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# Children spontaneously discover efficient sorting algorithms in a seriation task

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## Abstract

Efficient algorithms can enhance problem-solving in many cognitive domains but can be difficult to discover and use. For example, classical studies of seriation suggest that children struggle to apply algorithmic strategies in a simple sorting problem. We investigate the spontaneous discovery of algorithmic solutions across development. We gave children a variant of the sorting problem with hidden object ranks. Children sort animated bunnies into the right order, from the shortest to the tallest, when the bunnies are standing behind a wall so their heights are not visible. Children performed far above chance on this difficult sorting task, potentially because higher demands in memory and reasoning incentivized strategic behaviors. Children independently discovered at least two efficient algorithmic solutions to the sorting problem, Selection sort and Shaker sort. This result suggests that children are far more competent at sorting tasks than previous research would suggest. Additionally, older children were more efficient sorters than younger children. This suggests that competent performance on sorting tasks improves throughout development.

**Keywords:** sorting; seriation; algorithm

## Introduction

Imagine that you are about to bake cookies. One approach would be for every individual cookie, you mix the ingredients, blend them, and bake. Another approach is that you could go through the same process only once with the entire batch. Both methods produce similar final results, but the latter one is more structured and systematic, thus making it a more efficient strategy. Strategies are special cases of procedural knowledge that can take the forms of algorithms (Alexander, Graham, & Harris, 1998; Chi, 2013). The acquisition and use of cognitive algorithms are important because structured forms of problem-solving can enhance performance at many tasks (Pressley et al., 1990; Bjorklund, 1990; Harris & Graham, 1996). Some example domains in which people often rely on algorithmically structured knowledge include memory strategies (Ornstein & Naus, 1985; Wellman, 1988; Baker-Ward, Ornstein, & Holden, 1984; Ornstein, Haden, & San Souci, 2008), mathematical cognition (Anderson, 1996; Braithwaite, Pyke, & Siegler, 2017; Fuson et al., 1997), and grammatical rules (Anderson, 1996; Byrnes, 1992).

Previous research has shown that children start to be strategic even when they are very young (Wellman, 1988; Wellman, Ritter, & Flavell, 1975), and they become increasingly capable of using more efficient strategies as they age (Gholson, 1980; Ornstein, Baker-Ward, & Naus, 1988; Baker-Ward et al., 1984; Siegler, 1998). Children are also very competent in choosing adaptively among various algorithms, depending on specific constraints or demands of the tasks (Shrager & Siegler, 1998; Siegler, 1987; Crowley, Shrager, & Siegler, 1997).

The goal of this paper is to investigate the spontaneous discovery of structured algorithmic strategies by children. In particular, we examined the algorithmic structure in children's behaviors when solving a difficult sorting task. Sorting is considered a fundamental problem in computer science. It is a rich and interesting problem because it requires the use of strategies for executing an appropriate sequence of actions to achieve the correct final ordering of objects (Cormen, Leiserson, Rivest, & Stein, 2009). Sorting is the basic building block of many mathematical and programming problems. Therefore, its solutions have been studied extensively, and various algorithms with different spatial and temporal efficiency can be used to solve a sorting task (Cormen et al., 2009).

Our sorting task tests a skill Piaget called seriation. The Piagetian version of the task studies the behaviors of children by asking them to arrange a disordered set of sticks of different lengths into the correct order (Piaget, 1965; Piaget & Inhelder, 1969). Since then, similar tasks have been used with children to show developmental changes in seriation abilities (McGonigle-Chalmers & Kusel, 2019; Chapman, 1988; Kingma, 1986; Kingma & Reuvekamp, 1984). Children go through predictable stages before they are fully capable of efficient seriation. They start with chaotic and trial-and-error strategies, and move towards systematic, efficient ones. Seriation is an important skill for children to master because it is crucial for the development of early math skills (Schminke, Maertens, & Arnold, 1973). For instance, it is theorized to be the foundation of the comprehension of relationships between numbers (Ginsburg, 1977) and is predictive of the comprehension of the number line (Kingma & Reuvekamp, 1984). Therefore, learning accurate and efficient algorithms to perform the seriation task is closely linked to the development of more general numerical abilities and mathematics cognition.

However, traditional studies of seriation examine a version of the sorting problem that is limited in an important way. Specifically, classic seriation tasks allow participants to directly observe the ranks of the items to be sorted (e.g., the heights of the sticks). Direct visual access reduces the intense memory and reasoning demands that would otherwise motivate the application of a structured algorithmic solution.

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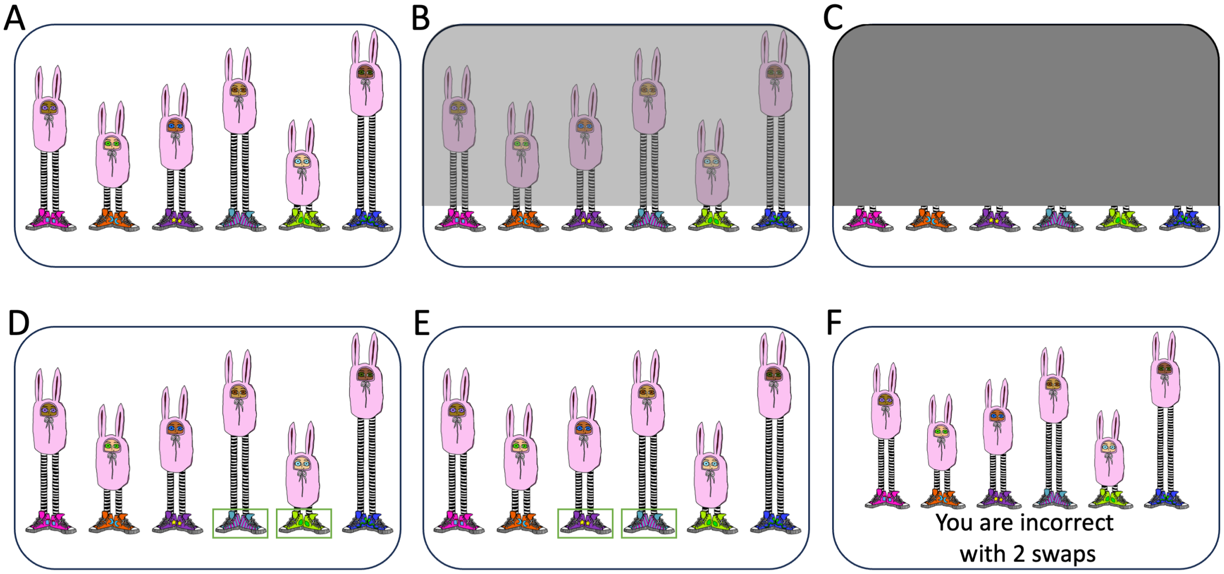

Figure 1: Study design. (**A**) Practice trial 1: heights fully visible. (**B**) Practice trial 2: heights partially visible. (**C**) Practice trial 3 and test trials: heights not visible. (**D**) An example of a comparison that resulted in a swap. (**E**) An example of a comparison that resulted in a non-swap. (**F**) Trial summary feedback page.

As a result, classic tasks may underestimate children's capacity to apply algorithmic reasoning. On the other hand, more complex tasks might reveal limits to children's capabilities that have also been underexplored by previous tasks.

Here, we employ a seriation task that requires children to sort objects with unobservable object ranks. This version of the sorting task increases the demands on memory and reasoning, and therefore more strongly motives the application of a structured solution. Children were asked to arrange an out-of-order set of objects with hidden heights. The objects don't have salient visual features that signal their ordering to the children, contrary to a standard Piagetian task in which children can visually differentiate a long stick from a short one. Therefore, this task has a high demand on many domain-general cognitive abilities of children, including working memory capacity and their ability to make inferences based on specific constraints and circumstances.

Our results show that children are able to sort objects, even when the ranks of the objects are visually hidden and must be inferred. We also show that while older children can sort more accurately and efficiently, using at least two structured algorithms, the age at which children begin to use identifiable algorithms is younger than Piaget suggested—as young as 4 years old.

## Methods

### Participants

A total of 103 children from the ages of 4 to 9 participated in this study in a museum or a lab setting. 22 participants were removed from the analysis because they failed to complete the experiment. The analysis is done on the remaining 81 participants (Mean age = 7.02, SD = 1.57).

### Design

**Task** We designed a sorting game that children can play on a touchscreen computer. The game contains 3 practice trials and 6 test trials. Figure 1 shows our experimental design. In every trial, children see 6 animated bunnies. The bunnies have different heights, and the objective is to sort them into the correct order, with the shortest bunny on the left and the tallest bunny on the right. In addition to having different heights, the bunnies all wear different colored shoes. The initial positions of the bunnies and the colors of their shoes were randomly generated for every trial.

During each trial, children performed a sequence of comparisons until they were satisfied with the ordering. In each selection, they choose two bunnies, and the two selected bunnies will perform a pairwise comparison. If the bunnies are out of relative rank order (i.e. the leftmost selected bunny is taller than the rightmost selected bunny), they will change places, resulting in a swap, as shown in Figure 1D. If the selected bunnies are already in the right relative order, as shown in Figure 1E, with the shorter one on the left, and the taller one on the right, then their positions will not change, resulting in a non-swap. The number of swaps used on a given trial is the sum of swaps and non-swaps.

**Procedure** In the first practice trial, children could see the bunnies, as shown in Figure 1A. They advanced to the next practice trial only if they successfully put the bunnies in the right order. However, children received no instructions on which bunnies they should choose to compare in order to achieve the goal. In the second practice trial, depicted in Figure 1B, children could see the bunnies standing behind a gray glass window. They could see the different colored shoes fully, but they could not see the height of the bunnies clearly. The purpose of these two practice trials is to let children get familiar with the demand and the goal of a sorting task. The second practice trial also prepares them for the following trials, in which they cannot see the height of the bunnies at all. In the final practice trial and all the test trials, as shown in Figure 1C, children were not able to see the bunnies' different heights. They were told that these bunnies were standing behind a gray curtain and that they would only see the bunnies' shoes. At the end of each trial, children verbally signaled to the experimenter when they thought the bunnies were in the right order, and they received a summary of whether the bunnies were in the correct final order and the number of swaps they had performed in the trial.

## Results

### Children are able to sort objects, even when the ordering of the objects is hidden

Our results show that all children were able to sort the bunnies in the practice trials in which they were visible (practice trials 0 and 1). This indicates that children, as young as 4 years old, have the ability to sort. Our result is contrary to conventional beliefs about children's seriation abilities, which suggest that children cannot perform seriation until 7 years old (Piaget, 1965; Piaget & Inhelder, 1969; Flavell, 1963). Figure 2A shows the average percentage accuracy of all participants in each test trial. In this figure, percentage accuracy is defined as, out of all the testing trials that were included in the analysis, the percentage of trials that have the correct final ordering. We exclude a trial if it contains less than 5 swaps, or if that trial contains more than 3 standard deviations from the mean (46) number of swaps. A substantial portion of children performed the sorting task correctly even when the heights of the bunnies were not visible (35.7% accuracy on average), in all testing trials. Our results in children's accuracy are consistent with the idea that the high demands in memory and reasoning of our sorting tasks motivated children to discover and use more structured and efficient algorithmic solutions.

### Older children are more accurate sorters

Older children performed better on our sorting task: their accuracy across 6 test trials is higher than that of younger children. Figure 2B shows children's percentage accuracy as a function of age. As it shows, the percentage of trials

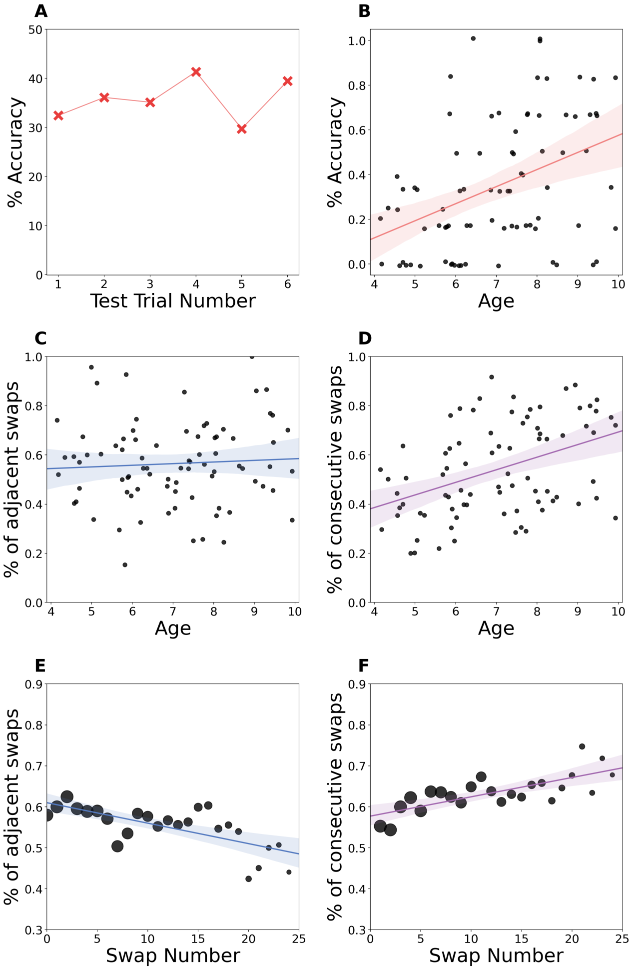

Figure 2: Results. (**A**) Percentage accuracy for all test trials. A significant portion of participants performed the sorting task accurately, even when the order of the objects was hidden. (**B**) Percentage accuracy as a function of age. The positive slope of our fit line shows that older children have higher accuracy. (**C**) Percentage of swaps that are adjacent swaps as a function of age. The flat trend indicates that older and younger children used a similar percentage of adjacent swaps. (**D**) Percentage of swaps that are consecutive swaps as a function of age. The positive trend demonstrates that older children used a greater percentage of consecutive swaps. (**E**) Percentage of swaps that are adjacent swaps as a function of swap number. The negative trend demonstrates that the deeper participants get into a trial, the less likely they are to use adjacent swaps. (**F**) Percentage of swaps that are consecutive swaps as a function of swap number. The positive trend demonstrates that the deeper participants get into a trial, the more likely they are to use consecutive swaps.

that older children performed correctly is significantly greater than that of younger children (β = 0.076, 95% CI = [0.039, 0.11], $p < 0.001$). This result is consistent with the theory that since our sorting task demands many domain-general cognitive abilities from participants, older children perform better than younger children because they have higher memory and reasoning capacities.

It is worth noting that older children were not using more swaps than younger children (β = -0.68, 95% CI = [-2.08, 0.71], p = 0.336). One possible explanation is that older children's swapping behaviors might be more efficient than that of younger children, and this allows them to have a higher accuracy while not using more swaps.

## Older children's sorting behavior shifts from spatial proximity to temporal proximity

To facilitate a greater understanding of the developmental results, we analyzed the behavioral structure of participants' responses. We looked at two types of swaps participants can perform: 1) adjacent swaps, and 2) consecutive swaps. An adjacent swap is when participants select two adjacent positions in a swap. A consecutive swap is when participants select a position that they also selected in the swap immediately prior. Both types of swaps are essential building blocks of many identifiable sorting algorithms. For instance, if a participant is using Shaker sort (details below), then they will frequently use adjacent swaps and if a participant is using Selection sort, they will frequently use consecutive swaps and adjacent swaps.

These two types of swaps, however, are different in their demands on participants' cognitive abilities. To perform adjacent swaps, participants don't need to remember which positions they selected in the previous swap. However, to perform consecutive swaps, participants have to remember at least one selected position from the previous swap to select the same position again. Therefore, consecutive swaps have a higher demand for participants' abilities to memorize and track what has been selected, or to use a strategy that encodes this regularity. Consecutive swaps are also more demanding because they require participants to click on two bunnies that are not necessarily close to each other. Furthermore, consecutive swaps demand a higher level of conceptual understanding of object relations in the sense that for participants to select two bunnies that are not adjacent, they would have to understand that two objects that are not placed immediately next to each other may also be related and have properties that can be compared.

We performed a generalized linear regression analysis using a binomial link function with random subject effects to assess the effect of age on the types of swaps participants used. The results show that while age does not have a significant effect on the percentage of adjacent swaps participants used (β = 0.036, 95% CI = [-0.80, 0.88], p = 0.543; Figure 2 C), older children are more likely to perform consecutive swaps (β = 0.23, 95% CI = [0.12, 0.34], $p < 0.001$; Figure 2 D). Our results provide supporting evidence for the theory that consecutive swaps require increased cognitive resources.

We also performed an additional set of generalized linear regression analyses using a binomial link function with random subject effects to assess whether being further into the experiment or a trial influenced the types of swaps participants used. The results show that children were less likely to perform adjacent swaps when they were further into a trial (β = -0.13, 95% CI = [-0.18, -0.07], $p < 0.001$; Figure 2 E). In contrast, children were more likely to perform consecutive swaps when they were further into a trial (β = 0.55, 95% CI = [0.28, 0.41], $p < 0.001$; Figure 2 F). Our results show that the more experience and practice children have, the more likely they realize the flexibility and utility of consecutive swaps and start incorporating them into their sorting strategies. As a result, children who heavily relied on adjacent swaps gradually transitioned to increase their use of consecutive swaps, which may contribute to the increase in efficiency that we see in Figure 2B.

Together, these results point to behavioral differences between more efficient sorters and less efficient sorters. They also offer a possible explanation as to why older children were more efficient, and why children's efficiency increased during the experiment.

## Children discover and use various sorting algorithms

We analyzed the algorithmic structure of the strategies children used to understand these behavioral differences in more detail. We performed a pattern-matching analysis on the sequences of swaps that they performed in each trial to identify specific algorithms. As Thompson, Van Opheusden, Sumers, and Griffiths (2022) demonstrated, participants who use different algorithms exhibit different identifiable signature behavioral patterns. Children appeared to engage more frequently with exploratory behaviors instead of following a specific algorithm precisely as compared to adults. Therefore, it is challenging to align their full sequences of swaps with a specific algorithm without substantial errors in classification. Even when children are using algorithms in problem-solving, they also frequently deviate from the algorithms that they are using, producing incomplete implementations of strategies. We identified key signature patterns for the two sorting algorithms that children discovered to overcome this difficulty in data analysis. We classify a participant as using a particular algorithm if the participant produced a sequence of swaps that contains the signature pattern anywhere throughout the trial.

The two most common algorithms were Selection Sort (also the most common algorithm used by adults) and Shaker Sort (less common among adults, but attested). Figure 3 shows the signature sequences of swaps generated by using these two algorithms. Selection sort generates a fixed sequence of 15 swaps that are guaranteed to establish the correct order. It can also be separated into forward Selection Sort and backward Selection Sort, depending on whether the participant starts from the left or the right. The left panel of Figure 3A shows an example of the sequence of swaps that would

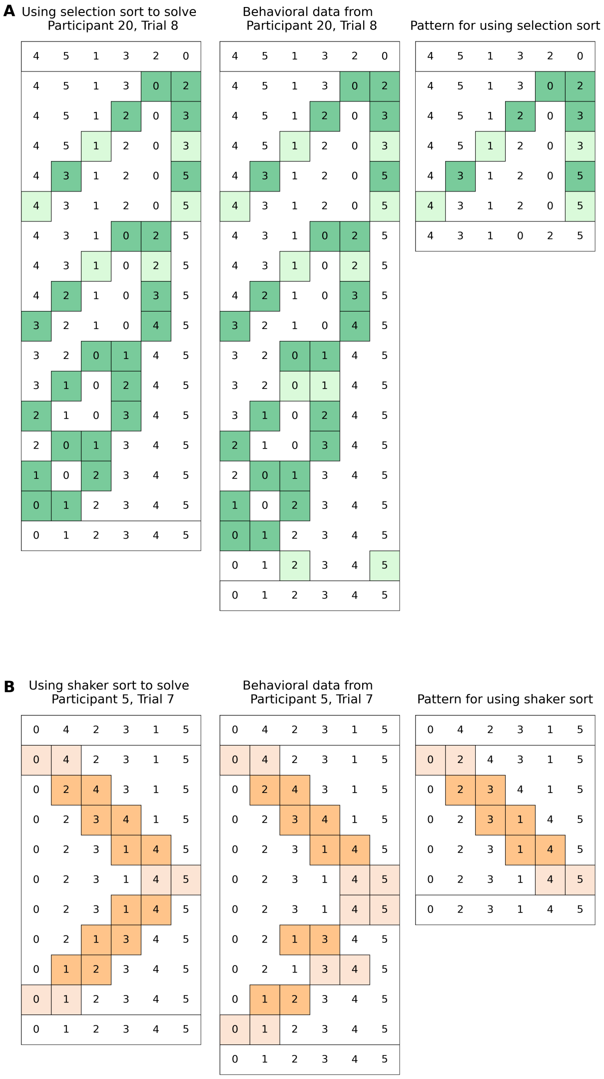

Figure 3: Sorting algorithms and the pattern we used to identify them. (**A**) Selection sort and the pattern we use to identify it. Left panel: sequences of swaps generated by using the Selection sort. Middle panel: behavioral pattern generated by a participant using the Selection sort. Right panel: The pattern we used to identify the use of Selection sort. (**B**) Shaker sort and the pattern we use to identify it. Left panel: sequences of swaps generated by using the Shaker sort. Middle panel: behavioral pattern generated by a participant using the Shaker sort. Right panel: The pattern we used to identify the use of Shaker sort.

be generated if a participant was using the selection sort algorithm. In this figure, the top row represents the initial ordering of the bunnies, such that 0 represents the shortest bunny and 6 represents the tallest bunny. The bottom row represents the final ordering. Every row represents a swap, with two selected positions shaded. Darker positions indicate a swap and lightly shaded positions indicate a non-swap. The numbers of every row represent the ordering of the bunnies after each swap. The middle panel of Figure 3A shows behavioral patterns generated by a participant when they faced a trial with the same initial orderings as shown in the left panel. Consistent with our expectation, children's actual behavior shows redundant swap attempts and deviations from the most efficient implementation of the algorithm, but nonetheless clear signs of algorithmic structure. The pattern we used to identify the use of Selection Sort is shown in the right panel of Figure 3A.

Shaker Sort is archived by selecting two consecutive bunnies at a time and moving the selection from left to right, and then from right to left, until the participant arrives at the correct ordering (imagine shaking a horizontal bottle, side to side). It can also be separated into forward Shaker Sort and backward Shaker Sort, depending on which side the participant started at. Shaker Sort also guarantees accuracy if applied for enough iterations (shakes!). However, the number of iterations required does depend on the specific initial ordering of the bunnies. Figure 3B shows an example of using the Shaker Sort algorithm with maximum efficiency, as well as behavioral results from a participant using this algorithm. The pattern we used to identify the use of shaker sort is shown in the right panel of Figure 3B.

Our pattern-matching analysis shows that for a significant percentage of trials (28.1%), participants performed a sequence of swaps that were consistent with them using at least 1 identifiable sorting algorithm. Out of 445 trials, there are 50 trials in which participants' behavior aligns with the pattern of using Selection Sort; there are 85 trials in which participants' behavior aligns with the pattern of using Shaker Sort; there are 10 trials in which participants' behavior aligns with the pattern of using both algorithms.

Our results indicate that not only do children perform well at this more challenging version of a sorting task, but they also independently discover and apply at least two efficient sorting algorithms.

## Conclusions and Discussion

These results demonstrate that young children are able to perform well at a sorting task, even when the ordering of the objects is hidden from view. Accuracy improved with children's age. Our analysis also provides supporting evidence that the improvement in sorting performance is potentially in part due to increased application of a type of swap (consecutive swaps) that allows them to sort more efficiently. Lastly, we show that a significant portion of children used at least one identifiable and efficient sorting algorithm.

Our experiments show that children have demonstrated surprisingly high competence in these sorting tasks. In a similar study done by Thompson et al. (2022), adult participants were asked to sort 6 images with hidden ranks. Participants in the asocial condition, in which they received no guiding information on efficient sorting algorithms, yielded a 64.5% average accuracy on these sorting tasks. Children in our experiment produced an average accuracy of 35.7%. These results support the theory that children start to discover and use efficient algorithms from a very young age. While Piaget (1965) and Piaget and Inhelder (1969) pointed out that children develop the ability to spontaneously seriate around the age of 7, our results show that spontaneous systematic sorting behaviors can be observed in children as young as 4 years old. Our pattern-matching analysis also provides a possible explanation.

We found that children can spontaneously discover and use systematic algorithms far younger than Piaget and subsequent psychologists thought. These algorithm uses potentially went undetected because they don't always yield correct performance. For example, some children showed a tendency to use Selection Sort but deviated from the algorithm during implementation, resulting in an error in the final ordering. Another example would be children using the Shaker Sort algorithm but only making one pass, resulting in the final ordering sometimes being correct and sometimes incorrect, depending on the initial ordering of the bunnies. Multiple domain-general cognitive abilities, such as memory and motor capacities, might influence whether children are able to implement sorting algorithms and achieve the correct final ordering.

We note that our pattern-matching analysis allowed us to identify participants who used identifiable algorithms, even when they may not have produced the correct final ordering from a complete implementation of the algorithm. Our motivation to apply a flexible classifier was that in order for a participant to produce the signature pattern to the degree of accuracy required by our analysis, the participant must have a working understanding of the structure of the algorithm and the sequences of procedures it implies.

Previous research has shown that infants can seriate only under particular circumstances and only after they have observed a demonstration (Calvert, Richards, & Kent, 2014; Fragaszy, Galloway, Johnson-Pynn, & Brakke, 2002; Greenfield, Nelson, & Saltzman, 1972). Studies have also shown that young children's seriation performances can be improved by receiving training (Coxford, 1964; Kidd, Pasnak, Gadzichowski, Ferral-Like, & Gallington, 2008; Kingma, 1987). However, these training methods all involve letting children observe the proper procedure of putting a set of objects in order, and they work best when children can already seriate, to some degree, before the training.

Our experiment makes a unique contribution in showing that children can spontaneously discover and use efficient sorting algorithms, without receiving guidance or observing demonstrations of these sorting algorithms. However, children's learning of algorithms needs to be incentivized by particular circumstances. For instance, our sorting task adds cognitive demands that require children to have a conceptual understanding of the task by removing visual access to the ranks of objects. This characteristic might motivate children to use more structured sorting behaviors that reduce cognitive demands, and thus promote the discovery of algorithmic sorting solutions. Since seriation is a fundamental building block for many early math abilities and seriation performance is predictive of the development of future math skills, our work has important implications for educational interventions that aim to promote young children's math abilities.

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