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Publication Date
2013-11-01

DOI
10.1016/j.jsc.2013.05.010

Peer reviewed

Corrigendum to "Rational rotation-minimizing frames on polynomial space curves of arbitrary degree" [J. Symbolic Comput. 45 (2010) 844–856]

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May 27, 2013

Abstract

The existence of rational rotation–minimizing frames on polynomial space curves is characterized by the satisfaction of a certain identity among rational functions. Part 2 of Remark 5.1 in the original paper states an inequality among the degrees of the denominators of these rational functions, but the proof given therein was incomplete. A formal proof of this inequality, which is essential to the complete categorization of rational rotation–minimizing frames on polynomial space curves, appears to be a rather formidable task. Since all known examples and special cases suggest that the inequality is correct, it is restated here as a conjecture rather than a definitive result, and some preliminary steps towards the proof are presented.

Keywords: rotation–minimizing frames; Pythagorean–hodograph curves; spatial motion planning; quaternions; polynomial identities.

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1 Introduction

Let $\mathbf{r}(t)$ be a spatial Pythagorean–hodograph (PH) curve, generated [2] from a primitive[1] quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ according to $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$, where $\mathcal{A}^*(t)$ is the conjugate of $\mathcal{A}(t)$. The *parametric speed* of $\mathbf{r}(t)$ is $\sigma(t) = |\mathbf{r}'(t)| = |\mathcal{A}(t)|^2$ $= u^2(t) + v^2(t) + p^2(t) + q^2(t)$. An adapted orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on $\mathbf{r}(t)$, where $\mathbf{f}_1$ is the curve tangent, is a *rotation–minimizing frame* (RMF) if its angular velocity $\boldsymbol{\omega}$ satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$ [1]. For a *rational* RMF, it is sufficient and necessary [7] that the condition

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2} \tag{1}$$

be satisfied by relatively prime polynomials $a(t), b(t)$. PH curves that satisfy (1) are called *RRMF curves*. Solutions defining true space curves were identified in [3] for $\mathcal{A}(t)$ quadratic, and in [4] for $\mathcal{A}(t)$ of any degree, under the assumption that $a^2 + b^2 = \sigma$.

Part 2 of Remark 5.1 in [4] stated that $\deg(a^2+b^2) \le \deg(\sigma)$ is *necessary* for the satisfaction of (1), and a proof of this claim was briefly sketched. Subsequently, the authors identified non–planar RRMF quintics in [5] that satisfy (1) with $\deg(a^2+b^2) < \deg(\sigma)$, and used this claim to give a complete classification of all RRMF quintics.

The existence of solutions to (1) with $\deg(a^2+b^2) \neq \deg(\sigma)$ and the complete classification of RRMF quintics in [5] prompted the authors to re–examine the claim in Remark 5.1 of [4], that solutions must satisfy $\deg(a^2+b^2) \le \deg(\sigma)$, and in this context it became apparent that the proof is incomplete. Concerted efforts to definitively prove this inequality have thus far proved unsuccessful. However, all known examples and special cases suggest that it is correct. Part 2 of Remark 5.1 in [4] is therefore restated as follows.

Conjecture 1 *Let* $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ *and* $a(t) + \mathrm{i}\,b(t)$ *be primitive, and satisfy* (1). *Then* $\deg(a^2 + b^2) \le \deg(u^2 + v^2 + p^2 + q^2)$.

2 A rational function bound

The remainder of this note gives some preliminary results on Conjecture 1. Let $\mathbb{H}$ denote the skew field of quaternions, $e(t), f(t), g(t), h(t), c(t), d(t) \in \mathbb{R}[t]$, and $\mathcal{Q}(t) = e(t) + f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \in \mathbb{H}[t]$. As in [4], it is convenient to introduce the notations

$$[\,\mathcal{Q}\,] = [\,e,f,g,h\,] = \frac{ef' - e'f - gh' + g'h}{e^2 + f^2 + g^2 + h^2} \quad \text{and} \quad [\,c,d\,] = \frac{cd' - c'd}{c^2 + d^2}\,. \tag{2}$$

[1]A quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ is said to be *primitive* if $\gcd(u,v,p,q) = 1$. Similarly, a complex polynomial $a(t) + \mathrm{i}\,b(t)$ is primitive if $\gcd(a,b) = 1$.

Lemma 2.1 Let $a(t), b(t), c(t), d(t), e(t), f(t), g(t), h(t) \in \mathbb{R}[t]$, $\mathcal{C} \in \mathbb{H}$, and $r = \alpha + \mathrm{i}\,\beta \in \mathbb{C}$ with $\beta \neq 0$. Then the following results hold.[2]

(a) Condition (1) remains unchanged if $\mathcal{A}(t)$ is replaced by $\mathcal{C}\mathcal{A}(t)$ for any $\mathcal{C} \neq 0$.

(b) $[\,e, f, g, h\,] \pm [\,c, d\,] = [\,E, F, G, H\,]$ where $E + F\,\mathbf{i} + G\,\mathbf{j} + H\,\mathbf{k} = (e + f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k})(c \pm \mathrm{i}\,d)$. In particular, $[\,e, f\,] \pm [\,c, d\,] = [\,C, D\,]$, where $C + \mathrm{i}\,D = (e + \mathrm{i}\,f)(c \pm \mathrm{i}\,d)$. In addition,

$$[\,(t - r)\,\mathcal{A}(t)\,] \;=\; \frac{\beta}{(t-\alpha)^2 + \beta^2}\,\frac{u^2 + v^2 - p^2 - q^2}{u^2 + v^2 + p^2 + q^2} \;+\; [\,u, v, p, q\,]\,.$$

(c) If $c + \mathrm{i}\,d = (t - r)^m$, then $[\,c, d\,] = m\,\beta\,[(t-\alpha)^2 + \beta^2]^{-1}$. Also, if $[\,a, b\,] = 0$, then a, b are linearly dependent over $\mathbb{R}$.

(d) If (a, b) and (c, d) are primitive with $[\,a, b\,] = [\,c, d\,]$ then $a + \mathrm{i}\,b = z\,(c + \mathrm{i}\,d)$ for $z \in \mathbb{C}$.

Proof:

(a) First, note that $uv' - u'v - pq' + p'q$ is the $\mathbf{i}$ component of $-\,\mathcal{A}'^*(t)\,\mathcal{A}(t)$. Now if $\mathcal{A}(t) \to \mathcal{C}\mathcal{A}(t)$, then $\mathcal{A}'^*(t)\mathcal{A}(t) \;\to\; \mathcal{A}'^*(t)\,\mathcal{C}^*\mathcal{C}\mathcal{A}(t) \;=\; |\mathcal{C}|^2\mathcal{A}'^*(t)\,\mathcal{A}(t)$ and $|\mathcal{A}(t)|^2 \;\to\; |\mathcal{C}|^2\,|\mathcal{A}(t)|^2$. Thus, condition (1) clearly remains unchanged when $\mathcal{A}(t) \to \mathcal{C}\mathcal{A}(t)$.

(b) This can be verified by straightforward calculation.

(c) If $m = 1$, then $c = t - \alpha, d = -\beta$ and thus $[\,c, d\,] = \beta\,[(t-\alpha)^2 + \beta^2]^{-1}$. Now, induction on m and the second part of item (b) verifies the first part of this item. Suppose now that $[\,a, b\,] = 0$. Then, $ab' = a'b$, and thus the Wronskian $W(a, b)$ vanishes, which implies that a, b are linearly dependent (over $\mathbb{R}$).

(d) Assume first that $a + \mathrm{i}\,b$ and $c + \mathrm{i}\,d$ are monic. In this case, note that $\deg(a) > \deg(b)$ and $\deg(c) > \deg(d)$. Now since $[\,a, b\,] = [\,c, d\,]$ and $(a + \mathrm{i}\,b)(c - \mathrm{i}\,d) = ac + bd + \mathrm{i}(bc - ad)$, item (b) shows that $[\,ac + bd, bc - ad\,] = 0$, and hence $ac + bd$ and $bc - ad$ are linearly dependent. But $\deg(ac + bd) > \deg(bc - ad)$, and hence $bc - ad = 0$. The latter implies that $a = c$ and $b = d$. Finally, let $z_1, z_2 \in \mathbb{C}$ be such that $z_1(a + \mathrm{i}\,b)$ and $z_2(c + \mathrm{i}\,d)$ are monic. Item (a) shows that $[\,z_1(a + \mathrm{i}\,b)\,] = [\,z_2(c + \mathrm{i}\,d)\,]$ and thus $z_1(a + \mathrm{i}\,b) = z_2(c + \mathrm{i}\,d)$. Therefore, $a + \mathrm{i}\,b = z_1^{-1}z_2(c + \mathrm{i}\,d)$, as required. ∎

Note that item (d) verifies Conjecture 1 in the case $(u, v) = (0, 0)$ or $(p, q) = (0, 0)$.

Now for $t \in \mathbb{R}$, let $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ be monic, primitive, and of degree $n \geq 1$. Then Theorem 2.1 of [6] shows that constants $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \in \mathbb{H}$ exist, such that

$$\mathcal{A}(t) \;=\; (t - \mathcal{C}_1)(t - \mathcal{C}_2)\,\cdots\,(t - \mathcal{C}_n)\,. \tag{3}$$

[2] Henceforth the imaginary unit i and quaternion element $\mathbf{i}$ are considered equivalent.

We derive a slightly different factorization of $\mathcal{A}(t)$, suited to the present context. Writing $\mathcal{C}_i = \mathrm{Re}(\mathcal{C}_i)+\mathrm{Im}(\mathcal{C}_i)$, we note that $r_i = \mathrm{Re}(\mathcal{C}_i)+\mathrm{i}\,|\mathrm{Im}(\mathcal{C}_i)| \in \mathbb{C}$ is a root of $|\mathcal{A}(t)|^2$. Moreover, since $\mathcal{C}_i$ and r_i are *similar* in the sense that a quaternion $\mathcal{S}_i \neq 0$ exists such that $\mathcal{S}_i\,\mathcal{C}_i = r_i\,\mathcal{S}_i$ (see Proposition 1.3 of [8]), using (3) we obtain

$$\mathcal{A}(t) \;=\; \mathcal{S}_1^{-1}\,(t-r_1)\,\mathcal{S}_1\,\mathcal{S}_2^{-1}\,(t-r_2)\,\mathcal{S}_2\,\cdots\,\mathcal{S}_n^{-1}\,(t-r_n)\,\mathcal{S}_n\,. \tag{4}$$

We are now ready to bound $[\,\mathcal{A}(t)\,]$, for any primitive quaternion polynomial. In view of item (a) of Lemma 2.1, we may suppose that $\mathcal{A}(t) = u(t)+v(t)\,\mathbf{i}+p(t)\,\mathbf{j}+q(t)\,\mathbf{k}$ is monic. Let $\mathcal{A}(t)$ be of degree n, and have the factorization (4). Then we define

$$\mathcal{B}_j(t) \;=\; u_j(t)+v_j(t)\,\mathbf{i}+p_j(t)\,\mathbf{j}+q_j(t)\,\mathbf{k} \;=\; \mathcal{S}_j\prod_{i=j+1}^{n}(t-\mathcal{C}_i)$$

for $j=1,2,\ldots,n-1$, and $\mathcal{B}_n=\mathcal{S}_n$. Then, writing $r_i=\alpha_i+\mathrm{i}\,\beta_i$ and $\phi_i=\beta_i/[\,(t-\alpha_i)^2+\beta_i^2\,]$, repeated application of items (a) and (b) in Lemma 2.1 yields

$$[\,\mathcal{A}(t)\,] \;=\; \sum_{i=1}^{n}\phi_i\,\frac{u_i^2+v_i^2-p_i^2-q_i^2}{u_i^2+v_i^2+p_i^2+q_i^2}\,.$$

Since $u_i^2+v_i^2+p_i^2+q_i^2$ is a *positive* polynomial, we note that

$$-1 \;\le\; \frac{u_i^2+v_i^2-p_i^2-q_i^2}{u_i^2+v_i^2+p_i^2+q_i^2} \;\le\; 1\,,$$

and thus

$$\sum_{i=1}^{n}\frac{-\,|\beta_i|}{(t-\alpha_i)^2+\beta_i^2} \;\le\; [\,\mathcal{A}(t)\,] \;\le\; \sum_{i=1}^{n}\frac{|\beta_i|}{(t-\alpha_i)^2+\beta_i^2}\,. \tag{5}$$

Based on the above results, we conclude with a special case of Conjecture 1.

Corollary 2.1 *Let $\mathcal{A}(t)=u(t)+v(t)\,\mathbf{i}+p(t)\,\mathbf{j}+q(t)\,\mathbf{k}$ and $a(t)+\mathrm{i}\,b(t)$ be monic primitive polynomials of degrees n and k satisfying (1) with $|\mathcal{A}(t)|^2$ having precisely two distinct zeros $\alpha\pm\mathrm{i}\,\beta\in\mathbb{C}$ with $\beta>0$. Then $k\le n$.*

Proof: Assume that $k\ge 1$. We have $[\,u,v,p,q\,]=[\,a,b\,]$, and thus a^2+b^2 has precisely the (two distinct) zeros $\alpha\pm\mathrm{i}\,\beta$. Therefore, either $a+\mathrm{i}\,b=(t-r)^k$ or $a+\mathrm{i}\,b=(t-\overline{r})^k$, $r=\alpha+\mathrm{i}\,\beta$. Suppose first that $a+\mathrm{i}\,b=(t-r)^k$. Then, in view of item (c) of Lemma 2.1 and formula (5), we see that $k\le n$. The case $a+\mathrm{i}\,b=(t-\overline{r})^k$ is treated similarly. ∎

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