DISCRETIZATION OF C*-ALGEBRAS

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Abstract. We investigate how a C*-algebra could consist of functions on a noncommutative set: a *discretization* of a C*-algebra A is a $*$-homomorphism $A \to M$ that factors through the canonical inclusion $C(X) \subseteq \ell^\infty(X)$ when restricted to a commutative C*-subalgebra. Any C*-algebra admits an injective but nonfunctorial discretization, as well as a possibly noninjective functorial discretization, where M is a C*-algebra. Any subhomogenous C*-algebra admits an injective functorial discretization, where M is a W*-algebra. However, any functorial discretization, where M is an AW*-algebra, must trivialize $A = B(H)$ for any infinite-dimensional Hilbert space H.

1. Introduction

In operator algebra it is common practice to regard C*-algebras as noncommutative analogues of topological spaces, and to regard W*-algebras as noncommutative analogues of measurable spaces. What would it mean to make precise how a C*-algebra is a 'noncommutative ring of continuous functions'? Several natural approaches to this question cannot faithfully represent examples as simple as matrix algebras $\mathbb{M}_n(\mathbb{C})$ [35, 7, 36, 4]. Such obstructions suggest more carefully considering what 'noncommutative sets' in the foundations of noncommutative geometry should be, before attempting to topologize them.

This article explores the idea of embedding the C*-algebra in an appropriate noncommutative algebra of 'bounded functions on the noncommutative set underlying its spectrum', just like any topological space embeds in a discrete one. More precisely, consider the case of a commutative C*-algebra A. A representation of A as operating on a Hilbert space H is equivalent to a $*$-homomorphism $A \to B(H)$. Similarly, representing A as continuous complex-valued functions on a compact Hausdorff space X can equivalently be viewed as a $*$-homomorphism $A \to \ell^\infty(X)$ to the algebra of bounded functions on the set X. More generally, representating A as (discrete) functions on a set X can equivalently be viewed as a $*$-homomorphism to the algebra $\mathbb{C}^X$ of all functions on X.

In the spirit of noncommutative geometry, we thus seek a category $\mathbf{A}$ of $*$-algebras to play the role of the dual to the category of 'noncommutative sets'. This category should contain the commutative algebras $\ell^\infty(X)$ (or $\mathbb{C}^X$) as a full subcategory, dual to the category of sets. In keeping with the programme of taking commutative subalgebras seriously [18, 35, 7, 36, 6, 19, 16, 20, 17], we posit that a representation of a C*-algebra as an algebra of functions on a noncommutative set should be an algebra homomorphism $\phi\colon A \to M$ for some M in $\mathbf{A}$, whose restriction

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to every commutative C*-subalgebra $C \simeq C(X)$ of A factors through the natural inclusion $C(X) \subseteq \ell^\infty(X)$ via a morphism $\ell^\infty(X) \to M$ in $\mathbf{A}$. We call such a map ϕ a *discretization* of A.

$$\begin{array}{ccc} A & \xrightarrow{\quad\phi\quad} & M \\ \uparrow & & \uparrow \\ C(X) & \hookrightarrow & \ell^\infty(X) \end{array}$$

Section 2 makes this definition precise, relative to a parameterizing category $\mathbf{A}$ that can then remain unspecified. This approach to terminology gives most flexibility in investigating the open problem of finding a suitable noncommutative extension of the functor $C(X) \mapsto \ell^\infty(X)$ before us. We show that every C*-algebra admits a discretization into a C*-algebra M that is injective but nonfunctorial. We also show that there is a universal candidate for a functorial discretization into the category of C*-algebras, but it remains open whether this functorial discretization is injective for every C*-algebra.

In Section 3 we show that a sizeable class of C*-algebras that are 'close to being commutative' does indeed have injective functorial discretizations, namely the *subhomogeneous* algebras: subalgebras of $\mathbb{M}_n(C)$ for some commutative C*-algebra C. The discretization is achieved by profinite completion, suggesting that profinite completion for subhomogeneous algebras is a noncommutative substitute for the 'underlying set functor' that sends a compact Hausdorff space to its underlying discrete space.

On the other hand, in Section 4 we show that no subcategory of W*-algebras, or even AW*-algebras, can be dual to noncommutative sets in the sense of injectively discretizing every C*-algebra. In particular, every functor from C*-algebras to AW*-algebras taking each C*-algebra to a discretization must trivialize $A = B(H)$ for any infinite-dimensional Hilbert space H. A number of related examples and obstructions are discussed, including separable algebras A for which the same trivialization occurs. Viewing $*$-homomorphisms out of a C*-algebra as representing it by functions on a noncommutative set dates back at least to Akemann [1] and Giles and Kummer [14], who took the representation to be the canonical homomorphism $A \to A^{**}$ into the bidual. They noted [2, p. 10] that their theory was not functorial. Our obstructions amplify this observation by suggesting that W*-algebras indeed cannot play the role of 'noncommutative $\ell^\infty(X)$-algebras' for C*-algebras as large as $B(H)$.

The article concludes with a discussion in Section 5 of the implications of our obstructions, with an eye toward future work on the problem of finding injective functorial discretizations of all C*-algebras.

2. Discretization

We assume throughout this article that all rings, algebras, and subalgebras are unital, and that all homomorphisms preserve units. Write $\mathrm{Spec}(C)$ for the Gelfand spectrum of a commutative C*-algebra C. Write **Cstar** for the category of C*-algebras with $*$-homomorphisms and **Wstar** for the subcategory of W*-algebras with normal $*$-homomorphisms.

Recall that a *pro-C*-algebra* [31, 32] is a topological $*$-algebra that is a directed (or "inverse") limit in the category of topological $*$-algebras of a system of

C*-algebras. Pro-C*-algebras with continuous $*$-homomorphisms form a category **proCstar**. The algebra $\mathbb{C}^X$ of all complex-valued functions on a set X equipped with its topology of pointwise convergence is a pro-C*-algebra, as it is the directed limit of the finite-dimensional C*-algebras $\mathbb{C}^S$ for all finite subsets $S \subseteq X$.

Lemma 2.1. *The functors $X \mapsto \ell^\infty(X)$ and $X \mapsto \mathbb{C}^X$ are contravariant equivalences between the category of sets and full subcategories of* **Wstar** *and* **proCstar**.

Proof. The proof for the functor ℓ^∞ can be found in [40, Section 6.1]. We sketch an argument that covers both functors.

It is rather clear that each of the above assignments forms a contravariant functor into the specified category. It only remains to show that each is naturally bijective on Hom-sets. Fix $x \in X$. Let $\mathrm{ev}_x \colon \mathbb{C}^X \to \mathbb{C}$ denote the continuous $*$-homomorphism given by evaluation at x, whose restriction to $\ell^\infty(X)$ is normal. The maps $X \to \mathbf{proCstar}(\mathbb{C}^X, \mathbb{C})$ and $X \to \mathbf{Wstar}(\ell^\infty(X), \mathbb{C})$, given in each case by $x \mapsto \mathrm{ev}_x$, are both bijections; this follows by verifying that the kernel of either kind of morphism $\mathbb{C}^X \to \mathbb{C}$ or $\ell^\infty(X) \to \mathbb{C}$ is generated as an ideal by a characteristic function χ_S, which entails that $S = X \setminus \{x\}$ for some $x \in X$.

Now the argument that the functors in question are bijective on Hom-sets is purely formal, and can be proved by essentially the same argument as the one given in the algebraic context in [21, Theorem 4.7]. □

The previous lemma leads naturally to the following notion, in keeping with the programme of taking commutative subalgebras seriously. As mentioned in the introduction, the definition is made relative to a category **A** of complex algebras that is a candidate to contain 'algebras of bounded functions on noncommutative sets.'

Definition 2.2. Let **A** denote a category of $\mathbb{C}$-algebras containing the algebras $\ell^\infty(X)$ for any set X with their normal *-homomorphisms. Given a C*-algebra A, a *(bounded)* **A**-*discretization* is a homomorphism $\phi \colon A \to M$ whose restriction to each commutative C*-subalgebra C of A factors through the natural inclusion $C \to \ell^\infty(\mathrm{Spec}(C))$ via a morphism $\phi_C \colon \ell^\infty(\mathrm{Spec}(C)) \dashrightarrow M$ in **A**.

$$\begin{array}{ccc} A & \xrightarrow{\quad\phi\quad} & M \\ \uparrow & & \uparrow{\scriptstyle \phi_C} \\ C & \hookrightarrow & \ell^\infty(\mathrm{Spec}(C)) \end{array}$$

We call a discretization ϕ *faithful* when it is injective and all ϕ_C can be chosen injective. We call ϕ *compatible* if the morphisms ϕ_C can be chosen such that ϕ_C factors through ϕ_D via the induced morphism $\ell^\infty(\mathrm{Spec}(C)) \to \ell^\infty(\mathrm{Spec}(D))$ for commutative C*-subalgebras $C \subseteq D \subseteq A$.

When **A** is **Cstar** or **Wstar** above, we will speak of C*- or W*-discretizations instead of **A**-discretizations.

Proposition 2.3. *Every C*-algebra has a faithful C*-discretization.*

Proof. Write L for the functor $C \mapsto \ell^\infty(\mathrm{Spec}(C))$. Given a finite family $S = \{C_1, \ldots, C_n\}$ of commutative C*-subalgebras of A, write A_S for the colimit in **Cstar** of the diagram whose objects are A, the C_i, and the $L(C_i)$, along with the inclusions of each C_i into both A and $L(C_i)$. This can be constructed up to isomorphism as an iterated amalgamated free product:

$$A_S \simeq (\cdots((A *_{C_1} L(C_1)) *_{C_2} L(C_2))\cdots) *_{C_n} L(C_n).$$

Thus the natural maps from A and the $L(C_i)$ into A_S are all embeddings; see [8, Theorem 3.1] or [29, Theorem 4.2].

The finite families S of commutative C*-subalgebras of A form a directed set under inclusion. Consider the directed colimit $M = \operatorname{colim}_S A_S$. By construction the mediating map $\phi\colon A \to M$ is a C*-discretization. For finite subfamilies $S \subseteq T$ of commutative C*-subalgebras of A, the induced map $A_S \to A_T$ is injective because A_T is formed from A_S by iterated pushouts. Thus the natural maps $A_S \to M$ are injective [37, Theorem 1], from which it follows that ϕ is faithful. □

The discretization $\phi\colon A \to M$ constructed in the proof above is not compatible: for commutative C*-subalgebras $C \subsetneq D \subseteq A$, the algebra M is obtained by gluing together distinct copies of $L(C)$ and $L(D)$ without regard to the natural inclusion $L(C) \to L(D)$. In Theorem 2.5 below we modify the construction to ensure compatibility, with the caveat that we no longer know that the discretization is even injective. This universally constructed C*-discretization will in fact satisfy the following natural condition.

Definition 2.4. Let $\mathbf{A}$ be a category as in Definition 2.2. A *functorial* $\mathbf{A}$*-discretization* is a functor $F\colon \mathbf{Cstar} \to \mathbf{A}$ together with natural homomorphisms $\eta_A\colon A \to F(A)$ such that η_C for each commutative C*-algebra C turns into the natural inclusion $C \to \ell^\infty(\operatorname{Spec}(C))$ by a natural isomorphism $F(C) \simeq \ell^\infty(\operatorname{Spec}(C))$.

A functorial discretization automatically gives compatible discretizations $A \to F(A)$ for every C*-algebra A: writing $i_C\colon C \to A$ for the inclusion of a commutative C*-subalgebra gives the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\quad\eta_A\quad} & F(A) \\ {\scriptstyle i_C}\uparrow & & \uparrow{\scriptstyle F(i_C)} \\ C & \xrightarrow{\quad\eta_C\quad} & F(C) \simeq \ell^\infty(\operatorname{Spec}(C)) \end{array}$$

Compatibility follows by applying F to successive inclusions $C \subseteq D \subseteq A$.

Write $\mathbf{cCstar}$ for the full subcategory of $\mathbf{Cstar}$ of commutative C*-algebras. Write $\mathcal{C}(A)$ for the small subcategory of $\mathbf{cCstar}$ consisting of the commutative C*-subalgebras of a C*-algebra A with their inclusion morphisms; we also view this as a partially ordered set.

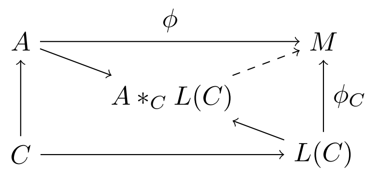
Theorem 2.5. *The functor* $F\colon \mathbf{Cstar} \to \mathbf{Cstar}$ *given by*

$$F(A) = \operatorname{colim}_{C\in\mathcal{C}(A)} A *_C \ell^\infty(\operatorname{Spec}(C))$$

equipped with the naturally induced $*$-homomorphisms $\eta_A\colon A \to F(A)$ is a functorial C-discretization. For each C*-algebra A, the C*-discretization $A \to F(A)$ is universal among all compatible C*-discretizations of A. Thus F is universal among all functorial C*-discretizations.*

Proof. We follow the idea of [35, Theorem 2.15] but with arrows reversed. Write $L = \ell^\infty \circ \operatorname{Spec}\colon \mathbf{cCstar} \to \mathbf{Cstar}$. The assignment $A \mapsto \mathcal{C}(A)$ is a functor to the category of small categories. Given a C*-algebra A, the assignment $C \mapsto A *_C L(C)$ is functorial $\mathcal{C}(A) \to \mathbf{Cstar}$. So $F(A) = \operatorname{colim}_{C \in \mathcal{C}(A)} A *_C L(C)$ defines a functor $F\colon \mathbf{Cstar} \to \mathbf{Cstar}$. Moreover, the induced $*$-homomorphisms $\eta_A\colon A \to F(A)$ are natural by construction. Finally, if A is commutative so that $A \in \mathcal{C}(A)$, then one naturally has an isomorphism $F(A) \simeq \ell^\infty(\operatorname{Spec}(A))$ that turns η_A into the inclusion $A \to \ell^\infty(\operatorname{Spec}(A))$. Thus F is a functorial C*-discretization.

To verify universality of η_A, fix a compatible C*-discretization $\phi\colon A \to M$. Each $C \in \mathcal{C}(A)$ then makes the following outer square commute.



The morphisms ϕ and ϕ_C factor uniquely through the pushout $A *_C L(C)$. Compatibility of the ϕ_C means that these uniquely determined morphisms form a cocone from the diagram of the $A *_C L(C)$ to M. Thus we obtain a $*$-homomorphism $F(A) = \operatorname{colim}_{C \in \mathcal{C}(A)} A *_C L(C) \to M$ through which ϕ factors uniquely, as desired.

Finally, if (F', η') is any functorial C*-discretization, then by the local universality of the previous paragraph the natural morphisms $\eta'_A\colon A \to F'(A)$ factor uniquely through $\eta_A\colon A \to F(A)$, from which it readily follows that F' factors through a unique natural transformation $F \Rightarrow F'$ whose composite with η is η'. □

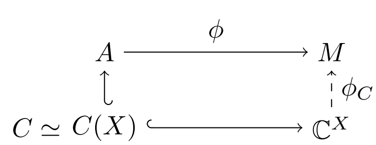
Whereas the 'incompatible' discretization of Proposition 2.3 is faithful, it is not clear whether the natural C*-discretizations $A \to F(A)$ of the last theorem are faithful or even injective. Abstract nonsense alone does not answer this question.

Question 2.6. *Is the universal functorial C*-discretization $\eta_A\colon A \to F(A)$ of Theorem 2.5 injective or faithful for every C*-algebra A? Equivalently, does every C*-algebra have an injective or faithful compatible C*-discretization?*

Remark 2.7. The definitions and results above carefully used the Gelfand spectrum $\operatorname{Spec}(C)$ of a commutative C*-algebra C. Henceforth we loosen notation, and write $C = C(X)$ for an arbitrary commutative C*-algebra, and $C \simeq C(X)$ for an arbitrary commutative C*-subalgebra of a C*-algebra A.

Recall from Lemma 2.1 that sets may also be encoded algebraically through the algebra of discrete (possibly unbounded) functions as $X \mapsto \mathbb{C}^X$. The rest of the paper will also discuss 'unbounded' discretizations.

Definition 2.8. Let $\mathbf{A}$ denote a category of $\mathbb{C}$-algebras containing the algebras $\mathbb{C}^X$ for any set X with the $*$-homomorphisms that are continuous with respect to the topology of pointwise convergence. Given a C*-algebra A, an *unbounded* $\mathbf{A}$-*discretization* is a homomorphism $\phi\colon A \to M$ whose restriction to each commutative C*-subalgebra $C \simeq C(X)$ of A factors through the inclusion $C(X) \to \mathbb{C}^X$ via a morphism $\phi_C\colon \mathbb{C}^X \dashrightarrow M$ in $\mathbf{A}$.



Define *injective*, *faithful*, and *functorial* unbounded discretizations analogous to the bounded case. For $\mathbf{A} = \mathbf{proCstar}$ we refer to *unbounded pro-C*-discretizations*.

3. Functorial discretizations through profinite completion

For a compact Hausdorff space X, the natural inclusion $C(X) \to \ell^\infty(X)$ is a W*-discretization of the corresponding commutative C*-algebra. Also, if A is a finite-dimensional C*-algebra, then the identity map $A \to A$ is a W*-discretization. This section provides a common generalization of these two examples: Theorems 3.3 and 3.5 below show that the profinite completion of a C*-algebra is a functorial discretization that is faithful for a large class of algebras.

For a C*-algebra A, let $\mathcal{F}(A)$ denote the family of closed ideals I of A for which A/I is finite-dimensional. Then $\mathcal{F}(A)$ is closed under finite intersections, as is readily verified by embedding $A/(I \cap J) \to A/I \oplus A/J$ for ideals $I, J \in \mathcal{F}(A)$. Thus the finite-dimensional C*-algebras A/I for $I \in \mathcal{F}(A)$ form an inversely directed system. We may take the directed limit of this system either in the category **Cstar** to obtain a C*-algebra, or in the category of topological algebras to obtain a pro-C*-algebra. We denote these two directed limits by

$$P_b(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \mathbf{Cstar},$$
$$P_u(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \mathbf{proCstar}.$$

Given a $*$-homomorphism $f \colon A \to B$ and $J \in \mathcal{F}(B)$, the induced embedding $A/f^{-1}(J) \hookrightarrow B/J$ ensures that $f^{-1}(J) \in \mathcal{F}(A)$. Universality provides a composite $*$-homomorphism

$$P_b(A) = \lim_{I \in \mathcal{F}(A)} A/I \ \to \ \lim_{J \in \mathcal{F}(B)} A/f^{-1}(J) \ \to \ \lim_{J \in \mathcal{F}(B)} B/J = P_b(B)$$

making the assignments P_b and P_u functorial.

Notice that the diagram over which the limit $P_b(A)$ is computed consists of W*-algebras with normal $*$-homomorphisms. The subcategory **Wstar** of **Cstar** is closed under limits since the forgetful functor $\mathbf{Wstar} \to \mathbf{Cstar}$ is right adjoint to the universal enveloping W*-algebra functor [12]. Thus $P_b(A)$ is a W*-algebra, and for $f \colon A \to B$ in **Cstar** the induced morphism $P_b(f) \colon P_b(A) \to P_b(B)$ is a normal $*$-homomorphism. Thus P_b is a functor $\mathbf{Cstar} \to \mathbf{Wstar}$.

Each of the two functors P_b and P_u is a kind of profinite completion [13].

Definition 3.1. We call $P_b \colon \mathbf{Cstar} \to \mathbf{Wstar}$ the *bounded profinite completion*, and $P_u \colon \mathbf{Cstar} \to \mathbf{proCstar}$ the *unbounded profinite completion*.

Let $b(P) \subseteq P$ denote the set of *bounded* elements of a pro-C*-algebra P: those elements whose spectrum forms a bounded subset of $\mathbb{C}$. This is a C*-algebra that lies densely in P [31, Proposition 1.11].

Proposition 3.2. *If A is a C*-algebra, then $P_b(A) \simeq b(P_u(A))$: the W*-algebra $P_b(A)$ is $*$-isomorphic to the algebra of bounded elements of the pro-C*-algebra $P_u(A)$.*

Proof. Suppose that a C*-algebra B forms a cone over the diagram of finite-dimensional algebras A/I for $I \in \mathcal{F}(A)$. Then B also forms a cone over this diagram in the category **proCstar**, and this cone factors uniquely through a morphism $B \to P_u(A)$. But the image of this morphism lands in the C*-algebra $b(P_u(A))$ [31, Corollary 1.13]. Thus $b(P_u(A))$ satisfies the universal property of

$\lim_{I\in\mathcal{F}(A)} A/I$ computed in **Cstar**. It follows that the map $P_b(A) \to P_u(A)$ induced by the universal property of the latter is an isomorphism onto $b(P_u(A))$. $\square$

Henceforth we identify $P_b(A)$ with the dense subalgebra $b(P_u(A)) \subseteq P_u(A)$. Invoking the universal property of limits once again, for each C*-algebra A there is a $*$-homomorphism $\eta_A \colon A \to P_b(A) \subseteq P_u(A)$ that is natural in A. This map makes P_b and P_u into functorial discretizations.

Theorem 3.3. *Bounded profinite completion is a functorial W*-discretization. Unbounded profinite completion is an unbounded functorial pro-C*-discretization.*

Proof. For a commutative C*-algebra $C = C(X)$, each $I \in \mathcal{F}(C)$ is of the form $I = I_S = \{f \in C \mid f(S) = 0\}$ for some finite subset $S \subseteq X$. The surjection $C \twoheadrightarrow C/I \simeq C(S)$ is Gelfand dual to the inclusion $S \hookrightarrow X$. Thus

$$\begin{aligned} P_b(C(X)) &= \lim_{S\subseteq X} C(S) \simeq \ell^\infty(X), \\ P_u(C(X)) &= \lim_{S\subseteq X} C(S) \simeq \mathbb{C}^X, \end{aligned}$$

and under these isomorphisms the natural map $\eta_C \colon C \to P_b(C) \subseteq P_u(C)$ corresponds to the natural inclusion $C(X) \hookrightarrow \ell^\infty(X) \subseteq \mathbb{C}^X$.

It remains to verify that these functors behave as expected on morphisms. Fix a $*$-homomorphism $f \colon B = C(Y) \to C = C(X)$, which is Gelfand dual to a continuous function $\widehat{f} \colon X \to Y$. For any finite set $S \subseteq X$, the restriction of $\widehat{f}$ to $S \to \widehat{f}(S)$ is Gelfand dual to $C(\widehat{f}(S)) \simeq B/f^{-1}(I_S) \to C/I_S \simeq C(S)$. Taking the directed limit in **Wstar** over finite subsets $S \subseteq X$, we see that the induced map $P_b(f) \colon P_b(B) \to P_b(C)$ corresponds to $\ell^\infty(\widehat{f})$ under the isomorphisms $P_b(B) \simeq \ell^\infty(Y)$ and $P_b(C) \simeq \ell^\infty(X)$. This completes the proof for P_b; the analogous argument in **proCstar** also holds for P_u. $\square$

Example 3.4. Let $A = \mathbb{M}_n(C(X))$ for a compact Hausdorff space X. Then $P_b(A) = \mathbb{M}_n(\ell^\infty(X))$ and $P_u(A) = \mathbb{M}_n(\mathbb{C}^X)$.

Proof. Write $C = C(X)$, and recall that every closed ideal $J \subseteq \mathbb{M}_n(C)$ is of the form $\mathbb{M}_n(I)$ for some closed ideal $I \subseteq C$ [27, Corollary 17.8]. Such an ideal J has finite codimension in A if and only if I has finite codimension in C. Thus

$$\begin{aligned} P_b(A) = \ &\lim_{J\in\mathcal{F}(A)} A/J \ = \ \lim_{I\in\mathcal{F}(C)} \mathbb{M}_n(C)/\mathbb{M}_n(I) \\ \simeq \ &\lim_{I\in\mathcal{F}(C)} \mathbb{M}_n(C/I) \ \simeq \ \mathbb{M}_n(\ell^\infty(X)) \end{aligned}$$

and similarly $P_u(A) \simeq \mathbb{M}_n(\mathbb{C}^X)$. $\square$

Let us emphasize that, even though the profinite completion functors yield discretizations of *all* C*-algebras, there are many C*-algebras A for which $P_b(A) = P_u(A) = 0$ is trivial. Indeed, if A is any C*-algebra with no finite-dimensional representations, then by construction of the profinite completions we necessarily have $P_b(A) = P_u(A) = 0$. Example include: the algebra $B(H)$ of bounded operators on an infinite-dimensional Hilbert space H; the CCR algebra [30]; the Calkin algebra $B(H)/K(H)$; and the (separable) Cuntz algebra $\mathcal{O}_n$ generated by $n \geq 2$ isometries [11]. Thus it is interesting to see which algebras have injective or faithful discretizations to their profinite completion. This is addressed in the next theorem.

Recall that a C*-algebra A is *residually finite-dimensional* when it has a faithful family of finite-dimensional representations. Similarly, A is *subhomogeneous* when there is an integer $n \geq 1$ such that every irreducible representation of A has dimension at most n; this is equivalent [9, Proposition IV.1.4.3] to A being isomorphic to a C*-subalgebra of $\mathbb{M}_k(C)$ for a commutative C*-algebra C and an integer $k \geq 1$. For a point x in a set X, we let $\delta_x = \chi_{\{x\}} \in \ell^\infty(X) \subseteq \mathbb{C}^X$ denote the indicator function of the singleton $\{x\}$.

Theorem 3.5. *For a C*-algebra A, the functorial discretizations P_b and P_u are:*

(i) injective if and only if A is residually finite-dimensional;
(ii) faithful if A is subhomogeneous.

Proof. (i) If A is residually finite-dimensional, every nonzero $a \in A$ allows $I_a \in \mathcal{F}(A)$ with $a \notin I_a$ (meaning that a has nonzero image in A/I_a). Thus a is not in the kernel of $\eta_A\colon A \to \lim_{I \in \mathcal{F}(A)} A/I = P_b(A) \subseteq P_u(A)$. Hence η_A is injective. (See also [13, Lemma 1.10].) The converse follows directly from the definition.

(ii) Consider a commutative C*-subalgebra $C(X) \subseteq A$, and $x \in X$. Because the homomorphisms $\ell^\infty(X) \simeq P_b(C(X)) \to P_b(A)$ and $\mathbb{C}^X \simeq P_u(C(X)) \to P_u(A)$ are respectively normal and continuous, it suffices to show that $\delta_x \in \ell^\infty(X) \subseteq \mathbb{C}^X$ is not in their kernel. Indeed, the kernel I of either morphism is an ideal generated by a characteristic function χ_S for some $S \subseteq X$, so that I contains exactly those δ_x with $x \in S$. Hence if all $\delta_x \notin I$, then $S = \varnothing$ and therefore $I = 0$.

Evaluation at x is a pure state on $C(X)$, which extends [9, II.6.3.2] to a pure state ρ_x on A. Because A is subhomogeneous, the GNS construction applied to ρ_x yields a finite-dimensional representation $\pi\colon A \to B(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ for some integer $n \geq 1$, with cyclic vector $v_x \in \mathbb{C}^n$. Let $I \in \mathcal{F}(A)$ denote the kernel of π. The induced $*$-homomorphism $\psi\colon \ell^\infty(X) \to A/I \hookrightarrow \mathbb{M}_n(\mathbb{C})$ has image isomorphic to $C(S)$ for some finite subset $S \subseteq X$; in fact, this set S is characterized as those pure states on $C(X)$ that are induced by vector states of the representation π. Now $\pi(f)v_x = f(x)v_x$ for $f \in C(X)$ by construction of π. Thus $x \in S$, so that δ_x is not in the kernel of ψ. It follows that δ_x has nonzero image in each of the limit algebras $P_b(A)$ and $P_u(A)$, as desired. □

Remark 3.6. For C*-algebras A that are residually finite-dimensional but not subhomogeneous, the natural map $A \to P_b(A)$ is technically an injective discretization, but it does not satisfy all desiderata for an 'algebra of bounded functions on the noncommutative underlying set' of A. Consider the C*-sum $A = \overline{\bigoplus_{k=1}^\infty \mathbb{M}_k(\mathbb{C})}$. Let $I_n \subseteq A$ denote the kernel of the projection $A \twoheadrightarrow \mathbb{M}_1(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_n(\mathbb{C})$ onto the first n components. By an argument similar to that in [23, Lemma 7.5], the kernel of any finite-dimensional representation of A must contain some I_n. It follows that the I_n form a cofinal chain in $\mathcal{F}(A)$, so that the profinite completion

$$A \to P_b(A) \simeq \lim_{n\to\infty} A/I_n \simeq A$$

is an isomorphism. But this is far from the behavior one would expect when comparing to the commutative example $C = \overline{\bigoplus_{k=1}^\infty \mathbb{C}} \simeq \ell^\infty(\mathbb{N}) \simeq C(\beta\mathbb{N})$; the profinite completion $C \to P_b(C)$ corresponds under this isomorphism to the embedding $C \simeq C(\beta\mathbb{N}) \to \ell^\infty(\beta\mathbb{N})$, indicating that C is 'far below' $P_b(C)$ as a subalgebra.

Almost all faithful discretizations of C*-algebras we know are supplied by Theorem 3.5 above. We conclude this section by describing another significant example of a faithful compatible discretization that is not of this form.

Example 3.7. For an infinite-dimensional Hilbert space H, consider the C*-subalgebra $A = \mathbb{C} \oplus K(H)$ of $B(H)$ generated by the identity and the compact operators. The embedding $A \hookrightarrow B(H)$ is a faithful compatible W*-discretization.

Proof. Any commuting set of self-adjoint compact operators on H has an orthonormal basis of H of simultaneous eigenvectors, so the same remains true for commuting sets of self-adjoint operators in A. Let $C \simeq C(X) \subseteq A$ be a commutative C*-subalgebra. For $x \in X$ let $p_x \in B(H)$ denote the projection onto the simultaneous eigenspace $\{v \in H \mid f \cdot v = f(x)v \text{ for all } f \in C\}$. Now each $p_x \neq 0$ and $\sum p_x = 1$ in $B(H)$. It follows that the W*-subalgebra W_C generated by the p_x is isomorphic to $\ell^\infty(X)$, and the fact that $fp_x = p_x f = f(x) \cdot p_x$ for all $f \in C$ guarantees that the natural inclusion $C \subseteq W_C$ corresponds under this isomorphism to the natural inclusion $C(X) \subseteq \ell^\infty(X)$. Thus the discretization is faithful.

Compatibility for commutative C*-subalgebras $C \subseteq D \subseteq A$ is readily established from the simple observation that a simultaneous eigenspace for D restricts to a simultaneous eigenspace for C. □

The example above is a faithful compatible W*-discretization for which we do not know of any extension to an unbounded discretization.

4. Obstructions to discretizations with many projections

Can the bounded faithful functorial W*-discretization for subhomogeneous C*-algebras of Theorem 3.5 be extended to general C*-algebras through some method other than profinite completion? Perhaps surprisingly, we prove in this section that the answer is no: any W*-discretization of the algebra $B(H)$ for an infinite-dimensional Hilbert space H is necessarily zero. In fact, the obstruction is even more serious: if we replace the category of W*-algebras ('noncommutative measurable spaces') with the category of AW*-algebras [22, 5] ('noncommutative complete Boolean algebras' [20]), the obstruction persists.

The next definition is crucial to our obstructions, and relies on the following notions from measure theory. An *atom* of a measure space (X, μ) is a measurable subset $U \subseteq X$ with $\mu(U) > 0$, such that $\mu(V) < \mu(U)$ implies $\mu(V) = 0$ for any measurable subset $V \subseteq U$. An atom of a regular Borel measure on a locally compact Hausdorff space is necessarily a singleton [24, 2.IV]. A measure is *diffuse* if it has no atoms. We will say that a positive linear functional $\psi \colon C(X) \to \mathbb{C}$ of a commutative C*-algebra, given by $\psi(f) = \int f \, d\mu$ for a regular Borel measure μ on X, is *diffuse* when μ is diffuse.

Definition 4.1. Let A be a C*-algebra. A pair of commutative C*-subalgebras C and D is *relatively diffuse* when every extension of a pure state of D to a state of A restricts to a diffuse state on C.

Example 4.2. Consider the separable Hilbert space $H = L^2[0,1]$, and the C*-algebra $A = B(H)$. Write D for the discrete maximal abelian W*-subalgebra generated by the projections q_n onto the Fourier basis vectors $e_n = \exp(2\pi i n -)$ for $n \in \mathbb{Z}$, and C for the continuous maximal abelian W*-subalgebra $L^\infty[0,1]$. Then C and D are relatively diffuse.

Proof. There is a canonical conditional expectation $E\colon A \to D$ that sends $f \in A$ to its diagonal part $\sum q_n f q_n$. For $f \in C$ then $E(f) = \int_0^1 f(t)\,\mathrm{d}t$ because

$$\langle f e_n, e_n \rangle = \int_0^1 f(t) \cdot e^{2\pi i n t} \cdot \overline{e^{2\pi i n t}}\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t.$$

Because ψ is a pure state of D now $\psi = \psi \circ E$ by the solution of the Kadison-Singer problem [28]. Hence $\psi(f) = \psi(E(f)) = \psi(\int_0^1 f(t)\,\mathrm{d}t) = \int_0^1 f(t)\,\mathrm{d}t$. □

Example 4.3. For $H = L^2[0,1]$, consider any separable C*-subalgebra $C \subseteq L^\infty[0,1] \subseteq B(H)$ for which the state $f \mapsto \int_0^1 f(t)\,dt$ is diffuse (such as $C = C[0,1]$). Then there is a separable C*-subalgebra $A \subseteq B(H)$ containing C and a commutative C*-subalgebra D generated by projections, with C and D relatively diffuse.

Proof. Let e_n and E be as in Example 4.2. Because C is separable, we can fix a sequence $\{f_i\}_{i=1}^\infty$ of elements whose linear span is dense in C. For each f_i and for each integer $j \geq 1$, the positive solution to the paving conjecture [28] ensures that there is a finite set of projections $p_k = p_k^{(i,j)}$ in the discrete maximal abelian subalgebra of $B(H)$ relative to the Fourier basis e_n with $\sum p_k = 1$ and $\|p_k(f_i - E(f_i))p_k\| \leq 1/j$. Let D be the commutative C*-subalgebra of $B(H)$ generated by the $p_k^{(i,j)}$ for all i, j, and k. Let A be the C*-subalgebra of $B(H)$ generated by C and D; as both C and D are countably generated, the same is true of A, whence A is separable. An argument familiar in the literature on the Kadison-Singer problem (as in [3, p310]) shows that any extension of a pure state ψ_0 on D to a state ψ on A satisfies $\psi(f) = \psi_0(E(f))$ for all $f \in C$. The same computation as in Example 4.2 shows that $\psi(f) = \int_0^1 f(t)\,dt$, which is diffuse on C by hypothesis. □

Remark 4.4. It is possible to modify Examples 4.2 and 4.3 so that the conclusions can be reached without using the full force of Kadison-Singer. In either case, identify the algebra $C = C(\mathbb{T})$ of continuous functions on the unit circle with the subalgebra $\{f \mid f(0) = f(1)\} \subseteq C[0,1] \subseteq B(H)$. The algebra of Fourier polynomials—or more generally, the Wiener algebra $A(\mathbb{T})$—is a dense subalgebra of C and lies in the algebra $M_0 \subseteq B(H)$ of operators that are l_1-bounded in the sense of Tanbay [38] with respect to the Fourier basis $\{e_n \mid n \in \mathbb{Z}\}$. Thus C lies in the norm closure M of M_0, and it was shown in [38] (without the full force of Kadison-Singer) that every element of M is compressible (that is, the operator $f - E(f)$ satisfies paving with respect to the basis e_n for any $f \in M$). The computations in either example given above may now proceed in the same manner.

The relatively diffuse subalgebras C and D in the examples above had pure states of D inducing a *unique* diffuse state on C. We thank the referee for the following example which allows for possibly non-unique extensions.

Example 4.5. Let A and D be as in Example 4.2, but consider the commutative C*-subalgebra of A generated by the bilateral shift $e_n \mapsto e_{n+1}$, and let C be its bicommutant. Then C and D are relatively diffuse.

Proof. Write C_0 for the C*-subalgebra generated by the shift $u\colon H \to H$; its Gelfand spectrum is the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ [15, Problem 84]. Let $f_n \in C(\mathbb{T})$ be a decreasing sequence converging to the characteristic function $\delta_\lambda = \chi_{\{\lambda\}}$ of some $\lambda \in \mathbb{T}$. Then, since the bounded sequence (f_n) converges pointwise to δ_λ, the sequence $(f_n(u))$ in C_0 converges strongly to the projection $\delta_\lambda(u)$ in

C. But $\lim_n \langle f_n(u)(e_0), e_0\rangle = \langle \delta_\lambda(e_0), e_0\rangle$ vanishes because u has no eigenvectors. Hence $\|E(f_n(u))\| = \|\langle f_n(u)(e_0), e_0\rangle 1_H\| \to 0$. Thus a state ψ of A that is pure on D satisfies $\psi(f_n) = \psi(E(f_n)) \to 0$, and is therefore diffuse on C. □

Relatively diffuse pairs of commutative C*-subalgebras are inherited along $*$-homomorphisms, as follows.

Lemma 4.6. *Let $\phi\colon A \to B$ be a morphism in* **Cstar**. *If two commutative C*-subalgebras $C, D \subseteq A$ are relatively diffuse, then so are $\phi(C), \phi(D) \subseteq B$.*

Proof. Fix a pure state ψ_0 on $\phi(D)$, and let ψ be any extension to a state on B. Then $\psi \circ \phi$ is a state on A that extends $\psi_0 \circ \phi$ from D; observe that the latter is a pure state on D as it is a composition of a $*$-homomorphism with a pure state. By hypothesis, the restriction of $\psi \circ \phi$ to C is diffuse. As the restriction of ϕ to $C \twoheadrightarrow \phi(C)$ is Gelfand dual to the inclusion $\mathrm{Spec}(\phi(C)) \hookrightarrow \mathrm{Spec}(C)$ of a closed subspace, the measure on $\mathrm{Spec}(\phi(C))$ corresponding to $\psi|_{\phi(C)}$ is the restriction of the measure on $\mathrm{Spec}(C)$ corresponding to $\psi_0|_C$, which is diffuse. It follows that the restriction of ψ to C' is diffuse. □

The major result below and its many corollaries will refer to commutative diagrams of the following kind, where A is a C*-algebra with relatively diffuse commutative C*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$.

$$\begin{array}{ccc} C \simeq C(X) & \hookrightarrow & \ell^\infty(X) \\ \downarrow & & \downarrow \phi_C \\ A & \xrightarrow{\ \phi\ } & M \\ \uparrow & & \uparrow \phi_D \\ D \simeq C(Y) & \hookrightarrow & \ell^\infty(Y) \end{array} \tag{4.1}$$

Theorem 4.7. *If a C*-algebra A has relatively diffuse commutative C*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$, and if there is a C*-algebra M with $*$-homomorphisms ϕ, ϕ_C and ϕ_D making the diagram* (4.1) *commute, then for any $x \in X$ and $y \in Y$:*

$$\phi_C(\delta_x)\phi_D(\delta_y) = 0.$$

Proof. Let $x \in X$ and $y \in Y$, and write $p = \phi_C(\delta_x)$ and $q = \phi_D(\delta_y)$. Fix any state σ on the C*-algebra qBq, and let ψ denote the state on A given by $\psi(a) = \sigma(q\phi(a)q)$. For $g \in D$, observe $\psi(g) = \sigma(\phi_D(\delta_y g \delta_y)) = \sigma(\phi_D(g(y)\delta_y)) = g(y)\sigma(q) = g(y)$, so that ψ restricts to a pure state on D. By hypothesis, the restriction of ψ to C is of the form $f \mapsto \int f\, d\mu$ for some diffuse Radon measure μ on X. Thus for each integer $n \geq 1$ we may find an open neighborhood U_n of x with $\mu(U_n) \leq \frac{1}{n}$. Urysohn's lemma provides a continuous function $f_n\colon X \to [0,1]$ that vanishes on $X \setminus U_n$ and satisfies $f_n(x) = 1$. Since $\delta_x \leq f_n$ in $\ell^\infty(X)$ we have $p = \phi_C(\delta_x) \leq \phi_C(f_n)$. Positivity of $b \mapsto \sigma(qbq)$ yields

$$\sigma(qpq) \leq \sigma(q\phi_C(f_n)q) = \psi(f_n) = \int f_n\, d\mu \leq \mu(U_n) \leq \frac{1}{n}.$$

As $n \to \infty$ we find that $\sigma(pqp) = 0$ for all states σ on B, making $qpq = 0$. It follows that $\|qp\|^2 = \|qpq\| = 0$ and thus $pq = (qp)^* = 0$. □

Write **AWstar** for the category of AW*-algebras with $*$-homomorphisms whose restriction to the projection lattices preserve arbitrary least upper bounds[1]; **Wstar** is a full subcategory. We call **AWstar**-discretizations *AW*-discretizations*.

Corollary 4.8. *If a C*-algebra A has two relatively diffuse commutative C*-subalgebras, then any AW*-discretization $\phi\colon A \to M$ satisfies $M = 0$. Consequently, every functorial AW*-discretization $F\colon$ **Cstar** $\to$ **AWstar** has $F(A) = 0$ for such A.*

Proof. Let $C \simeq C(X)$ and $D \simeq C(Y)$ be the relatively diffuse commutative C*-subalgebras, and let $\phi_C\colon \ell^\infty(X) \to M$ and $\phi_D\colon \ell^\infty(Y) \to M$ be the discretizing morphisms as in Definition 2.2, yielding a commuting diagram (4.1). For $x \in X$ and $y \in Y$, set $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$. As $\sum \delta_x = 1_C$ and $\sum \delta_y = 1_D$ (in the sense of least upper bounds of orthogonal projections), and as ϕ_C and ϕ_D are morphisms in **AWstar**, we have $\sum p_x = 1 = \sum q_y$ in M. By Theorem 4.7, each p_x is orthogonal to all of the q_y, so that p_x is orthogonal to $\sum q_y = 1 \in M$. Therefore $p_x = 0$ for all $x \in X$, whence $1 = \sum p_x = 0$ in M and $M = 0$. □

Example 4.9. If there is a morphism $B(H) \to A$ in **Cstar** for some infinite-dimensional Hilbert space, then A has no nontrivial AW*-discretization.

Proof. First note that H as above is unitarily isomorphic to $L^2[0,1] \otimes H$, so $a \mapsto a \otimes 1$ is a $*$-homomorphism $B(L^2[0,1]) \to B(L^2[0,1]) \otimes B(H) \simeq B(H)$. Example 4.2 along with Lemma 4.6 show that A contains a relatively diffuse commutative C*-subalgebras, so that Corollary 4.8 applies. □

In particular, by the last example the Calkin algebra $A = B(H)/K(H)$ has no nontrivial AW*-discretization for $H = L^2[0,1]$.

Theorem 4.7 has the following consequence for purely ring-theoretic discretizations, with much tamer conclusion than those of Corollaries 4.8 or 4.11.

Corollary 4.10. *If a C*-algebra A has relatively diffuse C*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$, and if there is a commutative diagram of the form* (4.1) *where M is a ring and ϕ, ϕ_C, ϕ_D are ring homomorphisms, then for every $x \in X$ and $y \in Y$:*

$$\phi_C(\delta_x)\phi_D(\delta_y) = \phi_D(\delta_y)\phi_C(\delta_x) = 0.$$

Proof. Invoking Theorem 4.7 in the case where

$$M_1 = (A *_{C(X)} \ell^\infty(X)) *_{C(Y)} \ell^\infty(Y)$$

is the colimit in **Cstar** of the diagram (4.1) with M deleted, we obtain that the images of δ_x and δ_y are orthogonal in M_1. Now let $R \circledast_S T$ denote the amalgamated free product of rings (which coincides with the amalgamated free product of $\mathbb{C}$-algebras when S is a unital subalgebra of algebras R and T), and let

$$M_0 = (A \circledast_{C(X)} \ell^\infty(X)) \circledast_{C(Y)} \ell^\infty(Y)$$

be the colimit in the category of rings of the diagram (4.1) with M deleted. There is a natural map $M_0 \to M_1$ induced by the universal property of M_0. It is a folk result that this is an embedding [10, 34]. Thus the images of δ_x and δ_y in M_0 are already orthogonal. But the morphisms ϕ, ϕ_C, and ϕ_D of (4.1) factor universally through M_0, so the images of δ_x and δ_y in M are orthogonal. □

[1]See [20, Lemma 2.2] for further characterizations of these morphisms.

We conclude this section with an obstruction for unbounded discretizations into topological algebras. Write **TAlg** for the category of Hausdorff topological $\mathbb{C}$-algebras with continuous homomorphisms. Recall [39, Chapter 10] that a family $(a_i)_{i\in I}$ of elements in a Hausdorff topological ring R is *summable* if the net (a_J) indexed by finite subsets $J \subseteq I$ converges, where $a_J = \sum_{j\in J} a_j$; in that case we write $\sum a_i$ for the limit.

Corollary 4.11. *Let A be a C*-algebra with relatively diffuse C*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$. Then every unbounded* **TAlg***-discretization of A is zero. More precisely: if there is a commutative diagram*

$$\begin{array}{ccc} C \simeq C(X) & \hookrightarrow & \mathbb{C}^X \\ \downarrow & & \downarrow \phi_C \\ A & \xrightarrow{\phi} & M \\ \uparrow & & \uparrow \phi_D \\ D \simeq C(Y) & \hookrightarrow & \mathbb{C}^Y \end{array}$$

where M is a Hausdorff topological ring, ϕ_C and ϕ_D are continuous homomorphisms, and ϕ is a homomorphism, then $M = 0$.

Proof. It suffices to prove the second, more general claim. Because the natural embedding $C(X) \hookrightarrow \mathbb{C}^X$ has image in the subring $\ell^\infty(X) \subseteq \mathbb{C}^X$ and similarly for $C(Y)$, we may apply Corollary 4.10 to conclude that the idempotents $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$ satisfy $p_x q_y = 0$ for all $x \in X$ and $y \in Y$.

The orthogonal set of idempotents $\{\delta_x \mid x \in X\}$ is summable with $\sum \delta_x = 1$ in $\mathbb{C}^X$, so the family of images $(p_x)_{x\in X}$ under the continuous homomorphism ϕ_C is also summable in M with $\sum p_x = 1$. Similarly, we have $(q_y)_{y\in Y}$ summable in M with $\sum q_y = 1$.

Now consider the net $(p_I q_J)$ indexed by the directed set of all 'rectangular' subsets $I \times J \subseteq X \times Y$ with both $I \subseteq X$ and $J \subseteq Y$ finite. As both (p_I) and (q_J) converge to 1, we have $p_I q_J \to 1^2 = 1$. But each $p_I q_J = \sum_I \sum_J p_x q_y = 0$, so we have $1 = \lim p_I q_J = 0$. Thus $M = 0$. □

Just as in Example 4.9, if there is a morphism $B(H) \to A$ in **Cstar** with H an infinite-dimensional Hilbert space, then every unbounded **TAlg**-discretization of A is trivial.

Remark 4.12. Similar to the C*-discretization in Proposition 2.3, one could construct a pro-C*-discretization by replacing the pushouts $A *_C \ell^\infty(\mathrm{Spec}(C))$ in **Cstar** with the pushouts $A *_C \mathbb{C}^{\mathrm{Spec}(C)}$ in **proCstar**. However, the previous corollary shows that this construction must trivialize for algebras A that have relatively diffuse commutative C*-subalgebras.

We close with one further example of a separable algebra having no injective W*-discretizations. We only sketch its proof, as the complete argument would require us to modify several results above to account for possibly nonunital commutative subalgebras, a technicality that we have avoided for the sake of readability.

Example 4.13. Let $H = L^2[0,1]$ and $C = C[0,1] \subseteq L^\infty[0,1] \subseteq B(H)$. Then $A = C + K(H)$ is a separable C*-algebra of type I for which every AW*-discretization and every unbounded **TAlg**-discretizations has nonzero kernel. (It does, however, have nonzero non-injective such discretizations that factor through the commutative C*-algebra $A/K(H)$.)

Proof. Let e_n, and q_n be as in Example 4.2. Within $B(H)$, write $C_0(\mathbb{Z}) \simeq D \subseteq K(H)$ for the nonunital commutative C*-subalgebra generated by the q_n. If one alters Definition 4.1 to allow for possibly nonunital C*-subalgebras, then C and D are relatively diffuse. Indeed, each pure state ψ_0 on D is supported on some projection $p = q_n$, and every extension of ψ_0 to a state ψ on A satisfies $\psi(f) = \psi(pfp) = (\int_0^1 f\,dt)\psi(p) = \int_0^1 f\,dt$ for all $f \in C[0,1]$. A suitable modification of Theorem 4.7 holds for such C and D, with hardly a change to the proof.

Now if $\phi\colon A \to M$ is an AW*-discretization or an unbounded **TAlg**-discretization, then we claim that $K(H) \subseteq \ker(\phi)$. Indeed, the same method of proof of Corollaries 4.8 and 4.11 shows that D is contained in $\ker(\phi)$ (noting that C is still a unital subalgebra), and $K(H)$ is the ideal generated by D. □

5. Conclusion

In contrast to the obstructions [35, 7, 4], based on the Kochen-Specker theorem [25] from quantum physics, the fact that profinite completion faithfully discretizes all finite-dimensional C*-algebras shows that the results in Section 4 are truly infinite-dimensional obstructions and are therefore independent of the Kochen-Specker theorem.

From the perspective of discretization as discussed in this paper, the search for a suitable candidate **A** for a category of algebras dual to 'noncommutative sets' remains open. Having ruled out various candidates, we now briefly discuss the implications, including possible avenues to avoid these obstructions.

Within the category **Cstar**, there remains the interesting open Question 2.6 of whether every C*-algebra has a functorial (or equivalently, compatible) C*-discretization that is injective or faithful. This question is addressed in recent work of Kornell [26] that takes a radically different approach: passing to a model of set theory in which every subset of $\mathbb{R}$ is measurable, so that the Axiom of Choice fails.

A positive answer to Question 2.6 would still not entail a candidate category of algebras dual to 'noncommutative sets'. That would require isolating a suitable subcategory **A** of **Cstar** containing the algebras $\ell^\infty(X)$ and their normal $*$-homomorphisms as a full subcategory (dual to 'classical' sets). One of the most notable feature of the algebras $\ell^\infty(X)$ and $\mathbb{C}^X$ is their abundance of projections. But using this structure as a guide makes Corollaries 4.8 and 4.11 particularly troubling. Suppose that A, $C(X)$, and $C(Y)$ are as in Theorem 4.7. Let $\phi\colon A \to M$ be the discretization of Proposition 2.3. On the one hand, that proposition demonstrates that $\ell^\infty(X)$ and $\ell^\infty(Y)$ embed faithfully into M. On the other hand, for all $x \in X$ and $y \in Y$, Theorem 4.7 implies that the images of $\delta_x \in \ell^\infty(X)$ and $\delta_y \in \ell^\infty(Y)$ are orthogonal in M. So it is not contradictory to faithfully embed both $\ell^\infty(X)$ and $\ell^\infty(Y)$ into a common discretization making all $\delta_x\delta_y$ vanish.

Thus Corollaries 4.8 and 4.11 merely indicate that globally 'gluing' projections via the structure of an AW*-algebra or via convergence of nets of finite sums is inadequate for discretization. This suggests exploring new structures imposing a suitable 'global coherence' on projections in noncommutative $*$-algebras beyond AW*-algebras or topological algebras. To speculate only about a single possibility:

the notion of contramodule [33] formalizes 'infinite summation' operations that cannot be interpreted as convergence of sums in any topology.

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