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A Class of Incomplete Character Sums *

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Abstract

Using ℓ-adic cohomology of tensor inductions of lisse $\overline{\mathbb{Q}}_\ell$-sheaves, we study a class of incomplete character sums.
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Introduction

Through out this paper, p is a fixed prime number, $\mathbb{F}_p$ is a finite field with p elements, and $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$. For any power q of p, let $\mathbb{F}_q$ be the subfield of $\mathbb{F}$ with q elements. Let ℓ be a prime number distinct from p, let $\chi_1, \ldots, \chi_k : \mathbb{F}_{q^d}^* \to \overline{\mathbb{Q}}_\ell^*$ be a family of multiplicative characters and $\psi : \mathbb{F}_{q^d} \to \overline{\mathbb{Q}}_\ell^*$ an additive character on an extension field $\mathbb{F}_{q^d}$ of $\mathbb{F}_q$. We extend χ_i $(i = 1, \ldots, k)$ to $\mathbb{F}_{q^d}$ by setting $\chi_i(0) = 0$. Let $f_1(t_1, \ldots, t_n), \ldots, f_k(t_1, \ldots, t_n), f_{k+1}(t_1, \ldots, t_n) \in \mathbb{F}_{q^d}[t_1, \ldots, t_n]$ be a family of polynomials with coefficients in $\mathbb{F}_{q^d}$. Motivated by a number of applications in [6], we are interested in estimating the following type of incomplete character sum

$$S' = \sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \chi_1\big(f_1(x_1, \ldots, x_n)\big) \cdots \chi_k\big(f_k(x_1, \ldots, x_n)\big)\psi\big(f_{k+1}(x_1, \ldots, x_n)\big),$$

where the summation is over those $x_1, \ldots, x_n$ in the subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^d}$. Note that for the classical complete character sum

$$S = \sum_{x_1,\ldots,x_n \in \mathbb{F}_{q^d}} \chi_1\big(f_1(x_1, \ldots, x_n)\big) \cdots \chi_k\big(f_k(x_1, \ldots, x_n)\big)\psi\big(f_{k+1}(x_1, \ldots, x_n)\big),$$

*We would like to thank the referee for suggestion of applying tensor induction to higher rank sheaves. Originally we only treat rank one sheaves using transfer. The research of Lei Fu is supported by NSFC.

the summation is over all $x_1, \ldots, x_n$ in the field $\mathbb{F}_{q^d}$. Under suitable hypothesis, one can give a sharp estimate of the following form for the complete sum

$$|S| \leq C(n, \deg(f_i))q^{\frac{nd}{2}},$$

where $C(n, \deg(f_i))$ is a constant depending only on n and the degrees of the functions f_i's. We believe that there is also a sharp estimate for the incomplete sum S'. Namely, under suitable hypothesis, there should be an estimate of the form

$$|S'| \leq C(n, d, \deg(f_i))q^{\frac{n}{2}},$$

where $C(n, d, \deg(f_i))$ is a constant depending only on n, d and the degrees of f_i. This is indeed true in the one variable case $n = 1$, see [6]. In the present paper, we use tensor induction and ℓ-adic cohomology theory to study this problem.

Let us recall the general method of studying (complete) character sum using ℓ-adic cohomology theory. Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$, let $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ be the base change of X_0, let $\mathcal{L}_0$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0, and let $\mathcal{L}$ be the inverse image of $\mathcal{L}_0$ in X. The family of complete character sums corresponding to $\mathcal{L}_0$ is the family of sums

$$S_m(X_0, \mathcal{L}_0) = \sum_{x \in X_0(\mathbb{F}_{q^m})} \mathrm{Tr}(F_x, \mathcal{L}_{0,\bar{x}}),$$

where $X_0(\mathbb{F}_{q^m}) = \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{F}_q}(\mathrm{Spec}\,\mathbb{F}_{q^m}, X_0)$ is the set of $\mathbb{F}_{q^m}$-points in X_0, $\mathcal{L}_{0,\bar{x}}$ is the stalk of $\mathcal{L}_0$ at a geometric point over x, and $F_x \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^m})$ is the geometric Frobenius element at x, that is, the inverse of the Frobenius substitution. By Grothendieck's trace formula ([1, Rapport 3.2]), we have

$$S_m(X_0, \mathcal{L}_0) = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}\big(F^m, H_c^i(X, \mathcal{L})\big),$$

where $H_c^i(X, \mathcal{L})$ are ℓ-adic cohomology groups with compact support and F is the geometric Frobenius correspondence. Deligne's theorem [3, Corollarie 3.3.4] gives an estimate for the Archimedean absolute values of the eigenvalues of F on $H_c^i(X, \mathcal{L})$. If we have some information about the Betti numbers $\dim H_c^i(X, \mathcal{L})$, then we can get an estimate for $S_m(X_0, \mathcal{L}_0)$.

Let $\mathbb{F}_{q^d}$ be an extension of $\mathbb{F}_q$ of degree d, let $X_1 = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d}$ be the base change of X_0, and let $\mathcal{L}_1$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_1. For any $\mathbb{F}_q$-point $x : \mathrm{Spec}\,\mathbb{F}_q \to X_0$ of X_0, let $x_1 : \mathrm{Spec}\,\mathbb{F}_{q^d} \to X_1$ be its base change. It is a $\mathbb{F}_{q^d}$-point of X_1. In this paper, we are concerned with the following incomplete character sum

$$S_1'(X_1,\mathcal{L}_1)=\sum_{x\in X_0(\mathbb{F}_q)}\mathrm{Tr}(F_{x_1},\mathcal{L}_{1,\bar{x}_1}),$$

where the summation is over those $\mathbb{F}_{q^d}$-points of X_1 arising from $\mathbb{F}_q$-points of X_0 by base change. In section 1, we describe the tensor induction $\otimes$-$\mathrm{Ind}(\mathcal{L}_1)$ of $\mathcal{L}_1$ ([4, 10.3-5]), which is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1 on X_0. We show that the above incomplete sum $S_1'(X_1,\mathcal{L}_1)$ coincides with the following complete character sum

$$S_1(X_0,\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))=\sum_{x\in X_0(\mathbb{F}_q)}\mathrm{Tr}(F_x,(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))_{\bar{x}}).$$

In principle, this reduces the study of our incomplete sum to the study of the complete sum for the new sheaf $\otimes$-$\mathrm{Ind}(L_1)$. We describe explicitly the tensor induction for Kummer type sheaves in §2, and for Artin-Schreier type sheaves in §3. In general, it is hard to estimate the Betti numbers $\dim H^i_c(X,\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))$. In §4, we study the case where $\dim(X)=1$. In this case, the Grothendieck-Ogg-Shararevich formula for Euler characteristic numbers of sheaves on curves gives us more information about the Betti numbers. We can thus get relatively complete results (Theorems 4.5-6).

1 Tensor Induction

Let G be a pro-finite group, let H be an open subgroup of G of finite index d, and let $\rho: H\to \mathrm{GL}(V)$ be a (continuous) representation of H, where V is a finitely dimensional vector space over $\overline{\mathbb{Q}}_\ell$. Choose representatives $Hg_1,\ldots,Hg_d$ for the family of right cosets $H\backslash G$. For any $g\in G$, the set $\{Hg_1g,\ldots,Hg_dg\}$ is also a family of representatives of right cosets. So we have

$$Hg_ig=Hg_{\tau(i)}$$

for a permutation $i\mapsto\tau(i)$ of $\{1,\ldots,d\}$. Define

$$\otimes\text{-}\mathrm{Ind}(\rho):G\to\mathrm{GL}(\otimes^d V)$$

by

$$\Big((\otimes\text{-}\mathrm{Ind}(\rho)(g)\Big)(v_1\otimes\cdots\otimes v_d)=\rho(g_1gg_{\tau(1)}^{-1})(v_{\tau(1)})\otimes\cdots\otimes\rho(g_dgg_{\tau(d)}^{-1})(v_{\tau(d)})$$

for any $v_1,\ldots,v_d\in V$. One can show that $\otimes$-$\mathrm{Ind}(\rho)$ is a (continuous) representation of G, and its isomorphism class is independent of the choice of representatives of right cosets. We call $\otimes$-$\mathrm{Ind}(\rho)$ the *tensor induction* of ρ. For more details about the tensor induction, see [2, §13] and [4, 10.3-5]. In the following, we summarize the properties of tensor induction which are used in the paper.

Proposition 1.1. *Let G be a pro-finite group, let H be an open subgroup of G of finite index d, and let $\rho : H \to \mathrm{GL}(V)$ be a representation of H.*

(i) Suppose that H is normal in G. Let $\{Hg_1, \ldots, Hg_d\}$ be a family of representatives for $H\backslash G$, and let $\rho^{(i)} : H \to \mathrm{GL}(V)$ $(i = 1, \ldots, d)$ be the representations defined by $\rho^{(i)}(h) = \rho(g_i h g_i^{-1})$ for any $h \in H$. Then for any $h \in H$, we have

$$\otimes\text{-Ind}(\rho)(h) = \otimes_{i=1}^d \rho^{(i)}(h).$$

(ii) Suppose that H is normal in G and G/H is a cyclic group. For any $\sigma \in G$ such that σH is a generator of G/H, we have

$$\mathrm{Tr}(\otimes\text{-Ind}(\rho)(\sigma)) = \mathrm{Tr}(\rho(\sigma^d)).$$

Proof. (i) follows directly from the definition of the tensor induction. To prove (ii), use $He, H\sigma, \cdots, H\sigma^{d-1}$ as representatives for right cosets. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then $\{e_{j_1} \otimes \cdots \otimes e_{j_d} | 1 \leq j_1, \ldots, j_d \leq n\}$ is a basis of $\otimes^d V$. We have

$$\otimes\text{-Ind}(\rho)(\sigma)(e_{i_1} \otimes \cdots \otimes e_{i_d}) = e_{i_2} \otimes \cdots \otimes e_{i_d} \otimes \rho(\sigma^d)(e_{i_1}).$$

Suppose $\rho(\sigma^d)(e_i) = \sum_{j=1}^n a_{ij} e_j$. In the expansion of $\otimes\text{-Ind}(\rho)(\sigma)(e_{i_1} \otimes \cdots \otimes e_{i_d})$ as a linear combination of $\{e_{j_1} \otimes \cdots \otimes e_{j_d} | 1 \leq j_1, \ldots, j_d \leq n\}$, the coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_d}$ is nonzero only when $i_1, \ldots, i_d$ are equal to a common i, and in this case, the coefficient is a_{ii}. It follows that

$$\mathrm{Tr}(\otimes\text{-Ind}(\rho)(\sigma)) = \sum_i a_{ii} = \mathrm{Tr}(\rho(\sigma^d)).$$

□

Let X_0 be a scheme of finite type over $\mathbb{F}_q$, and let $X_1 = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d}$ be its base change to an extension field $\mathbb{F}_{q^d}$ of degree d over $\mathbb{F}_q$. Suppose X_0 is geometrically connected, that is, the base change $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ is connected. Fix a geometric point for X. Take its images in X_0 and X_1 as the base points of X_0 and X_1, respectively, and let $\pi_1(X_0)$ and $\pi_1(X_1)$ be the étale fundamental groups with respect to these base points. Then $\pi_1(X_1)$ is a normal subgroup of $\pi_1(X_0)$. We have an isomorphism

$$\pi_1(X_0)/\pi_1(X_1) \stackrel{\cong}{\to} \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q),$$

and $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ is a cyclic group. Let $x : \mathrm{Spec}\, \mathbb{F}_{q^m} \to X_0$ be an $\mathbb{F}_{q^m}$-points in X_0. We can talk about the geometric Frobenius element F_x in $\pi_1(X_0)$ corresponding to x. It is defined up to conjugation in

$\pi_1(X_0)$. Consider the case where $m = 1$. Then the image of F_x in the quotient group $\pi_1(X_0)/\pi_1(X_1)$ is a generator. Let $x_1 : \mathrm{Spec}\,\mathbb{F}_{q^d} \to X_1$ be the base change of x. One can verify that F_x^d and F_{x_1} define the same conjugacy class in $\pi_1(X_1)$.

Let $\mathcal{L}_1$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_1 of rank r. It defines a representation

$$\rho_1 : \pi_1(X_1) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell).$$

Regard $\pi_1(X_1)$ as a subgroup $\pi_1(X_0)$, and consider the tensor induction

$$\otimes\text{-Ind}(\rho_1) : \pi_1(X_0) \to \mathrm{GL}(r^d, \overline{\mathbb{Q}}_\ell).$$

We define the *tensor induction* $\otimes$-Ind$(\mathcal{L}_1)$ of $\mathcal{L}_1$ to be the lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 of rank r^d corresponding to the last representation. By the above discussion and Proposition 1.1 (ii), if $x : \mathrm{Spec}\,\mathbb{F}_q \to X_0$ is an $\mathbb{F}_q$-point in X_0 and $x_1 : \mathrm{Spec}\,\mathbb{F}_{q^d} \to X_1$ is its base change, then we have

$$\mathrm{Tr}\big(F_x, (\otimes\text{-Ind}(\mathcal{L}_1))_{\bar{x}}\big) = \mathrm{Tr}(F_{x_1}, \mathcal{L}_{1,\bar{x}_1}). \tag{1}$$

Proposition 1.2. *Keep the notation above.*

(i) The incomplete character sum

$$S_1'(X_1, \mathcal{L}_1) = \sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}(F_{x_1}, \mathcal{L}_{1,\bar{x}_1})$$

coincides with the complete character sum

$$S_1(X_0, \otimes\text{-Ind}(\mathcal{L}_1)) = \sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}\big(F_x, (\otimes\text{-Ind}(\mathcal{L}_1))_{\bar{x}}\big).$$

(ii) Let $\rho_1 : \pi_1(X_1) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell)$ be the representation corresponding to the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_1$, and let $\sigma \in \pi_1(X_0)$ be an element such that its image under the homomorphism $\pi_1(X_0) \to \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ is the Frobenius substitution. For each $0 \le i \le d-1$, let $\rho_1^{(i)} : \pi_1(X_1) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell)$ be the representation defined by

$$\rho_1^{(i)}(g) = \rho_1(\sigma^i g \sigma^{-i}),$$

and let $\pi : X_1 \to X_0$ be the projection. Then the representation corresponding the sheaf $\pi^*(\otimes\text{-Ind}(\mathcal{L}_1))$ on X_1 is $\otimes_{i=0}^{d-1} \rho_1^{(i)}$.

(iii) Let $\mathcal{L}_1$ and $\mathcal{M}_1$ be two lisse $\overline{\mathbb{Q}}_\ell$-sheaves. Then we have

$$\big(\otimes\text{-Ind}\big)(\mathcal{L}_1 \otimes \mathcal{M}_1) \cong \Big(\otimes\text{-Ind}(\mathcal{L}_1)\Big) \otimes \Big(\otimes\text{-Ind}(\mathcal{M}_1)\Big).$$

Proof. (i) follows from Equation (1). (ii) follows from Proposition 1.1 (i). (iii) follows from the definition of tensor induction. ∎

2 Tensor Induction of Kummer type sheaves

The Kummer covering

$$[q-1]:\mathbb{G}_{m,\mathbb{F}_q}\to\mathbb{G}_{m,\mathbb{F}_q},\quad x\mapsto x^{q-1}$$

on $\mathbb{G}_{m,\mathbb{F}_q}=\operatorname{Spec}\mathbb{F}_q[t,t^{-1}]$ defines a $\mathbb{F}_q^*$-torsor

$$1\to\mathbb{F}_q^*\to\mathbb{G}_{m,\mathbb{F}_q}\stackrel{[q-1]}{\to}\mathbb{G}_{m,\mathbb{F}_q}\to 1.$$

Let $\chi:\mathbb{F}_q^*\to\overline{\mathbb{Q}}_\ell^*$ be a multiplicative character. Pushing-forward the above torsor by χ^{-1}, we get a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{K}_\chi$ on $\mathbb{G}_{m,\mathbb{F}_q}$ of rank 1. We call $\mathcal{K}_\chi$ the *Kummer sheaf* associated to χ. For any $x\in\mathbb{G}_{m,\mathbb{F}_q}(\mathbb{F}_{q^m})=\mathbb{F}_{q^m}^*$, we have

$$\operatorname{Tr}(F_x,\mathcal{K}_{\chi,\bar{x}})=\chi(\mathrm{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(x)),$$

where $\mathrm{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ denotes the norm for the field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$.

Let $f\in\Gamma(X_0,\mathcal{O}_{X_0}^*)$ be a section of the subsheaf of units in the structure sheaf $\mathcal{O}_{X_0}$. Then the $\mathbb{F}_q$-algebra homomorphism

$$\mathbb{F}_q[t,t^{-1}]\to\Gamma(X_0,\mathcal{O}_{X_0}),\quad t\mapsto f$$

defines an $\mathbb{F}_q$-morphism $X_0\to\mathbb{G}_{m,\mathbb{F}_q}$ which we still denote by f. Denote $f^*\mathcal{K}_\chi$ by $\mathcal{K}_{\chi,f}$. It is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank one on X_0. For any $x\in X_0(\mathbb{F}_{q^m})$, we have

$$\operatorname{Tr}(F_x,(\mathcal{K}_{\chi,f})_{\bar{x}})=\chi\big(\mathrm{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f(x))\big).$$

For any $f_1,f_2,f\in\Gamma(X_0,\mathcal{O}_{X_0}^*)$ and any multiplicative characters $\chi,\chi_1,\chi_2:\mathbb{F}_q^*\to\overline{\mathbb{Q}}_\ell^*$, we have

$$\mathcal{K}_{\chi,f_1f_2}\cong\mathcal{K}_{\chi,f_1}\otimes\mathcal{K}_{\chi,f_2},\quad \mathcal{K}_{\chi_1\chi_2,f}\cong\mathcal{K}_{\chi_1,f}\otimes\mathcal{K}_{\chi_2,f}.$$

Confer [1, Sommes trig. 1.4-8].

Suppose that X_0 is a normal scheme. Then $\mathcal{K}_{\chi,f}$ can be described as follows. Let K be the function field of X_0. Fix a separable closure of $\overline{K}$ of $K(X_0)$. Let the base point of X_0 be the canonical morphism $\operatorname{Spec}\overline{K}\to X_0$. Then we have a canonical isomorphism

$$\pi_1(X_0)\cong\varprojlim_L\operatorname{Gal}(L/K),$$

where L goes over those finite Galois extensions of K contained in $\overline{K}$ such that the normalization of X_0 in L is etale over X_0. Let $\xi\in\overline{K}$ be a root of the polynomial $T^{q-1}-f$. Then any root of $T^{q-1}-f$

is of the form $a\xi$ for some $a\in\mathbb{F}_q^*$, $K[\xi]$ is a Galois extension, and the normalization of X_0 in $K[\xi]$ is etale over X_0. It follows that we have a canonical epimorphism

$$\pi_1(X_0)\to \mathrm{Gal}(K[\xi]/K).$$

On the other hand, we have a canonical monomorphism

$$\mathrm{Gal}(K[\xi]/K)\hookrightarrow \mathbb{F}_q^*,\quad g\mapsto \frac{g(\xi)}{\xi}.$$

The representation corresponding to $\mathcal{K}_{\chi,f}$ is the composite

$$\pi_1(X_0)\to \mathrm{Gal}(K[\xi]/K)\hookrightarrow \mathbb{F}_q^*\stackrel{\chi^{-1}}{\to}\overline{\mathbb{Q}}_\ell^*.$$

Let $\mathbb{F}_{q^d}$ be an extension of $\mathbb{F}_q$ of degree d, let $\chi:\mathbb{F}_{q^d}^*\to\overline{\mathbb{Q}}_\ell^*$ be a multiplicative character, let $f\in\mathbb{F}_{q^d}(t_1,\ldots,t_n)$ be a rational function, and let X_0 be an open subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$ so that its inverse image in $\mathbb{A}^n_{\mathbb{F}_{q^d}}$ is contained in the complement of the union of hypersurfaces $\{f=0\}\cup\{f=\infty\}$. Let X_1 be the base change of X_0 from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, and let $\pi:X_1\to X_0$ be the projection. Consider the sheaf $\mathcal{K}_{\chi,f}$ on X_1. Let us describe the character $\pi_1(X_1)\to\overline{\mathbb{Q}}_\ell^*$ corresponding to the rank one lisse sheaf $\pi^*(\otimes\text{-Ind}(\mathcal{K}_{\chi,f}))$ on X_1. The function field of X_0 (resp. X_1) is the rational function field $K_0=\mathbb{F}_q(t_1,\ldots,t_n)$ (resp. $K_1=\mathbb{F}_{q^d}(t_1,\ldots,t_n)$). Let ξ be a root of the polynomial $T^{q^d-1}-f$ in a separable closure $\overline{K}_1$ of $K_1=\mathbb{F}_{q^d}(t_1,\ldots,t_n)$. Then $\mathcal{K}_{\chi,f}$ corresponds to the character

$$\pi_1(X_1)\to \mathrm{Gal}(K_1[\xi]/K_1)\hookrightarrow \mathbb{F}_{q^d}^*\stackrel{\chi^{-1}}{\to}\overline{\mathbb{Q}}_\ell^*.$$

Denote this character by $\rho_1:\pi_1(X_1)\to\overline{\mathbb{Q}}_\ell^*$. More explicitly, we have

$$\rho_1(g)=\chi^{-1}\Big(\frac{g(\xi)}{\xi}\Big)$$

for any $g\in\pi_1(X_1)$. Note that $\pi:X_1\to X_0$ is a Galois etale covering space, and we have canonical isomorphisms

$$\mathrm{Aut}(X_1/X_0)^\circ\cong \mathrm{Gal}\Big(\mathbb{F}_{q^d}(t_1,\ldots,t_n)/\mathbb{F}_q(t_1,\ldots,t_n)\Big)\cong \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q).$$

Choose an element $\sigma\in\pi_1(X_0)$ so that its image in $\mathrm{Gal}\Big(\mathbb{F}_{q^d}(t_1,\ldots,t_n)/\mathbb{F}_q(t_1,\ldots,t_n)\Big)$ is given by

$$\sigma(a)=a^q \text{ for any } a\in\mathbb{F}_{q^d},\quad \sigma(t_i)=t_i\ (i=1,\ldots,n).$$

Then the image of σ in $\pi_1(X_0)/\pi_1(X_1)$ is a generator. For any $0\le i\le d-1$, let $\rho_1^{(i)}:\pi_1(X_1)\to\overline{\mathbb{Q}}_\ell^*$ be the character defined by

$$\rho_1^{(i)}(g)=\rho_1(\sigma^i g\sigma^{-i})$$

for any $g \in \pi_1(X_1)$. We have

$$\begin{aligned}\rho_1^{(i)}(g) &= \rho_1(\sigma^i g \sigma^{-i}) \\ &= \chi^{-1}\Big(\frac{\sigma^i g\sigma^{-i}(\xi)}{\xi}\Big) \\ &= \chi^{-1}\Big(\sigma^i\Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big)\Big).\end{aligned}$$

Note that $\sigma^{-i}(\xi)$ and $g\sigma^{-i}(\xi)$ are roots of the polynomial $T^{q^d-1} - \sigma^{-i}(f)$, and hence $\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}$ lies in $\mathbb{F}_{q^d}^*$. We thus have

$$\sigma^i\Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big) = \Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big)^{q^i}.$$

So we have

$$\rho_1^{(i)}(g) = (\chi^{q^i})^{-1}\Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big)$$

for any $g \in \pi_1(X_1)$. This shows that $\rho_1^{(i)} : \pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$ corresponds to the sheaf $\mathcal{K}_{\chi^{q^i},\sigma^{-i}(f)}$. We also have

$$\sigma^i\Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big) = \Big(\frac{g\sigma^{-i}(\xi)}{\sigma^{-i}(\xi)}\Big)^{q^i} = \frac{g\Big(\big(\sigma^{-i}(\xi)\big)^{q^i}\Big)}{\big(\sigma^{-i}(\xi)\big)^{q^i}},$$

and hence

$$\rho_1^{(i)}(g) = \chi^{-1}\left(\frac{g\Big(\big(\sigma^{-i}(\xi)\big)^{q^i}\Big)}{\big(\sigma^{-i}(\xi)\big)^{q^i}}\right)$$

for any $g \in \pi_1(X_1)$. Note that $\big(\sigma^{-i}(\xi)\big)^{q^i}$ is a root of the polynomial $T^{q^d-1} - f(t_1^{q^i}, \ldots, t_n^{q^i})$. This shows that $\rho_1^{(i)} : \pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$ also corresponds to the sheaf $\mathcal{K}_{\chi, f(t_1^{q^i},\ldots,t_n^{q^i})}$. By Proposition 1.2 (ii), $\pi^*(\otimes\text{-Ind}(\mathcal{K}_{\chi,f}))$ corresponds to the character $\prod_{i=0}^{d-1} \rho_1^{(i)}$. So we get the following.

Proposition 2.1. *Let $\chi : \mathbb{F}_{q^d}^* \to \overline{\mathbb{Q}}_\ell^*$ be a multiplicative character, let $f \in \mathbb{F}_{q^d}(t_1, \ldots, t_n)$, and let X_0 be an open subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$ so that its inverse image in $\mathbb{A}^n_{\mathbb{F}_{q^d}}$ is contained in the complement of the union of hypersurfaces $\{f = 0\} \cup \{f = \infty\}$. Let X_1 be the base change of X_0 from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, and let $\pi : X_1 \to X_0$ be the projection. Then on X_1, we have isomorphisms*

$$\pi^*(\otimes\text{-Ind}(\mathcal{K}_{\chi,f})) \cong \bigotimes_{i=0}^{d-1} \mathcal{K}_{\chi^{q^i},\sigma^{-i}(f)} \cong \mathcal{K}_{\chi, \prod_{i=0}^{d-1} f(t_1^{q^i},\ldots,t_n^{q^i})},$$

where $\sigma^{-i}(f)$ is the rational function obtained from f by taking the q^i-th root for each coefficient of the numerator and the denominator of f.

3 Tensor Induction of Artin-Schreier type sheaves

The Artin-Schreier covering

$$\wp : \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}, \quad x \mapsto x^q - x$$

defines an $\mathbb{F}_q$-torsor

$$0 \to \mathbb{F}_q \to \mathbb{A}^1_{\mathbb{F}_q} \xrightarrow{\wp} \mathbb{A}^1_{\mathbb{F}_q} \to 0.$$

Let $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial additive character. Pushing-forward this torsor by ψ^{-1}, we get a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\psi$ of rank 1 on $\mathbb{A}^1_{\mathbb{F}_q}$, which we call the *Artin-Schreier sheaf*. For any $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}) = \mathbb{F}_{q^m}$, we have

$$\mathrm{Tr}(F_x, \mathcal{L}_{\psi,\bar{x}}) = \psi(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(x)),$$

where $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ denotes the trace for the field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$.

Let $f \in \Gamma(X_0, \mathcal{O}_{X_0})$ be a section of the structure sheaf $\mathcal{O}_{X_0}$. Then the $\mathbb{F}_q$-algebra homomorphism

$$\mathbb{F}_q[t] \to \Gamma(X_0, \mathcal{O}_{X_0}), \quad t \mapsto f$$

defines an $\mathbb{F}_q$-morphism $X_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ which we still denote by f. Denote $f^*\mathcal{L}_\psi$ by $\mathcal{L}_{\psi,f}$. It is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank one on X_0. For any $x \in X_0(\mathbb{F}_{q^m})$, we have

$$\mathrm{Tr}(F_x, (\mathcal{L}_{\chi,f})_{\bar{x}}) = \psi\big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f(x))\big).$$

For any $f_1, f_2 \in \Gamma(X_0, \mathcal{O}^*_{X_0})$, we have

$$\mathcal{L}_{\psi,f_1+f_2} \cong \mathcal{L}_{\psi,f_1} \otimes \mathcal{L}_{\psi,f_2}.$$

Confer [1, Sommes trig. 1.4-8].

Suppose that X_0 is a normal scheme. Then $\mathcal{L}_{\psi,f}$ can be described as follows. Let K be the function field of X_0. Fix a separable closure of $\overline{K}$ of $K(X_0)$, and let the base point of X_0 be the canonical morphism $\mathrm{Spec}\,\overline{K} \to X_0$. Let $\zeta \in \overline{K}$ be a root of the polynomial $T^q - T - f$. Then any root of $T^q - T - f$ is of the form $\zeta + a$ for some $a \in \mathbb{F}_q$, $K[\zeta]$ is a Galois extension and the normalization of X_0 in $K[\zeta]$ is etale over X_0. It follows that we have a canonical epimorphism

$$\pi_1(X_0) \to \mathrm{Gal}(K[\zeta]/K).$$

On the other hand, we have a canonical monomorphism

$$\mathrm{Gal}(K[\zeta]/K) \hookrightarrow \mathbb{F}_q, \quad g \mapsto g(\zeta) - \zeta.$$

The representation corresponding to $\mathcal{L}_{\psi,f}$ is the composite

$$\pi_1(X_0) \to \mathrm{Gal}(K[\zeta]/K) \hookrightarrow \mathbb{F}_q \stackrel{\psi^{-1}}{\to} \overline{\mathbb{Q}}_\ell^*.$$

Let $\mathbb{F}_{q^d}$ be an extension of $\mathbb{F}_q$ of degree d, let $\psi : \mathbb{F}_{q^d} \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial additive character, let $f \in \mathbb{F}_{q^d}(t_1, \ldots, t_n)$ be a rational function, and let X_0 be an open subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$ so that its inverse image is contained in the complement of the hypersurface $f = \infty$. Let X_1 be the base change of X_0 from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, and let $\pi : X_1 \to X_0$ be the projection. Consider the sheaf $\mathcal{L}_{\psi,f}$ on X_1. Let us describe the character $\pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$ corresponding to the rank one lisse sheaf $\pi^*(\mathrm{tran}(\mathcal{L}_{\psi,f}))$ on X_1. The function field of X_0 (resp. X_1) is the rational function field $K_0 = \mathbb{F}_q(t_1, \ldots, t_n)$ (resp. $K_1 = \mathbb{F}_{q^d}(t_1, \ldots, t_n)$). Let ζ be a root of the polynomial $T^{q^d} - T - f$ in a separable closure $\overline{K}_1$ of $K_1 = \mathbb{F}_{q^d}(t_1, \ldots, t_n)$. Then $\mathcal{L}_{\psi,f}$ corresponds to the character

$$\pi_1(X_1) \to \mathrm{Gal}(K_1[\zeta]/K_1) \hookrightarrow \mathbb{F}_{q^d} \stackrel{\psi^{-1}}{\to} \overline{\mathbb{Q}}_\ell^*.$$

Denote this character by $\rho_1 : \pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$. More explicitly, we have

$$\rho_1(g) = \psi^{-1}\big(g(\zeta) - \zeta\big)$$

for any $g \in \pi_1(X_1)$. We have canonical isomorphisms

$$\mathrm{Aut}(X_1/X_0)^\circ \cong \mathrm{Gal}\Big(\mathbb{F}_{q^d}(t_1, \ldots, t_n)/\mathbb{F}_q(t_1, \ldots, t_n)\Big) \cong \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q).$$

Choose $\sigma \in \pi_1(X_0)$ so that its image in $\mathrm{Gal}\Big(\mathbb{F}_{q^d}(t_1, \ldots, t_n)/\mathbb{F}_q(t_1, \ldots, t_n)\Big)$ is given by

$$\sigma(a) = a^q \text{ for any } a \in \mathbb{F}_{q^d}, \quad \sigma(t_i) = t_i \ (i = 1, \ldots, n).$$

For any $0 \leq i \leq d-1$, let $\rho_1^{(i)} : \pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$ be the character defined by

$$\rho_1^{(i)}(g) = \rho_1(\sigma^i g \sigma^{-i})$$

for any $g \in \pi_1(X_1)$. We have

$$\begin{aligned} \rho_1^{(i)}(g) &= \rho_1(\sigma^i g \sigma^{-i}) \\ &= \psi^{-1}\big(\sigma^i g \sigma^{-i}(\zeta) - \zeta\big) \\ &= \psi^{-1}\sigma^i\big(g\sigma^{-i}(\zeta) - \sigma^{-i}(\zeta)\big). \end{aligned}$$

Note that $\sigma^{-i}(\zeta)$ and $g\sigma^{-i}(\zeta)$ are roots of the polynomial $T^{q^d} - T - \sigma^{-i}(f)$, and hence $g\sigma^{-i}(\zeta) - \sigma^{-i}(\zeta)$ lies in $\mathbb{F}_{q^d}$. We thus have

$$\sigma^i\big(g\sigma^{-i}(\zeta) - \sigma^{-i}(\zeta)\big) = \big(g\sigma^{-i}(\zeta) - \sigma^{-i}(\zeta)\big)^{q^i}.$$

So we have

$$\rho_1^{(i)}(g) = \psi^{-1}\Big(\big(g\sigma^{-i}(\zeta) - \sigma^{-i}(\zeta)\big)^{q^i}\Big)$$

for any $g \in \pi_1(X_1)$. This shows that $\rho_1^{(i)} : \pi_1(X_1) \to \overline{\mathbb{Q}}_\ell^*$ corresponds to the sheaf $\mathcal{L}_{\psi\circ\sigma^i, \sigma^{-i}(f)}$. By Proposition 1.2 (ii), $\pi^*(\otimes\text{-Ind}(\mathcal{L}_{\psi,f}))$ corresponds to the character $\prod_{i=0}^{d-1} \rho_1^{(i)}$. So we get the following.

Proposition 3.1. *Let $\psi : \mathbb{F}_{q^d} \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial additive character, let $f \in \mathbb{F}_{q^d}(t_1, \ldots, t_n)$ be a rational function, and let X_0 be an open subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$ so that its inverse image in $\mathbb{A}^n_{\mathbb{F}_{q^d}}$ is contained in the complement of the hypersurface $f = \infty$. Let X_1 be the base change of X_0 from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, and let $\pi : X_1 \to X_0$ be the projection. Then on X_1, we have an isomorphism*

$$\pi^*(\otimes\text{-Ind}(\mathcal{L}_{\psi,f})) \cong \bigotimes_{i=0}^{d-1} \mathcal{L}_{\psi\circ\sigma^i, \sigma^{-i}(f)},$$

where $\sigma^{-i}(f)$ is the rational function obtained from f by taking the q^i-th root for each coefficient of the numerator and the denominator of f, and $\psi \circ \sigma^i$ is the additive character $a \mapsto \psi(a^{q^i})$ for any $a \in \mathbb{F}_{q^d}$.

Remark 3.2. Let $\psi_q : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial additive character of $\mathbb{F}_q$, and let $\psi : \mathbb{F}_{q^d} \to \overline{\mathbb{Q}}_\ell^*$ be the additive character $\psi = \psi_q \circ \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$. In the notation of Proposition 3.1, we have $\psi \circ \sigma^i = \psi$ and

$$\begin{aligned}\pi^*(\otimes\text{-Ind}(\mathcal{L}_{\psi,f})) &\cong \bigotimes_{i=0}^{d-1} \mathcal{L}_{\psi\circ\sigma^i, \sigma^{-i}(f)} \\ &\cong \bigotimes_{i=0}^{d-1} \mathcal{L}_{\psi, \sigma^{-i}(f)} \\ &\cong \mathcal{L}_{\psi, \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(f)},\end{aligned}$$

where $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(f) = \sum_{i=0}^{d-1} \sigma^i(f)$.

4 Character Sums on Curves

In this section, X_0 is a smooth geometrically connected curve over $\mathbb{F}_q$ of genus g, $\overline{X}_0$ is the smooth compactification of X_0, X and $\overline{X}$ are the base changes of X_0 and $\overline{X}_0$ from $\mathbb{F}_q$ to $\mathbb{F}$, respectively. Let us recall the general method of studying character sums on X_0 using ℓ-adic cohomology theory. Let $\mathcal{L}_0$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 of rank r. Suppose the inverse image $\mathcal{L}$ on X has neither nonzero constant subsheaf nor nonzero constant quotient sheaf. Then by [3, 1.4.1], we have

$$H^0_c(X, \mathcal{L}) = 0, \quad H^2_c(X, \mathcal{L}) = 0.$$

By the Grothendieck-Ogg-Shafarevich formula ([5, X 7.1]), we have

$$\begin{aligned}\dim H^1_c(X,\mathcal{L}) &= -\sum_{i=0}^{2}(-1)^i\dim H^i_c(X,\mathcal{L})\\ &= r\big(2g-2+\#(\overline{X}-X)\big)+\sum_{x\in\overline{X}-X}\mathrm{sw}_x(\mathcal{L}),\end{aligned}$$

where sw_x denotes the Swan conductor at x for any Zariski closed point x in $\overline{X}$. Suppose furthermore that $\mathcal{L}_0$ is punctually pure of weight w. (Confer [3, Définition 1.2.2]). Then by [3, Corollarie 3.3.4], all the eigenvalues of F on $H^1_c(X,\mathcal{L})$ have Archimedean absolute value $\leq q^{\frac{w+1}{2}}$. By the Grothendieck trace formula [1, Rapport 3.2], we have

$$\begin{aligned}S_m(X_0,\mathcal{L}_0) &= \sum_{x\in X_0(\mathbb{F}_{q^m})}\mathrm{Tr}(F_x,\mathcal{L}_{0,\bar{x}})\\ &= \sum_{i=0}^{2}(-1)^i\mathrm{Tr}\big(F^m,H^i_c(X,\mathcal{L})\big)\\ &= -\mathrm{Tr}\big(F^m,H^1_c(X,\mathcal{L})\big).\end{aligned}$$

So we have

$$|S_m(X_0,\mathcal{L}_0)|\leq\Big(r\big(2g-2+\#(\overline{X}-X)\big)+\sum_{x\in\overline{X}-X}\mathrm{sw}_x(\mathcal{L})\Big)q^{\frac{m(w+1)}{2}}.$$

Finally since $\mathcal{L}_0$ is punctually pure, $\mathcal{L}$ is semi-simple ([3, Théorème 3.4.1 (iii)]). The condition that $\mathcal{L}$ has has neither nonzero constant subsheaf nor nonzero constant quotient sheaf is equivalent to the condition that $\mathcal{L}$ has no nonzero constant subsheaf. We thus get the following.

Proposition 4.1. *Let X_0 be a smooth geometrically connected curve over $\mathbb{F}_q$ of genus g, $\overline{X}_0$ its smooth compactification, and $\mathcal{L}_0$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 of rank r. Suppose $\mathcal{L}_0$ is punctually pure of weight w, and its inverse image $\mathcal{L}$ on X has no nonzero constant subsheaf. Then we have*

$$\begin{aligned}|S_m(X_0,\mathcal{L}_0)| &= \Big|\sum_{x\in X_0(\mathbb{F}_{q^m})}\mathrm{Tr}(F_x,\mathcal{L}_{0,\bar{x}})\Big|\\ &\leq \Big(r\big(2g-2+\#(\overline{X}-X)\big)+\sum_{x\in\overline{X}-X}\mathrm{sw}_x(\mathcal{L})\Big)q^{\frac{m(w+1)}{2}}.\end{aligned}$$

Theorem 4.2. *Let X_0 be a smooth geometrically connected curve over $\mathbb{F}_q$ of genus g, $\overline{X}_0$ its smooth compactification, $\mathbb{F}_{q^d}$ an extension of $\mathbb{F}_q$ of degree d, $X_1=X_0\otimes_{\mathbb{F}_q}\mathbb{F}_{q^d}$, and $\mathcal{L}_1$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_1 of rank r. Suppose $\mathcal{L}_1$ is punctually pure of weight w, and suppose the inverse image of $\otimes$-$\mathrm{Ind}(\mathcal{L}_1)$ on X has no nonzero constant subsheaf. Let*

$$S_1'(X_1,\mathcal{L}_1)=\sum_{x\in X_0(\mathbb{F}_q)}\mathrm{Tr}(F_{x_1},\mathcal{L}_{1,\bar{x}_1}),$$

where $x_1 : \mathrm{Spec}\,\mathbb{F}_{q^d} \to X_1$ *is the base change of* $x : \mathrm{Spec}\,\mathbb{F}_q \to X_0$ *for any* $x \in X_0(\mathbb{F}_q)$. *Then we have*

$$|S_1'(X_1, \mathcal{L}_1)| \leq \Big(\big(2g - 2 + \#(\overline{X} - X)\big) + d \sum_{x \in \overline{X} - X} \mathrm{sw}_x(\mathcal{L})\Big) \cdot r^d \cdot q^{\frac{dw+1}{2}}.$$

Proof. By Proposition 1.2 (i) and Grothendieck's trace formula ([1, Rapport 3.2]), we have

$$\begin{aligned} S_1'(X_1, \mathcal{L}_1) &= \sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}\big(F_x, (\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))_{\bar{x}}\big) \\ &= \sum_{i=0}^{2} (-1)^i \mathrm{Tr}\big(F, H_c^i(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))\big). \end{aligned}$$

Since the inverse image of $\otimes$-Ind$(\mathcal{L}_1)$ on X has neither nonzero constant subsheaf nor nonzero constant quotient sheaf, we have $H_c^i(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) = 0$ for $i \neq 1$. So we have

$$S_1'(X_1, \mathcal{L}_1) = -\mathrm{Tr}\big(F, H_c^1(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))\big).$$

One can verify $\otimes$-Ind$(\mathcal{L}_1)$ is punctually pure of weight dw. By [3, Corollarie 3.3.4], all eigenvalues of F on $H_c^1(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))$ have Archimedean absolute value $\leq q^{\frac{dw+1}{2}}$. So we have

$$|S_1'(X_1, \mathcal{L}_1)| \leq \big(\dim H_c^1(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))\big) q^{\frac{dw+1}{2}}.$$

On the other hand, by the Grothendieck-Ogg-Shafarevich formula ([5, X 7.1]), we have

$$\begin{aligned} \dim H_c^1(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) &= -\sum_{i=0}^{2} (-1)^i \dim H_c^i(X, \otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) \\ &= r^d \big(2g - 2 + \#(\overline{X} - X)\big) + \sum_{x \in \overline{X} - X} \mathrm{sw}_x(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)). \end{aligned}$$

To prove our theorem, it suffices to show that

$$\sum_{x \in \overline{X} - X} \mathrm{sw}_x(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) \leq d r^d \sum_{x \in \overline{X} - X} \mathrm{sw}_x(\mathcal{L}_1).$$

Choose $\sigma \in \pi_1(X_0)$ so that its image under the canonical epimorphism $\pi_1(X_0) \to \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ is the Frobenius substitution. Let $\rho_1 : \pi_1(X_1) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell)$ be the representation corresponding to $\mathcal{L}_1$ and let $\rho_1^{(i)} : \pi_1(X_1) \to \mathrm{GL}(r, \overline{\mathbb{Q}}_\ell)$ $(0 \leq i \leq d-1)$ be the representations defined by

$$\rho_1^{(i)}(g) = \rho_1(\sigma^i g \sigma^{-i})$$

for any $g \in \pi_1(X_1)$. Then by Proposition 1.2 (ii), $\pi^*(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1))$ corresponds to the representation $\otimes_{i=0}^{d-1} \rho_1^{(i)}$, where $\pi : X_1 \to X_0$ is the projection. Note that σ induce an automorphism of X over X_0, and an automorphism of $\overline{X}$ over $\overline{X}_0$. For any $x \in \overline{X} - X$ and any $0 \leq i \leq d-1$, we have

$$\mathrm{sw}_x(\rho_1^{(i)}) = \mathrm{sw}_{\sigma^i(x)}(\rho_1).$$

Combined with Lemma 4.3 below, we get

$$\begin{aligned}
\mathrm{sw}_x(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) &= \mathrm{sw}_x\Big(\otimes_{i=0}^{d-1}\rho_1^{(i)}\Big) \\
&\leq r^d \max\{\mathrm{sw}_x(\rho_1^{(i)}) | 0 \leq i \leq d-1\} \\
&= r^d \max\{\mathrm{sw}_{\sigma^i(x)}(\rho_1) | 0 \leq i \leq d-1\} \\
&= r^d \max\{\mathrm{sw}_{\sigma^i(x)}(\mathcal{L}_1) | 0 \leq i \leq d-1\} \\
&\leq r^d \sum_{i=0}^{d-1} \mathrm{sw}_{\sigma^i(x)}(\mathcal{L}_1).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{x\in\overline{X}-X} \mathrm{sw}_x(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) &\leq r^d \sum_{x\in\overline{X}-X}\sum_{i=0}^{d-1} \mathrm{sw}_{\sigma^i(x)}(\mathcal{L}_1) \\
&= dr^d \sum_{x\in\overline{X}-X} \mathrm{sw}_x(\mathcal{L}_1).
\end{aligned}$$

□

Lemma 4.3. *Let K be a local field whose residue field is of characteristic prime to ℓ, and let U and V be $\overline{\mathbb{Q}}_\ell$-representations of* $\mathrm{Gal}(\overline{K}/K)$*. Then we have the following estimation for the Swan conductor of tensor product*

$$\mathrm{sw}(U\otimes V) \leq \dim(U\otimes V)\cdot \max(\mathrm{sw}(U), \mathrm{sw}(V)).$$

Proof. Let $G^{(\lambda)}$ ($\lambda \geq 0$) be the filtration of $\mathrm{Gal}(\overline{K}/K)$ in upper numbering, let $G^{(\lambda+)}$ be the closure of $\bigcup_{\epsilon>0} G^{(\lambda+\epsilon)}$, and let

$$U = \bigoplus_\lambda U_\lambda, \quad V = \bigoplus_\lambda V_\lambda$$

be the decompositions of U and V as representations of the wild inertia subgroup $G^{(0+)}$ such that

$$\begin{aligned}
U_\lambda^{G^{(\lambda+)}} &= U_\lambda, \qquad U_\lambda^{G^{(\lambda)}} = 0 \text{ (if } \lambda > 0), \\
V_\lambda^{G^{(\lambda+)}} &= V_\lambda, \qquad V_\lambda^{G^{(\lambda)}} = 0 \text{ (if } \lambda > 0).
\end{aligned}$$

For any $\lambda, \mu \geq 0$, we have

$$(U_\lambda \otimes V_\mu)^{G^{(\max(\lambda,\mu)+)}} = U_\lambda \otimes V_\mu.$$

On the other hand, we have

$$U\otimes V = \bigoplus_{\lambda,\mu} U_\lambda \otimes V_\mu.$$

It follows that

$$\begin{aligned}
\mathrm{sw}(U\otimes V) &\leq \sum_{\lambda,\mu}\max(\lambda,\mu)\mathrm{dim}(U_\lambda)\mathrm{dim}(V_\mu)\\
&\leq \max(\mathrm{sw}(U),\mathrm{sw}(V))\sum_{\lambda,\mu}\mathrm{dim}(U_\lambda)\mathrm{dim}(V_\mu)\\
&= \mathrm{dim}(U\otimes V)\cdot\max(\mathrm{sw}(U),\mathrm{sw}(V)).
\end{aligned}$$

□

Remark 4.4. In the proof of Theorem 4.2, suppose there exists closed points $\infty_1,\ldots,\infty_m\in\overline{X}-X$ such that $\sigma(\infty_j)=\infty_j$ $(i=1,\ldots,m)$. Then we have

$$\begin{aligned}
\mathrm{sw}_{\infty_j}(\otimes\text{-}\mathrm{Ind}(\mathcal{L}_1)) &= \mathrm{sw}_{\infty_j}\Big(\otimes_{i=0}^{d-1}\rho_1^{(i)}\Big)\\
&\leq r^d\max\{\mathrm{sw}_{\infty_j}(\rho_1^{(i)})|0\leq i\leq d-1\}\\
&= r^d\max\{\mathrm{sw}_{\sigma^i(\infty_j)}(\rho_1)|0\leq i\leq d-1\}\\
&= r^d\mathrm{sw}_{\infty_j}(\mathcal{L}_1).
\end{aligned}$$

The estimate in Theorem 4.2 can be improved as

$$|S_1'(X_1,\mathcal{L}_1)|\leq\Big(\big(2g-2+\#(\overline{X}-X)\big)+d\sum_{x\in\overline{X}-X,\ x\neq\infty_1,\ldots,\infty_m}\mathrm{sw}_x(\mathcal{L}_1)+\sum_{j=1}^m\mathrm{sw}_{\infty_j}(\mathcal{L}_1)\Big)\cdot r^d\cdot q^{\frac{dw+1}{2}}.$$

From now on, let σ be the automorphism of $\mathbb{F}_{q^d}(t)$ defined by $\sigma(t)=t$ and $\sigma(a)=a^q$ for any $a\in\mathbb{F}_{q^d}$. We will apply Theorem 4.2 to Kummer type and Artin-Schreier type sheaves. Note that these sheaves have rank 1.

Theorem 4.5. *Let $f(t)\in\mathbb{F}_{q^d}(t)$ be a rational function. Write $f(t)=\prod_{j=1}^k f_j(t)^{n_j}$, where $f_j(t)\in\mathbb{F}_{q^d}[t]$ are irreducible polynomials and n_j are nonzero integers. Let $\chi:\mathbb{F}_{q^d}^*\to\overline{\mathbb{Q}}_\ell^*$ be a multiplicative character for $\mathbb{F}_{q^d}$. Suppose that the rational function $\prod_{i=0}^{d-1}f(t^{q^i})$ is not of the form $h(t)^{\mathrm{ord}(\chi)}$ in $\mathbb{F}(t)$, where $\mathrm{ord}(\chi)$ is the smallest integer d such that $\chi^d=1$. Then we have*

$$\Big|\sum_{a\in\mathbb{F}_q,\ f(a)\neq0,\infty}\chi(f(a))\Big|\leq\Big(d\sum_{j=1}^k\deg(f_j)-1\Big)\sqrt{q}.$$

Proof. Let X_0 be the complement in $\mathbb{A}^1_{\mathbb{F}_q}$ of the hypersurface $\prod_{i=0}^{d-1}\prod_{j=1}^k\sigma^{-i}(f_j)=0$. Consider the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{K}_{\chi,f}$ on $X_1=X_0\otimes_{\mathbb{F}_q}\mathbb{F}_{q^d}$. We have

$$\sum_{a\in\mathbb{F}_q,\ f(a)\neq0,\infty}\chi(f(a))=\sum_{x\in X_0(\mathbb{F}_q)}\mathrm{Tr}(F_{x_1},(\mathcal{K}_{\chi,f})_{\bar{x}_1}).$$

Note that $\mathcal{K}_{\chi,f}$ is tamely ramified everywhere on $\overline{X} = \mathbb{P}^1_{\mathbb{F}}$, and hence $\mathrm{sw}_x(\mathcal{K}_{\chi,f}) = 0$ for all $x \in \overline{X} - X = \mathbb{P}^1_{\mathbb{F}} - X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$. The set $\overline{X} - X$ consists of ∞ and the roots of $\prod_{i=0}^{d-1} \prod_{j=1}^{k} \sigma^{-i}(f_j)$. There are $d \sum_{j=1}^{k} \deg(f_j)$ such roots. The genus of X_0 is 0, and $\mathcal{K}_{\chi,f}$ is punctually pure of weight 0 and of rank 1. Our estimate follows directly from the estimate in Theorem 4.2, provided that we can prove $\otimes$-Ind$(\mathcal{K}_{\chi,f})$ is geometrically nonconstant. Indeed, by Proposition 2.1, we have $\pi^*(\otimes\text{-Ind}(\mathcal{K}_{\chi,f})) \cong \mathcal{K}_{\chi, \prod_{i=0}^{d-1} f(t^{q^i})}$, where $\pi : X_1 \to X_0$ is the projection. Since $\prod_{i=0}^{d-1} f(t^{q^i})$ is not of the form $h(t)^{\mathrm{ord}(\chi)}$ in $\mathbb{F}(t)$, the sheaf $\mathcal{K}_{\chi, \prod_{i=0}^{d-1} f(t^{q^i})}$ is geometrically nonconstant. □

Theorem 4.6. *Let $f(t), g(t) \in \mathbb{F}_{q^d}(t)$ be rational functions. Write $f(t) = \prod_{j=1}^{k} f_j(t)^{n_j}$, where $f_j(t) \in \mathbb{F}_{q^d}[t]$ are irreducible polynomials and n_j are nonzero integers. Let $D_1 = \sum_{j=1}^{k} \deg(f_j)$, let $D_2 = \max(\deg(g), 0)$, let D_3 be the degree of the denominator of $g(t)$, and let D_4 be the sum of degrees of those irreducible polynomials dividing the denominator of g but distinct from $f_j(t)$ $(j = 1, \ldots, k)$. Let $\chi : \mathbb{F}^*_{q^d} \to \overline{\mathbb{Q}}^*_\ell$ be a multiplicative character of $\mathbb{F}_{q^d}$, and let $\psi_q : \mathbb{F}_q \to \overline{\mathbb{Q}}^*_\ell$ be a nontrivial additive character of $\mathbb{F}_q$. Suppose $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g) = \sum_{i=0}^{d-1} \sigma^i(g)$ is not of the form $r(t)^q - r(t)$ in $\mathbb{F}(t)$. Then we have the estimate*

$$\left| \sum_{a \in \mathbb{F}_q,\ f(a) \neq 0, \infty,\ g(a) \neq \infty} \chi(f(a)) \psi_q\Big(\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g(a))\Big) \right| \leq (d(D_1 + D_3 + D_4) + D_2 - 1)\sqrt{q}.$$

Proof. Let $u(t)$ be the denominator of $g(t)$, let X_0 be the complement in $\mathbb{A}^1_{\mathbb{F}_q}$ of the hypersurface $\prod_{i=0}^{d-1} \Big(\sigma^{-i}(u) \prod_{j=1}^{k} \sigma^{-i}(f_j)\Big) = 0$, and let $\mathcal{L}_1$ be the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_1 = \mathcal{K}_{\chi,f} \bigotimes \mathcal{L}_{\psi,g}$ on $X_1 = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d}$, where $\psi = \psi_q \circ \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$. Then we have

$$\sum_{a \in \mathbb{F}_q,\ f(a) \neq 0, \infty,\ g(a) \neq \infty} \chi(f(a)) \psi_q\Big(\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g(a))\Big) = \sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}(F_{x_1}, \mathcal{L}_{1,\bar{x}_1}).$$

Note that $\mathcal{K}_{\chi,f}$ is tamely ramified everywhere on $\overline{X} = \mathbb{P}^1_{\mathbb{F}}$, and has no effect on the Swan conductor of $\mathcal{L}_1$. So for any $x \in \overline{X} - X = \mathbb{P}^1_{\mathbb{F}} - X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$, we have

$$\mathrm{sw}_x(\mathcal{L}_1) = \mathrm{sw}_x(\mathcal{L}_{\psi,g}).$$

Let v_x be the valuation of the field $\mathbb{F}(t)$ corresponding to the point $x \in \overline{X} - X$. If x is not a pole of g, then $\mathrm{sw}_x(\mathcal{L}_{\psi,g}) = 0$. Otherwise, we have

$$\mathrm{sw}_x(\mathcal{L}_{\psi,g}) \leq -v_x(g).$$

It follows that

$$\mathrm{sw}_\infty(\mathcal{L}_1) \leq D_2, \qquad \sum_{x \in \overline{X} - X,\ x \neq \infty} \mathrm{sw}_x(\mathcal{L}_1) \leq D_3.$$

The set $\overline{X} - X$ consists of ∞ and roots of $\prod_{i=0}^{d-1}\Big(\sigma^{-i}(u)\prod_{j=1}^{k}\sigma^{-i}(f_j)\Big)$. It follows that

$$\#(\overline{X} - X) \leq 1 + d(D_1 + D_4).$$

The genus of X_0 is 0, $\mathcal{L}_1$ is punctually pure of weight 0 and of rank 1. Our estimate follows directly from the estimate in Theorem 4.2 and Remark 4.4, provided that we can prove $\otimes$-Ind$(\mathcal{L}_1)$ is geometrically nonconstant. Let $\pi : X_1 \to X_0$ be the projection. It suffices to prove $\pi^*(\otimes\text{-Ind}(\mathcal{L}_1))$ is geometrically nonconstant. By Propositions 1.2 (iii), 2.1, 3.1 and Remarks 3.2, we have

$$\pi^*(\otimes\text{-Ind}(\mathcal{L}_1)) \cong \Big(\bigotimes_{i=0}^{d-1}\mathcal{K}_{\chi^{q^i},\sigma^{-i}(f)}\Big)\bigotimes\mathcal{L}_{\psi,\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g)}.$$

Since $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g)$ is not of the form $r(t)^q - r(t)$ in $\mathbb{F}(t)$, the sheaf $\mathcal{L}_{\psi,\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g)}$ is geometrically nonconstant, and it has wild ramification at poles of $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g)$, whereas $\bigotimes_{i=0}^{d-1}\mathcal{K}_{\chi^{q^i},\sigma^{-i}(f)}$ is at worst tamely ramified. So $(\bigotimes_{i=0}^{d-1}\mathcal{K}_{\chi^{q^i},\sigma^{-i}(f)})\bigotimes\mathcal{L}_{\psi,\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g)}$ is geometrically nonconstant. □

Remark 4.7. Let $f_1(t), \ldots, f_m(t), g_1(t), \ldots, g_n(t) \in \mathbb{F}_{q^d}(t)$ be rational functions, let $\chi_1, \ldots, \chi_m : \mathbb{F}_{q^d}^* \to \overline{\mathbb{Q}}_\ell^*$ be multiplicative characters, and let $\psi_1, \ldots, \psi_n : \mathbb{F}_{q^d} \to \overline{\mathbb{Q}}_\ell^*$ be additive characters. Then the sum

$$\sum_{a\in\mathbb{F}_q,\ f_i(a)\neq 0,\infty,\ g_j(a)\neq\infty} \chi_1(f_1(a))\cdots\chi_m(f_m(a))\psi_1(g_1(a))\cdots\psi_n(g_n(a))$$

can be reduced to the form of the sum in Theorem 4.6. Indeed, let $\chi : \mathbb{F}_{q^d}^* \to \overline{\mathbb{Q}}_\ell^*$ be a multiplicative character of order $q^d - 1$. We can find integers k_i $(i = 1, \ldots, m)$ such that $\chi_i = \chi^{k_i}$. Then we have $\chi_i(f_i(a)) = \chi(f_i(a)^{k_i})$. Let $f(t) = \prod_{i=1}^{m} f_i(t)^{k_i}$. We can find $\lambda_j \in \mathbb{F}_{q^d}$ $(j = 1, \ldots, n)$ such that $\psi_j(a) = \psi_q(\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\lambda_j a))$ for any $a \in \mathbb{F}_{q^d}$. Let $g(t) = \sum_{j=1}^{n}\lambda_j g_j$. Then we have

$$\begin{aligned}&\sum_{a\in\mathbb{F}_q,\ f_i(a)\neq 0,\infty,\ g_j(a)\neq\infty} \chi_1(f_1(a))\cdots\chi_m(f_m(a)))\psi_1(g_1(a))\cdots\psi_n(g_n(a))\\ &= \sum_{a\in\mathbb{F}_q,\ f_i(a)\neq 0,\infty,\ g_j(a)\neq\infty} \chi(f(a))\psi_q\Big(\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(g(a))\Big).\end{aligned}$$

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