Title
THE TWIN PARADOX IN SPECIAL RELATIVITY

Author
Muller, Richard A.

Publication Date
1971-10-01

Submitted to American Journal
of Physics

LBL-369
Preprint c.1

THE TWIN PARADOX IN SPECIAL RELATIVITY

Richard A. Muller

October 1971

AEC Contract No. W-7405-eng-48

DISCLAIMER

THE TWIN PARADOX IN SPECIAL RELATIVITY*

Richard A. Muller

Space Sciences Laboratory and Lawrence Berkeley Laboratory,
University of California, Berkeley, California 94720

Abstract

We resolve the twin paradox by calculating the relative ages of the twins first in the frame of the stationary twin, and then in the frame of the accelerating twin. If we account for the effects of acceleration by keeping track of the instantaneous Lorentz frame of the accelerating twin, both calculations agree.

The twin (or clock) paradox is one of the most perplexing of the paradoxes of Special Relativity. It can be stated as follows: Suppose we have a set of twins, John and Mary. John is a physics teacher, and he doesn't get to travel very much, but Mary is an astronaut who is chosen for the first trip to a nearby star. She boards a spaceship and travels to the star at a velocity v (nearly equal to c, the speed of light, although such a high v is not essential to the argument), and after a brief visit, returns to the earth at the same speed. Now the equations of Special Relativity say that if Mary is moving with respect to John, time intervals in her inertial frame are "dilated" with respect to time intervals in his frame, by an amount γ, where

$$\gamma = \frac{1}{[1-(\frac{v}{c})^2]^{1/2}} . \qquad (1)$$

Therefore, at the end of Mary's trip, she is younger than John. The paradox comes about from the fact that, except during the brief intervals when Mary is accelerating and decelerating, she is in an inertial frame, and therefore by the broad principle of "relativity" she can claim that John is moving, not she. Therefore his time intervals should be dilated, not hers, and when she returns she should find him younger than she is!

Although the paradox is resolved when one uses the equations of Special Relativity more carefully, it is surprising how few physicists are aware of the explanation, and think the solution lies somewhere in the realm of General Relativity.[1] One can resolve the paradox by using General Relativity, as is shown by Tolman[2] (who in

fact derives all the equation he needs using only the equivalence principle). But General Relativity is not really necessary. C. Darwin[3] uses only the relativistic Doppler shift formulas of Special Relativity (in a rather elegant way) to show that if the twins keep in touch with each other by sending light pulses back and forth, they agree in their calculations of each other's age. But he does not show explicitly what mistake in reasoning leads to the paradox. Others[4] have correctly pointed out that the paradox arises from the fact that one twin (Mary, in our story) accelerates, while the other (John) does not. What I wish to show here is that we can go one step farther. We can use the equations of Special Relativity, and take the accelerations completely into account simply by assuming that they take place in essentially negligible "proper time" (i.e., by assuming the acceleration does not cause either the accelerating twin to age, or his local clock to jump suddenly ahead or behind) and by keeping track of the effect that changes in his Lorentz frame must have on his measurements of distant events. This approach has been used by Schild,[5] but I think that by doing the mathematics explicitly we arrive at the clearest explanation for the professional physicist who has some familiarity with Lorentz transformations.

Let us assume, for simplicity, that Mary begins her trip at birth, at local time $t = 0$ and position $x = 0$, and that the distance to the star measured by a stationary observer on earth (John) is D_j. The distance to the star measured by Mary is

$$D_m = D_j/\gamma$$

due to the Lorentz contraction. We will calculate the relative ages of

John and Mary, first doing the calculation in John's frame, and then doing the calculation in Mary's frame. The answers from the two calculations will agree.

Let us do the calculation in John's frame. Our equations will be numbered with a subscript j to remind the reader that all the quantities referred to are those as measured by John. Mary travels a total distance (to the star and back) of $2\,D_j$, at a velocity v (-v during her return). This trip takes a time, $2\,D_j/v$. The Lorentz transformation for time gives the relationship between the time indicated on John's clock (T_j) and the time indicated on Mary's clock (T_m):

$$T_m = \gamma\,(T_j - \frac{vx}{c^2}), \qquad (3_j)$$

where x is the distance between them. During the trip out to the star,

$$x = v\,T_j. \qquad (4_j)$$

Substituting this in Eq. (3_j) we get:

$$\begin{aligned} T_m &= \gamma\,(T_j - \frac{v^2}{c^2}\,T_j) \\ &= \gamma\,(1 - \frac{v^2}{c^2})\,T_j \qquad (5_j) \\ &= T_j/\gamma. \end{aligned}$$

In terms of time <u>intervals</u>,

$$\Delta T_m = \Delta T_j/\gamma. \qquad (6_j)$$

This equation demonstrates that Mary's clock is running slower than John's, by a factor of $1/\gamma$. (Note that $\gamma \geq 1$.) We leave it to the reader to demonstrate that equation (6_j) is also true while Mary returns

from the star, i.e., that after she reverses her velocity, her clock is still running slower by $1/\gamma$. During the trip, John has aged by

$$A_j = 2\,D_j/v\,; \qquad (7_j)$$

Mary has aged by

$$A_m = \frac{2\,D_j}{\gamma v}\,. \qquad (8_j)$$

The difference in their ages is

$$A_j - A_m = (1 - \frac{1}{\gamma})\,\frac{2\,D_j}{v}\,. \qquad (9_j)$$

Now let us do the calculation in Mary's frame. All the quantities referred to now will be those measured by Mary. The Lorentz transformation becomes

$$T_j = \gamma\,(T_m - \frac{vx}{c^2}). \qquad (3_m)$$

And in terms of time intervals, we can show

$$\Delta T_j = \frac{\Delta T_m}{\gamma}, \qquad (4_m)$$

i.e., John's clock is running slower than Mary's by a factor of $1/\gamma$. To some readers it may seem that we have already hit a paradox, since a moment ago we concluded that Mary's clock is running slower than John's. But although these statements seem "paradoxical", there is no real paradox until we bring John and Mary back together and compare them to see which is older. Special Relativity says that you cannot relate the times of distant events (or the ages of separated people) in an unambiguous way, since the relationship between them depends on the Lorentz frame in which you do the comparison. Einstein's theory destroyed the concept of "absolute simultaneity." But when the clocks (or people) are back together--when Mary has returned to earth--then you can compare them. Simultaneity is a valid concept if we are concerned with a single inertial frame. The fact that we get a different answer for the question of whose clock is running faster, depending on the frame in which we do the calculation, is not yet a paradox.

Mary's trip takes a time $2D_m/v$, during which time she has aged by

$$A_m = \frac{2D_m}{v} \qquad (6_m)$$

(Recall that we assume that she ages a negligible amount during her accelerations and decelerations.) Since John's clock is running slower by $1/\gamma$, he has aged

$$A_j = \frac{2D_m}{\gamma v}.$$

If time dilation were the only factor to take into account, Mary would claim that she is older by an amount

$$A_m - A_j = (1 - \frac{1}{\gamma})\frac{2D_m}{v} = (1 - \frac{1}{\gamma})\frac{2D_j}{\gamma v}, \qquad (8_m)$$

and we would have a paradox.

But there is another factor, and it is in this factor that the key to the resolution of the paradox lies. When Mary reaches the star, and heads back, she changes inertial frames. And observers in different inertial frames disagree on when distant events took place. Before the turnaround, the relationship between John's clock and Mary's clock (as measured in Mary's frame) was

$$T_j = \gamma(T_m - \frac{vD_m}{c^2}). \qquad (9_m)$$

Immediately after the turnaround, the relationship between their clocks is

$$T_j = \gamma(T_m + \frac{vD_m}{c^2}). \qquad (10_m)$$

Equations (9_m) and (10_m) seem to contradict each other. We have Mary turn around in essentially zero proper time (so that T_m doesn't change) but this seems to imply that T_j does suddenly change. What's going on here?

What's going on is that observers in different inertial frames, even if it is the same observer (Mary, before and after turnaround) disagree on the time of distant events just as observers in different frames disagree on the distance to far objects. Recall that at the beginning of her journey, Mary and John disagreed about the distance to the star--it looked closer to Mary because of the Lorentz contraction. Who was right then? Was the distance to the star D_m or D_j? The answer is that they were both right, because distances depend on the frame in which you measure them. Likewise John's age, as measured by Mary, changes when Mary changes Lorentz frames. The amount by which it changes is given by the difference between equations (9_m) and (10_m), i.e.

$$\frac{2vD_m\gamma}{c^2} = \frac{2vD_j}{c^2}. \qquad (11_m)$$

The effect that a change in Lorentz frames has on the time of a distant event is proportional to the distance to that event. Thus we are justified in ignoring the effects of acceleration only at the beginning and end of Mary's trip, when the distance between her and John is zero.

When Mary returns to earth, the age difference between her and John is the contribution of time dilation (Eq. 8_m) minus the amount due to the change in her Lorentz frames (Eq. 11_m):

$$A_m - A_j = (1 - \frac{1}{\gamma}) \frac{2D_j}{\gamma v} - \frac{2vD_j}{c^2}$$
$$= \frac{2D_j}{\gamma v} - \frac{2D_j}{\gamma^2 v} - \frac{2vD_j}{c^2}$$
$$= \frac{2D_j}{\gamma v} - \frac{2D_j}{v}(\frac{1}{\gamma^2} + \frac{v^2}{c^2}).$$

But from Eq. (1), $1/\gamma^2 + v^2/c^2 = 1$, so

$$A_m - A_j = \frac{2D_j}{\gamma v} - \frac{2D_j}{v}$$
$$= (\frac{1}{\gamma} - 1)\frac{2D_j}{v}. \qquad (12_m)$$

Notice that equation (12_m) is identical to equation (9_j)! Upon her return, both Mary and John agree! John is older. If we assume that the star is 7.5 light-years away, and that Mary traveled at 3/4 the speed of light ($v/c = 3/4$), then

$$\frac{2D_j}{v} = 20 \text{ years}$$
$$\frac{1}{\gamma} = \sqrt{1 - 0.56} = 0.66.$$

And John is older by

$$A_j - A_m = (1 - 0.66)\,20 = 6.8 \text{ years}.$$

Several of the important features of this discussion are illustrated in Fig. 1, which is a plot of the relationship between A_j and A_m as determined in John's frame and Mary's frame for the above example. Notice the cross-over point for the two curves; this is where Mary has come to a halt (so that she is in the same inertial frame as John) and has not yet reversed her initial velocity. For a

moment John and she are in the same inertial frame, and during that moment their measurements agree(but not their ages).

The twin paradox, and this method of resolving it, illustrates in a dramatic way one of the revolutionary aspects of Einstein's theory: that there is no universal time system that all observers can agree to. and that the time at which an event occurs depends on the Lorentz frame in which it is observed.

Acknowledgments

I am grateful to several of my colleagues, especially Luis Alvarez and Sherwood Parker, for stimulating discussions of this problem, and for their helpful comments on this paper.

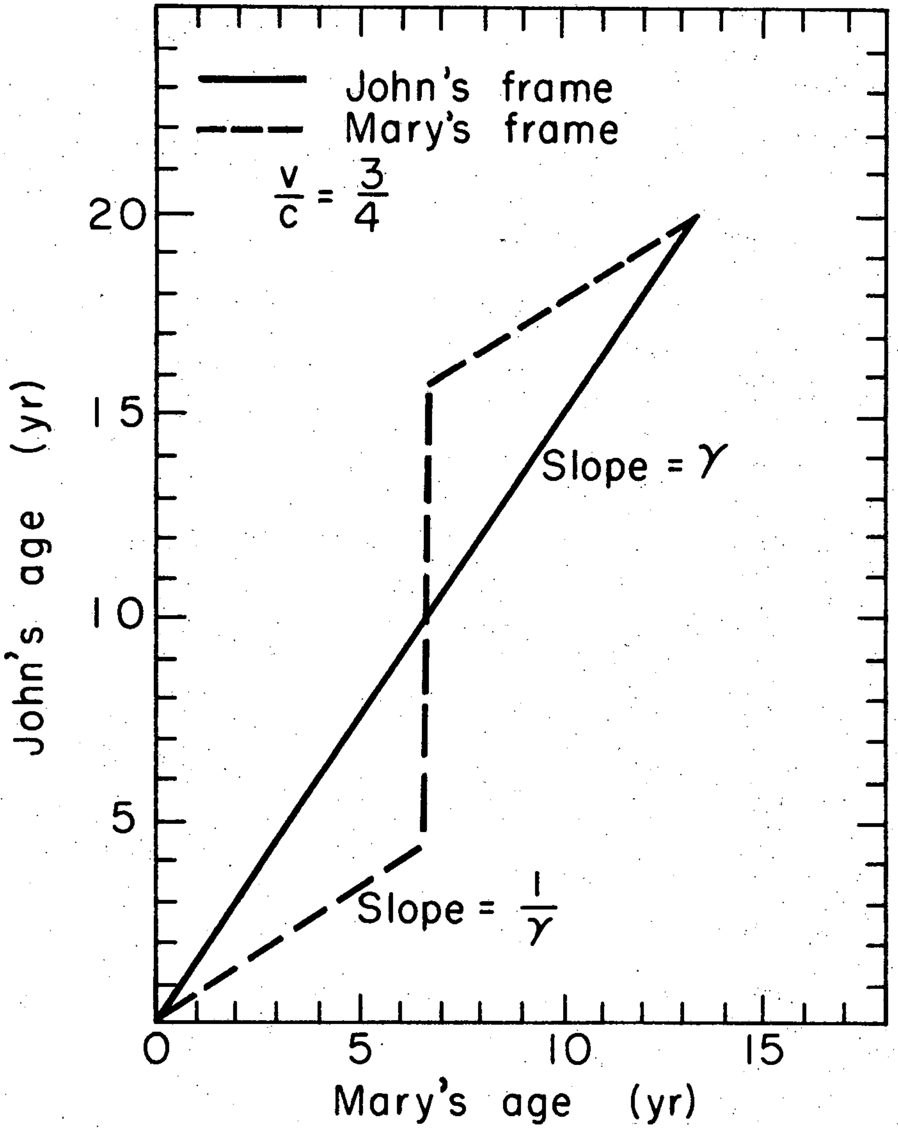
Footnotes and References

*Work done under the auspices of the National Aeronautics and Space Administration and the U. S. Atomic Energy Commission.

1. Einstein stated that the resolution of the paradox lay outside the realm of special relativity, in Naturwissenschaften 6, 697 (1918).
2. R. C. Tolman, Relativity, Thermodynamics and Cosmology (Oxford, 1966), p. 194-197.
3. C. G. Darwin, Nature 180, 976 (1957).
4. See, for example, W. H. McCrea, Nature 167, 680 (1951).
5. A. Schild, American Mathematical Monthly 66, 1 (1959).

Figure Caption

Fig. 1. The relative ages of the twins John and Mary as determined in each of their frames. They agree about each other's age only when they are in the same Lorentz frame, or separated by zero distance.



XBL 7110-1497

Fig. 1

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720