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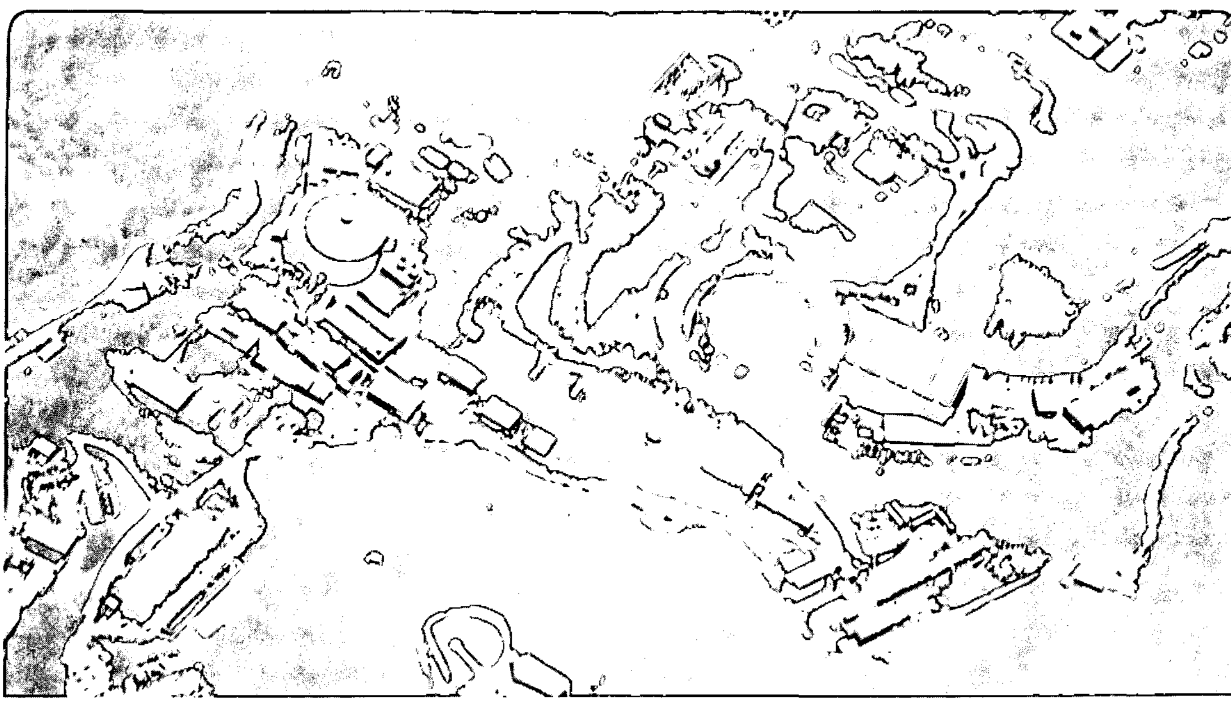

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How many zeros of a random polynomial are real?[1]

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How many zeros of a random polynomial are real?

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November 17, 1993

Abstract

We give an elementary derivation of the Kac integral formula for the expected number of real zeros of a random polynomial with independent standard normally distributed coefficients. We show that the expected number of real zeros is the length of the moment curve $(1, t, \ldots, t^n)$ projected onto the surface of the unit sphere, divided by π. The probability density of a real zero is proportional to how fast this curve is traced out. We generalize the Kac formula to polynomials with coefficients that have an arbitrary multivariate normal distribution. We show, for example, that for a particularly nice definition of random polynomial, the expected number of real zeros is *exactly* the square root of the degree.

If the random polynomials have an arbitrary density function, the expected number of zeros is a *weighted* length of the moment curve. We also calculate the the distribution of the real zeros of random power series and Fourier series, random sums of orthogonal polynomials, and random Dirichlet series. Extensions to systems of equations are also discussed.

Mathematics Subject Classification. Primary 34F05; Secondary 30B20.

Key words and phrases. Random Polynomials, Buffon needle problem, Integral Geometry, Random Power Series.

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Contents

1 Introduction

What is the expected number of real zeros E_n of a "random" polynomial of degree n? If the coefficients are independent standard normals, we show that as $n \to \infty$,

$$E_n = \frac{2}{\pi}\log(n) + 0.6257358072... + \frac{2}{n\pi} + O(1/n^2) .$$

The $\frac{2}{\pi}\log n$ term was derived by Kac in 1943 [12], who produced an integral formula for the expected number of real zeros. Papers on zeros of random polynomials include [2], [8], [11], [18] and [19]. There is also the comprehensive book of Bharucha-Reid and Sambandham [1].

We will derive the Kac (sometimes known as the Kac-Rice) formula for the expected number of real zeros with an elementary geometric argument that is related to the Buffon needle problem. We present the argument in a manner such that precalculus level mathematics is sufficient for understanding (and enjoying) the introductory arguments, while elementary calculus and linear algebra are sufficient prerequisites for much of the paper.

These results can be generalized to arbitrary normal distributions. Several authors [3] [15] [23] have studied random polynomials with independent normally distributed coefficients, each with mean zero, but with the variance of the i^{th} coefficient of a polynomial of degree n being equal to $\binom{n}{i}$. This is, in some sense, the most natural definition of random polynomial. For this particular random polynomial, the expected number of real zeros is exactly the square root of the degree.

We also compute the density of the real zeros of other collections of random functions. Specifically, we consider power series, Fourier series, sums of orthogonal polynomials, and Dirichlet series.

Fortunately, the methods discussed in this paper work equally well for random functions in several variables, so we are able to generalize many of our results to systems.

2 Elementary geometry and random polynomials

Section 2.1 is restricted to elementary geometry. Polynomials are never mentioned. The relationship is revealed in Section 2.2.

2.1 How fast do equators sweep out area?

We will denote (the surface of) the unit sphere centered at the origin in $\Re^{n+1}$ by S^n. Our figures correspond to the case $n = 2$. Higher dimensions provide no further complications.

Definition 2.1 *If $P \in S^n$ is any point, the* **associated equator** *$P_\perp$ is the set of points of S^n on perpendicular to the line from the origin to P.*

This generalizes our familiar notion of the Earth's equator, which is equal to (north pole)$_\perp$ and also equal to (south pole)$_\perp$. See Figure 1 below. Notice that $P_\perp$ is always a unit sphere ("great hypercircle") of dimension $n - 1$.

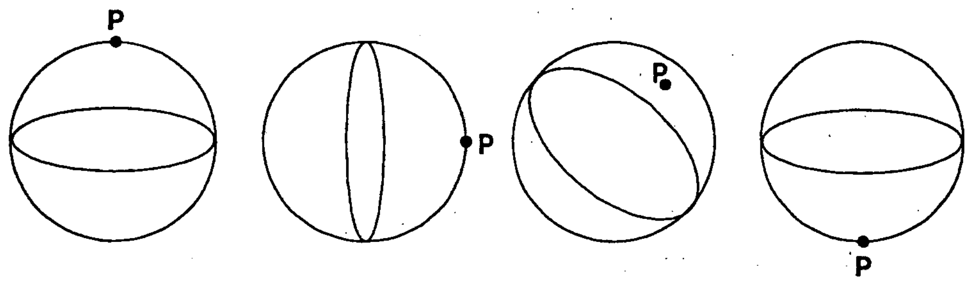


Figure 1 Points P and associated equators $P_\perp$.

Let $\gamma(t)$ be a (rectifiable) curve on the sphere S^n.

Definition 2.2 *Let* $\gamma_\perp$, **the equators of a curve,** *be the set* $\{P_\perp | P \in \gamma\}$.

Assume that γ has a finite length $|\gamma|$. Let $|\gamma_\perp|$ to be the area "swept out" by $\gamma_\perp$ – we will provide a precise definition shortly. We wish to relate $|\gamma|$ to $|\gamma_\perp|$.

If the curve γ is a small section of a great circle, then $\cup\gamma_\perp$ is a lune, the area bounded by two equators as illustrated in Figure 2. If γ is an arc of length θ, then our lune covers θ/π of the area of the sphere. The simplest case is $\theta = \pi$. We thus obtain the formula valid for arcs of great circles, that

$$\frac{|\gamma_\perp|}{\text{area of } S^n} = \frac{|\gamma|}{\pi}. \tag{1}$$

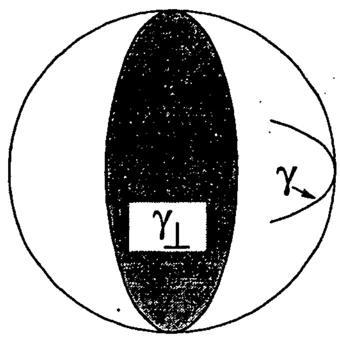


Figure 2 The lune $\cup\gamma_\perp$ when γ is a great circular arc

If γ is not a section of a great circle we may approximate it by a union of small great circular arcs, and the argument is seen to still apply.

The alert reader may notice something wrong. What if we continue our γ so that it is more than just half of a great-circle or what if our curve γ spirals many times around a point? Clearly γ may have quite a large length, but $|\gamma_\perp|$ remains small. The correct definition for $|\gamma_\perp|$ is the area swept out by $\gamma(t)_\perp$, as t varies, *counting multiplicities.* We now give the precise definitions.

Definition 2.3 *The* **multiplicity** *of a point* $Q \in \cup\gamma_\perp$ *is the number of equators in* $\gamma_\perp$ *that contain* Q, *i.e. the cardinality of* $\{t \in \Re | Q \in \gamma(t)_\perp\}$.

Definition 2.4 *We define* $|\gamma_\perp|$ *to be volume of* $\cup\gamma_\perp$ *counting multiplicity. More precisely, we define* $|\gamma_\perp|$ *to be the integral of the multiplicity over* $\cup\gamma_\perp$.

The sphere is then divided into regions of points that have the same multiplicity, as is schematically illustrated in Figure 3.

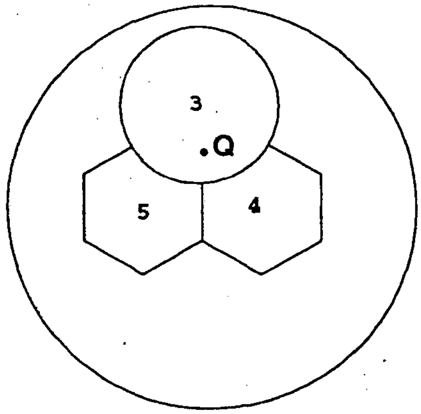


Figure 3 Schematic regions on a sphere with multiplicities 3, 4, and 5. Q has multiplicity 3.

Lemma 2.1 *If γ is a rectifiable curve, then*

$$\frac{|\gamma_\perp|}{\textit{area of } S^n} = \frac{|\gamma|}{\pi}.$$

As an example, consider a point P on the surface of the Earth. If we assume that the point P is receiving the direct ray of the sun — for our purposes, we consider the sun to be fixed in space relative to the Earth during the course of a day, with rays arriving in parallel — then $P_\perp$ is the great circle that divides day from night. This great circle is known to astronomers as the **terminator**. During the Earth's daily rotation, the point P runs through all the points on a circle γ of fixed latitude. Similarly, the Earth's rotation generates the collection of terminators $\gamma_\perp$.

The multiplicity in $\gamma_\perp$ is two on a region between two latitudes. This is a fancy mathematical way of saying that unless you are too close to the poles, you witness both a sunrise and a sunset every day! The summer solstice is a convenient example. P is on the tropic of Cancer and Equation (1) becomes

$$\frac{2 \times \text{(The surface area of the Earth between the Arctic/Antarctic Circles)}}{\text{The area of the Earth}} = \frac{\text{The length of the Tropic of Cancer}}{\pi \times \text{(The radius of the Earth)}}$$

or equivalently

$$\frac{\text{(The surface area of the Earth between the Arctic/Antarctic Circles)}}{\text{The area of the Earth}} = \frac{\text{The length of the Tropic of Cancer}}{\text{The length of the Equator}}.$$

Equations appropriate for other days of the year may be derived by the reader.

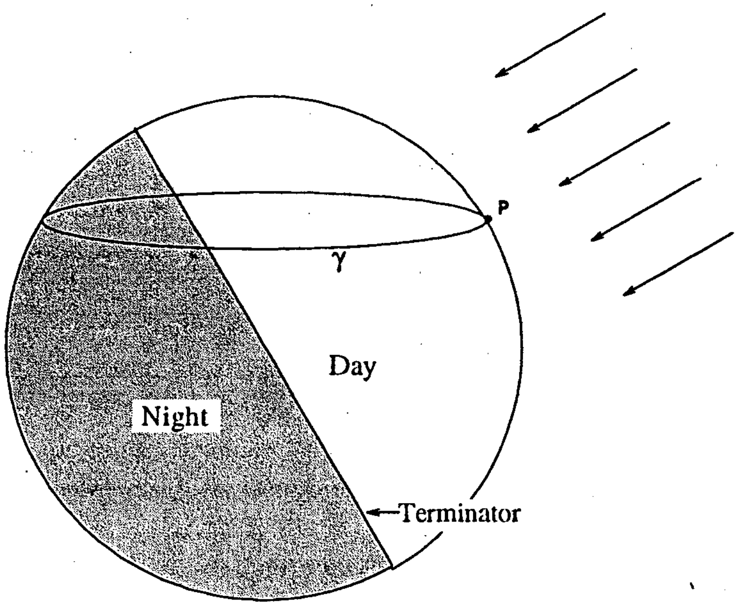


Figure 4 On the summer solstice, the direct ray of the sun reaches P on the Tropic of Cancer γ.

2.2 The expected number of real zeros of a random polynomial

What does the geometric argument in the previous section and Formula (1) in particular have to do with the number of real zeros of a random polynomial? Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

be a non-zero polynomial. Define the two vectors

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ and } v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}.$$

The curve in $\Re^{n+1}$ traced out by $v(t)$ as t runs over the real line is the **moment curve**.

The condition that t is a zero of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ is precisely the condition that a is perpendicular to $v(t)$. Another way of saying this is that $v(t)_\perp$ is the set of polynomials which have t as a zero.

Define unit vectors

$$\mathbf{a} \equiv a/\|a\|, \ \gamma(t) \equiv v(t)/\|v(t)\|.$$

As before, $\gamma(t)_\perp$ corresponds to the polynomials which have t as a zero.

Let $\gamma = \{\gamma(t) : t \in \Re\}$. When $n = 2$, the curve γ is the intersection of the elliptical (squashed) cone and the unit sphere. In particular, the curve is not planar – See Figure 5. If we include the point at infinity, γ becomes a simple closed curve when n is even. (In projective space, the curve is closed for all n.) The number of times that a point a on our sphere is covered by an equator in $\gamma_\perp$, i.e. the multiplicity of a in $\gamma_\perp$ is exactly the number of real zeros of the corresponding polynomial.

So far, we have not discussed *random* polynomials. It is well known that if the a_i are independent standard normals, then the vector $\mathbf{a}$ is uniformly distributed on the sphere S^n since the joint density function in spherical coordinates is a function of the radius alone.

What is $E_n \equiv$ the expected number of real zeros of a random polynomial? A random polynomial is identified with a uniformly distributed random point on the sphere, so E_n is the area of the sphere with our convention of counting multiplicities.

Equation (1) (read backwards!) states that

$$E_n = \frac{1}{\pi}|\gamma|.$$

Our question about the expected number of real zeros of a random polynomial is reduced to finding the length of the curve γ. We compute this length in Section 3.1.

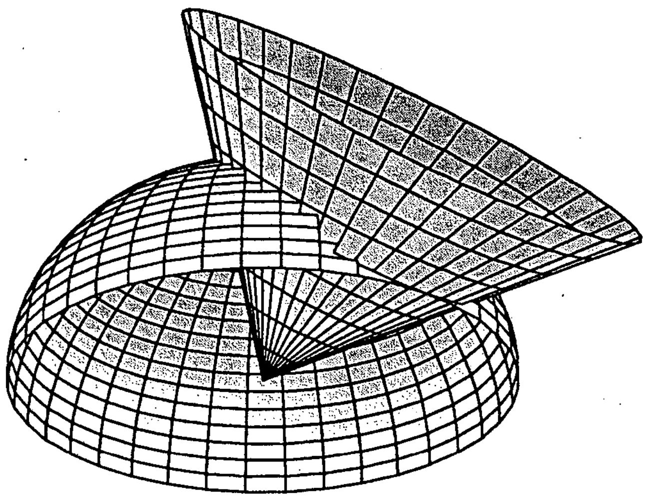


Figure 5 When $n = 2$, γ is the intersection of the sphere and cone. The intersection is a curve that includes the north pole and a point on the equator.

Similar geometric considerations have also lead to the calculation of the distribution of the real eigenvalues and real generalized eigenvalues of random matrices [7, 4, 5, 6].

3 Calculations

We now obtain concrete results. The reader will also find more little surprises in the sections to follow.

3.1 Calculating the length of γ

We invoke calculus to obtain the integral formula for the length of γ, and hence the expected number of zeros of a random polynomial, E_n. The result was first obtained by Kac in 1943.

Theorem 3.1 *The expected number of real zeros of a degree n polynomial with independent standard normal coefficients is*

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}}dt$$

$$= \frac{4}{\pi}\int_0^1\sqrt{\frac{1}{(1-t^2)^2} - \frac{(n+1)^2t^{2n}}{(1-t^{2n+2})^2}}dt. \quad (2)$$

Proof The standard arclength formula is

$$|\gamma| = \int_{-\infty}^{\infty} \|\gamma'(t)\| dt.$$

To calculate the integrand, we first consider any differentiable $v(t) : \Re \longrightarrow \Re^{n+1}$. It is not hard to show that

$$\gamma'(t) = \left(\frac{v(t)}{\sqrt{v(t)\cdot v(t)}}\right)' = \frac{[v(t)\cdot v(t)]v'(t) - [v(t)\cdot v'(t)]v(t)}{[v(t)\cdot v(t)]^{3/2}},$$

and therefore,

$$\|\gamma'(t)\|^2 = \left(\frac{v(t)}{\sqrt{v(t)\cdot v(t)}}\right)' \cdot \left(\frac{v(t)}{\sqrt{v(t)\cdot v(t)}}\right)' = \frac{[v(t)\cdot v(t)][v'(t)\cdot v'(t)] - [v(t)\cdot v'(t)]^2}{[v(t)\cdot v(t)]^2}.$$

We may proceed in two different ways.

Method I (Direct approach):

If $v(t)$ is the moment curve then we may calculate $\|\gamma'(t)\|$ with the help of the following observations and some messy algebra:

$$v(t)\cdot v(t) = 1 + t^2 + t^4 + \ldots + t^{2n} = \frac{1-t^{2n+2}}{1-t^2};$$

$$v'(t)\cdot v(t) = t + 2t^3 + 3t^5 + \ldots + nt^{2n-1} = \frac{1}{2}\frac{d}{dt}\left(\frac{1-t^{2n+2}}{1-t^2}\right) = \frac{t\left(1-t^{2n} - n\,t^{2n} + n\,t^{2n+2}\right)}{\left(t^2-1\right)^2};$$

$$v'(t)\cdot v'(t) = 1 + 4t^2 + 9t^4 + \ldots + n^2t^{2n-2} = \frac{1}{4t}\frac{d}{dt}t\frac{d}{dt}\left(\frac{1-t^{2n+2}}{1-t^2}\right) = \frac{t^{2n+2} - t^2 - 1 + t^{2n}\left(n\,t^2 - n - 1\right)^2}{\left(t^2-1\right)^3};$$

we arrive at the following formula, first arrived at by Kac.

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sqrt{(t^{2n+2}-1)^2 - (n+1)^2t^{2n}(t^2-1)^2}}{(t^2-1)(t^{2n+2}-1)} dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}} dt.$$

Method II (Sneaky version):

By introducing a logarithmic derivative, we can avoid the messy algebra in Method I. Let $v(t) : \Re \longrightarrow \Re^{n+1}$ be any differentiable curve. Then it easy to check that

$$\frac{\partial^2}{\partial x \partial y} \log[v(x)\cdot v(y)]\Big|_{y=x=t} = \|\gamma'(t)\|^2. \tag{3}$$

Thus we have an alternative expression for $\|\gamma'(t)\|^2$.

Remember that we obtained $\gamma(t)$ from $v(t)$ by projecting. The length of $v(t)$ is unimportant, any scaling is equally valid. The logarithmic derivative is tailored to be invariant under scaling by any $\lambda(t)$:

$$\frac{\partial^2}{\partial x\partial y}\log[\lambda(x)v(x)\cdot\lambda(y)v(y)] = \frac{\partial^2}{\partial x\partial y}\{\log[v(x)\cdot v(y)] + \log(\lambda(x)) + \log(\lambda(y))\} = \frac{\partial^2}{\partial x\partial y}\log[v(x)\cdot v(y)].$$

The simplicity with which lengths of curves can be measured is one way to motivate the scale independent Fubini-Study metric [10].

When $v(t)$ is the moment curve,

$$v(x)\cdot v(y) = 1 + xy + x^2y^2 + \ldots + x^ny^n = \frac{1-(xy)^{n+1}}{1-xy}.$$

the Kac formula is then

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{\frac{\partial^2}{\partial x\partial y}\log\frac{1-(xy)^{n+1}}{1-xy}\Big|_{y=x=t}}\, dt.$$

This version of the Kac formula first appeared in [15]. □

3.2 The density of zeros

We focused, up until now, on the length of $\gamma = \{\gamma(t) | -\infty < t < \infty\}$, and concluded that it equals the expected number of zeros on the real line. What we really did, however, was observe that the density of zeros is the speed of the curve divided by π. Thus

$$\rho_n(t) \equiv \frac{1}{\pi}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}$$

is the expected number of real zeros per unit length at the point $t \in \Re$. This is a true density: integrating $\rho_n(t)$ over any interval (or measurable set) produces the expected number of real zeros on that (or set). The *probability* density for a random real zero is $\rho_n(t)/E_n$. It is straightforward [12, 13] to see that as $n \to \infty$, the real zeros are concentrated near the point $t = \pm 1$.

The asymptotic behavior of both the density and expected number of real zeros is displayed in the subsection below.

3.3 The asymptotics of the Kac formula

This section contains a long calculation that is unrelated to the rest of the paper. A short argument could have shown that $E_n \sim \frac{2}{\pi}\log n$ [12], but since several researchers, including Christensen, Sambandham, Stevens and Wilkins have sharpened Kac's original estimate, we show here how successive terms of the asymptotic series may be derived, although we will only derive a few terms of the series explicitly. The constant C_1 and the next term $\frac{2}{n\pi}$ were unknown to previous researchers. See [1, pp. 90–91] for a summary of previous estimates of the constant.

Theorem 3.2 *As* $n \to \infty$,

$$E_n = \frac{2}{\pi}\log(n) + C_1 + \frac{2}{n\pi} + O(1/n^2) ,$$

where

$$C_1 = \frac{2}{\pi}\left(\log(2) + \int_0^\infty \left\{\sqrt{\frac{1}{x^2} - \frac{4e^{-2x}}{(1-e^{-2x})^2}} - \frac{1}{x+1}\right\} dx\right) = 0.6257358072\ldots .$$

Proof

We now study the asymptotic behavior of the density of zeros. To do this, we make the change of variables $t = 1 + x/n$, so

$$E_n = 4\int_0^\infty \hat{\rho}_n(x)\, dx ,$$

where

$$\hat{\rho}_n(x) = \frac{1}{n\pi}\sqrt{\frac{n^4}{x^2(2n+x)^2} - \frac{(n+1)^2(1+x/n)^{2n}}{[(1+x/n)^{2n+2}-1]^2}}$$

is the (transformed) density of zeros. Using

$$\left(1+\frac{x}{n}\right)^n = e^x\left(1 - \frac{x^2}{2n}\right) + O(1/n^2) ,$$

and Mathematica, we see that for any fixed x, as $n \to \infty$, the density of zeros is given by

$$\hat{\rho}_n(x) = \hat{\rho}_\infty(x) + \left[\frac{x(2-x)}{2n}\hat{\rho}_\infty(x)\right]' + O(1/n^2) , \tag{4}$$

where

$$\hat{\rho}_\infty(x) \equiv \frac{1}{2\pi}\left[\frac{1}{x^2} - \frac{4e^{-2x}}{(1-e^{-2x})^2}\right]^{1/2}.$$

This asymptotic series cannot be integrate term by term. We solve this problem by considering the asymptotic series for the Kac power series (Section 4.6):

$$\frac{\chi[x>1]}{2\pi x} - \frac{1}{2\pi(2n+x)} = \frac{\chi[x>1]}{2\pi x} - \frac{1}{4n\pi} + O(1/n^2)\,, \tag{5}$$

where we have introduced the factor

$$\chi[x>1] \equiv \begin{cases} 1 & if \quad x>1 \\ 0 & if \quad x \le 1 \end{cases}$$

to avoid to pole at $x=0$. Subtracting (5) from (4), we obtain

$$\hat{\rho}_n(x) - \left\{\frac{\chi[x>1]}{2\pi x} - \frac{1}{2\pi(2n+x)}\right\} = \left\{\hat{\rho}_\infty(x) - \frac{\chi[x>1]}{2\pi x}\right\} + \left\{\left[\frac{x(2-x)}{2n}\hat{\rho}_\infty(x)\right]' + \frac{1}{4\pi n}\right\} + O(1/n^2)\,.$$

We then integrate term by term from 0 to ∞ to get

$$\int_0^\infty \hat{\rho}_n(x)\,dx - \frac{1}{2\pi}\log(2n) = \int_0^\infty \left\{\hat{\rho}_\infty(x) - \frac{\chi[x>1]}{2\pi x}\right\}\,dx + \frac{1}{2n\pi} + O(1/n^2)\,.$$

The theorem immediately follows from this formula and one final trick: we replace $\chi[x>1]/x$ with $1/(x+1)$ in the definition of C_1 so we can express it as a single integral of an *elementary* function.

□

4 Generalizations and examples

Reviewing the discussion in Section 2, we see that we could omit some members of our basis set $\{1, x, x^2, \ldots, x^n\}$ and ask how many zeros are expected to be real of an nth degree polynomial with, say, its cubic term deleted. The proof would hardly change. Or we can change the function space entirely and ask how many zeros of the random function

$$a_0 + a_1\sin(x) + a_2 e^{|x|}$$

are expected to be real – the answer is 0.63662. The only assumption is that the coefficients are standard normals. If $f_0, f_1, \ldots, f_n$ is any collection of rectifiable functions, we may define the analogue of the moment curve

$$v(t) = \begin{pmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}. \tag{6}$$

The function $\frac{1}{\pi}\|\gamma'(t)\|$ is the density of a real zero; its integral over $\Re$ is the expected number of real zeros.

We may relax the assumption that the coefficient vector $a = (a_0, \ldots, a_n)^T$ contains independent standard normals, by allowing for any multivariate distribution with zero mean. If the a_i are normally distributed, $E(a) = 0$ and $E(aa^T) = C$, then a is a (central) multivariate normal distribution with covariance matrix C. It is easy to see that a has this distribution if and only if $C^{-1/2}a$ is a vector of standard normals. Since

$$a \cdot v(t) = C^{-1/2}a \cdot C^{1/2}v(t),$$

the density of real zeros with coefficients from an arbitrary central multivariate normal distribution is

$$\frac{1}{\pi}||\mathbf{w}'(t)||, \text{ where } w(t) = C^{1/2}v(t), \text{ and } \mathbf{w}(t) = w(t)/||w(t)||. \tag{7}$$

The expected number of real zeros is the integral of $\frac{1}{\pi}||\mathbf{w}'(t)||$.

We now state our general result.

Theorem 4.1 *Let $f_0(t), \ldots, f_n(t)$ be any collection of differentiable functions and $a_0, \ldots, a_n$ be the elements of a multivariate normal distribution with mean zero and covariance matrix C. The expected number of real zeros on an interval (or measurable set) I of the equation*

$$a_0 f_0(t) + a_1 f_1(t) + \ldots + a_n f_n(t) = 0$$

is

$$\int_I \frac{1}{\pi}||\mathbf{w}'(t)||dt,$$

where $\mathbf{w}$ is defined by Equations (6) and (7). In logarithmic derivative notation this is

$$\frac{1}{\pi}\int_I \left(\frac{\partial^2}{\partial x \partial y}\left(\log v(x)^T C v(y)\right)\Big|_{y=x=t}\right)^{1/2} dt,$$

where v is defined by Equation (6).

Geometrically, changing the covariance is the same as changing the inner product on the space of functions.

We now enumerate several examples of Theorem 4.1. We consider examples for which $v(x)^T C v(y)$ is a nice enough function of x and y that the density of zeros can be easily described. For a survey of the literature related to the first, third, and fifth examples, see [1], which also includes the results of numerical experiments. In our discussion of random series, proofs of convergence are omitted. Interested readers may refer to [22]. We also suggest the classic book of J-P. Kahane [14], where other problems about random series of functions are considered.

4.1 The Kac formula

If the coefficients of random polynomials are independent, standard normal random variables, we saw in the previous section that from

$$v(x)^T C v(y) = \frac{1-(xy)^{n+1}}{1-xy}, \tag{8}$$

we can derive the Kac integral formula.

4.2 A random polynomial with a simple answer

Consider random polynomials

$$a_0 + a_1 x + \ldots + a_n x^n = 0,$$

where the a_i are independent normals with variances $\binom{n}{i}$. Such random polynomials have been studied by [15] and [23], because of their mathematical properties, and by [3] because of their relationship to quantum physics.

The binomial theorem simplifies the computation of

$$v(x)^T C v(y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^k = (1+xy)^n.$$

We see that the density of zeros is given by

$$\rho(t) = \frac{\sqrt{n}}{\pi(1+t^2)}.$$

This is a Cauchy distribution, that is, arctan(t) is uniformly distributed on $[-\pi/2, \pi/2]$. Integrating the density shows that the expected number of real zeros is $\sqrt{n}$. As we shall see in the following section, this simple expected value and density is reflected in the geometry of γ.

Exercise: Show that if we have two independent random polynomials, $p(t)$ and $q(t)$, each of degree n, and each distributed as in this example, then the expected number of *fixed points* of the rational mapping

$$p(t)/q(t) : \Re \cup \{\infty\} \rightarrow \Re \cup \{\infty\}$$

is exactly $\sqrt{n+1}$. (Hint: Consider $p(t) - tq(t) = 0$.)

4.3 Random trigonometric sums and Fourier series

Consider the trigonometric sum

$$\sum_{k=0}^{n} a_k \cos \nu_k \theta + b_k \sin \nu_k \theta,$$

where a_k and b_k, are independent normal random variables with means zero and variances σ_k^2. Notice that

$$v(x)^T C v(y) = \sum_{k=0}^{n} \sigma_k^2 (\sin \nu_k x \sin \nu_k y + \cos \nu_k x \cos \nu_k y) = \sum_{k=0}^{n} \sigma_k^2 \cos \nu_k (x-y),$$

and we see that the integrand is a constant. Thus the real zeros of the random trigonometric sum are uniformly distributed on the real line, and the expected number of zeros on the interval $[a,b]$ is

$$\frac{b-a}{\pi} \sqrt{\frac{\sum \nu_k^2 \sigma_k^2}{\sum \sigma_k^2}}.$$

This formula also holds for a variety of Fourier series. The similarity between this formula and the Pythagorean theorem is more than superficial, as we will see in the following section. Several authors, including Christensen, Das, Dunnage, Jamrom, Maruthachalam, Qualls and Sambandham have derived results about the expected number of zeros of these and other trigonometric sums.

4.4 Spijker's lemma on the Riemann sphere

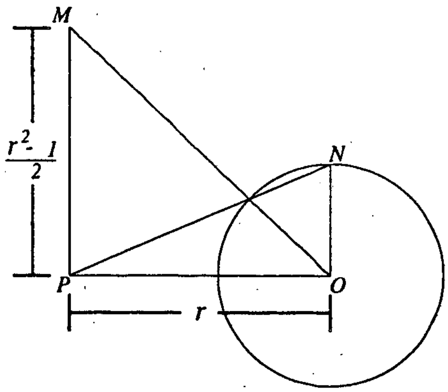
Any curve in $\Re^n$ can be interpreted as $v(t)$ for some space of random functions. Let

$$r(t) = \frac{a(t) + ib(t)}{c(t) + id(t)}$$

be any rational function, where a, b, c, and d are real polynomials of a real variable t. Let γ be the stereographic projection of $r(t)$ onto the Riemann sphere. It is not difficult to show that γ is the projection of the curve

$$(f_0(t), f_1(t), f_2(t))$$

onto the unit (Riemann) sphere, where $f_0 = 2(ac+bd)$, $f_1 = 2(bc-ad)$, $f_2 = a^2+b^2-c^2-d^2$. The geometry is illustrated in the figure below.



Therefore the length of γ is π times the expected number of zeros of the random function

$$a_0 f_0 + a_1 f_1 + a_2 f_2,$$

where the a_i are independent standard normals. For example, if a, b, c, and d are polynomials of degrees no more than n, then any such function has degree at most $2n$, so the length of γ can be no more than $2n\pi$. By taking a Möbius transformation, we arrive at Spijker's lemma:
The image of any circle in the complex plane under a complex rational mapping, with numerator and denominator having degrees no more than n, has length no longer than $2n\pi$.
This example was obtained from Wegert and Trefethen [25].

4.5 Random sums of orthogonal polynomials

Consider the vector space of polynomials of the form $\sum_{k=0}^{n} a_k P_k(x)$ where a_k are independent standard normal random variables, and where $\{P_k(x)\}$ is a set of normalized orthogonal polynomials with any nonnegative weight function on any interval. The Darboux-Christoffel formula [9] 8.902 states that

$$\sum_{k=0}^{n} P_k(x)P_k(y) = \left(\frac{q_n}{q_{n+1}}\right) \frac{P_n(y)P_{n+1}(x) - P_n(x)P_{n+1}(y)}{x-y},$$

where q_n (*resp.* q_{n+1}) is defined to be the lead coefficient of P_n (*resp.* P_{n+1}). With this formula and a bit of work, we see that

$$\rho(t) = \frac{\sqrt{3}}{6\pi}\sqrt{2G'(t) - G^2(t)},$$

where

$$G(t) = \frac{d}{dt} \log \frac{d}{dt}\left(\frac{P_{n+1}(t)}{P_n(t)}\right).$$

This is equivalent to formula (5.21) in [1]. Interesting asymptotic results have been derived by Das and Bhatt. The easiest example to consider is the random Chebyshev polynomials, for which the the density of zeros is an elementary function of n and t.

4.6 Kac power series

Consider a random power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots,$$

where a_k are independent standard normal random variables. This has radius of convergence one with probability one. Thus we will assume that $-1 < x < 1$. In this case,

$$v(x)^T C v(y) = \frac{1}{1-xy}.$$

The logarithmic derivative reveals a density of zeros of the form

$$\rho(t) = \frac{1}{\pi(1-t^2)}.$$

We see that the expected number of zeros on any subinterval $[a,b]$ of $(-1,1)$ is

$$\frac{1}{2\pi}\log\frac{(1-a)(1+b)}{(1+a)(1-b)}.$$

This result may also be derived from the original Kac formula by letting $n \to \infty$.

4.7 Kac power series with correlated coefficients

What effect does correlation have on the density of zeros? We will consider a simple generalization of the previous example. Consider random power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

where a_k are standard normal random variables, but assume that the correlation between a_k and a_{k+1} equals some constant r for all k. Thus the covariance matrix is tridiagonal with one on the diagonal and r on the superdiagonal and subdiagonal. In order to assure that this matrix be positive definite, we will assume that $|r| \leq \frac{1}{2}$. By the Gershgorin Theorem the spectral radius of the covariance matrix is less than or equal to $1+2r$, and therefore the radius of convergence of the random sequence is independent of r. Thus we will, as in the previous example, assume that $-1 < x < 1$. We see that

$$v(x)^T C v(y) = \frac{1+r(x+y)}{1-xy},$$

so

$$\rho(t) = \frac{1}{\pi}\sqrt{\frac{1}{(1-t^2)^2} - \frac{r^2}{(1+2rt)^2}}.$$

Notice that the correlation between coefficients has decreased the density of zeros throughout the interval.

4.8 Random entire functions

Consider a random power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

where a_k are independent central normal random variables with variances $1/k!$, i.e., the covariance matrix is diagonal with the numbers $1/k!$ down the diagonal. This series has infinite radius of convergence with probability one. Now clearly

$$v(x)^T C v(y) = e^{xy},$$

so $\rho(t) = 1/\pi$. In other words, the real zeros are uniformly distributed on the real line, with a density of $1/\pi$ zeros per unit length.

4.9 Random Dirichlet series

Consider a random Dirichlet series

$$f(x) = a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \ldots.$$

Assume that a_k are independent standard normal random variables. This converges with probability one if $x > 1/2$. We see that

$$v(x)^T C v(y) = \sum_{k=1}^{\infty} \frac{1}{k^{x+y}} = \zeta(x+y),$$

and that the expected number of zeros on any interval $[a,b]$ is

$$\frac{1}{2\pi} \int_a^b \sqrt{[\log \zeta(2t)]''} dt.$$

5 The geometry of γ

The second and third examples above are points of departure for lots of interesting mathematics. We explore these connections in the following subsections.

5.1 A curve with more symmetries than meet the eye

We return to the second example, and explore why the distribution of the real zeros was so simple. Take the curve $v(t)$ and make the change of variables $t = \tan\theta$ and scale to obtain

$$\gamma(\theta) = (\cos^n \theta)(v(\tan\theta)).$$

Doing so shows that

$$\gamma(\theta) = \begin{pmatrix} \binom{n}{0}^{1/2} \cos^n \theta \\ \binom{n}{1}^{1/2} \cos^{n-1} \theta \sin\theta \\ \binom{n}{2}^{1/2} \cos^{n-2} \theta \sin^2 \theta \\ \vdots \\ \binom{n}{n}^{1/2} \sin^n \theta \end{pmatrix},$$

i.e. $\gamma_k(\theta) = \binom{n}{k}^{1/2} \cos^{n-k}\theta \sin^k \theta$, where the dimension index k runs from 0 to n.

We have chosen to denote this curve the *super-circle*. The binomial expansion of $(\cos^2\theta + \sin^2\theta)^n = 1$ tells us that our super-circle lives on the unit sphere. Indeed when $n = 2$, the super-circle is merely a small-circle on the unit sphere in $\Re^3$. When $n = 1$, the super-circle is the unit circle in the plane.

What is not immediately obvious is that every point on this curve "looks" exactly the same. We will display an orthogonal matrix $Q(\phi)$ that rotates $\Re^{n+1}$ in such a manner that each and every point on the super-circle $\gamma(\theta)$ is sent to $\gamma(\theta+\phi)$. To do this, we show that γ is a solution to a "nice" ordinary differential equation.

By a simple differentiation of the kth component of $\gamma(\theta)$, we see that

$$\frac{d}{d\theta}\gamma_k(\theta) = \alpha_k \gamma_{k-1}(\theta) - \alpha_{k+1}\gamma_{k+1}(\theta), \quad k = 0, \ldots, n,$$

where $\alpha_k \equiv \sqrt{k(n+1-k)}$. In matrix–vector notation this means that

$$\frac{d}{d\theta}\gamma(\theta) = A\gamma(\theta), \text{ where } A = \begin{pmatrix} 0 & -\alpha_1 & & & & \\ \alpha_1 & 0 & -\alpha_2 & & & \\ & \alpha_2 & 0 & -\alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{n-1} & 0 & -\alpha_n \\ & & & & \alpha_n & 0 \end{pmatrix}. \qquad (9)$$

i.e. A has the α_i on the subdiagonal, the $-\alpha_i$ on the superdiagonal, and 0 everywhere else including the main diagonal.

The solution to the ordinary differential equation (9) is

$$\gamma(\theta) = e^{A\theta}\gamma(0). \tag{10}$$

The matrix $Q(\phi) \equiv e^{A\phi}$ is orthogonal because A is anti-symmetric, and indeed $Q(\phi)$ is the orthogonal matrix that we promised would send $\gamma(\theta)$ to $\gamma(\theta+\phi)$. We suspect that Equation (10) with the specification that $\gamma(0) = (1, 0, \ldots, 0)^T$ is the most convenient description of the super-circle. Differentiating (10) any number of times shows explicitly that

$$\frac{d^j\gamma}{d\theta^j}(\theta) = e^{A\theta}\frac{d^j\gamma}{d\theta^j}(0).$$

In particular, the speed is invariant. A quick check shows that it is $\sqrt{n}$. If we let θ run from $-\pi/2$ to $\pi/2$, we trace out a curve of length $\pi\sqrt{n}$.

The ideas here may also be expressed in the language of invariant measures for polynomials [15]. This gives a deeper understanding of the symmetries that we will only sketch here. Rather than representing a polynomial as

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

, we homogenize the polynomial and consider

$$\hat{p}(t_1, t_2) = a_0 t_2^n + a_1 t_1 t_2^{n-1} + \ldots + a_{n-1} t_1^{n-1} + a_n t_1^n.$$

For any angle α, a new "rotated" polynomial may be defined by

$$\hat{p}_\alpha(t_1, t_2) = \hat{p}(t_1\cos\alpha + t_2\sin\alpha, -t_1\sin\alpha + t_2\cos\alpha).$$

It is not difficult to show directly that if the a_i are independent and normally distributed with variance $\binom{n}{i}$, then so are the coefficients of the rotated polynomial. The symmetry of the curve and the symmetry of the polynomial distribution are equivalent. An immediate consequence of the rotational invariance is that the distribution of the real zeros must be Cauchy.

5.2 Geodesics on flat tori

Consider the third example of the previous section. Fix a finite interval $[a, b]$. For simplicity assume that

$$\sqrt{\sum_{k=0}^{n} \sigma_k^2} = 1 .$$

The curve $\gamma(\theta)$ is given by

$$(\sigma_0\cos\nu_0\theta, \sigma_0\sin\nu_0\theta, \ldots, \sigma_n\cos\nu_n\theta, \sigma_n\sin\nu_n\theta) .$$

This curve is a geodesic on the flat $(n+1)$-dimensional torus

$$(\sigma_0\cos\theta_0, \sigma_0\sin\theta_0, \ldots, \sigma_n\cos\theta_n, \sigma_n\sin\theta_n) .$$

Therefore if we lift to the universal covering space of the torus, γ becomes a straight line in $\Re^{n+1}$. By the Pythagorean theorem, the length of γ is

$$(b-a)\sqrt{\sum_{k=0}^{n} \nu_k^2\sigma_k^2} ,$$

which equals the expected number of zeros on the interval $[a,b]$.

Now replace $[a,b]$ with $(-\infty,\infty)$. If ν_i/ν_j is rational for all i and j, then γ is closed, otherwise it is dense in some subtorus.

Exercise: Show that if we choose the ν_k and the σ_k correctly, the second example becomes a special case of the third example. Thus the super-circle of the second example is a spherical helix.

5.3 The Kac matrix

Historically, Mark Kac was the first mathematician to obtain an exact formula for the expected number of real zeros of a random polynomial. Ironically, he also has his name attached to a certain matrix that is important to understanding the second example, yet we have no evidence that he ever made the connection.

The $n+1$ by $n+1$ Kac matrix is defined as the tridiagonal matrix

$$S_n = \begin{pmatrix} 0 & n & & & & \\ 1 & 0 & n-1 & & & \\ & 2 & 0 & n-2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & n-1 & 0 & 1 \\ & & & & n & 0 \end{pmatrix}.$$

The history of this matrix is documented in [24] where several proofs that S_n has eigenvalues $-n, -n+2, -n+4, \ldots, n-2, n$. One of the proofs is denoted as "mild trickery by Kac." We will derive the eigenvalues by employing a different trick.

Theorem 5.1 *The eigenvalues of S_n are the integers $2k-n$ for $k=0,1,\ldots,n$.*

Proof Define

$$f_k(x) \equiv \sinh^k(x)\cosh^{n-k}(x), \quad k=0,\ldots,n,$$

$$g_k(x) \equiv (\sinh(x)+\cosh(x))^k(\sinh(x)-\cosh(x))^{n-k}, \quad k=0,\ldots,n$$

If V is the vector space of functions with basis $\{f_k(x)\}$, then the $g_k(x)$ are clearly in this vector space. Also, $\frac{d}{dx}f_k(x) = kf_{k-1}(x)+(n-k)f_{k+1}(x)$, so that the Kac matrix is the representation of the operator d/dx in V. We actually wrote $g_k(x)$ in a more complicated way than we needed to so that we could emphasize that $g_k(x) \in V$. Actually, $g_k(x) = \exp((2k-n)x)$ is an eigenfunction of d/dx with eigenvalue $2k-n$ for $k=0,\ldots,n$. The eigenvector is obtained by expanding out the above expression for $g_k(x)$.

The matrix above may be symmetrized in which case it looks exactly like the matrix A in (9) without any minus signs. Indeed iS_n is similar to the matrix in (9) that so nicely explains the symmetry when the coefficients have variances that are binomial coefficients.

6 Extensions to other distributions

This paper began by considering random polynomials with standard normal coefficients, and then we realized quickly that any multivariate normal distribution with mean zero (the so-called "central distributions") hardly presented any further difficulty. We now generalize to arbitrary distributions, with a particular focus on the non-central multivariate normal distributions. The basic theme is the same: the density of zeros is equal to the rate at which the equators of a curve sweeps out volume. Previous investigators include Ali, Dunnage, Hirata, Ibragimov, Logan, Maslova, Mishra, Pratihari, Samal, Sambandham, Stevens, and Shepp. See [1] for a survey of results. There is also the interesting recent work of Odlyzko and Poonen on zeros of polynomials with coefficients that are quantized [20].

6.1 Arbitrary distributions

Given $f_0(t), f_1(t), \ldots, f_n(t)$ we now ask for the expected number of real zeros of the random equation

$$a_0 f_0(t) + a_1 f_1(t) + \cdots + a_n f_n(t) = 0,$$

where we will assume that the a_i have an arbitrary joint probability density function $\sigma(a)$.

Define $v(t) \in \Re^{n+1}$ by

$$v(t) = \begin{pmatrix} f_0(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

and let

$$\gamma(t) \equiv v(t)/||v(t)||. \tag{11}$$

Instead of working on the sphere, let us work in $\Re^{n+1}$ by defining $\gamma(t)_\perp$ to be the *hyperplane* through the origin perpendicular to $\gamma(t)$.

Fix t and choose an orthonormal basis such that $e_0 = \gamma(t)$ and $e_1 = \gamma'(t)/||\gamma'(t)||$. As we change t to $t + dt$, the volume swept out by the hyperplanes will form an infinitesimal wedge. (See figure.)

This wedge is the Cartesian product of a two dimensional wedge in the plane span(e_0,e_1) with $\Re^{n-1}$, the entire span of the remaining $n-1$ basis directions. The volume of the wedge is

$$||\gamma'(t)||\ dt \int_{\Re^n \equiv \{e_0 \cdot a = 0\}} |e_1 \cdot a| \sigma(a) da^n \ ,$$

where the domain of integration is the n-dimensional space perpendicular to e_0, and a^n denotes n-dimensional Lebesgue measure in that space. Intuitively $||\gamma'(t)||dt$ is rate at which the wedge is being swept out. The width of the wedge is infinitesimally proportional to $|e_1 \cdot a|$, where a is in this perpendicular hyperspace. The factor $\sigma(a)$ scales the volume in accordance with our chosen probability measure.

Theorem 6.1 *If a has a joint probability density $\sigma(a)$, then the density of the real zeros of $a_0 f_0(t) + \cdots + a_n f_n(t) = 0$ is*

$$\rho(t) = ||\gamma'(t)|| \int_{\gamma(t)\cdot a = 0} \frac{|\gamma'(t) \cdot a|}{||\gamma'(t)||} \sigma(a) da^n = \int_{\gamma(t)\cdot a=0} |\gamma'(t) \cdot a|\ \sigma(a) da^n \ ,$$

where da^n is standard Lesbesgue measure in the subspace perpendicular to $\gamma(t)$.

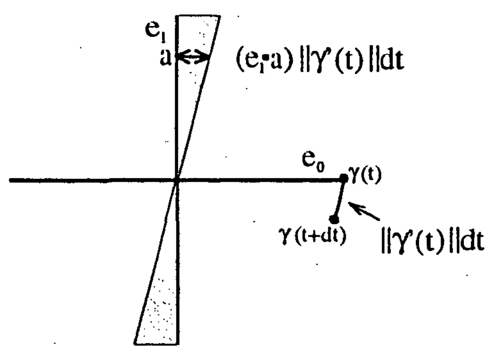


Figure 6 "Infinitesimal" wedge area

6.2 Non-central multivariate normals

We apply the results in the previous subsection to the case of multivariate normal distributions. We begin by assuming that our distribution has mean m and covariance matrix I. We then show that the restriction on the covariance matrix is readily removed. Thus we assume that

$$\sigma(a) = (2\pi)^{-(n+1)/2} e^{-\sum (a-m_i)^2/2}, \text{ and } m = (m_0, \ldots, m_n)^T.$$

Theorem 6.2 *Assume that $(a_0, \ldots, a_n)^T$ has the multivariate normal distribution with mean m and covariance matrix I. Let $\gamma(t)$ be defined as in (11). Let $m_0(t)$ and $m_1(t)$ be the components of m in the $\gamma(t)$ and $\gamma'(t)$ directions, respectively. The density of the real zeros of the equation $\sum a_i f_i(t) = 0$ is*

$$\rho_n(t) = \frac{1}{\pi}\|\gamma'(t)\| e^{-\frac{1}{2}m_0(t)^2}\left\{e^{-\frac{1}{2}m_1(t)^2} + \sqrt{\frac{\pi}{2}}m_1(t)\operatorname{erf}\left[\frac{m_1(t)}{\sqrt{2}}\right]\right\}.$$

For polynomials with identically distributed normal coefficients, this formula is equivalent to [1, Section 4.3C].

Proof

Since we are considering the multivariate normal distribution, we may rewrite $\sigma(a)$ in coordinates $x_0, \ldots, x_n$ in the directions $e_0, \ldots, e_n$ respectively. Thus

$$\sigma(x_0, \ldots, x_n) = (2\pi)^{-(n+1)/2} e^{-\frac{1}{2}\sum (x - m_i(t))^2},$$

where $m_i(t)$ denotes the coordinate of m in the e_i direction. The n-dimensional integral formula that appears in Theorem 6.1 reduces to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |x_1|\, e^{-\frac{1}{2}(m_0(t)^2)} e^{-\frac{1}{2}(x_1 - m_1(t))^2}\, dx_1$$

after integrating out the $n-1$ directions orthogonal to the wedge. From this, the formula in the theorem is obtained by direct integration. □

We can now generalize these formulas to allow for arbitrary covariance matrices as we did with Theorem 4.1. We phrase this corollary in a manner that is self-contained: no reference to definitions anywhere else in the paper is necessary.

Corollary 6.1 *Let $v(t) = (f_0(t), f_1(t), \ldots, f_n(t))^T$ and let $a = (a_0, \ldots, a_n)$ be a multivariate distribution with mean $m = (m_0, \ldots, m_n)^T$ and covariance matrix C. Equivalently consider random functions of the form $\sum a_i f_i(t)$ with mean $\mu(t) = m_0 f_0(t) + \ldots + m_n f_n(t)$ and covariance matrix C. The expected number of real zeros of the equation $\sum a_i f_i(t) = 0$ on the interval $[a, b]$ is*

$$\frac{1}{\pi}\int_a^b \|\gamma'(t)\| e^{-\frac{1}{2}m_0^2(t)}\left\{e^{-\frac{1}{2}m_1^2(t)} + \sqrt{\frac{\pi}{2}}m_1(t)\operatorname{erf}\left[\frac{m_1(t)}{\sqrt{2}}\right]\right\} dt,$$

where

$$w(t) = C^{1/2}v(t), \quad \gamma(t) = \frac{w(t)}{\|w(t)\|}, \quad m_0(t) = \frac{\mu(t)}{\|w(t)\|}, \text{ and } m_1(t) = \frac{m_0'(t)}{\|\gamma'(t)\|}.$$

Proof There is no difference between the equation $a \cdot v = 0$ and $C^{-1/2}a \cdot C^{1/2}v = 0$. The latter equation describes a random equation problem with coefficients from a multivariate normal with mean $C^{-1/2}m$ and covariance matrix I. Since $\mu(t)/\|w(t)\| = \gamma(t) \cdot C^{-1/2}m$ and $m_0'(t)/\|\gamma'(t)\| = \gamma'(t) \cdot C^{-1/2}m/\|\gamma'(t)\|$, the result follows immediately from Theorem 6.2.

□

6.3 Examples

We explore two cases in which non-central normal distributions have particularly simple zero densities:

- **Case I:** $m_0(t) = m$ and $m_1(t) = 0$
- **Case II:**$m_0(t) = m_1(t)$

Case I: $m_0(t) = m$ and $m_1(t) = 0$

If we can arrange for $m_0 = m$ to be a constant then $m_1(t) = 0$ and the density is

$$\rho(t) = \frac{1}{\pi}\|\gamma'(t)\|e^{-\frac{1}{2}m^2} .$$

In this very special case, the density function for the mean m case is just a constant factor ($e^{-\frac{1}{2}m^2}$) times the mean zero case.

This can be arranged if and only if the function $\|w(t)\|$ is in the linear space spanned by the f_i. The next few examples show when this is possible. In parentheses, we indicate the subsection of this paper where the reader may find the zero mean case for comparison.

Example 1 (4.2) A random polynomial with a simple answer, even degree: Let $f_i(t) = t^i, i = 0, \ldots, n$ and $C = \text{diag}[\binom{n}{i}]$. so that $\|w(t)\| = (1+t^2)^{n/2}$. The constant weight case occurs when our space has mean $\mu(t) = (1+t^2)^{n/2}$.

As a simple application when $n = 2$, if a_0, a_1, and a_2 are independent standard Gaussians, then the random polynomial

$$(a_0 + m) + a_1\sqrt{2}t + (a_2 + m)t^2,$$

is expected to have

$$\sqrt{2}e^{-m^2}$$

real zeros. The density is

$$\rho(t) = \frac{1}{\pi}\frac{\sqrt{2}}{(1+t^2)}e^{-m^2/2}.$$

Note that as $m \to \infty$, we are looking at perturbations to the equation $t^2 + 1 = 0$ with no real zeros, so we expect the number of real zeros to converge towards 0.

Example 2 (4.3) Trigonometric sums : $\mu(t) = m\sqrt{\sigma_0^2 + \cdots + \sigma_n^2}$.

Example 3 (4.6) Kac power series : $\mu(t) = m(1-t^2)^{-1/2}$.

Example 4 (4.8) Entire functions : $\mu(t) = me^{t^2/2}$.

Example 5 (4.9) Dirichlet Series:

$$\mu(t) = m\sqrt{\zeta(2t)} = \sum_{k=1}^{\infty}\frac{m_k}{k^t},$$

where $m_k = 0$ if k is not a square, and $m_k = m\prod_i \frac{(2n_i-1)!!}{(2n_i)!!}$ if k has the prime factorization $\prod_i p_i^{2n_i}$.

Case II:$m_0(t) = m_1(t)$

We may pick a $\mu(t)$ for which $m_0(t) = m_1(t)$ by solving the first order ordinary differential equation $m_0(t) = m_0'(t)/||\gamma'(t)||$. The solution is

$$\mu(t) = m||w(t)|| \ \exp\left[\int_K^t ||\gamma'(x)||dx\right] .$$

There is really only one integration constant since the result of shifting by K can be absorbed into the m factor. If the resulting $\mu(t)$ is in the linear space spanned by the f_i, then we choose this as our mean.

Though there is no reason to expect this, it turns out that if we make this choice of $\mu(t)$, then the density may be integrated in closed form. The expected number of zeros on the interval $[a, b]$ is

$$\int_a^b \rho(t)dt = \frac{1}{4}\text{erf}^2(m_0(t)/\sqrt{2}) - \frac{1}{2\pi}\Gamma[0, m_0^2(t)]\Big|_a^b .$$

Example 6 (4.6) Kac power series : Consider a power series with independent, identically distributed normal coefficients with mean m. In this case $\mu(t) = \frac{m}{1-t}$, where $m =$ (mean/standard deviation), so $m_0(t) = m\sqrt{\frac{1+t}{1-t}}$. A quick check shows that $m_1(t) = m_0(t)$.

Example 7 (4.8) Entire functions : In this case $\mu(t) = me^{t+t^2/2}$, so $m_0(t) = me^t$.

Example 8 (4.9) Dirichlet Series: This we leave as an exercise. Choose $K > 1/2$.

One final example

Theorem 6.3 *Consider a random polynomial of degree n with coefficients that are independent and identically distributed normal random variables. Define $m \neq 0$ to be the mean divided by the standard deviation. Then as $n \to \infty$,*

$$E_n = \frac{1}{\pi}\log(n) + \frac{C_1}{2} + \frac{1}{2} - \frac{\gamma}{\pi} - \frac{2}{\pi}\log|m| + O(1/n) ,$$

where C_1 is defined in Section 3.3, and where $\gamma = 0.5772156649...$ is Euler's constant. Furthermore, the expected number of positive zeros is asymptotic to

$$\frac{1}{2} - \frac{1}{2}\text{erf}^2(m/\sqrt{2}) + \frac{1}{\pi}\Gamma[0, m^2].$$

sketch of proof

We break up the domain of integration into four subdomains: $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$ and $[1, \infty)$. Observe that the expected number of zeros on the first and second intervals are the same, as are the expected number of zeros on the third and fourth intervals. Thus we will focus on the first and third interval, doubling our final answer.

The asymptotics of the density of zeros is easy to analyze on [0,1], because it converges quickly to that of the power series (Example 6, above). Doubling this gives us the expected number of positive zeros.

On the interval $(-\infty, -1]$, one can parallel the proof in Section 3.3. We make the change of variables $-t = 1+x/n$. The weight due to the nonzero mean can be shown to be $1+O(1/n)$. Therefore, the asymptotic series for the density of the zeros is the same up to $O(1/n)$. We subtract the asymptotic series for the density of the zeros of the non-central Kac power series, and then integrate term by term.

□

The $\frac{1}{\pi}\log(n)$ term was first derived by Sambandham.

7 Systems of equations

The results that we have derived about random equations in one variable may be generalized to systems of m equations in m unknowns. What used to be a curve $v(t) : \Re \to \Re^N$ is now an m-dimensional surface $v(t) : \Re^m \to \Re^N$ defined in the same way. The random coefficients now form an $m \times N$ matrix. We assume that the rows of this matrix are independent and identically distributed multivariate normals with mean zero and covariance matrix C.

Theorem 7.1 *Let $f_0(t), \ldots, f_N(t)$ be any collection of real valued rectifiable functions defined on $\Re^m$, let U be a measurable subset of $\Re^m$, and let the vectors $(a_{k0}, \ldots, a_{kN})$, $k = 1, \ldots, m$ be independent and identically distributed. Assume each vector is a multivariate normal random vector with mean zero and covariance matrix C. The expected number of real zeros of the system of equations*

$$a_{k0}f_0(t) + a_{k1}f_1(t) + \ldots + a_{kN}f_N(t) = 0, \qquad k = 1, \ldots, m\ ,$$

that lie in the set U, is

$$\pi^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)\int_U \left(\det\left[\frac{\partial^2}{\partial x_i \partial y_j}\left(\log v(x)^T C v(y)\right)\Big|_{y=x=t}\right]_{ij}\right)^{1/2} dt,$$

where $v(t) = (f_0(t), \ldots, f_N(t))$.

Proof If we project the set $\{v(t) |\ t \in U\}$ onto the unit sphere, the volume of the resulting set, divided by the volume of projective space of dimension m, equals the expected number of zeros in the region U. The Fubini-Study metric thus gives us the desired formula. See [15, 16] for details. □

The reader might prefer to phrase questions about random systems in terms of measures on Grassman manifolds. We remark that the factor before the integral is the reciprocal of the volume of the projective space of dimension m.

We now apply this formula to extend our previous examples.

7.1 The Kac formula

Consider systems of polynomial equations with independent standard normal coefficients. The most straightforward generalization occurs if the components of v are all the monomials $\{\prod_{k=1}^m x_k^{i_k}\}$, where for all k, $i_k \le d$. In other words, the Newton polyhedron is a hypercube.

Clearly,

$$v(x)^T v(y) = \prod_{i=1}^{m} \sum_{k=0}^{d} (x_i y_i)^k$$

from which we see that the matrix in the formula above is diagonal, and the density of the zeros on $\Re^m$ breaks up as a product of densities on $\Re$. Thus if $E_d^{(m)}$ represents the expected number of zeros for the system,

$$E_d^{(m)} = \pi^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)(\pi E_d^{(1)})^m.$$

The asymptotics of the univariate Kac formula shows that as $d \to \infty$,

$$E_d^{(m)} \sim \pi^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)(2\log d)^m.$$

The same asymptotic formula holds for a wide range of newton polyhedra, including the usual definition of degree: $\sum_{k=1}^m i_k \le d$ [16].

7.2 A random polynomial with a simple answer

Consider a system of m random polynomials each of the form

$$\sum_{i_1,\ldots,i_m} a_{i_1\ldots i_m} \Pi_{k=1}^{m} x_k^{i_k},$$

where $\sum_{k=1}^{m} i_k \leq d$, and where the $a_{i_1\ldots i_m}$ are independent normals with mean zero and variances equal to multinomial coefficients:

$$\binom{d}{i_1,\ldots,i_m} = \frac{d!}{(d-\sum_{k=1}^{m} i_k)! \prod_{k=1}^{m} i_k!}.$$

The multinomial theorem simplifies the computation of

$$v(x)^T C v(y) = \sum_{i_1,\ldots,i_m} \binom{d}{i_1,\ldots,i_m} \prod_{k=1}^{m} x^{i_k} y^{i_k} = (1 + x \cdot y)^d.$$

We see that the density of zeros is

$$\rho(t) = \pi^{-\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right) \frac{d^{m/2}}{(1+t \cdot t)^{(m+1)/2}}.$$

In other words, the zeros are uniformly distributed on real projective space, and the expected number of zeros is $d^{m/2}$.

Mike Shub and Steve Smale [23] have generalized this result as follows. Consider m independent equations of degrees $d_1, \ldots, d_m$, each defined as in this example. Then the expected number of real zeros of the system is

$$\sqrt{\Pi_{k=1}^{m} d_k}.$$

The result has also been generalized to underdetermined systems of equations [15, 17]. That is to say, we may consider the expected volume of a random real projective variety. The degrees of the equations need not be the same. The key result is as follows. *The expected volume of a real projective variety is the square root of the product of the degrees of the equations defining the variety, divided by the volume of the real projective space of the same dimension as the variety.* For a detailed discussion of random real projective varieties, see [17].

7.3 Random trigonometric sums

The generalization of these sums leads to sums of random harmonic polynomials, or random hyperspherical harmonics. The random harmonic polynomials form irreducible representations of the orthogonal groups, and therefore there is an essentially unique invariant normal measure. If we are given a system of m independent random harmonic polynomials in $m+1$ homogeneous variables, and the degrees of the polynomials are d_1, ..., d_m, then the expected number of real zeros for the system is

$$\sqrt{\prod_{k=1}^{m} \frac{d_k(d_k+m-1)}{m}}.$$

By considering the eigenspaces of the Laplacian, we can classify all invariant normal measures on systems of polynomials. Using the formula above, we can calculate the expected number of real zeros for any such measure. See [16] for details.

7.4 Kac power series

For a power series in m variables with independent standard normal coefficients, we see as in the first example, that the density of zeros on $\Re^m$ breaks up as the product of m densities:

$$\rho(t) = \pi^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)\prod_{k=1}^{m}\frac{1}{(1-t_k^2)}\,.$$

Notice that the power series converges with probability one on the unit hypercube, and that at the boundaries of this domain the density of zeros becomes infinite.

7.5 Random entire functions

Consider a random power series

$$f(x) = \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_m}\Pi_{k=1}^m x_k^{i_k},$$

where the $a_{i_1\ldots i_m}$ are independent normals with mean zero and variance $\left(\prod_{k=1}^m i_k!\right)^{-1}$. Clearly

$$v(x)^T Cv(y) = e^{x\cdot y},$$

so the zeros are uniformly distributed on $\Re^m$ with density

$$\pi^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right)$$

zeros per unit volume.

8 The Buffon needle problem revisited

In 1777, Buffon showed that if you drop a needle of length L on a plane containing parallel lines spaced a distance D from each other, then the expected number of intersections of the needle with the lines is

$$\frac{2L}{\pi D}.$$

Buffon assumed $L = D$, but the restriction is not necessary. In fact the needle may be bent into any reasonable plane curve and the formula still holds. This is perhaps the most celebrated theorem in integral geometry and is considered by many to be the first [21].

Let us translate the Buffon needle problem to the sphere as was first done by Barbier in 1860 – see [25] for a history. Consider a sphere with a fixed great circle. Draw a "needle" (a small piece of a great circle) on the sphere at random and consider the expected number of intersections of the needle with the great circle. If we instead fix the needle, and vary the great circle, it is clear that that the answer would be the same.

Any curve on the sphere can be approximated by a series of small needles. The expected number of intersections of the curve with a great circle is the sum of the expected number of intersection of each needle with a great circle. Thus the expected number of intersections of a fixed curve with a random great circle is given by

$$(\textit{some constant})L,$$

where L is the length of the curve. To find the constant consider the case where the fixed curve is itself a great circle. Then the average number of intersections is clearly 2 and L is clearly 2π. Thus the formula for the expected number of intersections of the curve with a random great circle must be

$$\frac{L}{\pi}.$$

Of course the theorem generalizes to curves on a sphere of any dimension.

To relate Barbier's result to random polynomials, we consider the curve γ on the unit sphere in $\Re^{n+1}$. By Barbier, the length of γ is π times the expected number of intersections of γ with a random great circle. What are these intersections? Consider a polynomial $p(x) = \sum_0^n a_n x^n$, and let $\mathbf{p}_\perp$ be the great circle perpendicular to the vector $\mathbf{p} \equiv (a_0, \ldots, a_n)$. Clearly $\gamma(t) \in \mathbf{p}_\perp$ for the values of t where $\gamma(t) \perp \mathbf{p}$. As we saw in Section 2, these are the values of t for which $p(t) = 0$. Thus the number of intersections of γ with $\mathbf{p}_\perp$ is exactly the number of real zeros of p, and the expected number of intersections is therefore the expected number of real zeros.

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