The best way to treat the constitutive equations is the derivation of their general form experimentally [6], as it is then apparent how to obtain approximations for the range of strain of interest. Until such expressions have been found for solid propellants, we shall postulate that it is possible to approximate the constitutive equations by a truncated series of a given order. This method of approximating the constitutive equations by a truncated series for a second order theory has previously been used numerous times (see for example Rivlin [3], also Green [2]).

In problems involving strains which are still relatively small and the solution as given by classical theory still approximately valid, it is natural to seek a series solution using the classical solution as a first approximation (note that this method is not directly applicable to the problem of elastic stability). Several such series solutions have been developed; for example, Rivlin's second order theory [3] and Green's successive approximation method [2]. The principal advantages of such series solutions are (1) the effects of the non-linearities (i.e., deviations from the classical solution) are clearly illustrated and (2) the higher order solutions may be found immediately if a general solution of the classical problem is known (where body forces are included). The solution for each term of the series merely involves the solution of the classical problem with a nonhomogeneous part depending upon the previous terms. (Note: we shall extend the solution method to include those problems for which the classical solution is not known). We shall illustrate yet a third series solution method, i.e., solution by perturbation. We have chosen this method for two reasons (1) the method of application is entirely straightforward and (2) we believe that it reduces the algebraic complexities in many problems. The reduction in algebraic complexity is chiefly achieved in two ways, first by deriving the governing equations for each individual problem and second by judiciously selecting the appropriate level of approximation at which to apply certain of the boundary conditions. To be able to apply perturbation theory two conditions must exist [5]; first there must exist a state, near to the one sought, that has an exact solution. Second, the passage from this state to the one in question must take place in a smooth manner (thus excluding stability problems). These conditions are met for the class of problems considered as the actual solution may be reached after a smooth transition from the solution of the classical elastic equations. Although the foundation upon which our perturbation theory is based depends upon the existence of the classical solution its application does not require knowledge of this solution, as was pointed out above.

We shall consider three general types of material response: compressible, incompressible, and near-incompressible. A near-incompressible material is defined as one for which in the classical region Poisson's ratio is very nearly equal to one-half. We shall discriminate materals that are nearly incompressible for three reasons: (1) to effect a reduction in algebra, (2) to isolate the effects of compressibility, and (3) to illustrate some of the difficulties that arise as $\nu \to 0.5$.

We shall exhibit a general solution scheme but we shall not derive general equations as it appears to be simpler to derive the governing equations separately for each individual class of problems. Thus our procedure will be to illustrate the method for several simple examples occurring in rocket motor design, including a consideration of the thick-walled cylinder subjected to various loads and thermal effects.

The following treatment consists of two parts: (1) a preliminary consideration of the general finite elastic theory (including some extension necessary for the following work) and (2) illustration of the general solution scheme.

1. Summary of Notations and Formulas (We shall employ curvilinear tensor notation; see [1] and [4])

Notations

x^i	Material coordinates (i.e., coordinates fixed in the body)
y^i	Spatial coordinates (i.e., coordinates fixed in space)
g_{ij}	Covariant components of the metric tensor for the material coordinates in the undeformed space
h_{ij}	Covariant components of the metric tensor for the spatial coordinates
G_{ij}	Covariant components of the metric tensor for the material coordinate system (Green's deformation tensor) in the deformed space
$\Vert$	Indicates covariant differentiation with respect to the deformed material coordinate system (example $\tau^{ij}\Vert_i$)
$\bar{\Gamma}^i_{jk}$	Christoffel three index symbols of the second kind taken with respect to the deformed material coordinate system
τ^{ij}	Contravariant tensor components of stress in the deformed body, referred to the deformed base vectors
σ_{ij}	Physical components of stress in the deformed body per unit of deformed area, referred to the deformed base vectors
s^{ij}	Contravariant tensor components of stress in the deformed body per unit of undeformed area, referred to the deformed base vectors
η^i	Contravariant tensor components of the displacements in the direction of the undeformed material base vectors ($\eta^{(i)}$ are the physical components)
ξ^i	Contravariant tensor components of the displacement in the direction of the deformed material base vectors ($\xi^{(i)}$ are the physical components)

E^{ij}	Contravariant tensor components of strain
θ_i	Strain invariants
I_i	Alternate form of strain invariants
A	Helmholtz free energy per unit of initial volume
E	Internal energy per unit of initial volume
S	Entropy per unit of initial volume
K	Thermal coefficient of conduction (assumed as temperature and strain independent)
C_E	Specific heat at constant deformation
T_a	Absolute temperature
T	Relative temperature ratio
T_o	Reference temperature
α	Linear coefficient of thermal expansion
$C = 0(a)$	Indicates C is of the order of magnitude of a
B	Bulk modulus
ν	Poisson's ratio
μ	Shear modulus
ρ	Density per unit of deformed volume
ρ_o	Density per unit of undeformed volume
V	Deformed volume
V_o	Undeformed volume
F^i	Contravariant components of the body force per unit of mass, referred to the deformed base vectors.

Governing equations:

Equilibrium equations

$$\tau^{ij}\|_i + \rho F^j = 0 \tag{1.1}$$

Strain-displacement relations

$$E^i_j = \frac{1}{2}(G^{*i}_j - \delta^i_j) \tag{1.2}$$

where $$G^{*i}_{j} = g^{ik}G_{jk} \tag{1.3}$$

also $$\theta_1 = \frac{1}{1!}\,\delta^{i}_{j}E^{j}_{i} = O(E^{i}_{j}) \tag{1.4}$$

$$\theta_2 = \frac{1}{2!}\,\delta^{ij}_{km}E^{k}_{i}E^{m}_{j} = O(E^{i}_{j})^2 \tag{1.5}$$

$$\theta_3 = \frac{1}{3!}\,\delta^{ijk}_{omn}E^{o}_{i}E^{m}_{j}E^{n}_{k} = O(E^{i}_{j})^3 \tag{1.6}$$

$$I_1 = 2\theta_1 + 3 \tag{1.7}$$

$$I_2 = 4\theta_2 + 4\theta_1 + 3 \tag{1.8}$$

$$I_3 = 8\theta_3 + 4\theta_2 + 2\theta_1 + 1 = \left(\frac{dV}{dV_o}\right)^2 = \frac{|G_{ij}|}{|g_{ij}|}. \tag{1.9}$$

Form of the Helmholtz free energy

$$A = E - T_a S, \tag{1.10}$$

For an isotropic body

$$A = A(\theta_1, \theta_2, \theta_3, T) \tag{1.11}$$

where $$T = \frac{T_a - T_o}{T_o}. \tag{1.12}$$

Stress-strain laws for compressible material

$$\tau^{ij} = \frac{1}{\sqrt{I_3}}\frac{\partial A}{\partial E_{ij}} = \frac{s^{ij}}{\sqrt{I_3}} \tag{1.13}$$

Noting the above representation for A, see (1.11), we may alternatively write Equation (1.13) for an isotropic compressible material as

$$\tau^{ij} = \frac{1}{\sqrt{I_3}}\left[g^{ij}\frac{\partial A}{\partial \theta_1} + (\tfrac{1}{2}B^{ij} - g^{ij})\frac{\partial A}{\partial \theta_2} + \frac{1}{4}(g^{ij} - B^{ij} + I_3 G^{ij})\frac{\partial A}{\partial \theta_3}\right] \tag{1.14}$$

where

$$B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs}. \tag{1.15}$$

Heat conduction equation

$$(KG^{ij} T_a\|_{j})\|_{i} = \rho(C_E \overset{\circ}{T}_a - \frac{T_a}{\rho_o}\frac{\partial s_{ij}}{\partial T_a}\overset{\circ}{E}_{ij}) \tag{1.16}$$

where

$$C_E = -\frac{\partial^2 A}{\partial T_a^2} T_a .$$

Physical components

$$\sigma_{ij} = \sqrt{\frac{G_{jj}}{G^{ii}}}\,\tau^{ij} \qquad \text{(no sum)} \tag{1.17}$$

$$\eta^{(i)} = \sqrt{g_{ii}}\,\eta^{i} \qquad \text{(no sum)} \tag{1.18}$$

$$\xi^{(i)} = \sqrt{G_{ii}}\,\xi^{i} \qquad \text{(no sum)} \tag{1.19}$$

2. Derivation of Constitutive Equations for Incompressible Thermoelasticity.

Incompressible thermoelasticity describes materials that are mechanically incompressible (i.e., due to stresses) but thermally compressible. We shall consider an incompressible body subjected to certain temperatures and forces such that at some internal point the state of the body is characterized by an absolute temperature T_a (or relative temperature ratio T, see [4]) and displacements η^i. We shall now subject the body to virtual displacements $\delta\eta^i$ (the restrictions placed on $\delta\eta^i$ will be considered below) at constant temperature (i.e., $\delta T = 0$). If we now consider the virtual work performed on some arbitrary volume V during the virtual displacements, we find that

$$V.W. = \iiint_V \tau^{ij} \delta E_{ij} dV \qquad \text{(see [1], p.71).} \qquad (2.1)$$

This virtual work in an elastic body must equal the change in value of the Helmholtz free energy function (see [4]), $A^* = A^*(E_{ij}, T)$ where A^* is the Helmholtz free energy per unit of deformed volume. Thus,

$$V.W. = \iiint_V \delta A^* dV \qquad (2.2)$$

In the formulation of δA^* the variations of δE_{ij} are not arbitrary but must satisfy the additional constraint

$$I_3 - \left(\frac{dV}{dV_o}\right)^2 = 0, \text{ see Eq.(1.9),}$$

where $\left(\frac{dV}{dV_o}\right)$ is a function of T only. Defining a Lagrange multiplier h' we form

$$L = A^* + \frac{h'}{2}\left[I_3 - \left(\frac{dv}{dV_o}\right)^2\right]$$

Thus $\delta A^* = \delta L$ and $I_3 - \left(\frac{dV}{dV_o}\right)^2 = 0$ where we may now treat the δE_{ij} as independent. We obtain

$$\delta L = \delta A^* = \frac{\partial A^*}{\partial E_{ij}}\,\delta E_{ij} + \frac{h'}{2}\,\frac{\partial I_3}{\partial E_{ij}}\,\delta E_{ij}, \text{ where } I_3 = \left(\frac{dv}{dV_o}\right)^2. \qquad (2.3)$$

From Eqs. (2.1), (2.2), and (2.3), we obtain

$$\iiint_V \left[\tau^{ij} - \frac{\partial A^*}{\partial E_{ij}} - \frac{h'}{2}\,\frac{\partial I_3}{\partial E_{ij}}\right]\delta E_{ij}\,dV = 0.$$

Treating the δE_{ij} as arbitrary and in view of the arbitrariness of the volume integral we obtain

$$\tau^{ij} = \frac{h'}{2}\,\frac{\partial I_3}{\partial E_{ij}} + \frac{\partial A^*}{\partial E_{ij}}.$$

Expressing the above equation in terms of the Helmholtz free energy per unit of undeformed volume A,

$$A^* = \frac{dV_o}{dV}A = \frac{1}{\sqrt{I_3}}\,A, \text{ see (1.9)},$$

thus

$$\tau^{ij} = \frac{h'}{2}\,\frac{\partial I_3}{\partial E_{ij}} + \frac{1}{\sqrt{I_3}}\,\frac{\partial A}{\partial E_{ij}}$$

also

$$\frac{\partial I_3}{\partial E_{ij}} = 2G^{ij} I_3, \text{ (see [1], p. 75)}, \tag{2.4}$$

and

$$\tau^{ij} = hG^{ij} + \frac{1}{\sqrt{I_3}} \frac{\partial A}{\partial E_{ij}}, \text{ where } I_3 = \left(\frac{dV}{dV_o}\right)^2. \tag{2.5}$$

Letting

$$\sqrt{I_3} - 1 = \frac{dV - dV_o}{dV_o} = f(T) \tag{2.6}$$

we may write

$$\tau^{ij} = hG^{ij} + \frac{1}{1+f(T)} \frac{\partial A}{\partial E_{ij}}, \text{ where } I_3 = 1+2f(T)+f^2(T) \tag{2.7}$$

For an isotropic material $A = A(\theta_1, \theta_2, \theta_3, T)$, see Equation (1.11), or noting the relationship between θ_1 and I_3, Equation (1.9), we may write

$$A = A(\theta_2, \theta_3, I_3 - 1 - 2f(T) - f^2(T), T)$$

If we let

$$\overline{I}_3 = I_3 - 1 - 2f(T) - f^2(T) \tag{2.8}$$

then

$$A = A(\theta_2, \theta_3, \overline{I}_3, T) \tag{2.9}$$

The condition of incompressibility is now expressed by $\overline{I}_3 = 0$.

Let

$$A\big|_{\overline{I}_3=0} = \overline{A}(\theta_2, \theta_3, T) \tag{2.9a}$$

now $$\frac{\partial A}{\partial E_{ij}} = \frac{\partial A}{\partial \theta_2}\frac{\partial \theta_2}{\partial E_{ij}} + \frac{\partial A}{\partial \theta_3}\frac{\partial \theta_3}{\partial E_{ij}} + \frac{\partial A}{\partial \bar{I}_3}\frac{\partial \bar{I}_3}{\partial E_{ij}}$$

but $$\frac{\partial \bar{I}_3}{\partial E_{ij}} = \frac{\partial I_3}{\partial E_{ij}} = 2I_3G^{ij}, \text{ see (2.4)},$$

$$\left.\frac{\partial A}{\partial \theta_2}\right|_{\bar{I}_3=0} = \frac{\partial \bar{A}}{\partial \theta_2}$$

$$\left.\frac{\partial A}{\partial \theta_3}\right|_{\bar{I}_3=0} = \frac{\partial \bar{A}}{\partial \theta_3}$$

$$\frac{\partial A}{\partial \bar{I}_3} = \text{scalar} = h''$$

thus $$\left.\frac{\partial A}{\partial E_{ij}}\right|_{\bar{I}_3=0} = \frac{\partial \bar{A}}{\partial \theta_2}\frac{\partial \theta_2}{\partial E_{ij}} + \frac{\partial \bar{A}}{\partial \theta_3}\frac{\partial \theta_3}{\partial E_{ij}} + 2h''I_3G^{ij} \tag{2.10}$$

or $$\left.\frac{\partial A}{\partial E_{ij}}\right|_{\bar{I}_3=0} = \frac{\partial \bar{A}}{\partial E_{ij}} + 2h''I_3G^{ij}, \tag{2.11}$$

hence

$$\tau^{ij} = HG^{ij} + \frac{1}{\sqrt{I_3}}\frac{\partial \bar{A}}{\partial E_{ij}} \quad \text{and} \quad \bar{I}_3 = 0, \tag{2.12}$$

Alternatively

$$\tau^{ij} = HG^{ij} + \frac{1}{1+f(T)} \frac{\partial \bar{A}}{\partial E_{ij}} \quad \text{and} \quad I_3 = 1+2f(T)+f^2(T), \tag{2.13}$$

where $$H = h + 2\sqrt{I_3}\, h'' ,$$

or from Equation (1.14)

$$\tau^{ij} = HG^{ij} + \frac{1}{1+f(T)} \left[\left(\tfrac{1}{2}B^{ij} - g^{ij}\right)\frac{\partial \bar{A}}{\partial \theta_2} + \tfrac{1}{4}\left(g^{ij} - B^{ij} + I_3 G^{ij}\right)\frac{\partial \bar{A}}{\partial \theta_3} \right], \tag{2.14}$$

where $$B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs} .$$

3. Governing Equations for Thick-walled Cylinders.

Consider the behavior of an infinitely long thick-walled cylinder subjected to axially symmetrical forces and temperature distribution. In order to illustrate two fundamental approaches to the formulation of the finite elastic equations we shall consider two distinctly different types of displacement fields; first we shall consider the effects of pressure and temperature which will cause motion only in the radial direction, and secondly we shall consider the effects of a vertical gravity loading which shall give rise to both a longitudinal and radial motion. In the first problem we shall select our material coordinate system such that in the initial state it coincides with the cylindrical coordinate system (r, θ, z) and in the second problem we shall select our material coordinate system such that in the final state it coincides with the cylindrical coordinate system (r, θ, z). Thus if we compute the stress at the point $r = r_o$, $\theta = \theta_o$ and $z = z_o$, we have for the first problem found the state of stress of the material that initially occupied the point (r_o, θ_o, z_o) while in the second problem we have found the state of stress for the material that occupies (r_o, θ_o, z_o) in its deformed position.

Pressure and Temperature Loading of a Thick-walled Cylinder

We shall describe the initial state in cylindrical coordinates (r,θ,z). The material coordinates shall be selected such that in the initial state they coincide with these cylindrical coordinates, i.e.,

$$(x^1, x^2, x^3) = (r, \theta, z) \tag{3.1}$$

then
$$(ds_o)^2 = g_{ij}dx^i dx^j$$

where
$$g_{ij} = \begin{bmatrix} 1 & & \\ & (x^1)^2 & \\ & & 1 \end{bmatrix}. \tag{3.2}$$

The spatial coordinate system will be chosen as a cylindrical coordinate system also, thus

$$(ds)^2 = h_{\alpha\beta}dy^\alpha dy^\beta$$

where
$$h_{\alpha\beta} = \begin{bmatrix} 1 & & \\ & (y^1)^2 & \\ & & 1 \end{bmatrix}.$$

The displacement field will be

$$\eta^1 = \eta^{(1)} = u(x^1) = u(r)$$

$$\eta^2 = 0$$

$$\eta^3 = 0.$$

In the deformed state the material coordinates will be related to the spatial coordinate system as follows

$$y^1 = x^1 + u(x^1) = r + u(r)$$

$$y^2 = x^2 = \theta$$

$$y^3 = x^3 = z.$$

The Green's deformation tensor is given by

$$G_{ij} = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

or
$$G_{ij} = \begin{bmatrix} (1+\frac{du}{dr})^2 & & \\ & (r+u)^2 & \\ & & 1 \end{bmatrix} \qquad (3.3)$$

also
$$G^{*i}_{\ \ j} = g^{ik} G_{jk}$$

or $$G^{*i}_{\ j} = \begin{bmatrix} (1+\frac{du}{dr})^2 & & \\ & (1+\frac{u}{r})^2 & \\ & & 1 \end{bmatrix} \qquad (3.4)$$

The Christoffel symbols of the second kind may be found from

$$\bar{\Gamma}^{i}_{jk} = \frac{G^{in}}{2}\left[\frac{\partial G_{jn}}{\partial x^k} + \frac{\partial G_{kn}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^n}\right]$$

$$\bar{\Gamma}^{1}_{11} = \frac{\frac{d^2u}{dr^2}}{(1 + \frac{du}{dr})}$$

$$\bar{\Gamma}^{3}_{13} = \bar{\Gamma}^{1}_{12} = \bar{\Gamma}^{1}_{13} = \bar{\Gamma}^{2}_{11} = \bar{\Gamma}^{3}_{11} = \bar{\Gamma}^{1}_{33} = 0$$

$$\bar{\Gamma}^{2}_{12} = \frac{(1 + \frac{du}{dr})}{(r + u)}$$

$$\bar{\Gamma}^{1}_{22} = - \frac{(r + u)}{(1 + \frac{du}{dr})} .$$

The equations of equilibrium in material coordinates, Equation (1.1), are

$$\tau^{i}_{j}||_{i} + \rho F_{j} = 0$$

The non-trivial equation for this problem is, for j = 1,

$$\tau^i_{1||i} = 0$$

or
$$\frac{\partial \tau^1_1}{\partial x^1} + \tau^1_1 \left[\overline{\Gamma}^1_{11} + \overline{\Gamma}^2_{21} + \overline{\Gamma}^3_{31} \right] - \tau^1_1\, \overline{\Gamma}^1_{11} - \tau^2_2\, \overline{\Gamma}^2_{21} - \tau^3_3\, \overline{\Gamma}^3_{31} = 0$$

Noting that $\tau^1_1 = \sigma_{11} = \sigma_r$, $\tau^2_2 = \sigma_{22} = \sigma_\theta$,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{u}{r}\frac{d\sigma_r}{dr} + \frac{du}{dr}\frac{\sigma_r - \sigma_\theta}{r} = 0. \qquad (3.5)$$

The strain components, Equation (1.2), are

$$E^i_j = \frac{1}{2}(G^{*i}_{\ j} - \delta^i_j)$$

or
$$E^i_j = \begin{bmatrix} \frac{du}{dr} + \frac{1}{2}(\frac{du}{dr})^2 & & \\ & \frac{u}{r} + \frac{1}{2}(\frac{u}{r})^2 & \\ & & 0 \end{bmatrix} \qquad (3.6)$$

leading to the invariants

$$\theta_1 = E^1_1 + E^2_2 \qquad (3.7)$$

$$\theta_2 = E^1_1\, E^2_2$$

$$\theta_3 = 0$$

$$I_3 = 1 + 2E^1_1 + 2E^2_2 + 4E^1_1 E^2_2$$

The heat conduction equation, Equation (1.16), is

$$(KG^{ij}T_a\|_j)\|_i = \rho(C_E\dot{T}_a - \frac{T_a}{\rho_o}\frac{\partial s^{ij}}{\partial T_a}\dot{E}_{ij}).$$

For our problem we note K and T_o are constants and consider only steady state conditions. Thus,

$$G^{ij}T\|_{ij} = 0$$

or

$$G^{11}\frac{d^2T}{d(x^1)^2} - \frac{dT}{dx^1}\left[G^{11}\bar{\Gamma}^1_{11} + G^{22}\bar{\Gamma}^1_{22} + G^{33}\bar{\Gamma}^1_{33}\right] = 0$$

or

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \left\{(\frac{u}{r} + \frac{du}{dr})\frac{d^2T}{dr^2} - (r\frac{d^2u}{dr^2} - 2\frac{du}{dr})\frac{1}{r}\frac{dT}{dr}\right\}$$

$$+ \left\{\frac{u}{r}\frac{du}{dr}\frac{d^2T}{dr^2} - \left[u\frac{d^2u}{dr^2} - (\frac{du}{dr})^2\right]\frac{1}{r}\frac{dT}{dr}\right\} = 0 \qquad (3.8)$$

Governing Equations for Vertical Slump of a Thick-walled Cylinder

Selecting material coordinates (x^i) such that in the deformed body they coincide with the cylindrical coordinates (r,θ,z), the material metric tensor is

$$G_{ij} = \begin{bmatrix} 1 & & \\ & (x^1)^2 & \\ & & 1 \end{bmatrix} \qquad (3.9)$$

Let (y^i) be cylindrical coordinates, therefore

$$h_{ij} = \begin{bmatrix} 1 & & \\ & (y^1)^2 & \\ & & 1 \end{bmatrix}$$

The displacement field will be

$$\xi^1 = \xi^{(1)}(x^1) = u(r)$$

$$\xi^2 = 0$$

$$\xi^3 = \xi^{(3)}(x^1) = w(r),$$

Thus,

$$y^1 = x^1 - u^1 = r - u(r)$$

$$y^2 = x^2$$

$$y^3 = x^3 - u^3 = z - w(r).$$

Now

$$g_{ij} = h_{kn} \frac{\partial y^k}{\partial x^i} \frac{\partial y^n}{\partial x^j}$$

or

$$g_{ij} = \begin{bmatrix} (1-\frac{du}{dr})^2 + (\frac{dw}{dr})^2 & 0 & -\frac{dw}{dr} \\ 0 & (r-u)^2 & 0 \\ -\frac{dw}{dr} & 0 & 1 \end{bmatrix} \tag{3.10}$$

Also g^{ij} may be found from $g^{ij}g_{ik} = \delta^i_k$

$$g^{ij} = \begin{bmatrix} \frac{1}{(1-\frac{du}{dr})^2} & 0 & \frac{\frac{dw}{dr}}{(1-\frac{du}{dr})^2} \\ 0 & \frac{1}{(r-u)^2} & 0 \\ \frac{\frac{dw}{dr}}{(1-\frac{du}{dr})^2} & 0 & 1 + \frac{\frac{dw}{dr}}{(1-\frac{du}{dr})^2} \end{bmatrix} \tag{3.11}$$

and

$$I_3 = \frac{|G_{ij}|}{|g_{ij}|} = \frac{1}{(1-\frac{u}{r})^2(1-\frac{du}{dr})^2}. \tag{3.12}$$

The equilibrium equations are the usual equations for cylindrical coordinates,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

(3.13)

and

$$\frac{d\tau_{rz}}{dr} + \frac{1}{r}\tau_{rz} + \rho g = 0$$

From Equations (1.9) and (3.12),

$$\frac{\rho}{\rho_o} = \frac{1}{\sqrt{I_3}} = (1 - \frac{u}{r})(1 - \frac{du}{dr})$$

(3.14)

thus

$$\frac{d\tau_{rz}}{dr} + \frac{1}{r}\tau_{rz} + \rho_o(1 - \frac{u}{r})(1 - \frac{du}{dr})g = 0.$$

The deformation tensor is

$$G^{*i}_{\ j} = \begin{bmatrix} \dfrac{1}{(1 - \frac{du}{dr})^2} & 0 & \dfrac{\frac{dw}{dr}}{(1 - \frac{du}{dr})^2} \\ 0 & \dfrac{r^2}{(r - u)^2} & 0 \\ \dfrac{\frac{dw}{dr}}{(1 - \frac{du}{dr})^2} & 0 & 1 + \dfrac{(\frac{dw}{dr})^2}{(1 - \frac{du}{dr})^2} \end{bmatrix}$$

leading to strain displacement equations

$$E^i_j = \frac{1}{2} \begin{bmatrix} \dfrac{1}{(1 - \frac{du}{dr})^2} - 1 & 0 & \dfrac{\frac{dw}{dr}}{(1 - \frac{du}{dr})^2} \\ 0 & \dfrac{1}{(1 - \frac{u}{r})^2} - 1 & 0 \\ \dfrac{\frac{dw}{dr}}{(1 - \frac{du}{dr})^2} & 0 & \dfrac{(\frac{dw}{dr})^2}{(1 - \frac{du}{dr})^2} \end{bmatrix} \qquad (3.15)$$

SECOND ORDER ELASTIC THEORY

4. <u>General Theory</u>.

In the remainder of this section, we shall assume that the magnitude of strains is such that we need retain only first and second order terms in our final equations. It must be emphasized that when the resulting solutions are applied to a specific problem, the numerical results must justify the above assumption.

For example the strain-displacement relations, Equation (3.15), for a second order theory, would be written as follows:

$$E^i_j = \begin{bmatrix} \frac{du}{dr} + \frac{3}{2}\left(\frac{du}{dr}\right)^2 & 0 & \frac{1}{2}\frac{dw}{dr} + \frac{du}{dr}\frac{dw}{dr} \\ 0 & \frac{u}{r} + \frac{3}{2}\left(\frac{u}{r}\right)^2 & 0 \\ \frac{1}{2}\frac{dw}{dr} + \frac{du}{dr}\frac{dw}{dr} & 0 & \left(\frac{dw}{dr}\right)^2 \end{bmatrix}$$

The form of the Helmholtz free energy function, which determines the form of the constitutive equations will now be determined for a second order elastic theory. The energy datum is taken at the reference temperature T_o. For a compressible material, assuming that T is of the order of magnitude of E^i_j, we expand A in a power series in θ_i and T and retain terms of third order or less. The range of validity of the second order theory needs to be considered in light of the relative size of the coefficients of the neglected terms as compared to the terms retained in the expansion, a comparison which can best be made by finding the total expression for A by experimental means. Accordingly,

$$A = k_1(\theta_2 + k_2\theta_1^{\,2} + k_3\theta_1\theta_2 + k_4\theta_1^{\,3} + k_5\theta_3 + k_6T\theta_1 + k_7T\theta_1^{\,2} + k_8T\theta_2 + k_9T^2\theta_1 + k_{10}T^2 + k_{11}T^3). \quad (4.1)$$

In classical elasticity only the first and second order terms are retained, i.e.,

$$A = k_1(\theta_2 + k_2\theta_1^{\,2} + k_6T\theta_1 + k_{10}T^2)$$

which when compared to the usual notation [4] is written

$$A = -2\mu\left[\theta_2 - \frac{1-\nu}{2(1-2\nu)}\theta_1^{\,2} + \frac{(1+\nu)\alpha T_o}{1-2\nu}\theta_1 T + k_{10}T^2\right].$$

Comparing terms we find (note k_{10} and k_{11} need not be evaluated)

$$k_1 = -2\mu$$

$$k_2 = -\frac{1-\nu}{2(1-2\nu)} \quad (4.2)$$

$$k_6 = \frac{1+\nu}{1-2\nu}\alpha T_o$$

or $$\mu = -\frac{k_1}{2}$$

$$\nu = \frac{1 + 2k_2}{1 + 4k_2} \qquad (4.3)$$

$$\alpha = -\frac{k_6}{2T_o(1 + 3k_2)} .$$

In incompressible elasticity we need the function $\bar{A}(\theta_2, \theta_3, T)$, Equation (2.9a), which, in light of the above reasoning, shall be taken as

$$\bar{A} = c_1(\theta_2 + c_2\theta_3 + c_3T\theta_2 + c_4T^2 + c_5T^3) \qquad (4.4)$$

It is also necessary to have an expression for $\frac{dV - dV_o}{dV_o}$ as a function of T. For a second order incompressible theory, Equation (2.6) is written

$$\frac{dV - dV_o}{dV_o} = \beta_1 T + \beta_2 T^2 = f(T) \qquad (4.5)$$

For classical thermoelasticity we have $\frac{dV - dV_o}{dV_o} = 3\alpha T_o T$; thus $\beta_1 = 3\alpha T_o$. Hence, the incompressibility condition $\bar{I}_3 = 0$, from Equation (2.8), yields

$$I_3 - \left[1 + f(T)\right]^2 = 0 \qquad (4.6)$$

Substituting for I_3,

$$1 + 2\theta_1 + 4\theta_2 + 8\theta_3 - 1 - 2f(T) - f^2(T) = 0$$

or (4.7)

$$\theta_1 + 2\theta_2 + 4\theta_3 = \beta_1 T + (\beta_2 + \tfrac{1}{2}\beta_1{}^2)T^2$$

Now let $$(\beta_2 + \tfrac{1}{2}\beta_1{}^2) = \beta_2{}^* \qquad (4.8)$$

then $$\theta_1 + 2\theta_2 + 4\theta_3 = \beta_1 T + \beta_2{}^* T^2. \qquad (4.9)$$

In order to relate the coefficients appearing in Equations (4.1) and (4.4) it is necessary to express A in terms of $\theta_2, \theta_3, \bar{I}_3$ and T. Noting that

$$\bar{I}_3 = 2\theta_1 + 4\theta_2 + 8\theta_3 - 2\beta_1 T - 2\beta_2{}^* T^2$$

we may write Equation (4.1) as

$$A = k_1 \Big\{ \theta_2 + k_5\theta_3 + \left[-4\beta_1 k_2 + k_3\beta_1 - 2k_6 + k_8\right] T\theta_2 + \frac{k_2}{4}\bar{I}_3{}^2$$

$$+ \left[k_2\beta_1 + \frac{k_6}{2}\right]\bar{I}_3 T + \frac{k_4}{8}\bar{I}_3{}^3 + \left[\frac{3\beta_1 k_4}{4} + \frac{k_7}{4}\right]\bar{I}_3{}^2 T$$

$$+ \left[k_2\beta_2{}^* + \frac{3k_4\beta_1{}^2}{2} + \beta_1 k_7 + \frac{k_9}{2}\right]\bar{I}_3 T^2 + \left[-2k_2 + \frac{k_3}{2}\right]\bar{I}_3\theta_2 + h(T)\Big\}.$$

Thus

$$A\Big|_{\bar{I}_3=0} = \bar{A} = k_1\left\{\theta_2 + k_5\theta_3 + \left[-4\beta_1 k_2 + k_3\beta_1 - 2k_6 + k_8\right]T\theta_2 + h(T)\right\}$$

which, when compared with Equation (4.4), yields

$$\begin{aligned} c_1 &= k_1 \\ c_2 &= k_5 \\ c_3 &= \left[-4\beta_1 k_2 + k_3\beta_1 - 2k_6 + k_8\right]. \end{aligned} \qquad (4.10)$$

To obtain the constitutive equations we need merely apply either Equation (1.14) or (2.14). For example, for the pressure and temperature loading of a compressible cylinder we will make use of Equation (1.14) where, (see Equations (3.2), (3.3) and (3.7)),

$$g^{11} = 1$$

$$B^{11} = 2\theta_1 + 3 - G_{11}$$

$$G_{11} = 2E_1^1 + 1$$

$$G^{11} = \frac{1}{1 + 2E_1^1}$$

$$\theta_1 = E_1^1 + E_2^2$$

$$\theta_2 = E_1^1 E_2^2$$

$$\theta_3 = 0$$

Thus,

$$\tau^{11} = \frac{k_1}{\sqrt{I_3}} \Big[2k_2(E_1^1 + E_2^2) + k_3 E_1^1 E_2^2 + 3k_4(E_1^1 + E_2^2)^2 + k_6 T$$

$$+ 2k_7 T(E_1^1 + E_2^2) + k_9 T^2 + k_3 E_2^2(E_1^1 + E_2^2) + k_8 E_2^2 T + E_2^2 \Big].$$

Utilizing Equations (1.2), (1.17), and (2.4), we can write

$$\sigma_r = G_{11}\tau^{11} = (2E_1^1 + 1)\tau^{11} ,$$

or

$$\sigma_r = k_1 \Big[(1+2k_2)E_2^2 + 2k_2 E_1^1 + k_6 T + E_1^1 E_2^2 (2k_3 + 6k_4 + 1) + (E_1^1)^2 (3k_4 + 2k_2)$$

$$+ (E_2^2)^2 (k_3 + 3k_4 - 1 - 2k_2) + E_1^1 T(2k_7 + k_6) + E_2^2 T(2k_7 + k_8 - k_6) + k_9 T^2 \Big]$$

Likewise

$$\sigma_r - \sigma_\theta = k_1 \Big\{ (E_2^2 - E_1^1) + \Big[(E_1^1)^2 - (E_2^2)^2\Big](4k_2 + 1 - k_3) + (E_1^1 - E_2^2)T(2k_6 - k_8) \Big\}.$$

Making use of Equation (3.6),

$$\sigma_r = k_1 \Big[(1+2k_2)\frac{u}{r} + 2k_2 \frac{du}{dr} + k_6 T + (k_3 + 3k_4 - \frac{1}{2} - k_2)\left(\frac{u}{r}\right)^2$$

$$+ (k_2 + 3k_4 + 2k_2)\left(\frac{du}{dr}\right)^2 + (2k_3 + 6k_4 + 1)\frac{u}{r}\frac{du}{dr} + (2k_7 + k_6)T\frac{du}{dr}$$

$$+ (2k_7 + k_8 - k_6)\, T\,\frac{u}{r} + k_9 T^2 \Big] \qquad (4.11)$$

and

$$\sigma_r - \sigma_\theta = k_1\left(\frac{u}{r} - \frac{du}{dr}\right)\left[1 + \left(k_3 - \frac{1}{2} - 4k_2\right)\left(\frac{u}{r} + \frac{du}{dr}\right) + (k_8 - 2k_6)T\right]. \qquad (4.12)$$

5. Outline of Method.

In order to cast the governing equations for pressure and temperature problems into a form susceptible to a perturbation solution (i.e., into a form where the relative magnitude of the various terms is apparent) we shall make the following change of variables; (A somewhat similar method of introducing a perturbation parameter for thick cylindrical shell theory is given in [9]).

$$v = \frac{u}{u_{max}} \tag{5.1}$$

$$\rho = \frac{r}{a}, \tag{5.2}$$

where a is the inner radius of the cylinder.

Thus

$$\left|\frac{v}{\rho}\right| \leq 1$$

and

$$\frac{u}{r} = \frac{v u_{max}}{\rho a}$$

Let

$$\frac{u_{max}}{a} = \delta \quad \text{(For a second order theory } |\delta| < |\text{.)} \tag{5.3}$$

Therefore,

$$\frac{u}{r} = \delta \frac{v}{\rho}$$

Also let

$$\sigma_r = k_1 \delta s_\rho$$

(5.4)

$$\sigma_\theta = k_1 \delta s_\theta$$

$$T = \delta T^* \qquad (5.5)$$

Making the above change of variables the governing equations (3.5), (3.8), (4.11) and (4.12) may be written

$$\frac{ds_\rho}{d\rho} + \frac{s_\rho - s_\theta}{\rho} = -\delta\left[\frac{v}{\rho}\frac{ds_\rho}{d\rho} + \frac{dv}{d\rho}\frac{s_\rho - s_\theta}{\rho}\right]$$

$$\frac{d^2T^*}{d\rho^2} + \frac{1}{\rho}\frac{dT^*}{d\rho} = -\delta\left[\left(\frac{v}{\rho} + \frac{dv}{d\rho}\right)\frac{d^2T^*}{d\rho^2} - \left(\rho\frac{d^2v}{d\rho^2} - 2\frac{dv}{d\rho}\right)\frac{1}{\rho}\frac{dT^*}{d\rho}\right] - \delta^2\left\{\frac{v}{\rho}\frac{dv}{d\rho}\frac{d^2T^*}{d\rho^2}\right.$$

$$\left. - \left[v\frac{d^2v}{d\rho^2} - \left(\frac{dv}{d\rho}\right)^2\right]\frac{1}{\rho}\frac{dT^*}{d\rho}\right\}$$

$$s_\rho = (1+2k_2)\frac{v}{\rho} + 2k_2\frac{dv}{d\rho} + k_6T^* + \delta\left[(k_3+3k_4-\frac{1}{2}-k_2)\left(\frac{v}{\rho}\right)^2 + 3(k_4\right.$$

$$+ k_2)\left(\frac{dv}{d\rho}\right)^2 + (2k_3+6k_4+1)\frac{v}{\rho}\frac{dv}{d\rho} + (2k_7+k_6)T^*\frac{dv}{d\rho} + (2k_7$$

$$\left. + k_8 - k_6)T^*\frac{v}{\rho} + k_9T^{*2}\right] \qquad (5.6)$$

$$s_\rho - s_\theta = \frac{v}{\rho} - \frac{dv}{d\rho} + \delta\left\{(k_3 - \frac{1}{2} - 4k_2)\left[(\frac{v}{\rho})^2 - (\frac{dv}{d\rho})^2\right] + (k_8 - 2k_6)T^*(\frac{v}{\rho} - \frac{dv}{d\rho})\right\}.$$

As $\delta \to 0$ the above equations approach the classical elastic equations; thus, we have established the conditions that are prerequisite to a solution by perturbation, i.e., (1) a near state for which an exact solution is known (as $\delta \to 0$ we have the state as defined by the classical field equations) and (2) the transition from this near state to the desired state should proceed in a smooth manner (as previously indicated we shall not consider stability problems). It should be noted that the perturbation parameter δ is merely a device used in obtaining a solution and as such will not appear in the final solution. To effect a solution by perturbation we expand the dependent variables in a perturbation series, i.e.,

$$T^* = T^{*(0)} + \delta T^{*(1)} + \ldots$$

$$v = v^{(0)} + \delta v^{(1)} + \ldots$$

$$s_\rho = s_\rho^{(0)} + \delta s_\rho^{(1)} + \ldots \qquad (5.7)$$

$$s_\theta = s_\theta^{(0)} + \delta s_\theta^{(1)} + \ldots,$$

where $v^{(0)}$ is the solution of the near state (first approximation), $v^{(1)}$ is the first corrective term, etc. Substitution of the above expressions into the governing equations and equating coefficients of δ we obtain the following systems of equations;

First system

$$\frac{d^2T^{*(0)}}{d\rho^2} + \frac{1}{\rho}\frac{dT^{*(0)}}{d\rho} = 0$$

$$\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} = 0$$

$$(5.8)$$

$$s_\rho^{(0)} - (1 + 2k_2)\frac{v^{(0)}}{\rho} - 2k_2\frac{dv^{(0)}}{d\rho} - k_6T^{*(0)} = 0$$

$$s_\rho^{(0)} - s_\theta^{(0)} - \frac{v^{(0)}}{\rho} + \frac{dv^{(0)}}{d\rho} = 0 ,$$

Second system

$$\frac{d^2T^{*(1)}}{d\rho^2} + \frac{1}{\rho}\frac{dT^{*(1)}}{d\rho} = - \left(\frac{v^{(0)}}{\rho} + \frac{dv^{(0)}}{d\rho}\right)\frac{d^2T^{*(0)}}{d\rho^2} + \left(\rho\frac{d^2v^{(0)}}{d\rho^2}\right.$$

$$\left. - 2\frac{dv^{(0)}}{d\rho}\right)\frac{1}{\rho}\frac{dT^{*(0)}}{d\rho}$$

$$\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} = \frac{1}{\rho}\left(\frac{v^{(0)}}{\rho} - \frac{dv^{(0)}}{d\rho}\right)^2 \qquad (5.9)$$

$$s_\rho^{(1)} - (1 + 2k_2)\frac{v^{(1)}}{\rho} - 2k_2\frac{dv^{(1)}}{d\rho} - k_6T^{*(1)} = (k_3 + 3k_4 - \frac{1}{2}$$

$$- k_2)\left(\frac{v^{(0)}}{\rho}\right)^2$$

Continued

(5.9 Continued)

$$+ 3(k_4 + k_2)\left(\frac{dv^{(0)}}{d\rho}\right)^2 + (2k_3 + 6k_4 + 1)\frac{v^{(0)}}{\rho}\frac{dv^{(0)}}{d\rho} + (2k_7 + k_6)T^{*(0)}\frac{dv^{(0)}}{d\rho}$$

$$+ (2k_7 + k_8 - k_6)T^{*(0)}\frac{v^{(0)}}{\rho} + k_9(T^{*(0)})^2$$

$$s_\rho^{(1)} - s_\theta^{(1)} - \left(\frac{v^{(1)}}{\rho} - \frac{dv^{(1)}}{d\rho}\right) = (k_3 - 4k_2 - \tfrac{1}{2})\left[\left(\frac{v^{(0)}}{\rho}\right)^2 - \left(\frac{dv^{(0)}}{d\rho}\right)^2\right]$$

$$+ (k_8 - 2k_6)T^{*(0)}\left(\frac{v^{(0)}}{\rho} - \frac{dv^{(0)}}{d\rho}\right).$$

The first set of equations is that of classical elasticity. The left hand side of the two systems of equations are identical. Whereas the first system is homogeneous (in the absence of body forces and temperature sources) the second system has non-homogeneous parts composed of non-linear terms derived from the solution of the first system of equations; hence, the non-homogenous parts of the second system are known functions once we have solved the first system of equations.

It is to be pointed out that, although the perturbation solution method leads to an infinite series, for a second order theory we need only consider the first two terms of this series. If we used perturbation to solve a problem described by a third order theory we would need three terms, etc., since neglecting the third and higher terms of the series introduces errors of the same magnitude as those introduced by neglecting third and higher powers of strain in the formulation of a second order theory.

6. Compressible Response.

Uniaxial Test

The uniaxial test is analyzed since it is the most common test performed upon material samples. Although it is preferable to obtain elastic properties using an experimental approach as outlined in [6], it is possible to obtain results from a uniaxial test. For a description of a uniaxial test where thermal effects are included, see [7]. The solution of the uniaxial problem may be considered from a perturbation standpoint.

We shall select material coordinates x^i to coincide with rectangular cartesian coordinates in the initial state. Choosing the spatial coordinates y^i as rectangular cartesian coordinates we thus find

$$g_{ij} = \delta_{ij} = h_{ij}.$$

The displacement field will be given by

$$\eta^i = e_i x^i \quad \text{(no sum)},$$

and as

$$y^i = x^i + \eta^i$$

or

$$y^i = (1 + e_i)x^i \quad \text{(no sum)}$$

Hence,

$$G_{ij} = G^{*i}_{\ j} = \begin{bmatrix} (1+e_1)^2 & & \\ & (1+e_2)^2 & \\ & & (1+e_3)^2 \end{bmatrix}$$

and

$$E^i_j = \begin{bmatrix} e_1 + \frac{1}{2}e_1{}^2 & & \\ & e_2 + \frac{1}{2}e_2{}^2 & \\ & & e_3 + \frac{1}{2}e_3{}^2 \end{bmatrix},$$

Utilizing Eqs. (1.15), (1.7), (1.9), (4.1) and (1.5) we find

$$g^{11} = 1$$

$$B^{11} = I_1 - G_{11} = I_1 - (1 + e_1)^2$$

$$G^{11} = \frac{1}{(1 + e_1)^2}$$

$$I_1 = 2\theta_1 + 3 ,$$

$$I_3 = 1 + 2\theta_1 + 4\theta_2 + 8\theta_3 ,$$

$$\frac{\partial A}{\partial \theta_1} = k_1(2k_2\theta_1 + k_3\theta_2 + 3k_4\theta_1{}^2 + k_6T + 2k_7T\theta_1 + k_9T^2)$$

$$\frac{\partial A}{\partial \theta_2} = k_1(1 + k_3\theta_1 + k_8T)$$

$$\frac{\partial A}{\partial \theta_3} = k_1k_5 ,$$

$$\theta_1 = e_1 + e_2 + e_3 + \frac{1}{2} e_1{}^2 + \frac{1}{2} e_2{}^2 + \frac{1}{2} e_3{}^2$$

$$\theta_2 = e_1e_2 + e_1e_3 + e_2e_3.$$

From Eq. (1.13)

$$s^{11} = k_1\Big[2k_2e_1 + (2k_2 + 1)(e_2 + e_3) + (k_2 + 3k_4)e_1{}^2 + (k_2 + \frac{1}{2} + 3k_4$$

$$+ k_3)(e_2^2 + e_3^2) + (2k_3 + 6k_4)(e_1e_2 + e_1e_3) + (3k_3 + k_5$$

$$+ 6k_4)e_2e_3 + k_6T + 2k_7e_1T + (2k_7 + k_8)T(e_2 + e_3) + k_9T^2\Big]$$

likewise

$$s^{22} = k_1\Big[2k_2e_2 + (2k_2 + 1)(e_1 + e_3) + (k_2 + 3k_4)e_2^2 + (k_2 + \frac{1}{2} + 3k_4$$

$$+ k_3)(e_1^2 + e_3^2) + (2k_3 + 6k_4)(e_1e_2 + e_2e_3) + (3k_3 + k_5 + 6k_4)e_1e_3$$

$$+ k_6T + 2k_7e_2T + (2k_7 + k_8)T(e_1 + e_3) + k_9T^2\Big].$$

For a uniaxial test of an isotropic material

$e_2 = e_3$ and

$$s^{11} = k_1 \Big[2k_2 e_1 + 2(2k_2 + 1)e_2 + (k_2 + 3k_4)e_1{}^2 + (2k_2 + 1 + 12k_4 + 5k_3$$

$$+ k_5)e_2^2 + 2(2k_3 + 6k_4)e_1 e_2 + k_6 T + 2k_7 e_1 T + 2(2k_7 + k_8)Te_2 + k_9 T^2 \Big]$$

$$s^{22} = k_1 \Big[(4k_2 + 1)e_2 + (2k_2 + 1)e_1 + (2k_2 + \tfrac{1}{2} + 12k_4 + 3k_3)e_2^2$$

$$+ (k_2 + \tfrac{1}{2} + 3k_4 + k_3)e_1{}^2 + (5k_3 + 12k_4 + k_5)e_1 e_2 + k_6 T$$

$$+ (4k_7 + k_8)Te_2 + (2k_7 + k_8)Te_1 + k_9 T^2 \Big].$$

From Eqs. (1.13), (1.17) and the above

$$\sigma_{11} = \frac{1 + e_1}{(1 + e_2)^2} s^{11}$$

or

$$\sigma_{11} = k_1 \Big[2k_2 e_1 + 2(2k_2 + 1)e_2 + 3(k_2 + k_4)e_1{}^2 + (12k_4 + 5k_3 - 6k_2$$

$$- 3 + k_5)e_2^2 + 2(2k_3 + 6k_4 + 1)e_1 e_2 + k_6 T + (2k_7 + k_6)e_1 T$$

$$+ 2(2k_7 + k_8 - k_6)Te_2 + k_9 T^2 \Big]. \qquad (6.1)$$

Let σ_{1n} be the nominal stress (i.e., per unit of initial area) then $\sigma_{1n} = (1 + e_1)s^{11}$

or $\sigma_{1n} = k_1\Big[2k_2e_1 + 2(2k_2+1)e_2 + 3(k_2 + k_4)e_1^2 + (2k_2 + 1 + 12k_4 + k_5$

$+ 5k_3)e_2^2 + 2(2k_2 + 1 + 2k_3 + 6k_4)e_1e_2 + k_6T + (2k_7 + k_6)e_1T$

$$+ 2(2k_7 + k_8)Te_2 + k_9T^2\Big] \qquad (6.2)$$

also $\sigma_{22} = s_{22} = 0$, hence

$$0 = k_1\Big[(4k_2 + 1)e_2 + (2k_2 + 1)e_1 + (2k_2 + \tfrac{1}{2} + 12k_4 + 3k_3)e_2^2$$

$$+ (k_2 + \tfrac{1}{2} + 3k_4 + k_3)e_1{}^2 + (5k_3 + 12k_4 + k_5)e_1e_2 + k_6T$$

$$+ (4k_7 + k_8)Te_2 + (2k_7 + k_8)Te_1 + k_9T^2\Big]. \qquad (6.3)$$

Thermal Stressing of a Thick-walled Cylinder

The first of Eqs. (5.8) may be written

$$\frac{d}{d\rho}\left[\rho\ \frac{dT^{*(0)}}{d\rho}\right] = 0$$

whose solution is

$$T^{(0)} = C^{(0)}\ln\rho + D^{(0)} \qquad (6.4)$$

Noting the above results and using the third and fourth of Eqs. (5.8) we may write the second of Eq. (5.8) as

$$\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}\left[\rho v^{(0)}\right]\right\} = -\frac{k_6}{2k_2}\, C^{(0)}\, \frac{1}{\rho}$$

whose solution is

$$v^{(0)} = \overline{A}^{(0)}\rho + B^{(0)}\frac{1}{\rho} - \frac{k_6}{4k_2}\, C^{(0)}\rho(\ln\rho - \tfrac{1}{2}). \qquad (6.5)$$

Thus,

$$T^{*(0)} = C^{(0)}\ln\rho + D^{(0)}$$

$$v^{(0)} = A^{(0)}\rho + B^{(0)}\frac{1}{\rho} - \frac{k_6}{4k_2}\, C^{(0)}\rho\ln\rho \qquad (6.6)$$

$$s_\rho^{(0)} = (1 + 4k_2)A^{(0)} + k_6 D^{(0)} - \frac{k_6}{2}C^{(0)} + B^{(0)}\frac{1}{\rho^2} - \frac{k_6}{4k_2}C^{(0)}\ln\rho.$$

From the above solution to the first system of equations we are able to calculate the nonhomogenous part of the second system of equations. Thus, we are able to integrate the second system. The resulting expression for $T^{*(1)}$ is

$$T^{*(1)} = C^{(1)}\ln\rho + D^{(1)} + B^{(0)}C^{(0)}\frac{1}{\rho^2}$$

while the expressions for the remaining dependent variables may be similarly obtained. The constants $A^{(0)}$, $B^{(0)}$, $C^{(0)}$, $D^{(0)}$, $A^{(1)}$, $B^{(1)}$ etc., are evaluated from the boundary conditions.

In order to avoid the rather involved algebra we shall consider the problem of a uniform temperature drop T of a thick-walled cylinder bonded to a rigid case. The inner and outer cylinder radii are a and b respectively. The boundary conditions are

$$\text{at } r = a,\ \rho = 1 \quad \begin{cases} T^{*(0)} = T^* = \dfrac{T}{\delta} \\ T^{*(1)} = 0 \\ s_\rho^{(0)} = 0 \\ s_\rho^{(1)} = 0 \end{cases} \tag{6.7}$$

$$\text{and at } r = b,\ \rho = \frac{b}{a} = c \quad \begin{cases} T^{*(0)} = T^* \\ T^{*(1)} = 0 \\ v^{(0)} = 0 \\ v^{(1)} = 0. \end{cases}$$

Applying the above boundary conditions to Eq. (6.6) we obtain

$$C^{(0)} = 0$$

$$D^{(0)} = T^*$$

$$A^{(0)} = \frac{k_6 T^*}{c^2 - (1 + 4k_2)} \qquad (6.8)$$

$$B^{(0)} = - c^2 A^{(0)}$$

Let
$$X_1 = \frac{k_6}{c^2 - (1 + 4k_2)}$$

then
$$T^{(0)} = T^*$$

$$v^{(0)} = T^* X_1 (\rho - \frac{c^2}{\rho}) \qquad (6.9)$$

$$s_\rho^{(0)} = T^* \left\{ X_1 \left[(1+4k_2) - \frac{c^2}{\rho^2} \right] + k_6 \right\}.$$

The second of Eq. (6.7) yields $C^{(1)} = D^{(1)} = 0$. The above first order solution and Eqs. (5.9) yields

$$\frac{d}{d\rho}\left\{\frac{1}{\rho} \frac{d}{d\rho}\left[\rho v^{(1)}\right]\right\} = \frac{(T^* X_1)^2}{k_2} (4k_2 - 1 - 2k_3)\frac{c^4}{\rho^5}.$$

The solution is

$$v^{(1)} = \frac{(T^* X_1)^2}{8k_2} (4k_2 - 1 - 2k_3)\frac{c^4}{\rho^3} + A^{(1)}\rho + B^{(1)} \frac{1}{\rho}. \qquad (6.10)$$

The third of Eq. (5.9) becomes

$$s_{\rho}^{(1)} = (1 + 4k_2)A^{(1)} + B^{(1)} \frac{1}{\rho^2} - (X_1 T^*)^2 \Bigg[\left(\frac{1 + 2k_3 + 4k_2}{8k_2}\right)\frac{c^4}{\rho^4}$$

$$- (1 + 8k_2 - 2k_3)\frac{c^2}{\rho^2} - (3k_3 + 12k_4 + \frac{1}{2} + 2k_2)\Bigg]$$

$$+ X_1 (T^*)^2 \left[(4k_7 + k_8) + (2k_6 - k_8)\frac{c^2}{\rho^2} \right] + k_9 (T^*)^2. \qquad (6.11)$$

Applying the second boundary condition in Eq. (6.7) to the above equations we find that

$$A^{(1)} = X_2 (T^*)^2 \qquad (6.12)$$

$$B^1 = X_3 (T^*)^2$$

where

$$X_2 = \frac{c^2 X_4 - X_5}{c^2 - (1 + 4k_2)} \qquad (6.13)$$

$$X_3 = \frac{c^2 [X_5 - X_4 (1 + 4k_2)]}{c^2 - (1 + 4k_2)} \qquad (6.14)$$

$$X_4 = - \frac{X_1^2 (4k_2 - 1 - 2 k_3)}{8k_2} \qquad (6.15)$$

$$X_5 = X_1^2\left[\frac{1 + 2k_3 + 4k_2}{8k_2} c^4 - (1 + 8k_2 - 2k_3)c^2 - (3k_3 + 12k_4 + \frac{1}{2}\right.$$

$$\left. + 2k_2)\right] - X_1\left[(4k_7 + k_8) + (2k_6 - k_8)c^2\right] - k_9 \qquad (6.16)$$

$$X_1 = \frac{k_6}{c^2 - (1 + 4k_2)} \qquad (6.17)$$

$$c = \frac{b}{a}.$$

Making use of Eqs. (6.10), (6.11), (6.12), (5.1), (5.2), (5.3), (5.4), and (5.7), we may express our results as follows

$$\sigma_r = k_1 T\left\{X_1\left[(1 + 4k_2) - \frac{b^2}{r^2}\right] + k_6\right\} + k_1 T^2\left\{(1 + 4k_2)X_2 + X_3 \frac{a^2}{r^2}\right.$$

$$- X_1^2\left[\left(\frac{1 + 2k_3 + 4k_2}{8k_2}\right)\frac{b^4}{r^4} - (1 + 8k_2 - 2k_3)\frac{b^2}{r^2} - (3k_3 + 12k_4 + \frac{1}{2}\right.$$

$$\left.\left. + 2k_2)\right] + X_1\left[(4k_7 + k_8) + (2k_6 - k_8)\frac{b^2}{r^2}\right] + k_9\right\} \qquad (6.18)$$

$$u = TX_1(r - \frac{b^2}{r}) + T^2\left[X_2 r + X_3 \frac{a^2}{r} - X_4 \frac{b^4}{r^3}\right]. \qquad (6.19)$$

The very important fact that the results do not depend upon δ (as δ is only a numerical indicator of the relative sizes of the various terms appearing in the governing equations) is clearly illustrated in the above results.

Pressure Loading of a Thick-walled Cylinder

Setting $T^{*(0)} = 0$ in Eq. (6.6) we find that

$$v^{(0)} = A^{(0)}\rho + B^{(0)}\frac{1}{\rho}$$

(6.20)

$$s_\rho^{(0)} = (1 + 4k_2)A^{(0)} + B^{(0)}\frac{1}{\rho^2}.$$

Using the first and fourth of Eq. (5.9) we may write the third Eq. (5.9) as

$$\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}\left[\rho v^{(1)}\right]\right\} = \frac{1}{k_2}(4k_2 - 2k_3 - 1)(B^{(0)})^2\frac{1}{\rho^5}.$$

The solution is

$$v^{(1)} = A^{(1)}\rho + B^{(1)}\frac{1}{\rho} + \frac{1}{8k_2}(4k_2 - 2k_3 - 1)\frac{(B^{(0)})^2}{\rho^3} \quad (6.21)$$

hence from Eq. (5.9)

$$s_\rho^{(1)} = (1 + 4k_2)A^{(1)} + B^{(1)}\frac{1}{\rho^2} + (3k_3 + 12k_4 + 2k_2 + \tfrac{1}{2})(A^{(0)})^2$$

(6.22)

$$- \frac{4k_2 + 2k_3 + 1}{8k_2}\frac{(B^{(0)})^2}{\rho^4} + (2k_3 - 8k_2 - 1)A^{(0)}B^{(0)}\frac{1}{\rho^2}.$$

Let us now consider the specific problem of a thick-walled cylinder subjected to an internal pressure P_i, the boundary conditions will be:

At $$r = a \ (\rho = 1); \ \sigma_r = -P_i$$

or $$s_\rho = s_\rho^{(0)} = -\frac{P_i}{k_1\delta}$$

Let $$\overline{P} = \frac{P_i}{k_1\delta}$$

then $$s_\rho^{(0)} = -\overline{P}$$

and $$s_\rho^{(1)} = 0 \qquad (6.23)$$

and at $$r = b(\rho = \frac{b}{a} = c); \quad s_\rho^{(0)} = 0 = s_\rho^{(1)}. \qquad (6.24)$$

Applying the above boundary conditions to Eq. (6.20) we find

$$A^{(0)} = \frac{\overline{P}}{(c^2-1)(1 + 4k_2)}$$

$$B^{(0)} = -\frac{c^2\overline{P}}{c^2-1}. \qquad (6.25)$$

Noting the above results and applying the remaining boundary conditions to Eq. (6.22) we find

$$A^{(1)} = (\bar{P})^2 X_1 \qquad (6.26)$$

$$B^{(1)} = (\bar{P})^2 X_2$$

where
$$X_1 = -\frac{1}{(c^2-1)^2(1+4k_2)^2}\left[\frac{(3k_3+12k_4+2k_2+\frac{1}{2})}{(1+4k_2)} + \frac{c^2(4k_2+2k_3+1)(1+4k_2)}{8k_2}\right] \qquad (6.27)$$

$$X_2 = \frac{c^2}{(c^2-1)^2(1+4k_2)}\left[\frac{(c^2+1)(4k_2+2k_3+1)(1+4k_2)}{8k_2} + (2k_3-8k_2-1)\right]. \qquad (6.28)$$

Using the above notation we may write our solution from Eqs. (6.20), (6.21), and (6.22) as

$$\sigma_r = \frac{P_i}{c^2-1}\left[1-\frac{b^2}{r^2}\right] - \frac{(P_i)^2(4k_2+2k_3+1)}{8k_1k_2(c^2-1)^2}\left[c^2-(c^2+1)\frac{b^2}{r^2}+\frac{b^4}{r^4}\right]$$

$$u = \frac{P_i}{k_1(1+4k_2)(c^2-1)}\left[r-(1+4k_2)\frac{b^2}{r}\right] + \frac{P_i^2}{k_1^2}\left[X_1 r + X_2\frac{a^2}{r} + \frac{(4k_2-2k_3-1)}{8k_2(c^2-1)^2}\frac{b^4}{r^3}\right]. \qquad (6.29)$$

We shall now consider an externally case bonded thick-walled cylinder subjected to internal pressure. Let the elastic properties of the thin case be $\bar{\nu}$ and $\bar{E}$ and denote its thickness by t. The motion of the case is given by

$$\bar{u} = \frac{(1 - \bar{\nu}^2)}{\bar{E}} \frac{b^2 P'}{t} \tag{6.30}$$

where P' is the interface pressure. We shall consider the situation when the case is very rigid in comparison to the thick-walled cylinder, i.e., we must consider the material of the thick-walled cylinder as rather compressible. (See Section 8 for a discussion of compressibility effects.) The displacement at the interface will be of second order as compared to the displacement at the inner radius (in the application to a specific problem one must verify this assumption). We may thus illustrate one of the methods for reducing the algebraic complexities of a given problem, i.e., introducing various boundary conditions at different levels of approximation. We employ the following boundary conditions:

At $\rho = 1$; $\quad s_\rho^{(0)} = -\bar{P} \quad$ where $\quad \bar{P} = \dfrac{P_i}{k_1 \delta}$ (6.31)

$$s_\rho^{(1)} = 0$$

and at

$$\rho = c\ ; \qquad v^{(0)} = 0$$

$$v^{(1)} = \bar{w} \qquad \text{where} \qquad \bar{w} = \frac{\bar{u}}{a\delta^2}\,. \tag{6.32}$$

The interface pressure P' is found from the continuity condition

$$\sigma_r(b) = -P' \tag{6.33}$$

i.e. $$k_1 \delta \left[s_\rho^{(0)} + \delta s_\rho^{(1)} \right]_{r=b} = -P'$$

The above boundary conditions when applied to Eqs. (6.20), (6.21), and (6.22) give

$$A^{(0)} = -\frac{\bar{P}}{1 + 4k_2 - c^2}$$

$$B^{(0)} = \frac{c^2 \bar{P}}{1 + 4k_2 - c^2}$$

$$A^{(1)} = \bar{P}^2 X_6$$

$$B^{(1)} = \bar{P}^2 X_7$$

where (6.34)

$$X_6 = \frac{\bar{X}_1 - cX_2 - \dfrac{c\bar{u}k_1^2}{aP_i^2}}{1 + 4k_2 - c^2}$$

$$X_7 = \frac{c(1 + 4k_2)\left[\dfrac{\bar{u}k_1^2}{aP_i^2} + X_2 \right] - c^2 \bar{X}_1}{1 + 4k_2 - c^2}$$

$$X_1(r) = -\frac{1}{(1 + 4k_2 - c^2)^2}\left[(3k_3 + 12k_4 + 2k_2 + \frac{1}{2}) - \frac{(4k_2 + 2k_3 + 1)}{8k_2}\frac{b^4}{r^4}\right.$$

$$\left. - (2k_3 - 8k_2 - 1)\frac{b^2}{r^2}\right] \tag{6.35}$$

$$\bar{X}_1 = X_1(r = a)$$

$$X_2 = -\frac{c(4k_2 - 2k_3 - 1)}{8k_2(1 + 4k_2 - c^2)^2}$$

$$\bar{u} = \frac{(1 - \bar{\nu}^2)}{\bar{E}}\frac{b^2 P'}{t}.$$

We now evaluate P' from Eqs. (6.30), (6.32), and (6.33):

$$P' = \frac{-4k_1k_2cP_i + X_4(P_i)^2}{X_5} \tag{6.36}$$

where

$$X_4 = \frac{X_3 c}{1 + 4k_2 - c^2} + 4k_2c\bar{X}_1 - (1 + 4k_2)(c^2 - 1)X_2$$

$$X_5 = \frac{k_1c\left[k_1(1+4k_2)(c^2-1)(1-\bar{\nu}^2)b - \bar{E}t(1+4k_2-c^2)\right]}{\bar{E}t} \tag{6.37}$$

$$X_3 = k_3 + 12k_4 + 10k_2 + 1 - \frac{2k_3 + 1}{8k_2}. \tag{6.37}$$

Using the above notation we find

$$\sigma_r = \frac{P_i}{1+4k_2-c^2}\left[\frac{b^2}{r^2} - (1+4k_2)\right] + \frac{P_i^2}{k_1}(1+4k_2)X_6 + \frac{P_i^2 X_7}{k_1}\frac{a^2}{r^2} - \frac{P_i^2}{k_1}X_1(r)$$

$$u = \frac{P_i}{k_1(1+4k_2-c^2)}\left[\frac{b^2}{r} - r\right] + \frac{P_i^2 X_6}{k_1^2} r + \frac{P_i^2 X_7}{k_1^2}\frac{a^2}{r}$$

$$+ \frac{(4k_2 - 2k_3 - 1)P_i^2}{8k_2 k_1^2(1+4k_2-c^2)}\frac{b^4}{r^3}. \tag{6.38}$$

Vertical Gravity Loading of a Thick-walled Cylinder

Consider now the deformation of a vertical case bonded thick-walled cylinder subjected to gravity loading. We shall assume that the case is sufficiently rigid so that longitudinal case motion may be neglected. The classical solution [8] of this problem yields the results that the radial displacement u is zero, hence $u^{(0)}$ is zero and $u = \delta u^{(1)}$. From this result we see that the radial motion will be an order smaller than the vertical motion, i.e., $u = \delta u^{(1)}$ compared to $w = w^{(0)} + \delta w^{(1)}$. Therefore, we need to include only the first order terms of the radial displacement. The solution of the classical problem serves as a valuable guide to the relative magnitude of various terms occurring in the second order solution. The set of governing equations, Eqs. (3.13), (3.14), (3.15), and (4.1) is:

$$E^i_j = \begin{bmatrix} \frac{du}{dr} & 0 & \frac{1}{2}\frac{dw}{dr} \\ 0 & \frac{u}{r} & 0 \\ \frac{1}{2}\frac{dw}{dr} & 0 & \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \end{bmatrix} \tag{6.39}$$

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6.40}$$

$$\frac{d\tau_{rz}}{dr} + \frac{1}{r}\tau_{rz} + \rho_o g = 0 \tag{6.41}$$

$$A = k_1(\theta_2 + k_2\theta_1^2 + k_3\theta_1\theta_2 + k_4\theta_1^3 + k_5\theta_3). \tag{6.42}$$

The deformed boundaries are located at $r = a + u|_{\text{inner radius}}$ and $r = b + u|_{\text{outer radius}}$. The displacements enter into the boundary conditions since material coordinates were selected to coincide with the cylindrical coordinates in the final state. Inasmuch as we are only retaining first order terms of u (corresponding to the classical solution) we consider the boundaries as located at r = a and r = b respectively. From Eq. (6.42) we find

$$\frac{\partial A}{\partial \theta_1} = k_1\left[2k_2\theta_1 + k_3\theta_2 + 3k_4\theta_1^{\,2}\right]$$

$$\frac{\partial A}{\partial \theta_2} = k_1\left[1 + k_3\theta_1\right]$$

$$\frac{\partial A}{\partial \theta_3} = k_1 k_5 .$$

Kinematic variables needed to complete the formulation of the solution are recorded below:

$$\theta_1 = \frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$\theta_2 = -\frac{1}{4}\left(\frac{dw}{dr}\right)^2$$

$$\theta_3 = 0$$

$$g^{11} = \left(1 + 2\,\frac{du}{dr}\right)$$

$$B^{11} = 2\left(1 + \frac{u}{r} + 2\,\frac{du}{dr}\right)$$

$$G^{11} = 1$$

$$\frac{1}{\sqrt{I_3}} = 1 - \frac{u}{r} - \frac{du}{dr} .$$

Placing the above results into Eq. (1.14) we find

$$\sigma_r = k_1 \left[2k_2 \frac{du}{dr} + (2k_2 + 1)\frac{u}{r} + (k_2 - \frac{k_3}{4}) (\frac{dw}{dr})^2 \right] \quad (6.43)$$

$$\sigma_\theta = k_1 \left[(2k_2 + 1)\frac{du}{dr} + 2k_2 \frac{u}{r} + (k_2 - \frac{k_3}{4} + \frac{1}{2} - \frac{k_5}{4}) (\frac{dw}{dr})^2 \right] \quad (6.44)$$

$$\tau_{rz} = - \frac{k_1}{2} \frac{dw}{dr}. \quad (6.45)$$

The above equations, because of their simplicity, may be integrated directly without resorting to a perturbation solution. Identical results may be obtained by perturbation. Integration of Eq. (6.41) gives

$$\tau_{rz} = - \rho_o g \frac{r}{2} + \frac{A}{r}$$

and applying the boundary condition $\tau_{rz}(a) = 0$ we find

$$\tau_{rz} = \frac{\rho_o g}{2} (\frac{a^2}{r} - r), \quad (6.46)$$

Substitution of the above expression into Eq. (6.45) and integrating we obtain

$$w = - \frac{\rho_o g}{k_1}(a^2 \ln r - \frac{r^2}{2}) + B$$

Applying the boundary condition w(b) = 0, we obtain

$$w = \frac{\rho g}{k_1} \left[a^2 \ln \frac{b}{r} - \frac{b^2 - r^2}{2} \right]. \quad (6.47)$$

Substitution of the above results into Eqs. (6.43), (6.44), and (6.40) we obtain

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = \frac{\rho_o^2 g^2}{2k_2k_1^2}\left[\left(\frac{1}{2} - \frac{k_5}{4}\right)\left(\frac{a^4}{r^3} - 2\frac{a^2}{r} + r\right) + 2\left(k_2 - \frac{k_3}{4}\right)\left(\frac{a^4}{r^3} - r\right)\right],$$

which upon integration is

$$u = Ar + B\frac{1}{r} - \frac{\rho_o^2 g^2}{2k_2k_1^2}\left\{\left(\frac{1}{2} - \frac{k_5}{4}\right)\left[\frac{a^2}{2}\frac{\ln r}{r} + \frac{a^2 r}{2}(2\ln r - 1) - \frac{r^2}{8}\right]\right.$$

$$\left. + 2\left(k_2 - \frac{k_3}{4}\right)\left(\frac{a^4}{2}\frac{\ln r}{r} + \frac{r^3}{8}\right)\right\}. \qquad (6.48)$$

Thus, Eq. (6.43) becomes

$$\sigma_r = k_1\left[(4k_2 + 1)A + \frac{B}{r^2}\right] - \frac{\rho_o^2 g^2}{k_1}\left[\left(\frac{1}{2} - \frac{k_5}{4}\right)\left(\frac{a^2}{2r^2} + 2a^2\ln r - \frac{r^2}{2}\right)\right.$$

$$\left. + 2a^2\left(k_2 - \frac{k_3}{4}\right)\right] - \frac{\rho_o^2 g^2}{2k_2k_1}\left\{\left(\frac{1}{2} - \frac{k_5}{4}\right)\left[\frac{a^2}{2}\frac{\ln r}{r^2} + \frac{a^2}{2}(2\ln r - 1)\right.\right.$$

$$\left.\left. - \frac{r^2}{8}\right] + 2\left(k_2 - \frac{k_3}{4}\right)\left(\frac{a^4}{2}\frac{\ln r}{r^2} + \frac{r^2}{8}\right)\right\}, \qquad (6.49)$$

The constants A and B may be evaluated from appropriate boundary conditions, which, for a rigid case are

$$\sigma_r(a) = 0$$

$$u(b) = 0 \qquad (6.50)$$

7. Incompressible Response.

No real material is truly incompressible. Therefore, whenever we represent a material as incompressible, we are only approximating its true behavior. It is very desirable to utilize such an approximation, when justified, because of the resulting substantial reduction in algebraic difficulties. The assumption of incompressibility is very often made in finite elasticity as most of the materials capable of finite elastic deformations are very nearly incompressible. The question of whether the actual stress and strain state is well approximated by the incompressible state (for a given nearly incompressible material) will depend upon the type of boundary conditions (as shall be illustrated later). We must avoid those problems that approach the physically contradictory problem of enforcing displacement boundary conditions which prescribes a net change in volume of an incompressible material. For this type of problem a singularity arises and thus for problems of this type the solution will be very sensitive to the actual amount of compressibility present. A further consideration of this problem will be presented in Section 8, including some numerical indications of the range of validity of such a theory.

Uniaxial Test

Equations (2.12), (2.14), and (4.4) become, upon specialization for uniaxial stress field in an incompressible material,

$$\sigma_{11} = H + k_1\left[2e_2 + (k_5 - 3)e_2^2 + 2e_1e_2 + 2c_3Te_2\right] \qquad (7.1)$$

$$\sigma_{22} = 0 = H + k_1\left[e_2 + e_1 + \frac{1}{2}e_2^2 - \frac{1}{2}e_1^2 + (k_5 - 1)e_1e_2 + c_8Te_2 + c_8Te_1\right] \qquad (7.2)$$

and from Eq. (4.9)

$$\beta_1 T + \beta_2^* T^2 = e_1 + 2e_2 + \frac{1}{2} e_1^2 + 3e_2^2 + 4e_1 e_2 . \qquad (7.3)$$

Uniform Temperature Drop of a Thick-walled Cylinder

Noting Eq. (4.5) we may write Eq. (2.13) as

$$\tau^{ij} = HG^{ij} + \frac{1}{1 + \beta_1 T + \beta_2 T^2} \frac{\partial \bar{A}}{\partial E_{ij}} .$$

In a manner similar to the derivation given in Section 4, we find

$$\sigma_r = H + k_1 \left[\frac{u}{r} + \frac{1}{2}\left(\frac{u}{r}\right)^2 + 2 \frac{u}{r}\frac{du}{dr} + (c_3 - \beta_1) T \frac{u}{r} \right]$$

$$(7.4)$$

$$\sigma_r - \sigma_\theta = k_1 \left[\frac{u}{r} - \frac{du}{dr} + \frac{1}{2}\left(\frac{u}{r}\right)^2 - \frac{1}{2}\left(\frac{du}{dr}\right) + (c_3 - \beta_1) T \left(\frac{u}{r} - \frac{du}{dr}\right) \right].$$

The incompressibility condition as given by Eq. (4.9) becomes

$$0 = \frac{du}{dr} + \frac{u}{r} - \beta_1 T + \frac{1}{2}\left(\frac{u}{r}\right)^2 + 2 \frac{u}{r}\frac{du}{dr} + \frac{1}{2}\left(\frac{du}{dr}\right)^2 - \beta_2^* T^2 . \qquad (7.5)$$

Applying the perturbation scheme, as outlined in Section 5 we find the following two systems of equations (where $H = k_1 \delta (h^{(0)} + \delta h^{(1)} + \ldots)$):

First system

$$\frac{d^2T^{*(0)}}{d\rho^2} + \frac{1}{\rho}\frac{dT^{*(0)}}{d\rho} = 0$$

$$\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} = 0 \qquad (7.6)$$

$$s_\rho^{(0)} = h^{(0)} + \frac{v^{(0)}}{\rho}$$

$$s_\rho^{(0)} - s_\theta^{(0)} = \frac{v^{(0)}}{\rho} - \frac{dv^{(0)}}{d\rho}$$

$$\frac{dv^{(0)}}{d\rho} + \frac{v^{(0)}}{\rho} - \beta_1 T^{*(0)} = 0$$

Second system

$$\frac{d^2T^{*(1)}}{d\rho^2} + \frac{1}{\rho}\frac{dT^{*(1)}}{d\rho} = -\left(\frac{v^{(0)}}{\rho} + \frac{dv^{(0)}}{d\rho}\right)\frac{d^2T^{*(0)}}{d\rho^2} + \left(\rho\frac{d^2v^{(0)}}{d\rho^2}\right.$$

$$\left. - 2\frac{dv^{(0)}}{d\rho}\right)\frac{1}{\rho}\frac{dT^{*(0)}}{d\rho}$$

$$\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} = \frac{1}{\rho}\left(\frac{v^{(0)}}{\rho} - \frac{dv^{(0)}}{d\rho}\right)^2$$

$$s_\rho^{(1)} = h^{(1)} + \frac{v^{(1)}}{\rho} + \frac{1}{2}\left(\frac{v^{(0)}}{\rho}\right)^2 + 2\frac{v^{(0)}}{\rho}\frac{dv^{(0)}}{d\rho} + (c_3 - \beta_1)T^{*(0)}\frac{v^{(0)}}{\rho}$$

(7.7)

$$s_\rho^{(1)} - s_\theta^{(1)} = \frac{v^{(1)}}{\rho} - \frac{dv^{(1)}}{d\rho} + \frac{1}{2}\left(\frac{v^{(0)}}{\rho}\right)^2 - \frac{1}{2}\left(\frac{dv^{(0)}}{d\rho}\right)^2 + (c_3 - \beta_1)T^{*(0)}\left(\frac{v^{(0)}}{\rho} - \frac{dv^{(0)}}{d\rho}\right)$$

$$\frac{dv^{(1)}}{d\rho} + \frac{v^{(1)}}{\rho} - \beta_1 T^{*(1)} = -\frac{1}{2}\left(\frac{v^{(0)}}{\rho}\right)^2 - 2\,\frac{v^{(0)}}{\rho}\,\frac{dv^{(0)}}{d\rho} - \frac{1}{2}\left(\frac{dv^{(0)}}{d\rho}\right)^2 + \beta_2^*(T^{*(0)})^2$$

The solution of the first system of equations, recalling that we are considering a uniform temperature drop, is given by

$$v^{(0)} = A^{(0)}\,\frac{1}{\rho} + \frac{\beta_1}{2}\,T^*\rho$$

$$s_\rho^{(0)} = B^{(0)} + \frac{A^{(0)}}{\rho^2} + \frac{\beta_1 T^*}{2} \tag{7.8}$$

$$s_\rho^{(0)} - s_\theta^{(0)} = 2\,\frac{A^{(0)}}{\rho^2}$$

$$h^{(0)} = B^{(0)}$$

The boundary conditions for zero pressure at the inner radius and a rigid case at the outer radius are

at

$$\rho = 1 \quad s_\rho^{(0)} = 0 = s_\rho^{(1)} \tag{7.9}$$

and at

$$\rho = c \quad v^{(0)} = 0 = v^{(1)}$$

Comparing the above results with Eqs. (6.18) and (6.19) we see the substantial reduction in algebra effected by the assumption of incompressibility. The validity of this assumption will be discussed in Section 8.

Pressurization of a Thick-walled Cylinder

From Eq. (7.8) we see that the solution to the first system is

$$v^{(0)} = \frac{A^{(0)}}{\rho}$$

$$s_\rho^{(0)} = B^{(0)} + \frac{A^{(0)}}{\rho^2} \tag{7.13}$$

$$h^{(0)} = B^{(0)}.$$

Noting these results the solution of Eq. (7.7) is

$$v^{(1)} = \frac{A^{(1)}}{\rho} - \frac{(A^{(0)})^2}{2\rho^3}$$

$$(7.14)$$

$$s_\rho^{(1)} = B^{(1)} + \frac{A^{(1)}}{\rho^2} - \frac{3}{2}\,\frac{(A^{(0)})^2}{\rho^4}.$$

Considering the internally pressurized unbonded cylinder we have for boundary conditions

at

$$\rho = 1 \quad s_\rho^{(0)} = -\frac{P_i}{k_1 \delta} = -\overline{P}$$

$$s_\rho^{(1)} = 0 \tag{7.15}$$

and at $\quad \rho = c \qquad s_\rho^{(0)} = s_\rho^{(1)} = 0$

The above boundary conditions yield

$$A^{(0)} = -\frac{c^2\bar{P}}{c^2 - 1}$$

$$B^{(0)} = \frac{\bar{P}}{c^2 - 1}$$

$$B^{(1)} = -\frac{3c^2\bar{P}^2}{2(c^2 - 1)^2}$$

$$A^{(1)} = \frac{3c^2(c^2 + 1)\bar{P}^2}{2(c^2 - 1)^2}.$$

Thus our final results are

$$u = -\frac{P}{k_1(c^2 - 1)}\,\frac{b^2}{r} + \frac{b^2P^2}{k_1^{\,2}\,2(c^2 - 1)^2 r}\left[3(c^2 + 1) - \frac{b^2}{r^2}\right]$$

$$\tag{7.16}$$

$$\sigma_r = \frac{P}{c^2 - 1}\left[1 - \frac{b^2}{r^2}\right] - \frac{3c^2P^2}{2k_1(c^2 - 1)^2}\left[1 - \frac{b^2 + a^2}{r^2} + \frac{b^2a^2}{r^4}\right].$$

Once again comparing the above solution with Eq. (6.29) we see the substantial reduction of labor made possible by the incompressibility assumption.

8. Near-Incompressible Response.

The following considerations were motivated by two characteristics of the incompressible assumption, as indicated in the introductory remarks to the previous section: simplification in the mathematical description of the problem and introduction of large errors for certain types of boundary conditions. We hope to achieve two goals in the following investigation: to be able to extend this simplification of description to a larger class of problems (i.e., to those materials which exhibit compressibility) by introducing a corrective term that will account for the actual compressibility and to obtain a numerical indication of the error introduced in a given problem by the incompressibility assumption. We shall first obtain a series representation (by means of perturbation) of the classical field equations, where the first approximation is the set of classical incompressible field equations and subsequent terms account for the actual compressibility of the material. Subsequently we shall solve a uniform temperature drop problem, first with boundary conditions that exemplify the merits of the solution method, and second for boundary conditions that will cause the solution method to break down for certain values of ν and c (as the actual state will not be near the incompressible state). Lastly we shall derive the governing equations for a near-incompressible second order elastic theory.

Writing the classical stress (τ^i_j) and strain (ϵ^i_j) tensors in terms of their deviatoric ($\bar{s}^i_j$ and e^i_j) and isotropic(Θ and θ) components one obtains

$$\tau^i_j = \overline{s}^i_j + \frac{\Theta}{3}\,\delta^i_j \tag{8.1}$$

$$\epsilon^i_j = e^i_j + \frac{\theta}{3}\,\delta^i_j$$

where

$$\Theta = \tau^i_i$$

$$\theta = \epsilon^i_i. \tag{8.2}$$

The classical constitutive equations may be written as

$$\overline{s}^i_j = 2\mu e^i_j \tag{8.3}$$

$$\Theta = 3B(\theta - 3\alpha T_o T).$$

We shall now define a parameter ϵ as follows

$$\epsilon = \frac{\overline{B}}{B}. \tag{8.4}$$

Where $\overline{B}$ is of the order of magnitude of μ; thus ϵ is small since $B >> \mu$ for a nearly incompressible material. For classical elasticity it is most convenient to let $\overline{B} = \mu$ then $\epsilon = \frac{\mu}{B} = \frac{3(1-2\nu)}{2(1+\nu)}$.

Thus we may write

$$\Theta = \frac{3\overline{B}}{\epsilon}(\theta - 3\alpha T_o T) \tag{8.5}$$

Combining Eqs. (8.4), (8.1), and (8.5) we obtain

$$\epsilon^i_j = \frac{1}{2\mu}(\tau^i_j - \frac{\Theta}{3}\,\delta^i_j) + \frac{\epsilon}{9\overline{B}}\,\Theta\delta^i_j + \alpha T_o T\delta^i_j. \tag{8.6}$$

Solving for τ^i_j we find

$$\tau^i_j = 2\mu\epsilon^i_j + \frac{\Theta}{3}\delta^i_j - \epsilon\frac{2\mu}{9\overline{B}}\Theta\delta^i_j - 2\mu\alpha T_o T\delta^i_j \tag{8.7}$$

and inverting Eq. (8.5) we obtain

$$\theta = \epsilon\frac{\Theta}{3\overline{B}} + 3\alpha T_o T. \tag{8.8}$$

We have now established the condition for a perturbation solution for a class of problems determined by the size of ν and the type of boundary conditions. Expanding our dependent variables in series in ϵ, substituting into the above equations, and equating coefficients, we obtain the following systems of constitutive equations

$$\tau^{i(0)}_j = 2\mu\epsilon^{i(0)}_j + \frac{\Theta^{(0)}}{3}\delta^i_j - 2\mu\alpha T_o T^{(0)}\delta^i_j \tag{8.9}$$

$$\theta^{(0)} = 3\alpha T_o T^{(0)}$$

and for $n \geq 1$

$$\tau^{i(n)}_j = 2\mu\epsilon^{i(n)}_j + \frac{\Theta^{(n)}}{3}\delta^i_j - 2\mu\alpha T_o T^{(n)}\delta^i_j - \frac{2\mu}{9\overline{B}}\Theta^{(n-1)}\delta^i_j$$

$$\theta^{(n)} = 3\alpha T_o T^{(n)} + \frac{\Theta^{(n-1)}}{3\overline{B}}. \tag{8.10}$$

The equilibrium equations may likewise be written in series form, i.e.,

$$\tau^{i}_{j}\Big|^{(0)}_{i} + \rho F^{(0)}_{j} = 0 \tag{8.11}$$

$$\tau^{i}_{j}\Big|^{(n)}_{i} + \rho F^{(n)}_{j} = 0 \qquad n \geq 1$$

Substitution of the constitutive equations (8.9) and (8.10) into the above equations, we obtain the following displacement equations of equilibrium

$$\nabla^2 u^{(0)}_{j} + \frac{1}{\mu}\left(\frac{1}{3}\Theta\Big|^{(0)}_{j} + \rho F^{(0)}_{j}\right) + \alpha T_o T\Big|^{(0)}_{j} = 0 \tag{8.12}$$

where $$u^{i}\Big|^{(0)}_{i} = 3\alpha T_o T^{(0)}$$

and for

$$n \geq 1 \quad \nabla^2 u^{(n)}_{j} + \frac{1}{\mu}\left(\rho F^{(n)}_{j} + \frac{1}{3}\,\Theta\Big|^{(n)}_{j} + \frac{\mu}{9\overline{B}}\,\Theta\Big|^{(n-1)}_{j}\right) + \alpha T_o T\Big|^{(n)}_{j} = 0 \tag{8.13}$$

where $$u^{i}\Big|^{(n)}_{i} = \frac{\Theta^{(n-1)}}{3\overline{B}} + 3\alpha T_o T^{(n)}.$$

The solution of each system of the above equations is equivalent to the solution of a nonhomogeneous incompressible problem.

We shall now consider the uniform temperature drop of a thick-walled cylinder in light of the above equations introducing a dimensionless temperature $T^{(0)} = \frac{\Delta T}{T_o}$. Solving the first system of equations we obtain (where $\overline{B} = \mu$)

$$u^{(0)} = \frac{A^{(0)}}{r} + \frac{3\alpha}{2} r\Delta T$$

$$\sigma_r^{(0)} = -2\mu \frac{A^{(0)}}{r^2} + \frac{C^{(0)}}{3} + \mu\alpha\Delta T \qquad (8.14)$$

$$\Theta^{(0)} = C^{(0)}.$$

Noting the above results, the solution to the second system of equations becomes

$$u^{(1)} = \frac{A^{(1)}}{r} + \frac{C^{(0)}}{6\mu} r$$

$$(8.15)$$

$$\sigma_r^{(1)} = - 2\mu \frac{A^{(1)}}{r^2} + \frac{C^{(1)}}{3} + \frac{C^{(0)}}{9}.$$

Let us first consider the specific problem where we have no external case; thus, the boundary conditions become (note that no restrictions are placed on volume change)

at $r = a$ $\sigma_r^{(0)} = \sigma_r^{(1)} = 0$

and at $r = b$ $\sigma_r^{(0)} = \sigma_r^{(1)} = 0$

Using the above boundary conditions we find

$$u^{(0)} = \frac{3\alpha\Delta T}{2} r$$

$$u^{(1)} = -\frac{\alpha\Delta T r}{2} = (-\frac{1}{3}) u^{(0)} \tag{8.16}$$

$$\sigma_r^{(0)} = \sigma_r^{(1)} = 0.$$

Noting Eq. (8.16) and the general equations (8.13) for the n^{th} term we see that $u^{(n)} = -\frac{1}{3} u^{(n-1)}$, and noting that $\epsilon = \frac{\mu}{B}$ we find that

$$u = \frac{3\alpha\Delta T}{2} r \left[1 - \frac{\mu}{3B} + (\frac{\mu}{3B})^2 - \ldots\right]$$
$$\sigma_r = 0. \tag{8.17}$$

For illustration let us consider the exact solution, which may be written as

$$u = \frac{3\alpha\Delta T}{2} r (\frac{1}{1 + \frac{\mu}{3B}}) \tag{8.18}$$

or for small $\frac{\mu}{3B}$

$$u = \frac{3\alpha\Delta T}{2} r \left[1 - \frac{\mu}{3B} + (\frac{\mu}{3B})^2 \ldots\right] \tag{8.19}$$

Let us now consider the range of validity of the above approximation. To use only the first term (i.e., assume incompressibility) let us make $\frac{\mu}{3B} \leq 0.05$(which is equivalent to $\nu \geq 0.428$). To use only the first two terms let us make $(\frac{\mu}{3B})^2 \leq 0.05$ which is equivalent to $\nu \geq 0.226$). Thus with only one corrective term to the incompressible solution we may consider very compressible materials. We shall now consider the specific problem where the thick-walled cylinder is bonded to a rigid case. Thus the boundary conditions are

$$\begin{aligned} &\text{at} \quad r = a \quad \sigma_r^{(0)} = \sigma_r^{(1)} = 0 \\ &\text{and at} \quad r = b \quad u^{(0)} = u^{(1)} = 0 \end{aligned} \tag{8.20}$$

Note that we now have a boundary condition $u(b) = 0$ that tends to specify a volume change that is physically impossible for as $a \to 0$ the condition that $u(b) = 0$ means that the volume must remain a constant whereas a change in temperature demands a volume change, thus we shall find the solution to be very sensitive to the actual amount of compressibility as $a \to 0$. Noting the above boundary condition we find that

$$\begin{aligned} u^{(0)} &= \frac{3\alpha\Delta T}{2}(r - \frac{b^2}{r}) \\ \sigma_r^{(0)} &= 3\mu\alpha\Delta T(\frac{b^2}{r^2} - c^2) \\ u^{(1)} &= \frac{\alpha\Delta T(1 + 3c^2)}{2}(\frac{b^2}{r} - r) \\ \sigma_r^{(1)} &= \alpha\mu\Delta T(1 + 3c^2)(c^2 - \frac{b^2}{r^2}), \end{aligned} \tag{8.21}$$

As above we find that

$$u = \frac{3\alpha\Delta T}{2}\left(r - \frac{b^2}{r}\right)\left\{1 - \frac{\mu}{B}\left(\frac{1}{3} + c^2\right) + \left[\frac{\mu}{B}\left(\frac{1}{3} + c^2\right)\right]^2 + \cdots\right\}$$

$$\sigma_r = 3\alpha\mu\Delta T\left(\frac{b^2}{r^2} - c^2\right)\left\{1 - \frac{\mu}{B}\left(\frac{1}{3} + c^2\right) + \cdots\right\}. \tag{8.22}$$

For $\left|\frac{\mu}{B}\left(\frac{1}{3} + c^2\right)\right| < 1$ the above series may be summed to yield

$$u = \frac{3\alpha\Delta T\left(r - \frac{b^2}{r}\right)}{2\left[1 + \frac{\mu}{B}\left(\frac{1}{3} + c^2\right)\right]}$$

$$\sigma_r = \frac{3\alpha\mu\Delta T\left(\frac{b^2}{r^2} - c^2\right)}{1 + \frac{\mu}{B}\left(\frac{1}{3} + c^2\right)}. \tag{8.23}$$

For $\left|\frac{\mu}{B}\left(\frac{1}{3} + c^2\right)\right| \geq 1$ we see that the above series does not converge. Letting ν_c be the critical value of ν that determines the boundary of the region of convergence, we find that for

$c \doteq 2 \qquad \nu_c = .393$

$c = 10 \qquad \nu_c = .495$

We shall now consider the range of validity of the above approximation, Eq. (8.21). To use only the first term (i.e., assume incompressibility) let us make

$\frac{\mu}{B}\left(\frac{1}{3} + c^2\right) \leq .05$ then $c = 2 \qquad \nu \geq .494$

$c = 10 \qquad \nu \geq .49975$

To use only the first two terms let us make

$$\left[\frac{\mu}{B}\left(\frac{1}{3} + c^2\right)\right]^2 \leq .05 \quad \text{then} \quad c = 2 \quad \nu \geq .4746$$

$$c = 10 \quad \nu \geq .4989$$

It is to be noted that, for certain problems, although the series may not converge for $\nu \leq \nu_c$ it is possible that if the series may be summed for $\nu > \nu_c$ the resulting expression may be valid for $\nu \leq \nu_c$ (as was the case above), this of course would need to be investigated for each individual problem.

Let us now establish a near-incompressible second order elastic theory. We shall base it upon the assumption that the compressibility effects are of the same order of magnitude as the second order non-linear effects. For illustrative purposes we shall consider the governing equations for the pressurization of a thick-walled cylinder.

Proceeding as before we can separate the problem into two systems, i.e.,

$$s_\rho = s_\rho^{(0)} + \delta s_\rho^{(1)}. \tag{8.24}$$

We shall now express the governing equations for $s_\rho^{(0)}$ (the classical equations) as was done for near-incompressible classical elasticity, see Eqs. (8.9) and (8.10).
Thus,

$$s_\rho^{(0)} = s_\rho^{(0)(0)} + \epsilon s_\rho^{(0)(1)} \tag{8.25}$$

where

$$\epsilon = \frac{\bar{B}}{B}.$$

Let us now choose $\overline{B}$ such that $\epsilon = \delta$, whence

$$s_\rho^{(0)} = s_\rho^{(0)(0)} + \delta s_\rho^{(0)(1)} \tag{8.26}$$

Noting these expressions for the first system variables and referring to Eq. (5.9) we may write the equation for the second system as

$$\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} = \frac{1}{\rho}\left(\frac{v^{(0)(0)}}{\rho} - \frac{dv^{(0)(0)}}{d\rho}\right)^2 \quad \text{etc.} \tag{8.27}$$

Now let us write $s_\rho^{(1)} = s_\rho^{(1)(0)} + \delta s_\rho^{(1)(1)}$ where $s_\rho^{(1)(0)}$ is the effect if the material were incompressible and $s_\rho^{(1)(1)}$ is the compressible effect. Thus,

$$s_\rho = s_\rho^{(0)(0)} + \delta s_\rho^{(1)(0)} + \delta^2 s_\rho^{(1)(1)} \tag{8.28}$$

For a second order theory we may neglect $s_\rho^{(1)(1)}$. To obtain the governing equations for $s_\rho^{(1)(0)}$ we use the equations derived in incompressible finite elasticity except that in the constitutive equations I_3 is no longer equal to 1. Hence,

$$s_\rho^{(1)(0)} = h^{(1)(0)} + \frac{v^{(1)(0)}}{\rho} - \frac{1}{2}\left(\frac{v^{(0)(0)}}{\rho}\right)^2 + \frac{dv^{(0)(0)}}{d\rho}\,\frac{v^{(0)(0)}}{\rho}$$

$$s_\rho^{(1)(0)} - s_\theta^{(1)(0)} = \frac{v^{(1)(0)}}{\rho}\,\frac{dv^{(1)(0)}}{d\rho} - \frac{1}{2}\left(\frac{v^{(0)(0)}}{\rho}\right)^2 + \frac{1}{2}\left(\frac{dv^{(0)(0)}}{d\rho}\right)^2 \tag{8.29}$$

$$\frac{dv^{(1)(0)}}{d\rho} + \frac{v^{(1)(0)}}{\rho} = -\frac{1}{2}\left(\frac{dv^{(0)(0)}}{d\rho}\right)^2 - \frac{1}{2}\left(\frac{v^{(0)(0)}}{\rho}\right)^2 - 2\,\frac{dv^{(0)(0)}}{d\rho}\,\frac{v^{(0)(0)}}{\rho}$$

$$\frac{ds_\rho^{(1)(0)}}{d\rho} + \frac{s_\rho^{(1)(0)} - s_\theta^{(1)(0)}}{\rho} = \frac{1}{\rho}\left(\frac{v^{(0)(0)}}{\rho} - \frac{dv^{(0)(0)}}{d\rho}\right)^2. \tag{8.29}$$

The system of equations governing $s_\rho^{(0)(1)}$ is obtained from the second term equations for near-incompressible classical elasticity, Eq. (8.10),

$$s_\rho^{(0)(1)} = -\frac{dv^{(0)(1)}}{d\rho} + \frac{\Theta^{(0)(1)}}{3} + \frac{k_1}{9\overline{B}}\,\Theta^{(0)(0)}$$

$$s_\rho^{(0)(1)} - s_\theta^{(0)(1)} = \frac{v^{(0)(1)}}{\rho} - \frac{dv^{(0)(1)}}{d\rho}$$

(8.30)

$$\frac{v^{(0)(1)}}{\rho} + \frac{dv^{(0)(1)}}{d\rho} = \frac{k_1}{3\overline{B}}\,\Theta^{(0)(0)}$$

$$\frac{ds_\rho^{(0)(1)}}{d\rho} + \frac{s_\rho^{(0)(1)} - s_\theta^{(0)(1)}}{\rho} = 0.$$

As both $s_\rho^{(1)(0)}$ and $s_\rho^{(0)(1)}$ are second order terms, we shall combine them, letting

$$\overline{s}_\rho^{(1)} = s_\rho^{(1)(0)} + s_\rho^{(0)(1)}. \tag{8.31}$$

Adding the system of equations (8.29) and (8.30) and rearranging terms slightly we obtain the following governing system of equations for the second order solution:

(Let $\frac{\Theta^{(0)(0)}}{3} = \bar{h}^{(0)}$)

$$\bar{s}_{\rho}^{(1)} = -\frac{d\bar{v}^{(1)}}{d\rho} + \bar{h}^{(1)} + \frac{1}{2}\left(\frac{\bar{v}^{(0)}}{\rho}\right)^{2} + \frac{k_1}{3\bar{B}}\bar{h}^{(0)}$$

$$\bar{s}_{\rho}^{(1)} - \bar{s}_{\theta}^{(1)} = \frac{\bar{v}^{(1)}}{\rho} - \frac{d\bar{v}^{(1)}}{d\rho} \tag{8.32}$$

$$\frac{d\bar{v}^{(1)}}{d\rho} + \frac{\bar{v}^{(1)}}{\rho} = \left(\frac{\bar{v}^{(0)}}{\rho}\right)^{2} + \frac{k_1}{\bar{B}} h^{(0)}$$

$$\frac{d\bar{s}_{\rho}^{(1)}}{d\rho} + \frac{\bar{s}_{\rho}^{(1)} - \bar{s}_{\theta}^{(1)}}{\rho} = \frac{4}{\rho}\left(\frac{\bar{v}^{(0)}}{\rho}\right)^{2}$$

where $s_{\rho}^{(0)(0)} = \bar{s}_{\rho}^{(0)}$. The governing equations for the first term are identical to those of incompressible elasticity and are given by Eq. (7.6).

Uniaxial Test

Because of the dependence of the near-incompressible theory, as developed above, upon the notion of perturbation it will be expedient to consider the uniaxial test in the same manner. The uniaxial test for a compressible or incompressible solid may be similarly treated.

Let
$$e_1 = e_1^{(0)} + e_1^{(1)} \tag{8.33}$$

where $e_1^{(1)}$ is the deviation from classical incompressibility, etc.

$$\sigma_1 = (\sigma_1^{(0)} + \sigma_1^{(1)} \tag{8.33}$$

Using Eqs. (8.9), (8.10), (7.1) and (7.2) we proceed as above and obtain

$$\sigma_1^{(0)} = -\frac{3k_1}{2} e_1^{(0)}$$

$$\sigma_1^{(1)} = k_1 \left[e_2^{(1)} - e_1^{(1)} + \frac{3}{4}(k_2 - \frac{5}{2})(e_1^{(0)})^2 \right]$$

$$0 = e_1^{(1)} + 2e_2^{(1)} - \frac{k_1}{B} e_2^{(0)} - \frac{3}{4}(e_1^{(0)})^2. \tag{8.34}$$

Pressure Loading of a Thick-walled Cylinder

The solution of the system of Eqs. (7.6) and (8.32) with the boundary condition $\sigma_r(a) = -P$, $\sigma_r(b) = 0$ yields

$$u = -\frac{P}{k_1(c^2-1)}\frac{b^2}{r} + \frac{1}{B}\frac{P}{2(c^2-1)}r + \frac{b^2P^2}{2k_1^{\,2}(c^2-1)r}\left[3(c^2+1) - \frac{b^2}{r^2}\right]$$

$$\sigma_r = \frac{P}{c^2-1}\left[1 - \frac{b^2}{r^2}\right] - \frac{3c^2P^2}{2k_1(c^2-1)^2}\left[1 - \frac{b^2+a^2}{r^2} + \frac{b^2a^2}{r^4}\right] \tag{8.35}$$

It will be noted that the above solution retains the algebraic simplicity of the incompressible solution, Eq. (7.16). The second term in the expression for u is the corrective term for compressibility.

9. An Approximate Solution Scheme.

As pointed out previously in the Introduction, the perturbation solution method depends upon the existence of the classical solution, not upon the knowledge of this solution. Thus the two sets of equations developed by the perturbation method govern the problem whether or not we are able to solve the classical problem. We might use some approximate solution scheme to solve the two systems of equations but practically this is rather poor, because the nonhomogeneous parts of the second system depend upon the solution of the first system; thus any errors in the solution of the first system of equations tend to be magnified. In order to avoid this magnification of error it is necessary to slightly modify the system of equations before we apply an approximate solution scheme to each.

The modified system of equations that we shall develop will not only allow us to solve approximately each system of equations without reflecting the errors of the solution to the first system of equations into the second system of equations but will actually contain in the second solution a correction to improve the approximate solution of the first system.

We may use any approximate solution method applicable in classical elasticity to solve each of the systems of equations, e.g., minimum potential energy, collocation, numerical schemes, etc., because we may view each system merely as a mathematical problem identical to some classical elastic problem. It is apparent that we may use any approximate method that is valid for classical elasticity.

We shall only consider the form of the equilibrium equation for a thick-walled cylinder and the form of the σ_r boundary condition. We will not derive any of the other thick-walled cylinder equations, as this one example will serve to illustrate the method. Although at this point we will solve no examples, we will point to two previously obtained solutions which may be viewed in this light.

Let us now consider the derivation of the equilibrium equation for a thick-walled cylinder. From Eq. (5.6)

$$\frac{ds_\rho}{d\rho} + \frac{s_\rho - s_\theta}{\rho} = -\delta\left[\frac{v}{\rho}\frac{ds_\rho}{d\rho} + \frac{dv}{d\rho}\frac{s_\rho - s_\theta}{\rho}\right] \tag{9.1}$$

Referring to Eq. (5.7)

$$\begin{aligned} s_\rho &= s_\rho^{(0)} + \delta s_\rho^{(1)} \\ s_\theta &= s_\theta^{(0)} + \delta s_\theta^{(1)} \\ v &= v^{(0)} + \delta v^{(1)} \end{aligned} \tag{9.2}$$

and substituting the above expressions into Eq. (9.1) we obtain

$$\left[\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho}\right] + \delta\left[\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} + \frac{v^{(0)}}{\rho}\frac{ds_\rho^{(0)}}{d\rho}\right.$$

$$\left. + \frac{dv^{(0)}}{d\rho}\frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho}\right] + \ldots = 0 \tag{9.3}$$

As before the first system equilibrium equation is

$$\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} = 0 \tag{9.4}$$

If, for example, we only approximately solve the problem, we might, instead of satisfying the above equation, have

$$\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} = \overline{g}(\rho) \tag{9.5}$$

Assuming that the error function $\overline{g}(\rho)$ is small (compared to $s_\rho^{(0)}$ and $s_\theta^{(0)}$) we may write $\overline{g}(\rho) = \epsilon g(\rho)$ where $g(\rho)$ is of the order of magnitude of $s_\rho^{(0)}$ and $s_\theta^{(0)}$ and $\epsilon \ll 1$. If ϵ is of the order of magnitude of δ we shall take $\epsilon = \delta$, thus $\overline{g}(\rho) = \delta g(\rho)$ (note if $\epsilon = 0(\delta^2)$ then, as we shall see, the second system of equations remains unchanged) or

$$\frac{ds_\rho^{(0)}}{d\rho} + \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} = \delta g(\rho) \tag{9.6}$$

Noting the above results Eq. (9.3) becomes

$$\delta g(\rho) + \delta \left[\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} + \frac{v^{(0)}}{\rho} \frac{ds_\rho^{(0)}}{d\rho} + \frac{dv^{(0)}}{d\rho} \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} \right]$$

$$+ \delta^2 \left[. \; . \; . \right] + \; . \; . \; . \; . = 0$$

The second system equilibrium equation is now

$$\frac{ds_\rho^{(1)}}{d\rho} + \frac{s_\rho^{(1)} - s_\theta^{(1)}}{\rho} = - \left[\frac{v^{(0)}}{\rho} \frac{ds_\rho^{(0)}}{d\rho} + \frac{dv^{(0)}}{d\rho} \frac{s_\rho^{(0)} - s_\theta^{(0)}}{\rho} + g(\rho) \right] \tag{9.7}$$

A similar consideration may be applied to the remaining field equations. An example of such a solution (i.e., when the field equations are only approximately solved) is the finite near-incompressible theory of the previous section, for there we approximately solve the first system by an incompressibility approximation, introduce an error function into the second approximation by means of the compressibility terms and lastly approximately solve the second system.

Similarly we might have a boundary condition of the form

$$s_\rho \big|_{\rho=c} = \overline{P}$$

from Eq. (9.2)

$$s_\rho^{(0)} \big|_{\rho=c} + \delta s_\rho^{(1)} \big|_{p=c} + \; . \; . \; . \; . = \overline{P} \tag{9.8}$$

The boundary condition of the first system of equations is

$$s_\rho^{(0)} \big|_{\rho=c} = \overline{P} \tag{9.9}$$

Let us assume that we only approximately satisfy this boundary condition, i.e., we set

$$s_\rho^{(0)}\big|_{\rho=c} = \bar{\bar{P}} \qquad \text{where} \qquad \bar{P} - \bar{\bar{P}} = \delta P \tag{9.10}$$

and $\bar{\bar{P}}$ is of the order of magnitude of $s_\rho^{(0)}$. Thus Eq. (9.8) becomes

$$\delta s_\rho^{(1)}\big|_{p=c} + \ldots = \bar{P} - \bar{\bar{P}} = \delta P$$

and the second boundary condition becomes

$$s_\rho^{(1)}\big|_{p=c} = P \tag{9.11}$$

Equation (6.38) is a solution of this type, i.e., we solved the first system using the approximation that the case was rigid, then we corrected the error in the second approximation.

10. Numerical Examples.

As an example of the numerical results that may be expected when one evaluates the analytical solutions presented in the previous sections we shall consider the pressurization problem for a thick-walled cylinder bonded to an elastic case (see Section 6 and Fig. 1). The resulting solution merely involves algebraic operations, thus the results may be obtained extremely rapidly by means of an electronic computer.*

* Sincere appreciation is expressed to R.E. Nickell who programmed the solution.

The nonlinearities in a finite elastic problem are due to two causes; (1) nonlinear geometric effects and (2) nonlinear constitutive equations. The nonlinear effects introduced by the constitutive equations depend both upon the magnitude of the strains and also upon the relative magnitudes of the elastic constants that appear in the constitutive equation, whereas the nonlinear effects introduced by geometry depend only upon the magnitude of the strains. Thus one must determine the range of validity of a given approximation for each particular material used. For our material, as we shall see later, the magnitude of the second order constants (k_1k_3, k_1k_4 and k_1k_5) is substantially larger than the magnitude of the first order constants (k_1 and k_1k_2); thus we would expect the range of validity of the first order solution to be rather restricted. The proper way to judge the range of validity for a given approximation is to inspect the size of the subsequent term in the series, thus as we have obtained the second order solution we are able to investigate the range of validity of the first order solution. Likewise to be able to consider the accuracy of our second order solution we would need to inspect the third order solution; as this has not been done it may be possible that some of the results presented herein for large pressures may fall beyond the domain of the second order solution.

In deriving the solution in Section 6 the following assumptions were made in order to consider the case as rigid in the first approximation. (If these assumptions were not true we could not set $v^{(0)}(c) = 0$); (1) the case is very rigid in comparison to the thick-walled cylinder and (2) the thick-walled cylinder is relatively compressible. In order to satisfy the above assumptions we have restricted our calculations to large $\overline{E}$, small $\frac{b}{t}$, small $-k_1$, and small $-k_2$ (k_2 is a measure of the compressibility, see Eq. (4.2)).

As the finite elastic response of propellants has not been suitably characterized it was necessary for us to obtain the elastic constants for a typical propellant by fitting the second order equations governing a uniaxial test from Section 6 to an experimental curve. The resulting elastic properties should be viewed as <u>very tentative</u> as the method of curve fitting was not entirely satisfactory and as there were some questions as to the accuracy of the experimental data. Depending upon how we choose to fit the uniaxial expressions of Section 6 to the experimental data we could obtain a substantial range in values of the elastic constants, these ranges are indicated in Figures 7, 10, 11 and 14 by the vertical dashed lines. The seemingly best fit yielded the following values

$$k_1 = -558 \text{ psi}$$

$$k_2 = -8.01$$

$$k_3 = -25.8$$

$$k_4 = 72.8$$

$$k_5 = 4.11$$

The pressurization solution as presented in Section 6 depends upon the following parameters $\overline{E}$, $\overline{\nu}$, $\frac{b}{t}$, c, k_1, k_2, k_3 and k_4. We selected the following values for our "standard" solution.

$$\overline{E} = 30 \times 10^6 \text{ psi}$$

$$\overline{\nu} = 0.3$$

$$\frac{b}{t} = 100$$

$$c = 2$$

$$k_1 = -558 \text{ psi} \qquad k_3 = -25.8$$

$$k_2 = -8.01 \qquad k_4 = 72.8$$

The results as presented in Figures 2 - 14 are for the above parametric values unless the values of the parameter are specifically indicated; i.e., in Figure 5, for example, we are studying the effect upon the "standard" solution when $\overline{E}$ varies from the value given above.

The case was so stiff compared to the thick-walled cylinder that we nearly obtained hydrostatic compression of the thick-walled cylinder as may be seen in Figure 3, 4, 9, 10 and 13 where we see that the stresses are nearly equal to the applied pressure. For small pressures the solution differs only slightly from the first order solution (shown by the dotted lines in Figure 2) but as the pressure is increased the nonlinearities become more and more important, thus in Figures 12 and 14 we see that for small pressure the solution is nearly independent of the second order constants. Also in Figure 9, whereas for low pressures $\sigma_\theta(a)$ is a linear function of the first order constant k_1, we see that for higher pressures it becomes a nonlinear function of k_1. As the strains are much larger at the inner surface of the thick-walled cylinder than at the outer (we found $\frac{u(b)}{b} < .001$) we should expect the nonlinearities to be far more pronounced at the inner surface than at the outer. This prediction is readily verified by comparing Figures 3 and 4 and by noting from Figure 13 that $\sigma_r(b)$ is essentially independent of the second order constants even for large values of the applied pressure.

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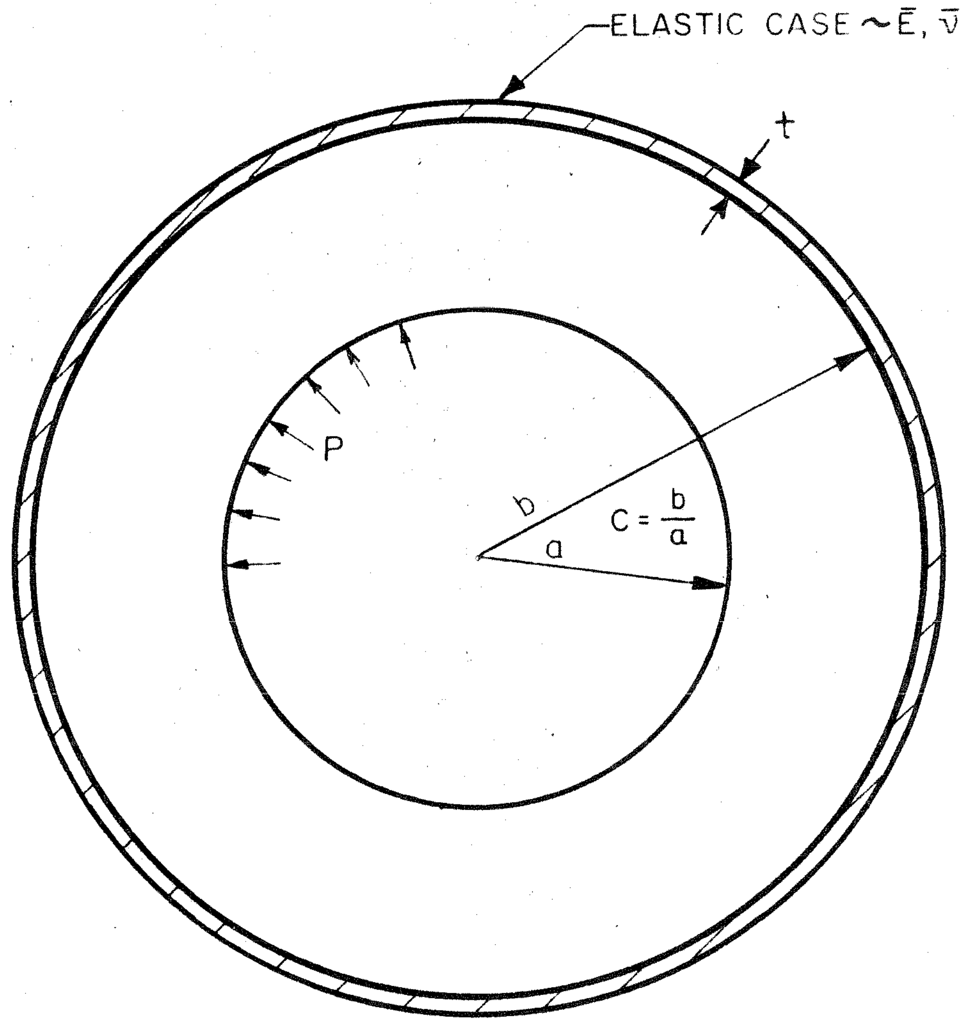

FIG. I PRESSURIZATION OF A THICK-WALLED CYLINDER BONDED TO AN ELASTIC CASE
(SEE SECTIONS 6 AND 10)

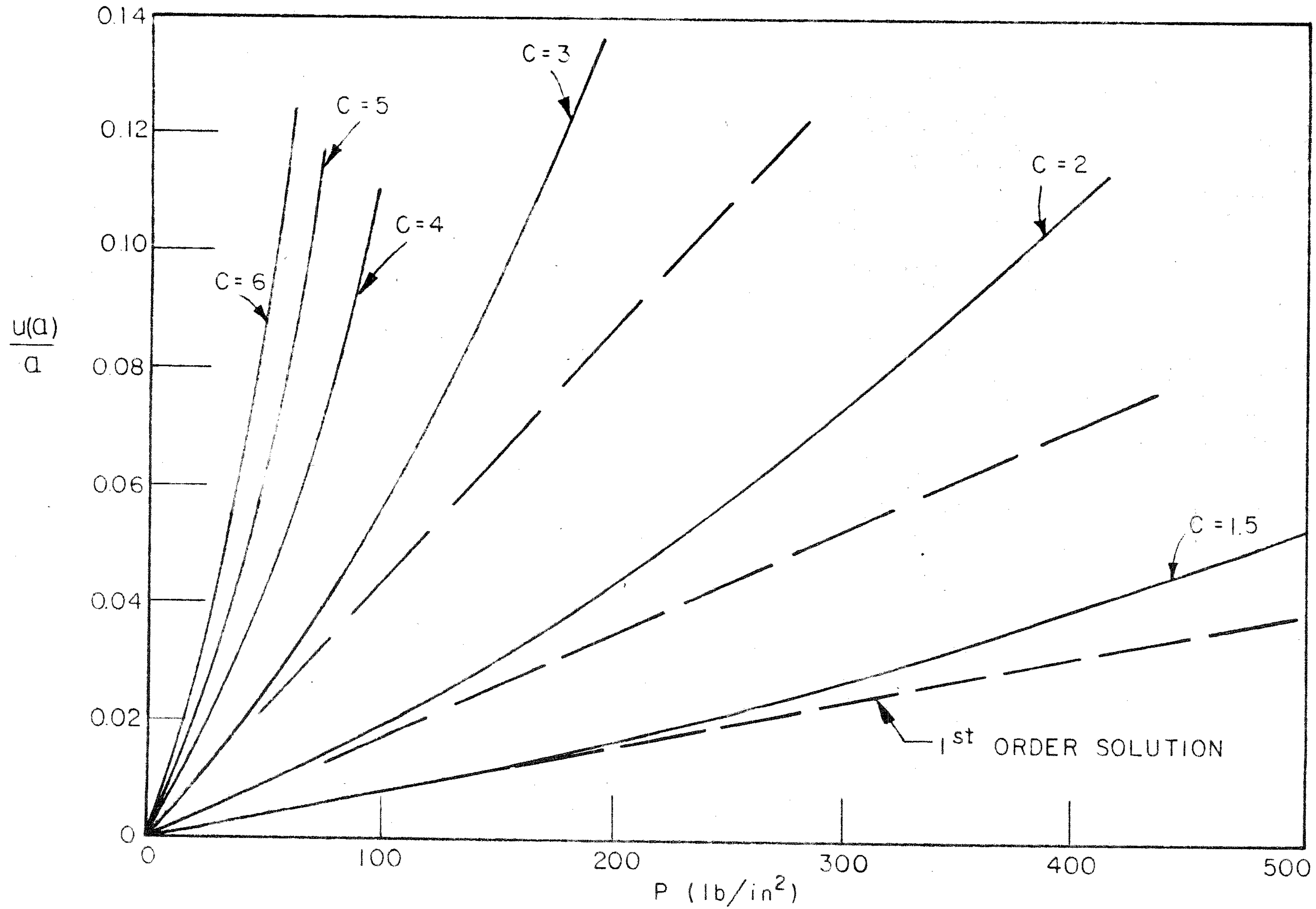

FIG.2 INNER BORE DISPLACEMENT – PRESSURE AS A FUNCTION OF CYLINDRICAL RADII RATIO.

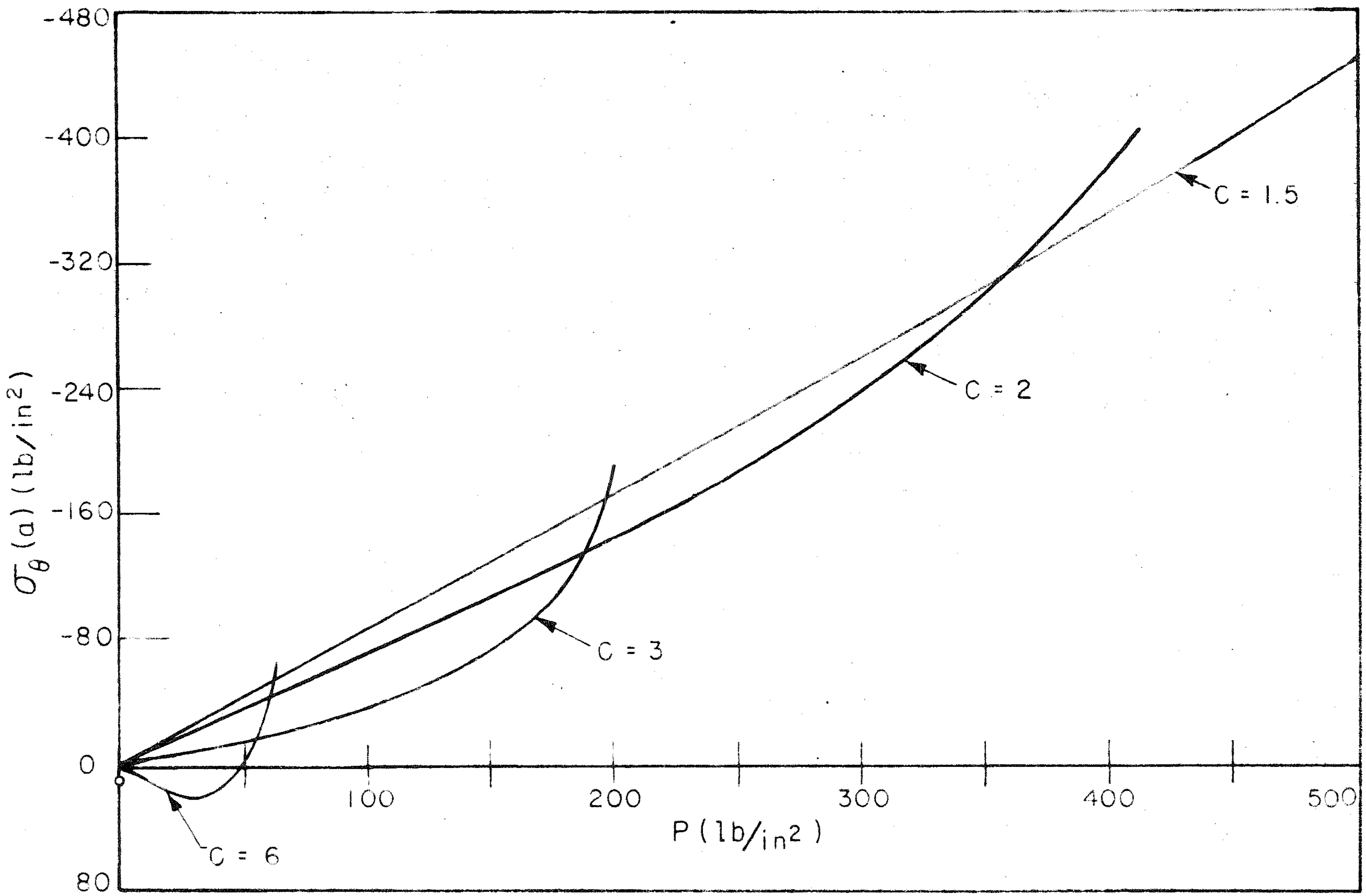

FIG. 3 INNER BORE TANGENTIAL STRESS — PRESSURE AS A FUNCTION OF CYLINDRICAL RADII RATIO

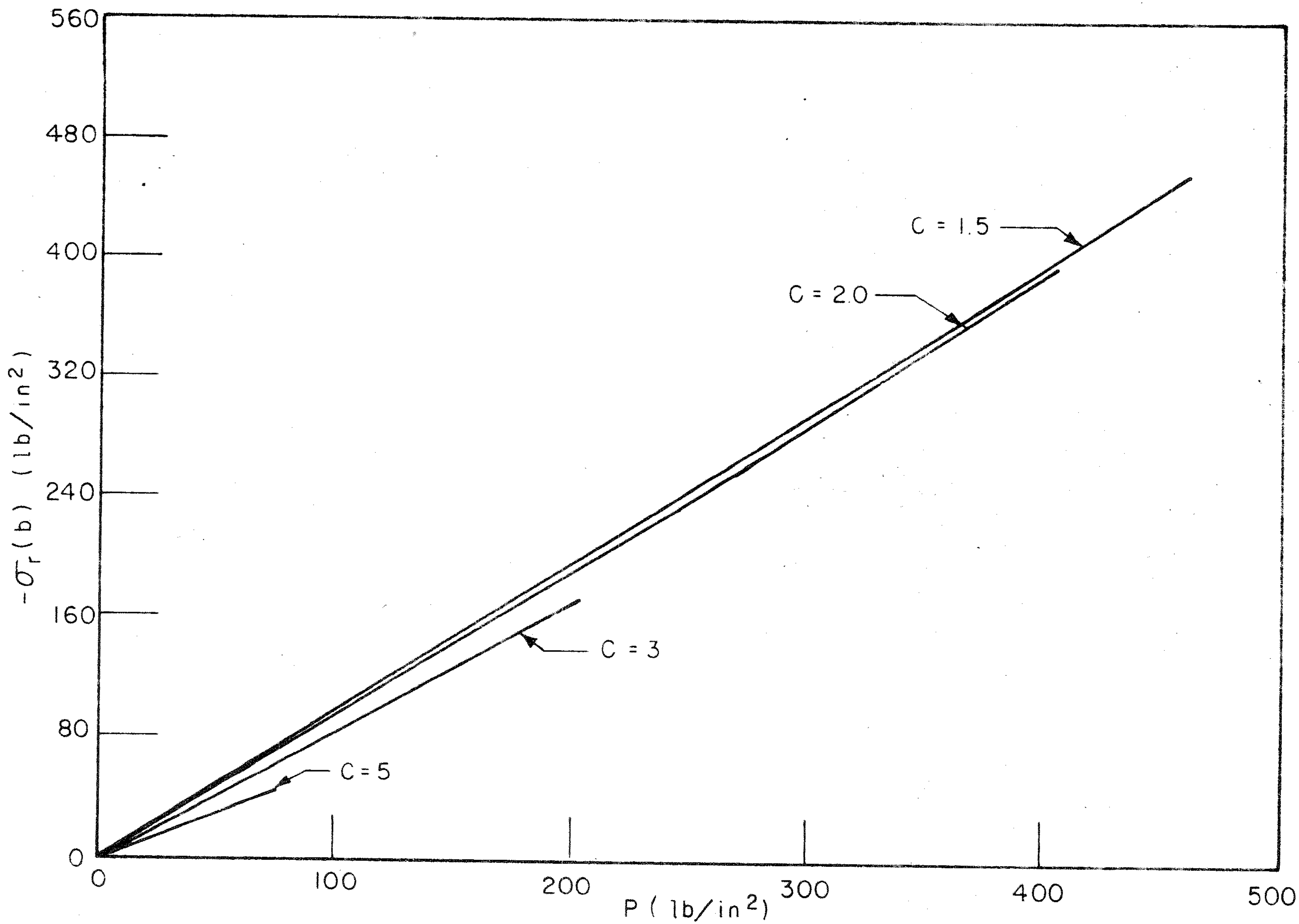

FIG. 4 CASE PRESSURE — PRESSURE AS A FUNCTION OF CYLINDRICAL RADII RATIO

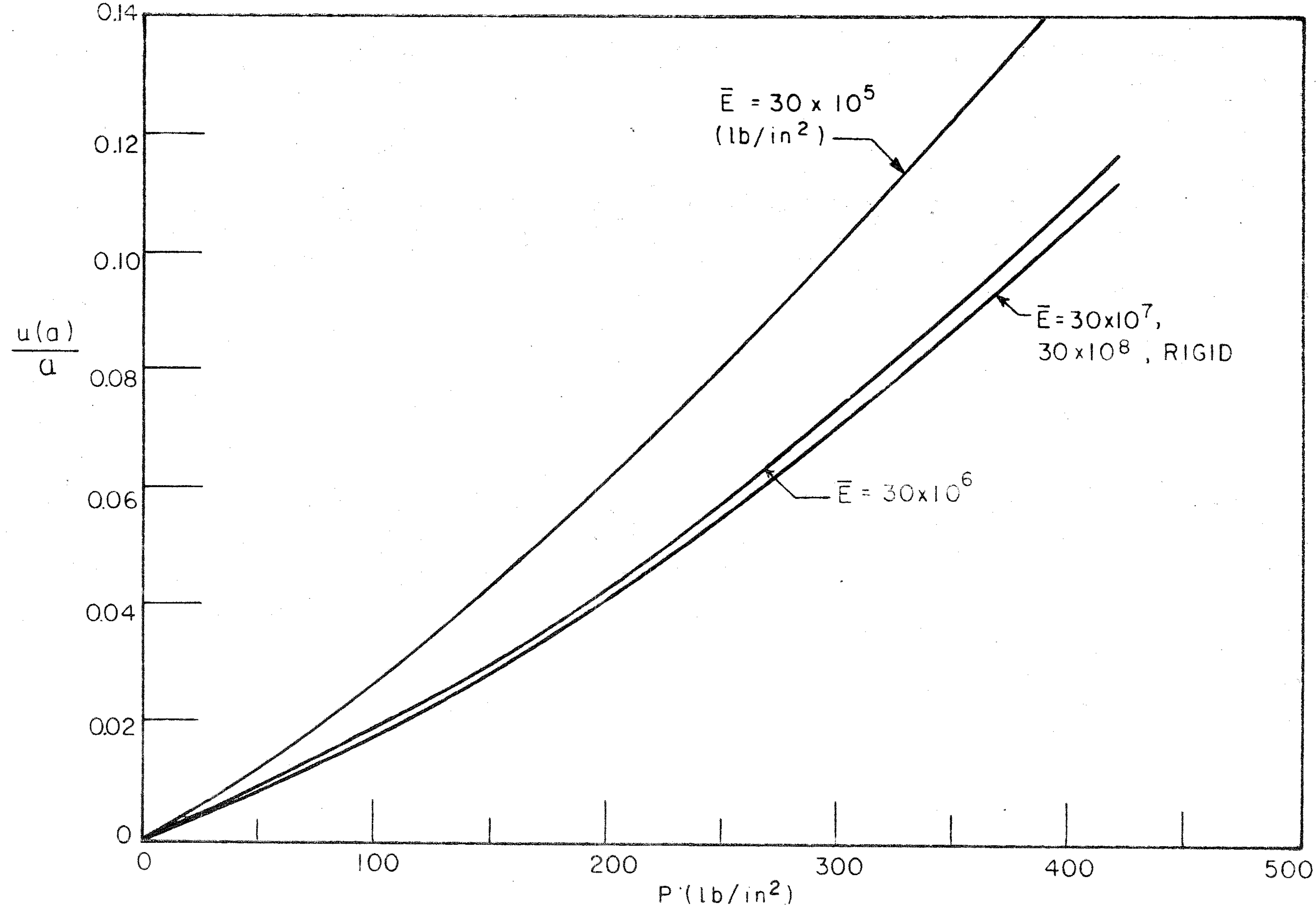

FIG.5 INNER BORE DISPLACEMENT - PRESSURE AS A FUNCTION OF CASE STIFFNESS (b/t = 100)

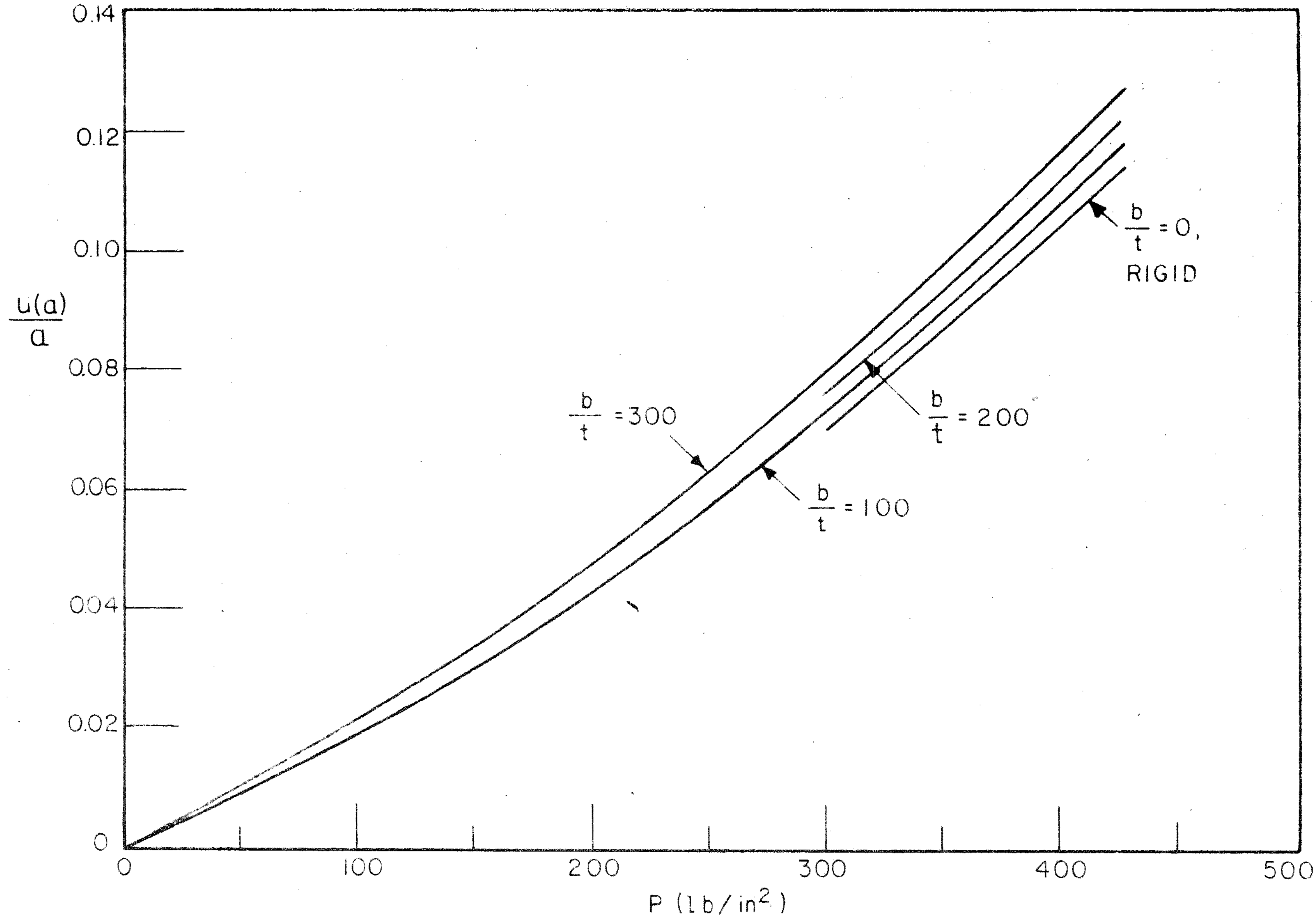

FIG. 6 INNER BORE DISPLACEMENT — PRESSURE AS A FUNCTION OF CASE STIFFNESS ($\bar{E} = 30 \times 10^6$)

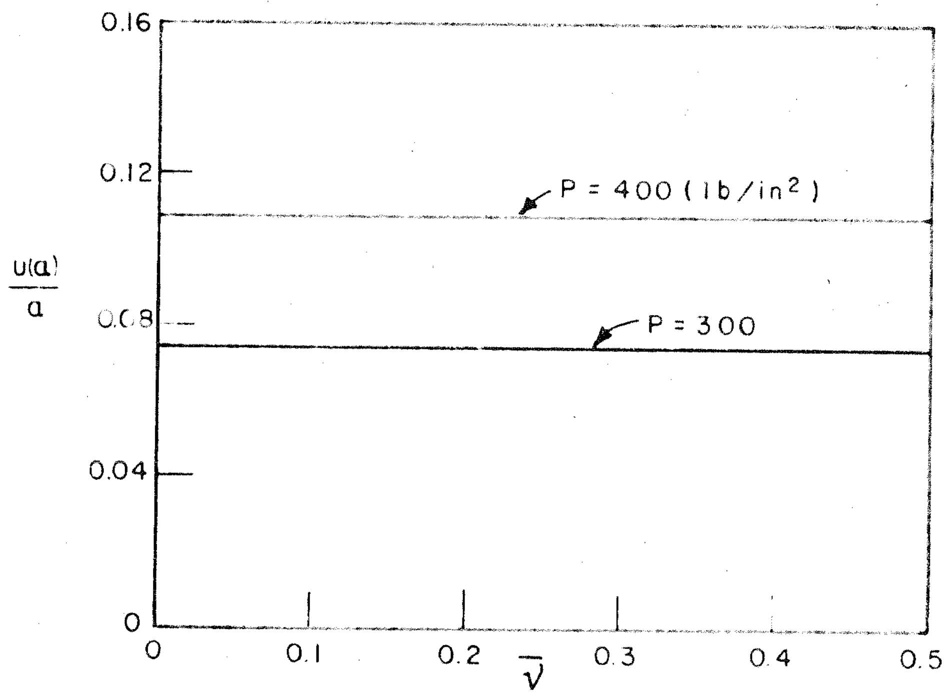

FIG. 7 INNER BORE DISPLACEMENT - CASE POISSON'S RATIO

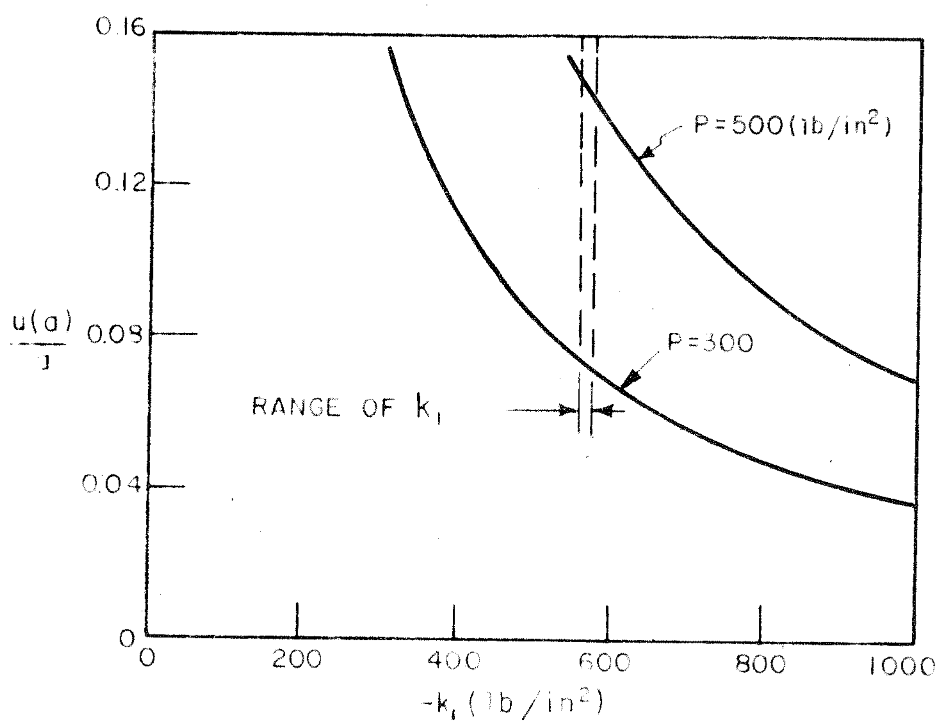

FIG. 8 INNER BORE DISPLACEMENT - ELASTIC MATERIAL PARAMETER

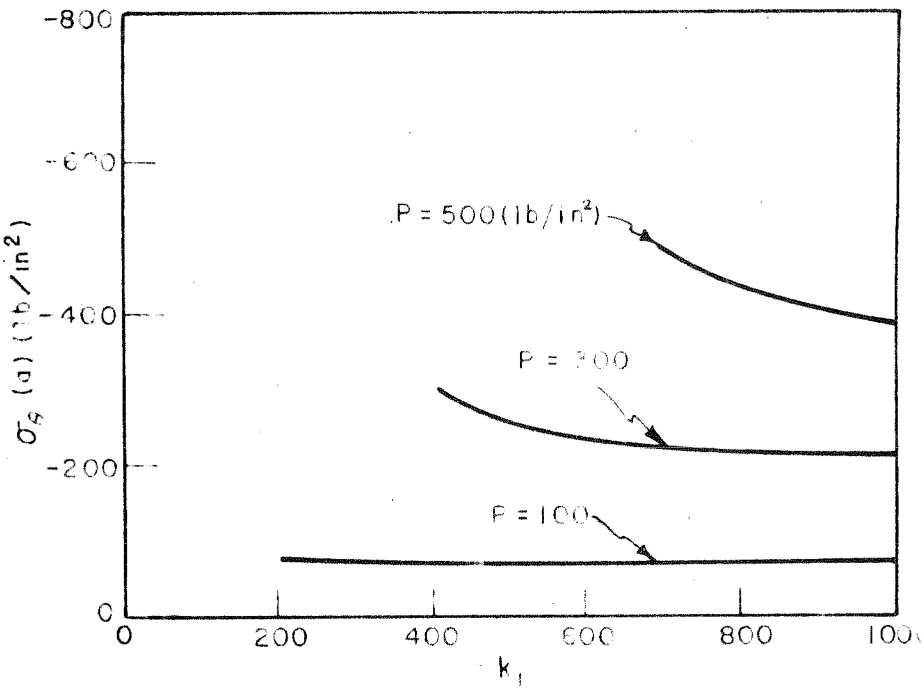

FIG. 9 INNER BORE TANGENTIAL STRESS — ELASTIC MATERIAL PARAMETER

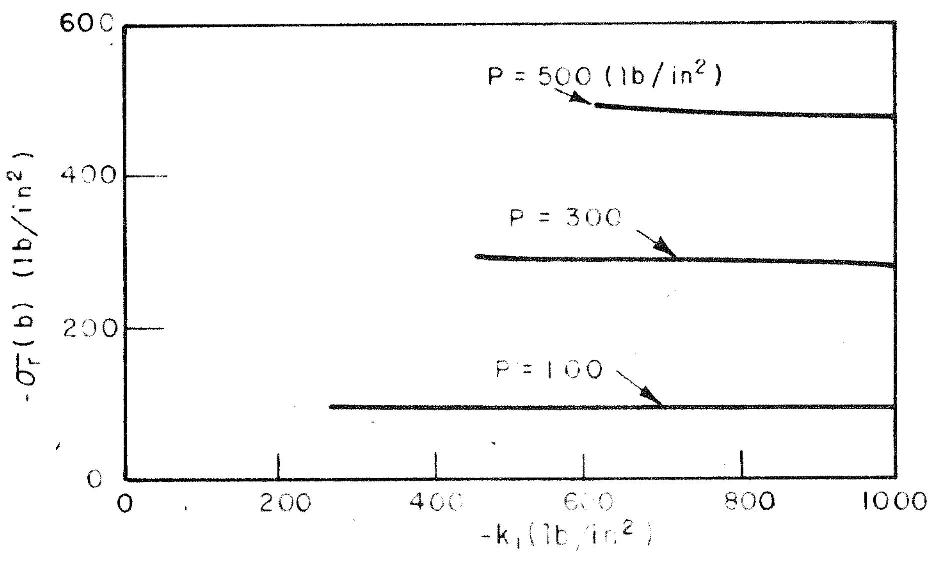

FIG. 10 CASE PRESSURE - ELASTIC MATERIAL PARAMETER

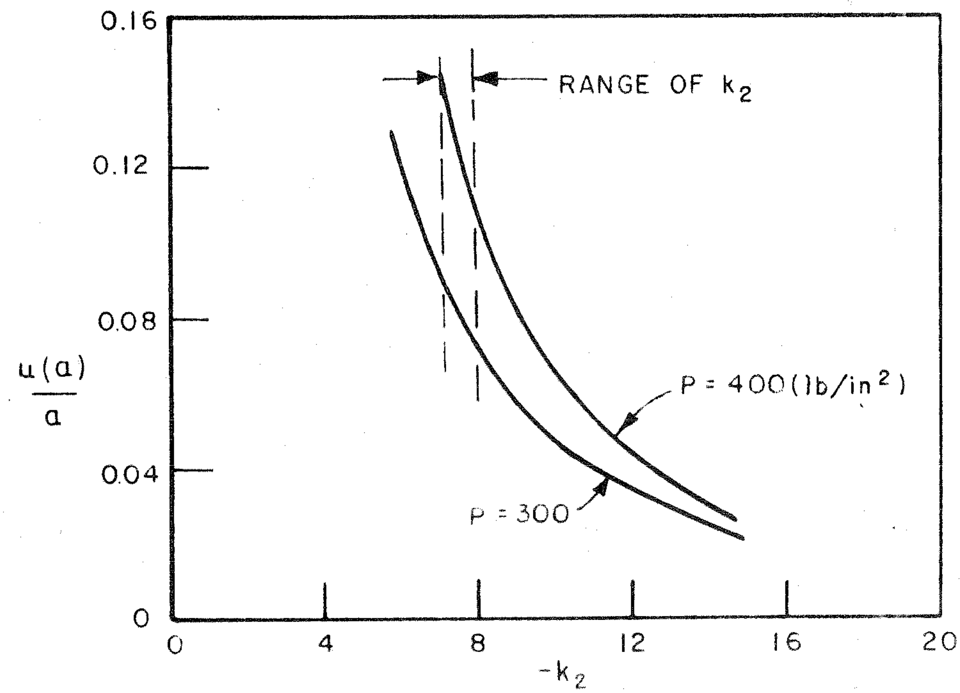

FIG. 11 INNER BORE DISPLACEMENT - ELASTIC MATERIAL PARAMETER

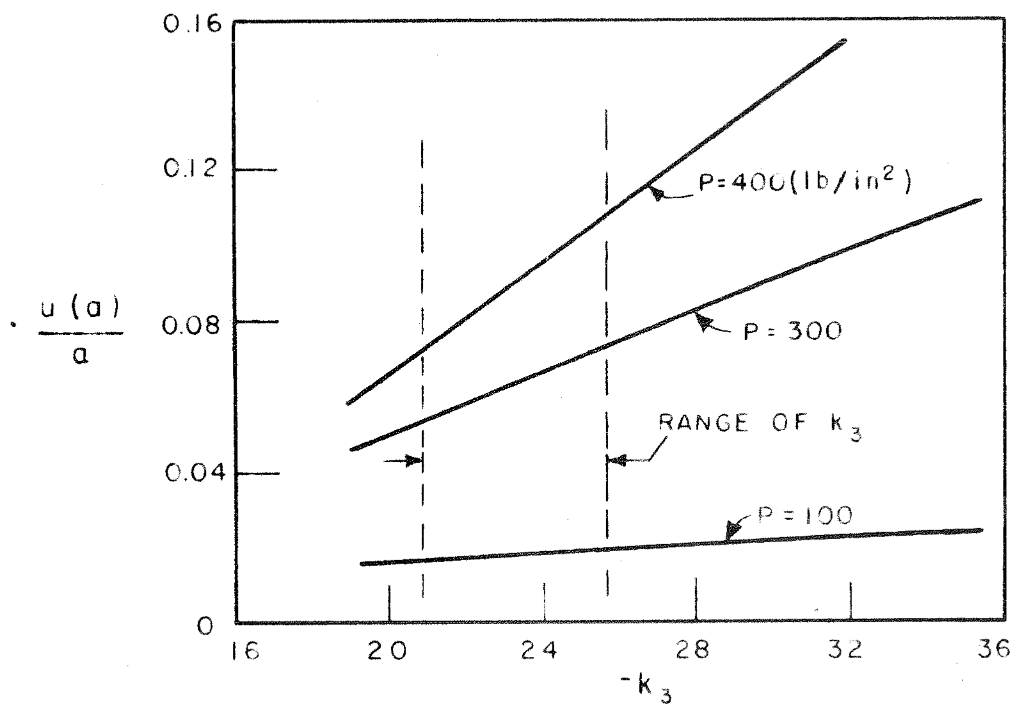

FIG. 12 INNER BORE DISPLACEMENT - ELASTIC MATERIAL PARAMETER

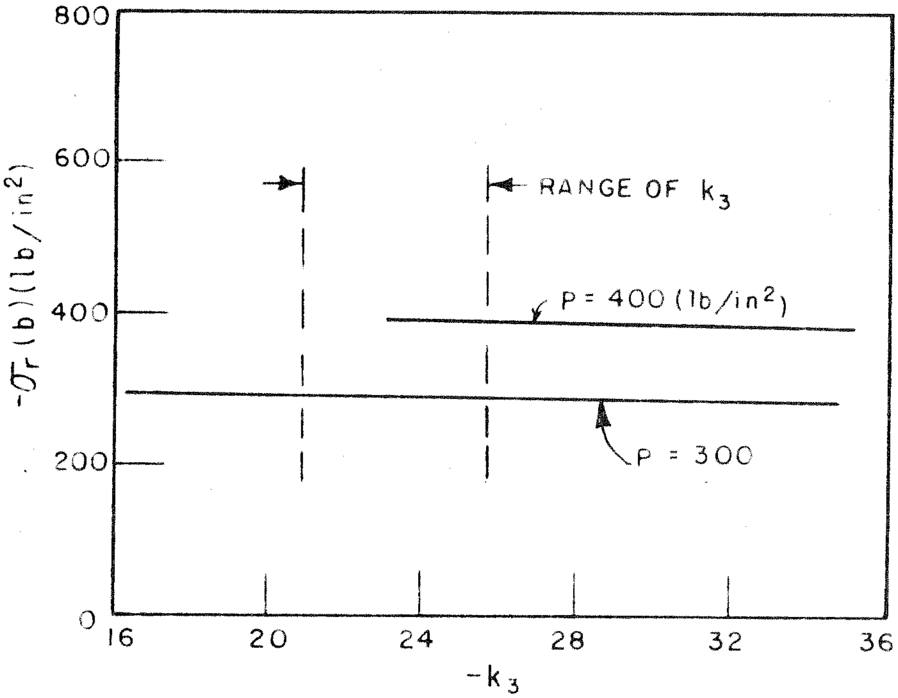

FIG. 13 CASE PRESSURE – ELASTIC MATERIAL PARAMETER

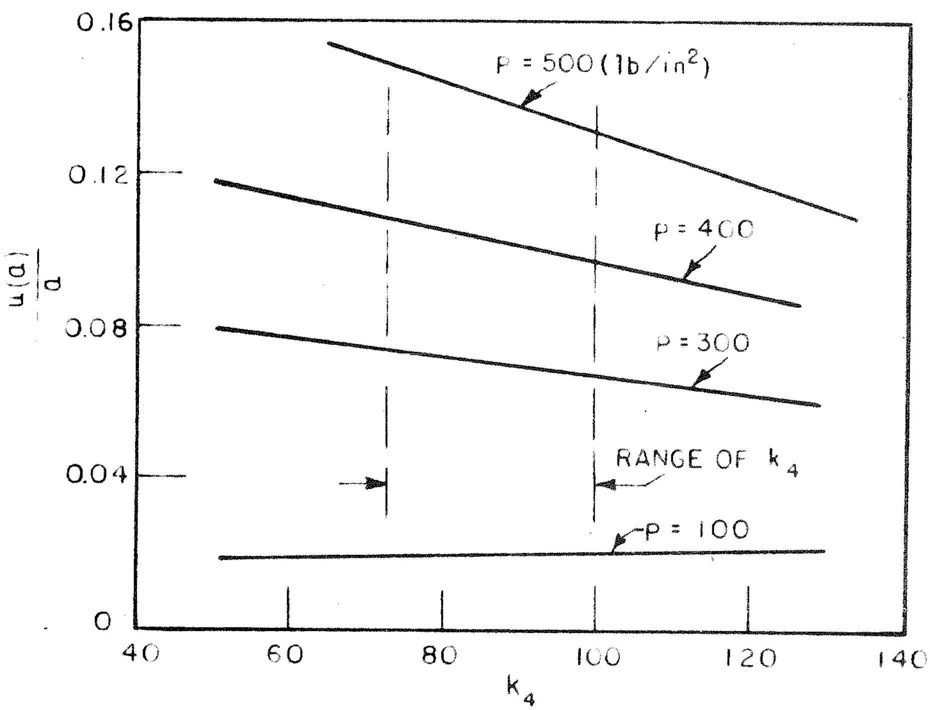

FIG. 14 INNER BORE DISPLACEMENT-ELASTIC MATERIAL PARAMETER

PART IV

THERMAL DEFORMATIONS OF VISCOELASTIC MATERIALS

by

R. L. TAYLOR

INTRODUCTION

It is well known that the response of stressed viscoelastic materials is influenced greatly by temperature changes in the transition range between the glassy and rubbery states. Any meaningful thermoviscoelastic analysis should reflect this behavior. One means of accounting for this type of response for a selected class of materials is the use of the time-temperature equivalence postulate: a change in temperature is equivalent to a shift in time. If the changes in response can be specified by a single time-temperature shift function, the material has been classified as "thermorheologically simple" by Schwarzl and Staverman [1]. In this part a solution method for a class of problems involving thermorheologically simple materials will be discussed.

GENERAL THEORY

1. Field Equations for Thermorheologically Simple Viscoelastic Materials.

With reference to an orthogonal curvilinear coordinate system, the linearized equilibrium equations in the absence of body forces and inertia terms are

$$\tau^i_j(x,t)_{,i} = 0 \qquad (1.1)$$

where x denotes the curvilinear coordinate triad, (x^1, x^2, x^3), while t denotes time, τ^i_j are the mixed tensor components of stress, a repeated index appearing in a contravariant and covariant position implies summation, and a covariant index preceded by a comma implies covariant differentiation [5].

The linearized strain-displacement relations are

$$\epsilon_{ij}(x,t) = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (1.2)$$

where ϵ_{ij} and u_i are the covariant components of the linear strain tensor and the displacement vector respectively. In the subsequent analysis the mixed strain tensor will be needed. This is accomplished in the usual manner by raising an index to obtain the associated mixed strain tensor.
Thus,

$$\epsilon^k_j = g^{ki}\,\epsilon_{ij} = \frac{1}{2}(u^k_{,j} + g^{ki}\,u_{j,i}) \qquad (1.3)$$

where g^{ki} are the contravariant components of the metric tensor and u^k those of the displacement vector, respectively.

As is customary in isotropic viscoelastic analysis the stress and strain tensors are decomposed into their deviatoric and spherical components for convenience. Accordingly,

$$\tau^i_j = s^i_j + \delta^i_j \sigma \tag{1.4}$$

where $$\sigma = \frac{1}{3}\tau^i_i$$

and

$$\epsilon^i_j = e^i_j + \delta^i_j \epsilon$$
$$\epsilon = \frac{1}{3}\epsilon^i_i \tag{1.5}$$

In the above δ^i_j is the Kronecker delta, defined as

$$\delta^i_j = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases} \tag{1.6}$$

Considering first the constitutive equations in the absence of thermal effects, it is known that the linear constitutive laws admit the differential-operator representation

$$P_1(D) s^i_j(x,t) = Q_1(D) e^i_j(x,t)$$

and

$$P_2(D)\sigma(x,t) = Q_2(D)\epsilon(x,t) \tag{1.7}$$

where $$D \equiv \frac{\partial}{\partial t} \; ; \; P_i(D) = \sum_{n=0}^{N_i} p_i^{(n)} D^n$$

$$Q_i(D) = \sum_{m=0}^{M_i} q_i^{(m)} D^m$$

and

$$p_i^{(N_i)} \neq 0 \, , \quad q_i^{(M_i)} \neq 0 \quad (i = 1, 2)$$

Thus P_i and Q_i are polynomial differential time operators of degree N_i and M_i respectively. The material properties are introduced through the $p_i^{(n)}$ and $q_i^{(m)}$ coefficients. The use of differential-operator representation is practically limited to media possessing finite and discrete relaxation spectra and retardation times. For this reason, it is convenient to consider the constitutive laws in integral form.

First, consider a mechanical test conducted at constant temperature. If an experiment is performed in which a constant strain (e.g., a constant deviator strain $e_{(0)}{}^i_j$) is introduced at time zero and the material is assumed undisturbed for $t < 0$, then the time history of stress can be determined by measurement. Considering the stress-time history for the $\epsilon_{(0)}{}^i_j$ deviator strain, the deviator-stress history s^i_j may be expressed through the relation

$$s^i_j(t) = G_1(t) \; e_{(0)}{}^i_j \tag{1.8}$$

where $G_1(t)$ is defined as the deviatoric (shear) relaxation function and $\epsilon_{(0)}{}^{i}_{j}$ is the constant strain introduced at $t = 0$. Subsequently, if the strain is a prescribed function of time, then the stress history may be determined by using the Boltzmann superposition principle. Thus, for example,

$$s^i_j(t) = \sum_{k=0}^{n-1} G_1(t_{(n)} - t_{(k)}) \left[e_{(k+1)}{}^{i}_{j} - e_{(k)}{}^{i}_{j} \right] \qquad (1.9)$$

where $e_{(k)}{}^{i}_{j}$ is the magnitude of the strain at the time $t_{(k)}$. If we now pass to the limit by letting $t_{(k+1)} - t_{(k)}$ tend to zero, the the sum becomes an integral and the relaxation integral laws in the absence of thermal effects are given by

$$s^i_j(x,t) = \int_0^t G_1(t-t') \frac{\partial}{\partial t'} e^i_j(x,t')dt'$$

and (1.10)

$$\sigma(x,t) = \int_0^t G_2(t-t') \frac{\partial}{\partial t'} \epsilon(x,t')dt'$$

Similarly, the creep integral laws are given by

$$e^i_j(x,t) = \int_0^t J_1(t-t') \frac{\partial}{\partial t'} s^i_j(x,t')dt'$$

and (1.11)

$$\epsilon(x,t) = \int_0^t J_2(t-t') \frac{\partial}{\partial t'} \sigma(x,t')dt'$$

where G_i and J_i, $(i = 1,2)$, are relaxation moduli and creep compliances respectively (in shear and dilatation).

Equations (1.10) and (1.11) vary slightly from the integral laws presented in Gross [3] in that instantaneous elastic and steady creep do not occur explicitly. However, in the above form, if the limits are taken from 0^- to t^+ then the discontinuities in the integrals between 0^- and 0^+ and t^- and t^+ will give rise to the two additional terms included in [3]. In evaluating any of the integrals which follows (unless otherwise indicated) these discontinuities must first be removed by evaluating the integrals from 0^- to 0^+ and t^- to t^+.

The above constitutive laws were presented on the basis that the body remained isothermal for all time. Thus the material properties G_i, J_i, $p_i^{(n)}$ and $q_i^{(n)}$ must be regarded as having been determined for the temperature at which the material is being stressed. Mechanical properties of visco-elastic materials in the transition range between the glassy and the rubbery state show marked dependence upon the temperature. One method of accounting for temperature dependence is through the use of the time-temperature equivalence hypothesis in which a "reduced time" is introduced to account for both time and temperature variations. To exemplify the use of the "reduced time," consider the variation of the relaxation moduli G_i with the temperature. Following the notation of Muki and Sternberg [6] let $G_i(t)$ be the relaxation modulus at the constant base temperature T_o. We desire to account for variations of temperature from T_o, say for any uniform temperature T. Let $\overline{G}_i(t,T)$ be the relaxation modulus at the temperature T, thus

$$\overline{G}_i(t,T_o) = G_i(t) = L_i(\log t) \qquad (1.12)$$

The postulate of a single time-temperature equivalence function (i.e., the thermorheologically simple material) now assumes the form

$$\overline{G}_i(t,T) = L_i\left[\log t + \log \phi(T)\right] = G_i\left[t\phi(T)\right] \qquad (1.13)$$

where $\phi(T)$ is defined as the shift function. For uniform temperature, the product $t\phi(T)$ is defined as the "reduced time" ξ.

Once the shift function $\phi(T)$ is known, $\overline{G}_i(t,T)$ may be determined for other temperatures.

We next suppose the material to be subjected to non-uniform temperature $T(x,t)$. In extending the above concept, the reduced time must be generalized consistent with the postulated time-temperature equivalence and also the thermal expansion must be included in the constitutive law describing the dilatational behavior of the material. Moreland and Lee [4] stated these modifications and obtained general constitutive laws for a thermorheologically simple material as follows:

$$s^i_j(x,t) = \int_0^t G_1(\xi - \xi')\frac{\partial}{\partial t'} e^i_j(x,t')dt'$$

$$\sigma(x,t) = \int_0^t G_2(\xi - \xi')\frac{\partial}{\partial t'}\left[\epsilon(x,t') - \alpha_0\theta(x,t')\right]dt' \qquad (1.14)$$

where the reduced time is now determined from

$$\xi = f(x,t) = \int_0^t \phi\left[T(x,t')\right]dt', \quad \xi' = f(x,t') \qquad (1.15)$$

while the "pseudo-temperature" $\theta(x,t)$ is defined by

$$\theta(x,t) = \frac{1}{\alpha_o}\int_{T_o}^{T(x,t)} \alpha(T')dT', \quad \alpha_o = \alpha(T_o) \tag{1.16}$$

$\alpha(T)$ is the temperature-dependent coefficient of thermal expansion. If α is temperature independent, then

$$\theta(x,t) = T(x,t) - T_o$$

where $T(x,t)$ is the solution of the Fourier heat conduction equation. The creep integral laws modified for the effects of temperature are given by

$$\begin{aligned} e^i_j(x,t) &= \int_0^t J_1(\xi - \xi')\frac{\partial}{\partial t'} s^i_j(x,t')dt' \\ \epsilon(x,t) &= \int_0^t J_2(\xi - \xi')\frac{\partial}{\partial t'} \sigma(x,t')dt' + \alpha_o\theta(x,t) \end{aligned} \tag{1.17}$$

In order to make use of the Laplace transform it is convenient to remove from the field equations the explicit dependence upon the physical time. From Eq. (1.15), $f(x,t)$ can be inverted formally with respect to time to yield

$$t = g(x,\xi) \tag{1.18}$$

The explicit dependence upon time t in any function of space and time $F(x,t)$ is now removed by substituting Eq. (1.18) for t, thus,

$$F(x,t) = F\Big[x, g(x,\xi)\Big] \tag{1.19}$$

In order to avoid any ambiguity, we will define, following [6],

$$\hat{F}(x,\xi) = F(x,t) \tag{1.20}$$

It must be emphasized that $\hat{F}(x,\xi)$ is not the same function as $F(x,t)$ but has first been subjected to the transformation given by Eq. (1.18).

Substituting the results of Eq. (1.20) and making the appropriate changes in variables in the constitutive equations leads to the relaxation integral laws:

$$\begin{aligned} \hat{s}^i_j(x,\xi) &= \int_0^{\xi} G_1(\xi - \xi')\frac{\partial}{\partial \xi'}\, \hat{e}^i_j(x,\xi')d\xi' \\ \hat{\sigma}(x,\xi) &= \int_0^{\xi} G_2(\xi - \xi')\frac{\partial}{\partial \xi'}\left[\hat{\epsilon}(x,\xi') - \alpha_o\hat{\theta}(x,\xi')\right]d\xi' \end{aligned} \tag{1.21}$$

and to the creep integral laws

$$\begin{aligned} \hat{e}^i_j(x,\xi) &= \int_0^{\xi} J_1(\xi - \xi')\frac{\partial}{\partial \xi'}\, \hat{s}^i_j(x,\xi')d\xi' \\ \hat{\epsilon}(x,\xi) &= \int_0^{\xi} J_2(\xi - \xi')\frac{\partial}{\partial \xi'}\, \hat{\sigma}(x,\xi') + \alpha_o\hat{\theta}(x,\xi) \end{aligned} \tag{1.22}$$

The differential operator form of the constitutive equations modified for the effects of temperature and consistent with the time-temperature equivalence hypothesis may be determined most easily by taking the Laplace transform with respect to the reduced time in Eqs. (1.21) and (1.22). Substituting the relation between transforms of relaxation and creep functions, and the transforms of the corresponding differential operators, and finally inverting the resulting transforms results in the differential operator constitutive laws. These results are presented in [4] and [6] and are not repeated here since all subsequent discussion is based on integral laws.

2. Displacement Equations of Equilibrium.

The field equations for a thermorheologically simple material were presented in the previous section. In order to determine the stresses and displacements in the interior of a body when displacements or stresses are prescribed on its surface, it is desirable, when possible, to reduce the number of dependent variables. To this end, the strain-displacement relations are first substituted into the constitutive equations yielding,

$$s^i_j(x,t) = \int_0^t G_1(\xi-\xi')\frac{\partial}{\partial t'}\left\{\frac{1}{2}\left[u^i_{;j} + g^{ik}u_{j,k}\right] - \frac{1}{3}\delta^i_j u^k_{,k}\right\} dt'$$

$$\sigma(x,t) = \int_0^t G_2(\xi-\xi')\frac{\partial}{\partial t'}\left\{\frac{1}{3} u^k_{,k} - \alpha_o\theta\right\} dt' \tag{2.1}$$

Combining the deviatoric and spherical components of stress to obtain the components of the stress tensor in terms of the displacement components, we obtain

$$\tau^i_j = \int_0^t G_1(\xi-\xi')\frac{\partial}{\partial t'}\left\{\frac{1}{2}\left[u^i_{,j} + g^{ik}u_{j,k}\right] - \frac{1}{3}\delta^i_j\, u^k_{,k}\right\} dt'$$

$$+ \frac{1}{3}\delta^i_j \int_0^t G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[u^k_{,k} - 3\alpha_o\theta\right] dt' \tag{2.2}$$

Substituting the expression for stress into the equilibrium equation and assuming that the integration and differentiation may be interchanged, yields after regrouping terms,

$$\int_0^t \left\{ \left[G_1(\xi-\xi') + 2G_2(\xi-\xi') \right] \frac{\partial}{\partial t'}(u^i_{,j}) + 3G_1(\xi-\xi') \frac{\partial}{\partial t'}(g^{ik} u_{j,k}) \right\}_{,i} dt'$$

$$-6 \int_0^t \left\{ G_2(\xi-\xi') \frac{\partial}{\partial t'}(\alpha_o \theta) \right\}_{,j} dt' + 2 \int_0^t \left[(G_1(\xi-\xi') - G_2(\xi-\xi') \right]_{,i} \frac{\partial}{\partial t'}(u^i_{,j}) dt' \tag{2.3}$$

$$-2 \int_0^t \left[G_1(\xi-\xi') - G_2(\xi-\xi') \right]_{,j} \frac{\partial}{\partial t'}(u^i_{,i}) dt' = 0$$

The covariant derivatives of the material properties will no longer vanish due to the non-homogeneity introduced by the variable temperature field; thus, the last two integrals include terms which do not exist in the homogeneous linear viscoelastic analysis which has been presented to date in the literature.

APPLICATIONS

3. Specialization for Axisymmetric Plane Strain of Infinite Cylinders.

For axisymmetric plane strain, the only non-zero displacement component is the radial component u^1. Furthermore, the only variations in the radial displacement will be in the radial coordinate x^1. Thus two of the displacement equations of equilibrium are satisfied identically and the third is given by

$$\int_0^t \left\{ \left[G_1(\xi-\xi') + 2G_2(\xi-\xi') \right] \frac{\partial}{\partial t'}(u^i_{,1}) + 3G_1(\xi-\xi') \frac{\partial}{\partial t'}(g^{ik} u_{1,k}) \right\}_{,i} dt'$$

$$- 6 \int_0^t \left\{ G_2(\xi-\xi') \frac{\partial}{\partial t'}(\alpha_o \theta) \right\}_{,1} dt' = 0 \tag{3.1}$$

The evaluation of the covariant derivatives yields upon regrouping and expressing in terms of the contravariant displacement component,

$$\int_0^t \Bigg(\frac{\partial}{\partial x^1} \left\{ \left[2G_1(\xi-\xi') + G_2(\xi-\xi') \right] \frac{\partial^2}{\partial t' \partial x^1}(u^1) \right\}$$

$$+ \left[2G_1(\xi-\xi') + G_2(\xi-\xi') \right] \frac{\partial^2}{\partial t' \partial x^1}\left(\frac{1}{x^1} u^1\right) - 3 \frac{\partial}{\partial x^1} G_1(\xi-\xi') \frac{\partial}{\partial t'}\left(\frac{u^1}{x^1}\right) \tag{3.2}$$

$$- 3 \frac{\partial}{\partial x^1} \left[G_2(\xi-\xi') \frac{\partial}{\partial t'} \alpha_o \theta(t') \right] \Bigg) dt' = 0$$

The solution for u^1 proves to be very difficult since in general the variable coefficients are dependent upon the x^1 coordinate in a very complex manner. For this reason, we turn our attention to approximations which might give some significant results. The analysis to follow will pertain to instances in which the temperature is varying slowly enough that it can be assumed to be uniform throughout the entire cylinder. This forms one limit case to the analysis. The other limit, that of the temperature varying with radius but time independent has been investigated previously for cylinders by Moreland and Lee [4]. The elastic-viscoelastic analogy is extended for the above two limit cases in [8]. The intermediate case where temperature varies in both space and time, while it may be the most important part of any analysis, remains for the present intractable.

4. Slowly Varying Uniform Temperature Fields.

For slowly varying temperatures independent of the spatial coordinates, the reduced time also becomes independent of the spatial coordinates and, consequently, we may write Eq. (3.1) in the simple form

$$\int_0^t \left\{ \left[2G_1(\xi-\xi') + G_2(\xi-\xi') \right] \frac{\partial}{\partial t'} u^i_{,i1}(x,t') \right\} dt' = 0 \qquad (4.1)$$

For which a solution exists when

$$u^i_{,i1} = 0 \qquad (4.2)$$

The evaluation of the covariant derivatives leads to

$$\frac{\partial u^1}{\partial x^1 \partial x^1} + \frac{1}{x^1}\frac{\partial u^1}{\partial x^1} - \frac{u^1}{x^1 x^1} = 0 \qquad (4.3)$$

In the above instance, the contravariant radial displacement tensor component is the same as the physical component, hence, we may write the above equation in the more familiar notation of u and r (for u' and x' respectively) as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = 0 \tag{4.4}$$

for which the general integral is

$$u(r,t) = C_1(t)\ r + C_2(t)\frac{1}{r} \tag{4.5}$$

5. <u>Infinite Cylinder Rigidly Encased</u>.

Considering now an infinite cylinder with a stress-free inner boundary and fixed outer boundary (i.e., enclosed by a rigid case) where a and b are the inner and outer radii respectively the constants of integration will be evaluated from

$$u(b,t) = 0$$

$$\tau_1^1 = \sigma_r(a,t) = 0 \tag{5.1}$$

Satisfying the first boundary condition gives from Eq. (4.5)

$$C_2(t) = -\ b^2 C_1(t) \tag{5.2}$$

and hence

$$u^1(r,t) = u(r,t) = C_1(t)\left[r - \frac{b^2}{r}\right] = -\ C_1(t)\left[\frac{b^2 - r^2}{r}\right] \tag{5.3}$$

Upon noting that for $r = a$,

$$u^k_{,k} = \frac{\partial u}{\partial r} + \frac{u}{r} = 2C_1(t)$$

and (5.4)

$$u^1_{,1} = g^{1k}u_{1,k} = \frac{\partial u}{\partial r} = C_1(t)\left[1 + \frac{b^2}{a^2}\right]$$

The second boundary condition yields, upon using Eq. (2.2),

$$\sigma_r(a,t) = 0 = \int_0^t \left\{G_1(\xi-\xi')\frac{\partial}{\partial t'}\left[C_1(t')(1 + \frac{b^2}{a^2}) - \frac{2}{3}C_1(t')\right]\right. \tag{5.5}$$

$$\left. + \frac{1}{3}G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[2C_1(t') - 3\alpha_o\theta(t')\right]\right\} dt'$$

or

$$\int_0^t \left[\left\{a^2\left[G_1(\xi-\xi') + 2G_2(\xi-\xi')\right] + 3b^2G_1(\xi-\xi')\right\}\frac{\partial}{\partial t'}\left(C_1(t')\right)\right. \tag{5.6}$$

$$\left. - 3a^2G_2(\xi-\xi')\frac{\partial}{\partial t'}\left(\alpha_o\ \theta(t')\right)\right] dt' = 0$$

Subjecting $C_1(t)$ and $\theta(t)$ to the transformation Eq. (1.18) and taking the Laplace transform+ yields

$$\hat{C}_1^* = \frac{3a^2G_2^*\ \alpha_o\hat{\theta}^*}{a^2(G_1^* + 2G_2^*) + 3b^2G_1^*} \tag{5.7}$$

+ An asterisk denotes the Laplace transform of the function with respect to the reduced time.

If the ratio of radii is defined as

$$c = \frac{b}{a} , \tag{5.8}$$

then after representing the functional dependence of the material properties by

$$pR^*(p) = \frac{G_2^*}{(G_1^* + 2G_2^*) + 3c^2 G_1^*} , \tag{5.9}$$

$R(\xi)$ may be determined by clearing fractions and inverting using the convolution integral. Thus, $R(\xi)$ is given by

$$\int_0^{\xi} R(\xi')\frac{\partial}{\partial(\xi-\xi')}\Big[(1 + 3c^2)G_1(\xi-\xi') + 2G_2(\xi-\xi')\Big]d\xi' = G_2(\xi) \tag{5.10}$$

Now, the formal inversion of the problem may be performed, yielding

$$C_1(t) = 3\int_0^{t} R(\xi-\xi')\frac{\partial}{\partial t'}\Big(\alpha_o\theta(t')\Big)dt' \tag{5.11}$$

Numerical inversions of similar functions are given in [2], [6], and [7]. Finally we may write

$$u(r,t) = - 3\left(\frac{b^2 - r^2}{r}\right)\int_0^{t} R(\xi-\xi')\frac{\partial}{\partial t'}\Big(\alpha_o\theta(t')\Big)dt' \tag{5.12}$$

where R(ξ) is defined by Eq. 5.10.

By combining Eq. (5.7) and the transform of Eq. (5.3) and clearing fractions before inversion we may write the solution in a different form, one which will not require the evaluation of R(ξ):

$$\int_0^t \frac{\partial}{\partial t'} \left\{ \left[G_1(\xi-\xi') + 2G_2(\xi-\xi') \right] + 3c^2 G_1(\xi-\xi') \right\} u(r,t')dt' \tag{5.13}$$

$$= -3\left(\frac{b^2 - r^2}{r}\right) \int_0^t \frac{\partial}{\partial t'} \left[G_2(\xi-\xi') \right] \alpha_o \theta(t')dt'$$

The above expression is a Volterra integral equation, for which numerical solution techniques exist, one of which will be discussed subsequently.

The solution of the corresponding thermoelasticity problem is

$$u(r,t) = -\left(\frac{b^2 - r^2}{r}\right) \frac{(1 + \nu)\alpha_o \theta(t)}{1 + (1 - 2\nu)c^2} \tag{5.14}$$

Thus, the solution is observed to be dependent only upon Poisson's ratio. It is easily verified that the material properties of the thermoviscoelastic problems also occur in combinations such that the solution is dependent only upon the time dependent Poisson's ratio. Since data available on the time and temperature dependence of Poisson's ratio are limited, it appears to be preferable to express the solution in terms of the extension modulus E and the bulk modulus K, quantities for which data are more readily available. If it is assumed that the volumetric behavior is purely elastic and, hence, time independent, Eq. (5.13) may be modified so that the material properties G_1 and G_2 are expressed in terms of E and K, the extension modulus and bulk modulus respectively. After performing this modification and also removing the discontinuities in the integrals at times 0 and t, the time dependence of u(r,t) is determined by solving the non-homogeneous Volterra equation of the second kind:

$$\left(1 + \frac{c^2 E_G}{3K}\right) u(r,t) - \frac{c^2}{3K} \int_{o^+}^{t^-} \frac{\partial}{\partial t'} \left[E(\xi - \xi') \right] u(r,t') dt'$$

$$= -\frac{3}{2}\left(\frac{b^2 - r^2}{r}\right) \left[\left(1 - \frac{E_G}{9K}\right) \alpha_o \theta(t) + \frac{1}{9K} \int_{o^+}^{t^-} \frac{\partial}{\partial t'} \left[E(\xi - \xi') \right] \alpha_o \theta(t') dt' \right] \tag{5.15}$$

where E_G is the initial or glassy modulus. A common technique for solving Volterra equations is through the use of the Laplace transform; in the above example this requires a functional knowledge of E. However, if a numerical scheme of integration is introduced, the measured data are sufficient to determine the behavior of the system. Lee and Rogers [2] have proposed a finite difference solution which may be used. The time interval of interest is divided into n increments t_1, i = 1,2,...,n + 1 with $t_1 = 0$ and $t_{n+1} = t$ (the reduced time is also divided into n intervals ξ_i, i = 1,2,..., n + 1). As will be shown, the increments need not be the same over each interval. The integrals with limits 0^+ to t^- are also separated into n intervals

$$\int_{o^+}^{t^-} \left[\quad \right] dt' = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \left[\quad \right] dt'$$

The solution to the example problem now takes the form

$$(1 + \frac{c^2 E_G}{3K}) u(r, t_{n+1}) = \frac{c^2}{3K} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial t'} \left[E(\xi_{n+1} - \xi') \right] u(r,t') dt' \tag{5.16}$$

$$- \frac{3}{2} (\frac{b^2 - r^2}{r}) \left[(1 - \frac{E_G}{9K}) \alpha_o \theta(t_{n+1}) + \frac{1}{9K} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial t'} \left[E(\xi_{n+1} - \xi') \right] \alpha_o \theta(t') dt' \right]$$

If the functions $u(r,t')$ and $\theta(t')$ occurring under the integrals are approximated by

$$u(r,t') = \frac{1}{2} \left[u(r,t_{i+1}) + u(r,t_i) \right]$$

$$t_i < t' < t_{i+1} \tag{5.17}$$

$$\theta(t') = \frac{1}{2} \left[\theta(t_{i+1}) + \theta(t_i) \right]$$

the integrals may be evaluated, and lead to the result

$$(1 + \frac{c^2 E_G}{3K}) u(r,t_{n+1}) \cong \frac{c^2}{3K} \sum_{i=1}^{n} \frac{1}{2} \left[u(r,t_{i+1}) + u(r,t_i) \right] \left[E(\xi_{n+1} - \xi_{i+1}) \right.$$

$$\left. - E(\xi_{n+1} - \xi_i) \right] - \frac{3}{2} (\frac{b^2 - r^2}{r}) \left\{ (1 - \frac{E_G}{9K}) \alpha_o \theta(t_{n+1}) + \frac{1}{2} \alpha_o \left[\theta(t_{i+1}) \right. \right.$$

$$\left. \left. + \theta(t_i) \right] \left[E(\xi_{n+1} - \xi_{i+1}) - E(\xi_{n+1} - \xi_i) \right] \right\} \tag{5.18}$$

The terms in this approximation occur in a form such that the error propagation is quickly attenuated. Using this intuitive argument, it may be anticipated that the above representation will be stable. The above simple formulation introduces no complications when the intervals $t_{i+1} - t_i$ (or $\xi_{i+1} - \xi_i$) are varied. This may not be the case if more elaborate difference schemes are introduced.

The fact that an initial value problem has now replaced the original boundary value problem enhances the solution technique since the value at each succeeding time interval is dependent only on the preceding times and not on later ones as might be the case in other boundary value problems. The solution is now in a form for which the digital computer may be used to perform the final numerical steps. Some examples of the solution method are discussed [2].

6. Numerical Solutions of Infinite Cylinder Rigidly Encased.

A numerical analysis has been performed for two uniform temperature fields. The cylinder analyzed has a radii ratio $c = 4$. The bulk modulus was selected as 74,600 psi, the coefficient of thermal expansion $\alpha_o = 6 \times 10^{-5}/^\circ F$ and the extension relaxation function and shift function as shown in Fig. 1.

First, the problem in which the temperature is suddenly dropped 80°F at all points of the cylinder is investigated. Physically, this requires the cylinder to have a distributed sink such that heat may be instantaneously dissipated. However, the solution to this hypothetical problem may be utilized for the solution of other physically important problems. The solution has been carried out using Eq. (5.18); the time history of the inner boundary tangential strain $\frac{u}{a}$ is shown in Fig. 2. In addition the method of solution presented by Eqs. (5.9), (5.11), and (5.12) was carried out. After expressing

Eq. (5.9) in terms of the extension and bulk moduli, the first step in this solution is to perform the inversion. This may be performed by expressing the modified Eq. (5.9) as a convolution integral and using the finite difference technique of numerical integration. The functional dependence of $R(\xi)$ is shown in Fig. 3. For the constant uniform temperature field Eq. (5.12) may be integrated to yield

$$u(r,t) = -3\left(\frac{b^2 - r^2}{r}\right)R(\xi)\alpha_o\theta \qquad (6.1)$$

where θ represents the constant uniform temperature change.

For a temperature drop of 80°F and the properties of the cylinder cited previously, the inner boundary tangential strain is again as shown in Fig. 2. From Eq. (6.1), one observes that a constant uniform temperature drop may be used to generate the function $R(\xi)$.

The second example investigated is the slow temperature decrease of the rigidly encased cylinder used in the first example. The dependence of ξ upon t for a uniform cooling of 2°F/100 min. was determined from Eq. (1.15) and is shown in Fig. 4. The solution of Eq. (5.18) for this temperature decrease is shown in Fig. 5. The initial departure of the strain from a straight line is due to the crude time intervals selected for desk calculator computation. Using the kernel function $R(\xi)$ in Eq. (5.12), the solution was repeated, yielding again the results shown in Fig. 5. Once the function $R(\xi)$ is known the determination of the circumferential strain history requires the evaluation of the single integral which appears in Eq. (5.12). Thus, using this method, it appears to be easier to obtain solutions for various uniform temperature variations.

While the above calculations are for a cylindrical inner boundary, appropriate strain concentration factors may be used to approximate the maximum strain for different shaped inner configurations, utilizing information in [8], since the material is homogeneous.

7. Infinite Cylinder Bonded to a Thin Elastic Case.

We next turn our attention to an infinite cylinder bonded to a thin elastic case. Designating the displacement, stresses, and the temperature in the cylinder with subscript I's and those of the case by II's, the mechanical properties of the elastic case are specified by E_{II}, ν_{II}, and α_{II}; its thickness by h, where it is assumed that $h/b \ll 1$; and its uniform temperature $\theta_{II}(t)$. The temperature of the case is allowed to differ from that of the cylinder since in many instances the temperature surrounding the system may drop suddenly, so that initially the case is at one temperature the cylinder at essentially another. This instance may prove to be one of particular interest in studying bond failures between the case and cylinder.

The boundary conditions are given by a stress free inner boundary

$$\sigma_{rI}(a,t) = 0 \qquad (7.1)$$

and the continuity conditions at the interface

$$\begin{aligned} \sigma_{rI}(b,t) &= \sigma_{rII}(b,t) \\ u_I(b,t) &= u_{II}(b,t) \end{aligned} \qquad (7.2)$$

The general integral for u_I is given, as in the previous section, by Eq. (4.5). For a thin case, the solution for the displacement of an infinite case is

$$u_{II}(b,t) = -\left(\frac{1-\nu_{II}^2}{E_{II}}\right)\frac{b^2}{h}\sigma_{rII}(b,t)+(1+\nu_{II})b\alpha_{II}\theta_{II}(t) \qquad (7.3)$$

Satisfying the boundary conditions in an analogous manner as presented in the previous section and expressing the solution in terms of $E_I(t)$ and K_I, the displacement at the inner boundary of the cylinder is determined from

$$162\,\frac{h}{b}\left(\frac{b^2}{b^2-a^2}\right)\frac{E_{II}K_I^2}{(1+\nu_{II})}\frac{u_I(t)}{a} + 108(1-\nu_{II})K_I^2\int_0^t \frac{\partial}{\partial t'}\left[E_I(\xi-\xi')\right]\frac{u_I(t')}{a}\,dt'$$

$$+\frac{2h}{b}\left(\frac{b^2}{b^2-a^2}\right)\frac{E_{II}}{1+\nu_{II}}\int_0^t \frac{\partial}{\partial t'}\left[\int_0^{t'}\frac{\partial}{\partial t''}\left(E_I(\xi'')\right)E_I(\xi-\xi'-\xi'')\,dt''\right]\frac{u_I(t')}{a}\,dt'$$

$$= -\,81\,\frac{h}{b}\left(\frac{E_{II}}{1+\nu_{II}}\right)K_I^2\left[3\alpha_{oI}\theta_I(t)-2(1+\nu_{II})\left(\frac{b^2}{b^2-a^2}\right)\alpha_{II}\theta_{II}(t)\right]$$

$$(7.4)$$

$$+\,18K_I\int_0^t\frac{\partial}{\partial t'}E_I(\xi-\xi')\left\{\left[9K_I(1-\nu_{II})-3\,\frac{h}{b}\right]\alpha_{oI}\theta_I(t')\right.$$

$$\left.-\,2(1+\nu_{II})\frac{h}{b}\left(\frac{E_{II}}{1+\nu_{II}}\right)\left(\frac{b^2}{b^2-a^2}\right)\alpha_{II}\theta_{II}(t')\right\}dt'$$

(Continued)

$$-3 \int_{0}^{t} \frac{\partial}{\partial t'} \left(\int_{0}^{t'} \frac{\partial}{\partial t''} \left(E_I(\xi'') \right) E_I(\xi - \xi' - \xi'') dt'' \right) \left\{ \left[6K_I (1+\nu_{II}) \right. \right. \qquad (7.4) \text{ continued}$$

$$\left. \left. -2 \frac{h}{b} \left(\frac{E_{II}}{1+\nu_{II}} \right) \right] \alpha_{oI} \theta_I (t') - 2(1+\nu_{II}) \frac{h}{b} \left(\frac{E_{II}}{1+\nu_{II}} \right) \left(\frac{b^2}{b^2 - a^2} \right) \alpha_{II} \theta_{II} (t') \right\} dt'$$

A solution to this equation may be obtained by employing the finite difference technique of the preceding section.

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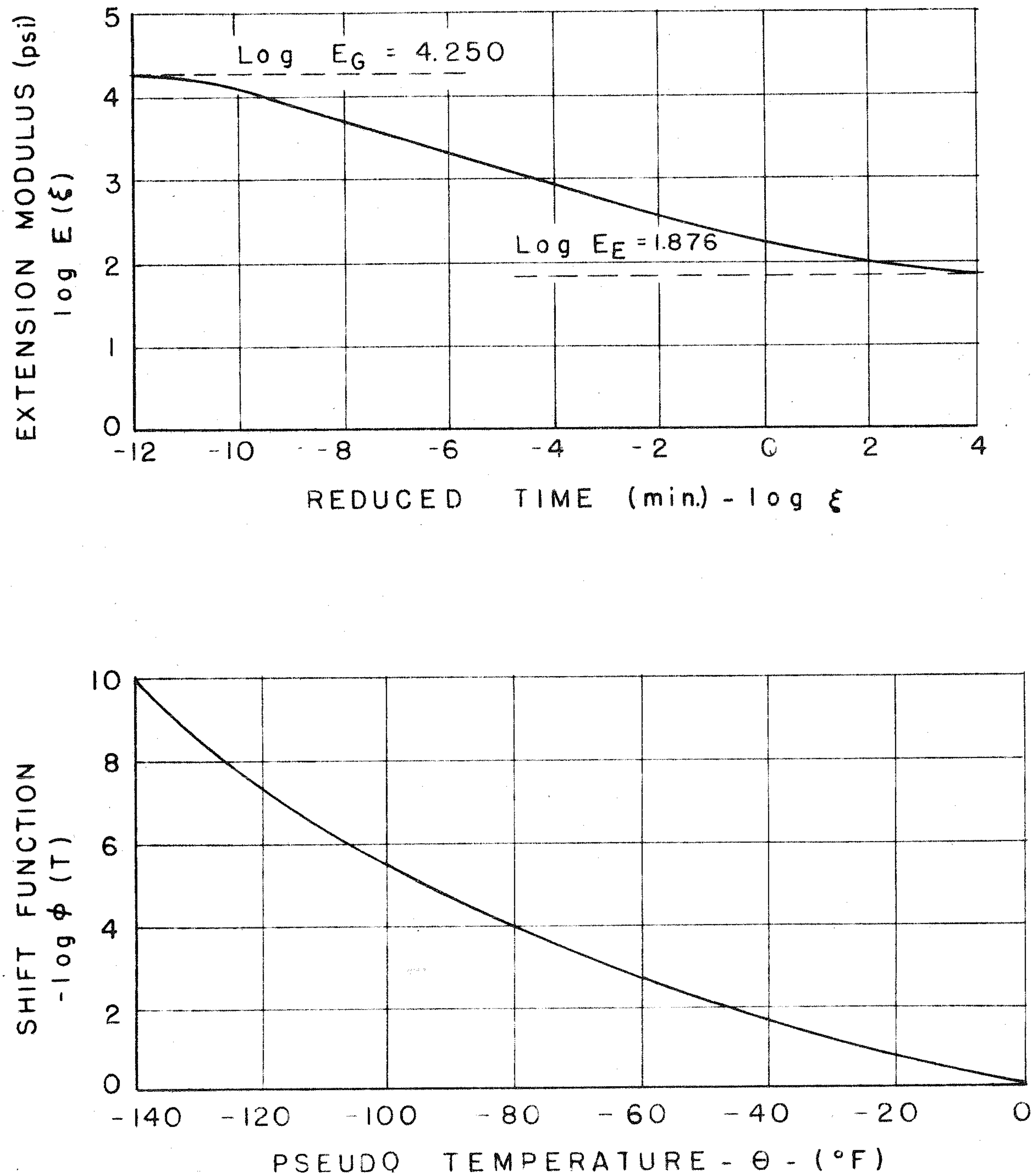

FIG. 1. - THERMOMECHANICAL TIME HISTORY FOR VISCOELASTIC MATERIAL OF EXAMPLE PROBLEMS, REFERENCE TEMPERATURE 77°F.

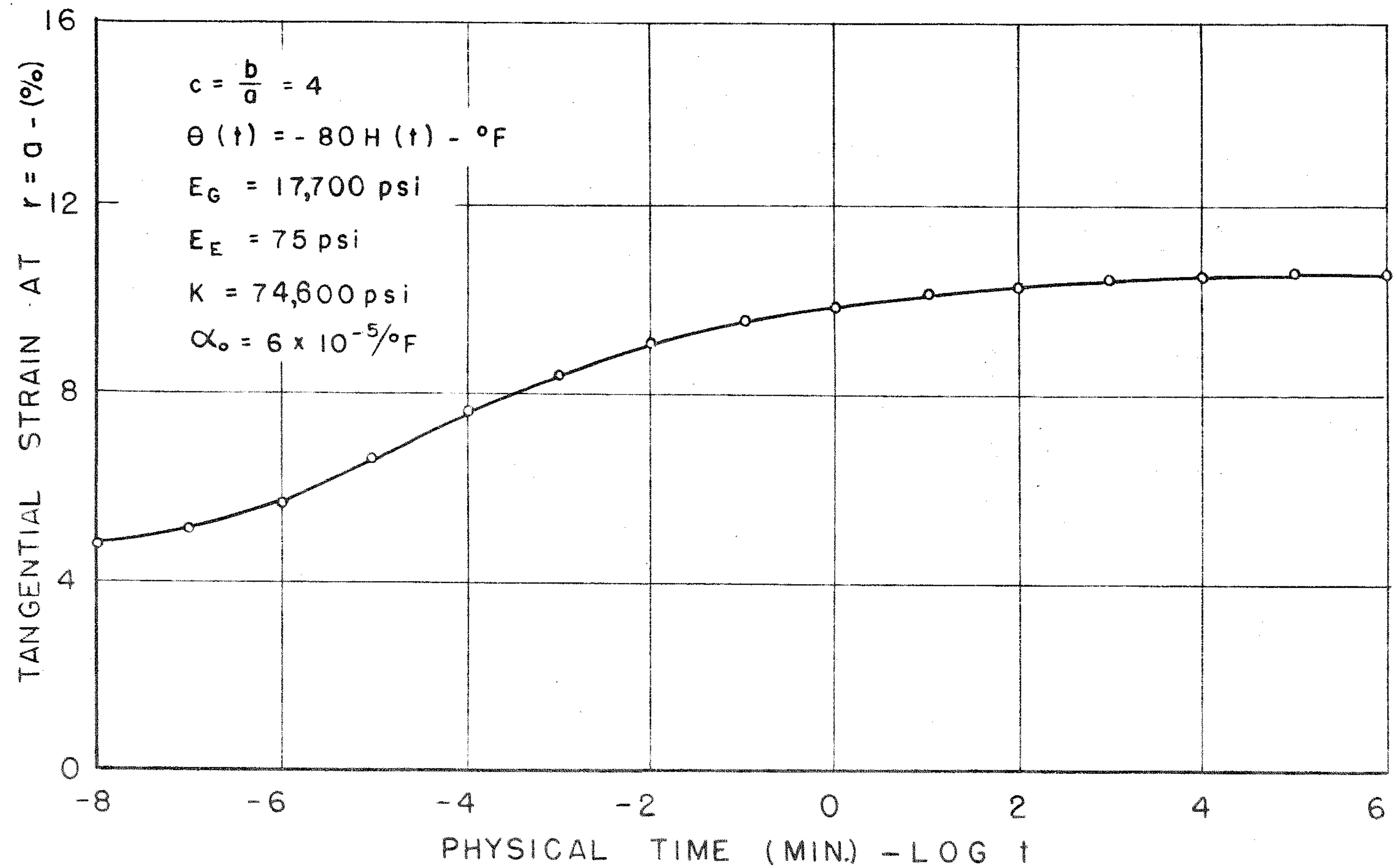

FIG. 2. TANGENTIAL STRAIN TIME HISTORY FOR A CYLINDER SUBJECTED TO A CONSTANT TEMPERATURE DECREASE AT $t = 0$

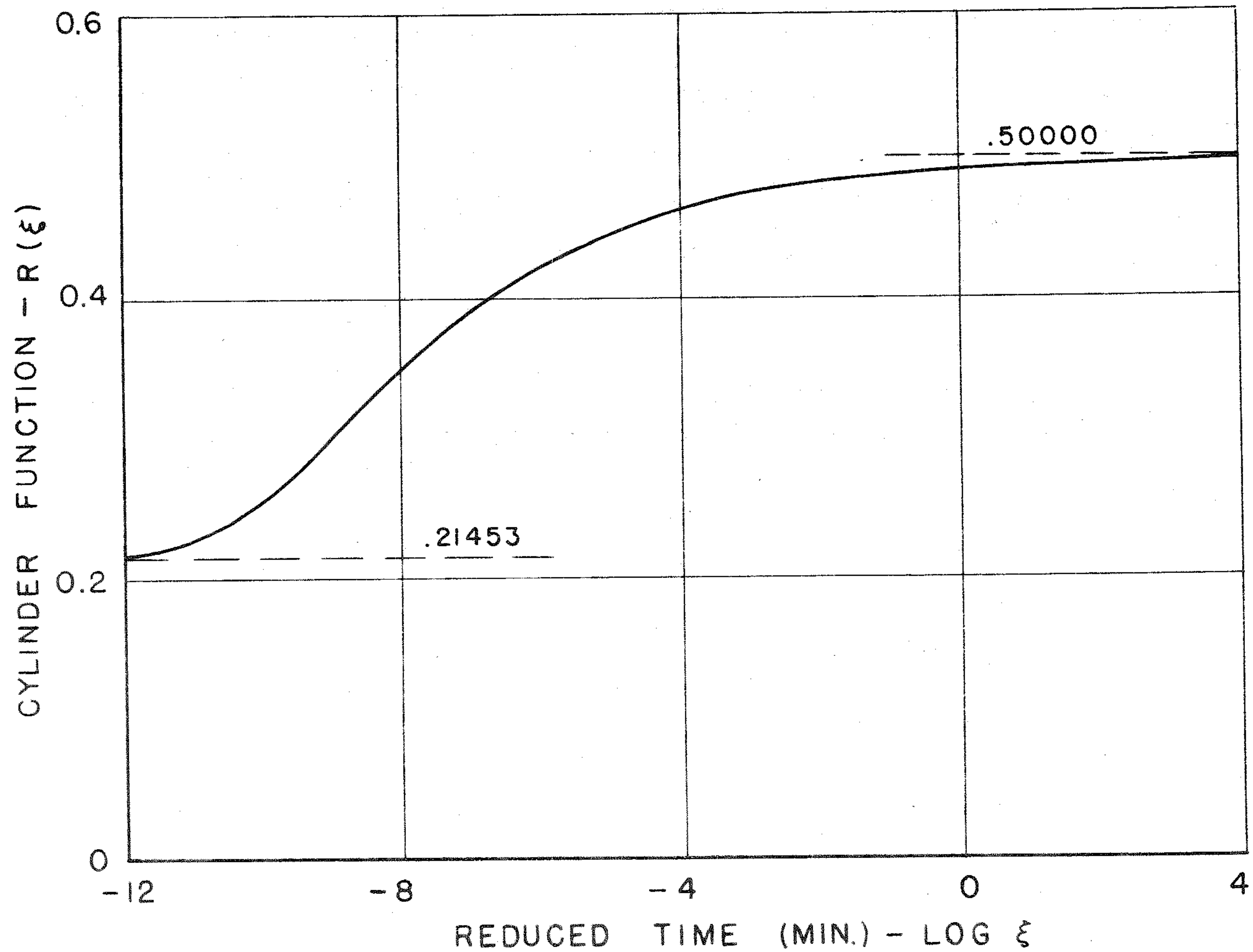

FIG. 3. CYLINDER FUNCTION FOR EQ. 5.12, c = 4

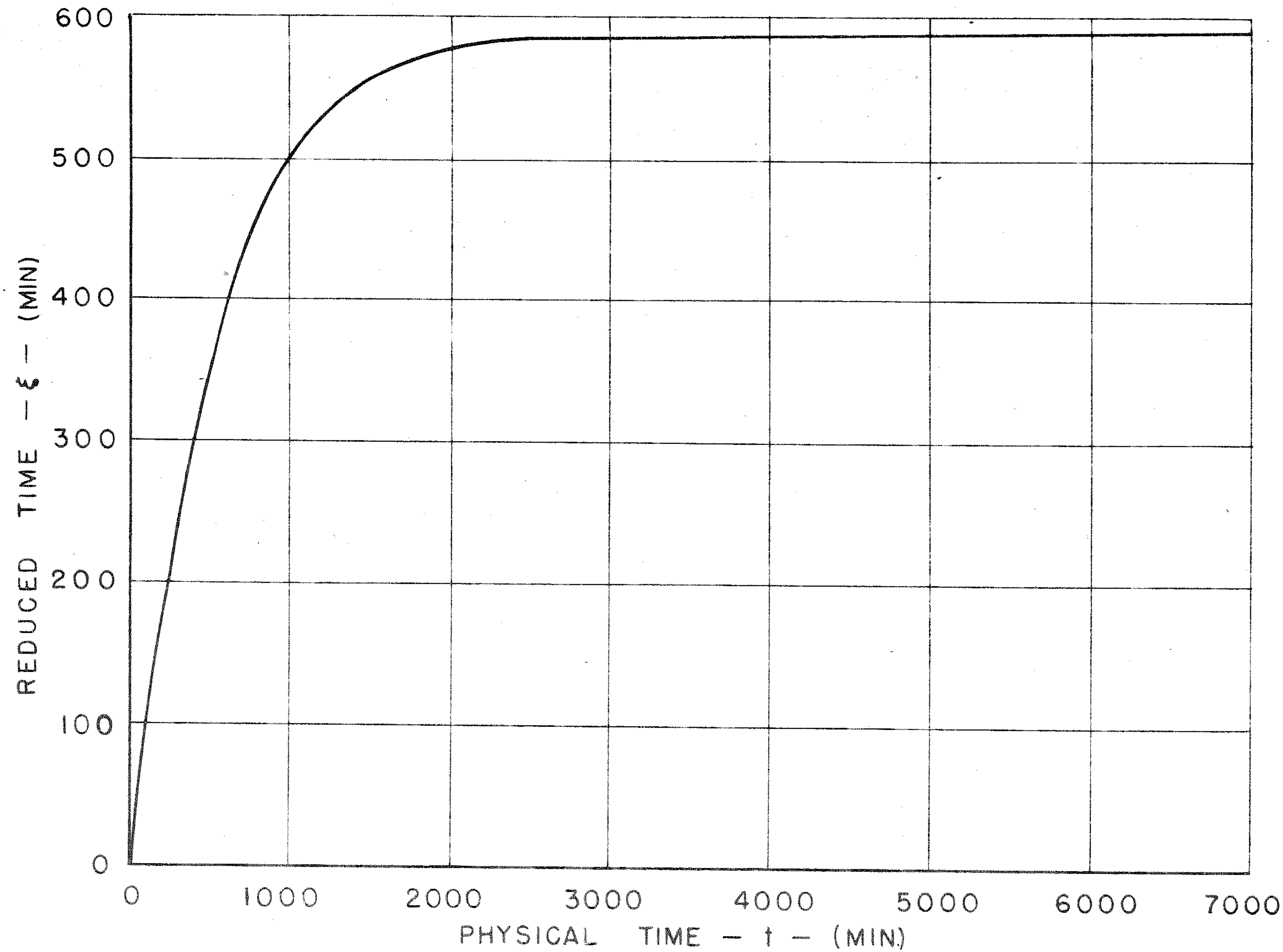

FIG 4. REDUCED TIME VS. PHYSICAL TIME FOR UNIFORM COOLING OF 2°F MIN AND SHIFT FUNCTION IN FIG. 1.

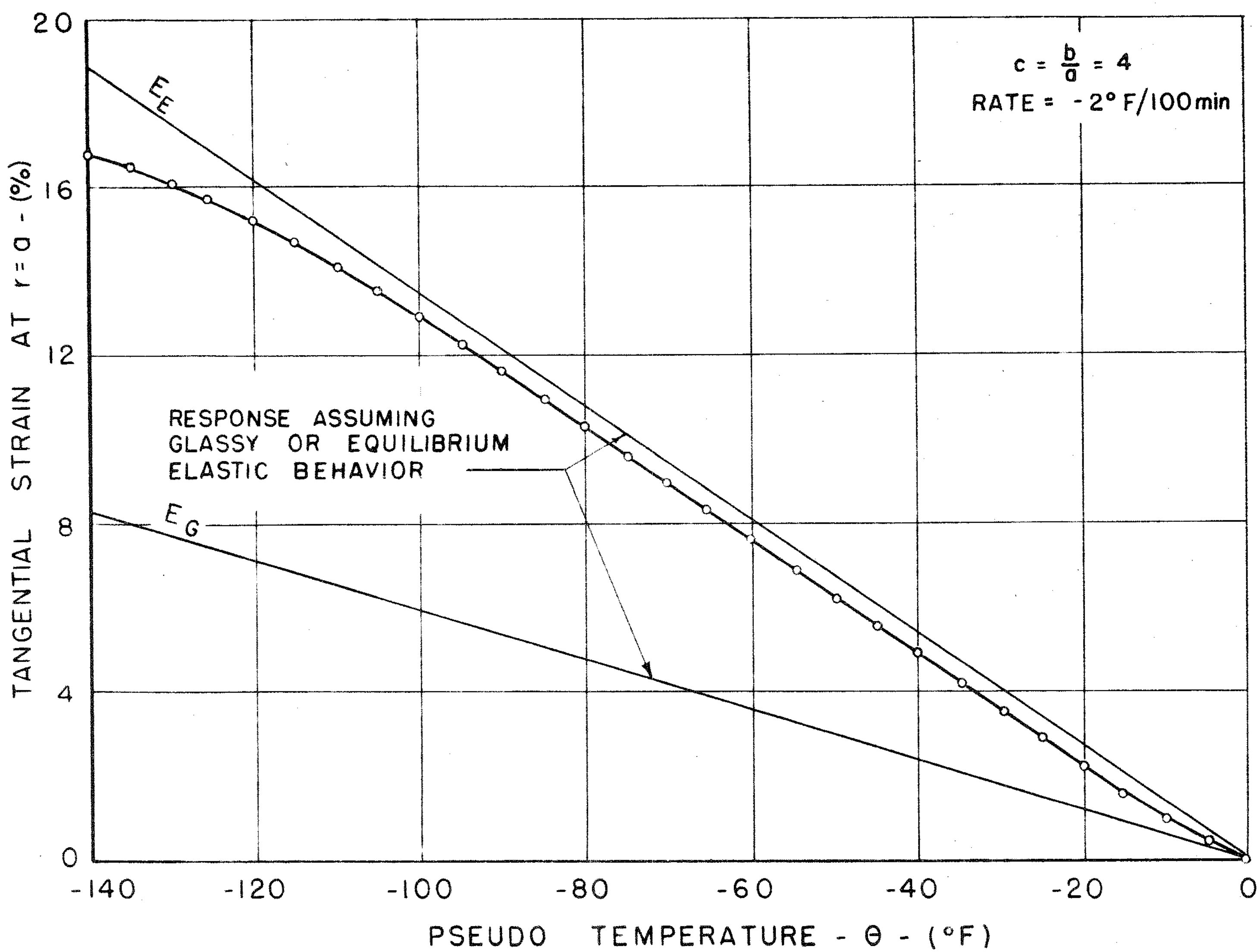

FIG. 5. — UNIFORM COOLING OF A THICK WALL CYLINDER