Title
T-PRODUCTS AND SPECTRAL FUNCTION SUM RULES IN A THEORY OF CURRENTS

Authors
Bierter, Willy
Bitar, Khalil M.

Publication Date
1968-09-01

UCRL-18446
Cy. 2

University of California

Ernest O. Lawrence Radiation Laboratory

T-PRODUCTS AND SPECTRAL FUNCTION SUM RULES IN A THEORY OF CURRENTS

Willy Bierter and Khalil M. Bitar

September 1968

Berkeley, California

DISCLAIMER

Submitted to Physical Review Letters

UCRL-18446
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

T-PRODUCTS AND SPECTRAL FUNCTION SUM RULES IN A THEORY OF CURRENTS

Willy Bierter and Khalil M. Bitar

September 1968

T-PRODUCTS AND SPECTRAL FUNCTION SUM RULES IN A THEORY OF CURRENTS*

Willy Bierter and Khalil M. Bitar[+ ‡]

Department of Physics and Lawrence Radiation Laboratory
University of California, Berkeley, California

ABSTRACT

We present a new method for deriving spectral function sum rules in a theory of currents. Applying this method to Sugawara's theory we obtain the Weinberg sum rules along with higher moment sum rules that may be used to test that theory.

First we briefly discuss a method[1] for deriving spectral function sum rules. We center our considerations on the time ordered product:

$$M_{\mu\nu}^{ab}(q) \quad = \quad -i \int d^4x \; e^{iq\cdot x} \; \langle A|T[J_\mu{}^a(x) \; J_\nu{}^b(0)]|B\rangle \quad . \tag{1}$$

* This work was supported in part by the U.S. Atomic Energy Commission.

+ Address after September 23: Institute for Advanced Study, Princeton, New Jersey 08540.

‡ Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant No. AF-AFOSR-68-1471.

In a theory of currents, where the Hamiltonian H is known, $M_{\mu\nu}^{ab}(q)$ can be expressed as a sum of terms each of which is given by well-specified equal time commutators. This is easily seen, by expanding the current $J_\mu{}^a(x)$ in powers of the variable x_0, to yield

$$
\begin{aligned}
M_{\mu\nu}^{ab}(q) &= \frac{1}{q_0}\int d^3x\, e^{-i\vec{q}\vec{x}} \langle A|[J_\mu{}^a(0,\vec{x}), J_\nu{}^b(0)]|B\rangle \\
&+ \frac{1}{q_0^2}\int d^3x\, e^{-i\vec{q}\vec{x}} \langle A|\Big[[J_\mu{}^a(0,\vec{x}), H], J_\nu{}^b(0)\Big]|B\rangle \\
&\vdots \\
&+ \frac{n!}{q_0^{n+1}}\int d^3x\, e^{-i\vec{q}\vec{x}} \langle A|\underbrace{[\cdots}_{n+1}[J_\mu{}^a(0,\vec{x}), H],\cdots,H], J_\nu{}^b(0)]|B\rangle \quad .
\end{aligned}
\tag{2}
$$

(The Bjorken limit[2] is that where $q_0 \to \infty$ with $\vec{q}$ fixed. In this paper we do not go to this limit.) Clearly, if H is known all terms on the right-hand side of (2) can be calculated (although some terms are very singular) and one has an expansion for $M_{\mu\nu}^{ab}(q)$ in powers of $\frac{1}{q_0}$. If we write now $M_{\mu\nu}$ in terms of integrals over higher order spectral functions we obtain sum rules for these functions by simply comparing terms of equal powers of $\frac{1}{q_0}$ on both sides of (2).

This is most easily demonstrated for the two-point spectral functions as we shall proceed to show. We consider the vector (V_μ) and axial vector (A_μ) currents in the Sugawara theory.[3] In this theory one has $SU(3) \otimes SU(3)$ currents obeying the equal-time commutation relations of the algebra of fields[4] with an energy-momentum tensor $\Theta_{\mu\nu}$ constructed as follows:

$$\Theta_{\mu\nu} = \frac{1}{2C}\{[V_\mu{}^a(x),\ V_\nu{}^a(x)]_+ - g_{\mu\nu}[V_\mu{}^a(x),\ V_a{}^\mu(x)]_+ + (V \leftrightarrow A)\} \quad . \quad (3)$$

Adopting Θ_{00} as the Hamiltonian density one obtains the following equations of motion:

$$\partial^\mu V_\mu{}^a(x) = \partial^\mu A_\mu{}^a(x) = 0$$

$$\partial_\mu V_\nu{}^a(x) - \partial_\nu V_\mu{}^a(x) = \frac{1}{2C} f_{abc}\{[V_\mu{}^b(x), V_\nu{}^c(x)]_+ + [A_\mu{}^b(x),\ A_\nu{}^c(x)]_+\}$$

$$\partial_\mu A_\nu{}^a(x) - \partial_\nu A_\mu{}^a(x) = \frac{1}{2C} f_{abc}\{[V_\mu{}^b(x),\ A_\nu{}^c(x)]_+ + [A_\mu{}^b(x),\ V_\nu{}^c(x)]_+\} \quad . \quad (4)$$

Following Weinberg[5] we write

$$\langle 0|V_\mu{}^a(x)\ V_\nu{}^b(0)|0\rangle = \frac{1}{(2\pi)^3}\,\delta_{ab}\int d^4p\ \Theta(p_0)\ e^{-ipx}\ \rho_V(p^2)[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}]$$

$$\langle 0|A_\mu{}^a(x)\ A_\nu{}^b(0)|0\rangle = \frac{1}{(2\pi)^3}\,\delta_{ab}\int d^4p\ \Theta(p_0)\ e^{-ipx}\{\rho_A(p^2)[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}] + F_\pi{}^2\ \delta(p^2)\ p_\mu p_\nu\} \quad . \quad (5)$$

It then follows that

$$-i\int d^4x\ e^{iqx}\langle 0|T[V_\mu{}^a(x)\ V_\nu{}^b(0)]|0\rangle = \delta_{ab}\int dm^2\ \rho_V(m^2)[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}]\Delta_F(q^2;m^2) \quad (6a)$$

and

$$-i\int d^4x\ e^{iqx}\ \langle 0|T[A_\mu{}^a(x)\ A_\nu{}^b(0)]|0\rangle = \delta_{ab}\int dm^2\{\rho_A(m^2)[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}] + F_\pi{}^2\ \delta(m^2)\ q_\mu q_\nu\}\Delta_F(q^2;m^2) \quad (6b)$$

with

$$\Delta_F(q^2;m^2) = \frac{1}{q^2 - m^2 + i\epsilon} \quad .$$

From (6) we see that the only q_0 dependence is given in the factors q_μ, q_ν and $\Delta_F(q^2;m^2)$. Thus expanding (6) in powers of $\frac{1}{q_0}$ is straightforward and the coefficients are integrals over the spectral functions ρ_V and ρ_A. Through Eq. (2), with $|A\rangle$ and $|B\rangle$ taken as the vacuum states, the integrals are then given by the symmetric part (in the internal symmetry space) of equal time commutators of the form $[\cdots[J_\mu{}^a(0,\vec{x}), H],\cdots H], J_\nu{}^b(x)]$.

We consider the case $\mu = \nu = i$ and isolate in (6) the term proportional to $\frac{1}{q_0^2}$. Through Eq. (2) we get

$$\int dm^2 \rho_V(m^2)[-1 + \frac{q_i q_i}{m^2}] = (\text{sym.}) \int d^3x\ e^{-i\vec{q}\vec{x}} \langle 0 | \Big[[V_i{}^a(0,\vec{x}),H],V_i{}^b(0)\Big]|0\rangle, \tag{7a}$$

$$\int dm^2 \{\rho_A(m^2)[-1 + \frac{q_i q_i}{m^2}] + F_\pi{}^2 q_i q_i\} = (\text{sym.}) \int d^3x\ e^{-i\vec{q}\vec{x}} \langle 0 | \Big[[A_i{}^a(0,\vec{x}),H], A_i{}^b(0)\Big]|0\rangle \quad . \tag{7b}$$

Using the Sugawara Hamiltonian we find that

$$\Big[[V_i{}^a(0,\vec{x}), H], V_i{}^b(0)\Big] = \Big[[A_i{}^a(0,\vec{x}), H], A_i{}^b(0)\Big] \quad , \tag{8}$$

which leads to the equality of the left-hand sides of (7a) and (7b). Since q_i is an independent variable, this leads to the two Weinberg sum rules:

$$\int dm^2\, m^{-2}[\rho_V(m^2) - \rho_A(m^2)] = F_\pi^{\;2} \,, \tag{9a}$$

$$\int dm^2\, [\rho_V(m^2) - \rho_A(m^2)] = 0. \tag{9b}$$

Obviously these are not the only sum rules[6] that we can derive, for now we can proceed further and isolate higher terms, e.g. the $\frac{1}{q_0^{\,4}}$ term (we put here $q_i = 0.$[7]). From (6) we then have

$$-\int dm^2 m^2 \rho_V(m^2) = (\text{sym.})\int d^3x\; e^{-i\vec{q}\vec{x}}\langle 0|[[[V_i^{\,a}(0,\vec{x}),H],H],H],V_i^{\,b}(0)]|0\rangle \,, \tag{10a}$$

$$-\int dm^2 m^2 \rho_A(m^2) = (\text{sym.})\int d^3x e^{-i\vec{q}\vec{x}}\langle 0|[[[A_i^{\,a}(0,\vec{x}),H],H],H],A_i^{\,b}(0)]|0\rangle \,. \tag{10b}$$

Again using the explicit form (4) of the equations of motion we get

$$[[[V_i^{\,a}(0,\vec{x}),H],H],H],\; V_i^{\,b}(0)] = [[[A_i^{\,a}(0,\vec{x}),H],H],H],A_i^{\,b}(0)], \tag{11}$$

which implies of course the following sum rule:

$$\int dm^2\, m^2[\rho_V(m^2) - \rho_A(m^2)] = 0 \quad . \tag{12}$$

As a matter of fact, due to the symmetric roles which V_μ and A_μ play in the equations of motion and $\Theta_{\mu\nu}$, it seems that one can explicitly show that

$$\underbrace{[\cdots}_{n+1}[V_i^{\,a}(0,\vec{x}),H],\cdots,H],V_i^{\,b}(0)] = \underbrace{[\cdots}_{n+1}[A_i^{\,a}(0,\vec{x}),H],\cdots,H],A_i^{\,b}(0)], \tag{13}$$

which leads to the result

$$\int dm^2 (m^2)^r [\rho_V(m^2) - \rho_A(m^2)] = 0 \quad . \tag{14}$$

Moreover Eq. (13) implies through Eq. (12), if the expansion (2) converges, the result that for any state $|A\rangle$ and $|B\rangle$:

$$M^{ab}_{\mu\nu}[V^a(x), V^b(0)] = M^{ab}_{\mu\nu}[A^a(x), A^b(0)] \tag{15}$$

or

$$T[V_\mu{}^a(x)\ V_\nu{}^b(0)] = T[A_\mu{}^a(x)\ A_\nu{}^b(0)] \quad . \tag{16}$$

Discussion:

That we are able to derive Weinberg's first sum rule (9a) from Sugawara's theory is expected, since this is a consequence of the algebra of fields[4,6] incorporated explicitly into the theory. The second and higher moment sum rules, however, are a consequence of our method and the Sugawara Hamiltonian. As can be seen from (9a, 9b), (12) and (14) we obtain the following result:

$$\rho_V(m^2) = \rho_A(m^2) \quad ; \quad F_\pi = 0 \quad .$$

This is also to be expected since we are dealing with a theory of perfect symmetry. For a study of theories with symmetry breaking our techniques apply as well but not as simply. For example, we find that at the $SU(2) \otimes SU(2)$ level, if we break the symmetry by introducing PCAC according to Bardakci, Frishman and Halpern,[8] the right-hand side of (12) becomes proportional to singular vacuum expectation values of operators of the form $[V_i(0)\ V_i(0)\ \sigma(0)]$ and $[A_i(0)\ A_i(0)\ \sigma(0)]$. Similar results are also obtained for sum rules derived independently for ρ_V and ρ_A by a direct comparison of the same powers of $\frac{1}{q_0}$

in Eqs. (2) and (6). In these cases one can use the sum rules to estimate the nature of the singularities of such expectation values if a knowledge of the high m^2-behavior of the spectral functions is available.

Since our higher moment sum rules are derived from a study of differences like $[\cdots[V,H],\cdots,H],V] - [\cdots[A,H],\cdots,H],A]$, it is then hoped that, in some theory with broken symmetry, such differences could be expressed in manageable quantities leading to meaningful such sum rules.

It would be interesting to study these higher moment sum rules in the broken-symmetry theory recently proposed by Sugawara.[9] However, one will probably find the breaking also expressed in terms of singular functions that are vacuum expectation values of products of currents at the same space-time point.

For sum rules obtained from the Bjorken limit of (2) we refer the reader to Ref. 10 and 11. We only mention in passing that a direct use of Sugawara's Hamiltonian in the sum rule derived by Bjorken[2] for the process $e^+ + e^- \rightarrow$ hadrons leads to a quartic divergence for the integral $\int_0^\infty dq^2\, q^4\, \sigma_{tot}(q^2)$, which is his result as well.

The methods discussed above can be extended to the study of the higher n-point functions of the currents and of course also for other Hamiltonians than the Sugawara Hamiltonian, e.g. those derived from effective Lagrangians.

One of us (W.B.) is indebted to Dr. Geoffrey Chew for the kind hospitality extended to him by the Theoretical Group at the Lawrence Radiation Laboratory and a research fellowship from the "Swiss National Fund" is gratefully acknowledged.

FOOTNOTES AND REFERENCES

1. K. M. Bitar, unpublished.

2. J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

3. H. Sugawara, Phys. Rev. 170, 1659 (1968).

4. T. D. Lee, S. Weinberg and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

5. S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

6. We can also get these sum rules by using $\mu = 0$, $\nu = j$ and isolating in (6) the $\frac{1}{q_0}$ term for (9a) and the $\frac{1}{q_0^3}$ for (9b); or by taking $\mu = i$, $\nu = j$ and isolating in (6) the $\frac{1}{q_0^2}$ term for (9a). That we get Weinberg's first sum rule from a $\frac{1}{q_0}$ term which is proportional to the equal time commutator of the currents implies that this sum rule is a direct consequence of the algebra of fields and in particular the equality of the Schwinger terms for the vector and axial vector commutators. That we get the second sum rule (9b) only by going to $\frac{1}{q_0^2}$ or higher power implies that one needs to go beyond the algebra of fields to a specific Lagrangian or Hamiltonian to obtain it. Indeed in Ref. 4 an effective Lagrangian was needed; here this is replaced by the Sugawara Hamiltonian. This may be an indication of some equivalence between this Hamiltonian and the effective Lagrangians of Ref. 4 and others.

7. If we keep $q_i \neq 0$ we obtain along with our sum rule (12) again the first Weinberg sum rule (9a).

8. K. Bardakci, Y. Frishman and M. B. Halpern, Phys. Rev. 170, 1353 (1968).

9. H. Sugawara, Enrico Fermi Institute preprint EFI 68-52.

10. D. Gross. Phys. Rev. Letters 21, 308 (1968).

11. C. G. Callan and D. Gross, Phys. Rev. Letters 21, 311 (1968).