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ON EFFICIENT PROVISION OF SOCIAL OVERHEAD GOODS*

Introduction

My colleague, Trout Rader occasionally remarks "Consumer efficiency requires that price equals marginal cost. But what is marginal cost?"[1] This question focuses the problem recognized by Harold Hotelling [6] that although under rather general circumstances, (see T. Rader [9]) marginal cost pricing is necessary for Pareto optimal resource allocation, it is not in general sufficient. If, for example, marginal costs are constant with respect to output for each industry but depend on the amount of "set-up costs" incurred, the "price equals marginal cost" rule provides no guidance for choosing the optimal amount of set-up costs.

In this paper we consider a partially decentralized economy in which the supply of certain commodities called "social overhead goods" is determined by a collective authority. All other commodities are produced by private firms. It will be assumed that for any fixed supply of social overhead goods, the production possibility sets for the private firms are convex. It will also be assumed that consumers have convex preferences. The private sector will operate competitively, treating the supply of social overhead goods as an environmental parameter. For this model we show that there is a sense in which we can paraphrase Rader's remark as follows: "For the private sector, every Pareto optimum is a 'competitive equilibrium' but equilibrium is Pareto optimal only if the central authority does the right thing. But how does the central authority choose the right thing to do?" We will explore some possible methods for aiding the central authority in its choice of activities.

* This research was partially funded by NSF Research Grant GS 3070. The author thanks Professors Trout Rader, James Barr, and Robert Parks of Washington University for helpful comments. All errors have been deliberately selected from previous works of Professor Rader.

1 It is only fair to acknowledge that remarks such as this do not constitute Professor Rader's entire repertoire of cocktail repartee.

I. A Model of a Partially Decentralized Economy

A. The Technology

There are finite sets, J of ordinary commodities, K of social overhead goods, and M of firms consisting of $\bar{\jmath}$, $\bar{k}$ and $\bar{m}$ elements respectively. Production possibilities for the central authority are described by an input correspondence L from $E_{\bar{k}}^{+}$ (the non-negative orthant in Euclidean $\bar{k}$ space) to $E_{\bar{\jmath}}^{+}$ such that for any $a \in E_{\bar{k}}^{+}$, $L(a)$ is interpreted as the set of all input vectors of ordinary commodities which can be used to produce the vector a of social overhead goods. Let $A \equiv \{a | a \in E_{\bar{k}}^{+}$ and $L(a) \neq \phi\}$. For each firm, $m \in M$, there is a correspondence Y_m from A to the set of subsets of $E_{\bar{\jmath}}$ such that for $a \in E_{\bar{k}}^{+}$, $Y_m(a)$ is the production set of firm m when the vector of social overhead goods is a. (We adopt the convention of G. Debreu [4] in interpreting the production set of a firm as the set of technically feasible input-output vectors for that firm with positive elements denoting outputs and negative elements denoting inputs.) We also specify a vector $w \in E_{\bar{\jmath}}^{+}$ of total initial holdings for the economy. In our subsequent analysis, we will wish to assume that the sets $Y_m(a)$ and $L(a)$ are convex sets for any fixed $a \in A$. This need not, however, imply that the set of feasible aggregate outputs is convex.

Perhaps the most appealing examples of a conceivable "real world technology" which would fit the description of this model would be a world in which production of goods and services is subject to non-increasing returns and quasi-concavity for any given state of knowledge. The social overhead goods are quantities of knowledge of various kinds. There is a knowledge producing sector which is financed by taxation and which makes its output freely accessible to all firms. The model could also be interpreted to encompass the railroad example or the bridge example of Hotelling if, for a given stock of track or a given size of bridge, the maintenance and operation of the railroad or the bridge is subject to constant or increasing costs. In these cases, the railroad track or the bridge would be social overhead goods. The services of the railroad or the bridge would be ordinary commodities. Notice that there is nothing in the structure of the model that excludes the possibility that some social overhead goods be useful to only one firm.

For any real world situation, there may be a number of possible ways of choosing a set of social overhead goods so that for fixed quantities of social overhead goods, the production sets of firms are convex in the space of ordinary commodities. For example if a firm has a quasi-concave production function with increasing returns when all inputs are varied proportionately, it may be possible to designate any of several subsets of its inputs as social overhead goods in such a way that the production function is concave in the remaining goods. In this paper we will not consider the question of the optimal membership of the set of social overhead goods, but only the problem of efficient allocation once such a set is chosen. Perhaps some light will be thrown on the former question from the study of the latter.

We present a simple example which may be helpful to illustrate the nature of the technologies considered.

Example 1. There are two commodities produced for final demand. There is one factor of production, "labor", available in fixed supply. (We will represent quantities of the first two commodities by the first two components of a vector in E_3 and quantities of labor by the third component.) The vector of initial holdings for the economy is $(0, 0, l)$ where $l > 0$. There are two social overhead goods, quantities of which are represented by the vector (a_1, a_2). The input requirements for producing either social overhead good will be one unit of labor per unit of social overhead good. Thus

$$L(a_1, a_2) = (0, 0, a_1 + a_2) \quad \text{for all} \quad (a_1, a_2) \geqslant 0.$$

There is a single firm producing each of the ordinary goods. Where (a_1, a_2) is fixed, each firm produces subject to constant returns to scale using labor as its only input. We suppose that for $i = 1, 2$, the labor requirements per unit of output of good i will be $c_i a_i^{-\beta}$ (where $c_i > 0$ and $\beta > 0$ are parameters).

It is clear from the linear structure of this technology that where a_1 and a_2 are fixed, the set of feasible outputs of commodities 1 and 2 will be

$$\{(y_1, y_2) \in E_2^+ | [c_1 a_1^{-\beta}] y_1 + [c_2 a_2^{-\beta}] y_2 \leqslant l - a_1 - a_2\}.$$

In general this will be a triangular region like the shaded portion of Figure 1. But for alternative values of (a_1, a_2) there will be different triangular regions. The set of *all* feasible outputs of commodities 1 and 2 will be bounded on the northeast by the envelope curve generated by the family of linear equations $[c_1 a_1^{-\beta}] y_1 + [c_2 a_2^{-\beta}] y_2 = l - a_1 - a_2$ where $(a_1, a_2) \geqslant 0$. We can solve for this envelope curve. The curve is described by the equation

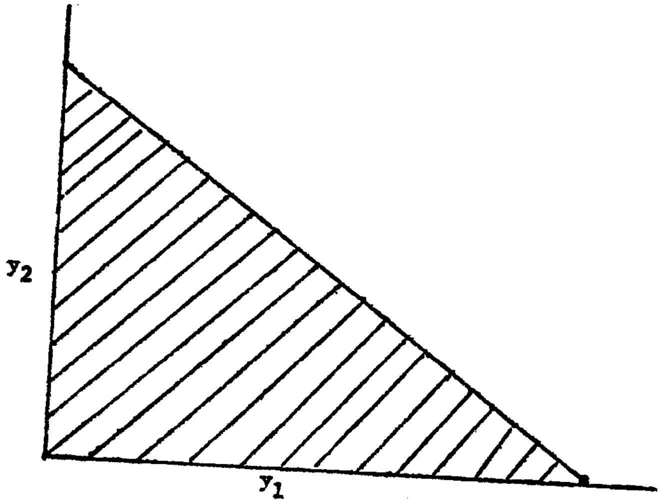

Figure 1

$$l = [c_1{}^\alpha y_1{}^\alpha + c_2{}^\alpha y_2{}^\alpha][\beta^{-\beta\alpha} + \beta^\alpha]$$

where

$$\alpha = \frac{1}{\beta + 1}, \quad y_1 \geqslant 0 \quad \text{and} \quad y_2 \geqslant 0.$$

The set of feasible outputs of the two consumer goods is represented by the shaded area in Figure 2. Note that this set is *not* convex.

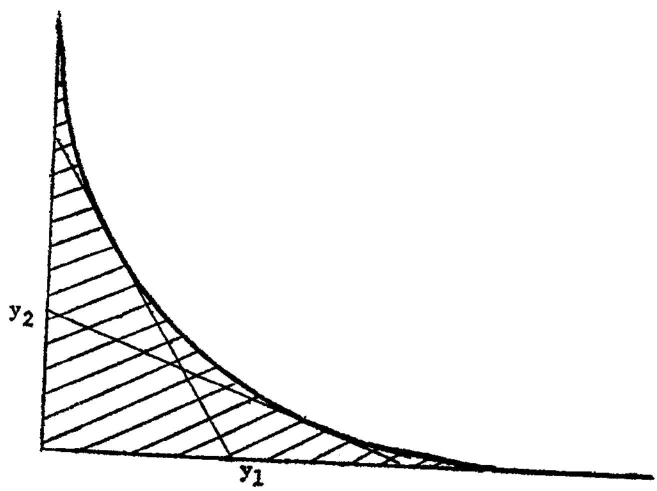

Figure 2

B. Consumers and Feasible Resource Allocations

There is a set N consisting of $\bar{n}$ consumers. Each consumer n has a preference relation $R_n \subset E_j^+ \times E_j^+$, and a vector of initial holdings w^n. Also, $\sum_N w^n = w$ where w is the total initial resource holdings in the economy. A consumer allocation is a vector $x = (x^1, \ldots x^{\bar{n}}) \in E_{j\bar{n}}^+$ where for $n \in N$, $x^n \in E_j$ is interpreted as the consumption vector of consumer n.

For $a \in A$, define the set $V(a) \equiv \{(x, y, z) | z \in L(a), y \in \pi_M Y_m(a), x \in E_{j\bar{n}}^+ \text{ and } \sum_N x^n = \sum_M y^m - z + w\}$. Define the set $V \equiv \bigcup_{a \in A} V(a)$. We call $V(a)$ the set of all resource allocations consistent with a, and V the set of all feasible resource allocations. Let $X(a) \equiv \{x |$ for some $y, z, (x, y, z) \in V(a)\}$ and let $X \equiv \bigcup_{a \in A} X(a)$. We can think of $X(a)$ as the set of all consumer allocations consistent with a, while X is the set of all feasible consumer allocations. If for every $a \in A$, the sets $Y_m(a)$ and $L(a)$ are all convex then for every $a \in A$, the sets $V(a)$ and $X(a)$ will be convex. But, in general the sets V and X need not be convex.[2]

C. Optimality and Partially Decentralized Market Equilibrium

We define the Pareto ordering R on $E_{\bar{n}j}^+ \times E_{\bar{n}j}^+$ so that $xR\hat{x}$ if and only if $x^n R_n \hat{x}^n$ for all $n \in N$. Also $xP\hat{x}$ if and only if $xR\hat{x}$ and not $\hat{x}Rx$. An allocation x is said to be *conditionally optimal subject to a* if $a \in A$, $x \in X(a)$, and for every $\hat{x} \in X(a)$, not $\hat{x}Px$.

A competitive price vector for ordinary commodities will be a vector $p \in E_j$. Since we will wish to consider allocation methods in which ordinary commodities are produced and sold competitively while social overhead goods are financed by taxation, we need to define a *wealth distribution function* which is somewhat more general than that defined by Debreu [6] for a "private ownership economy". In particular, a *wealth distribution function* is a function W from $E_j \times E_{\bar{k}} \times E_{\bar{m}j} \times E_{\bar{n}j}^+$ to $E_{\bar{n}}^+$ such that

$$W(p, z, y, w^1, \ldots, w^{\bar{n}}) = (W_1(p, z, y, w^1), \ldots, W_{\bar{n}}(p, z, y, w^n))$$

and

[2] This theorem and subsequent theorems could be substantially generalized. For example we could allow the "survival sets" to be only some proper subset of the non-negative orthant. We could also allow some ordinary commodities to be "public goods". The simplifications made here are largely for expositional simplicity. For a more general framework in which much of the analysis in this paper could be applied, see T. Bergstrom [2].

$$\sum_N W_i(p, z, y, w^i) = p\left(\sum_M y^m - z + w\right).$$

Thus the wealth of a consumer depends on prices, activities of firms, the resources used by the central authority and on his initial holdings. The sum of individual wealths is the total value of initial holdings plus the total profits of firms less the value of inputs used to produce social overhead goods.

For $\bar{a} \in A$, a *partially decentralized market equilibrium subject to* $\bar{a}$ *with wealth distribution function* W is a price vector $\bar{p} \in E_J$ and a point $(\bar{x}, \bar{y}, \bar{z}) \in V(\bar{a})$ that:

(1) $\bar{z}$ minimizes $\bar{p}z$ on $L(\bar{a})$,
(2) for every $m \in M$, $\bar{y}^m$ maximizes $\bar{p}y$ on $Y_m(\bar{a})$,
(3) for every $n \in N$, $\bar{x}^n$ maximizes R_n on $E_J^+ \cap \{x | \bar{p}x \leqslant W_n(\bar{p}, \bar{z}, \bar{y}, w^n)\}$.

In market equilibrium subject to $\bar{a}$, the central authority minimizes the cost (at prices $\bar{p}$) of producing $\bar{a}$. Firms maximize profits on the production sets $Y_m(\bar{a})$ where ordinary commodities are competitively priced at $\bar{p}$ and no charge is imposed for the use of social overhead goods. Individuals maximize preferences subject to budget constraints which may depend among other things on taxes. Finally market excess demands are zero. (This follows since $(\bar{x}, \bar{y}, \bar{z}) \in V(\bar{a})$.)

II. The Structure of the Model

A. Boundedness of Production Possibilities

To ensure the existence of a Pareto optimal allocation we would like the set of feasible allocations to be compact. We offer here a set of assumptions which guarantee compactness of V and hence of X. Clearly this could be done in a number of ways, but the result below should be adequate for our purposes.

The set $J_0 \subset J$ is said to be a set of *limiting factors in fixed supply* if (1) for every $a \in A$ and every $m \in M$, $y \in Y_m(a)$ implies $y_j \leqslant 0$ for every $j \in J_0$, (2) for every $a \in A$, there exists a bounding vector $B^+(a) \in E_J$ such that for every $m \in M$, if $y \in Y_m(a)$ and $y_j \leqslant w_j$ for all $j \in J_0$, then $y \leqslant B^+(a)$. (3) There is an upper bound $a^+ \in E_K$ for the set $\{a |$ for some $z \in L(a)$, $z_j \leqslant w_j$ for all $j \in J_0\}$.

THEOREM 1. The set $V \equiv \bigcup_{a \in A} V(a)$ is compact if:
(1) the correspondences $Y_m(a)$ for $m \in M$ and $L(a)$ are upper semi-continuous,

(2) for all $a \in A$, $L(a) \subset E_j^+$,
(3) for all $m \in M$, $a \leqslant a'$ implies $Y_m(a) \subset Y_m(a')$,
(4) there is a non-empty set J_0 of limiting factors in fixed supply.

Proof. The presence of limiting factors in fixed supply implies that for every $a \in A$, and every $m \in M$, the set $\tilde{Y}_m(a) = \{y | y \in Y_m(a)$ and $\{y + \sum_{i \in M, i \neq m} Y_i + w - L(a)\} \cap E_j^+ \neq \phi\}$ has both upper and lower bounds $B^+(a)$ and $B^-(a)$. Also the set $\tilde{A} \equiv \{a | \{\sum_M Y_m(a) + w - L(a)\} \cap E_j^+ \neq \phi\}$ is bounded from above by a^+ and from below by 0. Since $a \leqslant a'$ implies $Y_m(a) \subset Y_m(a')$, it must be that the bounds $B^-(a^+)$ and $B^+(a^+)$ for the sets $\tilde{Y}_m(a^+)$ are also bounds for the sets $\tilde{Y}_m(a)$ for every $a \in \tilde{A}$. Let $\tilde{V} = V \cap \{(x, y, z) |$ for all $m \in M$, $B^-(a^+) \leqslant y^m \leqslant B^+(a^+)\}$. It is easily verified that $\tilde{V}$ is bounded. From the definition of $\tilde{Y}_m$, it follows that $\tilde{V} = V$. Hence V is bounded.

We now show that $\tilde{A}$ is compact. Let $\{a(q)\}$ be a sequence in $\tilde{A}$ such that $\lim a(q) = \bar{a}$. Since $\{a(q)\} \subset \tilde{A}$, there exists a sequence $\{x(q)\}$ such that for every q, $x(q) \in \{\sum_M Y_m(a(q)) + w - L(a(q))\} \cap E_j^+$. From the results of the previous paragraph it follows that the latter set is bounded. Hence there is a convergent subsequence $\{x(q^*)\}$ such that $\lim x(q^*) = \bar{x} \in E_j^+$. Upper semi-continuity of the correspondences Y_m and L imply that $\bar{x} \in \{\sum_M Y_m(\bar{a}) + w - L(\bar{a})\} \cap E_j^+$. Hence $a \in \tilde{A}$. Since $\tilde{A}$ is bounded, $\tilde{A}$ must be compact.

Let $\{(x(q), y(q), z(q))\}$ be a sequence in V converging to $(\bar{x}, \bar{y}, \bar{z})$. Then for every q, there is an $a(q) \in \tilde{A}$ such that $z(q) \in L(a(q))$ and $y^m(q) \in Y_m(a(q))$ for all $m \in M$. Since $\tilde{A}$ is compact, the sequence $\{a(q)\}$ has a subsequence convergent to some $\bar{a} \in A$. Upper semi-continuity implies that $\bar{z} \in L(\bar{a})$ and $\bar{y}^m \in Y_m(\bar{a})$ for all $m \in M$. Also, $\sum_N \bar{x}^n = \sum_M \bar{y}^m + w - \bar{z} \in V$. Therefore V is closed. Since V is also bounded, V must be compact. Q.E.D.

COROLLARY 1. If the hypothesis of Theorem 1 is satisfied and if for every $n \in N$, R_n is transitive, reflexive and $R_n(x^n) \equiv \{x | x R_n x^n\}$ is closed for all $x \in E_j^+$ then there exists a Pareto optimal allocation.

Proof. The relation R inherits the properties assumed for the relations R_n. By Theorem 1, V is compact. Hence X is compact. But it is known that a relation with the assumed properties has at least one maximal element on any compact set. See T. Rader [9].

B. *Existence and Optimality of Equilibrium*

The following assumptions about preferences, technology and wealth distribution functions will be useful for adapting the well-known theorems on existence and optimality of competitive equilibrium to our model.

Assumption P. For all $n \in N$, R_n is a complete quasi-ordering on E_J^+. R_n is continuous, weakly convex, locally nonsatiated and monotonic.

Assumption T. Assumption T is satisfied for a if (1) the set $\{\sum_M Y_m(a) - L(a)\}$ is nonempty, closed, and convex and (2) the set $\{\sum_M Y_m(a) - L(a) + w\} \cap E_J^+$ is bounded and has a non-empty interior.

Assumption W. The wealth distribution function W is continuous and homogeneous of degree one in p and $\sum_N W_n(p, z, y, w^i) = p(\sum_M y^m - z + w)$, for all (p, z, y) in the domain of W. If $\sum_N W_n > 0$, then $W_n > 0$ for all $n \in N$.

THEOREM 2. If assumption P is true, and assumption T is satisfied for $\bar{a} \in A$, then there exists a partially decentralized market equilibrium subject to $\bar{a}$ with any wealth distribution function satisfying assumption W.

The proof is a mathematically trivial adaptation of Debreu's proof [4] of the existence of competitive equilibrium.

THEOREM 3. If $(\bar{x}, \bar{y}, \bar{z})$ is a partially decentralized market equilibrium subject to $\bar{a}$, and if R_n is locally nonsatiated for all $n \in N$, then the allocation $\bar{x}$ is conditionally optimal subject to $\bar{a}$.

The proof is a straightforward modification of the usual proof that competitive equilibrium is Pareto optimal.

Remark 1. Where $(\bar{x}, \bar{y}, \bar{z})$ is a partially decentralized market equilibrium, subject to $\bar{a} \in A$, $\bar{x}$ is not necessarily Pareto optimal even if assumptions P, T and W hold and $\bar{x}$ is on the boundary of the set X.

To see this, we need only consider a counterexample. Consider an economy where the technology is that of Example 1, and where there is only one consumer with indifference curves represented by the dotted lines in Figure 3. The set X of feasible aggregate outputs is the shaded area below the solid curved line.[3] The point $\bar{x}$ is a partially decentralized

[3] In a strict sense this is not quite true if we allow the possibility that some "labor" be consumed directly as "leisure". For the purposes of our example we will suppose that all labor must be used in production. Thus we can represent the set X diagrammatically in two dimensions by examining its projection onto the x_1 and x_2 axes.

market equilibrium subject to $\bar{a}$ where $X(\bar{a})$ is the area below the straight line passing through $\bar{x}$. Although $\bar{x}$ is on the boundary of X, $\bar{x}$ is not Pareto optimal since $x^* P \bar{x}$.

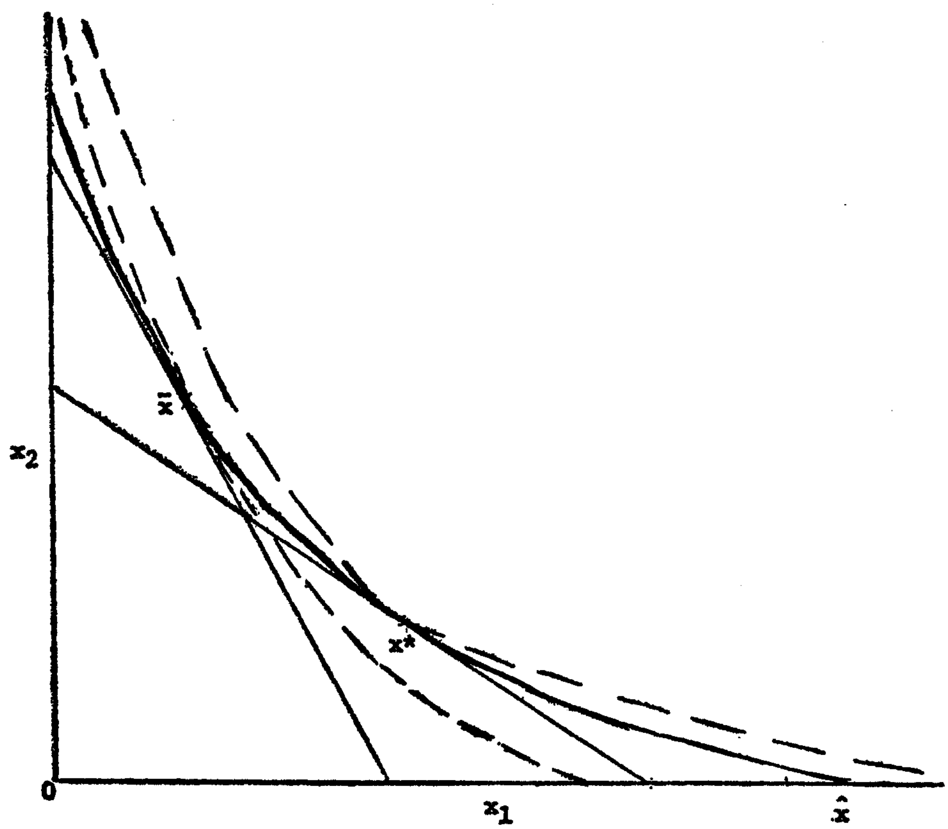

Figure 3

There is, however, a theorem which closely parallels the traditional theorem that any Pareto optimum is a competitive equilibrium.

For $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^{\bar{n}}) \in E_{J\bar{n}}^+$, we say that there is *conflict of interest* at $\bar{x}$ if for any $n' \in N$, there exists an allocation $(\hat{x}^1, \ldots, \hat{x}^{\bar{n}}) \in E_{\bar{n}J}^+$ such that $\sum_N \hat{x}^n = \sum_N \bar{x}^n$ and $\hat{x}^n P_n x_n$ for all $n \neq n'$, $n \in N$.

Theorem 4. If assumption P is true and $\{\sum_M Y_m(a) - L(a)\}$ is convex for all $a \in A$, then for any Pareto optimal allocation $\bar{x}$ such that there is conflict of interest at $\bar{x}$ and $\sum_M \bar{x}^m \gg 0$, there exist vectors $\bar{y}$, $\bar{z}$, and $\bar{a}$ and a non-zero price vector $\bar{p} \in E_J^+$ such that $\bar{p}$ and $(\bar{x}, \bar{y}, \bar{z})$ constitute a partially decentralized market equilibrium subject to $\bar{a}$ with some distribution of wealth.

Proof. Since $\bar{x}$ is a Pareto optimum it is feasible and hence $\bar{x} \in X(\bar{a})$ for some $\bar{a} \in A$. Our assumptions and the fact that $\bar{x}$ is Pareto optimal imply that $\{\sum_N x^n | (x^1, \ldots, x^{\bar{n}}) R \bar{x}\} - \{\sum_M Y_m(\bar{a}) - L(\bar{a}) - w\}$ is a convex set which does not contain the zero vector in its interior. It then follows (as a consequence of Minkowski's separation theorem) that there exists a non-zero $\bar{p} \in E_J$ such that:

(1) If $z \in L(\bar{a})$, $\bar{p}z \leqslant \bar{p}\bar{z}$.
(2) For all $m \in M$, if $y^m \in Y_m(\bar{a})$, $\bar{p}y^m \leqslant \bar{p}\bar{y}^m$.
(3) For all $n \in N$, if $x^n R_n \bar{x}^n$, then $\bar{p}x^n \geqslant \bar{p}\bar{x}^n$.

Since preferences are monotonic, $\bar{p} > 0$. To complete our theorem we need only to show that (3) can be strengthened to: (3)′ For all $n \in N$, if $x^n P_n \bar{x}^n$, then $\bar{p}x^n > \bar{p}\bar{x}^n$. From a well known theorem of Debreu [4, p. 63] it must be that if (3) holds and if $\bar{p}\bar{x}^n > 0$ for all $n \in N$, then (3)′ holds. By assumption, $\sum_N \bar{x}^n \gg 0$. Since $\bar{p} > 0$, $\bar{p} \sum_N \bar{x}^n > 0$. Hence $\bar{p}\bar{x}^n > 0$ for some $n \in N$. Suppose $\bar{p}\bar{x}^{n'} = 0$ for some $n' \in N$. Since there is conflict of interest at $\bar{x}$, there exists an allocation $\hat{x} \in E_{\bar{n}J}^+$ such that $\sum_N \hat{x}^n = \sum_N \bar{x}^n$ and $\hat{x}^n P_n \bar{x}^n$ for all $n \neq n'$, $n \in N$. But Debreu's theorem and result (3) then imply that $\bar{p} \sum_N \hat{x}^n > \bar{p} \sum_N \bar{x}^n$. This contradicts the assumption that $\sum_N \hat{x}^n = \sum_N \bar{x}^n$. Hence $\bar{p}\bar{x}^n > 0$ for all $n \in N$. Therefore condition (3)′ holds. If we now let $W_n = \bar{p}\bar{x}^n$ for every $n \in N$, the theorem is proved. Q.E.D.

At this point we might reconsider Rader's remark and our paraphrase. Corollary 1 and Theorem 4 tell us that (under certain conditions) a Pareto optimum exists and that every Pareto optimum is a partially decentralized market equilibrium. Where firms have differentiable cost functions, profit maximization requires that any ordinary commodity produced in positive quantities must be priced at marginal cost under the market equilibrium price vector $\bar{p}$. In this sense "price equals marginal cost" is necessary for Pareto optimality. But since not every partially decentralized market equilibrium is Pareto optimal, the marginal cost rule is not sufficient. Alternatively we can say that every Pareto optimum is a partially decentralized market equilibrium, but such an equilibrium is Pareto optimal only if the central authority chooses an appropriate vector of social overhead goods. The remainder of this paper concerns itself with the question of how the central authority can make an "appropriate choice".

III. On Finding Pareto Optimal Allocations

For a central authority which accepted the Pareto criterion as a partial welfare guide, it would be useful to find necessary and/or sufficient conditions that an allocation be Pareto optimal. It would be especially pleasant if these conditions provided at least crude practical guidelines for choice concerning the quantity of social overhead goods. We present some results which are suggestive but only partially satisfactory. That interesting necessary and sufficient conditions are hard to come by is foreshadowed by the work of Samuelson [11] and Chipman and Moore [3].

A. A Necessary Condition for Pareto Optimality

THEOREM 5. If preferences of at least one consumer are locally non-satiated, then a necessary but not sufficient condition that an allocation $\bar{x}$ be Pareto optimal is that $\bar{x}$ belongs to the boundary of $X \equiv \bigcup_{a \in A} X(a)$.

Necessity is easily demonstrated. To see that this condition is not sufficient, observe that in the example of Figure 3, the point $\bar{x}$ belongs to the boundary of X but is not Pareto optimal.

Define the set A^* of potentially efficient supplies of social overhead goods so that $A^* = A \cap \{a | X(a)$ is not contained in the interior of $X\}$. Then we have the following easy consequence of Theorem 5.

COROLLARY 2. If x is Pareto optimal, then $\bar{x} \in X(\bar{a})$ for some $\bar{a} \in A^*$.

This result together with Theorem 4 suggests that a central authority which subscribes to the Pareto criterion need only consider points a in A^*. This fact may provide a criterion of considerable usefulness for eliminating inefficient outputs of social overhead goods. For instance in Example 1, it is easy to show that the set $A^* = \{(a_1, a_2) \in E_2{}^+ | a_1 + a_2 \leqslant (\beta/(1 + \beta))l\}$. Thus we would know that it can never be efficient to produce more than $(\beta/(1 + \beta))l$ units of either social overhead good. Similar results could be obtained in the analysis of more general economies. Of course we are left with the problem of choosing from among points in A^*.

B. Benefit-Cost Analysis

One might think that the model we have constructed would provide the most congenial of framework for the application of benefit-cost analysis. We will suppose that the central authority is able to observe prices $\bar{p}$ for ordinary commodities which with the allocation $(\bar{x}, \bar{y}, \bar{z})$ constitutes a partially decentralized market equilibrium subject to some vector $\bar{a}$ of social overhead goods. We now formalize three forms of benefit-rule.

Benefit-Cost Rule I. Reject any project which results in a vector a of social overhead goods if for all

$$y \in \pi_M Y_m(a) \quad \text{and all} \quad z \in L(a), \quad \bar{p} \sum_M (y_m - \bar{y}_m) < \bar{p}(z - \bar{z}).$$

The interpretation of this rule is that any project will be rejected if the value at prices $\bar{p}$ of every possible change in outputs due to the change from $\bar{a}$ to a is less than the consequent change in the cost of inputs used by the central authority. This would seem to correspond rather closely to the traditional benefit-cost comparison.

If one examines the definition of the set $X(a)$, it is easy to see that benefit-cost rule I could be restated in the equivalent form: I′. Reject any project which results in a vector of social overhead goods a such that for all $x \in X(a)$, $\bar{p} \sum_N x^n < \bar{p} \sum_N \bar{x}^n$. The next result provides partial justification for the use of this rule.

THEOREM 6. Where $\bar{p}$ and $(\bar{x}, \bar{y}, \bar{z})$ constitute a partially decentralized market equilibrium subject to $\bar{a}$, a sufficient condition that no $x \in X(a)$ be Pareto superior to $\bar{x}$ is that

$$\bar{p} \sum_N x^n < \bar{p} \sum_N \bar{x}^n \quad \text{for all} \quad x \in X(a).$$

The proof is a trivial adaptation of the proof that competitive equilibrium is Pareto optimal.

Theorem 6 tells us that no project rejected by benefit-cost rule I could result in an allocation Pareto superior to $\bar{x}$. We cannot, of course assert that $\bar{x}$ is itself Pareto superior to all allocations in $X(a)$. One might hope that we could establish dominance of $\bar{a}$ over a in some weaker sense. For example, we might make the following conjecture.

False Conjecture. If $\bar{p} \sum_N x^n < \bar{p} \sum_N \bar{x}^n$ for all $x \in X(a)$, then for every $x \in X(a)$ there exists an allocation $\hat{x} \in X(\bar{a})$ such that $\hat{x}$ is Pareto superior to $\bar{x}$.

Chipman and Moore [3] provide counterexamples to this conjecture. They do, however, demonstrate that the conjecture would hold if in addition to our assumption P we assume that all consumers have identical and homothetic preferences.

Benefit-cost rule I must be thought of as a conservative rule. It has merit in that it rejects only projects which cannot dominate the initially observed allocation in the Pareto sense. On the other hand it does lend a privileged position to the originally observed equilibrium in the sense that allocations attainable with $\bar{a}$ need not dominate all allocations which would be attainable if the rejected project were adopted. Also, of course, the rule tells us only which projects to reject. We might consider versions of the benefit-cost rule which tell us what to accept. (We again suppose that there is an observed $\bar{p}$ and $(\bar{x}, \bar{y}, \bar{z})$ which constitute a partially decentralized market equilibrium subject to $\bar{a}$.) Perhaps the simplest and most commonly suggested rule is:

Benefit-Cost Rule II. Choose $\hat{a}$ so that for some $x \in X(\hat{a})$, $\bar{p} \sum_N \hat{x}^n \geqslant \bar{p} \sum_N x^n$ for all $x \in X$.

Looking at our definitions of $X(a)$ and X, it is clear that this notion corresponds closely to the idea of maximizing the difference between benefits and costs of alternative vectors of social overhead goods evaluated at prices $\bar{p}$. This rule, however would not generally result in Pareto optimal resource allocation. This is very quickly seen from the example of Figure 4. Here if we let $\bar{p}$ be the price vector normal to the tangent line through $\bar{x}$, we notice that the allocation $\hat{x}$ satisfying benefit-cost rule II would be a point on the horizontal axis. Recalling the construction of the figure from Example 1, this would require that $\hat{a} = (0, (\beta/(1 + \beta))l)$. But in the figure, x^* is the only Pareto optimal allocation and it is easily seen that $x^* \notin X(\hat{a})$. Thus it appears that benefit-cost rule II is quite unsatisfactory from the Pareto viewpoint.

We might consider a benefit-cost rule which places stronger informational demands on the central authority. For example we might consider:

Benefit-Cost Rule III. Choose $\hat{a}$ so that where $\hat{p}$ and $(\hat{x}, \hat{y}, \hat{z})$ constitute a partially decentralized market equilibrium subject to a, $\hat{p} \sum_N \hat{x}^n \geqslant \hat{p} \sum_N x^n$ for all $x \in X$.

It is easily shown (applying the method of proof from the theorem that competitive equilibrium is Pareto optimal) that any allocation $\hat{a}$ satisfying benefit-cost rule III will be Pareto optimal. But the rule is of very limited usefulness because in general there need not exist any $\hat{a}$ and $\hat{p}$ which satisfy the rule. We could be guaranteed the existence of such vectors only if X were a convex set. But in general X need not be convex. In fact it is clear that the non-convexity in Example 1 is such that there will be no $\hat{a}$ and $\hat{p}$ satisfying benefit-cost rule III.

We must conclude that for an economy of the type we consider, benefit-cost rules of the traditional sort are of little use as a guide to efficient provision of social overhead goods. Practitioners of this technique might argue that we have stacked the deck against them by considering a technology which results in a non-convex set of feasible allocations. On the other hand, one might argue that the most interesting case for government intervention can be made precisely where there are substantial indivisibilities or increasing returns.

In the next section we consider an alternative method of choosing quantities of social overhead goods. This method is shown to have desirable properties at least for certain types of economies.

IV. Quasi-Linear Technology and Lindahl-Hotelling Equilibrium

There is an analogy between social overhead goods of the type we consider and pure public goods as considered by Lindahl [8], and Samuelson [12], and Bergstrom [2]. The supply of social overhead goods, like the supply of public goods, will in general affect the utility of several consumers. Where it is presumed that given the supply of social overhead goods, the prices and supplies of all consumer goods are determined by partially decentralized market equilibrium, the supply of social overhead goods instead of affecting consumer utilities directly, affects them indirectly through effects on the prices of consumer goods. Social overhead goods, like public goods exhaust ordinary economic resources in their production. These resources must be withdrawn from the private sector by some form of taxation. We consider an equilibrium notion similar to the Lindahl equilibrium for the supply of pure public goods (Lindahl [8], Foley [5], Bergstrom [2]). In particular, we divide the costs of each social overhead good among the consumers in such a way that, knowing their shares of the costs and knowing the effect of the supply of social overhead goods on market prices for ordinary goods, all consumers agree on the same quantity of social overhead goods. We will show that for at least one interesting class of economies, an equilibrium of this sort exists, and that equilibria are Pareto optima and vice versa.

A. Quasi-Linear Technology

Matters are considerably simplified if the set of feasible outputs given the quantity of social overhead goods is the intersection of a half-space with the non-negative orthant. We define a technology to be quasi-linear if for every $a \in A$, there exists a unique vector $p(a) \gg 0$, normalized so that $p_{\bar{j}}(a) = 1$, such that $\sum_M Y_m(a) \cap \{y | y_j \geqslant 0$ for all j such that $1 \leqslant j \leqslant \bar{j} - 1\} = \{y | p(a)y \leqslant 0$ and $y_j \geqslant 0$ for $j \leqslant \bar{j} - 1\}$. We call $p(a)$ the supporting price vector at a.

The most well known example of such a technology is one which satisfies the conditions of the celebrated Samuelson non-substitution theorem. The Samuelson theorem assumes that there are $\bar{j} - 1$ ordinary commodities which are produced by firms whose production sets are convex and display constant returns to scale. Each firm may use any of the ordinary commodities as inputs. The $\bar{j}$th commodity is present in fixed supply and is required as an input for positive production by any firm. For convenience, assume that there is only one firm producing each producible commodity. This involves no loss of generality since one could think of the production set of one of these "firms" as the sum of the production sets of all firms producing the same good.

The main result of this section is Theorem 7 which states that for an economy with a quasi-linear technology, and for any $a \in A$ there exists a unique (normalized) price vector $p(a)$ which sustains a partially decentralized market equilibrium subject to a. This price vector is determined independently of preferences and wealth distribution. It is this property of quasi-linear technologies which makes them especially convenient for our purposes. Theorem 8 provides a sufficient condition that $p(a)$ be a continuous function on A.

LEMMA 1. (Samuelson non-substitution theorem). The following technology is quasi-linear. There is a set of firms $M = \{1, \ldots, \bar{j} - 1\}$ such that:

(1) For all $m \in M$, and all $a \in A$;
 (a) $Y_m(a)$ is a closed convex cone containing $E_{\bar{j}}^-$
 (b) if $y \in Y_m(a)$ then $y_i \leqslant 0$ for all $i \neq m$ and if $y_m > 0$, then $y_{\bar{j}} < 0$.
(2) There exists $y \in \sum Y_m(a)$ such that $y \gg 0$.

A proof of this proposition is presented by Arrow [1].

LEMMA 2. If the conditions of Lemma 1 are satisfied and if also $L(\bar{a})$ is a closed subset of $E_{\bar{j}}^+$ and if the initial allocation w is such that $w_j = 0$ for $j \neq \bar{j}$, then where $p(a)$ is the supporting price vector at a,

$$E_{\bar{j}}^+ \cap \left\{ \sum_M Y_m(a) - L(a) + w \right\} = E_{\bar{j}}^+ \cap \{x | p(a)x \leqslant W(a)\}$$

where

$$W(a) = \max_{z \in L(a)} \{p(a)(w - z)\}.$$

Proof. Since $L(a)$ is a closed subset of E_j^+ and $p(a) \gg 0$, $\max_{z \in L(a)} p(w - z)$ exists. Suppose that $x = y - z + w \in E_j^+ \cap \{\sum_M Y_m(a) - L(a) + w\}$. Then $x \geqslant 0$ and $y = x + z - w$ where $y_j \geqslant 0$ for all $j \leqslant \bar{j} - 1$. Since technology is quasi-linear, $p(a)y \leqslant 0$. It follows that $p(a)x \leqslant p(a)(w - z) \leqslant W(a)$. Thus the first of the two sets is contained in the second. Suppose $x \geqslant 0$ and $p(a)x \leqslant W(a)$. Let $y = x + \bar{z} - w$ where $\bar{z}$ maximizes $p(a)(w - z)$ on $L(a)$. Then $y_j \geqslant 0$ for $j \leqslant \bar{j} - 1$ and $p(a)y = p(a)x - W(a) \leqslant 0$. Therefore $y \in \sum_M Y_m(a)$ and $x = y - \bar{z} + w \in \sum_M Y_m(a) - L(a) + w$. Hence the second set is contained in the first. Q.E.D.

THEOREM 7. Let preferences be locally non-satiated, transitive and have closed upper contour sets. Suppose that the technology satisfies the conditions of Lemma 2. Let $W(a) = \max_{z \in L(a)} p(a)(w - z)$. If $(W_1, \ldots, W_n)$ is a wealth distribution such that $\sum_N W_n = W(a)$ and $W_n \geqslant 0$ for all $n \in N$, then there exists a partially decentralized market equilibrium subject to a with wealth distribution $(W_1, \ldots, W_{\bar{n}})$ and with an equilibrium price vector equal to the supporting price vector $p(a)$. Furthermore, if $W(a) > 0$, $p(a)$ is the only possible (normalized) equilibrium price vector for a.

Proof. Let $p = p(a)$ be the supporting price vector at a. For every $n \in N$, choose $x^n(p)$ so that $x^n(p)$ maximizes R_n on $E_j^+ \cap \{x | px \leqslant W_n\}$. This is possible since R_n is transitive and has closed upper contour sets and since positivity of $p(a)$ and non-negativity of W_n imply that $E_j^+ \cap \{x | px = W_n\}$ is non-empty and compact. Local nonsatiation implies that $px^n(p) = W_n$ and hence that $p \sum_N x^n(p) = \sum_N W_n = W(a)$. Then, applying Lemma 2, $\sum_N x^n = \bar{y} - \bar{z} + w$ for some $\bar{y} \in \sum_M Y_m(a)$ and $\bar{z} \in L(a)$. But $p \sum_N x^n = p\bar{y} + p(w - \bar{z}) = W(a)$. Hence $p\bar{y} = W(a) - p(w - \bar{z}) \geqslant 0$. Since $p = p(a)$ is a supporting price vector, $py \leqslant 0$ for all $y \in \sum_M Y_m(a)$. Hence $p\bar{y} = 0 \geqslant py$ for all $y \in \sum_M Y_m(a)$, and $p(w - \bar{z}) = W(a)$ so that $p\bar{z} \leqslant pz$ for all $z \in L(a)$. But then since firms' production sets are independent and contain the origin, there exist for each $m \in M$, $\bar{y}^m \in Y_m(a)$ such that $\sum_M \bar{y}^m = \bar{y}$ and $p\bar{y}^m = 0 \geqslant py$ for all $y \in Y_m(a)$. Therefore $p = p(a)$ and $(x^1(p), \ldots, x^{\bar{n}}(p), \bar{y}^1, \ldots, \bar{y}^{\bar{m}}, \bar{z})$ constitute a decentralized market equilibrium subject to a, with wealth distribution $W_1, \ldots, W_n)$.

Since production is quasi-linear, $p(a)$ is the only normalized non-zero price vector at which profits are maximized at non-zero outputs. It is then easily seen that $p(a)$ is the only possible normalized equilibrium price vector. Q.E.D.

THEOREM 8. If technology is quasi-linear, and if $\sum_M Y_m(a)$ is a continuous correspondence, then the supporting price vector $p(a)$ is a continuous function on A.

Proof. Suppose $a(n) \to \bar{a}$, $p(a(n)) \to \bar{p}$. We wish to demonstrate that $\bar{p} = p(\bar{a})$. According to Lemma 1, $p(\bar{a})$ is unique and $\sum_M Y_m(\bar{a}) = \{y|p(\bar{a})y \leqslant 0$ and $y_j \geqslant 0$ for $j \leqslant \bar{\jmath} - 1\}$. Thus we need only to show that this set is equal to $\{y|\bar{p}y \leqslant 0$ and $j \leqslant \bar{\jmath} - 1\}$. Let $a(n) \to \bar{a}$ and $p(a(n)) \to \bar{p}$. Suppose that $y \in \sum_M Y_m(\bar{a})$. Since $\sum_M Y_m$ is lower semi-continuous, there exists a sequence $y(n) \to y$ such that $y(n) \in \sum_M Y_m(a(n))$. Thus $p(a(n))y(n) \leqslant 0$ and $y_j(n) \geqslant 0$ for $j \leqslant \bar{\jmath} - 1$. But since $p(a(n)) \to \bar{p}$, it must be that $\bar{p}y \leqslant 0$ and $y_j \geqslant 0$ for $j \leqslant \bar{\jmath} - 1$. This proves that the former set is contained in the latter.

Suppose that $\bar{p}y < 0$ and $y_j \geqslant 0$ for all $j < \bar{\jmath} - 1$. Then since $p(a(n)) \to \bar{p}$, $p(a(n))y < 0$ for sufficiently large values of n. But this implies $y \in \sum_M Y_m(a(n))$ for all sufficiently large values of n. Therefore since $\sum_M Y_m$ is upper semi-continuous, $y \in \sum_M Y_m(\bar{a})$. Since $p(a(n)) \gg 0$ and $p_{\bar{\jmath}}(a(n)) = 1$, it must be that $\bar{p} \geqslant 0$ and $\bar{p}_{\bar{\jmath}} = 1$. Hence there exists some vector $\mathring{y}$ such that $\bar{p}\mathring{y} < 0$ and $\mathring{y}_j \geqslant 0$ for $j \leqslant \bar{\jmath} - 1$.

Now suppose that $\bar{p}y = 0$ and $y_j \geqslant 0$ for $j \leqslant \bar{\jmath} - 1$. Using the results of the previous paragraph, we can construct a sequence $\{y(n)\}$ such that $y(n) \to y$, $\bar{p}y(n) < 0$, and $y_j(n) \geqslant 0$ for $j \leqslant \bar{\jmath} - 1$. But from the previous paragraph we know that $y(n) \in \sum_M Y_m(a(n))$ for all n. Since $\sum_M Y_m$ is upper semi-continuous, $y(n) \to y$ and $a(n) \to \bar{a}$, it must be that $y \in \sum_M Y_m(\bar{a})$. Hence $\sum_M Y_m(\bar{a}) = \{y|\bar{p}y \leqslant 0$ and $y_j \geqslant 0$ for $j \leqslant \bar{\jmath} - 1\}$. It follows that $\bar{p} = p(\bar{a})$ and hence that $p(a)$ is continuous. Q.E.D.

B. Induced Preferences for Social Overhead Goods

The preference level of a consumer facing competitive prices is determined once commodity prices and his income are specified. In a quasi-linear economy, satisfying Theorem 7, the equilibrium price vector $p(a)$ for ordinary commodities is completely determined by the vector a of social overhead goods. We therefore find it useful to define an induced preference relations R^*_n on income and social overhead goods in the following way. Where (a, W_n) and (a', W_n') both belong to $A \times E_1{}^+$, let $x(a, W_n)$ maximize R_n subject to $p(a)x \leqslant W_n$ and let $x(a', W_n')$ maximize R_n subject to $p(a')x \leqslant W_n'$. Then we say that $(a, W_n)R^*_n(a', W_n')$ if and only if $x(a, W_n)R_n x(a', W_n')$. Our next result lists some results about the relation R^*_n which we will find useful.

LEMMA 3.
(1) If R_n is transitive, R^*_n is transitive.
(2) If R_n is reflexive, R^*_n is reflexive.
(3) If R_n is complete, R^*_n is complete.
(4) If R_n is locally non-satiated, then R^*_n is locally non-satiated. In fact for every $a \in A$ and $W > W' \geqslant 0$, $(a, W)P^*_n(a, W')$.
(5) If $a \geqslant a'$ implies $\sum_M Y_m(a') \subset \sum_M Y_m(a)$ then R^*_n is monotonic.
(6) If $\sum_M Y_m(a)$ is a continuous correspondence on A and if R_n is continuous, then for any $a \in A$, and $W > 0$, R^*_n is continuous at (a, W).

The proofs of (1)–(5) are straightforward. To prove (6) one can simply apply Theorem 8 above along with Theorems 1.8 (4) and 4.8 (1) of Debreu [4].

We will occasionally wish to assume that R^*_n has convex upper contour sets. At this time I have no particularly appealing general assumptions on R_n and $\sum_M Y_m$ which imply this condition. Examples will be provided to show economically interesting cases where it does and does not apply.

V. Lindahl-Hotelling Equilibrium

In this section we deal with a rather limited class of economies. It is hoped that this discussion will suggest more general procedures. In particular we deal with economies satisfying the following.

Assumption A. Technology is quasi-linear. The only resource held initially is labor. Social overhead goods are produced subject to constant returns to scale using labor as the only input.

We now define *Lindahl-Hotelling Equilibrium.* Let $\bar{q} = (\bar{q}^1, \ldots, \bar{q}^{\bar{n}})$ where for $n \in N$, $\bar{q}^n \in E_{\bar{k}}{}^+$ is the vector of tax prices paid by consumer n for social overhead goods. Let $\bar{p} \in E_{\bar{k}}{}^+$ be a price vector for ordinary goods and let $\bar{a} \in A$. Let $(\bar{x}, \bar{y}, \bar{z}) \in V(\bar{a})$ such that

(1) For all $n \in N$, $(\bar{a}, \bar{p}w^n - \bar{q}^n\bar{a})R^*{}_n(a, \bar{p}w^n - \bar{q}^n a)$ for every $a \in A$.
(2) For all $(a,z)\in\{(a,z)|a\in A, z\in L(a)\}$, $\sum_N \bar{q}^n a - \bar{p}z \leqslant 0 = \sum_N \bar{q}^n\bar{a} - \bar{p}\bar{z}$.
(3) The price $\bar{p} = p(\bar{a})$ and the allocation $(\bar{x}, \bar{y}, \bar{z})$ constitute a partially decentralized market equilibrium with a wealth distribution such that $W_n = \bar{p}w^n - \bar{q}^n\bar{a}$ for every $n \in N$.

When these conditions are satisfied, we say that $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitutes a Lindahl-Hotelling equilibrium.

Although the definition is rather lengthy, its interpretation is fairly simple. The first condition requires that each consumer n, believing that if $a \in A$ is chosen his after tax income will be $\bar{p}w^n - \bar{q}^n a$ and that ordinary commodities will be priced competitively at $p(a)$, will find that the same choice of social overhead goods $\bar{a}$ maximizes his preferences on A. The central authority chooses $a \in A$ so as to maximize the difference between the summed individual evaluations of the social overhead good and the cost of the resources used as inputs. Tax payments add up to the value of inputs used by the central authority. Finally markets for ordinary commodities clear at the competitive prices $\bar{p}$ where a consumer's wealth is his income from factor endowments minus his tax payments.

THEOREM 9. If preferences (on ordinary commodities) are locally non-satiated and if assumption A is satisfied, then a Lindahl-Hotelling equilibrium is Pareto optimal.

Proof. Let $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitute a Lindahl-Hotelling equilibrium. Suppose that $\hat{x} \in X$ and that $\hat{x}^n R_n \bar{x}^n$ for all $n \in N$ and $\hat{x}^i P_i \bar{x}^i$ for some $i \in N$. Since $\hat{x} \in X$, $\hat{x} \in X(\hat{a})$ for some $\hat{a} \in A$. Condition (1) of the definition of Lindahl-Hotelling equilibrium implies that $p(\hat{a})\hat{x}^i > \bar{p}w^i - \bar{q}^i\hat{a}$.[4] Local nonsatiation implies that for all $n \in N$, $p(\hat{a})\hat{x}^n \geqslant \bar{p}w^n - \bar{q}^n\hat{a}$. Thus $p(\hat{a}) \sum_N \hat{x}^n > \bar{p}w - (\sum_N \bar{q}^n)\hat{a}$. According to Condition (3) of the definition, $(\sum_N \bar{q}^n)a \leqslant \bar{p}z$ for all $z \in L(\hat{a})$. But since labor is the only input used to produce social overhead goods (and the price of labor is normalized to one) $\bar{p}z = z_l = p(\hat{a})z$. Likewise since labor is the only good initially held, $\bar{p}w = p(\hat{a})w$. Since technology is quasi-linear, $p(\hat{a})y \leqslant 0$ for all $y \in \sum_M Y_m(\hat{a})$. It now follows that for all $z \in L(\hat{a})$ and all $y \in \sum_M Y_m(\hat{a})$, $p(\hat{a})(\sum_N \hat{x}^n - y + z - w) > 0$.

[4] If $p(\hat{a})\hat{x} < \bar{p}w^i - \bar{q}^i\hat{a}$, then since $\hat{x}^i P_i \bar{x}^i$, it must be that $(\hat{a}, \bar{p}w^i - \bar{q}^i\hat{a})P_i^*(\bar{a}, \bar{p}w^i - \bar{q}^i\bar{a})$. But this contradicts Condition (1).

Thus it cannot be that $\hat{x} \in X(\hat{a})$. Since this contradicts our previous assumption, we must conclude that there is no $\hat{x} \in X$ such that $\hat{x}P\bar{x}$.

Q.E.D.

Our next result specifies conditions which ensure that a Pareto optimum can be sustained as a Lindahl-Hotelling equilibrium. We shall make the following additional assumption.

Assumption C. For all $n \in N$ and all $(\bar{a}, \bar{w}) \in A \times E_1{}^+$, the set $\{(a, w) | (a, W) R^*{}_n(\bar{a}, \bar{W})\}$ is convex.

The strength of this assumption will be indicated in examples presented later, where we find some interesting economies to which it does and some to which it does not apply.

THEOREM 10. Let $\bar{x}$ be a Pareto optimal allocation at which there is conflict of interest. Let the economy satisfy the assumptions of Theorem 4 and in addition assumptions A and C. Assume also that if $a \geqslant a'$, $\sum_M Y_m(a) \supset \sum_M Y_m(a')$. Then there exists a Lindahl-Hotelling equilibrium $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ where $\bar{x}$ is the specified Pareto optimum.

Proof. We note from Theorem 4 that we can find $\bar{p}$, $\bar{a}$, $\bar{y}$, and $\bar{z}$ such that $(\bar{x}, \bar{y}, \bar{z})$ constitutes a partially decentralized market equilibrium subject to $\bar{a}$ where each consumer n is allowed an after tax wealth $\bar{p}\bar{x}^n$. We are left with the task of finding $\bar{q} \in E_{\bar{k}\bar{n}}$ such that $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitute a Lindahl-Hotelling equilibrium. To do this we construct the sets $S_1 \equiv \{a^1, \ldots, a^n, W_1, \ldots, W_n) |$ for all $n \in N$ $(a^n, W) R^*{}_n(\bar{a}, \bar{p}\bar{x}^n)\}$ and $S_2 \equiv \{(a, \ldots, a, W_1, \ldots, W_n) | \sum_N W_n \leqslant \max_{z \in L(a)} \bar{p}(w - z)\}$. Our assumptions ensure that both sets are convex, that $(\bar{a}, \ldots, \bar{a}, \bar{p}\bar{x}^1, \ldots, \bar{p}\bar{x}^n) \in S_1 \cap S_2$ and that $0 \notin$ Interior $S_1 - S_2$. We therefore apply the separating hyperplane theory to get a vector $(\bar{q}, \bar{v}) \in E_{\bar{k}\bar{n}+\bar{n}}$ which separates the two sets. Our assumptions ensure that $R^*{}_n$ is monotonic in a and strictly monotonic in W_n. From the construction of S_1 it is clear that $\bar{q} \geqslant 0$ and $\bar{v} \gg 0$. From the construction of S_2 it can be seen that $v_1 = v_2 = \ldots = v_n$. It then becomes a straightforward matter to verify that the vector $\bar{q} = (\bar{q}^1, \ldots, \bar{q}^n)$ satisfies our requirements. Q.E.D.

THEOREM 11. There exists a Lindahl-Hotelling equilibrium for an economy in which preferences satisfy assumption P, assumptions A and C are satisfied, the initial labor holdings of each consumer are positive, and $\sum_M Y_m(a)$ and $L(a)$ are continuous correspondences on A.

Proof. The theorem can be proved as an application of the proof offered by T. Bergstrom [2] for the existence of Lindahl equilibrium. Our assumptions imply that R^*_n shares all the properties required for R_n in a conventional existence proof. We can let $A \times E_{\bar{n}}^+$ play the formal role of the allocation space where choice is made among points $(a, W_1, \ldots, W_n) \in A \times E_{\bar{n}}^+$. We observe that the set

$$Z = \left\{(a, W_1, \ldots, W_{\bar{n}}) \middle| \sum_N W_n \leqslant p(a)(w - z) \quad \text{for some} \quad z \in L(a)\right\}$$

will be closed, bounded and convex. The set Z then plays the role of the set of feasible allocations. One can apply the proof previously mentioned to show that there exist Lindahl prices $(\bar{q}^1, \ldots, \bar{q}^n) \in E_{\bar{k}\bar{n}}$ and a point $(\bar{a}, \bar{W}_1, \ldots, \bar{W}_n) \in A \times E_{\bar{n}}^+$ such that for all $n \in N$, $(\bar{a}, \bar{W}_n)$ maximizes R^*_n subject to $\bar{q}^n a + W_n \leqslant w_j{}^n$ and such that $\sum_N \bar{q}^n \bar{a} + \sum_N \bar{W}_n \geqslant \sum_N \bar{q}^n a + \sum_N W_n$ for all $(a, W_1, \ldots, W_n) \in Z$. Noting the construction of Z we observe that for some $\bar{z} \in L(\bar{a}), \sum_N \bar{q}^n \bar{a} + p(\bar{a})(w - \bar{z}) \geqslant \sum_N \bar{q}^n a + p(a)(w - z)$ for all (a, z) such that $z \in L(a)$. We can then employ assumption A to show that for all (a, z) such that $z \in L(a), \sum_N \bar{q}^n a - p(\bar{a})z \leqslant 0 = \sum_N \bar{q}^n \bar{a} - \bar{p}\bar{z}$.

Consider the wealth distribution such that for all $n \in N, \bar{W}_n = \bar{p}w^n - \bar{q}^n \bar{a}$. We can apply Theorem 7 to find an allocation $\bar{x} \in E_{\bar{m}\bar{n}}$ and a $\bar{y} \in \sum_M Y_m(\bar{a})$ such that $(\bar{x}, \bar{y}, \bar{z})$ constitutes a partially decentralized market equilibrium subject to $\bar{a}$, with the stated wealth distribution. It now follows that $(p(\bar{a}), \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitutes a Lindahl-Hotelling equilibrium. Q.E.D.

VI. Some Examples of Lindahl-Hotelling Equilibrium

Example 2. There are $\bar{j} - 1$ produced ordinary commodities. The $\bar{j}$th ordinary commodity is a factor in fixed supply. There is one firm producing each ordinary commodity. The only ordinary commodity used as an input is the $\bar{j}$th. There are $\bar{j} - 1$ social overhead goods. For a given quantity a of social overhead goods, firms produce with constant returns to scale where the factor requirement in firm j is $c_j a_j{}^{-\beta}$ times its output, (where c_j and β are positive parameters). Thus for $j = 1, \ldots, \bar{j} - 1$, $Y_j(a) = \{y | y_j \leqslant (1/c_j) a_j^\beta (- y_{\bar{j}}), y_{\bar{j}} \leqslant 0$ and $y_m = 0$ for $m \neq j$ and

$m \neq \bar{j}\}$. Let the input correspondence be $L(a) = \{z|z_j = 0$ for $j < \bar{j}$ and $z_{\bar{j}} = \sum_{j=1}^{\bar{j}-1} a_j\}$. Let there be a set N of consumers such that each $n \in N$ has an initial resource endowment $w^n = (0, \ldots, 0, w_{\bar{j}}^n)$ where $w_{\bar{j}}^n > 0$. Preferences of consumer n are represented by the utility function $u_n(x) = \prod_{j=1}^{\bar{j}-1} x_j^{\alpha_{nj}}$ where $\sum_{j=1}^{\bar{j}-1} \alpha_{nj} = 1$ and $\alpha_{nj} \geqslant 0$ for $j = 1, \ldots, \bar{j} - 1$.

We observe that the technology is quasi-linear and that

$$E_{\bar{j}}{}^+ \cap \left\{\sum_{j=1}^{\bar{j}-1} Y_j(a) - L(a) + w\right\}$$

$$= E_{\bar{j}}{}^+ \cap \left\{x \,\middle|\, \sum_{j=1}^{\bar{j}-1} c_j a_j^{-\beta} x_j + x_{\bar{j}} \leqslant \sum_N w_{\bar{j}}^n - \sum_{j=1}^{\bar{j}-1} a_j\right\}.$$

Hence

$$p(a) = (c_1 a_1^{-\beta}, \ldots, c_{\bar{j}-1} a_{\bar{j}-1}^{-\beta}, 1).$$

We then solve for the demand function, $x^n(p(a), W^n)$, of consumer n and let $u^*_n(a, W^n) = u(x^n(p(a), W_n))$. Then u^*_n represents R^*_n. It is an easy matter to show that

$$u^*_n = c_n' W_n \prod_{j=1}^{\bar{j}-1} a_j^{\beta\alpha_{nj}}$$

where c_n' is a constant. We observe that u^*_n is strictly quasi-concave. If consumer n maximizes u^*_n subject to

$$\sum_{j=1}^{\bar{j}-1} q_j^n a_j + W_n = p(a) w^n = w_{\bar{j}}^n,$$

he will choose

$$a_j = \left(\frac{\beta}{1+\beta}\right) \frac{\alpha_{nj} w_{\bar{j}}^n}{q_j^n} \quad \text{and} \quad W_n = \frac{1}{1+\beta} w_{\bar{j}}^n.$$

In order that all consumers choose the same value of a_j, it must be that for all $n \in N$

$$\frac{q_j^n}{\sum_N q_j^n} = \frac{\alpha_{nj} w_{\bar{j}}^n}{\sum_N \alpha_{nj} w_{\bar{j}}^n}.$$

Now

$$\left(\sum_N q_j^n\right) a + \max_{z \in L(a)} p(a)(w - z) = \sum_{j=1}^{\bar{j}-1} \left(\sum_N q_j^n\right) a_j - \sum_{j=1}^{\bar{j}-1} a_j + \sum_N w_j^n$$

is maximized at $\bar{a} \gg 0$ if and only if $\sum_N q_j^n = 1$ for all $j = 1, \ldots, \bar{j} - 1$.

It then must be that the Lindahl-Hotelling solution is as follows.

(1) For $n \in N$,

$$\bar{W}_n = \frac{1}{1 + \beta} w_j^n \quad \text{and} \quad \sum_N \bar{W}_n = \frac{1}{1 + \beta} \sum_N w_j^n.$$

(2) For $j = 1, \ldots, \bar{j} - 1$ and $n \in N$,

$$\bar{q}_j^n = \frac{\alpha_{nj} w_j^n}{\sum_N \alpha_{nj} w_j^n}.$$

(3) For $j = 1, \ldots, \bar{j} - 1$,

$$\bar{a}_j = \frac{\beta}{1 + \beta} \sum_N \alpha_{nj} w_j^n.$$

(4) For $n \in N$,

$$\sum_{j=1}^{\bar{j}-1} \bar{q}_j^n \bar{a}_j = \frac{\beta}{1 + \beta} w_j^n.$$

(5) For $n \in N$,

$$j = 1, \ldots, \bar{j} - 1, \quad \bar{p}_j = c_j \bar{a}_j^{-\beta} \quad \text{and} \quad \bar{x}^n = \frac{1}{\bar{p}_j} \alpha_{nj} \bar{W}_n.$$

These results also imply that $\bar{q}_j^n = \bar{x}_j^n / (\sum_N \bar{x}_j^n)$, that $-(\sum_N \bar{x}_j^n) \cdot \partial p_j / \partial a_j = 1$, and that $\bar{a}_j = \beta \sum_N \bar{p}_j^n \bar{x}_j^n$.

In this example, the task of a central authority which knows the value of β and wishes to find the Lindahl-Hotelling solution is particularly simple. The equilibrium total tax revenue from consumer n is simply $(\beta/(1 + \beta)) w_j^n$. If the central authority were able to observe quantities demanded at any competitive price vector p it could then discover the values of α_{nj} for each consumer and each good and then calculate the Lindahl-Hotelling equilibrium. Notice that total government expenditures on social overhead goods will be the fraction $\beta/(1 + \beta)$

of national income and that expenditures on any particular a_j are proportional to the total expenditures on commodity j.

Example 3. Suppose the technology and preferences are as in Example 2 except that for each j, there is a different β_j. That is

$$Y_j(a) = \left\{ y \,\middle|\, y_j = \frac{1}{c_j} a_j^{\beta_j}(-y_j),\ y_m = 0 \text{ for } m \neq j,\ m \neq \bar{j} \text{ and } y_j \leqslant 0 \right\}.$$

We proceed as in Example 2 to find the solution which is:

$$\text{(1)} \qquad \bar{W}_n = \gamma_n w_j^n \quad \text{where} \quad \gamma_n = \frac{1}{1 + \sum_j \beta_j \alpha_{nj}}.$$

$$\text{(2)} \qquad \bar{q}_j^n = \frac{\alpha_{nj}\gamma_n w_j^n}{\sum_N \alpha_{nj}\gamma_n w_j^n} = \frac{\alpha_{nj}\bar{W}_n}{\sum_N \alpha_{nj}\bar{W}_n} = \frac{\bar{x}_j^n}{\sum_N \bar{x}_j^n}.$$

$$\text{(3)} \qquad \bar{a}_j = \beta_j \sum_N \alpha_{nj}\gamma_n w_j^n = \beta_j \sum_N \alpha_{nj}\bar{W}_n = \beta_j \sum_N \bar{p}_j \bar{x}_j^n.$$

$$\text{(4)} \qquad \sum_{j=1}^{j-1} \bar{q}_j^n a = (1 - \gamma_n) w_j^n.$$

$$\text{(5)} \qquad \bar{p}_j = c_j \bar{a}_j^{-\beta_j}, \quad \bar{x}_j^n = \frac{1}{\bar{p}_j} \alpha_{nj} \bar{W}^n.$$

The solution here is slightly more complicated than in the previous example since the fraction of income which is paid in taxes $1 - \gamma_n$ differs among consumers being larger the more the consumer likes commodities the technology of which is highly responsive to the quantities of social overhead goods. Nevertheless computation remains simple if the β_j's and the demand functions are known.

Example 4. Technology and initial endowments are the same as in Example 2. Preferences of all consumers are represented by a Bergson-C.E.S. utility function of the form

$$u_n(x) = \left[\sum_{j=1}^{j-1} \alpha_{nj}^{1-\theta} x_j^{\theta} \right]^{1/\theta} \quad \text{where} \quad \theta \leqslant 1, \quad \theta \neq 0, \quad (\alpha_{n1}, \ldots, \alpha_{nj-1}) > 0.$$

(The limiting form where $\theta \to 0$ is Example 2.) We assume, with no loss of generality, that $\sum_{j=1}^{J-1} \alpha_{nj} = 1$. Let $\sigma = 1/(1-\theta)$ be the elasticity of substitution. Then the solution for $x_j^n(p, W_n)$ is

$$x_j^n = W_n \frac{\alpha_{nj} p_j^{-\sigma}}{\sum_j \alpha_{nj} p_j^{1-\sigma}} \quad \text{and} \quad u_n(x^n(p, W_n)) = W_n \left[\sum_j \alpha_{nj} p_j^{1-\sigma} \right]^{-1/(1-\sigma)}.$$

Also, $R^*_n(a, W_n)$ is represented by

$$u^*_n(a, W_n) = W_n \left[\sum_j \alpha_{nj} a_j^{-\beta(1-\sigma)} \right]^{-1/(1-\sigma)}$$

$$= W_n \left[\sum_j \alpha_{nj} a_j^{\gamma} \right]^{\beta/\gamma} \quad \text{where} \quad \gamma = -\beta(1-\sigma).$$

Now u^*_n will be quasi-concave if and only if $\gamma \leqslant 1$, which occurs when $\sigma \leqslant 1 + (1/\beta)$. Thus the more difficult it is to substitute ordinary commodities for each other, and the lower the elasticity of p_j with respect to a_j, the more likely is u^*_n to be quasi-concave (and hence R^*_n to have convex upper contour sets). If $\sigma \leqslant 1$, then u^* must be quasi-concave. If $\beta \leqslant 1$, then u^* must be quasi-concave whenever $\sigma \leqslant 2$.

When we solve for the Lindahl-Hotelling equilibrium in the case where $\gamma \leqslant 1$, we find that:

(1) $$\bar{W}_n = \frac{1}{1+\beta} w_j^n.$$

(2) $$\bar{q}_j^n = \frac{\alpha_{nj} w_j^n}{\sum_N \alpha_{nj} w_j^n}.$$

(3) $$\bar{a}_j = \frac{\beta}{1+\beta} \left[\sum_N \alpha_{nj} w_j^n \right].$$

(4) $$\sum_j \bar{q}_j^n \bar{a}_j = \frac{\beta}{1+\beta} w_j^n.$$

(5) $$\bar{p} = c_j \bar{a}_j^{-\beta} \quad \text{and} \quad \bar{x}_j^n = x_j^n(\bar{p}, \bar{W}_n).$$

This is essentially the same type of solution as for Example 2. The only added difficulty for the central authority is that demand functions are a bit more difficult to estimate. The novel feature of this example is that only for certain parameter values is R^*_n convex.

Example 5. The technology is quasi-linear and there exist function $g_1(a), \ldots, g_J(a)$ and $\bar{a} \in A$ where $g_1(\bar{a}) = g_2(\bar{a}) = \cdots = g_{J-1}(\bar{a}) = 1$ such that $\sum_M Y_m(a) = \{y \mid \text{for some } y_j' \in \sum_M Y_m(\bar{a}),\ y_j = g_j(a) y_j' \text{ for all } j = 1, \ldots, J-1\}$. Let $p(\bar{a}) = \bar{p}$ be the supporting price vector for $\sum_M Y_m(\bar{a})$. Then the supporting price vector for $\sum_M Y_m(a)$ is

$$\left(\frac{\bar{p}_1}{g_1(a)}, \ldots, \frac{\bar{p}_{J-1}}{g_{J-1}(a)}, 1\right).$$

If preferences of all consumers are as in Example 4, then R^*_n is represented by a function of the form

$$u^*_n = W\left[\sum_{j=1}^{J-1} \alpha_{nj} \bar{p}_j^{\,1-\sigma} (g_j(a))^{\sigma-1}\right]^{1/(\sigma-1)}.$$

If $g_j(a)$ is a concave function and if $\sigma \leqslant 2$, then u^* will be quasi-concave in a and W since u^* is a concave and increasing function of $g_1, \ldots, g_n$.

VII. On Finding Lindahl-Hotelling Equilibrium

We present a result which in principle might enable a central authority with knowledge only of demand functions for ordinary commodities and the technological data determining the supporting price function $p(a)$ and the correspondence $L(a)$ to compute a Lindahl-Hotelling solution for the quantity of social overhead goods and to determine the tax rates to impose. Also, a central authority which knows only the technological data and is able to observe a particular partially decentralized market equilibrium could use this result to determine whether or not this equilibrium is a Lindahl-Hotelling equilibrium.

THEOREM 12. *Consider an economy with quasi-linear technology satisfying the conditions of Theorems 9 and 10. Assume also that the supporting price vector $p(a)$ is a differentiable function of a in A, that preferences are represented by a continuous utility function u_n for each $n \in N$, and that there are (single valued) differentiable demand functions, $x^n(p, W)$ for each consumer $n \in N$. Let $\bar{a} \in$ Interior A and $\bar{p} = p(\bar{a})$ be the supporting price vector at $\bar{a}$. Then $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitutes a Lindahl-Hotelling equilibrium where the wealth distribution is $(\bar{W}_1, \ldots, \bar{W}_n)$ if and only if:*

(1) $\bar{x} = (x^1(\bar{p}, \bar{W}^1), \ldots, x^{\bar{n}}(\bar{p}, \bar{W}^{\bar{n}}))$.

(2) $\bar{p}\bar{z} \leqslant \bar{p}z$ for all $z \in L(\bar{a})$.

(3) $\sum_N \bar{W}_n = \bar{p}(w - \bar{z})$.

(4) $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^m)$ where

$$\sum_M \bar{y}^m \in \sum_M Y_m(\bar{a}) \quad \text{and} \quad \sum_N \bar{x}^n = \sum_M \bar{y}^m - \bar{z} + w.$$

(5) For all $n \in N$, $k \in K$, $\bar{q}_k{}^n = -\sum_J \bar{x}_j{}^n \frac{\partial p_j(\bar{a})}{\partial a_k}$.

(6) For all $k \in K$, $-\sum_J \left(\sum_N \bar{x}_j{}^n\right) \frac{\partial p_j(\bar{a})}{\partial a_k} = \sum_N \bar{q}_k{}^n = \frac{-\partial W(\bar{a})}{\partial a_k}$.

Proof. It is clear that the first four conditions are necessary for Lindahl-Hotelling equilibrium and sufficient for partially decentralized market equilibrium.

Let $\tilde{u}_n(p(a), W^n) = u_n(x^n(p(a), W_n)$. (The function $\tilde{u}_n$ is commonly called the indirect utility function, see D. Katzner [7].) Let $u^*{}_n(a, W_n) = \tilde{u}_n(p(a), W_n)$. Then $u^*{}_n$ represents $R^*{}_n$. Then $(\bar{p}, \bar{q}, \bar{a}, \bar{x}, \bar{y}, \bar{z})$ constitutes a Lindahl-Hotelling equilibrium if and only if conditions (1)–(4) hold and (5′) for all $n \in N$, there exists $w^n \geqslant 0$ such that $\sum_N w^n = w$ and $(\bar{a}, \bar{W}^n)$ maximizes u^* subject to $\bar{q}^n a + W^n \leqslant w^n$.

(6′) $\left(\sum_N \bar{q}^n\right) \bar{a} + W(\bar{a}) \geqslant \sum_N \bar{q}^n a + W(a)$ for all $a \in A$.

Our convexity and differentiability assumptions and the assumption that $\bar{a} \in$ Interior A, guarantee that the maximization problems (5′) and (6′) will be solved if and only if the first-order calculus conditions apply. These conditions are:

(5″) For all $n \in N$ and $n \in K$,

$$\frac{\sum_J [\partial \tilde{u}_n(p(\bar{a}), \bar{W}^n)/\partial p_j](\partial p_j/\partial a_k)}{[\partial \tilde{u}_n(p(\bar{a}), \bar{W}^n)/\partial W^n]} = \bar{q}_k{}^n$$

and

(6″) For all $k \in K$,

$$\sum_N \bar{q}_k{}^n = \frac{\partial W(\bar{a})}{\partial a_k}.$$

But it is known that under the conditions of the theorem,

$$\frac{\partial \tilde{u}_n(p(\bar{a}), \bar{W}^n)}{\partial p_j} \div \frac{\partial \tilde{u}_n(p(\bar{a}), \bar{W}^n)}{\partial W^n} = -\bar{x}_j^n.$$

See D. Katzner [7]. Therefore (5″) is equivalent to condition (5) above and (5″) and (6″) are together equivalent to (5) and (6). The theorem follows immediately.

An immediate consequence of conditions (5) and (6) of Theorem 11 is

COROLLARY 4. If there are $\bar{j} - 1$ social overhead goods and if $\partial p_j(a)/\partial a_k = 0$ for all $k \neq j$, then

$$\bar{q}_k^n = \frac{\bar{x}_k^n}{\sum_N \bar{x}_k^n}\left(\sum_N \bar{q}_k^n\right).$$

Suppose that the central authority uses the result of Theorem 12 to adjust tax shares according to quantities demanded in the search for a Lindahl-Hotelling solution. If consumers are aware that this is being done, they may account for the effect of their consumption decision on their tax burdens. If this is the case we have a problem which is closely related to the well-known "free rider problem" of the public finance literature.

It may be informative to examine Example 2 in this light. Suppose that the central authority is aware of the production parameters β_j and c_j and knows that utility functions are of the Cobb-Douglas form, but does not know the values of the α_{nj}. It might choose to estimate these parameters by observing the quantities that consumers choose under certain price and income situations. In particular if a consumer maximized u_n subject to $\sum_j p_j x_j^n \leqslant W^n$, he will choose x_j^n so that $\alpha_{nj} = p_j x_j^n / W^n$ for each j. Suppose that using information about quantities demanded, the central authority found a solution where

$$\bar{q}_j^n = \frac{\bar{x}_j^n}{\sum_N \bar{x}_j^n}, \quad \bar{a}_j = \beta_j \bar{p}_j \sum_N \bar{x}_j^n, \quad \bar{W}^n = w_{\bar{j}}^n - \sum_j \beta_j \bar{p}_j \bar{x}_j^n, \quad \bar{p}_j = c_j \bar{a}_j^{-\beta}$$

and $\bar{x}_j^n$ is the quantity of j chosen by n for each $n \in N$, $j = 1, \ldots, \bar{j} - 1$. If for each $n \in N$, $\bar{x}^n$ were chosen by consumer n so as to maximize u_n subject to $\sum_j \bar{p}_j x_j \leqslant \bar{W}^n$, then there would be a Lindahl-Hotelling equilibrium.

But if a consumer knows that his tax burden will always be

$$\sum_j \bar{q}_j^n \bar{a}_j = \sum_j \frac{x_j^n}{\sum_N \bar{x}_j^n}\left(\beta_j \bar{p}_j \sum_N \bar{x}_j^n\right) = \sum_j \beta_j \bar{p}_j \bar{x}_j^n$$

then he will actually choose $\bar{x}^n$ to maximize u_n subject to $\sum_j \bar{p}_j(1 + \beta_j)x_j^n \leqslant w_j^n$. If the β_j's are identical this reduces to $\sum_j \bar{p}_j x_j^n \leqslant w_j^n/(1 + \beta) = \bar{W}^n$ so that the solution is a Lindahl-Hotelling equilibrium. But in general the two constraints and hence the quantities chosen will be different and the solution would not be a Lindahl-Hotelling equilibrium.

The central authority might anticipate this response by consumers, and correct its response accordingly. Of course the consumers, like the Three Little Pigs, may anticipate the correction, and so on. Whether there is a plausible equilibrium for such a process might be worth investigating.

If the central authority is willing to settle for cruder estimates of the nature of demand for certain large identifiable subgroups of the population, the free rider problem loses its importance since the behavior of a single individual would have only a very small effect on the estimated demand function for his population class.

Conclusion

We have presented a model in which non-convexities of production may be large relative to the size of the markets. It is suggested that, even here, a substantial proportion of the economy may be left to the competitive market while only certain activities are "socialized". Simply ascribing certain choices of activities to a central authority and asserting that if the central authority does the "right thing" the economy will perform efficiently is not, of course, a very useful scientific achievement. Unless we believe the central authority to be endowed not only with the "best of intentions" but with analytic and computational acumen superior to that of our profession, little has been contributed if we cannot specify some rules of practical value to guide its choices. This paper has been only partially successful in such a program but, it is hoped, represents a small step forward.

It might be tempting to interpret our Theorem 3 as saying that if the central authority is foolish or misguided in its activities, once its decision is made, it is still efficient for the remainder of the economy to operate competitively. Such a conclusion, however, I think would be misleading, since it might be that in such circumstances efficiency could be enhanced by removing some activities from the domain of the central authority. The choice of which sectors of economic activity should be "socialized" when the actual and potential efficiency of the central authority is limited remains here, as in political debates, undecided.

It is my opinion that the notion of Lindahl-Hotelling equilibrium is useful in suggesting a reasonable objective for the central authority and has some promise of actual empirical measurement. The most serious restriction in the discussion here is the limitation to quasi-linear economies. One could in principle, define a Lindahl-Hotelling equilibrium for non quasi-linear economies. Additional problems are then introduced since the effect of social overhead goods on equilibrium prices will in general depend on demand influences and on the income distributional effects of taxation.

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Summary

An economy is considered in which production possibility sets for "firms" depend on the supply of certain commodities called "social overhead goods." These goods are supplied by a "central authority" which uses ordinary commodities as inputs. Production possibilities, in the present of a fixed vector of social overhead goods, are described by convex sets, but the set of all feasible outputs need not be convex.

Alternative decision rules for the central authority are considered. It is assumed that decisions other than those concerning the supply of social overhead goods are made competitively.

Zusammenfassung

Zur effizienten Bereitstellung von Sozialgütern. Es wird eine Wirtschaft betrachtet, in der die Produktionsmöglichkeiten privater Firmen abhängig sind von dem Angebot bestimmter Waren, die „Sozialgüter" genannt werden. Diese Güter werden von einer „zentralen Autorität" angeboten, die gewöhnliche Waren als Inputs verwendet. Die Produktionsmöglichkeiten werden bei Vorhandensein eines fixen Vektors von Sozialgütern durch konvexen Kurvenverlauf beschrieben, die Kurve aller möglichen Outputs hingegen muß nicht konvexen Charakters sein.

Auch alternative Entscheidungsregeln für die zentrale Autorität werden in Betracht gezogen. Es wird angenommen, daß Entscheidungen, die nicht das Angebot von Sozialgütern betreffen, unter Konkurrenzbedingungen getroffen werden.

Краткое содержание

Об эффективном обеспечении общественными накладными товарами.

Здесь рассматривается экономика, в которой ряд производственных возможностей для «фирм» зависит от поставки определенных предметов, называемых «общественными накладными товарами». Эти товары поставляются «центральной властью», которая для затрат использует обыкновенные товары. Производственные возможности при наличии постоянного вектора общественных накладных товаров характеризуются выпуклыми множествами, но множество всех верояных выпусков не обязательно должно быть выпуклым. Обсуждаются альтернативные правила решений для центральных властей. Предполагается, что решения, не касающиеся поставки общественных накладных товаров принимаются в порядке конкурентоспособности.