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ISSN

Authors

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Publication Date

2010-07-01

DOI

Square integrable holomorphic functions on infinite-dimensional Heisenberg type groups

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Abstract   We introduce a class of non-commutative, complex, infinite-dimensional Heisenberg like Lie groups based on an abstract Wiener space. The holomorphic functions which are also square integrable with respect to a heat kernel measure μ on these groups are studied. In particular, we establish a unitary equivalence between the square integrable holomorphic functions and a certain completion of the universal enveloping algebra of the "Lie algebra" of this class of groups. Using quasi-invariance of the heat kernel measure, we also construct a skeleton map which characterizes globally defined functions from the $L^2(\nu)$-closure of holomorphic polynomials by their values on the Cameron–Martin subgroup.

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1 Introduction

The aim of this paper is to study spaces of holomorphic functions on an infinite-dimensional Heisenberg like group based on a complex abstract Wiener space. In particular, we prove Taylor, skeleton, and holomorphic chaos isomorphism theorems. The tools we use come from properties of heat kernel measures on such groups which have been constructed and studied in [4]. We will state the main results of our paper and then conclude this introduction with a brief discussion of how our results relate to the existing literature.

1.1 Statements of the main results

1.1.1 The Heisenberg like groups and heat kernel measures

The basic input to our theory is a complex abstract Wiener space, (W, H, μ), as in Notation 2.4 which is equipped with a continuous skew-symmetric bi-linear form $\omega : W \times W \to \mathbf{C}$ as in Notation 3.1. Here and throughout this paper, $\mathbf{C}$ is a finite dimensional complex inner product space. The space, $G := W \times \mathbf{C}$, becomes an infinite-dimensional "Heisenberg like" group when equipped with the following mul-

tiplication rule

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2)\right). \tag{1.1}$$

A typical example of such a group is the Heisenberg group of a symplectic vector space, but in our setting we have an additional structure of an abstract Wiener space to carry out the heat kernel measure analysis.

The group G contains the *Cameron–Martin* group, $G_{CM} := H \times \mathbf{C}$, as a subgroup. The Lie algebras of G and G_{CM} will be denoted by $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ respectively which, as sets, may be identified with G and G_{CM} respectively—see Definition 3.2, Notation 3.3, and Proposition 3.5 for more details.

Let $b(t) = (B(t), B_0(t))$ be a Brownian motion on $\mathfrak{g}$ associated to the natural Hilbertian structure on $\mathfrak{g}_{CM}$ as described in Eq. (4.1). The Brownian motion $\{g(t)\}_{t \geq 0}$ on G is then the solution to the stochastic differential equation,

$$dg(t) = g(t) \circ db(t) \text{ with } g(0) = \mathbf{e} = (0, 0). \tag{1.2}$$

The explicit solution to Eq. (1.2) may be found in Eq. (4.2). For each $T > 0$ we let $\nu_T := \mathrm{Law}(g(T))$ be the *heat kernel* measure on G at time T as explained in Definitions 4.1 and 4.2. Analogous to the abstract Wiener space setting, ν_T is left (right) quasi-invariant by an element, $h \in G$, iff $h \in G_{CM}$, while $\nu_T(G_{CM}) = 0$, see Theorem 4.5, Proposition 4.6, and [4, Proposition 6.3].

In addition to the above infinite-dimensional structures we will need corresponding finite dimensional approximations. These approximations will be indexed by $\mathrm{Proj}(W)$ which we now define.

Notation 1.1 *Let* $\mathrm{Proj}(W)$ *denote the collection of finite rank continuous linear maps,* $P : W \to H$, *such that* $P|_H$ *is an orthogonal projection. (Explicitly, P must be as in Eq. (2.17) below.) Further, let $G_P := PW \times \mathbf{C}$ (a subgroup of G_{CM}) and $\pi_P : G \to G_P$ be the projection map defined by $\pi_P(w, c) := (Pw, c)$.*

To each $P \in \mathrm{Proj}(W)$, G_P is a finite dimensional Lie group. The Brownian motions and heat kernel measures, $\{\nu_t^P\}_{t>0}$, on G_P are constructed similarly to those on G–see Definition 4.10. We will use $\{(G_P, \nu_T^P)\}_{P \in \mathrm{Proj}(W)}$ as finite dimensional approximations to (G, ν_T).

1.1.2 The Taylor isomorphism theorem

The Taylor map, $\mathcal{T}_T$, is a unitary map relating the "square integrable" holomorphic functions on G_{CM} with the collection of their derivatives at $\mathbf{e} \in G_{CM}$. Before we can state this theorem we need to introduce the two Hilbert spaces involved.

In what follows, $\mathcal{H}(G_{CM})$ and $\mathcal{H}(G)$ will denote the space of holomorphic functions on G_{CM} and G respectively. (See Sect. 5 for the properties of these function spaces which are used throughout this paper.) We also let $\mathbf{T} := \mathbf{T}(\mathfrak{g}_{CM})$ be the algebraic tensor algebra over $\mathfrak{g}_{CM}$, $\mathbf{T}'$ be its algebraic dual, J be the two-sided ideal in $\mathbf{T}$

generated by

$$\{h \otimes k - k \otimes h - [h, k] : h, k \in \mathfrak{g}_{CM}\}, \tag{1.3}$$

and $J^0 = \{\alpha \in \mathbf{T}' : \alpha(J) = 0\}$ be the backwards annihilator of J–see Notation 6.1. Given $f \in \mathcal{H}(G)$ we let $\alpha := \mathcal{T}f$ denote the element of J^0 defined by $\langle \alpha, 1 \rangle = f(\mathbf{e})$ and

$$\langle \alpha, h_1 \otimes \cdots \otimes h_n \rangle := \left(\tilde{h}_1 \ldots \tilde{h}_n f \right)(\mathbf{e})$$

where $h_i \in \mathfrak{g}_{CM}$ and $\tilde{h}_i$ is the left invariant vector field on G_{CM} agreeing with h_i at $\mathbf{e}$–see Proposition 3.5 and Definition 6.2. We call $\mathcal{T}$ the Taylor map since $\mathcal{T}f \in J^0(\mathfrak{g}_{CM})$ encodes all of the derivatives of f at $\mathbf{e}$.

Definition 1.2 (L^2–holomorphic functions on G_{CM}) For $T > 0$, let

$$\|f\|_{\mathcal{H}_T^2(G_{CM})} = \sup_{P \in \mathrm{Proj}(W)} \|f|_{G_P}\|_{L^2(G_P, v_T^P)} \quad \text{for all } f \in \mathcal{H}(G_{CM}), \text{ and} \tag{1.4}$$

$$\mathcal{H}_T^2(G_{CM}) := \left\{ f \in \mathcal{H}(G_{CM}) : \|f\|_{\mathcal{H}_T^2(G_{CM})} < \infty \right\}. \tag{1.5}$$

In Corollary 6.6 below, we will see that $\mathcal{H}_T^2(G_{CM})$ is not empty and in fact contains the space of holomorphic cylinder polynomials ($\mathcal{P}_{CM}$) on G_{CM} described in Eq. (1.7) below. Despite the fact that $v_T(G_{CM}) = 0$, $\mathcal{H}_T^2(G_{CM})$ should roughly be thought of as the v_T–square integrable holomorphic functions on G_{CM}.

Definition 1.3 (Non-commutative Fock space) Let $T > 0$ and

$$\|\alpha\|_{J_T^0(\mathfrak{g}_{CM})}^2 := \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{h_1, \ldots, h_n \in S} |\langle \alpha, h_1 \otimes \cdots \otimes h_n \rangle|^2 \quad \text{for all } \alpha \in J^0(\mathfrak{g}_{CM}),$$

where S is any orthonormal basis for $\mathfrak{g}_{CM}$. The *non-commutative Fock space* is defined as

$$J_T^0(\mathfrak{g}_{CM}) := \left\{ \alpha \in J^0(\mathfrak{g}_{CM}) : \|\alpha\|_{J_T^0(\mathfrak{g}_{CM})}^2 < \infty \right\}.$$

It is easy to see that $\|\cdot\|_{J_T^0(\mathfrak{g}_{CM})}$ is a Hilbertian norm on $J_T^0(\mathfrak{g}_{CM})$–see Definition 6.4 and Eq. (6.8). For a detailed introduction to such Fock spaces we refer to [13].

Remark 1.4 When $\omega = 0$, $G(\omega)$ is commutative and the Fock space, $J_T^0(\mathfrak{g}_{CM})$, becomes the standard commutative bosonic Fock space of symmetric tensors over $\mathfrak{g}_{CM}^*$.

The following theorem is proved in Sect. 6–see Theorem 6.10.

Theorem 1.5 (The Taylor isomorphism) *For all* $T > 0$, $\mathcal{T}(\mathcal{H}_T^2(G_{CM})) \subset J_T^0(\mathfrak{g}_{CM})$ *and the linear map,*

$$\mathcal{T}_T := \mathcal{T}|_{\mathcal{H}_T^2(G_{CM})} : \mathcal{H}_T^2(G_{CM}) \to J_T^0(\mathfrak{g}_{CM}), \tag{1.6}$$

is unitary.

Associated to this theorem is an analogue of Bargmann's pointwise bounds which appear in Theorem 6.11 below.

1.1.3 The skeleton isomorphism theorem

Similarly to how it has been done on a complex abstract Wiener space by Sugita [26,27], the quasi-invariance of the heat kernel measure ν_T allows us to define the skeleton map from $L^p(G, \nu_T)$ to a space of functions on the Cameron–Martin subgroup G_{CM}, a set of ν_T-measure 0.

Definition 1.6 A *holomorphic cylinder polynomial on G* is a holomorphic cylinder function (see Definition 4.3) of the form, $f = F \circ \pi_P : G \to \mathbb{C}$, where $P \in \text{Proj}(W)$ and $F : PW \times \mathbf{C} \to \mathbb{C}$ is a holomorphic polynomial. The space of holomorphic cylinder polynomials will be denoted by $\mathcal{P}$.

The "Gaussian" heat kernel bounds in Theorem 4.11 easily imply that $\mathcal{P} \subset L^p(\nu_T)$ for all $p < \infty$—see Corollary 5.10.

Definition 1.7 (Holomorphic L^p–functions) For $T > 0$ and $1 \leqslant p < \infty$, let $\mathcal{H}_T^p(G)$ denote the $L^p(\nu_T)$-closure of $\mathcal{P} \subset L^p(\nu_T)$.

From Corollary 4.8 below, if $T > 0$, $p \in (1, \infty]$, $f \in L^p(G, \nu_T)$, and $h \in G_{CM}$, then $\int_G |f(h \cdot g)| d\nu_T(g) < \infty$. Thus, if $f \in \mathcal{H}_T^2(G)$ we may define the *skeleton map* (see Definition 4.7) by

$$(S_T f)(h) := \int_G f(h \cdot g) d\nu_T(g).$$

It is shown in Theorem 5.12 that $S_T(\mathcal{H}_T^2(G)) \subset \mathcal{H}_T^2(G_{CM})$ for all $T > 0$.

Theorem 1.8 (The skeleton isomorphism) *For each $T > 0$, the skeleton map, $S_T : \mathcal{H}_T^2(G) \to \mathcal{H}_T^2(G_{CM})$, is unitary.*

Following Sugita's results [26,27] in the case of an abstract Wiener space, we call $S_T|_{\mathcal{H}_T^2(G)}$ the skeleton map since it characterizes $f \in \mathcal{H}_T^2(G)$ by its "values", $S_T f$, on G_{CM}. Sugita would refer to G_{CM} as the skeleton of $G(\omega)$ owing to the fact that $\nu_T(G_{CM}) = 0$ as we show in Proposition 4.6.

Theorem 1.8 is proved in Sect. 8 and relies on two key density results from Sect. 7. The first is Lemma 7.3 (an infinite-dimensional version of [7, Lemma 3.5]) which states that the finite rank tensors (see Definition 7.2) are dense inside of $J_T^0(\mathfrak{g}_{CM})$. The second is Theorem 7.1 which states that

$$\mathcal{P}_{CM} := \{p|_{G_{CM}} : p \in \mathcal{P}\} \tag{1.7}$$

is a dense subspace of $\mathcal{H}_T^2(G_{CM})$. Matt Cecil [2] has modified the arguments presented in Sect. 7 to cover the situation of path groups over graded nilpotent Lie groups. Cecil's arguments are necessarily much more involved because his Lie groups have nilpotency of arbitrary step.

1.1.4 The holomorphic chaos expansion

So far we have produced (for each $T > 0$) two unitary isomorphisms, the skeleton map S_T and the Taylor isomorphism $\mathcal{T}_T$,

$$\mathcal{H}_T^2(G) \xrightarrow[\cong]{S_T} \mathcal{H}_T^2(G_{CM}) \xrightarrow[\cong]{\mathcal{T}_T} J_T^0(\mathfrak{g}_{CM}).$$

The next theorem gives an explicit formula for $(\mathcal{T}_T \circ S_T)^{-1} : J_T^0(\mathfrak{g}_{CM}) \to \mathcal{H}_T^2(G)$.

Theorem 1.9 (The holomorphic chaos expansion) *If $f \in \mathcal{H}_T^2(G)$ and $\alpha_f := \mathcal{T}_T S_T f$, then*

$$f(g(T)) = \sum_{n=0}^{\infty} \left\langle \alpha_f, \int\limits_{0 \le s_1 \le s_2 \le \cdots \le s_n \le T} db(s_1) \otimes \cdots \otimes db(s_n) \right\rangle \qquad (1.8)$$

where $b(t)$ and $g(t)$ are related as in Eq. (1.2) *or equivalently as in Eq.* (4.2).

This result is proved in Sect. 9 and in particular, see Theorem 9.10. The precise meaning of the right hand side of Eq. (1.8) is also described there.

1.2 Discussion

As we noticed in Remark 1.4 when the form $\omega \equiv 0$ the Fock space $J_T^0(\mathfrak{g}_{CM})$ is the standard commutative bosonic Fock space [9]. In this case the Taylor map is one of three isomorphisms between different representations of a Fock space, one other being the Segal–Bargmann transform. The history of the latter is described in [13] beginning with works of Bargmann [1] and Segal [24]. For other relevant results see [8,14].

To put our results into perspective, recall that the classical Segal–Bargmann space is the Hilbert space of holomorphic functions on $\mathbb{C}^n$ that are square-integrable with respect to the Gaussian measure $d\mu_n(z) = \pi^{-n} e^{-|z|^2} dz$, where dz is the $2n$-dimensional Lebesgue measure. One of the features of functions in the Segal–Bargmann space is that they satisfy the pointwise bounds $|f(z)| \le \|f\|_{L^2(\mu_n)} \exp(|z|^2/2)$ (compare with Theorem 6.11). As it is described in [13], if $\mathbb{C}^n$ is replaced by an infinite-dimensional complex Hilbert space H, one of the first difficulties is to find a suitable version of the Gaussian measure. It can be achieved, but only on a certain extension W of H, which leads one to consider the complex abstract Wiener space setting. From Sugita's [26,27] work on holomorphic functions over a complex abstract Wiener space, it is known that the pointwise bounds control only the values of the holomorphic functions on H. This difficulty explains, in part, the need to consider two function spaces: one is of holomorphic functions on H (or G_{CM} in our case) versus the square-integrable (weakly) holomorphic functions on W (or G in our case).

The Taylor map has also been studied in other non-commutative infinite-dimensional settings. Gordina [10–12] considered the Taylor isomorphism in the context of Hilbert-Schmidt groups, while Cecil [2] considered the Taylor isomorphism for path groups over stratified Lie groups. The nilpotentcy of the Heisenberg

like groups studied in this paper allow us to give a more complete description of the square integrable holomorphic function spaces than was possible in [10–12] for the Hilbert–Schmidt groups.

Complex analysis in infinite dimensions in a somewhat different setting has been studied by Lempert (e.g.[20]), and for more results on Gaussian-like measures on infinite-dimensional curved spaces see papers by Pickrell (e.g.[22,23]). For another view of different representations of Fock space, one can look at results in the field of white noise, as presented in the book by Obata [21]. The map between an L^2-space and a space of symmetric tensors sometimes is called the Segal isomorphism as in [18,19]. For more background on this and related topics see [16].

2 Complex abstract Wiener spaces

Suppose that W is a complex separable Banach space and $\mathcal{B}_W$ is the Borel σ–algebra on W. Let W_{Re} denote W thought of as a real Banach space. For $\lambda \in \mathbb{C}$, let $M_\lambda : W \to W$ be the operation of multiplication by λ.

Definition 2.1 A measure μ on $(W, \mathcal{B}_W)$ is called a (mean zero, non-degenerate) *Gaussian measure* provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_W e^{iu(w)} d\mu(w) = e^{-\frac{1}{2}q(u,u)} \quad \text{for all } u \in W_{\mathrm{Re}}^*, \tag{2.1}$$

where $q = q_\mu : W_{\mathrm{Re}}^* \times W_{\mathrm{Re}}^* \to \mathbb{R}$ is an inner product on W_{Re}^*. If in addition, μ is invariant under multiplication by i, that is, $\mu \circ M_i^{-1} = \mu$, we say that μ is a *complex Gaussian measure* on W.

Remark 2.2 Suppose $W = \mathbb{C}^d$ and let us write $w \in W$ as $w = x + iy$ with $x, y \in \mathbb{R}^d$. Then the most general Gaussian measure on W is of the form

$$d\mu(w) = \frac{1}{Z} \exp\left(-\frac{1}{2} Q \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) dx\, dy$$

where Q is a real positive definite $2d \times 2d$ matrix and Z is a normalization constant. The matrix Q may be written in 2×2 block form as

$$Q = \begin{bmatrix} A & B \\ B^{\mathrm{tr}} & C \end{bmatrix}.$$

A simple exercise shows $\mu = \mu \circ M_i^{-1}$ iff $B = 0$ and $A = C$. Thus the general complex Gaussian measure on $\mathbb{C}^d$ is of the form

$$d\mu(w) = \frac{1}{Z} \exp\left(-\frac{1}{2}(Ax \cdot x + Ay \cdot y)\right) dx\, dy$$
$$= \frac{1}{Z} \exp\left(-\frac{1}{2}Aw \cdot \bar{w}\right) dx\, dy,$$

where A is a real positive definite matrix.

Given a complex Gaussian measure μ as in Definition 2.1, let

$$\|w\|_H := \sup_{u \in W_{\mathrm{Re}}^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}} \quad \text{for all } w \in W, \tag{2.2}$$

and define the *Cameron–Martin subspace*, $H \subset W$, by

$$H = \{h \in W : \|h\|_H < \infty\}. \tag{2.3}$$

The following theorem summarizes some of the standard properties of the triple (W, H, μ).

Theorem 2.3 *Let (W, H, μ) be as above, where μ is a complex Gaussian measure on $(W, \mathcal{B}_W)$. Then*

(1) *H is a dense complex subspace of W.*
(2) *There exists a unique inner product, $\langle \cdot, \cdot \rangle_H$, on H such that $\|h\|_H^2 = \langle h, h \rangle$ for all $h \in H$. Moreover, with this inner product H is a complete separable complex Hilbert space.*
(3) *There exists $C < \infty$ such that*

$$\|h\|_W \leqslant C \|h\|_H \quad \text{for any } h \in H. \tag{2.4}$$

(4) *If $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H and $u, v \in W_{\mathrm{Re}}^*$, then*

$$q(u,v) = \langle u, v \rangle_{H_{\mathrm{Re}}^*} = \sum_{j=1}^{\infty}\left[u\left(e_j\right)v(e_j) + u\left(ie_j\right)v\left(ie_j\right)\right]. \tag{2.5}$$

(5) *$\mu \circ M_\lambda^{-1} = \mu$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

Proof We will begin with the proof of item 5. From Eq. (2.1), the invariance of μ under multiplication by i ($\mu \circ M_i^{-1} = \mu$) is equivalent to assuming that $q(u \circ M_i, u \circ M_i) = q(u, u)$ for all $u \in W_{\mathrm{Re}}^*$. By polarization, we may further conclude that

$$q\left(u \circ M_i, v \circ M_i\right) = q\left(u, v\right) \quad \text{for all } u, v \in W_{\mathrm{Re}}^*. \tag{2.6}$$

Taking $v = u \circ M_i$ in this identity then shows that $q(u \circ M_i, -u) = q(u, u \circ M_i)$ and hence that

$$q\left(u, u \circ M_i\right) = 0 \quad \text{for any } u \in W_{\mathrm{Re}}^*. \tag{2.7}$$

Therefore if $\lambda = a + ib$ with $a, b \in \mathbb{R}$, we see that

$$q(u \circ M_\lambda, u \circ M_\lambda) = q(au + bu \circ M_i, au + bu \circ M_i)$$
$$= (a^2 + b^2)q(u, u) = |\lambda|^2 q(u, u), \qquad (2.8)$$

from which it follows that $q(u \circ M_\lambda, u \circ M_\lambda) = q(u, u)$ for all $u \in W_{\text{Re}}^*$ and $|\lambda| = 1$. Coupling this observation with Eq. (2.1) implies $\mu \circ M_\lambda^{-1} = \mu$ for all $|\lambda| = 1$. If $|\lambda| = 1$, from Eqs. (2.2) and (2.8), it follows that

$$\|\lambda w\|_H = \sup_{u \in W_{\text{Re}}^* \setminus \{0\}} \frac{|u(\lambda w)|}{\sqrt{q(u, u)}} = \sup_{u \in W_{\text{Re}}^* \setminus \{0\}} \frac{|u \circ M_\lambda(w)|}{\sqrt{q(u \circ M_\lambda, u \circ M_\lambda)}}$$
$$= \sup_{u \in W_{\text{Re}}^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u, u)}} = \|w\|_H \quad \text{for all } w \in W.$$

In particular, if $\|h\|_H < \infty$ and $|\lambda| = 1$, then $\|\lambda h\|_H = \|h\|_H < \infty$ and hence $\lambda H \subset H$ which shows that H is a complex subspace of W. From [4, Theorem 2.3] summarizing some well-known properties of Gaussian measures, we know that item 3. holds, H is a dense subspace of W_{Re}, and there exists a unique real Hilbertian inner product, $\langle \cdot, \cdot \rangle_{H_{\text{Re}}}$, on H such that $\|h\|_H^2 = \langle h, h \rangle_{H_{\text{Re}}}$ for all $h \in H$. Polarizing the identity $\|\lambda h\|_H = \|h\|_H$ implies $\langle \lambda h, \lambda k \rangle_{H_{\text{Re}}} = \langle h, k \rangle_{H_{\text{Re}}}$ for all $h, k \in H$. Taking $\lambda = i$ and $k = -ih$ then shows $\langle ih, h \rangle_{\text{Re}} = \langle h, -ih \rangle_{\text{Re}}$, and hence that $\langle ih, h \rangle_{\text{Re}} = 0$ for all $h \in H$. Using this information it is a simple matter to check that

$$\langle h, k \rangle_H := \langle h, k \rangle_{H_{\text{Re}}} + i \langle h, ik \rangle_{H_{\text{Re}}} \quad \text{for all } h, k \in H, \qquad (2.9)$$

is the unique complex inner product on H such that $\text{Re} \langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{H_{\text{Re}}}$.

So it only remains to prove Eq. (2.5). For a proof of the first equality in Eq. (2.5), see [4, Theorem 2.3]. To prove the second equality in this equation, it suffices to observe that $\{e_j, ie_j\}_{j=1}^\infty$ is an orthonormal basis for $(H_{\text{Re}}, \langle \cdot, \cdot \rangle_{H_{\text{Re}}})$ and therefore,

$$\langle u, v \rangle_{H_{\text{Re}}^*} = \sum_{j=1}^\infty \left[u(e_j) v(e_j) + u(ie_j) v(ie_j) \right] \quad \text{for any } u, v \in H_{\text{Re}}^*.$$

$\square$

Notation 2.4 *The triple, (W, H, μ), appearing in Theorem 2.3 will be called a com-plex abstract Wiener space (Notice that there is redundancy in this notation since μ is determined by H, and H is determined by μ).*

Lemma 2.5 *Suppose that $u, v \in W_{\text{Re}}^*$ and $a, b \in \mathbb{C}$, then*

$$\int_W e^{au+bv} d\mu = \exp\left(\frac{1}{2} \left(a^2 q(u, u) + b^2 q(v, v) + 2ab q(u, v) \right) \right). \qquad (2.10)$$

Proof Equation (2.10) is easily verified when both a and b are real. This suffices to complete the proof, since both sides of Eq. (2.10) are analytic functions of $a, b \in \mathbb{C}$.

□

Lemma 2.6 *Let (W, H, μ) be a complex abstract Wiener space, then for any $\varphi \in W^*$, we have*

$$\int_W e^{\varphi(w)} d\mu(w) = 1 = \int_W e^{\overline{\varphi(w)}} d\mu(w), \tag{2.11}$$

$$\int_W |\text{Re } \varphi(w)|^2 \, d\mu(w) = \int_W |\text{Im } \varphi(w)|^2 \, d\mu(w) = \|\varphi\|_{H^*}^2, \tag{2.12}$$

and

$$\int_W |\varphi(w)|^2 \, d\mu(w) = 2 \|\varphi\|_{H^*}^2. \tag{2.13}$$

More generally, if $\mathbf{C}$ is another complex Hilbert space and $\varphi \in L(W, \mathbf{C})$, then

$$\int_W \|\varphi(w)\|_{\mathbf{C}}^2 \, d\mu(w) = 2 \|\varphi\|_{H^* \otimes \mathbf{C}}^2. \tag{2.14}$$

Proof If $u = \text{Re } \varphi$, then $\varphi(w) = u(w) - iu(iw)$. Therefore by Eqs. (2.6), (2.7), and (2.10),

$$\int_W e^\varphi d\mu = \int_W e^{u - iu \circ M_i} d\mu$$

$$= \exp\left(\frac{1}{2}\left(q(u, u) - q\left(u \circ M_i, u \circ M_i\right) - 2iq\left(u, u \circ M_i\right)\right)\right) = 1.$$

Taking the complex conjugation of this identity shows $\int_W e^{\overline{\varphi(w)}} d\mu(w) = 1$. Also using Lemma 2.5, we have

$$\int_W |\text{Re } \varphi(w)|^2 \, d\mu(w) = q(u, u) \text{ and}$$

$$\int_W |\text{Im } \varphi(w)|^2 \, d\mu(w) = \int_W |u(iw)|^2 \, d\mu(w) = q\left(u \circ M_i, u \circ M_i\right) = q\left(u, u\right).$$

To evaluate $q(u, u)$, let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for H so that $\{e_k, ie_k\}_{k=1}^\infty$ is an orthonormal basis for $(H_{\mathrm{Re}}, \mathrm{Re}\langle\cdot, \cdot\rangle_H)$. Then by Eq. (2.5),

$$q(u, u) = \sum_{k=1}^\infty \left[|u(e_k)|^2 + |u(ie_k)|^2 \right] = \sum_{k=1}^\infty |\varphi(e_k)|^2 = \|\varphi\|_{H^*}^2.$$

To prove Eq. (2.14), apply [4, Eq. (2.13)] to find

$$\int_W \|\varphi(w)\|_{\mathbf{C}}^2 \, d\mu(w) = \sum_{k=1}^\infty \left[\|\varphi(e_k)\|_{\mathbf{C}}^2 + \|\varphi(ie_k)\|_{\mathbf{C}}^2 \right]$$

$$= 2\sum_{k=1}^\infty \|\varphi(e_k)\|_{\mathbf{C}}^2 = 2\|\varphi\|_{H^*\otimes\mathbf{C}}^2.$$

$\square$

Remark 2.7 (Heat kernel interpretation of Lemma 2.6) The measure μ formally satisfies

$$\int_W f(w)d\mu(w) = \left(e^{\frac{1}{2}\Delta_{H_{\mathrm{Re}}}} f \right)(0),$$

where $\Delta_{H_{\mathrm{Re}}} = \sum_{j=1}^\infty \partial_{e_j}^2$ and $\{e_j\}_{j=1}^\infty$ is an orthonormal basis for H_{Re}. If f is holomorphic or anti-holomorphic, then f is harmonic and therefore

$$\int_W f(w)d\mu(w) = \left(e^{\frac{1}{2}\Delta_{H_{\mathrm{Re}}}} f \right)(0) = f(0).$$

Applying this identity to $f(w) = e^{\varphi(w)}$ or $f(w) = \overline{e^{\varphi(w)}}$ with $\varphi \in W^*$ gives Eq. (2.11). If $u \in W_{\mathrm{Re}}^*$, we have

$$\int_W u^2(w)d\mu(w) = \left(e^{\frac{1}{2}\Delta_{H_{\mathrm{Re}}}} u^2 \right)(0) = \sum_{n=0}^\infty \frac{1}{2^n n!} \left(\Delta_{H_{\mathrm{Re}}}^n u^2 \right)(0)$$

$$= \frac{1}{2} \left(\Delta_{H_{\mathrm{Re}}} u^2 \right)(0) = \frac{1}{2} \sum_{j=1}^\infty \left(\partial_{e_j}^2 u^2 \right)(0)$$

$$= \sum_{j=1}^\infty u(e_j)^2 = \|u\|_{H_{\mathrm{Re}}}^2.$$

Eqs. (2.12) and (2.13) now follow easily from this identity.

2.1 The structure of the projections

Let $i : H \to W$ be the inclusion map and $i^* : W^* \to H^*$ be its transpose, i.e. $i^*\ell := \ell \circ i$ for all $\ell \in W^*$. Also let

$$H_* := \left\{ h \in H : \langle \cdot, h \rangle_H \in \mathrm{Ran}\left(i^*\right) \subset H^* \right\} \tag{2.15}$$

or in other words, $h \in H$ is in H_* iff $\langle \cdot, h \rangle_H \in H^*$ extends to a continuous linear functional on W. (We will continue to denote the continuous extension of $\langle \cdot, h \rangle_H$ to W by $\langle \cdot, h \rangle_H$.) Because H is a dense subspace of W, i^* is injective, and because i is injective, i^* has a dense range. Since $h \in H \to \langle \cdot, h \rangle_H \in H^*$ is a conjugate linear isometric isomorphism, it follows from the above comments that $H_* \ni h \to \langle \cdot, h \rangle_H \in W^*$ is a conjugate linear isomorphism too, and that H_* is a dense subspace of H.

Lemma 2.8 *There is a one to one correspondence between* $\mathrm{Proj}(W)$ *(see Notation* 1.1*) and the collection of finite rank orthogonal projections, P, on H such that $PH \subset H_*$.*

Proof If $P \in \mathrm{Proj}(W)$ and $u \in PW \subset H$, then, because $P|_H$ is an orthogonal projection, we have

$$\langle Ph, u \rangle_H = \langle h, Pu \rangle_H = \langle h, u \rangle_H \quad \text{for all } h \in H. \tag{2.16}$$

Since $P : W \to H$ is continuous, it follows that $u \in H_*$, i.e. $PW \subset H_*$.

Conversely, suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for PH and $\ell_j \in W^*$ such that $\ell_j|_H = \langle \cdot, e_j \rangle_H$. Then we may extend P uniquely to a continuous operator from W to H (still denoted by P) by letting

$$Pw := \sum_{j=1}^n \ell_j(w)e_j = \sum_{j=1}^n \langle w, e_j \rangle_H e_j \quad \text{for all } w \in W. \tag{2.17}$$

From [4, Eq. 3.43], there exists $C = C(P) < \infty$ such that

$$\|Pw\|_H \leqslant C \|w\|_W \text{ for all } w \in W. \tag{2.18}$$

$\square$

3 Complex Heisenberg like groups

In this section we review the infinite-dimensional Heisenberg like groups and Lie algebras which were introduced in [4, Section 3].

Notation 3.1 *Let (W, H, μ) be a complex abstract Wiener space, $\mathbf{C}$ be a complex finite dimensional inner product space, and $\omega : W \times W \to \mathbf{C}$ be a continuous skew symmetric bilinear quadratic form on W. Further, let*

$$\|\omega\|_0 := \sup \left\{ \|\omega(w_1, w_2)\|_{\mathbf{C}} : w_1, w_2 \in W \text{ with } \|w_1\|_W = \|w_2\|_W = 1 \right\} \tag{3.1}$$

be the uniform norm on ω which is finite by the assumed continuity of ω.

Definition 3.2 Let $\mathfrak{g}$ denote $W \times \mathbf{C}$ when thought of as a Lie algebra with the Lie bracket operation given by

$$[(A, a), (B, b)] := (0, \omega(A, B)) . \tag{3.2}$$

Let $G = G(\omega)$ denote $W \times \mathbf{C}$ when thought of as a group with the multiplication law given by

$$g_1 g_2 = g_1 + g_2 + \frac{1}{2} [g_1, g_2] \quad \text{for any } g_1, g_2 \in G \tag{3.3}$$

or equivalently by Eq. (1.1).

It is easily verified that $\mathfrak{g}$ is a Lie algebra and G is a group. The identity of G is the zero element, $\mathbf{e} := (0, 0)$.

Notation 3.3 *Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when viewed as a Lie subalgebra of $\mathfrak{g}$ and G_{CM} denote $H \times \mathbf{C}$ when viewed as a subgroup of $G = G(\omega)$. We will refer to $\mathfrak{g}_{CM}$ (G_{CM}) as the* Cameron–Martin subalgebra (subgroup) *of $\mathfrak{g}$ (G). (For explicit examples of such $(W, H, \mathbf{C}, \omega)$, see [4].)*

We equip $G = \mathfrak{g} = W \times \mathbf{C}$ with the Banach space norm

$$\|(w, c)\|_{\mathfrak{g}} := \|w\|_W + \|c\|_{\mathbf{C}} \tag{3.4}$$

and $G_{CM} = \mathfrak{g}_{CM} = H \times \mathbf{C}$ with the Hilbert space inner product,

$$\langle (A, a) , (B, b) \rangle_{\mathfrak{g}_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_{\mathbf{C}}. \tag{3.5}$$

The associate Hilbertian norm is given by

$$\|(A, a)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_{\mathbf{C}}^2}. \tag{3.6}$$

As was shown in [4, Lemma 3.3], these Banach space topologies on $W \times \mathbf{C}$ and $H \times \mathbf{C}$ make G and G_{CM} into topological groups.

Notation 3.4 (Linear differentials) *Suppose $f : G \to \mathbb{C}$, is a Frechét smooth function. For $g \in G$ and $h, k \in \mathfrak{g}$ let*

$$f'(g)h := \partial_h f(g) = \frac{d}{dt}\Big|_0 f(g + th)$$

and

$$f''(g) (h \otimes k) := \partial_h \partial_k f(g).$$

(*Here and in the sequel a prime on a symbol will be used to denote its derivative or differential.*)

As G itself is a vector space, the tangent space, $T_g G$, to G at g is naturally isomorphic to G. Indeed, if $v, g \in G$, then we may define a tangent vector $v_g \in T_g G$ by $v_g f = f'(g)v$ for all Frechét smooth functions $f : G \to \mathbb{C}$. We will identify $\mathfrak{g}$ with $T_e G$ and $\mathfrak{g}_{CM}$ with $T_e G_{CM}$. Recall that as sets $\mathfrak{g} = G$ and $\mathfrak{g}_{CM} = G_{CM}$. For $g \in G$, let $l_g : G \to G$ be the left translation by g. For $h \in \mathfrak{g}$, let $\tilde{h}$ be the *left invariant vector field* on G such that $\tilde{h}(g) = h$ when $g = e$. More precisely, if $\sigma(t) \in G$ is any smooth curve such that $\sigma(0) = e$ and $\dot{\sigma}(0) = h$ (e.g. $\sigma(t) = th$), then

$$\tilde{h}(g) =, l_{g*}h := \frac{d}{dt}\Big|_0 g \cdot \sigma(t). \tag{3.7}$$

As usual, we view $\tilde{h}$ as a first order differential operator acting on smooth functions, $f : G \to \mathbb{C}$, by

$$\left(\tilde{h}f\right)(g) = \frac{d}{dt}\Big|_0 f(g \cdot \sigma(t)). \tag{3.8}$$

The proof of the following easy proposition may be found in [4, Proposition 3.7].

Proposition 3.5 *Let* $f : G \to \mathbb{C}$ *be a smooth function,* $h = (A, a) \in \mathfrak{g}$ *and* $g = (w, c) \in G$. *Then*

$$\tilde{h}(g) :=, l_{g*}h = \left(A, a + \frac{1}{2}\omega(w, A)\right) \quad \text{for any } g = (w, c) \in G \tag{3.9}$$

and, in particular,

$$\widetilde{(A, a)}f(g) = f'(g)\left(A, a + \frac{1}{2}\omega(w, A)\right). \tag{3.10}$$

If $h, k \in \mathfrak{g}$, *then*

$$\left(\tilde{h}\tilde{k}f - \tilde{k}\tilde{h}f\right) = \widetilde{[h, k]}f. \tag{3.11}$$

The one parameter group in G, e^{th}, *determined by* $h = (A, a) \in \mathfrak{g}$, *is given by* $e^{th} = th = t(A, a)$.

4 Brownian motion and heat kernel measures

This section will closely follow [4, Section 4] except for the introduction of a certain factor of $1/2$ into the formalism which will simplify later formulas. Let $\{b(t) = (B(t), B_0(t))\}_{t \geqslant 0}$ be a Brownian motion on $\mathfrak{g} = W \times \mathbb{C}$ with the variance determined by

$$\mathbb{E}\left[\text{Re} \langle b(s), h\rangle_{\mathfrak{g}_{CM}} \cdot \text{Re} \langle b(t), k\rangle_{\mathfrak{g}_{CM}}\right] = \frac{1}{2}\text{Re} \langle h, k\rangle_{\mathfrak{g}_{CM}} s \wedge t \tag{4.1}$$

for all $s, t \in [0, \infty)$, $h = (A, a)$, and $k := (C, c)$, where $A, C \in H_*$ and $a, c \in \mathbf{C}$. (Recall the definition of H_* from Eq. (2.15).)

Definition 4.1 The associated *Brownian motion* on G starting at $\mathbf{e} = (0, 0) \in G$ is defined to be the process

$$g(t) = \left(B(t), B_0(t) + \frac{1}{2} \int_0^t \omega\left(B\left(\tau\right), dB\left(\tau\right)\right) \right). \tag{4.2}$$

More generally, if $h \in G$, we let $g_h(t) := h \cdot g(t)$, the Brownian motion on G starting at h.

Definition 4.2 Let $\mathcal{B}_G$ be the Borel σ–algebra on G and for any $T > 0$, let $\nu_T : \mathcal{B}_G \to [0, 1]$ be the distribution of $g(T)$. We will call ν_T the *heat kernel measure* on G.

To be more explicit, the measure ν_T is the unique measure on $(G, \mathcal{B}_G)$ such that

$$\nu_T(f) := \int_G f \, d\nu_T = \mathbb{E}\left[f(g(T))\right] \tag{4.3}$$

for all bounded measurable functions $f : G \to \mathbf{C}$. Our next goal is to describe the generator of the process $\{g_h(t)\}_{t \geqslant 0}$.

Definition 4.3 A function $f : G \to \mathbf{C}$ is said to be a *cylinder function* if it may be written as $f = F \circ \pi_P$ for some $P \in \text{Proj}(W)$ and some function $F : G_P \to \mathbf{C}$, where G_P is defined as in Notation 1.1. We say that f is a *holomorphic* (*smooth*) cylinder function if $F : G_P \to \mathbf{C}$ is holomorphic (smooth). We will denote the space of holomorphic (analytic) cylinder functions by $\mathcal{A}$.

Proposition 4.4 (Generator of g_h) *If $f : G \to \mathbf{C}$ is a smooth cylinder function, let*

$$Lf := \sum_{j=1}^{\infty} \left[\widetilde{(e_j, 0)}^2 + \widetilde{(ie_j, 0)}^2 \right] f + \sum_{j=1}^{d} \left[\widetilde{(0, f_j)}^2 + \widetilde{(0, if_j)}^2 \right] f, \tag{4.4}$$

where $\{e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{d}$ are complex orthonormal bases for $(H, \langle \cdot, \cdot \rangle_H)$ and $(\mathbf{C}, \langle \cdot, \cdot \rangle_{\mathbf{C}})$ respectively. Then Lf is well defined, i.e. the sums in Eq. (4.4) are convergent and independent of the choice of bases. Moreover, for all $h \in G$, $\frac{1}{4} L$ is the generator for $\{g_h(t)\}_{t \geqslant 0}$. More precisely,

$$M_t^f := f\left(g_h(t)\right) - \frac{1}{4} \int_0^t Lf\left(g_h\left(\tau\right)\right) d\tau \tag{4.5}$$

is a local martingale for any smooth cylinder function, $f : G \to \mathbf{C}$.

Proof After bearing in mind the factor of $1/2$ used in defining the Brownian motion $b(t)$ in Eq. (4.1), this proposition becomes a direct consequence of Proposition 3.29 and Theorem 4.4 of [4]. Indeed, the Brownian motions in this paper are equal in distribution to the Brownian motions used in [4] after making the time change, $t \to t/2$. It is this time change that is responsible for the $1/4$ factor (rather than $1/2$) in Eq. (4.5). □

4.1 Heat kernel quasi-invariance properties

In this subsection, we are going to recall one of the key theorems from [4]. We first need a little more notation.

Let C_{CM}^1 denote the collection of C^1-paths, $g : [0, 1] \to G_{CM}$. The length of g is defined as

$$\ell_{G_{CM}}(g) = \int_0^1 \left\| l_{g^{-1}(s)*}g'(s) \right\|_{\mathfrak{g}_{CM}} ds. \tag{4.6}$$

As usual, the Riemannian distance between $x, y \in G_{CM}$ is defined as

$$d_{G_{CM}}(x, y) = \inf \left\{ \ell_{G_{CM}}(g) : g \in C_{CM}^1 \ni g(0) = x \text{ and } g(1) = y \right\}. \tag{4.7}$$

Let us also recall the definition of $k(\omega)$ from [4, Eq. 7.6];

$$k(\omega) = -\frac{1}{2} \sup_{\|A\|_{H_{Re}}=1} \|\omega(\cdot, A)\|_{H_{Re}^* \otimes \mathbf{C}_{Re}}^2$$

$$= - \sup_{\|A\|_H=1} \|\omega(\cdot, A)\|_{H^* \otimes \mathbf{C}}^2 \geqslant - \|\omega\|_{H^* \otimes H^* \otimes \mathbf{C}}^2 > -\infty, \tag{4.8}$$

wherein we have used [4, Lemma 3.17] in the second equality. It is known by Fernique's or Skhorohod's theorem that $\|\omega\|_2^2 = \|\omega\|_{H^* \otimes H^* \otimes C}^2 < \infty$, see [4, Proposition 3.14] for details.

Theorem 4.5 *For all $h \in G_{CM}$ and $T > 0$, the measures, $\nu_T \circ l_h^{-1}$ and $\nu_T \circ r_h^{-1}$, are absolutely continuous relative to ν_T. Let $Z_h^l := \frac{d(\nu_T \circ l_h^{-1})}{d\nu_T}$ and $Z_h^r := \frac{d(\nu_T \circ r_h^{-1})}{d\nu_T}$ be the respective Randon–Nikodym derivatives, $k(\omega)$ is given in Eq. (4.8), and*

$$c(t) := \frac{t}{e^t - 1} \quad \text{for any } t \in \mathbb{R}$$

with the convention that $c(0) = 1$. Then for all $1 \leqslant p < \infty$, Z_h^l and Z_h^r are both in $L^p(\nu_T)$ and satisfy the estimate

$$\left\| Z_h^* \right\|_{L^p(\nu_T)} \leqslant \exp \left(\frac{c(k(\omega)T/2)(p-1)}{T} d_{G_{CM}}^2(\mathbf{e}, h) \right), \tag{4.9}$$

where $* = l$ or $* = r$.

Proof This is [4, Theorem 8.1] (also see [4, Corollary 7.3]) with the modification that T should be replaced by $T/2$. This is again due to the fact that the Brownian motions in this paper are equal in distribution to those in [4] after making the time change, $t \to t/2$. $\square$

It might be enlightening to note here that we call G_{CM} the Cameron–Martin subgroup not only because it is constructed from the Cameron–Martin subspace, H, but also because it has properties similar to H. In particular, the following statement holds.

Proposition 4.6 *The heat kernel measure does not charge G_{CM}, i.e. $\nu_T(G_{CM}) = 0$.*

Proof Note that for a bounded measurable function $f : W \times C \to \mathbb{C}$ that depends only on the the first component in $W \times C$, that is, $f(w, c) = f(w)$ we have

$$\int_G f(w)d\nu_T(w, c) = \mathbb{E}[f(B(T))] = \int_W f(w)d\mu_T(w).$$

Note that for the projection $\pi : W \times C \to W$, $\pi(w, c) = w$ we have $\pi_* \nu_T = \mu_T$ and therefore

$$\nu_T(G_{CM}) = \nu_T(\pi^{-1}(H)) = \pi_* \nu_T(H) = \mu_T(H) = 0.$$

$\square$

For later purposes, we would like to introduce the heat operator, $S_T := e^{TL/4}$, acting on $L^p(G, \nu_T)$. To motivate our definition, suppose $f : G \to \mathbb{C}$ is a smooth cylinder function and suppose we can make sense of $u(t, y) = (e^{(T-t)L/4}f)(y)$. Then working formally, by Itô's formula, Eq. (4.5), and the left invariance of L, we expect $u(t, hg(t))$ to be a martingale for $0 \leqslant t \leqslant T$ and in particular,

$$\mathbb{E}[f(hg(T))] = \mathbb{E}[u(T, hg(T))] = \mathbb{E}[u(0, hg(0))] = (e^{TL/4}f)(h). \quad (4.10)$$

Definition 4.7 For $T > 0$, $p \in (1, \infty]$, and $f \in L^p(G, \nu_T)$, let $S_T f : G_{CM} \to \mathbb{C}$ be defined by

$$(S_T f)(h) = \int_G f(h \cdot g)d\nu_T(g) = \mathbb{E}[f(hg(T))]. \quad (4.11)$$

The following result is a simple corollary of Theorem 4.5 and Hölder's inequality along with the observation that $p' - 1 = (p-1)^{-1}$, where p' is the conjugate exponent to $p \in (1, \infty]$.

Corollary 4.8 *If $p > 1$, $T > 0$, $f \in L^p(G, \nu_T)$, $h \in G_{CM}$, and*

$$Z_h^l \in L^{\infty-}(\nu_T) := \cap_{1 \leqslant q < \infty} L^q(\nu_T) \quad (4.12)$$

is as in Theorem 4.5, *then* $S_T f$ *is well defined and may be computed as*

$$(S_T f)(h) = \int_G f(g) Z_h^l(g) dv_T(g). \tag{4.13}$$

Moreover, we have the following pointwise "Gaussian" bounds

$$|(S_T f)(h)| \leqslant \|f\|_{L^p(v_T)} \exp\left(\frac{c(k(\omega)T/2)}{T(p-1)} d_{GCM}^2(\mathbf{e}, h)\right). \tag{4.14}$$

We will see later that when f is "holomorphic" and $p = 2$, the above estimate in Eq. (4.14) may be improved to

$$|(S_T f)(h)| \leqslant \|f\|_{L^2(v_T)} \exp\left(\frac{1}{2T} d_{GCM}^2(\mathbf{e}, h)\right) \text{ for any } h \in G_{CM}. \tag{4.15}$$

This bound is a variant of Bargmann's pointwise bounds (see [1, Eq. (1.7)] and [6, Eq. (5.4)]).

Lemma 4.9 *Let $T > 0$ and suppose that $f : G \to \mathbb{C}$ is a continuous and in $L^p(v_T)$ for some $p > 1$. Then $S_T f : G_{CM} \to \mathbb{C}$ is continuous.*

Proof For $q \in (1, p)$ and $h \in G_{CM}$ we have by Hölder's inequality and Theorem 4.5 that

$$\mathbb{E} |f(hg(T))|^q = v_T\left(|f|^q Z_h^l\right) \leqslant \|f\|_{L^p(v_T)}^{q/p} \cdot \left\|Z_{h_n}^l\right\|_{L^{\frac{p}{p-q}}(v_T)}$$

$$\leqslant \|f\|_{L^p(v_T)}^{q/p} \exp\left(\frac{c(k(\omega)T/2)q}{T(p-q)} d_{GCM}^2(\mathbf{e}, h)\right) \tag{4.16}$$

Hence if $\{h_n\}_{n=1}^\infty \subset G_{CM}$ is a sequence converging to $h \in G_{CM}$, it follows that

$$\sup_n \mathbb{E} |f(h_n g(T))|^q \leqslant \|f\|_{L^p(v_T)}^{q/p} \exp\left(\frac{c(k(\omega)T/2)q}{T(p-q)} \sup_n d_{GCM}^2(\mathbf{e}, h_n)\right) < \infty, \tag{4.17}$$

which implies that $\{f(h_n g(T))\}_{n=1}^\infty$ is uniformly integrable. Since by continuity of f, $\lim_{n\to\infty} f(h_n g(T)) = f(hg(T))$, we may pass to the limit under the expectation to find

$$\lim_{n\to\infty} S_T f(h_n) = \lim_{n\to\infty} \mathbb{E} f(h_n g(T)) = \mathbb{E}[f(hg(T))] = S_T f(h).$$

$\square$

4.2 Finite dimensional approximations

Notation 4.10 *For each $P \in \text{Proj}(W)$, let $g_P(t)$ denote the G_P-valued Brownian motion defined by*

$$g_P(t) = \left(PB(t), B_0(t) + \frac{1}{2} \int_0^t \omega \left(PB(\tau), dPB(\tau) \right) \right).$$ (4.18)

Also, for any $t > 0$, let $\nu_t^P := \text{Law}(g_P(t))$ be the corresponding heat kernel measure on G_P.

The following Theorem is a restatement of [4, Theorem 4.16].

Theorem 4.11 (Integrated heat kernel bounds) *Suppose that $\rho^2 : G \to [0, \infty)$ be defined as*

$$\rho^2(w, c) := \|w\|_W^2 + \|c\|_\mathbf{C}.$$ (4.19)

Then there exists a $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$ and $T > 0$

$$\sup_{P \in \text{Proj}(W)} \mathbb{E}\left[e^{\frac{\varepsilon}{T} \rho^2(g_P(T))} \right] < \infty \quad \text{and} \quad \int_G e^{\frac{\varepsilon}{T} \rho^2(g)} d\nu_T(g) < \infty.$$ (4.20)

Proposition 4.12 *Let $P_n \in \text{Proj}(W)$ such that $P_n|_H \uparrow I_H$ on H and let $g_n(T) := g_{P_n}(T)$. Further suppose that $\delta > 0$ is as in Theorem 4.11, $p \in [1, \infty)$, and $f : G \to \mathbb{C}$ is a continuous function such that*

$$|f(g)| \leqslant C e^{\varepsilon \rho^2(g)/(pT)} \quad \text{for all } g \in G$$ (4.21)

for some $\varepsilon \in (0, \delta)$. Then $f \in L^p(\nu_T)$ and for all $h \in G$ we have

$$\lim_{n \to \infty} \mathbb{E} |f(hg(T)) - f(hg_n(T))|^p = 0,$$ (4.22)

and

$$\lim_{n \to \infty} \mathbb{E} |f(g(T)h) - f(g_n(T)h)|^p = 0.$$ (4.23)

Proof If $q \in (p, \infty)$ is sufficiently close to p so that $qp^{-1}\varepsilon < \delta$, then

$$\sup_n \mathbb{E} |f(g_n(T))|^q \leqslant C^q \sup_n \mathbb{E}\left[e^{p^{-1}q\varepsilon\rho^2(g)/T} \right]$$

which is finite by Theorem 4.11. This shows that $\{|f(g_n(T))|^p\}_{n=1}^\infty$ is uniformly integrable. As a consequence of [4, Lemma 4.7] and the continuity of f, we also know

that $f(g_n(T)) \to f(g(T))$ in probability as $n \to \infty$. Thus we have shown Eqs. (4.22) and (4.23) hold when $h = \mathbf{e} = 0$. Now suppose that $g = (w, c)$ and $h = (A, a)$ are in G. Then for all $\alpha > 0$,

$$
\begin{aligned}
\rho^2(gh) &= \|w + A\|_W^2 + \left\| a + c + \frac{1}{2}\omega(w, A) \right\|_\mathbf{C} \\
&\leqslant \|w\|_W^2 + \|A\|_W^2 + 2\|A\|_W \|w\|_W + \|a\|_\mathbf{C} + \|c\|_\mathbf{C} + \frac{1}{2}\|\omega(w, A)\|_\mathbf{C} \\
&\leqslant \rho^2(g) + \rho^2(h) + C\|A\|_W \|w\|_W \\
&\leqslant \rho^2(g) + \rho^2(h) + \frac{C}{2}\left[\alpha^{-1}\|A\|_W^2 + \alpha\|w\|_W^2\right] \\
&\leqslant \left(1 + \frac{C\alpha}{2}\right)\rho^2(g) + \left(1 + \frac{C}{2\alpha}\right)\rho^2(h),
\end{aligned}
\tag{4.24}
$$

where $C := (2 + \frac{1}{2}\|\omega\|_0)$. As Eq. (4.24) is invariant under interchanging g and h the same bound also hold for $\rho^2(hg)$. By choosing $\alpha > 0$ sufficiently small so that $(1 + \frac{C\alpha}{2})\varepsilon < \delta$, we see that $g \to f(gh)$ and $g \to f(hg)$ satisfy the same type of bound as in Eq. (4.21) for $g \to f(g)$. Therefore, by the first paragraph, we have now verified Eqs. (4.22) and (4.23) hold for any $h \in G$. □

5 Holomorphic functions on G and G_{CM}

We will begin with a short summary of the results about holomorphic functions on Banach spaces that will be needed in this paper.

5.1 Holomorphic functions on Banach spaces

Let X and Y be two complex Banach space and for $a \in X$ and $\delta > 0$ let

$$
B_X(a, \delta) := \{x \in X : \|x - a\|_X < \delta\}
$$

be the open ball in X with center a and radius δ.

Definition 5.1 (Hille and Phillips [17, Definition 3.17.2, p. 112.]) Let $\mathcal{D}$ be an open subset of X. A function $u : \mathcal{D} \to Y$ is said to be *holomorphic (or analytic)* if the following two conditions hold.

(1) u is locally bounded, namely for all $a \in \mathcal{D}$ there exists an $r_a > 0$ such that

$$
M_a := \sup\{\|u(x)\|_Y : x \in B_X(a, r_a)\} < \infty.
$$

(2) The function u is complex Gâteaux differentiable on $\mathcal{D}$, i.e. for each $a \in \mathcal{D}$ and $h \in X$, the function $\lambda \to u(a + \lambda h)$ is complex differentiable at $\lambda = 0 \in \mathbb{C}$.

(Holomorphic and analytic will be considered to be synonymous terms for the purposes of this paper.)

The next theorem gathers together a number of basic properties of holomorphic functions which may be found in [17]. (Also see [15].) One of the key ingredients to all of these results is Hartog's theorem, see [17, Theorem 3.15.1].

Theorem 5.2 *If $u : \mathcal{D} \to Y$ is holomorphic, then there exists a function $u' : \mathcal{D} \to \mathrm{Hom}(X, Y)$, the space of bounded complex linear operators from X to Y, satisfying*

(1) *If $a \in \mathcal{D}$, $x \in B_X(a, r_a/2)$, and $h \in B_X(0, r_a/2)$, then*

$$\left\| u(x + h) - u(x) - u'(x) h \right\|_Y \leqslant \frac{4M_a}{r_a(r_a - 2\|h\|_X)} \|h\|_X^2. \qquad (5.1)$$

In particular, u is continuous and Frechét differentiable on $\mathcal{D}$.
(2) *The function $u' : \mathcal{D} \to \mathrm{Hom}(X, Y)$ is holomorphic.*

Remark 5.3 By applying Theorem 5.2 repeatedly, it follows that any holomorphic function, $u : \mathcal{D} \to Y$ is Frechét differentiable to all orders and each of the Frechét differentials are again holomorphic functions on $\mathcal{D}$.

Proof By [17, Theorem 26.3.2 on p. 766.], for each $a \in \mathcal{D}$ there is a linear operator, $u'(a) : X \to Y$ such that $du(a + \lambda h)/d\lambda|_{\lambda=0} = u'(a)h$. The Cauchy estimate in Theorem 3.16.3 (with $n = 1$) of [17] implies that if $a \in \mathcal{D}$, $x \in B_X(a, r_a/2)$ and $h \in B_X(0, r_a/2)$ (so that $x + h \in B_X(a, r_a)$), then $\|u'(x)h\|_Y \leqslant M_a$. It follows from this estimate that

$$\sup \left\{ \left\| u'(x) \right\|_{\mathrm{Hom}(X,Y)} : x \in B_X(a, r_a/2) \right\} \leqslant 2M_a/r_a. \qquad (5.2)$$

and hence that $u' : \mathcal{D} \to \mathrm{Hom}(X, Y)$ is a locally bounded function. The estimate in Eq. (5.1) appears in the proof of the Theorem 3.17.1 in [17] which completes the proof of item 1.

To prove item 2. we must show u' is Gâteaux differentiable on $\mathcal{D}$. We will in fact show more, namely, that u' is Frechét differentiable on $\mathcal{D}$. Given $h \in X$, let $F_h : \mathcal{D} \to Y$ be defined by $F_h(x) := u'(x)h$. According to [17, Theorem 26.3.6], F_h is holomorphic on $\mathcal{D}$ as well. Moreover, if $a \in \mathcal{D}$ and $x \in B(a, r_a/2)$ we have by Eq. (5.2) that

$$\|F_h(x)\|_Y \leqslant 2M_a \|h\|_X / r_a.$$

So applying the estimate in Eq. (5.1) to F_h, we learn that

$$\left\| F_h(x + k) - F_h(x) - F_h'(x) k \right\|_Y \leqslant \frac{4(2M_a \|h\|_X / r_a)}{\frac{r_a}{2} \left(\frac{r_a}{2} - 2\|k\|_X \right)} \cdot \|k\|_X^2 \qquad (5.3)$$

for $x \in B(a, r_a/4)$ and $\|k\|_X < r_a/4$, where

$$F_h'(x)k = \frac{d}{d\lambda} \big|_0 F_h(x + \lambda k) = \frac{d}{d\lambda} \big|_0 u'(x + \lambda k) h =: \left(\delta^2 u \right)(x; h, k).$$

Again by [17, Theorem 26.3.6], for each fixed $x \in \mathcal{D}$, $(\delta^2 u)(x; h, k)$ is a continuous symmetric bilinear form in $(h, k) \in X \times X$. Taking the supremum of Eq. (5.3) over those $h \in X$ with $\|h\|_X = 1$, we may conclude that

$$
\begin{aligned}
&\left\| u'(x+k) - u'(x) - \delta^2 u(x; \cdot, k) \right\|_{\mathrm{Hom}(X,Y)} \\
&= \sup_{\|h\|_X = 1} \left\| F_h(x+k) - F_h(x) - F_h'(x)k \right\|_Y \\
&\leqslant \frac{4(2M_a/r_a)}{\frac{r_a}{2}\left(\frac{r_a}{2} - 2\|k\|_X\right)} \|k\|_X^2 .
\end{aligned}
$$

This estimate shows u' is Frechét differentiable with $u''(x) \in \mathrm{Hom}(X, \mathrm{Hom}(X, Y))$ being given by $u''(x)k = (\delta^2 u)(x; \cdot, k) \in \mathrm{Hom}(X, Y)$ for all $k \in X$ and $x \in \mathcal{D}$. □

5.2 Holomorphic functions on G and G_{CM}

For the purposes of this section, let $G_0 = G$ and $\mathfrak{g}_0 = \mathfrak{g}$ or $G_0 = G_{CM}$ and $\mathfrak{g}_0 = \mathfrak{g}_{CM}$. Also for $g, h \in \mathfrak{g}$, let (as usual) $ad_g h := [g, h]$.

Lemma 5.4 *For each $g \in G_0$, $l_g : G_0 \to G_0$ is holomorphic in the $\| \cdot \|_{\mathfrak{g}_0}$–topology. Moreover, a function $u : G_0 \to \mathbb{C}$ defined in a neighborhood of $g \in G_0$ is Gâteaux (Frechét) differentiable at g iff $u \circ l_g$ is Gâteaux (Frechét) differentiable at 0. In addition, if u is Frechét differentiable at g, then*

$$
\left(u \circ l_g\right)'(0) h = u'(g)\left(h + \frac{1}{2}[g, h]\right). \tag{5.4}
$$

(See [13, Theorem 5.7] for an analogous result in the context of path groups.)

Proof Since

$$
l_g(h) = gh = g + h + \frac{1}{2}[g, h] = g + \left(Id_{\mathfrak{g}_0} + \frac{1}{2}ad_g\right)h,
$$

it is easy to see that l_g is holomorphic and l_g' is the constant function equal to $Id_{\mathfrak{g}_0} + \frac{1}{2}ad_g \in \mathrm{End}(\mathfrak{g}_0)$. Using $ad_g^2 = 0$ or the fact that $l_g^{-1} = l_{g^{-1}}$, we see that l_g' is invertible and that

$$
l_g'^{-1} = \left(Id_{\mathfrak{g}_0} + \frac{1}{2}ad_g\right)^{-1} = Id_{\mathfrak{g}_0} - \frac{1}{2}ad_g .
$$

These observations along with the chain rule imply the Frechét differentiability statements of the lemma and the identity in Eq. (5.4).

If u is Gâteaux differentiable at g, $h \in \mathfrak{g}_0$, and $k := h + \frac{1}{2}[g, h]$, then

$$
\frac{d}{d\lambda}\Big|_0 u \circ l_g(\lambda h) = \frac{d}{d\lambda}\Big|_0 u(g \cdot (\lambda h)) = \frac{d}{d\lambda}\Big|_0 u(g + \lambda k)
$$

and the existence of $\frac{d}{d\lambda}|_0 u(g+\lambda k)$ implies the existence of $\frac{d}{d\lambda}|_0 u \circ l_g(\lambda h)$. Conversely, if $u \circ l_g$ is Gâteaux differentiable at 0, $h \in \mathfrak{g}_0$, and

$$k := h - \frac{1}{2}[g,h] = \left(Id_{\mathfrak{g}_0} + \frac{1}{2}ad_g\right)^{-1} h,$$

then

$$l_g(\lambda k) = g + \lambda \left(Id_{\mathfrak{g}_0} + \frac{1}{2}ad_g\right) k = g + \lambda h.$$

So the existence of $\frac{d}{d\lambda}|_0 (u \circ l_g)(\lambda k)$ implies the existence of $\frac{d}{d\lambda}|_0 u(g+\lambda h)$. □

Corollary 5.5 *A function $u : G_0 \to \mathbb{C}$ is holomorphic iff it is locally bounded and $h \to u(ge^h) = u(g \cdot h)$ is Gâteaux (Frechét) differentiable at 0 for all $g \in G_0$. Moreover, if u is holomorphic and $h \in \mathfrak{g}_0$, then*

$$\left(\tilde{h}u\right)(g) = \frac{d}{d\lambda}|_0 u\left(ge^{\lambda h}\right) = u'(g)(h+[g,h])$$

is holomorphic as well.

Notation 5.6 *The space of globally defined holomorphic functions on G and G_{CM} will be denoted by $\mathcal{H}(G)$ and $\mathcal{H}(G_{CM})$ respectively.*

Notice that the space $\mathcal{A}$ of holomorphic cylinder functions as described in Definition 4.3 is contained in $\mathcal{H}(G)$. Also observe that a simple induction argument using Corollary 5.5 allows us to conclude that $\tilde{h}_1 \ldots \tilde{h}_n u \in \mathcal{H}(G_0)$ for all $u \in \mathcal{H}(G_0)$ and $h_1, \ldots, h_n \in \mathfrak{g}_0$.

Proposition 5.7 *If $f \in \mathcal{H}(G)$ and $h \in \mathfrak{g}$, then $\tilde{ih}f = i\tilde{h}f$, $\tilde{ih}\bar{f} = -i\tilde{h}\bar{f}$,*

$$\left[\left(i\tilde{h}\right)^2 + \tilde{h}^2\right] f = 0, \text{ and} \tag{5.5}$$

$$\left(\tilde{h}^2 + \tilde{ih}^2\right)|f|^2 = 4\left|\tilde{h}f\right|^2. \tag{5.6}$$

Proof The first assertions are directly related to the definition of f being holomorphic. Using the identity $\tilde{ih}f = i\tilde{h}f$ twice implies Eq. (5.5). Equation (5.6) is a consequence of summing the following two identities

$$\tilde{h}^2 |f|^2 = \tilde{h}\left(f \cdot \bar{f}\right) = \tilde{h}^2 f \cdot \bar{f} + f \cdot \tilde{h}^2 \bar{f} + 2\tilde{h}f \cdot \tilde{h}\bar{f}$$

and

$$\tilde{ih}^2 |f|^2 = \tilde{ih}\left(f \cdot \bar{f}\right) = \tilde{ih}^2 f \cdot \bar{f} + f \cdot \tilde{ih}^2 \bar{f} + 2\tilde{ih}f \cdot \tilde{ih}\bar{f}$$
$$= -\tilde{h}^2 f \cdot \bar{f} - f \cdot \tilde{h}^2 \bar{f} + 2\tilde{h}f \cdot \tilde{h}\bar{f},$$

and using $\overline{\tilde{h}\bar{f}} = \overline{\tilde{h}f}$. □

Corollary 5.8 *Let L be as in Proposition 4.4. Suppose that* $f : G \to \mathbb{C}$ *is a holomorphic cylinder function (i.e.* $f \in \mathcal{A}$*), then* $Lf = 0$ *and*

$$L |f|^2 = \sum_{h \in \Gamma} \left| \tilde{h} f \right|^2, \tag{5.7}$$

where Γ *is an orthonormal basis for* $\mathfrak{g}_{CM}$ *of the form*

$$\Gamma = \Gamma_e \cup \Gamma_f = \{(e_j, 0)\}_{j=1}^{\infty} \cup \{(0, f_j)\}_{j=1}^{d} \tag{5.8}$$

with $\{e_j\}_{j=1}^{\infty}$ *and* $\{f_j\}_{j=1}^{d}$ *being complex orthonormal bases for* H *and* $\mathbf{C}$ *respectively.*

Proof These assertions follow directly form Eqs. (4.4), (5.5), and (5.6). □

Formally, if $f : G \to \mathbb{C}$ is a holomorphic function, then $e^{TL/4} f = f$ and therefore we should expect $S_T f = f|_{G_{CM}}$ where S_T is defined in Definition 4.7. Theorem 5.9 below is a precise version of this heuristic.

Theorem 5.9 *Suppose* $p \in (1, \infty)$ *and* $f : G \to \mathbb{C}$ *is a continuous function such that* $f|_{G_{CM}} \in \mathcal{H}(G_{CM})$ *and there exists* $P_n \in \text{Proj}(W)$ *such that* $P_n|_H \uparrow I_H$, *then*

$$\|f\|_{L^p(\nu_T)} \leqslant \sup_n \|f\|_{L^p\left(G_{P_n}, \nu_T^{P_n}\right)}. \tag{5.9}$$

If we further assume that

$$\sup_n \|f\|_{L^p\left(G_{P_n}, \nu_T^{P_n}\right)} < \infty, \tag{5.10}$$

then $f \in L^p(\nu_T)$, $S_T f = f|_{G_{CM}}$, *and* f *satisfies the Gaussian bounds*

$$|f(h)| \leqslant \|f\|_{L^p(\nu_T)} \exp\left(\frac{c\,(k(\omega)T/2)}{T(p-1)} d_{G_{CM}}^2 (\mathbf{e}, h)\right) \text{ for any } h \in G_{CM}. \tag{5.11}$$

Proof According to [4, Lemma 4.7], by passing to a subsequence if necessary, we may assume that $g_{P_n}(T) \to g(T)$ almost surely. Hence an application of Fatou's lemma implies Eq. (5.9). In particular, if we assume Eq. (5.10) holds, then $f \in L^p(\nu_T)$ and so $S_T f$ is well defined.

Now suppose that $P \in \text{Proj}(W)$ and $h \in G_P$. Working exactly as in the proof of Lemma 4.9, we find for any $q \in (1, p)$ that

$$\mathbb{E} |f(hg_P(T))|^q \leqslant \|f\|_{L^p(G_P, \nu_T^P)}^{q/p} \exp\left(\frac{c\,(k_P(\omega)T/2)\,q}{T(q-p)} d_{G_P}^2 (\mathbf{e}, h)\right), \tag{5.12}$$

where $d_{G_P}(\cdot, \cdot)$ is the Riemannian distance on G_P and (see [4, Eq. (5.13)]),

$$k_P(\omega) := -\frac{1}{2} \sup \left\{ \|\omega(\cdot, A)\|_{(PH)^* \otimes \mathbf{C}}^2 : A \in PH, \|A\|_{PH} = 1 \right\}. \tag{5.13}$$

Observe that $k_P(\omega) \geqslant k(\omega)$ and therefore, as c is a decreasing function, $c(k(\omega)) \geqslant c(k_P(\omega))$. Let $m \in \mathbb{N}$ be given and $h \in G_{P_m}$. Then for $n \geqslant m$ we have from Eq. (5.12) that

$$
\begin{aligned}
\mathbb{E}\left|f\left(hg_{P_n}(T)\right)\right|^q &\leqslant \|f\|^{q/p}_{L^p\left(G_{P_n}, v_T^{P_n}\right)} \exp\left(\frac{c\left(k_{P_n}(\omega)\,T/2\right)q}{T(q-p)}d_{G_{P_n}}^2(\mathbf{e}, h)\right) \\
&\leqslant \|f\|^{q/p}_{L^p\left(G_{P_n}, v_T^{P_n}\right)} \exp\left(\frac{c\left(k(\omega)\,T/2\right)q}{T(q-p)}d_{G_{P_m}}^2(\mathbf{e}, h)\right)
\end{aligned}
$$

wherein in the last inequality we have used $c(k(\omega)) \geqslant c(k_P(\omega))$ and the fact that $d_{G_{P_n}}^2(\mathbf{e}, h)$ is decreasing in $n \geqslant m$. Hence it follows that $\sup_{n \geqslant m} \mathbb{E}|f(hg_{P_n}(T))|^q < \infty$ and thus that $\{f(hg_{P_n}(T))\}_{n \geqslant m}$ is uniformly integrable. Therefore,

$$
S_T f(h) = \mathbb{E}f(hg(T)) = \lim_{n \to \infty} \mathbb{E}f\left(hg_{P_n}(T)\right) = \lim_{n \to \infty} \int_{G_{P_n}} f(hx)dv_T^{P_n}(x). \quad (5.14)
$$

On the other hand by [4, Lemma 4.8] (with T replaced by $T/2$ because of our normalization in Eq. (4.1)), $v_T^{P_n}$ is the heat kernel measure on G_{P_n} based at $\mathbf{e} \in G_{P_n}$, i.e. $v_T^{P_n}(dx) = p_{T/2}^{P_n}(e, x)dx$, where dx is the Riemannian volume measure (equal to a Haar measure) on G_{P_n} and $p_T^{P_n}(x, y)$ is the heat kernel on G_{P_n}. Since $f|_{G_{P_n}}$ is holomorphic, the previous observations allow us to apply [5, Proposition 1.8] to conclude that

$$
\int_{G_{P_n}} f(hx)dv_T^{P_n}(x) = f(h) \quad \text{for all } n \geqslant m. \quad (5.15)
$$

As $m \in \mathbb{N}$ was arbitrary, combining Eqs. (5.14) and (5.15) implies that $S_T f(h) = f(h)$ for all $h \in G_0 := \cup_{m \in \mathbb{N}} G_{P_m}$. Recall from Lemma 4.9 that $S_T f : G_{CM} \to \mathbb{C}$ is continuous and from the proof of [4, Theorem 8.1] that G_0 is a dense subgroup of G_{CM}. Therefore we may conclude that in fact $S_T f(h) = f(h)$ for all $h \in G_{CM}$. The Gaussian bound now follows immediately from Corollary 4.8. □

Corollary 5.10 *Suppose that $\delta > 0$ is as in Theorem 4.11 and $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}}$ is holomorphic and $|f| \leqslant Ce^{\varepsilon \rho^2/(pT)}$ for some $\varepsilon \in [0, \delta)$. Then $f \in L^p(v_T)$, $S_T f = f$, and the Gaussian bounds in Eq. (5.11) hold.*

Proof By Theorem 4.11, the given function f verifies Eq. (5.10) for any choice of $\{P_n\}_{n=1}^{\infty} \subset \text{Proj}(W)$ with $P_n|_H \uparrow P$ strongly as $n \uparrow \infty$. Hence Theorem 5.9 is applicable. □

As a simple consequence of Corollary 5.10, we know that $\mathcal{P} \subset L^p(v_T)$ (see Definition 1.6) and that $(S_T p)(h) = p(h)$ for all $h \in G_{CM}$ and $p \in \mathcal{P}$.

Notation 5.11 *For $T > 0$ and $1 \leqslant p < \infty$, let $\mathcal{A}_T^p$ and $\mathcal{H}_T^p(G)$ denote the $L^p(v_T)$–closure of $\mathcal{A} \cap L^p(v_T)$ and $\mathcal{P}$, where $\mathcal{A}$ and $\mathcal{P}$ denote the holomorphic cylinder*

functions (see Definition 4.3) and holomorphic cylinder polynomials on G respectively.

Theorem 5.12 *For all $T > 0$ and $p \in (1, \infty)$, we have $S_T(\mathcal{H}_T^p(G)) \subset \mathcal{H}(G_{CM})$.*

Proof Let $f \in \mathcal{H}_T^p(G)$ and $p_n \in \mathcal{P}$ such that $\lim_{n \to \infty} \|f - p_n\|_{L^p(v_T)} = 0$. If $h \in G_{CM}$, then by Corollary 4.8

$$|S_T f (h) - p_n (h)| = |S_T (f - p_n) (h)|$$
$$\leqslant \|f - p_n\|_{L^p(v_T)} \exp\left(\frac{c (k (\omega) T/2)}{T (p - 1)} d_{G_{CM}}^2 (\mathbf{e}, h)\right).$$

This shows that $S_T f$ is the limit of $p_n|_{G_{CM}} \in \mathcal{H}(G_{CM})$ with the limit being uniform over any bounded subset of h's contained in G_{CM}. This is sufficient to show that $S_T f \in \mathcal{H}(G_{CM})$ via an application of [17, Theorem 3.18.1]. □

Remark 5.13 It seems reasonable to conjecture that $\mathcal{A}_T^2 = \mathcal{H}_T^2(G)$, nevertheless we do not know if these two spaces are equal! We also do not know if $S_T f = f$ for every $f \in \mathcal{A} \cap L^2(v_T)$. However, Theorem 5.9 does show that $S_T f = f$ for all $f \in \mathcal{A} \cap_{P \in \text{Proj}(W)} L^p(v_T^P)$ with $L^p(v_T^P)$–norms of f being bounded.

6 The Taylor isomorphism theorem

The main purpose of this section is to prove the Taylor isomorphism Theorem 1.5 (or Theorem 6.10). We begin with the formal development of the algebraic setup. In what follows below for a vector space V we will denote the algebraic dual to V by V'. If V happens to be a normed space, we will let V^* denote the topological dual of V.

6.1 A non-commutative Fock space

Notation 6.1 *For $n \in \mathbb{N}$ let $\mathfrak{g}_{CM}^{\otimes n}$ denote the n–fold algebraic tensor product of $\mathfrak{g}_{CM}$ with itself, and by convention let $\mathfrak{g}_{CM}^{\otimes 0} := \mathbb{C}$. Also let*

$$\mathbf{T} := \mathbf{T}(\mathfrak{g}_{CM}) = \mathbb{C} \oplus \mathfrak{g}_{CM} \oplus \mathfrak{g}_{CM}^{\otimes 2} \oplus \mathfrak{g}_{CM}^{\otimes 3} \oplus \dots$$

be the algebraic tensor algebra over $\mathfrak{g}_{CM}$, $\mathbf{T}'$ be its algebraic dual, and J be the two-sided ideal in $\mathbf{T}$ generated by the elements in Eq. (1.3). The backwards annihilator of J is

$$J^0 = \{\alpha \in \mathbf{T}' : \alpha(J) = 0\}. \tag{6.1}$$

For any $\alpha \in \mathbf{T}'$ and $n \in \mathbb{N} \cup \{0\}$, we let $\alpha_n := \alpha|_{\mathfrak{g}_{CM}^{\otimes n}} \in (\mathfrak{g}_{CM}^{\otimes n})'$.

After the next definition we will be able to give numerous examples of elements in J^0.

Definition 6.2 *(Left differentials)* For $f \in \mathcal{H}(G_{CM})$, $n \in \mathbb{N} \cup \{0\}$, and $g \in G_{CM}$, define $\hat{f}_n(g) := D^n f(g) \in (\mathfrak{g}_{CM}^{\otimes n})'$ by

$$\left(D^0 f \right)(g) = f(g) \quad \text{and}$$

$$\langle D^n f(g), h_1 \otimes \cdots \otimes h_n \rangle = \left(\tilde{h}_1 \ldots \tilde{h}_n f \right)(g) \tag{6.2}$$

for all and $h_1, \ldots, h_n \in \mathfrak{g}_0$, where $\tilde{h} f$ is given as in Eq. (3.8) or Eq. (3.10). We will write Df for $D^1 f$ and $\hat{f}(g)$ to be the element of $\mathbf{T}(\mathfrak{g}_{CM})'$ determined by

$$\left\langle \hat{f}(g), \beta \right\rangle = \left\langle \hat{f}_n(g), \beta \right\rangle \quad \text{for all } \beta \in \mathfrak{g}_{CM}^{\otimes n} \text{ and } n \in \mathbb{N}_0. \tag{6.3}$$

Example 6.3 As a consequence of Eq. (3.11), $\hat{f}(g) \in J^0$ for all $f \in \mathcal{H}(G_{CM})$ and $g \in G_{CM}$.

In order to put norms on J^0, let us equip $\mathfrak{g}_{CM}^{\otimes n}$ with the usual inner product determined by

$$\langle h_1 \otimes \cdots \otimes h_n, k_1 \otimes \cdots \otimes k_n \rangle_{\mathfrak{g}_{CM}^{\otimes n}} = \prod_{j=1}^{n} \langle h_j, k_j \rangle_{\mathfrak{g}_{CM}} \quad \text{for any } h_i, k_j \in \mathfrak{g}_{CM}. \tag{6.4}$$

For $n = 0$ we let $\langle z, w \rangle_{\mathfrak{g}_{CM}^{\otimes 0}} := z \bar{w}$ for all $z, w \in \mathfrak{g}_{CM}^{\otimes 0} = \mathbb{C}$. The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}_{CM}^{\otimes n}}$ induces a dual inner product on $(\mathfrak{g}_{CM}^{\otimes n})^*$ which we will denote by $\langle \cdot, \cdot \rangle_n$. The associated norm on $(\mathfrak{g}_{CM}^{\otimes n})^*$ will be denoted by $\| \cdot \|_n$. We extend $\| \cdot \|_n$ to all of $(\mathfrak{g}_{CM}^{\otimes n})'$ by setting $\|\beta\|_n = \infty$ if $\beta \in (\mathfrak{g}_{CM}^{\otimes n})' \setminus (\mathfrak{g}_{CM}^{\otimes n})^*$. If Γ is any orthonormal basis for $\mathfrak{g}_{CM}$, then $\|\beta\|_n$ may be computed using

$$\|\beta\|_{\mathfrak{g}_{CM}^{\otimes n}}^2 := \sum_{h_1, \ldots, h_n \in \Gamma} |\langle \beta, h_1 \otimes \cdots \otimes h_n \rangle|^2. \tag{6.5}$$

Definition 6.4 *(Non-commutative Fock space)* Given $T > 0$ and $\alpha \in J^0(\mathfrak{g}_{CM})$, let

$$\|\alpha\|_{J_T^0(\mathfrak{g}_{CM})}^2 := \sum_{n=0}^{\infty} \frac{T^n}{n!} \|\alpha_n\|_n^2. \tag{6.6}$$

Further let

$$J_T^0(\mathfrak{g}_{CM}) := \left\{ \alpha \in J^0(\mathfrak{g}_{CM}) : \|\alpha\|_{J_T^0(\mathfrak{g}_{CM})}^2 < \infty \right\}. \tag{6.7}$$

The space, $J_T^0(\mathfrak{g}_{CM})$, is then a Hilbert space when equipped with the inner product

$$\langle \alpha, \beta \rangle_{J_T^0(\mathfrak{g}_{CM})} = \sum_{n=0}^{\infty} \frac{T^n}{n!} \langle \alpha_n, \beta_n \rangle_n \quad \text{for any } \alpha, \beta \in J_T^0(\mathfrak{g}_{CM}). \tag{6.8}$$

6.2 The Taylor isomorphism

Lemma 6.5 *Let $f \in \mathcal{H}(G_{CM})$ and $T > 0$ and suppose that $\{P_n\}_{n=1}^{\infty} \subset \mathrm{Proj}(W)$ is a sequence such that $P_n|_{\mathfrak{g}_{CM}} \uparrow I_{\mathfrak{g}_{CM}}$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{P_n})} = \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})} = \|f\|_{\mathcal{H}_T^2(G_{CM})} = \lim_{n\to\infty} \|f\|_{L^2\left(G_{P_n}, v_T^{P_n}\right)},$$

(6.9)

where $\|\cdot\|_{\mathcal{H}_T^2(G_{CM})}$ is defined in Eq. (1.4).

Proof By Theorem 5.1 of [6], for all $P \in \mathrm{Proj}(W)$,

$$\|f\|_{L^2(G_P, v_T^P)} = \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_P)},$$

(6.10)

where

$$\left\| \hat{f}(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_P)}^2 = \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\{h_j\}_{j=1}^n \subset \Gamma_P} \left| \left\langle \hat{f}(\mathbf{e}), h_1 \otimes \cdots \otimes h_n \right\rangle \right|^2$$

(6.11)

and Γ_P is an orthonormal basis for $\mathfrak{g}_P$. In particular, it follows that

$$\|f\|_{\mathcal{H}_T^2(G_{CM})} = \sup_{P \in \mathrm{Proj}(W)} \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_P)}$$

(6.12)

and hence we must now show

$$\sup_{P \in \mathrm{Proj}(W)} \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_P)} = \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})}.$$

(6.13)

If Γ is an orthonormal basis for $\mathfrak{g}_{CM}$ containing Γ_P, it follows that

$$\left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_P)}^2 \leq \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\{h_j\}_{j=1}^n \subset \Gamma} \left| \left\langle \hat{f}\,(\mathbf{e}), h_1 \otimes \cdots \otimes h_n \right\rangle \right|^2 = \left\| \hat{f}\,(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})}^2,$$

which shows that $\sup_{P \in \mathrm{Proj}(W)} \|\hat{f}(\mathbf{e})\|_{J_T^0(\mathfrak{g}_P)} \leqslant \|\hat{f}(\mathbf{e})\|_{J_T^0(\mathfrak{g}_{CM})}$. We may choose orthonormal bases, Γ_{P_n}, for $\mathfrak{g}_{P_n}$ such that $\Gamma_{P_n} \uparrow \Gamma$ as $n \uparrow \infty$. Then it is easy to show that

$$
\begin{aligned}
\lim_{n \to \infty} \|f\|_{L^2\left(G_{P_n}, \nu_T^{P_n}\right)} &= \lim_{n \to \infty} \left\| \hat{f}(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{P_n})} \\
&= \lim_{n \to \infty} \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\{h_j\}_{j=1}^n \subset \Gamma_{P_n}} \left| \left\langle \hat{f}(\mathbf{e}), h_1 \otimes \cdots \otimes h_n \right\rangle \right|^2 \\
&= \sum_{n=0}^{\infty} \frac{T^n}{n!} \sum_{\{h_j\}_{j=1}^n \subset \Gamma} \left| \left\langle \hat{f}(\mathbf{e}), h_1 \otimes \cdots \otimes h_n \right\rangle \right|^2 \\
&= \left\| \hat{f}(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})}
\end{aligned}
$$

from which it follows that $\sup_{P \in \mathrm{Proj}(W)} \|\hat{f}(\mathbf{e})\|_{J_T^0(\mathfrak{g}_P)} \geqslant \|\hat{f}(\mathbf{e})\|_{J_T^0 t(\mathfrak{g}_{CM})}$. $\qquad\square$

For the next corollary, recall that $\mathcal{P}$ and $\mathcal{P}_{CM}$ denote the spaces of holomorphic cylinder polynomials on G and G_{CM} respectively, see Definition 1.6 and Eq. (1.7).

Corollary 6.6 *If $f : G \to \mathbb{C}$ is a continuous function satisfying the bounds in Proposition 4.12 with $p = 2$, then $f|_{G_{CM}} \in \mathcal{H}_T^2(G_{CM})$ and $\hat{f}(\mathbf{e}) \in J_T^0(\mathfrak{g}_{CM})$. In particular, for all $T > 0$, $\mathcal{P}_{CM} \subset \mathcal{H}_T^2(G_{CM})$ and for any $p \in \mathcal{P}$, $\hat{p}(\mathbf{e}) \in J_T^0(\mathfrak{g}_{CM})$. This shows that $\mathcal{H}_T^2(G_{CM})$ and $J_T^0(\mathfrak{g}_{CM})$ are non-trivial spaces.*

Definition 6.7 For each $T > 0$, the *Taylor map* is the linear map, $\mathcal{T}_T : \mathcal{H}_T^2(G_{CM}) \to J_T^0(\mathfrak{g}_{CM})$, defined by $\mathcal{T}_T f := \hat{f}(\mathbf{e})$.

Corollary 6.8 *The Taylor map, $\mathcal{T}_T : \mathcal{H}_T^2(G_{CM}) \to J_T^0(\mathfrak{g}_{CM})$, is injective. Moreover, the function $\| \cdot \|_{\mathcal{H}_T^2(G_{CM})}$ is a norm on $\mathcal{H}_T^2(G_{CM})$ which is induced by the inner product on $\mathcal{H}_T^2(G_{CM})$ defined by*

$$
\langle u, v \rangle_{\mathcal{H}_T^2(G_{CM})} := \left\langle \hat{u}(\mathbf{e}), \hat{v}(\mathbf{e}) \right\rangle_{J_T^0(\mathfrak{g}_{CM})} \quad \text{for any } u, v \in \mathcal{H}_T^2(G_{CM}). \quad (6.14)
$$

Proof If $\hat{f}(\mathbf{e}) = 0$, then $\|f\|_{\mathcal{H}_T^2(G_{CM})} = 0$ which then implies that $f|_{G_P} \equiv 0$ for all $P \in \mathrm{Proj}(W)$. As $f : G_{CM} \to \mathbb{C}$ is continuous and $\cup_{P \in \mathrm{Proj}(W)} G_P$ is dense in G_{CM} (see the end of the proof of Theorem 5.9), it follows that $f \equiv 0$. Hence we have shown $\mathcal{T}_T$ injective. Since $\| \cdot \|_{J_T^0(\mathfrak{g}_{CM})}$ is a Hilbert norm and, by Lemma 6.9, $\|f\|_{\mathcal{H}_T^2(G_{CM})} = \|\mathcal{T}_T f\|_{J_T^0(\mathfrak{g}_{CM})}$, it follows that $\| \cdot \|_{\mathcal{H}_T^2(G_{CM})}$ is the norm on $\mathcal{H}_T^2(G_{CM})$ induced by the inner product defined in Eq. (6.14). $\qquad\square$

Our next goal is to show that the Taylor map, $\mathcal{T}_T$, is surjective. The following lemma motivates the construction of the inverse of the Taylor map.

Lemma 6.9 *For every* $f \in \mathcal{H}(G_{CM})$,

$$f(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \hat{f}_n (\mathbf{e}), g^{\otimes n} \right\rangle \quad \text{for any } g \in G_{CM}, \tag{6.15}$$

where the above sum is absolutely convergent. By convention, $g^{\otimes 0} = 1 \in \mathbb{C}$. *(For a more general version of this Lemma, see Proposition 5.1 in [3].)*

Proof The function $u(z) := f(zg)$ is a holomorphic function of $z \in \mathbb{C}$. Therefore,

$$f(g) = u(1) = \sum_{n=0}^{\infty} \frac{1}{n!} u^{(n)}(0)$$

and the above sum is absolutely convergent. In fact, one easily sees that for all $R > 0$ there exists $C(R) < \infty$ such that $\frac{1}{n!}|u^{(n)}(0)| \leqslant C(R)R^{-n}$ for all $n \in \mathbb{N}$. The proof is now completed upon observing

$$u^{(n)}(0) = \left(\frac{d}{dt}\right)^n u(t)\,|_{t=0} = \left(\frac{d}{dt}\right)^n f(tg)\,|_{t=0}$$

$$= \left(\frac{d}{dt}\right)^n f(e^{tg})\,|_{t=0} = \left(\tilde{g}^n f\right)(\mathbf{e}) = \left\langle \hat{f}_n (\mathbf{e}), g^{\otimes n} \right\rangle.$$

$\square$

The next theorem is a more precise version of Theorem 1.5.

Theorem 6.10 (Taylor isomorphism theorem) *For all* $T > 0$, *the space* $\mathcal{H}_T^2(G_{CM})$ *equipped with the inner product* $\langle \cdot, \cdot \rangle_{\mathcal{H}_T^2(G_{CM})}$ *is a Hilbert space,* $T(\mathcal{H}_T^2(G_{CM})) \subset J_T^0(\mathfrak{g}_{CM})$, *and* $\mathcal{T}_T := T|_{\mathcal{H}_T^2(G_{CM})} : \mathcal{H}_T^2(G_{CM}) \to J_T^0(\mathfrak{g}_{CM})$ *is a unitary transformation.*

Proof Given Corollary 6.8, it only remains to prove $\mathcal{T}_T$ is surjective. So let $\alpha \in J_T^0(\mathfrak{g}_{CM})$. By Lemma 6.9, if $f = \mathcal{T}_T^{-1}\alpha$ exists it must be given by

$$f(g) := \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \alpha_n, g^{\otimes n} \right\rangle \quad \text{for any } g \in G_{CM}. \tag{6.16}$$

We now have to check that the sum is convergent, the resulting function f is in $\mathcal{H}(G_{CM})$, and $\hat{f}(\mathbf{e}) = \alpha$. Once this is done, we may apply Lemma 6.5 to conclude that $\|f\|_{\mathcal{H}_T^2(G_{CM})} = \|\alpha\|_{J_T^0(\mathfrak{g}_{CM})} < \infty$ and hence we will have shown that $f \in \mathcal{H}_T^2(G_{CM})$ and $\mathcal{T}_T f = \alpha$. For each $n \in \mathbb{N} \cup \{0\}$, the function $u_n(g) := \frac{1}{n!}\langle \alpha_n, g^{\otimes n} \rangle$ is a continuous complex n–linear form in $g \in G_{CM}$ and therefore holomorphic. Since $|\langle \alpha_n, g^{\otimes n} \rangle| \leqslant \|\alpha_n\|_n \|g\|_{\mathfrak{g}_{CM}}^n$, then for $R > 0$

$$\sup \left\{ |u_n(g)| : \|g\|_{\mathfrak{g}_{CM}} \leqslant R \right\} \leqslant \|\alpha_n\|_n R^n.$$

Therefore it follows that

$$\sum_{n=0}^{\infty} \sup\left\{|u_n(g)| : \|g\|_{\mathfrak{g}_{CM}} \leqslant R\right\} \leqslant \sum_{n=0}^{\infty} \frac{T^n}{n!} \|\alpha_n\|_n \frac{R^n}{T^n}$$

$$\leqslant \sqrt{\sum_{n=0}^{\infty} \frac{T^n}{n!} \|\alpha_n\|_n^2} \sqrt{\sum_{n=0}^{\infty} \frac{T^n}{n!}\left(\frac{R^n}{T^n}\right)^2}$$

$$= \|\alpha\|_{J_T^0(\mathfrak{g}_{CM})} e^{R^2/(2T)} < \infty. \qquad (6.17)$$

This shows $f(g) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(g)$ with the limit being uniform over g in bounded subsets of $\mathfrak{g}_{CM}$. Hence, the sum in Eq. (6.16) is convergent and (see [17, Theorem 3.18.1]) the resulting function, f, is in $\mathcal{H}(G_{CM})$. Since

$$f(zh) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \langle \alpha_n, h^{\otimes n} \rangle \quad \text{for any } z \in \mathbb{C} \quad \text{and} \quad h \in \mathfrak{g}_{CM},$$

it follows that

$$\langle \alpha_n, h^{\otimes n} \rangle = \left(\frac{d}{dz}\right)^n f(zh)|_{z=0} = \left(\frac{d}{dt}\right)^n f\left(e^{th}\right)|_{t=0} = \left\langle \hat{f}_n(\mathbf{e}), h^{\otimes n} \right\rangle.$$

This is true for all n and $h \in \mathfrak{g}_{CM}$, so we may use the argument following Eq. (6.13) in [3] (or see the proof of Theorem 2.5 in [7]) to show $\hat{f}(\mathbf{e}) = \alpha$. □

As a consequence of Eq. (6.17) we see that if $f \in \mathcal{H}_T^2(G_{CM})$ then

$$|f(g)| \leqslant \|f\|_{\mathcal{H}_T^2(G_{CM})} e^{\|g\|_{\mathfrak{g}_{CM}}^2/(2T)} \quad \text{for any } g \in G_{CM}. \qquad (6.18)$$

The next theorem, which is an analogue of Bargmann's pointwise bounds (see [1, Eq. (1.7)] and [6, Eq. (5.4)]), improves upon the estimate in Eq. (6.18).

Theorem 6.11 (Pointwise bounds) *If $f \in \mathcal{H}_T^2(G_{CM})$ and $g \in G_{CM}$, then for all $g \in G_{CM}$,*

$$|f(g)| \leqslant \|f\|_{\mathcal{H}_T^2(G_{CM})} e^{d_{CM}^2(\mathbf{e},g)/(2T)}, \qquad (6.19)$$

where $d_{CM}^2(\cdot, \cdot)$ is the distance function on G_{CM} defined in Eq. (4.7).

Proof Let $P_n \in \text{Proj}(W)$ be chosen so that $P_n|_{\mathfrak{g}_{CM}} \uparrow I_{\mathfrak{g}_{CM}}$ as $n \to \infty$ and recall that $G_0 := \cup_{n=1}^{\infty} G_{P_n}$ is a dense subgroup of G_{CM} as explained in the proof of Theorem 5.9. Let $g \in G_{P_m}$ for some $m \in \mathbb{N}$ and let $\sigma : [0, 1] \to G_{CM}$ be a C^1–curve such that $\sigma(0) = \mathbf{e}$ and $\sigma(1) = g$. Then for $n \geqslant m$, $\sigma_n(t) := \pi_{P_n}(\sigma(t))$ is a C^1 curve in G_n such that $\sigma_n(0) = \mathbf{e}$ and $\sigma_n(1) = g$. Therefore by [6, Eq. (5.4)], we have

$$|f(g)| \leqslant \left\| f|_{G_{P_n}} \right\|_{L^2\left(G_{P_n}, \nu_T^{P_n}\right)} \cdot e^{d_{G_{P_n}}^2(\mathbf{e},g)/(2T)} \leqslant \|f\|_{\mathcal{H}_T^2(G_{CM})} \cdot e^{\ell_{G_{CM}}^2(\sigma_n)/(2T)},$$

(6.20)

where $\ell_{G_{CM}}(\sigma_n)$ is the length of σ_n as in Eq. (4.6). In the proof [4, Theorem 8.1], it was shown that $\lim_{n\to\infty} \ell_{G_{CM}}(\sigma_n) = \ell_{G_{CM}}(\sigma)$. Hence we may pass to the limit in Eq. (6.20) to find, $|f(g)| \leqslant \|f\|_{\mathcal{H}_T^2(G_{CM})} \cdot e^{\ell_{G_{CM}}(\sigma)/(2T)}$. Optimizing this last inequality over all σ joining $\mathbf{e}$ to g then shows that Eq. (6.19) holds for all $g \in G_0$. This suffices to prove Eq. (6.19) as both sides of this inequality are continuous in $g \in G_{CM}$ and G_0 is dense in G_{CM}.   □

7 Density theorems

The following density result is the main theorem of this section and is crucial to the next section. Techniques similar to those used in this section have appeared in Cecil [2] to prove an analogous result for path groups over stratified Lie groups.

Theorem 7.1 (Density theorem) *For all $T > 0$, $\mathcal{P}_{CM}$ defined by Eq. (1.7) is a dense subspace of $\mathcal{H}_T^2(G_{CM})$.*

Proof This theorem is a consequence of Corollary 7.4 and Proposition 7.12 below. □

The remainder of this section will be devoted to proving the results used in the proof of the theorem. We will start by constructing some auxiliary dense subspaces of $J_T^0(\mathfrak{g}_{CM})$ and $\mathcal{H}_T^2(G_{CM})$.

7.1 Finite rank subspaces

Definition 7.2 A tensor, $\alpha \in J^0(\mathfrak{g}_{CM})$, is said to have *finite rank* if $\alpha_n = 0$ for all but finitely many $n \in \mathbb{N}$.

The next lemma is essentially a special case of [7, Lemma 3.5].

Lemma 7.3 (Finite Rank Density Lemma) *The finite rank tensors in $J_T^0(\mathfrak{g}_{CM})$ are dense in $J_T^0(\mathfrak{g}_{CM})$.*

Proof For $\theta \in \mathbb{R}$, let $\varphi_\theta : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ be defined by

$$\varphi_\theta(A, a) = \left(e^{i\theta} A, e^{i2\theta} a \right).$$

Since

$$
\begin{aligned}
[\varphi_\theta(A, a), \varphi_\theta(B, b)] &= \left[\left(e^{i\theta} A, e^{i2\theta} a \right), \left(e^{i\theta} B, e^{i2\theta} b \right) \right] \\
&= \left(0, \omega \left(e^{i\theta} A, e^{i\theta} B \right) \right) = \left(0, e^{i2\theta} \omega(A, B) \right) \\
&= \varphi_\theta \left[(A, a), (B, b) \right]
\end{aligned}
$$

we see that φ_θ is a Lie algebra homomorphism.

Now let $\Phi_\theta : \mathbf{T}(\mathfrak{g}_{CM}) \to \mathbf{T}(\mathfrak{g}_{CM})$ be defined by $\Phi_\theta 1 = 1$ and

$$\Phi_\theta (h_1 \otimes \cdots \otimes h_n) = \varphi_\theta h_1 \otimes \cdots \otimes \varphi_\theta h_n \quad \text{for all } h_i \in \mathfrak{g}_{CM} \text{ and } n \in \mathbb{N}.$$

If we write $\xi \wedge \eta$ for $\xi \otimes \eta - \eta \otimes \xi$, then

$$\begin{aligned} \Phi_\theta(\xi \wedge \eta - [\xi, \eta]) &= (\varphi_{e^{i\theta}} \xi) \wedge (\varphi_{e^{i\theta}} \eta) - \varphi_{e^{i\theta}}[\xi, \eta] \\ &= (\varphi_{e^{i\theta}} \xi) \wedge (\varphi_{e^{i\theta}} \eta) - [\varphi_{e^{i\theta}} \xi, \varphi_{e^{i\theta}} \eta]. \end{aligned}$$

From this it follows that $\Phi_\theta(J) \subset J$ and therefore if $\alpha \in J^0(\mathfrak{g}_{CM})$, then $\alpha \circ \Phi_\theta \in J^0(\mathfrak{g}_{CM})$. Letting Γ be an orthonormal basis as in Eq. (5.8), we have $\varphi_\theta h = e^{i2\theta} h$ or $\varphi_\theta h = e^{i\theta} h$ for all $h \in \Gamma$. Therefore it follows that

$$\begin{aligned} |\langle \alpha \circ \Phi_\theta, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 &= |\langle \alpha, \varphi_\theta k_1 \otimes \varphi_\theta k_2 \otimes \cdots \otimes \varphi_\theta k_n \rangle|^2 \\ &= |\langle \alpha, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 \end{aligned}$$

and hence that

$$\begin{aligned} \|\alpha \circ \Phi_\theta\|^2_{J_T^0(\mathfrak{g}_{CM})} &= \sum_{n=0}^\infty \frac{T^n}{n!} \sum_{k_1, k_2, \ldots, k_n \in \Gamma} |\langle \alpha \circ \Phi_\theta, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 \\ &= \sum_{n=0}^\infty \frac{T^n}{n!} \sum_{k_1, k_2, \ldots, k_n \in \Gamma} |\langle \alpha, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 = \|\alpha\|^2_{J_T^0(\mathfrak{g}_{CM})}. \end{aligned}$$

So the map $\alpha \in J_T^0(\mathfrak{g}_{CM}) \to \alpha \circ \Phi_\theta \in J_T^0(\mathfrak{g}_{CM})$ is unitary. Moreover, since

$$\begin{aligned} |\langle \alpha, \varphi_\theta k_1 \otimes \varphi_\theta k_2 \otimes \cdots \otimes \varphi_\theta k_n \rangle &- \langle \alpha, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 \\ &\leqslant 2 |\langle \alpha, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 \end{aligned}$$

we may apply the dominated convergence theorem to conclude

$$\begin{aligned} \lim_{\theta \to 0} &\|\alpha \circ \Phi_\theta - \alpha\|^2_{J_T^0(\mathfrak{g}_{CM})} \\ &= \sum_{n=0}^\infty \frac{T^n}{n!} \sum_{k_1, k_2, \ldots, k_n \in \Gamma} \lim_{\theta \to 0} |\langle \alpha, \varphi_\theta k_1 \otimes \varphi_\theta k_2 \otimes \cdots \otimes \varphi_\theta k_n \rangle \\ &\quad - \langle \alpha, k_1 \otimes k_2 \otimes \cdots \otimes k_n \rangle|^2 \\ &= 0, \end{aligned}$$

so that $\alpha \to \alpha \circ \Phi_\theta$ is continuous. (Notice that $\Phi_\theta \circ \Phi_\alpha = \Phi_{\theta+\alpha}$, so it suffices to check continuity at $\theta = 0$.)

Let

$$F_n(\theta) = \frac{1}{2\pi n} \sum_{k=0}^{n-1} \sum_{\ell=-k}^{k} e^{i\ell\theta} = \frac{1}{2\pi n} \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel [28, p. 143]. Then $\int_{-\pi}^{\pi} F_n(\theta)d\theta = 1$ for all n and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta)u(\theta)d\theta = u(0) \quad \text{for all } u \in C\left([-\pi, \pi], \mathbb{C}\right).$$

We now let

$$\alpha(n) := \int_{-\pi}^{\pi} \alpha \circ \Phi_\theta F_n(\theta)d\theta.$$

Then

$$\limsup_{n\to\infty} \|\alpha - \alpha(n)\|_{J_T^0(\mathfrak{g}_{CM})}^2 \leqslant \limsup_{n\to\infty} \left\| \int_{-\pi}^{\pi} [\alpha - \alpha \circ \Phi_\theta] F_n(\theta)d\theta \right\|_{J_T^0(\mathfrak{g}_{CM})}$$

$$\leqslant \limsup_{n\to\infty} \int_{-\pi}^{\pi} \|\alpha - \alpha \circ \Phi_\theta\|_{J_T^0(\mathfrak{g}_{CM})} F_n(\theta)d\theta = 0.$$

Moreover if $\beta := k_1, \ldots, k_m \in \mathfrak{g}_{CM}$ with $m > n$, then there exits $\beta_l \in \mathfrak{g}_{CM}^{\otimes m}$ such that

$$\Phi_\theta \beta = \sum_{l=m}^{2m} e^{il\theta} \beta_l.$$

From this it follows that

$$\langle \alpha(n), \beta \rangle := \int_{-\pi}^{\pi} \langle \alpha, \Phi_\theta \beta \rangle F_n(\theta)d\theta = \sum_{l=m}^{2m} \langle \alpha, \beta_l \rangle \int_{-\pi}^{\pi} e^{il\theta} F_n(\theta)d\theta = 0$$

from which it follows that $\alpha(n)_m \equiv 0$ for all $m > n$. Thus $\alpha(n)$ is a finite rank tensor for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \|\alpha - \alpha(n)\|_{J_T^0(\mathfrak{g}_{CM})}^2 = 0$. □

Corollary 7.4 *The vector space,*

$$\mathcal{H}_{T,fin}^2(G_{CM}) := \left\{ u \in \mathcal{H}_T^2(G_{CM}) : \hat{u}(\mathbf{e}) \in J_T^0(\mathfrak{g}_{CM}) \text{ has finite rank} \right\} \quad (7.1)$$

is a dense subspace of $\mathcal{H}_T^2(G_{CM})$.

Proof This follows directly from Lemma 7.3 and the Taylor isomorphism Theorem 6.10. □

7.2 Polynomial approximations

To prove Theorem 7.1, it suffices to show that every element $u \in \mathcal{H}^2_{T,\text{fin}}(G_{CM})$ may be well approximated by an element from $\mathcal{H}^2_T(G)$. In order to do this, let $\{e_j : j = 1, 2, \} \subset H_*$ be an orthonormal basis for H and for $N \in \mathbb{N}$, define $P_N \in \text{Proj}(W)$ as in Eq. (2.17), i.e.

$$P_N(w) = \sum_{j=1}^{N} \langle w, e_j \rangle_H \, e_j \quad \text{for all } w \in W. \tag{7.2}$$

Let us further define $\pi_N := \pi_{P_N}$ and

$$u_N := u \circ \pi_N \quad \text{for all } N \in \mathbb{N}. \tag{7.3}$$

We are going to prove Theorem 7.1 by showing $u_N \in \mathcal{P}$ and $u_N \to u$ in $\mathcal{H}^2_T(G_{CM})$.

Remark 7.5 A complicating factor in showing $u_N|_{G_{CM}} \to u$ in $\mathcal{H}^2_T(G_{CM})$ is the fact that for general ω and $P \in \text{Proj}(W)$, $\pi_P : G \to G_P \subset G_{CM}$ is *not* a group homomorphism. In fact we have,

$$\pi_P \left[(w, c) \cdot (w', c')\right] - \pi_P(w, c) \cdot \pi_P(w', c') = \Gamma_P(w, w') \tag{7.4}$$

where

$$\Gamma_P(w, w') = \frac{1}{2} \left(0, \omega(w, w') - \omega(Pw, Pw')\right) \tag{7.5}$$

So unless ω is "supported" on the range of P, π_P is not a group homomorphism. Since, $(w, b) + (0, c) = (w, b) \cdot (0, c)$ for all $w \in W$ and $b, c \in \mathbb{C}$, we may also write Eq. (7.4) as

$$\pi_P \left[(w, c) \cdot (w', c')\right] = \pi_P(w, c) \cdot \pi_P(w', c') \cdot \Gamma_P(w, w'). \tag{7.6}$$

Lemma 7.6 *To each $k := (A, a) \in \mathfrak{g}_{CM}$, $g = (w, c) \in G$, and $P \in \text{Proj}(W)$, let*

$$k^P(g) = k^P(w, c) := \pi_P k + \Gamma_P(w, A) \in \mathfrak{g}_P \tag{7.7}$$

where Γ_P is defined in Eq. (7.5) above. If $u : G_{CM} \to \mathbb{C}$ is a holomorphic function and $g \in G$, then

$$\left(\tilde{k}(u \circ \pi_P)\right)(g) = \left\langle Du(\pi_P(g)), k^P(g) \right\rangle \tag{7.8}$$

or equivalently put,

$$\left\langle \widehat{u \circ \pi_P}(g), k \right\rangle = \langle D\left(u \circ \pi_P\right)(g), k \rangle = \left\langle Du\left(\pi_P(g)\right), k^P(g) \right\rangle. \qquad (7.9)$$

Proof By direct computation,

$$\left(\tilde{k} \left(u \circ \pi_P \right) \right)(g) = \frac{d}{dt}\bigg|_0 u\left(\pi_P\left(g \cdot e^{tk} \right) \right)$$

$$= \frac{d}{dt}\bigg|_0 \left\langle Du\left(\pi_P(g) \right), [\pi_P(g)]^{-1} \cdot \pi_P\left(g \cdot e^{tk} \right) \right\rangle$$

where by Eq. (7.6),

$$\frac{d}{dt}\bigg|_0 \left([\pi_P(g)]^{-1} \cdot \pi_P\left(g \cdot e^{tk} \right) \right)$$

$$= \frac{d}{dt}\bigg|_0 \left(P(tA), a + \frac{1}{2}\omega\left(w, tA \right) - \omega\left(Pw, tPA \right) \right)$$

$$= \left(PA, a + \frac{1}{2}\omega\left(w, A \right) - \omega(Pw, PA) \right)$$

$$= \pi_P k + \Gamma_P(w, A).$$

$\square$

Notation 7.7 *Given $P \in \text{Proj}(W)$ and $k_j = (A_j, c_j) \in \mathfrak{g}_{CM}$, let $K_j := k_j^P :$ $G_{CM} \to \mathfrak{g}_{CM}$ and $\kappa_n : G_{CM} \to \oplus_{j=1}^n \mathfrak{g}_{CM}^{\otimes j}$ be defined by*

$$\kappa_n = \left(\tilde{k}_n + K_n \otimes \right) \left(\tilde{k}_{n-1} + K_{n-1} \otimes \right) \cdots \left(\tilde{k}_1 + K_1 \otimes \right) 1$$

$$= \left(\tilde{k}_n + K_n \otimes \right) \left(\tilde{k}_{n-1} + K_{n-1} \otimes \right) \cdots \left(\tilde{k}_2 + K_2 \otimes \right) K_1. \qquad (7.10)$$

In these expressions, $K_j \otimes$ denotes operation of left tensor multiplication by K_j.

Example 7.8 The functions κ_n are determined recursively by $\kappa_1 = K_1$ and then

$$\kappa_n = \left(K_n \otimes + \tilde{k}_n \right) \kappa_{n-1} = K_n \otimes \kappa_{n-1} + \tilde{k}_n \kappa_{n-1} \quad \text{for all } n \geq 2. \qquad (7.11)$$

The first four κ_n are easily seen to be given by, $\kappa_1 = K_1$,

$$\kappa_2 = K_2 \otimes K_1 + \tilde{k}_2 K_1 = K_2 \otimes K_1 + \Gamma_P\left(A_2, A_1 \right),$$

$$\kappa_3 = \left(K_3 \otimes + \tilde{k}_3 \right) \left(K_2 \otimes K_1 + \Gamma_P\left(A_2, A_1 \right) \right)$$

$$= K_3 \otimes K_2 \otimes K_1 + K_3 \otimes \Gamma_P\left(A_2, A_1 \right) + \Gamma_P\left(A_3, A_2 \right) \otimes K_1 + K_2 \otimes \Gamma_P\left(A_3, A_1 \right),$$

and

$$\kappa_4 = K_4 \otimes K_3 \otimes K_2 \otimes K_1$$
$$+ \begin{pmatrix} K_4 \otimes K_3 \otimes \Gamma_P (A_2, A_1) + K_4 \otimes \Gamma_P (A_3, A_2) \otimes K_1 + K_4 \otimes K_2 \otimes \Gamma_P (A_3, A_1) \\ + \Gamma_P (A_4, A_3) \otimes K_2 \otimes K_1 + K_3 \otimes \Gamma_P (A_4, A_2) \otimes K_1 + K_3 \otimes K_2 \otimes \Gamma_P (A_4, A_1) \end{pmatrix}$$
$$+ \Gamma_P (A_4, A_3) \otimes \Gamma_P (A_2, A_1) + \Gamma_P (A_3, A_2) \otimes \Gamma_P (A_4, A_1) + \Gamma_P (A_4, A_2) \otimes \Gamma_P (A_3, A_1).$$

At the end we will only use κ_n evaluated at $\mathbf{e} \in G_{CM}$. Evaluating the above expressions at $\mathbf{e}$ amounts to replacing K_j by $\pi_P k_j$ in all of the previous formulas.

Proposition 7.9 *If $u \in \mathcal{H}(G_{CM})$, then, with the setup in Notation 7.7, we have*

$$\langle \widehat{u \circ \pi_P}, k_n \otimes \cdots \otimes k_1 \rangle = \langle \hat{u} \circ \pi_P, \kappa_n \rangle \quad \text{for any } n \in \mathbb{N}, \tag{7.12}$$

where both sides of this equation are holomorphic functions on G_{CM}.

Proof The proof is by induction with the case $n = 1$ already completed via Equation (7.9). To proceed with the induction argument, suppose that Eq. (7.12) holds for some $n \in \mathbb{N}$. Then by induction and the product rule

$$\langle \widehat{u \circ \pi_P}, k_{n+1} \otimes k_n \otimes \cdots \otimes k_1 \rangle = \tilde{k}_{n+1} \langle \widehat{u \circ \pi_P}, k_{n+1} \otimes k_n \otimes \cdots \otimes k_1 \rangle$$
$$= \tilde{k}_{n+1} \langle \hat{u} \circ \pi_P, \kappa_n \rangle$$
$$= \langle \hat{u} \circ \pi_P, \tilde{k}_{n+1} \kappa_n \rangle + \langle \tilde{k}_{n+1} [\hat{u} \circ \pi_P], \kappa_n \rangle. \tag{7.13}$$

To evaluate $\tilde{k}_{n+1}[\hat{u} \circ \pi_P]$ let $v \in \mathbf{T}(\mathfrak{g}_{CM})$ and let $\tilde{v}$ denote the corresponding left invariant differential operator on G_{CM}. Then

$$\langle \tilde{k}_{n+1} [\hat{u} \circ \pi_P], v \rangle (g) = \left(\tilde{k}_{n+1} \langle [\hat{u} \circ \pi_P], v \rangle \right) (g)$$
$$= \left(\tilde{k}_{n+1} [(\tilde{v}u) \circ \pi_P] \right) (g)$$
$$= \langle D (\tilde{v}u) (\pi_P(g)), k_{n+1}^P(g) \rangle$$
$$= \left(\widetilde{k_{n+1}^P(g)} \tilde{v}u \right) (\pi_P(g))$$
$$= \langle \hat{u} (\pi_P(g)), k_{n+1}^P(g) \otimes v \rangle. \tag{7.14}$$

Combining Eqs. (7.13) and (7.14) shows,

$$\langle \widehat{u \circ \pi_P}, k_{n+1} \otimes k_n \otimes \cdots \otimes k_1 \rangle = \langle \hat{u} \circ \pi_P, \tilde{k}_{n+1} \kappa_n \rangle + \langle \hat{u} \circ \pi_P, k_{n+1}^P \otimes \kappa_n \rangle$$
$$= \langle \hat{u} \circ \pi_P, \tilde{k}_{n+1} \kappa_n + k_{n+1}^P \otimes \kappa_n \rangle = \langle \hat{u} \circ \pi_P, \kappa_{n+1} \rangle$$

wherein we have used Eq. (7.11) for the last equality. $\qquad \square$

The induction proof of the following lemma will be left to the reader with Example 7.8 as a guide.

Lemma 7.10 *Let $k_j = (A_j, c_j) \in \mathfrak{g}_{CM}$ for $1 \leqslant j \leqslant n$, $\lfloor \frac{n}{2} \rfloor = n/2$ if n is even and $(n-1)/2$ if n is odd, and κ_n be as in Eq. (7.10). Then*

$$\kappa_n(\mathbf{e}) = \pi_P k_n \otimes \cdots \otimes \pi_P k_2 \otimes \pi_P k_1 + R\left(P : k_n, \ldots, k_1\right), \qquad (7.15)$$

where

$$R\left(P : k_n, \ldots, k_1\right) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} R_j\left(P : k_n, \ldots, k_1\right) \qquad (7.16)$$

with $R_j(P : k_1, \ldots, k_n) \in \mathfrak{g}_{CM}^{\otimes(n-j)}$. Each remainder term, $R_j(P : k_1, \ldots, k_n)$, is a linear combination (with coefficients coming from $\{\pm 1, 0\}$) of homogenous tensors which are permutations of the indices and order of the terms in the tensor product of the form

$$\Gamma_P\left(A_1, A_2\right) \otimes \cdots \otimes \Gamma_P\left(A_{2j-1}, A_{2j}\right) \otimes k_{2j+1} \otimes \cdots \otimes k_n. \qquad (7.17)$$

Proposition 7.11 *Let $P_N \in \mathrm{Proj}(W)$ and $\pi_N := \pi_{P_N}$ be as in Notation 1.1 and suppose that $u \in \mathcal{H}(G_{CM})$ satisfies $\|\hat{u}_n(\mathbf{e})\|_n < \infty$ for all n. Then*

$$\lim_{N \to \infty} \left\| \hat{u}_n(\mathbf{e}) - \left[\widehat{u \circ \pi_N}(\mathbf{e}) \right]_n \right\|_n = 0 \quad \text{for } n = 0, 1, 2, \cdots \qquad (7.18)$$

Proof To simplify notation, let $\alpha_n := \hat{u}_n(\mathbf{e})$ and $\alpha_n(N) := [\widehat{u \circ \pi_N}(\mathbf{e})]_n$. Let Γ be an orthonormal basis for $\mathfrak{g}_{CM}$ of the form in Eq. (5.8) and let $\mathbf{k} := (k_1, k_2, \ldots, k_n) \in \Gamma^n$. Then

$$\langle \alpha - \alpha(N), k_1 \otimes \cdots \otimes k_n \rangle = \langle \alpha, k_1 \otimes \cdots \otimes k_n - \pi_N k_1 \otimes \cdots \otimes \pi_N k_n \rangle$$
$$+ \langle \alpha, R(P_N : \mathbf{k}) \rangle$$

where $R(P_N : \mathbf{k})$ is as in Lemma 7.10. Therefore, $\|\alpha_n - \alpha_n(N)\|_n \leqslant C_N + D_N$ where

$$C_N := \sqrt{\sum_{\mathbf{k} \in \Gamma^n} |\langle \alpha, R(P_N : \mathbf{k}) \rangle|^2} \quad \text{and}$$

$$D_N := \sqrt{\sum_{\mathbf{k} \in \Gamma^n} |\langle \alpha_n, k_1 \otimes \cdots \otimes k_n - \pi_N k_1 \otimes \cdots \otimes \pi_N k_n \rangle|^2}.$$

We will complete the proof by showing that, $\lim_{N\to\infty} C_N = 0 = \lim_{N\to\infty} D_N$. To estimate C_N, use Lemma 7.10 and the triangle inequality for $\ell_2(\Gamma^n)$ to find,

$$C_N = \sqrt{\sum_{\mathbf{k}\in\Gamma^n} \left| \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \langle \alpha, R_j\,(P_N : \mathbf{k}) \rangle \right|^2} \leqslant \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sqrt{\sum_{\mathbf{k}\in\Gamma^n} |\langle \alpha, R_j\,(P_N : \mathbf{k}) \rangle|^2}.$$

But $\sum_{\mathbf{k}\in\Gamma^n} |\langle \alpha, R_j(P_N : \mathbf{k}) \rangle|^2$ is bounded by a sum of terms (the number of these terms depends only on j and n and *not* N) of which a typical term (see Eq. (7.17)) is;

$$\sum_{\mathbf{k}\in\Gamma^n} \left| \langle \alpha_{n-j}, \Gamma_{P_N}\,(A_1, A_2) \otimes \cdots \otimes \Gamma_{P_N}\,(A_{2j-1}, A_{2j}) \otimes k_{2j+1} \otimes \cdots \otimes k_n \rangle \right|^2.$$

$$(7.19)$$

The sum in Eq. (7.19) may be estimated by,

$$\|\alpha_{n-j}\|_{n-j} \sum_{l_1,\ldots,l_{2j}=1}^{\infty} \left\| \Gamma_{P_N}\,(e_{l_1}, e_{l_2}) \right\|_{\mathfrak{g}_{CM}}^2 \cdots \left\| \Gamma_{P_N}\,(e_{l_{2j-1}}, e_{l_{2j}}) \right\|_{\mathfrak{g}_{CM}}^2 = \|\alpha_{n-j}\|_{n-j}^2 \varepsilon_N^j,$$

where

$$\varepsilon_N = \frac{1}{4} \sum_{k,l=1}^{\infty} \|\omega\,(e_k, e_l) - \omega\,(P_N e_k, P_N e_l)\|_{\mathbf{C}}^2$$

$$= \frac{1}{4} \sum_{\max(k,l)>N}^{\infty} \|\omega\,(e_k, e_l) - \omega\,(P_N e_k, P_N e_l)\|_{\mathbf{C}}^2$$

$$\leqslant \frac{1}{2} \sum_{\max(k,l)>N}^{\infty} \|\omega\,(e_k, e_l)\|_{\mathbf{C}}^2 \to 0 \text{ and } N \to \infty.$$

Thus we have shown $\lim_{N\to\infty} C_N = 0$

For $N \in \mathbb{N}$, let $\Gamma_N = \{(0, f_j)\}_{j=1}^d \cup \{(e_j, 0)\}_{j=1}^N$. Since $k_1 \otimes \cdots \otimes k_n = \pi_N k_1 \otimes \cdots \otimes \pi_N k_n$ if $\mathbf{k} := (k_1, k_2, \ldots, k_n) \in \Gamma_N^n$, it follows that

$$D_N^2 = \sum_{\mathbf{k}\in\Gamma^n\setminus\Gamma_N^n} |\langle \alpha_n, k_1 \otimes \cdots \otimes k_n - \pi_N k_1 \otimes \cdots \otimes \pi_N k_n \rangle|^2$$

$$\leqslant 2 \sum_{\mathbf{k}\in\Gamma^n\setminus\Gamma_N^n} |\langle \alpha_n, k_1 \otimes \cdots \otimes k_n \rangle|^2. \tag{7.20}$$

Because

$$\sum_{\mathbf{k}\in\Gamma^n} |\langle \alpha_n, k_1 \otimes \cdots \otimes k_n \rangle|^2 = \|\alpha_n\|_n^2 < \infty$$

and $\Gamma_N^n \uparrow \Gamma_N$ as $N \uparrow \infty$, the sum in Eq. (7.20) tends to zero as $N \to \infty$. Thus $\lim_{N \to \infty} D_N = 0$ and the proof is complete. $\qquad\square$

Proposition 7.12 *If $u \in \mathcal{H}_{T,fin}^2(G_{CM})$ and $u_N := u \circ \pi_N$ as in Eq. (7.3), then $u_N \in \mathcal{P}$ and $u_N|_{G_{CM}} \to u$ in $\mathcal{H}_T^2(G_{CM})$.*

Proof Suppose $m \in \mathbb{N}$ is chosen so that $\hat{u}_n(\mathbf{e}) = 0$ if $n > m$. According to Proposition 7.9,

$$\langle \hat{u}_N(\mathbf{e}), k_n \otimes \cdots \otimes k_1 \rangle = \langle \hat{u}(\mathbf{e}), \kappa_n(\mathbf{e}) \rangle$$

where $\kappa_n(\mathbf{e}) \in \bigoplus_{j=1}^{\lfloor \frac{n}{2} \rfloor} \mathfrak{g}_{CM}^{\otimes(n-j)}$. From this it follows that $\langle \hat{u}_N(\mathbf{e}), k_n \otimes \cdots \otimes k_1 \rangle = 0$ if $n \geqslant 2m + 2$. Therefore, u_N restricted to $P_N H \times \mathbb{C}$ is a holomorphic polynomial and since $u_N = u_N|_{P_N H \times \mathbb{C}} \circ \pi_N$, it follows that $u_N \in \mathcal{P}$. Moreover,

$$\lim_{N \to \infty} \left\| \hat{u}(\mathbf{e}) - \hat{u}_N(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})}^2 = \lim_{N \to \infty} \sum_{n=0}^{2m+2} \frac{T^n}{n!} \left\| \hat{u}_n(\mathbf{e}) - \left[\hat{u}_N(\mathbf{e}) \right]_n \right\|_n^2 = 0,$$

wherein we have used Proposition 7.11 to conclude $\lim_{N \to \infty} \| \hat{u}_n(\mathbf{e}) - [\hat{u}_N(\mathbf{e})]_n \|_n = 0$ for all n. It then follows by the Taylor isomorphism Theorem 6.10 that $\lim_{N \to \infty} \| u - u_N \|_{\mathcal{H}_T^2(G_{CM})} = 0$. $\qquad\square$

8 The skeleton isomorphism

This section is devoted to the proof of the skeleton Theorem 1.8. Let us begin by gathering together a couple of results that we have already proved.

Proposition 8.1 *If $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}}$ is holomorphic, then*

$$\| f \|_{L^2(\nu_T)} \leqslant \| f|_{G_{CM}} \|_{\mathcal{H}_T^2(G_{CM})} = \| \hat{f}(\mathbf{e}) \|_{J_T^0(\mathfrak{g}_{CM})}. \tag{8.1}$$

If $\| f|_{G_{CM}} \|_{\mathcal{H}_T^2(G_{CM})}^2 < \infty$, then $S_T f = f$ and f satisfies the Gaussian pointwise bounds in Eq. (6.19). (See Corollary 8.3 for a more sophisticated version of this proposition.)

Proof See Theorems 5.9 and 6.11. $\qquad\square$

Lemma 8.2 *Let $f : G \to \mathbb{C}$ be a continuous function such that $f|_{G_{CM}}$ is holomorphic and let $\delta > 0$ be as in Theorem 4.11. If there exists an $\varepsilon \in (0, \delta)$ such that $|f(\cdot)| \leqslant C e^{\varepsilon \rho^2(\cdot)/(2T)}$ on G, then*

$$\| f \|_{L^2(\nu_T)} = \| f \|_{\mathcal{H}_T^2(G_{CM})} = \left\| \hat{f}(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})} < \infty. \tag{8.2}$$

(It will be shown in Corollary 8.4 that f is actually in $\mathcal{H}_T^2(G)$.) In particular, Eq. (8.2) holds for all $f \in \mathcal{P}$.

Proof Let $\{P_n\}_{n=1}^{\infty} \subset \mathrm{Proj}(W)$ be a sequence such that $P_n|_{\mathfrak{g}_{CM}} \uparrow I_{\mathfrak{g}_{CM}}$ as $n \to \infty$. Then, by Lemma 6.5 and Proposition 4.12 with $h = 0$,

$$\infty > \|f\|_{L^2(\nu_T)} = \lim_{n \to \infty} \|f\|_{L^2\left(G_{P_n} \nu_T^{P_n}\right)} = \|f\|_{\mathcal{H}_T^2(G_{CM})} = \left\| \hat{f}(\mathbf{e}) \right\|_{J_T^0(\mathfrak{g}_{CM})}.$$

$\square$

We are now ready to complete the proof of the Skeleton isomorphism Theorem 1.8.

8.1 Proof of Theorem 1.8

Proof By Corollary 5.10, $S_T f = f|_{G_{CM}}$ for all $f \in \mathcal{P}$ and hence by Lemma 8.2, $\|S_T f\|_{\mathcal{H}_T^2(G_{CM})} = \|f\|_{L^2(\nu_T)}$. It therefore follows that $S_T|_{\mathcal{P}}$ extends uniquely to an isometry, $\bar{S}_T$, from $\mathcal{H}_T^2(G)$ to $\mathcal{H}_T^2(G_{CM})$ such that $\bar{S}_T(\mathcal{P}) = \mathcal{P}_{CM}$. Since $\bar{S}_T$ is isometric and $\mathcal{P}_{CM}$ is dense in $\mathcal{H}_T^2(G_{CM})$, it follows that $\bar{S}_T$ is surjective, i.e. $\bar{S}_T :$ $\mathcal{H}_T^2(G) \to \mathcal{H}_T^2(G_{CM})$ is a unitary map. To finish the proof we only need to show $S_T f = \bar{S}_T f$ for all $f \in \mathcal{H}_T^2(G)$. Let $p_n \in \mathcal{P}$ such that $p_n \to f$ in $L^2(\nu_T)$. Then $p_n = S_T p_n \to \bar{S}_T f$ in $\mathcal{H}_T^2(G_{CM})$ and hence by the Gaussian pointwise bounds in Eq. (6.19), $\bar{S}_T f(g) = \lim_{n \to \infty} p_n(g)$ for all $g \in G_{CM}$. Similarly, using the Gaussian bounds in Corollary 4.8, it follows that

$$|S_T f(g) - p_n(g)| = |S_T (f - p_n)(g)|$$
$$\leqslant \|f - p_n\|_{L^2(\nu_T)} \exp\left(\frac{c(k(\omega)T/2)}{T} d_{G_{CM}}^2(\mathbf{e}, g)\right) \quad (8.3)$$

and hence we also have, $S_T f(g) = \lim_{n \to \infty} p_n(g)$ for all $g \in G_{CM}$. Therefore, $S_T f = \bar{S}_T f$ as was to be shown. $\square$

Corollary 8.3 *If $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}} \in \mathcal{H}_T^2(G_{CM})$, then $f \in \mathcal{H}_T^2(G)$, $S_T f = f|_{G_{CM}}$, and $\|f\|_{L^2(\nu_T)} = \|f\|_{\mathcal{H}_T^2(G_{CM})}$.*

Proof By Proposition 8.1 we already know that $S_T f = f|_{G_{CM}}$. By Theorem 1.8, there exists $u \in \mathcal{H}_T^2(G)$ such that $f|_{G_{CM}} = S_T u$. Let $p_n \in \mathcal{P}$ be chosen so that $p_n \to u$ in $L^2(\nu_T)$ and hence $p_n|_{G_{CM}} = S_T p_n \to S_T u = S_T f$ in $\mathcal{H}_T^2(G_{CM})$ as $n \to \infty$. Hence it follows from Proposition 8.1 that

$$\|f - p_n\|_{L^2(\nu_T)} \leqslant \left\|(f - p_n)|_{G_{CM}}\right\|_{\mathcal{H}_T^2(G_{CM})} = \|S_T(f - p_n)\|_{\mathcal{H}_T^2(G_{CM})},$$

and therefore, $\lim_{n \to \infty} \|f - p_n\|_{L^2(\nu_T)} = 0$, i.e. $p_n \to f$ in $L^2(\nu_T)$. Since $p_n \to u$ in $L^2(\nu_T)$ as well, we may conclude that $f = u \in \mathcal{H}_T^2(G)$. $\square$

Corollary 8.4 *Suppose that $f : G \to \mathbb{C}$ is a continuous function such that $|f| \leqslant Ce^{\varepsilon \rho^2/(2T)}$ and $f|_{G_{CM}}$ is holomorphic, then $f \in \mathcal{H}_T^2(G)$ and $S_T f = f$.*

Proof This is a consequence of Lemma 8.2 and Corollary 8.3. $\square$

9 The holomorphic chaos expansion

This section is devoted to the proof of the holomorphic chaos expansion Theorem 1.9 (or equivalently Theorem 9.10). Before going to the proof we will develop the machinery necessary in order to properly define the right side of Eq. (1.8).

9.1 Generalities about multiple Itô integrals

Let $(\mathbb{H}, \mathbb{W})$ be a complex abstract Wiener space. Analogous to the notation used in Sect. 6.1, we will denote the norm on $\mathbb{H}^{*\otimes n}$ by $\|\cdot\|_n$.

Notation 9.1 *For $\alpha \in \mathbb{H}^{*\otimes n}$ and $P \in \mathrm{Proj}(\mathbb{W})$, let $\alpha_P := \alpha \circ P^{\otimes n} \in \mathbb{H}^{*\otimes n}$.*

Proposition 9.2 *Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{H}^{*\otimes n}$ and $P_k \in \mathrm{Proj}(\mathbb{W})$ with $P_k|_{\mathbb{H}} \uparrow I|_{\mathbb{H}}$. Then $\alpha_{P_k} \to \alpha$ in $\mathbb{H}^{*\otimes n}$.*

Proof Let $\Lambda := \cup_k \Lambda_k$ be an orthonormal basis for $\mathbb{H}$ where Λ_k is chosen to be an orthonormal basis for $\mathrm{Ran}(P_k)$ such that $\Lambda_k \subset \Lambda_{k+1}$ for all k. Since $P_k u = u$ or $P_k u = 0$ for all $u \in \Lambda$ and $k \in \mathbb{N}$, we have

$$|\langle \alpha, u_1 \otimes \cdots \otimes u_n - P_k u_1 \otimes \cdots \otimes P_k u_n \rangle|^2 \leq |\langle \alpha, u_1 \otimes \cdots \otimes u_n \rangle|^2$$

where

$$\sum_{u_1,\ldots,u_n \in \Lambda} |\langle \alpha, u_1 \otimes \cdots \otimes u_n \rangle|^2 = \|\alpha\|_n^2 < \infty.$$

An application of the dominated convergence theorem then implies,

$$\lim_{k \to \infty} \|\alpha - \alpha_{P_k}\|_n^2 = \lim_{k \to \infty} \sum_{u_1,\ldots,u_n \in \Lambda} |\langle \alpha, u_1 \otimes \cdots \otimes u_n - P_k u_1 \otimes \cdots \otimes P_k u_n \rangle|^2$$

$$= \sum_{u_1,\ldots,u_n \in \Lambda} \lim_{k \to \infty} |\langle \alpha, u_1 \otimes \cdots \otimes u_n - P_k u_1 \otimes \cdots \otimes P_k u_n \rangle|^2 = 0.$$

$\square$

Lemma 9.3 *Suppose that $\{b(t)\}_{t \geq 0}$ is a $\mathbb{W}$–valued Brownian motion normalized by*

$$\mathbb{E}\left[\ell_1(b(t)) \ell_2(b(s))\right] = \frac{1}{2} s \wedge t \, (\ell_1, \ell_2)_{\mathbb{H}_{\mathrm{Re}}^*} \quad \text{for all } \ell_1, \ell_2 \in \mathbb{W}_{\mathrm{Re}}^*. \tag{9.1}$$

If $P \in \mathrm{Proj}(\mathbb{W})$, $T > 0$, and $\{f_s\}_{s \geq 0}$ is a $(P\mathbb{H})^*$–valued continuous adapted process, such that $\mathbb{E} \int_0^T |f_s|_{(P\mathbb{H})^*}^2 ds < \infty$, then

$$\mathbb{E}\left| \int_0^T \langle f_s, d(Pb)(s) \rangle \right|^2 = \int_0^T \mathbb{E} |f_s|_{(P\mathbb{H})^*}^2 \, ds. \tag{9.2}$$

Proof Let $\{e_j\}_{j=1}^d$ be an orthonormal basis for $P\mathbb{H}$ and write

$$Pb(s) = \sum_{j=1}^d \left[X_j(s)e_j + Y_j(s)ie_j \right]$$

where $X_j(s) = \mathrm{Re}(Pb(s), e_j)$ and $Y_j(s) = \mathrm{Im}(Pb(s), e_j)$. From the normalization in Eq. (9.1) it follows that $\{\sqrt{2}X_j, \sqrt{2}Y_j\}_{j=1}^d$ is a sequence of independent standard Brownian motions, and therefore the quadratic covariations of these processes are given by:

$$dX_j dY_k = 0 \quad \text{and} \quad dX_j dX_k = dY_j dY_k = \frac{1}{2}\delta_{jk}dt \quad \text{for all } j, k = 1, \ldots, d. \quad (9.3)$$

Using Eq. (9.3) along with the identity,

$$\langle f_s, d(Pb)(s) \rangle = \sum_{j=1}^d \left[\langle f_s, e_j \rangle dX_j(s) + \langle f_s, ie_j \rangle dY_j(s) \right], \qquad (9.4)$$

it follows by the basic isometry property of the stochastic integral that

$$\mathbb{E} \left| \int_0^T \langle f_s, d(Pb)(s) \rangle \right|^2 = \frac{1}{2} \sum_{j=1}^d \mathbb{E} \left[\int_0^T |\langle f_s, e_j \rangle|^2 \, ds + \int_0^T |\langle f_s, ie_j \rangle|^2 \, ds \right]$$

$$= \mathbb{E} \int_0^T \sum_{j=1}^d |\langle f_s, e_j \rangle|^2 \, ds = \int_0^T \mathbb{E} |f_s|^2_{(P\mathbb{H})^*} \, ds.$$

$\square$

Definition 9.4 For $P \in \mathrm{Proj}(\mathbb{W})$, $n \in \mathbb{N}$, and $T > 0$, let

$$M_n^P(T) := \int_{0 \le s_1 \le s_2 \le \cdots \le s_n \le T} dPb(s_1) \otimes dPb(s_2) \otimes \cdots \otimes dPb(s_n).$$

Alternatively put, $M_0^P(T) \equiv 1$ and $M_n^P(t) \in (P\mathbb{H})^{\otimes n}$ is defined inductively by

$$M_n^P(t) := \int_0^t M_{n-1}^P(s) \otimes dPb(s) \quad \text{for all } t \ge 0. \qquad (9.5)$$

Corollary 9.5 *Suppose that $T > 0$, $\alpha \in \mathbb{H}^{*\otimes n}$, and $P \in \mathrm{Proj}(\mathbb{W})$, then $\langle \alpha, M_n^P(T) \rangle$ is a square integrable random variable and*

$$\mathbb{E} \left| \left\langle \alpha, M_n^P(T) \right\rangle \right|^2 = \frac{T^n}{n!} \|\alpha_P\|_n^2.$$

Proof The proof is easily carried out by induction with the case $n = 1$ following directly from Lemma 9.3. Similarly from Lemma 9.3, Eq. (9.5), and induction we have

$$
\begin{aligned}
\mathbb{E} \left| \left\langle \alpha, M_n^P(T) \right\rangle \right|^2 &= \mathbb{E} \left| \int_0^T \left\langle \alpha, M_{n-1}^P(s) \otimes dPb(s) \right\rangle \right|^2 \\
&= \int_0^T \sum_{j=1}^d \mathbb{E} \left| \left\langle \alpha, M_{n-1}^P(s) \otimes e_j \right\rangle \right|^2 ds \\
&= \sum_{j=1}^d \int_0^T \frac{s^{n-1}}{(n-1)!} \|\langle \alpha, (\cdot) \otimes e_j \rangle\|_{n-1}^2 ds = \frac{T^n}{n!} \|\alpha_P\|_n^2.
\end{aligned}
$$

$\square$

Notation 9.6 *We now fix $T > 0$ and for $P \in \mathrm{Proj}(\mathbb{W})$, let $\tilde{\alpha}_P = \langle \alpha, M_n^P(T) \rangle$, i.e.*

$$\tilde{\alpha}_P = \left\langle \alpha, \int_{0 \le s_1 \le s_2 \le \cdots \le s_n \le T} dPb(s_1) \otimes dPb(s_2) \otimes \cdots \otimes dPb(s_n) \right\rangle.$$

Lemma 9.7 *If $P, Q \in \mathrm{Proj}(\mathbb{W})$, then*

$$\left\| \tilde{\alpha}_P - \tilde{\alpha}_Q \right\|_{L^2}^2 := \mathbb{E} \left| \tilde{\alpha}_P - \tilde{\alpha}_Q \right|^2 = \frac{T^n}{n!} \|\alpha_P - \alpha_Q\|_n^2.$$

Proof Let $R \in \mathrm{Proj}(\mathbb{W})$ be the orthogonal projection onto $\mathrm{Ran}(P) + \mathrm{Ran}(Q)$. We then have $(\alpha_P)_R = \alpha_P$ and $(\alpha_Q)_R = \alpha_Q$ and therefore, by Corollary 9.5,

$$
\begin{aligned}
\mathbb{E} \left| \tilde{\alpha}_P - \tilde{\alpha}_Q \right|^2 &= \mathbb{E} \left| (\alpha_P)_R^{\tilde{}} - (\alpha_Q)_R^{\tilde{}} \right|^2 = \mathbb{E} \left| (\alpha_P - \alpha_Q)_R^{\tilde{}} \right|^2 \\
&= \frac{T^n}{n!} \left\| (\alpha_P - \alpha_Q)_R \right\|_n^2 = \frac{T^n}{n!} \|\alpha_P - \alpha_Q\|_n^2.
\end{aligned}
$$

$\square$

Proposition 9.8 *Let $\alpha \in \mathbb{H}^{*\otimes n}$ and $P_k \in \mathrm{Proj}(\mathbb{W})$ with $P_k|_{\mathbb{H}} \uparrow I|_{\mathbb{H}}$, then $\{\tilde{\alpha}_{P_k}\}_{k=1}^\infty$ is an L^2–convergent series. We denote the limit by $\tilde{\alpha}$. This limit is independent of the choice of orthogonal projections used in constructing $\tilde{\alpha}$.*

Proof For $k, l \in \mathbb{N}$, by Lemma 9.7,

$$\left\| \tilde{\alpha}_{P_l} - \tilde{\alpha}_{P_k} \right\|_{L^2} = \left\| \alpha_{P_l} - \alpha_{P_k} \right\|_n \to 0 \text{ as } l, k \to \infty,$$

because, as we have already seen, $\alpha_{P_l} \to \alpha$ in $\mathbb{H}^{*\otimes n}$. Therefore $\tilde{\alpha} := L^2 - \lim_{k \to \infty} \tilde{\alpha}_{P_k}$ exists.

Now suppose that $Q_l \in \text{Proj}(\mathbb{W})$ also increases to $I|_{\mathbb{H}}$. By Lemma 9.7 and the fact that both α_{P_l} and α_{Q_l} converge to α in $\mathbb{H}^{*\otimes n}$, we have

$$\left\| \tilde{\alpha}_{P_l} - \tilde{\alpha}_{Q_l} \right\|_{L^2} = \left\| \alpha_{P_l} - \alpha_{Q_l} \right\|_{\mathbb{H}^{*\otimes n}} \to 0 \text{ as } l \to \infty.$$

$\square$

By polarization of the identity, $\|\tilde{\alpha}\|_{L^2}^2 = T^n \|\alpha\|_n^2 / n!$, it follows that

$$\left(\tilde{\alpha}, \tilde{\beta} \right)_{L^2} = \frac{T^n}{n!} (\alpha, \beta)_{\mathbb{H}^{*\otimes n}} \text{ for all } \alpha, \beta \in \mathbb{H}^{*\otimes n}.$$

Moreover, if $\alpha \in \mathbb{H}^{*\otimes n}$ and $\beta \in \mathbb{H}^{*\otimes m}$ with $m \neq n$, by the orthogonality of the finite dimensional approximations, $\tilde{\alpha}_{P_l}$ and $\tilde{\beta}_{P_l}$, we have that $(\tilde{\alpha}, \tilde{\beta})_{L^2} = 0$.

Corollary 9.9 (Itô's isometry) *Suppose that* $\alpha = \{\alpha_n\}_{n=0}^{\infty} \in \oplus_{n=0}^{\infty} \frac{T^n}{n!} \mathbb{H}^{*\otimes n}$, *i.e.* $\alpha_n \in \mathbb{H}^{*\otimes n}$ *for all n such that*

$$\|\alpha\|_T^2 = \sum_{n=0}^{\infty} \frac{T^n}{n!} \|\alpha_n\|_n^2 < \infty.$$

Then $\tilde{\alpha} := \sum_{n=0}^{\infty} \tilde{\alpha}_n$ *is* $L^2(\mathbf{P})$*–convergent and the map,*

$$\bigoplus_{n=0}^{\infty} \frac{T^n}{n!} \mathbb{H}^{*\otimes n} \ni \alpha \mapsto \tilde{\alpha} \in L^2(\mathbf{P}),$$

is an isometry, where $\mathbf{P}$ *is the probability measure used in describing the law of* $\{b(t)\}_{t \geq 0}$.

9.2 The stochastic Taylor map

Let $b(t) = (B(t), B_0(t)) \in \mathfrak{g}$ and $g(t) \in G$ be the Brownian motions introduced at the start of Sect. 4. We are going to use the results of the previous subsection with $\mathbb{H} = \mathfrak{g}_{CM}, \mathbb{W} = \mathfrak{g}$, and $b(t) = (B(t), B_0(t))$. Let $f \in \mathcal{H}_T^2(G)$ and $\alpha_f := T_T S_T f \in J_T^0(\mathfrak{g}_{CM})$. The following theorem is a (precise) restatement of Theorem 1.9.

Theorem 9.10 *For any* $f \in \mathcal{H}_T^2(G)$

$$f(g(T)) = \tilde{\alpha}_f, \tag{9.6}$$

where $\tilde{\alpha}_f$ was introduced in Corollary 9.9. (The right hand side of Eq. (1.8) is to be interpreted as $\tilde{\alpha}_f$.)

Proof First suppose that f is a holomorphic polynomial and $P \in \text{Proj}(W)$ so that $\pi_P \in \text{Proj}(\mathfrak{g})$. Then by Itô's formula,

$$f(g_P(T)) = f(\mathbf{e}) + \int_0^T \langle Df(g_P(t)), d\pi_P b(t) \rangle.$$

Iterating this equation as in the proof of [3, Proposition 5.2], if $N \in \mathbb{N}$ is sufficiently large, then

$$f(g_P(T)) = f(\mathbf{e}) + \sum_{n=1}^N \int_{0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq T} \langle D^n f(\mathbf{e}), d\pi_P b(s_1) \otimes \cdots \otimes d\pi_P b(s_n) \rangle$$

$$= f(\mathbf{e}) + \sum_{n=1}^N \left[D^n f(\mathbf{e}) \right]^{\tilde{}}_{\pi_P}.$$

We now replace P by $P_k \in \text{Proj}(W)$ with $P_k \uparrow I$ in this identity. Using Propositions 4.12 and 9.8, we may now pass to the limit as $k \to \infty$ in order to conclude,

$$f(g(T)) = f(\mathbf{e}) + \sum_{n=1}^N \left[D^n f(\mathbf{e}) \right]^{\tilde{}} = \tilde{\alpha}_f. \tag{9.7}$$

Now suppose that $f \in \mathcal{H}_T^2(G)$. By Theorem 7.1 we can find a sequence of holomorphic polynomials $\{f_n\}_{n=1}^\infty \subset \mathcal{P}$ such that

$$\mathbb{E}|f(g(T)) - f_n(g(T))|^2 = \|f - f_n\|_{L^2(\nu_T)}^2 \to 0 \text{ as } n \to \infty.$$

The isometry property of the Taylor and skeleton maps (Theorem 6.10 and Corollary 8.3), shows that $\alpha_{f_n} \to \alpha_f$ in J_T^0 and therefore by Corollary 9.9 $\tilde{\alpha}_{f_n} \to \tilde{\alpha}_f$ as $n \to \infty$. Hence we may pass to the limit in Eq. (9.7) applied to the sequence $f_n(g(T)) = \tilde{\alpha}_{f_n}$, to complete the proof of Eq. (9.6). $\qquad\square$

10 Future directions and questions

In this last section, we wish to speculate on a number of ways that the results in this paper might be generalized.

(1) It should be possible to remove the restriction on **C** being finite dimensional, i.e. we expect much of what have done in this paper to go through when **C** is replaced by a separable Hilbert space. In doing so one would have to modify the finite dimensional approximations used in our construction to truncate **C** as well.

(2)   We also expect that the level of non-commutativity of G may be increased. To be more precise, under suitable hypothesis it should be possible to handle more general graded nilpotent Lie groups.

(3)   Open questions:

(a)   as we noted in Remark 5.13 we do not know if $\mathcal{A}_T^p = \mathcal{H}_T^p(G)$. It might be easier to try to answer this question for $p = 2$.

(b)   give an intrinsic characterization of $\mathcal{H}_T^2(G)$ as in Shigekawa [25] in terms of functions in $L^2(\nu_T)$ solving a weak form of the Cauchy–Riemann equations.

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