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I. INTRODUCTION

This article is concerned with the problem of strong coupling as it is manifested in the properties of the "ordinary" particles, pions and nucleons. It is not possible to separate such a discussion cleanly from strange particles, which also undergo strong interactions, but at the current level of understanding of the pion-nucleon interaction the complications due to strange particles can be minimized.

Although it seems likely that the masses of the pion $(\underline{\mu})$ and the nucleon $(\underline{M})$ are consequences of strong coupling, no attempt is made here to discuss the masses in this sense. We consider $\underline{\mu}$ and $\underline{M}$ to be given parameters. The same is true for the pion-nucleon $(\underline{g})$ and pion-pion $(\underline{\lambda})$ coupling constants, even though future developments may show these quantities to be not really fundamental. The problem, then, is to relate the cross sections for the various interactions involving pions and nucleons to the four constants $\underline{\mu}$, $\underline{M}$, $\underline{g}$, and $\underline{\lambda}$. We shall restrict ourselves to processes in which there are two ingoing and two outgoing particles. These processes are the following:

1. $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{\pi} + \underline{\pi}$ — Pion-pion scattering.
2. $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$ — Nucleon pair production in pion-pion collisions and nucleon-antinucleon annihilation to produce two pions.

3. $\underline{\pi} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$ or $\underline{\pi} + \underline{\overline{N}} \leftrightarrow \underline{\pi} + \underline{\overline{N}}$ — Pion-nucleon or pion-antinucleon scattering.

4. $\underline{N} + \underline{N} \leftrightarrow \underline{N} + \underline{N}$ — Nucleon-nucleon scattering.

5. $\underline{N} + \underline{\overline{N}} \leftrightarrow \underline{N} + \underline{\overline{N}}$ — Nucleon-antinucleon scattering.

The above processes are different manifestations of three fundamental matrix elements that can be represented as shown in Figure 1. Here the wiggly lines refer to pions and the straight lines to nucleons. Any pair of lines may represent the ingoing two particles, with the other pair representing the outgoing, and going opposite to the direction of an arrow simply means changing particle to antiparticle with a change of sign of the four-momentum. Thus Figure 1(b) includes both Process 2 and Process 3, while Figure 1(c) includes Processes 4 and 5.

It is instructive also to consider diagrams of the type of Figures 1(a) and 1(b) with one of the pions replaced by a photon. Such diagrams correspond to the following processes:

6. $\underline{\gamma} + \underline{\pi} \leftrightarrow \underline{\pi} + \underline{\pi}$ — Photopion production from pions and radiative capture of a pion by a pion.

7. $\underline{\gamma} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$ — Nucleon pair production in photopion collisions and nucleon-antinucleon annihilation to produce a pion plus a photon.

8. $\underline{\gamma} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$ or $\underline{\gamma} + \underline{\overline{N}} \leftrightarrow \underline{\pi} + \underline{\overline{N}}$ — Photopion production from nucleon (or antinucleon) and radiative capture of a pion by a nucleon.

These last three processes will require for their description at least one additional parameter, the elementary electric charge e.

The above four-particle reactions are to be discussed here by means of spectral representations, often referred to as "dispersion relations." Actually these representations have much more content and utility than the original dispersion relations of electromagnetic theory, but they developed out of attempts to generalize the Kramers-Kronig equations (1, 2)The systematic derivation of the new dispersion relations is complicated and not at all suitable to a review of this kind, therefore we restrict ourselves to a short qualitative description of the main ingredients. The current justification of dispersion relations rests on two fundamental physical assumptions that have become prominent only within the past ten years, even though their origin is much older. Extensive use is also made of standard symmetry principles and associated conservation laws that are recognized as important in all areas of particle physics; we shall take these principles for granted and make no special mention of them.[1] The two distinctive principles are:

A. Signals never propagate with a velocity faster than that of light, no matter how short the distance involved. This is the principle of "microscopic causality." In the language of local quantum field theory, it is expressed by saying that the commutator of two Heisenberg field operators, taken at different space-time points, vanishes if the separation between these points is spacelike. Without use of the framework of local field theory no precise way is known of formulating microcausality.

B. The totality of all possible <u>physical</u> states of the universe forms a complete set of basis vectors in the quantum mechanical sense. That is to say, an arbitrary state vector may be expressed as a linear superposition of vectors, each representing a possible physical state with a total energy-momentum four vector that is positive timelike. This "spectral condition," although it sounds extremely plausible, is not universally accepted, the conjecture having often been made that local field theory is inconsistent unless "ghost" states, with no direct physical interpretation, are included.

The usual starting point in the derivation of a dispersion relation is the reduction formula, first given in a general form by Lehmann, Zimmermann, & Symanzik[3] and for the special case of π-N scattering by Low[4]. The reduction formula allows one to write the amplitude for an arbitrary transition in terms of the Fourier transform of a matrix element of a commutator of two Heisenberg field operators. The energy variable occurs only in the imaginary exponent, multiplying the space-time coordinate, and the vanishing of the commutator outside the light cone then allows one to extend the energy dependence into the complex plane. Goldberger was the first to use such an approach,[5] which after this crucial step leads immediately to dispersion relations. Later it was realized that a rigorous justification of Goldberger's extension into the complex plane was not really easy to achieve except for zero-mass particles scattered in the forward direction. Symanzik[6] was the first to solve the nonzero mass problem, and Bogoliubov[7] then showed that the extension was justifiable even for nonforward scattering if the particle mass were formally made imaginary and sufficiently large in absolute value. However, the difficult problem remained of investigating the behavior of the amplitude as a function of particle mass; it was necessary, of course, to show that dispersion relations continue to hold as the mass is made real and equal to its actual physical value.

Bogoliubov developed methods of proof appropriate to certain special cases,[7] but these have now been superseded by the work of Bremermann, Oehme, & Taylor,[8] based on the theory of many complex variables, and by the work of Jost & Lehmann[9] and of Dyson,[10] based on more familiar but still tricky mathematics.

Although the details of the derivation cannot be given here, we write down the Dyson representation,[10] which expresses conditions A and B in a form suitable to the deduction of dispersion relations. No specific use of Dyson's representation is made in this article, but it serves to illustrate the kind of connection between physics and mathematics that characterizes dispersion relations. Consider the (four-dimensiona Fourier transform of the matrix element of the commutator of two local Heisenberg operators, $\underline{j}(\underline{x}/2)$ and $\underline{h}(-\underline{x}/2)$:

$$\underline{F}(\underline{q}) = \int \underline{d}^4\underline{x} \;\; \underline{e}^{i\underline{q}\underline{x}} \left\langle \underline{P}, \underline{\alpha} \,\middle|\, \left[\underline{j}(\underline{x}/2),\ \underline{h}(-\underline{x}/2)\right] \,\middle|\, \underline{Q}, \underline{\beta} \right\rangle . \tag{I. 1}$$

The matrix element here connects two physical states whose total energy-momentum four vectors are $\underline{P}$ and $\underline{Q}$, respectively. The indices $\underline{\alpha}$ and $\underline{\beta}$ refer to the other degrees of freedom needed to complete the specification of these states. According to the microcausality assumption A, the matrix element vanishes for $\underline{x}^2 = \underline{x}_0{}^2 - \vec{\underline{x}}^2 < 0$, that is, for spacelike separation of the two operator-arguments.

Assumption B comes into play if we insert a complete set of "intermediate" physical states between the operators $\underline{j}$ and $\underline{h}$. It may then be seen by using displacement operators, which shift the arguments of $\underline{j}$ and $\underline{h}$ to the origin, that $\underline{F}(\underline{q})$ vanishes unless $\left[\frac{1}{2}(\underline{P} + \underline{Q}) + \underline{q}\right]$ is the energy-momentum of a state $\left|\underline{n}\right\rangle$ for which both the matrix elements

$$\left\langle \underline{P}, \underline{\alpha} \,\middle|\, \underline{j}(0) \,\middle|\, \underline{n} \right\rangle \qquad \text{and} \qquad \left\langle \underline{n} \,\middle|\, \underline{h}(0) \,\middle|\, \underline{Q}, \underline{\beta} \right\rangle \tag{I. 2}$$

fail to vanish, or $\frac{1}{2}\left[(P+Q)-q\right]$ is the energy-momentum of a state for which both

$$\left\langle P, \alpha \left| h(0) \right| n \right\rangle \quad \text{and} \quad \left\langle n \left| j(0) \right| Q, \beta \right\rangle \tag{I. 3}$$

fail to vanish. The four-momenta of the states $\left| n \right\rangle$ are all positive timelike, and we designate by m_1 the smallest mass of a state satisfying (I. 2) and by m_2 the corresponding smallest mass for (I. 3). Assumption B thus leads to the property that $F(q)$ vanishes except for

$$\frac{P_0 + Q_0}{2} + q_0 \geq 0 \qquad \text{and} \qquad \left(\frac{P+Q}{2} + q\right)^2 \geq m_1^{\,2}$$

or (I. 4)

$$\frac{P_0 + Q_0}{2} - q_0 \geq 0 \qquad \text{and} \qquad \left(\frac{P+Q}{2} - q\right)^2 \geq m_2^{\,2} .$$

Dyson was able to prove that, for $F(q)$ to satisfy the condition (I. 4) and at the same time be the Fourier transform of a function that vanishes for spacelike argument, it is necessary and sufficient that $F(q)$ can be represented as

$$F(q) = \int d^4u \int_0^\infty d\kappa^2 \; \epsilon\,(q_0 - u_0)\, \delta\left[(q - u)^2 - \kappa^2\right] \phi(u, \kappa^2) . \tag{I. 5}$$

The integrations here extend over a region such that the vectors $\frac{P+Q}{2} + u$ and $\frac{P+Q}{2} - u$ both lie in the forward light cone, while κ is positive and larger than either $m_1 - \sqrt{\left(\frac{P+Q}{2} + u\right)^2}$ or $m_2 - \sqrt{\left(\frac{P+Q}{2} - u\right)^2}$. Within this region $\phi\,(u, \kappa^2)$ is arbitrary.

Previously Lehman & Jost had deduced a somewhat similar representation for the special case $m_1 = m_2$.[9] We shall not write down the Jost-Lehmann representation, but historically it represented a significant step in the understanding of dispersion relations. The alternative approach to the problem through the theory of many complex variables, exploited by Bremermann, Oehme, & Taylor[8] yields the same results as achieved through the Dyson representation. These results have been summarized recently by Goldberger.[11]

For the reader who wishes to see all the essential steps in a complete and yet economical derivation of the pion-nucleon dispersion relation, the following use of the published literature is recommended: (a) Read the first and about half of the second section of Reference (8), up to the point where dispersion relations have been obtained for imaginary mass. (b) Switch here to a recent paper by Lehman[12] which uses the Dyson representation not only to carry out the necessary extension in the mass variable but also to justify the use of Legendre polynomials in implementing dispersion relations. (c) If any strength remains, read the Dyson paper.[10]

It should be stated at this point that interest in dispersion relations as a tool for strong coupling physics was first aroused by the 1955 papers of Goldberger[5] and Karplus & Ruderman[13], although at that time the mathematical difficulties in giving a systematic derivation were not realized. At present it remains true that the methods of implementation of dispersion relations are elementary and quite unrelated to the sophisticated mathematical techiques required for their derivation. Such a situation may not persist indefinitely, but it motivates the decision to avoid in this review the mathematics of derivation.

In the following section certain important kinematical questions are dealt with, preparatory to a general statement of the rules for formulating dispersion relations. The rules are then given in Sections III, IV, and V in such a way as to cover not only those relations that have been rigorously derived but also many relations conjectured on the basis of perturbation theory.

II. KINEMATICAL PRELIMINARIES

A. Energy and Angle Variables. In order to describe scattering amplitudes for processes with two ingoing and two outgoing particles, one needs in addition to spin and charge variables at least two invariants that correspond to the energy and angle of scattering in the barycentric system. To maintain a maximum symmetry let us assign four-momenta, p_1, p_2, p_3, p_4, all of which correspond formally to ingoing particles. Two of these momenta will always be negative timelike, representing the actual outgoing particles, while the other two are positive timelike and represent the incoming particles. Energy-momentum conservation is stated through the condition

$$p_1 + p_2 + p_3 + p_4 = 0, \tag{II. 1}$$

while the particle masses are introduced through the four constraints,

$$p_i^2 = m_i^2 . \tag{II. 2}$$

For the purposes of dispersion relations it is convenient to define three invariants

$$s_1 = (p_1 + p_4)^2 = (p_2 + p_3)^2 ,$$
$$s_2 = (p_2 + p_4)^2 = (p_1 + p_3)^2 ,$$
$$s_3 = (p_3 + p_4)^2 = (p_1 + p_2)^2 ,$$

each of which is the square of the total energy in the barycentric system for a particular pairing of incoming and outgoing particles.[2] For example, when p_1 and p_2 are incoming and p_3 and p_4 outgoing, the total energy is $\sqrt{s_3}$. In this case s_1 and s_2 may be interpreted as squares of four-momentum transfers.

It is easy to show that the physical range of an s variable when it is the square of an energy does not overlap the range when it is the square of a momentum transfer. In particular, in the former case s is always positive and extends to $+\infty$, while in the latter it may be negative and extends to $-\infty$.

The three variables s_1, s_2, s_3 are not independent of one another, but with the constraints (II. 1) and (II. 2) they can be shown to satisfy the relation

$$s_1 + s_2 + s_3 = m_1^2 + m_2^2 + m_3^2 + m_4^2 . \tag{II. 3}$$

Any two of the s's may be considered as indepdndent variables, with the third determined by (II. 3). In the dispersion-relation approach it is necessary for the s variables to be extended not only to nonphysical regions of the real axis but also throughout the complex plane. Condition (II. 3) requires that in such extensions the sum of the imaginary parts of the three s variables shall vanish.

In the theory of dispersion relations the substitution rule plays an important part. This rule was discovered in perturbation theory[3] and relates the different channels corresponding to a single diagram. For our purposes this rule will be contained in the statement that a single analytic function describes all three channels contained in the same diagram. In particular, the physical amplitude for the process when Particles 1 and 2 are ingoing is the boundary value, of an analytic function as the variable s_3 approaches the positive real axis in its physical energy range, with one of the other two s variables held fixed at a physical value while the amplitudes for Particles 1 and 3 or 1 and 4 ingoing are obtained from corresponding limits of the same function taken with the variables s_2 or s_1, respectively. Condition (II. 3) is to be obeyed, so that one is dealing in the limiting process with a single complex variable. However, the general rule has meaning only if the two independent s variables can both be extended into the complex plane. The above statement of the substitution rule has been rigorously proved only in a few special cases, but the general form of perturbation theory makes it extremely plausible.

An invariance principle related to the substitution rule, that follows when there are two or more identical particles among the four involved in a particular process, is the so-called "crossing symmetry." Exchanging two identical particles at most changes the sign of the amplitude, but such an interchange means exchanging two of the $\underline{s}$ variables, leaving the third alone. For example, suppose Particles 1 and 3 are identical. Then, depending on whether these are bosons or fermions the amplitude is either symmetric or antisymmetric under an exchange of $\underline{p}_1$ and $\underline{p}_3$, which means interchanging $\underline{s}_1$ and $\underline{s}_3$, leaving $\underline{s}_2$ alone.[4] If $\underline{p}_1$ and $\underline{p}_3$ are both incoming or both outgoing (i.e., $\sqrt{\underline{s}_2}$ is the energy), the symmetry in question is familiar and directly related to the Pauli principle. If one is incoming and the other outgoing, however, the symmetry cannot be so identified and is a special characteristic of field theory. In this case, if one starts with physical values of the $\underline{s}$ variables, the exchange in question necessarily leads to nonphysical values because of the above-mentioned nonoverlapping nature of the energy and momentum-transfer ranges. Thus, the general crossing symmetry has meaning only when a continuation of the amplitude into unphysical regions is possible.

B. Charge and Spin Variables. In this discussion the possibility of degrees of freedom of spin and charge has so far been ignored. It will now be explained how internal degrees of freedom may always be absorbed into invariant matrices, whose coefficients are invariant functions of the s variables only. The number of such functions depends on the complexity of the internal degrees of freedom: For processes associated with Figure 1(a), three independent functions are required, Figure 1(b) requires four functions, and Figure 1(c) ten functions. Remarkably enough, replacing a pion with a photon in Figure 1(a) or 1(b) has a very different effect in the two cases. Figure 1(a) with a photon requires just a single invariant function, while Figure 1(b) with a photon requires twelve. We shall now write down for the simpler problems the invariant matrices required and point out the implications of crossing symmetry for the corresponding invariant functions.

The four-pion problem is one of the simplest because there are no spins and all three branches of the diagram correspond to the same process, π-π scattering.[5] Each pion has a charge degree of freedom, however, and this is described in the conventional way[6] by an index that takes values 1, 2, 3. For the pion with momentum p_1 we associate the charge index α, with p_2 the index β, with p_3 the index γ, and with p_4 the index λ. Let us assume that (p_1, α) and (p_2, β) are incoming, with $(-p_3, \gamma)$ and $(-p_4, \lambda)$ outgoing. The scattering amplitude may then be considered a matrix in a nine-dimensional charge space that is the product of two three-dimensional spaces. The requirement of charge independence leads to the conclusion that only three independent matrices are allowed, corresponding to the fact that only three values of total I spin occur for the two-pion system: $I = 0, 1, 2$. It is convenient to choose as the three fundamental matrices,

$$(\underline{X}_1)_{\underline{\gamma}\,\underline{\lambda},\,\underline{\alpha}\,\underline{\beta}} = \delta_{\underline{\alpha}\,\underline{\beta}}\,\delta_{\underline{\gamma}\,\underline{\lambda}},$$

$$(\underline{X}_2)_{\underline{\gamma}\,\underline{\lambda},\,\underline{\alpha}\,\underline{\beta}} = \delta_{\underline{\alpha}\,\underline{\gamma}}\,\delta_{\underline{\beta}\,\underline{\lambda}}, \qquad \text{(II. 4)}$$

$$(\underline{X}_3)_{\underline{\gamma}\,\underline{\lambda},\,\underline{\alpha}\,\underline{\beta}} = \delta_{\underline{\beta}\,\underline{\gamma}}\,\delta_{\underline{\alpha}\,\underline{\lambda}},$$

and to write the complete amplitude as

$$\underline{X}_1\underline{A}\,(\underline{s}_1,\underline{s}_2,\underline{s}_3) + \underline{X}_2\underline{B}\,(\underline{s}_1,\underline{s}_2,\underline{s}_3) + \underline{X}_3\underline{C}\,(\underline{s}_1,\underline{s}_2,\underline{s}_3)\,. \qquad \text{(II. 5)}$$

The operation of particle exchange involves both the charge and the momentum. Since all four particles are identical bosons we get the following crossing relations:

$$\begin{array}{ll} \underline{A} \longleftrightarrow \underline{A} & \\ \underline{B} \longleftrightarrow \underline{C} & \underline{s}_1 \longleftrightarrow \underline{s}_2\,, \\ & \\ \underline{A} \longleftrightarrow \underline{B} & \\ \underline{C} \longleftrightarrow \underline{C} & \underline{s}_2 \longleftrightarrow \underline{s}_3\,, \end{array} \qquad \text{(II. 6)}$$

plus other relations that are redundant in content. The first of the above two lines simply represents the Pauli principle, but the second puts on the pion-pion scattering amplitude a type of a dynamical requirement unknown outside field theory.

With the definite assignment of $(\underline{p}_1\underline{\alpha})$ and $(\underline{p}_2\underline{\beta})$ as incoming particles it is possible to express A, B, and C in terms of the conventional amplitudes $\underline{A}^{\underline{I}}$ for scattering in states of well-defined $\underline{I}$ spin. The relations turn out to be

$$\begin{aligned} \underline{A}^0 &= 3\underline{A} + \underline{B} + \underline{C}, \\ \underline{A}^1 &= \underline{B} - \underline{C}, \\ \underline{A}^2 &= \underline{B} + \underline{C} \end{aligned} \qquad \text{(II. 7)}$$

In the barycentric system, if the magnitude of the three-momentum of any pion is called q and the angle of scattering θ, the physical meaning of the s variables is

$$s_1 = -2q^2(1 + \cos\theta),$$
$$s_2 = -2q^2(1 - \cos\theta), \qquad \text{(II. 8)}$$
$$s_3 = 4(q^2 + \mu^2).$$

The exchange of s_1 and s_2 thus corresponds to changing $\cos\theta$ to $-\cos\theta$; and the first line of (II. 6), when applied to (II. 7), says no more and no less than that A^0 and A^2 are even functions of $\cos\theta$ while A^1 is an odd function. The second line of (II. 6), however, which relates to the exchange of s_2 and s_3, expresses a condition on the energy and angular dependence, considered together.

A final essentially kinematical feature of the pion-pion problem is the connection between the amplitudes A^I and conventional phase shifts. The formula here is ambiguous as to normalization, but the dependence on energy and angle is unique. Chew & Mandelstam[15] We choose to normalize so that

$$A^I(q^2, \cos\theta) = \sqrt{\frac{q^2 + \mu^2}{q}} \sum_\ell (2\ell + 1)\, e^{i\delta_\ell^I} \sin\delta_\ell^I P_\ell(\cos\theta), \qquad \text{(II. 9)}$$

where δ_ℓ^I is the phase shift for a state of angular momentum ℓ and isotopic spin I. The phase shifts are real for $s_3 < 16\mu^2$ $(q^2 < 3\mu^2)$ and complex at higher energies, where production of two additional pions becomes possible. Single-pion production is forbidden by a combination of charge-conjugation invariance and charge independence,[7] which in general forbids the production of any odd number of pions. As noted above, only even ℓ values occur in (II. 9) for $I = 0, 2$, while only odd ℓ values occur for $I = 1$.

The two processes described by Figure 1 (b), pion-nucleon scattering and nucleon pair production in pion-pion collisions, are physically quite different even though they are limits of the same analytic function. Since the scattering problem is the more familiar of the two, we shall adjust our notation to conform with existing literature on pion-nucleon scattering [Chew, et al.](18).

Let us then assign to the incident and outgoing pions the momenta p_1 and $-p_3$ and the charge indices α and β, respectively. The corresponding nucleon momenta are p_2 and $-p_4$, but the degrees of freedom of the nucleon charge and spin will be suppressed in the conventional way.[8] The invariant amplitude may then be written as a sum of four terms,

$$\delta_{\beta\alpha}\left[-A^0(s_1,s_2,s_3)+\frac{1}{2}\,i\gamma\cdot(p_1-p_3)B^0(s_1,s_2,s_3)\right]$$
$$+\frac{1}{2}\left[\tau_\beta,\tau_\alpha\right]\left[-A^1(s_1,s_2,s_3)+\frac{1}{2}\,i\gamma\cdot(p_1-p_3)\,B^1(s_1,s_2,s_3)\right], \tag{II.10}$$

with the crossing relations, following from symmetry under interchange of the two pions,

$$A^0\longleftrightarrow A^0, \qquad A^1\longleftrightarrow A^1, \qquad s_1\longleftrightarrow s_3. \tag{II.11}$$
$$B^0\longleftrightarrow -B^0, \qquad B^1\longleftrightarrow B^1.$$

The connection between the amplitudes $A^{0,1}$, $B^{0,1}$ and those corresponding to states of well-defined I spin, A^I, B^I, where $I=\frac{1}{2},\frac{3}{2}$, is given by the formulas

$$A^{1/2}=A^0+2A^1, \qquad B^{1/2}=B^0+2B^1$$
$$A^{3/2}=A^0-A^1, \qquad B^{3/2}=B^0-B^1. \tag{II.12}$$

while the three s variables are related to the barycentric-system momentum q and the scattering angle θ (or equivalently to the total energy in the barycentric system, W, and the square of the momentum transfer, Δ^2) by

$$
\begin{aligned}
s_3 &= \left(\sqrt{M^2+q^2}+\sqrt{\mu^2+q^2}\right)^2 = W^2 \\
s_2 &= -2q^2(1-\cos\theta) = -\Delta^2 \qquad\qquad (\text{II. }13) \\
s_1 &= 2M^2 + 2\mu^2 - W^2 + \Delta^2 .
\end{aligned}
$$

Finally we need the connection to phase shifts. This is given conveniently in terms of functions f_1^I and f_2^I defined by

$$f_1^I = \frac{1}{q}\sum_\ell \left[e^{i\delta_{\ell+}^I} \sin\delta_{\ell+}^I\, P_{\ell+1}'(\cos\theta) - e^{i\delta_{\ell-}^I} \sin\delta_{\ell-}^I\, P_{\ell-1}'(\cos\theta) \right]$$

$$f_2^I = \frac{1}{q}\sum_\ell \left[e^{i\delta_{\ell-}^I} \sin\delta_{\ell-}^I - e^{i\delta_{\ell+}^I} \sin\delta_{\ell+}^I \right] P_\ell'(\cos\theta) ,$$

where $P_\ell'(\cos\theta)$ is the first derivative of the Legendre polynomial. The quantities $\delta_{\ell\pm}^I$ are phase shifts for scattering in states of isotopic spin I, orbital angular momentum ℓ, and total angular momentum $\ell \pm \frac{1}{2}$. These phase shifts can be complex for $W > M + 2\mu$, where pion production becomes possible. The relation between the f's and our A and B amplitudes is given by

$$
\begin{aligned}
4\pi f_1 &= \frac{(W+M)^2-\mu^2}{4W^2}\left[A + (W-M)B\right], \qquad\qquad (\text{II. }15)\\
4\pi f_2 &= \frac{(W-M)^2-\mu^2}{4W^2}\left[-A + (W+M)B\right],
\end{aligned}
$$

where, as in the pion-pion problem, the choice of normalization is an arbitrary one.

Now let us consider the same fundamental amplitude from the point of view of nucleon pair production. Here one must distinguish between k, the barycentric system 3-momentum of an incident pion, and $\underline{K}$, that of an outgoing nucleon. Since $\underline{p}_1$ and $\underline{p}_3$ are now both ingoing, with $\underline{-p}_2$ and $\underline{-p}_4$ outgoing, we have

$$\begin{aligned} s_2 &= 4(k^2 + \mu^2) = 4(K^2 + M^2) \\ s_1 &= -k^2 - K^2 + 2kK\cos\Theta \\ s_3 &= -k^2 - K^2 - 2kK\cos\Theta , \end{aligned} \qquad \text{(II. 16)}$$

where Θ is the angle in the barycentric system between an incident pion and an outgoing nucleon.

Again there are two isotopic spin values, but this time $I = 0, 1$ and the amplitudes for these states turn out to be just the quantities we have already labeled with the superscripts 0, 1. The remaining requirement is the equivalent of Formulas (II. 14) and (II. 15). Since off-diagonal elements of the S matrix are involved the process cannot be described simply in terms of phase shifts, but a partial-wave decomposition is still appropriate. One finds [Fulco[20]] that for total angular momentum J, the orbital angular momentum of the $N\bar{N}$ system can be either $J + 1$ or $J - 1$, while the isotopic spin is 0 for J even and 1 for J odd. Fulco[20] has worked out the formulas connecting the transition amplitudes in states of definite J to the invariant amplitudes $A^{0,1}$ and $B^{0,1}$, but we shall not give these formulas here because of their complication.

Even more complicated are the internal degrees of freedom in the problem of nucleon-nucleon or nucleon-antinucleon scattering. The relevant formulas have been worked out by Goldberger, Nambu, & Oehme[21], among others, and involve ten independent scalar functions. In the NN channel, if we associate p_1 and p_2 with the incoming nucleons and $-p_3$ and $-p_4$ with the outgoing, we again have relations of the type (II. 8):

$$\underline{s}_1 = -2\underline{q}^2 (1 + \cos \theta)$$
$$\underline{s}_2 = -2\underline{q}^2 (1 - \cos \theta) \qquad \text{(II. 17)}$$
$$\underline{s}_3 = 4(\underline{q}^2 + \underline{M}^2),$$

where $\underline{q}$ is the barycentric 3-momentum and θ the angle of scattering. These relations switch over in the $\underline{N\bar{N}}$ channel, where $\underline{p}_4$ becomes an ingoing antinucleon and $-\underline{p}_2$ an outgoing antinucleon, to

$$\underline{s}_1 = 4(\underline{K}^2 + \underline{M}^2)$$
$$\underline{s}_2 = -2\underline{K}^2(1 - \cos \Theta) \qquad \text{(II. 18)}$$
$$\underline{s}_3 = -2\underline{K}^2(1 + \cos \Theta),$$

where now $\underline{K}$ is the barycentric momentum and Θ the angle of scattering. There is of course a second $\underline{N\bar{N}}$ channel where $\sqrt{\underline{s}_2}$ is the energy.

It is out of the question to go deeply into the $\underline{NN}$ and $\underline{N\bar{N}}$ problem in this review. Suffice it to say that the same general approach may be used as in the $\underline{\pi\pi}$ and $\underline{\pi N}$ problems. For the details of formulation, Reference 21 should be consulted. Later we describe the important results obtained to data.

The replacement of a pion by a photon in Figure 1(a) leads to the only problem in the group under consideration where a single invariant function suffices. The process in question is $\underline{\gamma} + \underline{\pi} \rightarrow \underline{2\pi}$, and it can be shown [Wong[22]] that G parity[7] allows only the $\underline{I} = 1$ state and therefore only odd $\underline{J}$ values of the two-pion system. Furthermore, gauge invariance eliminates all electric multipoles, so that for each $\underline{J}$ value there is just one transition amplitude. The relevant formulas can be found in Reference 22.

Putting a photon in place of a pion in Figure 1(b) gives rise to a complicated problem that requires twelve invariant functions [Chew et al.[23]]. The most familiar channel here is $\underline{\gamma} + \underline{N} \rightarrow \underline{\pi} + \underline{N}$, where all possible isotopic and angular momentum states of the final pion-nucleon system may be produced by both electric and magnetic transitions. Formulas for the invariant matrices as well as the connection between multipole transition amplitudes and invariant amplitudes are given in Reference 23.

III. POLES IN SCATTERING AMPLITUDES

One of the most important practical consequences to date of the dispersion-relation approach to strong-coupling physics is the recognition of the presence in scattering amplitudes of poles, whose residues have not only a simple physical meaning but also great practical utility. One might almost say that everything so far understood theoretically about strong-coupling phenomena flows from these poles.

There are three different aspects of "polology" that deserve emphasis: (A) The existence and positions of the poles can be predicted simply on the basis of particle masses and internal quantum numbers, spin, parity, etc. (B) The residues of poles in different amplitudes or in different regions of the same amplitude are often simply related. In particular, "fundamental" coupling constants are usually defined directly in terms of residues. (C) Poles dominate the behavior of the scattering amplitude in their immediate neighborhood. On these three pillars a very substantial theoretical structure can be erected.

To implement the third aspect of "polology" it is of course necessary to know something about the other singularities, generally branch points, of the scattering amplitude in the complex plane. A good definition of the subject of "dispersion relations" is that it is the study of the location and nature of these singularities. Of course if enough were known about all the singularities one could construct the complete function, but at present we are far from such a situation, at least in practice. We are just now achieving a comfortable familiarity with the poles and beginning to understand what to do about the nearest branch points.

A. Location of Poles

The existence of poles in a few particular amplitudes has been rigorously proved in the course of deriving dispersion relations by the methods discussed in the introduction. (See, for example, Symanzik[(6)].) Perturbation theory, however, suggests a broad rule that covers not only the poles rigorously derived but many others--some already established experimentally. The rule is the following, as applied to our problem of two incoming and two outgoing particles:[9] If the two incoming particles and the two outgoing particles in any of the three channels of a diagram can be "connected" by a stable[10] single particle of mass m_0, then there will be a pole when the s variable corresponding to the square of the total four-momentum in this channel is equal to ${m^2}_0$. By "connected" we mean that the initial two-particle state and the final two-particle state can both assume all the same quantum numbers as the single particle in question. From the requirement of stability for the intermediate particle it follows that poles, although on the real axis, are never in the physical energy region. If they were, the single particle responsible for the pole could decay via strong interactions into either of the two particle states to which it couples. It also can be shown that poles are always outside the physical momentum-transfer region.

Let us investigate the diagrams of Figure 1 from the point of view of poles. In Figure 1(a) there are no poles at all if we ignore electromagnetic effects because a two-pion state has quantum numbers different from any known particle except the photon. Of course there may exist a still undiscovered boson of mass less than 2μ, baryon number and strangeness zero, isotopic spin 0, 1, or 2 with the appropriate even or odd spin, and with even G parity. If so, there will be poles in the pion-pion scattering amplitude in addition to the photon pole,[11] which is to be ignored in a strictly strong-coupling approach.

Figure 1(b) similarly contains no pole from the channel where two pions are incoming or outgoing but from the two channels where one pion and one nucleon occur, poles arise at $\underline{s_1} = \underline{M^2}$ and $\underline{s_3} = \underline{M^2}$, respectively, corresponding to a single nucleon connecting initial and final states. Figure 1(c) has three poles, one from each channel. The two-nucleon channel gives rise to a pole at $\underline{s_3} = \underline{M_D}^2$, corresponding to the deuteron, while the nucleon-antinucleon channels give rise to poles at $\underline{s_1} = \underline{\mu}^2$ and $\underline{s_2} = \underline{\mu}^2$, both corresponding to the pion.

In the diagram obtained by replacing a pion of Figure 1(a) by a photon there are no poles, but Figure 1(b) with a photon has three, one from each channel. The channels containing one nucleon and a pion or one nucleon and a photon each give nucleon poles, while the channel containing $\underline{\gamma} + \underline{\pi}$ on one side and $\underline{N}\underline{\overline{N}}$ on the other gives a pion pole. Table I summarizes the location of poles in the pion-nucleon problem.

B. Residues and Coupling Constants

Now, what about the residues? Again the rigorous dispersion-relation derivations have given for a few special cases an answer to this question that agrees with the rule suggested by perturbation theory. This rule is as follows:[12] (a) Pretend (whether you believe it or not) that all four external particles and the connecting particle are elementary and associated with local fields in the conventional sense. Construct from these fields invariant trilinear "interactions," satisfying all known symmetry requirements, that represent the two-particle to one-particle transitions in question. Associate with each trilinear interaction a real coefficient which may be called a "coupling constant." (b) Calculate the contribution to the scattering amplitude by conventional second-order perturbation theory. There will

TABLE I

The Positions of Poles ($\underline{m}_0$) and Lowest Branch Points ($\underline{s}_0$) Arising From the Various Channels of Figure 1.

Channel	$\underline{m}_0$	$\underline{s}_0$
1. $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{\pi} + \underline{\pi}$	—	$(2\underline{\mu})^2$
2. $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\bar{N}}$	—	$(2\underline{\mu})^2$
3. $\underline{\pi} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$	$\underline{M}$	$(\underline{M}+\underline{\mu})^2$
4. $\underline{N} + \underline{N} \leftrightarrow \underline{N} + \underline{N}$	$\underline{M}_D$	$(2\underline{M})^2$
5. $\underline{N} + \underline{\bar{N}} \leftrightarrow \underline{N} + \underline{\bar{N}}$	$\underline{\mu}$	$(2\underline{\mu})^2$
6. $\underline{\gamma} + \underline{\pi} \leftrightarrow \underline{\pi} + \underline{\pi}$	—	$(2\underline{\mu})^2$
7. $\underline{\gamma} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\bar{N}}$	$\underline{\mu}$	$(2\underline{\mu})^2$
8. $\underline{\gamma} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$	$\underline{M}$	$(\underline{M}+\underline{\mu})^2$

be one Feynman diagram for each connecting particle, the poles appearing automatically from the propagators of the connectors. The residues of these poles may be identified with the residues of the corresponding poles in the complete scattering amplitude, which are thus in general proportional to the product of two coupling constants.

Two important properties of the residues may be inferred from the above recipe. First: The residues are real. Second: The residue of a pole in one s variable does not depend on the remaining s variables. Thus not only is the residue proportional to products of coupling constants; it also is completely determined by these constants.

Note that no statement is being made about the validity of perturbation theory or even about the legitimacy of the concept of an interaction proportional to the product of local fields. We are simply giving a recipe that is convenient because the rules of perturbation calculation are familiar. It is perfectly possible to formulate a recipe for the residues that avoids a specification of the form of the interaction and makes no use of the apparatus of perturbation theory.[13] Such a formulation, however, would require us to develop elaborate notation otherwise unnecessary in this review.

The most important coupling constant in our problem is that describing the three-pronged vertex of Figure 2. Except for trivial and known factors, the square of this constant determines the residue of all the poles of Figures 1(b) and 1(c) except that involving the deuteron. It also appears linearly in the residues of the poles of Figure 1(b) when a photon replaces a pion.[14] The vertex of Figure 2 in general depends on the three invariants, q^2, p^2, p'^2, and the pion-nucleon coupling constant may be defined[15] as the value of this vertex function when all three particles are on the mass shell, i.e., $p^2 = p'^2 = M^2$, $q^2 = \mu^2$. These conditions are guaranteed to be satisfied when residues are calculated according to the above rules because two of the three particles are "external" and we consider the internal particle momentum at the point where its propagator is infinite, i.e., on its mass shell. From this point of view it is immaterial whether we introduce the coupling constant through the pseudoscalar " interaction"

$$g \bar{\psi} \gamma_5 \tau_i \psi \phi_i \quad . \tag{III. 1}$$

or the pseudovector "interaction"

$$\frac{f}{\mu} \bar{\psi} \gamma_5 \gamma_\mu \tau_i \psi \frac{\partial \phi_i}{\partial x_\mu} \quad , \tag{III. 2}$$

where ψ is the nucleon field and ϕ the pion field. When all three particles are on the mass shell, the two forms are identical for

$$f = \frac{\mu}{2M} g. \tag{III. 3}$$

Much less familar is the coupling constant associated with the vertex of Figure 3, whose square determines the residue of the deuteron pole in Figure 1(c). Actually this vertex involves two scalar functions, associated with the presence in the deuteron of both S- and D-wave components, and the corresponding "coupling constant" also has two parts. It can be shown [Goldberger et al.[(21)]] that the S part is much larger than the D and bears a simple relation to the triplet effective range of the neutron-proton system. Since the latter has been rather accurately measured it is possible to calculate the residue of the deuteron pole.

In order to calculate the residues of Figure 1(b), with a photon replacing a pion, it is necessary to consider also the three-pronged vertices of Figure 4. The coupling constant for Figure 4(a) is just e, the charge of the pion; but--as for Figure 3--an analysis of the nucleon-photon vertex Figure 4(b) shows that two constants are required (actually four, because the photon distinguishes between neutron and proton), this time corresponding to the nucleon charge and anomalous magnetic moment. The anomalous moments are very important, but since they are accurately known there is no difficulty in calculating the required residues [Chew et al.[23]] .

(C) Extrapolation to the Neighborhood of a Pole--"Polology"

It is obvious that in the immediate vicinity of a pole, a scattering amplitude is completely determined by the pole's residue. Since these residues are fixed by a few constants, "polology" leads to many definite and interesting predictions about scattering amplitudes. The predictions, however, always involve some kind of extrapolation of experimental data because, as we have seen, poles invariably lie in nonphysical regions.

In order to formulate extrapolation procedures it is necessary to know something about the other singularities of the scattering amplitude. This question will not be reached until the next section, but here we may describe an extremely simple type of extrapolation [Chew[25]] that is legitimate when a sufficiently large region of the complex plane, including a physical range of the real axis as well as the neighborhood of the pole, is singularity-free. This situation is believed to prevail[16] for an s variable in the region of its momentum-transfer range when one of the other s variables is held fixed at a physical point in the energy range.

Inspection of the kinematical relations of Section II [e.g., Formula (II.17)] shows that when the energy is held fixed, the remaining s variables are linearly related to cos θ with real coefficients. We may therefore speak of a cos θ complex plane in which the poles are in one-to-one correspondence with those of the momentum-transfer s variables (which are really only a single variable because of (II.3)). The physical region in cos θ is of course the interval -1 to +1 on the real axis.

It is trivial to compute the location in the cos θ plane of the poles enumerated above. They all lie on the real axis but outside the physical interval; in Table II the positions are given.

In every case the position of the pole approaches the end of the physical region, cos $\theta = \pm 1$, as the energy becomes very large, but at a finite energy the distance from the end of the physical interval to the pole varies sharply from case to case. At currently accessible energies the only poles near enough to allow practical extrapolations are those associated with pions in channels Nos. 4, 5, and 8.[17]

Since the neighborhood of the physical region in the cos θ plane is free from singularities, the real and the imaginary parts of the scattering amplitude are separately analytic functions. Now, our poles all lie on the real axis and have real residues; thus they occur only in the extension of the real part of the amplitude.[18] A possible extrapolation procedure may then be based on the following: Consider the function $f_R(z)$, which is the real part of any one of the scalar amplitudes discussed above, evaluated at a fixed physical energy for one of the s variables. The dependence on the other two s variables is expressed through $z = \cos\theta$. Then in a region of the complex plane which includes the physical interval $-1 < z < +1$, as well as the position of the pole $z = z_0$, the function

$$g_R(z) = (z - z_0)\, f(z) \qquad \text{(III.4)}$$

TABLE II

The Positions of Poles Expressed in Terms of cos θ. The Channel Indicated is that whose Angular Distribution Contains the Pole. The Channel that Gives Rise to a Pole, in the Sense of Table I, is Always Different from the Channel that Contains this Pole in cos θ.

Channel	Position of Pole in cos θ
1. $\pi + \pi \quad \pi + \pi$	——
2. $\pi + \pi \quad N + \bar{N}$	$\pm \dfrac{M^2 + k^2 + K^2}{2kK}$
3. $\pi + \pi \quad N + \bar{N}$	$-\sqrt{\left(1 + \dfrac{M^2}{q^2}\right)\left(1 + \dfrac{\mu^2}{q^2}\right)} + \dfrac{\mu^2}{2q^2}$
4. $N + N \quad N + N$	$\pm\left(1 + \dfrac{\mu^2}{2q^2}\right)$
5. $N + \bar{N} \quad N + \bar{N}$	(a) $+\left(1 + \dfrac{\mu^2}{2K^2}\right)$ (b) $-\left(1 + \dfrac{M_D^2}{2K^2}\right)$
6. $\gamma + \pi \quad \pi + \pi$	——
7. $\gamma + \pi \quad N + \bar{N}$	$\pm\sqrt{\dfrac{P^2 + M^2}{P^2}}$
8. $\gamma + N \quad \pi + N$	(a) $+\sqrt{\dfrac{q^2 + \mu^2}{q^2}}$ (b) $-\sqrt{\dfrac{q^2 + M^2}{q^2}}$

is analytic. Further, $g_R(z_0) = \lambda$, where λ is the residue of the pole.[19] At the same time, if $f_I(z)$ is the imaginary part of the amplitude, then

$$g_I = (z - z_0) f_I(z) \tag{III. 5}$$

is analytic in at least as large a region, with $g_I(z_0) = 0$. The cross section, with an appropriate normalizing factor that does not contain z, is given by[20]

$$\sigma(z) = f_R^2(z) + f_I^2(z) \quad , \tag{III. 6}$$

therefore

$$G(z) = (z - z_0)^2 \sigma(z) \tag{III. 7}$$

is an analytic function throughout this same region with the value λ^2 at $z = z_0$.

The function $G(z)$ can be experimentally measured in the interval $-1 < z < +1$ and fitted with a polynomial in z, or--what is equivalent but more convenient--a polynomial in $z - z_0$:[21]

$$G(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \quad . \tag{III. 8}$$

The coefficient a_0 in this expansion evidently is equal to λ^2, so that we have a direct method of confronting the theory with experiment.

Sufficient experimental data exist already to have allowed application of this procedure by Cziffra & Moravcsik(26) to the "backward" pion pole in neutron-proton scattering at average neutron energies (lab) of 90 and 400 Mev. The broad spread in the incident-neutron energy spectrum prevents these data from yielding an accurate value for the pion-nucleon coupling constant, but the residue obtained agrees satisfactorily with other determinations of f^2, which are to be discussed below. Eventually, it may be expected that backward n-p scattering at a well-defined energy will yield an accurate determination of f^2, which in this case is related to the coefficient a_0 in (III. 8) by the formula [Chew(25)]

$$\underline{f}^4 = (\underline{q}^4/\underline{M}^4)\,(\underline{q}^2 + \underline{M}^2)\underline{a}_0 \,. \tag{III. 9}$$

Moravcsik, Taylor, & Uretsky (28) have investigated the pion pole in photopion production from nucleons by the same method. The data here are poor, but the existence of the pole can be established and a rough value for $\underline{f}^2$ obtained. The "forward" pion poles in nucleon-nucleon and nucleon-antinucleon scattering will probably be harder to exploit because the imaginary part of the amplitude tends to be larger than the real part near the forward direction at high energies. This familiar diffraction effect means that the interesting part of the cross section, containing the pole, is only a small fraction of what is measured.[22]

It is also possible to extrapolate to poles, starting from the energy region of the real axis. These energy extrapolations can be done either at fixed momentum transfer, fixed angle, or at fixed angular momentum, but in all cases one must contend with a branch point lying at the lower end of the physical interval, between the experimental data and the pole. It is possible to get around this branch point, but the necessary techniques are much less direct than in the angle extrapolations. Often the term "effective range" theory is used to describe techniques of extrapolation in the energy variable.[23]

IV. DISPERSION RELATIONS AT FIXED MOMENTUM TRANSFER

Let us now consider the extension of an $\underline{s}$ variable into the complex plane when one of the other $\underline{s}$ variables is held fixed in the physical momentum-transfer range of the real axis. This situation is the opposite of that discussed above in which the fixed variable was in the energy range. Holding the momentum transfer fixed is actually the more familiar condition historically and the one for which nearly all the rigorous derivations have been given.

As was our practice in discussing poles, we give without proof a prescription for extension into the complex plane that includes all the systematically derived results as well as others conjectured on the basis of perturbation theory. Consider any invariant scattering amplitude $\underline{A}(\underline{s}_1, \underline{s}_2, \underline{s}_3)$, after the internal degrees of freedom have been removed, and suppose that $\underline{s}_2$ is held fixed on the real axis in the momentum-transfer range. The remaining two variables are linearly related through (II. 3), and it is convenient to break our amplitude into two parts,

$$\underline{A}(\underline{s}_1, \underline{s}_2, \underline{s}_3) = \underline{A}_1^{\underline{s}_2}(\underline{s}_1) + \underline{A}_3^{\underline{s}_2}(\underline{s}_3) , \qquad \text{(IV. 1)}$$

each of which is a function of a single variable. The rule for this decomposition has, of course, not yet been given. It is closely tied to the extension rule, which is as follows:

The functions $\underline{A}_1^{\underline{s}_2}(\underline{z})$ and $\underline{A}_3^{\underline{s}_2}(\underline{z})$ are associated with the channels in which $\underline{s}_1$ and $\underline{s}_3$, respectively, act as energy variables. Each may contain simple poles[24] of the type described above, with residues that are independent of $\underline{s}_2$. The remainder of the function in each case can be represented by an integral along the real axis of the form

$$\frac{1}{\pi}\int_{s_0}^{\infty} ds' \; \frac{\rho^{s_2}(s')}{s' - z}, \qquad \text{(IV. 2)}$$

where $\rho^{s_2}(s')$ is real for s_2 sufficiently small in absolute value, and the lower limit s_0 is the square of the lightest mass of a multiparticle state that has the quantum numbers of the channel in question. Table I gives the values of s_0 for the various channels.

The above prescription evidently allows an extension to complex z, and corresponds to the statement that $A_2^{s_1}(z)$ and $A_3^{s_1}(z)$ are each real analytic functions in the cut plane with singularities confined to poles and branch points on the real axis.[25] The cut is chosen to run along the positive real axis from the lowest branch point to $+\infty$. Also implied by (IV. 2), although not necessarily true in practice, is the vanishing of our functions at infinity; but the latter requirement may be relaxed by the technique of subtraction.[26] To avoid complicating the formulas, it will be assumed in our general discussion that subtractions are unnecessary, although in practical applications it is necessary to be careful about this point.

For values of s_2 such that ρ^{s_2} is real,[27] it follows that $\rho^{s_2}(s')$ is just the imaginary part of the function in question as z approaches the positive real axis from above, that is, in the limit $z \to s' + i\epsilon$. The complete representation of the function is thus given by

$$A_{1,3}^{s_2}(z) = \frac{\lambda_{1,3}}{m_{01,3}^{\;2} - z} + \frac{1}{\pi}\int_{s_{01,3}}^{\infty} ds' \; \frac{\mathrm{Im}\, A_{1,3}^{s_2}(s')}{s' - z} . \qquad \text{(IV. 3)}$$

In order to illustrate the foregoing, consider the diagram 1(a), for which there are no poles, and let s_2 be fixed at a small real negative value. Each of the three independent π-π amplitudes, A, B, C, may then be broken into two parts as in (IV.1), and each of these parts has an integral representation of the type (IV.3). The complete function can then be written, for example, as

$$A(s_1, s_2, s_3) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds_3' \, \frac{\mathrm{Im}\, A_3^{s_2}(s_3')}{s_3' - s_3} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds_1' \, \frac{\mathrm{Im}\, A_1^{s_2}(s_1')}{s_1' - s_1} \tag{IV.4}$$

where s_1 and s_3 may be complex but obey the relation $s_1 + s_2 + s_3 = 4\mu^2$, so that either may be eliminated in terms of the other. Suppose we want to apply (IV.4) in the region where s_3 is positive real and larger than $4\mu^2$, i.e., in the physical-energy region for s_3. It is then appropriate to eliminate s_1, and the physical scattering amplitude may be defined by [28]

$$^3A^{s_2}(s_3) = \lim_{z \to s_3 + i\epsilon} \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \left\{ \frac{\mathrm{Im}\, A_3^{s_2}(s')}{s' - z} + \frac{\mathrm{Im}\, A_1^{s_2}(s')}{s' - (4\mu^2 - s_2 - z)} \right\} \tag{IV.5}$$

The denominator of the second term cannot vanish for $s_2 > -4\mu^2$, so that the imaginary part of the expression comes entirely from the vanishing of the first denominator, [29] and we have

$$\mathrm{Im}\, {}^3A^{s_2}(s_3) = \mathrm{Im}\, A_3^{s_2}(s_3) \, . \tag{IV.6}$$

By considering the physical energy region for s_1 in a similar way we would find

$$\mathrm{Im}\, {}^1A^{s_2}(s_1) = \mathrm{Im}\, A_1^{s_2}(s_1) \, . \tag{IV.7}$$

therefore both terms in the integrand of the dispersion integral can be expressed in terms of imaginary parts of complete scattering amplitudes. Since the imaginary part of (IV.5) is satisfied identically once relation (IV.6) is used, the final dispersion relation is usually written for the real part only:

$$\mathrm{Re}\,{}^3A^{s_2}(s_3) = \frac{P}{\pi}\int_{4\mu^2}^{\infty} ds' \left\{ \frac{\mathrm{Im}\,{}^3A^{s_2}(s')}{s' - s_3} + \frac{\mathrm{Im}\,{}^1A^{s_2}(s')}{s' - 4\mu^2 + s_2 + s_3} \right\} . \qquad \text{(IV.8)}$$

Entirely similar procedures may be used to obtain dispersion relations at fixed momentum transfers for any of the processes 1 - 8. When poles occur, these are simply to be added to the dispersion integrals. The general relation then has the form

$$\mathrm{Re}\,{}^3A^{s_2}(s_3) = \frac{\lambda_3}{m_{03}^2 - s_3} + \frac{\lambda_1}{m_{01}^2 - \left(\sum_{i=1}^{4} m_i^2 - s_2 - s_3\right)} + \frac{1}{\pi}\int_{s_{03}}^{\infty} ds_3' \, \frac{\mathrm{Im}\,{}^3A^{s_2}(s_3')}{s_3' - s_3}$$

$$+ \frac{1}{\pi}\int_{s_{01}}^{\infty} ds_1' \, \frac{\mathrm{Im}\,{}^1A^{s_2}(s_1')}{s_1' - \left(\sum_{i=1}^{4} m_i^2 - s_2 - s_3\right)} . \qquad \text{(IV.6)}$$

It is characteristic that in the second or "crossed" term of a dispersion relation the imaginary part of the amplitude for a different channel occurs. Sometimes crossing symmetry allows one to express this amplitude in terms of the channel originally chosen for investigation. In the π-π scattering problem, for example, the crossing relations (II.6) tell us that, under the exchange of s_1 and s_3, $A \longleftrightarrow C$. Thus the numerator of the crossed term can be written $\mathrm{Im}\,{}^3C^{s_2}(s')$, which may be more convenient for practical applications.

The possibility of using crossing symmetry often determines which s variable is to be held fixed. In pion-nucleon scattering one nearly always holds s_2 fixed, rather than s_1, because of the more useful relations that result. Holding s_1 fixed leads to a crossed term involving the channel $\pi + \pi \leftrightarrow N + \bar{N}$, about which little is known experimentally. In the nucleon-nucleon dispersion relations nothing can be done to avoid the nucleon-antinucleon channel, and as a result the relation has been difficult to apply.

In the dispersion relation (IV.6) it is nearly always true that, near the lower limits of the integrations, nonphysical values of s_3' and s_1' occur for a fixed value of s_2. In pion-pion scattering, for example, the minimum physical value of s_3 for a fixed (negative) s_2 is $4\mu^2 - s_2$ and thus larger than the lower limit of the dispersion integral except in the case of forward scattering, where $s_2 = 0$. These nonphysical intervals give rise to most of the difficulty in proving dispersion relations, because it must be shown that the imaginary parts of the amplitudes in question have a meaning throughout the entire region of integration. The conjecture was made very early that the needed extension of the imaginary part of the amplitude could be achieved through conventional Legendre polynomial expansions such as (II.9), but it is necessary for these expansions to converge for a range of $\cos\theta$ that exceeds the physical range -1 to +1 on the negative side. A proof of this convergence has recently been given by Lehmann (12) for the cases in which rigorous derivations are possible.

Rigorous derivations have been given only for the channels $\pi + \pi \leftrightarrow \pi + \pi$ with either s_1 or s_2 fixed, $\pi + N \leftrightarrow \pi + N$ with s_2 fixed, and $\gamma + N \leftrightarrow \pi + N$ with the momentum transfer to the nucleon fixed.[30] In addition, proofs have so far been possible only for rather small magnitudes of the fixed momentum transfer.

It is expected that future developments will extend the rigorous derivations, both in the number of channels and the range of momentum transfer. In the meantime most theorists are disposed to use perturbation theory as a guide to the actual limitations of the dispersion-relation approach, and perturbation theory leads to relations of the type (IV.6) for all our eight channels. There do appear to be some restrictions on the momentum-transfer range in which the above simple considerations are valid, but (as will be seen in the next section) these limitations do not cause any real difficulty.

By far the most useful[31] of the fixed momentum-transfer dispersion relations is that for pion-nucleon scattering in the forward direction. Not only is there no unphysical range here, but also--by luck--a direct measurement of the needed integrands can be achieved through the "optical theorem" relating the total cross section to the imaginary part of the forward amplitude. The formulas for this application were first worked out by Goldberger, Miyazawa, & Oehme(30), and their relation to the invariant amplitudes $\underline{A}^{0,1}$ and $\underline{B}^{0,1}$, introduced above in formula (II.10), may be found in Reference 19.

An enormous experimental effort has gone into testing the forward-direction pion-nucleon dispersion relations, and some doubts have been raised[(31)] concerning the extent to which they are satisfied by the data. As the errors involved have become better understood, however, the apparent discrepancies between theory and experiment have diminished and the current belief by most workers who have carefully studied the question is that the relations are experimentally satisfied.[32] Since the only quantity in these relations that is not directly measurable is the residue of the nucleon pole, we have here a relatively accurate means of determining the pion-nucleon coupling constant. The result is

$$\underline{f}^2 = 0.08 \pm .01 \quad . \tag{IV.9}$$

None of the other channels (except $\underline{\pi}\ \underline{\pi} \leftrightarrow \underline{\pi}\ \underline{\pi}$) on our list has dispersion relations without unphysical regions of integration, but serious efforts have been made nonetheless to investigate $\underline{N} + \underline{N} \quad \underline{N} + \underline{N}$ [Matsuyama[33] and Grisaru[34]] and $\underline{\gamma} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$ [Chew et al.[29]] because large parts of the dispersion integrals for these two channels can be determined experimentally. The methods so far used in such attempts, however, are "dirty" and certain to undergo radical improvement in the near future. We prefer not to discuss these methods here and refer the interested reader to the original articles. The conclusion from all investigations made to date of these two channels is that the experimental data probably satisfy the dispersion relations, with poles whose residues are roughly determined and correspond to values of $\underline{f}^2$ in agreement with (IV.9).

V. THE MANDELSTAM REPRESENTATION

The rule for extending the two independent s variables simultaneously into the complex plane has been given by Mandelstam.[35] This prescription is based mainly on perturbation theory, and a long time may elapse before the rule is given the rigorous basis that now underlies some of the fixed momentum-transfer dispersion relations. However, Mandelstam's representation has passed many significant theoretical tests of internal consistency, and so far all its experimental consequences seem satisfied. If the representation can be believed, it not only allows many important types of extrapolation to the neighborhood of poles, but it apparently leads to a complete dynamical description of strong-coupling physics in the conventional sense. That is, when the masses and internal quantum numbers of elementary particles, as well as the mutual coupling constants, are known, the representation seems in principle to allow the calculation of all physically interesting quantities.

We first write down the representation for the simplest case, that of pion-pion scattering, and then generalize. According to Mandelstam the invariant amplitude $A(s_1, s_2, s_3)$, where the arguments can be complex but satisfy (II. 3), may be expressed as follows:

$$A(s_1, s_2, s_3) = \frac{1}{\pi^2}\iint ds_1'\, ds_2'\, \frac{\rho_{12}(s_1', s_2')}{(s_1' - s_1)(s_2' - s_2)}$$
$$+ \frac{1}{\pi^2}\iint ds_1'\, ds_3'\, \frac{\rho_{13}(s_1', s_3')}{(s_1' - s_1)(s_3' - s_3)} + \frac{1}{\pi^2}\iint ds_2'\, ds_3'\, \frac{\rho_{32}(s_3', s_2')}{(s_2' - s_2)(s_3' - s_3)} \quad (V.1)$$

The weight functions $\rho_{ij}(s_i', s_j')$ are real and the integrations in each s' variable go over a region of the positive real axis extending to infinity. For π-π scattering the region in which the weight functions are nonzero is asymptotic to the limiting values $s_i = 4\mu^2$, $s_j = 4\mu^2$. This particular region is shown in Figure 5. The general recipe for calculating boundary curves is not simple Mandelstam[36], but the asymptotic limits are always given by the s_0 of Table I. That is, the absolute lower limit of any s' variable of integration, which occurs when the other s' variable with which it is paired goes to infinity, is equal to the lowest mass of a multiparticle state that has the quantum numbers of the channel in question. If single-particle states can occur, then simple poles with constant residues are to be added to (V. 1). Also subtractions[33] may be needed if the amplitude does not vanish at infinity for both independent variables.

It is easy to see that holding one s variable fixed at a real value outside its energy range and carrying out one of the two integrations in the Mandelstam representation leads to ordinary dispersion relations. If we wish to arrive at (IV. 6), for example, then the first and third terms of (V. 1) may be written as

$$\frac{1}{\pi}\int_{s_{10}}^{\infty} ds_1' \, \frac{g_{12}^{s_2}(s_1')}{s_1' - s_1} + \frac{1}{\pi}\int_{s_{30}}^{\infty} ds_3' \, \frac{g_{23}^{s_2}(s_3')}{s_3' - s_3} , \qquad \text{(V. 2)}$$

where the new weight functions,

$$g_{2j}^{s_2}(s'_j) = \frac{1}{\pi}\int_{s_2^0(s_j')}^{\infty} ds_2' \, \frac{\rho_{j2}(s_j', s_2')}{s_2' - s_2} , \quad j = 1, 3, \qquad \text{(V. 3)}$$

are real for $s_2 \quad s_{20}$, since the lower limit $s_2^0(s_j')$ is always larger than s_{20}. This form is then exactly that required for the ordinary dispersion relations at fixed s_2. For the second term of (V. 1), we make use of the identity

$$\frac{1}{(s_1' - s_1)(s_3' - s_3)} = \frac{1}{s_1' + s_3' - s_1 - s_3}\left(\frac{1}{s_1' - s_1} + \frac{1}{s_3' - s_3}\right)$$

$$= \frac{1}{s_1' + s_3' - \sum_{i=1}^{4} m_i^2 + s_2}\left(\frac{1}{s_1'-s_1} + \frac{1}{s_3'-s_3}\right) \qquad (V.4)$$

to arrive at a similar form. Thus the entire expression (V. 1) can be written in the form (V. 2) if $g_{12}^{s_2}(s_1')$ is augmented by the integral

$$\frac{1}{\pi}\int_{s_3^0(s_1')}^{\infty} ds_3' \frac{\rho_{13}(s_1', s_3')}{s_1' + s_3' - \sum_{i=1}^{4} m_i^2 + s_2}, \qquad (V.5)$$

and $g_{32}^{s_2}(s_3')$ by a corresponding integral over ds_1'. The complete connection between (V. 1) and (IV. 6) is therefore given by

$$\mathrm{Im}^3 A^{s_2}(s_3') = \frac{1}{\pi}\int_{s_2^0(s_3')}^{\infty} ds_2' \frac{\rho_{32}(s_3', s_2')}{s_2' - s_2} + \frac{1}{\pi}\int_{s_1^0(s_3')}^{\infty} ds_1' \frac{\rho_{13}(s_1', s_3')}{s_1' + s_3' - \sum_{i=1}^{4} m_i^2 + s_2}$$

(V. 6)

$$\mathrm{Im}^1 A^{s_2}(s_1') = \frac{1}{\pi}\int_{s_2^0(s_1')}^{\infty} ds_2' \frac{\rho_{12}(s_1', s_2')}{s_2' - s_2} + \frac{1}{\pi}\int_{s_3^0(s_1')}^{\infty} ds_3' \frac{\rho_{13}(s_1', s_3')}{s_1' + s_3' \sum_{i=1}^{4} m_i^2 + s_2}.$$

Poles that may appear in the Mandelstam representation are to be carried over without change into the one-dimensional representation, except that a pole in the fixed variable is generally suppressed by making a subtraction, since it becomes just a constant in the reduced equation.

The denominators of the second integrals in (V.6) can vanish if $\underline{s}_2$ is sufficiently large and negative, so that the "imaginary parts" defined by these expressions become complex. It can easily be shown, however, that the imaginary parts of the "imaginary parts" cancel out when both terms of (IV.6) are calculated because the apparent singularity was introduced artificially through the partial fractions of (V.4).

A second elementary application of the Mandelstam representation is to justify the procedure outlined in Sec. III for extrapolating to the neighborhood of poles. Here the fixed $\underline{s}$ variable is in the physical-energy range; i.e., real and larger than the $\underline{s}_0$ for this channel. It is easy to see by inspection of (V.1) that if the remaining two variables are replaced by cos $\underline{\theta}$, then the singularities in the cos $\underline{\theta}$ complex plane all lie on the real axis and are outside the physical interval, -1 cos $\underline{\theta}$ +1. Furthermore the nearest branch points[34] are determined by the $\underline{s}_0$ values and always lie beyond any poles that occur. Thus there is no impediment to a simple polynomial extrapolation from the physical region.

Many other applications of (V.1) are possible. For example, Cini, Fubini, & Stanghellini[(37)] have derived dispersion relations at fixed cos $\underline{\theta}$, and several workers have deduced dispersion relations for fixed angular momentum. The latter are particularly powerful because they allow a simple incorporation of the unitarity of the S matrix into the problem. When unitarity is added to (V.1) the dynamics of the system seem to be almost completely determined.

In order to get dispersion relations for a given partial-wave amplitude (i.e., for a definite angular momentum), it is necessary to make a projection of (V.1). Taking $\underline{\pi}\,\underline{\pi}$ scattering again as an example and using (II.8) to replace $\underline{s}_1$ and $\underline{s}_2$ by cos $\underline{\theta}$ and $\underline{s}_3$ by $\underline{q}^2$, we would expand $\underline{A}(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ as follows:

$$A(q^2, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, q^{2\ell} A_\ell (q^2) P_\ell (\cos\theta), \tag{V.8}$$

where

$$q^{2\ell} A_\ell (q^2) = \frac{1}{2} \int_{-1}^{+1} dx \; A(q^2, x)\, P_\ell(x) \, . \tag{V.9}$$

Since the dependence of (V.1) on s_1 and s_2 and hence on $\cos\theta$ is contained explicitly in the denominators, one may carry out the integration (V.9) and obtain an expression for $A(q^2)$ in which the singularities in the q^2 complex plane are clearly exhibited.

There are of course no poles in the $\pi\,\pi$ case, and all the branch points turn out to lie on the real axis [Chew & Mandelstam[15]]. There is a branch point at $q^2 = 0$, the threshold of the physical region, another at $q^2 = 3\mu^2$, the threshold for producing two additional pions, and so on. It is convenient, then, to choose a cut running along the positive real axis from 0 to ∞. On the negative real axis there is a corresponding set of branch points, the first occurring at $q^2 = -\mu^2$, the second at $q^2 = -4\mu^2$, and so forth, so that a second cut may be chosen to run along the negative real axis from $-\infty$ to $-\mu^2$.

In more complicated channels the branch points may not all lie on the real axis, but their positions can be determined by inspection of (V.1) after the projection is carried out. There are in general three cuts, corresponding to the three channels of a single diagram, but when two or more identical particles appear in the same diagram there may be a coincidence of the singularities arising from different channels. Such a coincidence occurs in the $\pi\,\pi$ problem just described, where the left-hand cut covers a superposition of two sets of branch points. The results for $\pi + N \leftrightarrow \pi + N$ have been given by McDowell[38], for $\pi + \pi \leftrightarrow N + \bar{N}$ by Fulco[20], for $N + N \leftrightarrow N + N$ by Noyes & Wong[39], and for $\gamma + \pi \leftrightarrow \pi + \pi$ by H. Wong.[22]

The partial-wave amplitudes may be expressed in each case in terms of integrals along the cuts, where the integrand is the discontinuity in going across the cut and may be written in terms of the imaginary part of the physical amplitude for the channel which gives rise to the cut in question. It is possible, therefore, to consider these relations as coupled integral equations which determine the dynamics of the system.

To illustrate the situation consider the S-wave part of the amplitude $\underline{A}$ in pion-pion scattering. This amplitude, by projection according to (V.9) from (V.1), satisfies the dispersion relation

$$\underline{A}_0(\underline{q}^2) = \frac{1}{\pi} \int_{\mu^2}^{\infty} d\underline{q}'^2 \frac{\rho_0(\underline{q}'^2)}{+\underline{q}'^2 + \underline{q}^2} + \frac{1}{\pi} \int_0^{\infty} d\underline{q}'^2 \frac{\mathrm{Im}\, \underline{A}_0(\underline{q}'^2)}{\underline{q}'^2 - \underline{q}^2}, \qquad (\mathrm{V.10})$$

where

$$\rho_0(\underline{q}'^2) = 2 \int_0^{+\underline{q}'^2 - \mu^2} \frac{d\underline{q}''^2}{\underline{q}'^2}\, \mathrm{Im}\, \underline{B}(\underline{q}''^2, 1 - 2\frac{\underline{q}'^2 - \mu^2}{\underline{q}''^2}) . \qquad (\mathrm{V.11})$$

The other partial $\underline{A}$ amplitudes, as well as those of $\underline{B}$ and $\underline{C}$, satisfy similar relations, and by taking the linear combinations (II.7) one can form dispersion relations for partial waves of well-defined isotopic spin. At this stage the imaginary part on the right-hand cut, at least for $0 < \underline{q}^2 < 3\mu^2$, can be very simply expressed in terms of the unitarity condition. That is, since according to (II.9),

$$\underline{A}_\ell^{\,I}(\underline{q}^2) = \frac{\underline{q}^2 + \mu^2}{\underline{q}^2}\, e^{i\delta_\ell^{\,I}} \sin \delta_\ell^{\,I} , \qquad (\mathrm{V.12})$$

with $\delta_\ell^{\,I}$ real in this interval, it follows that

$$\underline{\text{Im}}\ \underline{A}_\ell{}^I(\underline{q}^2) = \sqrt{\frac{\underline{q}^2}{\underline{q}^2+\underline{\mu}^2}}\ \left|\underline{A}_\ell{}^I(\underline{q}^2)\right|^2 . \qquad \text{(V. 13)}$$

Beyond $\underline{q}^2 = 3\,\underline{\mu}^2$, it is necessary to include inelastic processes in the expression for the imaginary part of the amplitude.

The expression (V. 11) for the contribution from the cut along the negative real axis involves the imaginary part of the pion-pion amplitude for $\cos\underline{\theta} < -1$. By inspection of the boundaries in Figure 5 it can be shown that the polynomial expansion of the imaginary part, for the values of $\cos\underline{\theta}$ required in (V. 11), converges for $\underline{q}'^2 < 9\underline{\mu}^2$. [Chew & Mandelstam(15)]. There is no difficulty then in representing the function $\rho_0(q'^2)$ up to this point; beyond it new techniques, such as suggested by Mandelstam(35), must be used. These techniques are too complicated to be described here.

Attempts are currently being made to solve the pion-pion equations in a low-energy approximation in which only S and P waves and only the lowest branch points are considered [Chew & Mandelstam(15)]. The latter simplification corresponds to the neglect of inelastic processes and allows the use of (V. 13) throughout the physical region. The former permits an elementary calculation of the contribution from the left-hand (unphysical) cut.

The equations to be solved contain one free parameter, which may be called the pion-pion coupling constant. It is introduced conveniently as the value of the amplitude at the point $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = \underline{\xi} = \frac{4}{3}\underline{\mu}^2$, where the three amplitudes $\underline{A}$, $\underline{B}$ and $\underline{C}$ are all real and equal to one another. Precisely, we may define

$$\frac{\lambda}{3} = \underline{A}(\underline{\xi},\underline{\xi},\underline{\xi}) = \underline{B}(\underline{\xi},\underline{\xi},\underline{\xi}) = \underline{C}(\underline{\xi},\underline{\xi},\underline{\xi}) . \qquad \text{(V. 14)}$$

It is clear that at least one arbitrary constant is needed in the elastic approximation because the equations permit as a solution an amplitude that is zero everywhere. It is not known whether an arbitrary constant would be necessary in a calcuation that included inelastic processes, such as $\pi + \pi \rightarrow N + \bar{N}$; however, it will be so difficult to calculate such high-energy effects accurately that in practice λ will surely play the role of an "independent" constant for a long time to come. At the time of this writing it is known only from the absence of a 2π bound state that $0 \leq \lambda \leq 1$, but experimental efforts are under way which should soon yield some information.

Attempts are also being made to solve the integral equations resulting from the application of Mandelstam's representation to Figures 1(b) and 1(c), and in the final section the relationships of the different channels and their current status of understanding are surveyed. Mandelstam has shown[36] that the results of conventional perturbation theory can be reproduced by iteration of his integral equations, therefore there is a strong inclination to believe that they represent a complete dynamical framework, given the masses and conventional coupling constants. Of course these highly nonlinear equations, if they can be solved at all, must be applied to large coupling constants for which the perturbation series is meaningless. Whether the equations have unique solutions in such a situation is not known. Perhaps they have no solutions at all except for certain definite values of the masses and coupling constants!

VI. SUMMARY AND CONCLUSION

The reader may wonder why so few concrete results have been given in this review. The reason is that in the author's opinion the results obtained to date are relatively insignificant compared with what will be forthcoming in the next year or two. The power of the generalized dispersion relations, when supplemented by unitarity, has only recently been recognized, and theoretical attempts to utilize this power are in their infancy.

It is true that a large literature on dispersion relations already exists, but this is based almost entirely on fixed momentum-transfer relations which contain only a part of the story. All questions investigated to date will surely be reexamined within the more general framework and a vast clarification is guaranteed. The current literature is filled with confusion about "subtractions" and extensions in the momentum-transfer variable that we see no point in propagating further in this review.

It is possible already to see the outline of a general line of attack on the pion-nucleon problem that should go quite a distance toward answering the conventional questions. The starting point must be pion-pion scattering, where, as explained above, one can hope to calculate the amplitude up to $\underline{q}^2 \sim 3\underline{\mu}^2$ in terms of a single constant λ. Next one would go to the two channels of Figure 1(b), $\underline{\pi} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$ and $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\bar{N}}$ which in such an approach must be considered simultaneously and for which the $\underline{\pi\pi}$ interaction must already be known.

One of the most misleading aspects of the history of pion-nucleon theory is the partial success of attempts to understand low-energy pion nucleon scattering without any inclusion of a pion-pion interaction Chew & Low[29]]. Such success appears now to be largely accidental; it had the beneficial effect of reviving interest in field theory for strong-coupling phenomena, but if pion-pion scattering at very low kinetic energies were as strong as it must be at higher energies, simple models of the pion-nucleon interaction would not work. These models, of course, have never even pretended to answer such basic questions as why the S-wave pion-nucleon phase shifts are small.

It is perhaps worth spelling out the interrelation of the three processes $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{\pi} + \underline{\pi}$, $\underline{\pi} + \underline{N} \leftrightarrow \underline{\pi} + \underline{N}$, and $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$ in the Mandelstam framework. If one derives dispersion relations for pion-nucleon partial waves then there are two left-hand unphysical cuts, one corresponding to pion-nucleon scattering itself and one to the channel $\underline{\pi} + \underline{\pi} — \underline{N} + \underline{\overline{N}}$. Keeping only the former leads to integral equations roughly of the kind proposed by Chew & Low(29), provided the inelastic branch points are ignored.[35]

The nearest portion of the other cut requires a knowledge of the amplitude for $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$ at energies for this process between $\underline{2\mu}$ and $\underline{4\mu}$. Such an energy region is unphysical, and fortunately so, because if the dispersion relations for this amplitude are derived the contribution from the right-hand cut in the corresponding interval is controlled entirely by pion-pion elastic scattering. Precisely, the unitarity condition for this interval is that the phase of a partial-wave amplitude for $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$ is the same as the phase of the corresponding elastic pion-pion amplitude. This information, together with a knowledge of the contribution from the left-hand cuts,[36] is sufficient to determine the amplitude for $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\overline{N}}$, provided always that higher branch points are neglected.

The coupled integral equations that must be solved in carrying out Mandelstam's program are complicated but apparently manageable with fast electronic computers. There is reason to hope, then, that low-energy pion-nucleon scattering can be roughly calculated in terms of the two constants $\underline{\lambda}$ and $\underline{g}^2$. The neglect of higher branch points of course limits the accuracy of the calculation and precludes a treatment of high-energy scattering by this method.

It is perhaps worth emphasizing the philosophy behind the approximation of neglecting high-energy singularities. The underlying motivation lies in the property of an analytic function that its behavior in a small region is dominated by near-by singularities. The dispersion relations make this feature very clear, since they resemble Coulomb's law for a static potential produced by point charges (poles) and line charges (branch cuts). Faraway charges produce at most a slowly varying potential in a local region; strong variations of potential are produced by near-by charges. It is obvious that in strong-coupling problems no calculation can be exact; some approximation must be made. A program of successive approximations based on the distance of singularities from the region of interest seems to the author more plausible than any other procedure yet proposed.

Many valuable theoretical by-products would flow from a successful integration of Mandelstam's equations for Figure 1(b). A knowledge of the amplitude $\underline{\pi} + \underline{\pi} \leftrightarrow \underline{N} + \underline{\bar{N}}$ would allow at long last a correct calculation of the two-pion exchange contribution to the nucleon-nucleon interaction. Precisely, one may derive [Noyes & Wong[39]] dispersion relations for partial-wave nucleon-nucleon elastic-scattering amplitudes, where the two left-hand cuts are coincident and associated with the process $\underline{N} + \underline{\bar{N}} \leftrightarrow \underline{N} + \underline{\bar{N}}$. The nearest contribution comes from the single-pion intermediate state in this process, which is determined entirely by g^2. The next contribution is from the 2π state and is known as soon as one knows the amplitude for $N + \overline{N} \longleftrightarrow 2\pi$. There is some reason to believe that inclusion of these two singularities will allow a rough calculation of low-energy nucleon-nucleon phase shifts without any new parameters. If the faraway left-hand singularities are represented by an adjustable constant,[37] one may hope to achieve an accurate theory. It should be emphasized that in the solution of the nucleon-nucleon integral equations, the deuteron pole will appear automatically; it does not have to be inserted as an independent entity. Thus one expects to calculate the binding energy and quadrupole moment of the deuteron, as well as the triplet effective range, in terms of "fundamental" constants.

A second application of the amplitude for $N + \overline{N} \longleftrightarrow 2\pi$ is to the problem of nucleon electromagnetic structure [Chew, et al.[40], Federbush, et al.[41]]. Here, in conjunction with the vertex function for $\gamma \longrightarrow 2\pi$, this amplitude determines the structure and magnitude of the anomalous magnetic moment. The pion-photon vertex function can easily be calculated once the pion-pion scattering amplitude is available.

Also immediately calculable in terms of π-π scattering is the amplitude for $\gamma + \pi \longleftrightarrow 2\pi$, which is needed in the problem $\gamma + N \longleftrightarrow \pi + N$. One of the two left-hand cuts in the latter case involves photopion production itself but the other requires $\gamma + \pi \longleftrightarrow N + \overline{N}$, which in turn involves $\gamma + \pi \longleftrightarrow 2\pi$. It should be possible, then, to put the theory of low energy photopion production from nucleons on a sound basis.

The chain does not end here. With a proper understanding of photopion production one can calculate photon-nucleon elastic scattering, and this latter amplitude may allow a calculation of the neutron-proton mass difference [Cini, et al.[42]]. Similarly the charged-neutral pion mass difference may be calculable in terms of $\underline{\gamma} + \underline{\pi} \leftrightarrow \underline{\gamma} + \underline{\pi}$, which in turn depends on $\underline{\gamma} + \underline{\pi} \rightarrow 2\underline{\pi}$. The chance of achieving quantitatively reliable results from mass calculations is, however, much smaller than that for scattering amplitudes.

Even if, as is unlikely, the calculations outlined here were to yield good results in terms of the four constants, $\underline{\mu}$, $\underline{M}$, $\underline{\lambda}$ and $\underline{g}^2$, it must not be supposed that all questions would have been answered. Why should these constants have the particular values that are observed? Why are nucleons and pions the only nonstrange strongly interacting "elementary" particles? Why is the pion pseudoscalor? There is no understanding yet of such questions, and if we start to consider the hyperons and K particles the number of puzzles multiplies. Exciting as the prospects are for dynamical calculations with the Mandelstam representation, it must be remembered that these calculations are based on conventional field theory, just as it was invented 30 years ago, and that the breakthrough which will tell us the origin of elementary particles has not been achieved.

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FOOTNOTES

Page 4

[1]It will be assumed that for the strong coupling phenomena with which we are concerned, charge conjugation invariance and parity conservation are separately valid, as well as charge independence.

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[2]We shall refer to each possible pairing as a "channel". For each diagram there are three channels.

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[3]See, for example, Jauch & Rohrlich (14), p. 161.

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[4]Note that such an exchange is consistent with the constraint (II.3).

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[5]For a more complete discussion of the π-π problem, see Chew & Mandelstam(15).

[6]See, for example, Bethe & de Hoffman (16), p. 49.

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[7]The so-called G parity of Lee & Yang (17), which for states containing only pions is even or odd depending on whether the total number of pions is even or odd. States with nonzero baryon number generally do not have well-defined G parity.

[8]See, for example, the review by Chew (19).

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[9]The more general rule is stated in Sec. III of Reference 24.

[10]Since we are neglecting weak and electromagnetic interactions, all the usually discussed "elementary" particles are to be counted as stable.

[11]The photon pole for charged particles manifests itself in the Coulomb part of the amplitude that becomes infinite at zero momentum transfer.

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[12]A generalization of this rule for processes involving more than four particles is given in Reference 24.

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[13]See, for example, Symanzik (6).

[14]The pion-nucleon coupling constant also occurs in the residues of poles for processes involving more than four particles. See, for example, Chew and Low (24).

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[15]When electromagnetic effects are considered, one must define three constants, one for the processes $\pi^+ + n \leftrightarrow p$ and $n \leftrightarrow p + \pi^-$, one for the process $\pi^0 + p \leftrightarrow p$, and one for $\pi^0 + n \leftrightarrow n$. These three constants are expected to differ by a few percent.

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[16]Lehmann (12) has given a rigorous proof of analyticity properties in the momentum-transfer variable that almost, but not quite, guarantees the domain of analyticity required here.

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[17]We shall also see in the next section that the nearest branch points lie relatively close to the nucleon poles and further add to the diffficulty of extrapolation in these cases.

[18]The imaginary part of the amplitude has only branch points, which in the next section will be seen to be further from the physical region than the nearest branch points in the real part.

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[19]The residue $\underline{\lambda}$ is proportional to $\underline{g}^2$ in the case of the poles Nos. 4 and 5a and to $\underline{eg}$ in the case of (8a).

[20]With internal degrees of freedom there will be generally more than one scalar amplitude, but since all have the same properties of analyticity, the procedure outlined is still valid.

[21]The question of how high an order of polynomial should be used depends on the energy, the angular interval of the experiment, and the accuracy, as well as the location of the nearest branch point. The most careful study of this question to date has been by Cziffra & Moravcsik (26) and by Frazer. (27)

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[22]Extrapolations to poles in angular distributions can be and are being carried out for many processes not considered here because they involve strange or complex particles or more than four particles all together. The basic principles involved are always the same.

[23]An example of the effective-range type of extrapolation is that proposed by Chew & Low (29) in connection with P-wave pion-nucleon scattering. The Chew-Low procedure was very crude, however, since (among other circumstances) there really is no pole in the amplitude they considered. These authors approximated a pair of neighboring branch points by a simple pole, a procedure that is exact only for an infinitely heavy nucleon. Also, singularities associated with the pion-pion interaction were ignored.

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[24]In practice as seen in Table I the particular channels with which we are concerned in the pion-nucleon problem have at most one pole each.

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[25]See, for example, Reference 19, Sec. 40.

[26]The necessity for subtractions in dispersion relations is discussed in a systematic way by Bogoliubov et al. (7), p. 5

[27]See the discussion below, following equation (V.8).

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[28]Note carefully the difference between ${}^{3}A^{s_2}(s_3)$ and $A_3^{s_2}(s_3)$. The former is the complete amplitude, while the latter is only one of two parts. Their imaginary parts are the same, but not their real parts.

[29]Here one may use the rule $\frac{1}{s' - (s_3 + i\epsilon)} = P \frac{1}{s' - s_3} + i\pi\delta(s' - s_3)$,

where P signifies that the principal value of the integral is to be taken.

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[30]See the review by Goldberger (11).

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[31]Pion-pion scattering has thus far eluded direct observation because of the relatively short lifetime of the particles.

[32]See, for example, Schnitzer & Salzman (32). References to other work on the verification of the forward π-N relation can be found in these articles.

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[33] These subtractions do not correspond to the introduction of new arbitrary constants if they are made in only one variable. See Mandelstam(36).

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[34] The nearest right-hand branch point (or left) is given by the equation $\underline{s}_1(\cos \underline{\theta}, \underline{s}_3) = \underline{s}_{10}$ and the nearest left-hand branch point (or right) by $\underline{s}_2(\cos \underline{\theta}, \underline{s}_3) = \underline{s}_{20}$, if the fixed variable is $\underline{s}_3$.

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[35] To evaluate the contribution of the left-hand pion-nucleon cut an extension to $\cos \underline{\theta} < -1$ is required, just as in the pion-pion problem, and it may be necessary to introduce a cutoff if this extension is carried out by Legendre polynomials.

[36] The two left-hand cuts here are coincident, both being associated with pion-nucleon scattering.

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[37] This constant may be thought of as equivalent to the hard-core radius of conventional potentials.

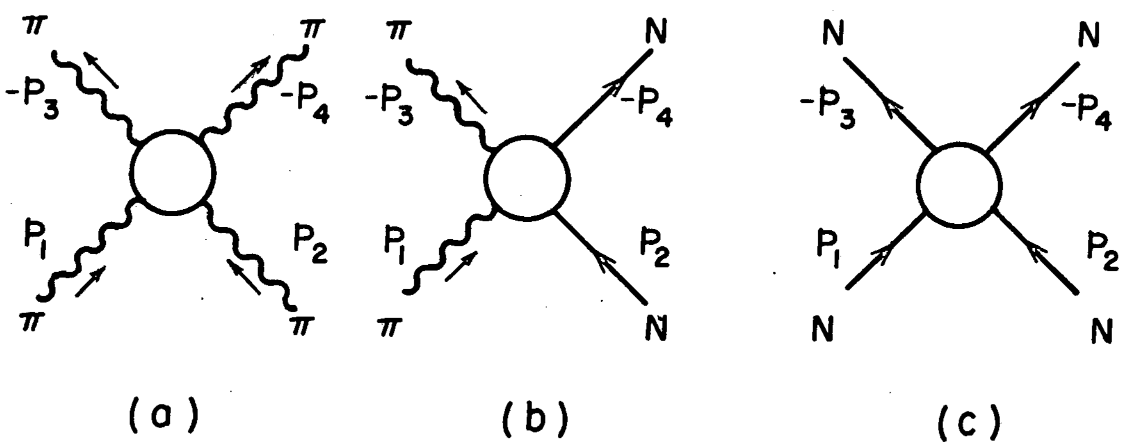

Figure 1. The three fundamental diagrams of the pion-nucleon system. Outgoing particles are assigned negative momenta in accordance with the convention of Section II.

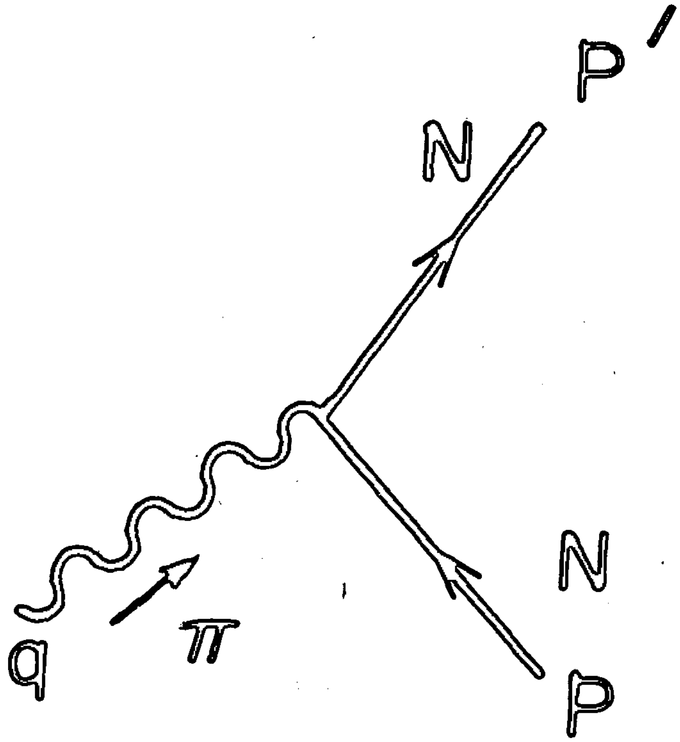

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Figure 2. The pion-nucleon vertex.

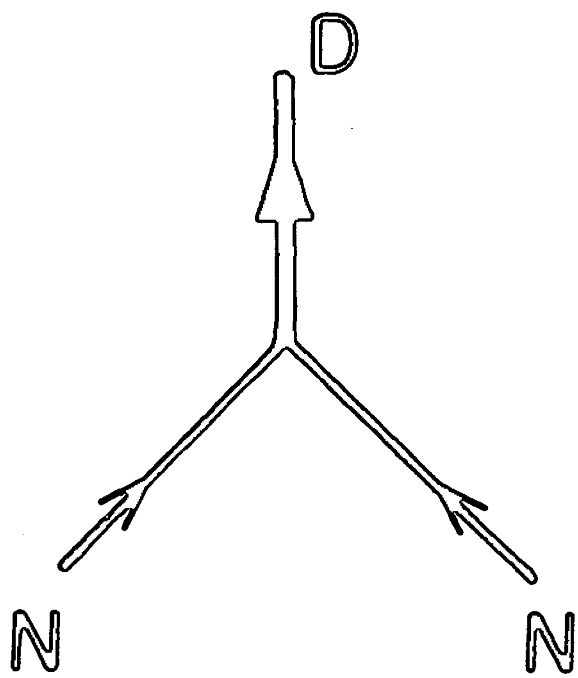

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Fig. 3. The nucleon-deuteron vertex.

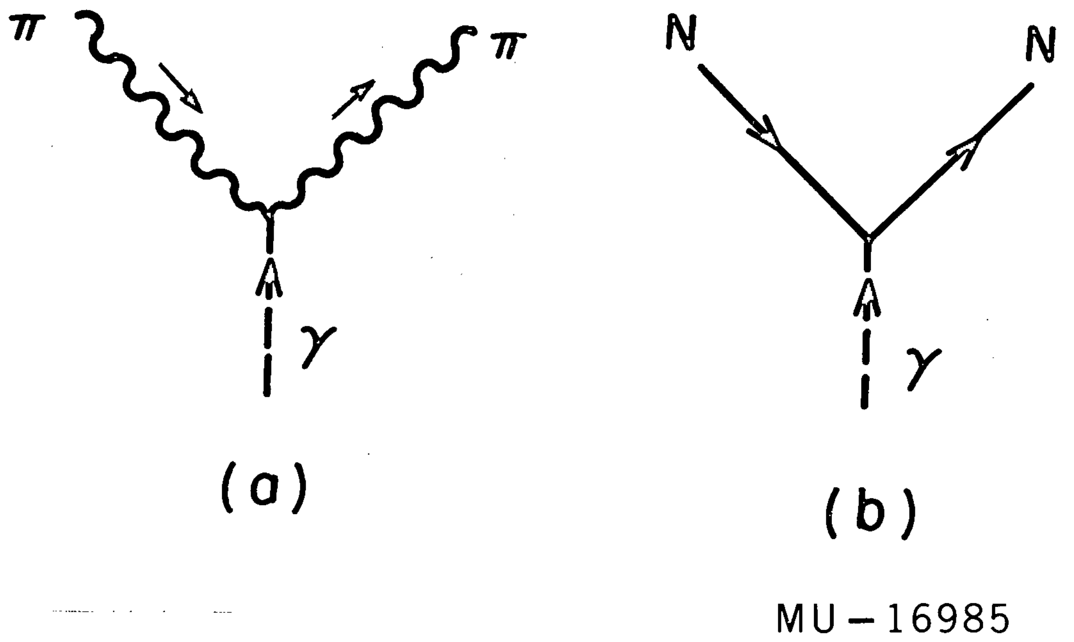

Fig. 4. The pion-photon vertex.

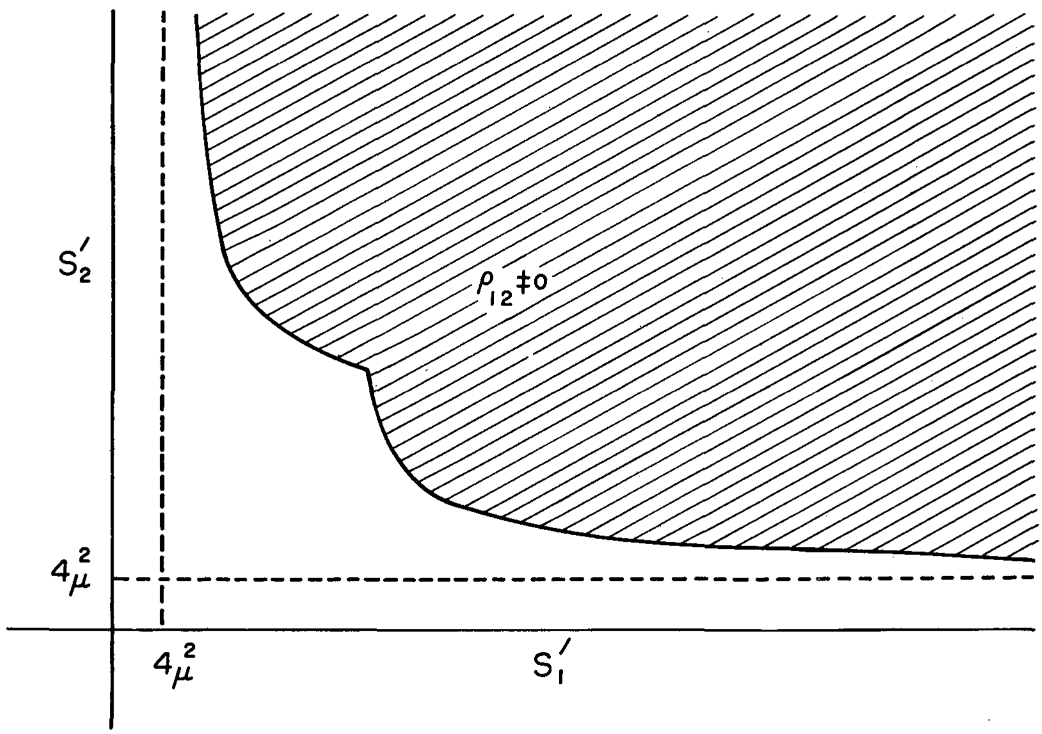

Fig. 5. The nucleon-photon vertex.