

# eScholarship

## Combinatorial Theory

### Title

Linear spaces embedded into projective spaces via Baer sublimes

### Permalink

<https://escholarship.org/uc/item/7r95048p>

### Journal

Combinatorial Theory, 2(1)

### ISSN

2766-1334

### Authors

Grundhöfer, Theo  
Stroppel, Markus J.  
Maldeghem, Hendrik Van

### Publication Date

2022

### DOI

10.5070/C62156876

### Supplemental Material

<https://escholarship.org/uc/item/7r95048p#supplemental>

### Copyright Information

Copyright 2022 by the author(s). This work is made available under the terms of a Creative Commons Attribution License, available at <https://creativecommons.org/licenses/by/4.0/>

Peer reviewed

# LINEAR SPACES EMBEDDED INTO PROJECTIVE SPACES VIA BAER SUBLINES

Theo Grundhöfer<sup>1</sup>, Markus J. Stroppel<sup>2</sup>, and Hendrik Van Maldeghem<sup>3</sup>

<sup>1</sup>*Universität Würzburg, Institut für Mathematik, D-97074 Würzburg, Germany  
grundhoefer@mathematik.uni-wuerzburg.de*

<sup>2</sup>*LExMath, Fakultät für Mathematik und Physik, Universität Stuttgart, D-70550 Stuttgart, Germany  
stroppel@mathematik.uni-stuttgart.de*

<sup>3</sup>*Vakgroep Wiskunde: Algebra en Meetkunde, Universiteit Gent, B-9000 Gent, Belgium  
Hendrik.VanMaldeghem@UGent.be*

Submitted: Jan 29, 2021; Accepted: Dec 3, 2021; Published: Mar 31, 2022

© The authors. Released under the CC BY license (International 4.0).

**Abstract.** Every nontrivial linear space embedded in a Pappian projective space such that the blocks of the linear space are projectively equivalent Baer sublimes with respect to a separable quadratic field extension is either a Baer subspace, or a Hermitian unital.

**Keywords.** Hermitian unital, Baer subplane, Baer subspace, Pappian projective space, embedding

**Mathematics Subject Classifications.** 51A45, 51E10

We consider linear spaces that are embedded into Pappian projective spaces  $\text{PG}(V)$  in such a way that all blocks of the linear space are projectively equivalent Baer sublimes of the projective space.

A *linear space* is an incidence structure  $(Q, \mathcal{B})$  such that any two points of  $Q$  are on a unique block  $B \in \mathcal{B}$ , and every block has at least two points. A linear space is called nontrivial if it has more than one block.

A *Hermitian unital* in a Pappian projective space consists of the absolute points of a unitary polarity of Witt index 1 of that space, with blocks induced by secant lines (see Section 1). The finite Hermitian unitals of order  $q$  are the classical examples of finite unitals (i. e., of  $2$ - $(q^3 + 1, q + 1, 1)$ -designs); they stem from unitary polarities of Pappian planes of order  $q^2$ . In any case, the blocks of a Hermitian unital are Baer sublimes (in the sense of Definition 1.3 below) with respect to a separable quadratic field extension. Conversely, results by Lefèvre-Percsy [LP82] and by Faina and Korchmáros [FK83] state that in finite Pappian planes the Hermitian unitals are characterized by that property.

In the present paper, we generalize those results in several directions: We consider Pappian projective spaces of arbitrary dimension, drop the finiteness assumption, and then characterize the class of Hermitian unitals together with Baer subspaces as the linear spaces having blocks

that are Baer sublines with respect to a separable quadratic field extension. (In planes, we do not even need the separability assumption.)

We state our result for the finite case first.

**Theorem A.** *Let  $V$  be a vector space over a finite field with  $\dim V \geq 3$ , and let  $(Q, \mathcal{B})$  be a nontrivial linear space such that  $Q$  is a spanning set of points in the projective space  $\text{PG}(V)$ , and every block  $B \in \mathcal{B}$  is a Baer subline of  $\text{PG}(V)$ .*

*Then either  $(Q, \mathcal{B})$  is a Baer subspace of  $\text{PG}(V)$ , or  $\dim V = 3$  and  $(Q, \mathcal{B})$  is a Hermitian unital in the projective plane  $\text{PG}(V)$ , in its standard embedding.*

We obtain Theorem A as a special case of Theorem B below; the statement is simpler because each field of order  $q^2$  has a unique subfield of order  $q$ . Also, the proof of the finite result is simpler; we complete it (in Proposition 4.6 below) before the proof of Theorem B is finished.

**Theorem B.** *Let  $C|R$  be a separable quadratic extension of fields. Consider a vector space  $V$  over  $C$ , of dimension at least three. Let  $(Q, \mathcal{B})$  be a nontrivial linear space such that  $Q$  is a subset spanning  $\text{PG}(V)$ , and every block  $B \in \mathcal{B}$  is a Baer subline of  $\text{PG}(V)$  with respect to  $C|R$ .*

*Then  $(Q, \mathcal{B})$  is either a Baer subspace (with respect to  $C|R$ ) of  $\text{PG}(V)$  or the Hermitian unital  $\mathcal{H}(C|R)$  in its standard embedding into  $\text{PG}(V)$ .*

In the planar case (viz., if  $\dim V = 3$ ), Theorem B is an immediate consequence of the following result, which involves the mapping  $\lambda$  defined in 1.3 in Section 1 below. The non-planar case is covered by Theorem D at the end of the paper.

As in [GSV19], we also consider, in the plane, generalized Hermitian unitals  $\mathcal{H}(C|R)$  where  $C|R$  is any (possibly inseparable) quadratic extension of fields; see Definition 1.1 below.

**Theorem C.** *Let  $C|R$  be a quadratic extension of fields. Let  $(Q, \mathcal{B})$  be a nontrivial linear space with  $Q \subseteq \text{PG}(2, C)$  such that every member of  $\mathcal{B}$  is a Baer subline of  $\text{PG}(2, C)$  with respect to  $C|R$ . If the mapping  $\lambda|_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{L}: B \mapsto \lambda(B)$  is injective then one of the following holds:*

- a.  *$(Q, \mathcal{B})$  contains O’Nan configurations, and is a Baer subplane with respect to  $C|R$ .*
- b.  *$(Q, \mathcal{B})$  does not contain any O’Nan configuration, and  $(Q, \mathcal{B})$  is the generalized Hermitian unital  $\mathcal{H}(C|R)$ , in its standard embedding.*

*If  $C|R$  is separable, then the mapping  $\lambda|_{\mathcal{B}}$  is injective.*

By an *O’Nan configuration*, we mean four blocks intersecting in six points (i.e., a  $(6_2 4_3)$  configuration). Naming this configuration in honor of O’Nan [O’N72] is customary in the context of unitals, see [BE08, p. 87]; the configuration is named after Veblen and Young in the axiomatics of projective spaces, or after Pasch in the context of ordered (Euclidean) geometry.

If one drops the assumption of injectivity for  $\lambda|_{\mathcal{B}}$  then there are additional examples (naturally, with inseparable  $C|R$ ), such as the projection into a line of an inseparable generalized Hermitian unital from its nucleus. If that projection is surjective, that is, if each element of  $R$  is a square in  $C$ , then one can endow every line with such a linear space; the union is again a linear space. This gives examples where  $\lambda|_{\mathcal{B}}$  is neither injective nor constant.

# 1. Hermitian unitals, Baer subspaces, and Möbius geometry

Let  $C|R$  be any quadratic (possibly inseparable) extension of fields; the classical example is  $\mathbb{C}|\mathbb{R}$ . The following is taken from [GSV19]. We can write  $C = R + \varepsilon R$ , with  $\varepsilon \in C \setminus R$ . There exist  $t, d \in R$  such that  $\varepsilon^2 - t\varepsilon + d = 0$ , since  $\varepsilon^2 \in R + \varepsilon R$ . The mapping

$$\sigma: C \rightarrow C: x + \varepsilon y \mapsto (x + ty) - \varepsilon y \quad \text{for } x, y \in R$$

is a field automorphism which generates  $\text{Aut}_R C$ : if  $C|R$  is separable, then  $\sigma$  has order 2 and generates the Galois group of  $C|R$ ; if  $C|R$  is inseparable, then  $\sigma$  is the identity.

For any vector space  $V$  over  $C$ , we consider the Pappian projective space  $\text{PG}(V)$ : points are one-dimensional subspaces  $[v] := vC$  of  $V$ ; the line set  $\mathcal{L}$  of  $\text{PG}(V)$  consists of all two-dimensional subspaces of  $V$ .

In particular, for any positive integer  $n$ , we consider the Pappian projective space  $\text{PG}(n, C) := \text{PG}(C^{n+1})$  of (projective) dimension  $n$ . We use homogeneous coordinates and write points as  $[X_0, \dots, X_n] := (X_0, \dots, X_n)C$ . Whenever we write  $[X_0, \dots, X_n]$  or  $[v]$ , we tacitly assume that this is a point, i.e., that  $(X_0, \dots, X_n)$  or  $v$ , respectively, is not trivial.

Assume that  $C|R$  is separable, and let  $h: V \times V \rightarrow C$  be a non-degenerate Hermitian or skew-Hermitian form on  $V$  of Witt index 1, with respect to  $\sigma$ . Mapping a point  $[w]$  of  $\text{PG}(V)$  to the hyperplane  $w^\perp := \{v \in V \mid h(v, w) = 0\}$  then gives a polarity of  $\text{PG}(V)$  (in the sense of Tits [Tit86, 8.3.2, p. 128]). The *Hermitian unital* defined by that polarity has point set  $U := \{[v] \mid v \leq v^\perp\}$ ; its blocks are induced by secant lines.

**Definition 1.1** (see [GSV19]). The *generalized Hermitian unital*  $\mathcal{H}(C|R)$  is the incidence structure  $(U, \mathcal{B})$  with the point set  $U := \{[X_0, X_1, X_2] \mid X_0^\sigma X_1 + X_2^\sigma X_2 \in R\varepsilon\}$ , and the set  $\mathcal{B}$  of *blocks* consists of the intersections of  $U$  with secant lines, i.e. with lines of  $\text{PG}(2, C)$  containing more than one point of  $U$ .

In the next proposition, we identify  $\mathcal{H}(C|R)$  in classical terms and motivate the name “generalized Hermitian unital”. The nucleus of a quadric is the projective subspace corresponding to the radical of the associated polar form.

**Proposition 1.2** (see [GSV19, 2.2, 2.3]). *If  $C|R$  is separable, then  $\mathcal{H}(C|R) = (U, \mathcal{B})$  is the Hermitian unital arising from the skew-Hermitian form  $h: C^3 \times C^3 \rightarrow C$  defined by*

$$h((X_0, X_1, X_2), (Y_0, Y_1, Y_2)) = \varepsilon^\sigma X_0^\sigma Y_1 - \varepsilon X_1^\sigma Y_0 + (\varepsilon^\sigma - \varepsilon) X_2^\sigma Y_2.$$

*If  $C|R$  is inseparable, then  $\mathcal{H}(C|R)$  is the projection of an ordinary quadric  $Q$  in a suitable projective space of dimension at least 3 from a subspace of codimension 1 in the nucleus of  $Q$ .*

*For every point  $p$  of  $U$ , there is a unique tangent to  $U$  in  $p$ ; i.e., a unique line of  $\text{PG}(2, C)$  meeting  $U$  just in  $p$ .*

**Definition 1.3.** Let  $E$  be any basis of  $V$  over  $C$ , and let  $\langle E \rangle_R$  denote the  $R$ -span of  $E$ . A *Baer subspace* (with respect to the extension  $C|R$ ) of  $\text{PG}(V)$  is the image  $\gamma(P)$  of the point set  $P := \{[X] \mid X \in \langle E \rangle_R \setminus \{0\}\}$  under an element  $\gamma \in \text{PGL}(V)$ . If  $\dim V > 2$ , we endow the point set  $\gamma(P)$  with the set  $\mathcal{L}_{\gamma(P)}$  of *blocks* that are obtained as intersections of  $\gamma(P)$

with secant lines. Thus every Baer subspace of  $\text{PG}(n, C)$  is isomorphic to the projective space  $\text{PG}(n, R) \cong (P, \mathcal{L}_P)$  over  $R$ .

A *Baer subplane* of  $\text{PG}(V)$  (with respect to  $C|R$ ) is a plane of a Baer subspace (with respect to  $C|R$ ). A *Baer subline* (with respect to  $C|R$ ) of  $\text{PG}(V)$  is a line of a Baer subspace (with respect to  $C|R$ ).

In particular, for  $n = 2$ , Baer subspaces and Baer subplanes of  $\text{PG}(2, C)$  are the same. Using dimensions of subspaces over  $R$ , one sees immediately: Each Baer subplane in  $\text{PG}(2, C)$  has the property that every line of  $\text{PG}(2, C)$  intersects it in at least one point, and dually, every point of  $\text{PG}(2, C)$  is contained in at least one line intersecting the Baer subplane in more than one point (and that intersection is then a Baer subline).

For any Baer subline  $B$ , let  $\lambda(B) := \langle B \rangle_C$  be the line of  $\text{PG}(n, C)$  containing  $B$ .

## Möbius geometry

We will use various models for the classical Möbius plane related to the extension  $C|R$ , as follows. Let  $\mathcal{M}$  be the geometry with point set  $\text{PG}(1, C)$  and blocks all Baer sublines with respect to  $C|R$ . Let  $X^2 + \alpha X + \beta \in R[X]$  be an irreducible polynomial over  $R$  having roots in  $C$ . Let  $\mathcal{O}$  be the quadric in  $\text{PG}(3, R)$  with equation  $X_0^2 + \alpha X_0 X_1 + \beta X_1^2 = X_2 X_3$ . Endowed with the set  $\mathcal{C}$  of all nontrivial plane intersections (that is, plane intersections containing at least two points), this becomes a geometry isomorphic to  $\mathcal{M}$ . This is the classical Möbius plane related to the extension  $C|R$ . A planar model is obtained by (“stereographically”) projecting  $\mathcal{O}$  onto a plane from a point  $p \in \mathcal{O}$ : The points are then all points of the affine plane  $\text{AG}(2, R)$  plus a point  $\infty$ , and the blocks are some conics completely contained in  $\text{AG}(2, R)$ , and all lines of  $\text{AG}(2, R)$  (with  $\infty$  added to each line, but to no conic). We refer to that model as the *affine model related to  $p$* .

*Remark 1.4.* The quadric  $\mathcal{O}$  in three-space has a nucleus if, and only if, the extension  $C|R$  is inseparable. In fact, the defining quadratic form  $X_0^2 + \alpha X_0 X_1 + \beta X_1^2 - X_2 X_3$  has degenerate polar form if, and only if, the characteristic is 2 and  $\alpha = 0$ .

**Proposition 1.5.** *Let  $(\mathcal{O}, \mathcal{C})$  be the Möbius plane related to the extension  $C|R$ , and assume that there exist a set  $X \subseteq \mathcal{O}$  of points and a set  $\mathcal{Y} \subseteq \mathcal{C}$  of circles in  $\mathcal{M}$  such that  $(X, \mathcal{Y})$  is a non-trivial linear space. Then  $C|R$  is inseparable.*

*Proof.* Let  $Y \in \mathcal{Y}$  be a block of that linear space and let  $p \in X \setminus Y$  be a point of it outside  $Y$ . We take an affine view of the stereographic projection leading to the affine model related to  $p$ : the plane at infinity is the tangent plane to  $\mathcal{O}$  in the point  $p$ , and the projection is a parallel projection. Without loss of generality, we assume that we project into the plane  $E$  containing  $Y$ .

The members of  $\mathcal{Y}$  joining  $p$  with a point of  $Y$  are projected to affine lines in  $E$ . Since two points of  $X$  are on a unique member of  $\mathcal{Y}$ , these lines are parallel, and they are all tangent to  $Y$ . Hence the conic  $Y$  has a nucleus  $n$  in the projective completion of  $E$ , and that nucleus is a point at infinity of the affine plane. Now each of these affine lines is a tangent to  $\mathcal{O}$  in three-space, and the line  $pn$  is also a tangent to  $\mathcal{O}$ . With respect to the polar form of the quadratic form defining  $\mathcal{O}$ , the point  $n$  is thus orthogonal to a non-planar set of points. This means that the polar form is degenerate, and  $C|R$  is inseparable (see Remark 1.4).  $\square$

Applying Proposition 1.5 to a line  $L$  of  $\text{PG}(V)$  containing more than one member of  $\mathcal{B}$ , we obtain:

**Corollary 1.6.** *Let  $C|R$  be a quadratic field extension, let  $V$  be a vector space of dimension at least 3 over  $C$ , and let  $(Q, \mathcal{B})$  be a nontrivial linear space with  $Q \subseteq \text{PG}(V)$  such that every member of  $\mathcal{B}$  is a Baer subline of  $\text{PG}(V)$  with respect to  $C|R$ . If  $\lambda|_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{L}: B \mapsto \lambda(B)$  is not an injective mapping, then the extension  $C|R$  is inseparable.  $\square$*

## 2. Proof of Theorem C

In this section, let  $C|R$  be a quadratic extension of fields, and let  $(Q, \mathcal{B})$  be a nontrivial linear space with  $Q \subseteq \text{PG}(2, C)$  such that every member of  $\mathcal{B}$  is a Baer subline of  $\text{PG}(2, C)$  with respect to  $C|R$ .

We assume in this section that the mapping  $\lambda|_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{L}: B \mapsto \lambda(B)$  is injective.

**Theorem 2.1.** *If  $(Q, \mathcal{B})$  contains at least one O’Nan configuration, then  $(Q, \mathcal{B})$  is a Baer subplane of  $\text{PG}(2, C)$  with respect to  $C|R$ .*

*Proof.* Let  $\{p_1, p_2\}$ ,  $\{p_3, p_4\}$  and  $\{p_5, p_6\}$  be the two-element sets of points of the O’Nan configuration which are not joined by a block of that configuration. As  $\text{PGL}(3, C)$  acts transitively on quadrangles, there is a unique Baer subplane  $(P, \mathcal{L}_P)$  of  $\text{PG}(2, C)$  (with respect to  $C|R$ ) containing the four points  $p_1, p_2, p_3, p_4$  and it obviously also contains  $p_5$  and  $p_6$ . We may assume that  $p_1, p_3, p_5$  are on a common block  $B \in \mathcal{B}$ . By our standing assumption of injectivity for  $\lambda$ , the block  $B$  is then the only block contained in  $\lambda(B)$ . Since Baer sublines with respect to  $C|R$  are determined by any three of their points, the member of  $\mathcal{L}_P$  containing  $p_1$  and  $p_3$  belongs to  $\mathcal{B}$ , just like the members of  $\mathcal{L}_P$  that contain  $\{p_2, p_3, p_6\}$ , or  $\{p_1, p_4, p_6\}$ , or  $\{p_2, p_4, p_5\}$ , respectively. Every member of  $\mathcal{L}_P$  distinct from those through  $p_1, p_2$ , or  $p_3, p_4$ , or  $p_5, p_6$ , intersects the union of foregoing four members of  $\mathcal{L}_P$  in at least three points and hence also belongs to  $\mathcal{B}$ . It now easily follows that  $P$  is entirely contained in  $Q$ . Since every point outside  $P$  is contained in a line intersecting  $P$  in at least two points, our injectivity assumption for  $\lambda$  yields  $P = Q$  and then  $\mathcal{L}_P = \mathcal{B}$ .  $\square$

### The case with no O’Nan configurations

For the rest of this Section, we assume that  $(Q, \mathcal{B})$  does not contain any O’Nan configuration.

**Lemma 2.2.** *Consider a triangle in  $\text{PG}(2, C)$  with vertices  $u, v, w$ , and let  $W$  and  $V$  be Baer sublines containing  $\{u, v\}$ , and  $\{u, w\}$ , respectively. Let  $c$  be a point of the line  $uv = \lambda(W)$ , but not contained in  $W$ . For each  $p \in W$ , let  $D_p$  be the Baer subline contained in the line  $pw$  and obtained by projecting  $V$  from  $c$ . Then there exists a unique Baer subline  $B$  of  $vw$ , with  $v, w \in B$ , such that the projection of  $\bigcup\{D_p \mid p \in W \setminus \{u\}\}$  from  $u$  into  $vw$  coincides with  $(vw \setminus B) \cup \{v, w\}$ .*

*Proof.* In suitable homogeneous coordinates we have  $u = [1, 0, 0]$ ,  $v = [0, 1, 0]$ ,  $w = [0, 0, 1]$ ,  $W = \{[x, y, 0] \mid x, y \in R\}$ , and  $V = \{[x, 0, z] \mid x, z \in R\}$ . Then  $c = [k, 1, 0]$  with  $k \in C \setminus R$ .

Let  $p = [r, 1, 0]$  with  $r \in R$ . Then  $D_p$  is the set  $\{[r, 1, s(r-k)] \mid s \in R\} \cup \{w\}$ . Its projection from  $u$  onto  $vw$  is the set  $\{[0, 1, s(r-k)] \mid s \in R\} \cup \{w\}$ . Since  $\{s(r-k) \mid r, s \in R\} = (C \setminus R) \cup \{0\}$ , the points not in such a projection are exactly the points of the Baer subline  $B = \{[0, y, z] \mid (y, z) \in R^2 \setminus \{(0, 0)\}\}$ , except for  $v$  and  $w$ .  $\square$

**Lemma 2.3.** *Let  $W \in \mathcal{B}$  be a block, and let  $w$  be a point in  $Q \setminus W$ . For every point  $p \in W$ , let  $B_p$  be the block containing  $w$  and  $p$ .*

- a. *There exists a unique point  $c \in \lambda(W)$  such that, for each  $p \in W$ , each  $q \in B_p$ , and each  $t \in W$ , the line  $cq$  contains a point of the block  $B_t$ .*
- b. *That unique point  $c$  does not belong to  $Q$ .*
- c. *The line joining that unique point  $c$  and  $w$  contains no point of  $Q$  apart from  $w$ .*

*Proof.* Let  $p \in W$  be arbitrary, and let  $q \in B_p \setminus \{w\}$  also be arbitrary. Select two points  $u$  and  $v$  in  $W \setminus \{p\}$ . There is a unique Baer subplane  $(P, \mathcal{L}_P)$  containing  $B_u$  and  $B_v$ , and since  $(Q, \mathcal{B})$  does not contain any O’Nan configuration, the point set  $P$  of that Baer subplane intersects  $W$  in just  $\{u, v\}$ . Let  $L$  be the unique line of  $\text{PG}(2, C)$  containing  $q$  and intersecting  $P$  in a Baer subline. By the previous sentence,  $L \neq pw$ . For  $x \in \{u, v\}$ , the line  $L$  intersects  $B_x$  in a point  $q_x \neq w$ . It follows that the unique Baer subline  $B \subseteq L$  containing  $\{q, q_u, q_v\}$  belongs to  $\mathcal{B}$ . Since the projection of a Baer subline is a Baer subline,  $B \cap p'w$  is nontrivial for each  $p' \in W$ .

So we have shown that the set  $A := \{q \in \text{PG}(2, C) \mid q \in B_p \setminus \{w\}, p \in W\}$  has a partition into blocks. No other block is contained in  $A$  because we assume that there are no O’Nan configurations in  $(Q, \mathcal{B})$ . Hence that partition is unique. We now show that the lines containing the blocks in that partition all intersect in a common point. Let  $c$  be the intersection of two of them, say  $\{c\} = L_1 \cap L_2$ , with  $L_i$  containing a block  $B_i \in \mathcal{B}$ , with  $B_i \subseteq A$ . The line pencil in  $c$  has a unique Baer subpencil containing  $L_1, L_2$  and  $cw$ . By uniqueness, this projects all blocks  $B_p$  with  $p \in W$ . Hence the intersection of any line through  $c$  and a point of  $A$  with  $A$  is a block of  $(Q, \mathcal{B})$ . Assertion a now follows. If  $c$  were in  $Q$  then it would belong to  $W = \lambda(W) \cap Q$ , and we would obtain an O’Nan configuration in  $(Q, \mathcal{B})$ , so Assertion b is true.

Aiming at a contradiction, assume that some point  $a \in cw \setminus \{w\}$  belongs to  $Q$ . Then  $cw$  contains a block  $D$  of  $(Q, \mathcal{B})$ .

Suppose first that the line  $au$  intersects some block  $B_p, p \in W \setminus \{u\}$ , say in the point  $b$ . Let  $(P, \mathcal{L}_P)$  be the Baer subplane containing  $B_u \cup B_p$ . Then  $P$  contains  $c$  and hence  $a \in cw \cap bu$ . It follows that each member of  $\mathcal{L}_P$  containing  $a$  but not  $w$  belongs to  $\mathcal{B}$ , yielding many O’Nan configurations, a contradiction.

Now we apply Lemma 2.2. By the choice of  $c$ , the blocks  $D_p$  just coincide with  $B_p$ . Projecting the points of  $\bigcup\{D_p \mid p \in W \setminus \{u\}\}$  from  $u$  into  $vw$  we obtain all points of  $vw$  except those in  $B \setminus \{v, w\}$ , for some Baer subline  $B$  with respect to  $C|R$ . For each  $a \in D \setminus \{w\}$ , the argument in the previous paragraph shows that  $ua$  meets  $vw$  in a point of  $B$ . So  $B$  is the unique Baer subline (with respect to  $C|R$ ) of  $vw$  containing the image of  $D$  under the projection. But the projection does not contain  $v$ , as  $c \notin D$ , contradicting  $v \in B$ . This contradiction completes the proof of Assertion c.  $\square$

**Lemma 2.4.** *Suppose  $|R| > 4$ . Then the point set  $Q$  of the generalized Hermitian unital  $\mathcal{H}(C|R)$  is determined by the union  $U$  of all blocks joining a fixed point  $w$  with each point of a block  $W$  not containing  $w$ , and a point  $a$  of  $Q \setminus U$ , together with the property that  $\mathcal{H}(C|R)$  is a linear space whose blocks are Baer sublines with respect to  $C|R$  in  $\text{PG}(2, C)$ .*

*Proof.* By Lemma 2.3, there is a unique point  $c$  with the property that each line of  $\text{PG}(2, C)$  joining  $c$  with any point of any block of  $(Q, \mathcal{B})$  through  $w$  and a point on  $W$  induces a block of  $(Q, \mathcal{B})$  intersecting every block through  $w$  and a point of  $W$ . We may assume that  $c$  lies on the line  $\lambda(W)$  carrying the block  $W$ . By Lemma 2.3, no point of  $cw \setminus \{w\}$  belongs to  $Q$ . In the Möbius plane induced on  $\lambda(W)$  by the Baer sublines with respect to  $C|R$ , we consider the derived affine plane  $\mathcal{A}$  at  $a'$ , where  $a'$  is the projection of  $a$  from  $w$  onto  $\lambda(W)$ . Then  $W$  is a conic in  $\mathcal{A}$ . Except for the possible nucleus  $n$  of  $W$ , every point  $b$  of  $\mathcal{A} \setminus W$  is on at most two tangents of  $W$ . By our assumption  $|R| > 4$ , the point lies on at least two secant lines  $M_1, M_2$  of  $W$  in  $\mathcal{A}$ . Let  $M_i$  intersect  $W$  in the points  $a_{i1}$  and  $a_{i2}$ . Let  $B_{ij}$  be the block of  $(Q, \mathcal{B})$  joining  $w$  with  $a_{ij}$ . Let  $P_i$  be the point set of the Baer subplane containing  $B_{i1} \cup B_{i2}$ ,  $i = 1, 2$ . Obviously  $c$  belongs to  $P_i$ . By the choice of  $M_i$ , we see that  $a'$  is not contained in  $P_i$ , hence  $a$  is not contained in  $P_i$  and the unique line  $K_i$  of  $\text{PG}(2, C)$  through  $a$  intersecting  $P_i$  in a Baer subline does not contain  $w$ . Hence  $K_i$  intersects  $B_{ij}$  in some point  $q_{ij} \neq w$ , and  $q_{i1} \neq q_{i2}$ . Then, by injectivity of  $\lambda_{\mathcal{B}}$ , the block of  $(Q, \mathcal{B})$  through  $q_{i1}$  and  $q_{i2}$  contains  $a$ , and contains a point  $b_i$  of  $bw$ . Obviously  $b_1 \neq b_2$  and so the intersection  $Q \cap bw$  is determined. If a nucleus  $n$  exists, it now also easily follows that  $Q \cap nw$  is determined.  $\square$

We remark that Lemma 2.4 remains true if  $|R| \leq 4$ ; the proof can be extended by considering the three small cases separately. As we do not need the more general result in the present paper, we omit that discussion.

**Theorem 2.5.** *If  $(Q, \mathcal{B})$  does not contain any O’Nan configuration, then  $(Q, \mathcal{B})$  is the generalized Hermitian unital  $\mathcal{H}(C|R)$ , and the embedding is standard.*

*Proof.* Assume first that  $Q$  is finite. Then  $q := |R|$  is one less than the (then also finite) number of points on any block in  $\mathcal{B}$ , and  $|C| = q^2$ . Consider a triangle in  $(Q, \mathcal{B})$  with vertices  $u, v, w$ . For every point  $p$  in the block  $W \in \mathcal{B}$  containing  $u$  and  $v$ , let  $B_p$  be the block containing  $w$  and  $p$ . By Lemma 2.3, there is a unique point  $c \in uv$  such that, for each  $p \in W$ , each  $x \in B_p$ , and each  $t \in W$ , the line  $cx$  intersects the block  $B_t$  nontrivially, and  $Q$  does not contain points of  $cw \setminus \{w\}$ . We now show that every other line through  $w$  contains  $q + 1$  points of  $Q$ . Considering all blocks of  $(Q, \mathcal{B})$  through  $u$  and a point of  $B_v$ , it follows from Lemma 2.2 (by projecting from  $w$ ) that the only lines through  $w$  which possibly only contain one point of  $Q$  (namely,  $w$  itself), are projected from  $w$  onto a Baer subline  $B^*$  of  $uv$  containing  $u$  and  $v$ . By Lemma 2.3,  $B^*$  also contains  $c$ . Varying  $v$  over  $W$ , we see that the only line through  $w$  not containing any point of  $Q$  except for  $w$  is  $cw$ . This shows that  $Q$  contains  $q^3 + 1$  points and  $(Q, \mathcal{B})$  is a unital. This unital is the Hermitian unital  $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$  (by [LP82] and [FK83], cp. [GSV21]) since all blocks are Baer sublines, and the embedding is standard (cp. [GSV19]).

Now assume that  $Q$  is infinite (in fact, we will only use  $|R| > 4$ ). Since  $(Q, \mathcal{B})$  does not contain O’Nan configurations, Lemma 2.3 holds. Consider a triangle in  $(Q, \mathcal{B})$  with vertices  $u, v, w$ . For every point  $p$  in the block  $W \in \mathcal{B}$  containing  $u$  and  $v$ , let  $B_p$  be the block containing



$w$  and  $p$ . Let  $c \in uv$  be the unique point such that, for each  $p \in W$ , each point  $q \in B_p$ , and each  $t \in W$ , the line  $cq$  intersects the block  $B_t$  nontrivially.

Let  $U$  be the union of  $\{c\}$  and all blocks of  $(Q, \mathcal{B})$  joining  $w$  with a point of  $W$ , and let  $A := \{t \in \text{PG}(2, C) \mid t \in cp, p \in B_u\}$ ; the only points of  $Q$  in  $A$  are those on the blocks joining  $w$  with a point of  $W$ . By Lemma 2.4, the theorem will be proved if we show that for every line  $L \neq cw$  through  $w$  some subgroup of the stabilizer  $G$  of  $U$  in  $\text{PGL}_3(C)$  acts transitively on  $L \setminus A$ . Clearly,  $G$  contains a group  $G(w, cw)$  of elations with axis  $cw$  and center  $w$  such that  $G(w, cw)$  acts transitively on  $B_u \setminus \{w\}$ , and also a group  $G(w, cu)$  of homologies with axis  $cu$ , center  $w$ , and acting transitively on  $B_u \setminus \{u, w\}$ . Set  $G^* = \langle G(w, cu), G(w, cw) \rangle$ .

In suitable affine coordinates, the action of  $G^*$  on  $uw \setminus \{w\}$  is given by the affine maps  $x \mapsto rx + t$ , with  $r, t \in R$  and  $r \neq 0$ . For any given  $k \in C \setminus R$ , the set  $\{rk + t \mid r, t \in R, r \neq 0\}$  coincides with  $C \setminus R$ , and we obtain transitivity of  $G^*$  on  $uw \setminus B_u$ .

Projecting from  $c$  (which is fixed by  $G^*$ ) onto  $L$  (which is also stabilized by  $G^*$ ) we obtain the transitivity we want.

So  $(Q, \mathcal{B})$  is the generalized Hermitian unital  $\mathcal{H}(C|R)$ , in its standard embedding.  $\square$

Theorem C is now a consequence of Theorem 2.1, Theorem 2.5, and Corollary 1.6. The planar case of Theorem B (where  $\dim V = 3$ ) follows.

### 3. Generalized Hermitian unitals in the plane

The following observation shows that generalized Hermitian unitals have geometric dimension two. In Lemma 4.7 below, we generalize this to Hermitian linear spaces (with respect to *separable extensions*) in projective spaces of arbitrary dimension.

Recall that a *full* subspace of a linear space is a subset  $F$  of the point set such that, for any two points in  $F$ , every point on the block joining them belongs to  $F$ .

**Lemma 3.1.** *Every full subspace of a generalized Hermitian unital is either the unital itself, or a block, or has at most one point.*

*Proof.* A full subspace of a generalized Hermitian unital  $\mathcal{U}(C|R)$  is also embedded in the projective plane  $\text{PG}(2, C)$  with all blocks being Baer sublines with respect to  $C|R$ . If the subspace is not contained in a block then it spans  $\text{PG}(2, C)$ . Hence Theorem 2.5 says that the subspace is the generalized Hermitian unital itself (since it does not contain O’Nan configurations), in its standard embedding.  $\square$

**Lemma 3.2.** *Let  $(Q, \mathcal{B})$  be a nontrivial linear space with  $Q \subseteq \text{PG}(2, C)$  such that each member of  $\mathcal{B}$  is a Baer subline of  $\text{PG}(2, C)$  with respect to  $C|R$ , and that the mapping  $\lambda$  is injective. Then  $(Q, \mathcal{B})$  is a Baer subplane if and only if it contains a triangle of blocks contained in a Baer subplane.*

*Proof.* Clearly every Baer subplane contains such a triangle of blocks. Conversely, let  $\{u, v, w\}$  be the vertices of such a triangle, and choose two points  $p, q$  on different sides (and different from the vertices). The block joining  $p$  and  $q$  in the Baer subplane meets the third side in a point of the subplane, and that point belongs to  $Q$ . This gives an O’Nan configuration in  $(Q, \mathcal{B})$ , and Theorem 2.1 applies.  $\square$

We also have the following property of Hermitian unitals.

**Lemma 3.3.** *Let  $\mathcal{H}(C|R) = (U, \mathcal{B})$  be a generalized Hermitian unital. Let  $w \in U$  be any point and  $W \in \mathcal{B}$  any block not containing  $w$ . Then every Baer subplane (of  $\text{PG}(2, C)$  with respect to  $C|R$ ) containing  $w$  and  $W$  contains one, and only one, block of  $\mathcal{H}(C|R)$  through  $w$  and some point of  $W$ .*

*Proof.* Let  $(P, \mathcal{L}_P)$  be a Baer subplane (with respect to  $C|R$ ) containing  $w$  and  $W$ . From Lemma 3.2 we infer that there is at most one block  $B \in \mathcal{L}_P \cap \mathcal{B}$  through  $w$  and meeting  $W$ .

By Lemma 2.3.a there are a point  $c$  on  $\lambda(W) \setminus W$  and a line  $L$  through  $c$  such that  $L$  intersects every block of  $\mathcal{H}(C|R)$  through  $w$  and a point  $t \in W$ , and that intersection is outside  $\{w, t\}$ .

The lines  $L$  and  $\lambda(W)$  meet in  $c \notin W$ , so  $L \cap P$  is not a block of  $(P, \mathcal{L}_P)$  but consists of just one point  $s$ . The block  $B \in \mathcal{L}_P$  joining  $w$  and  $s$  meets  $\lambda(W)$  in a point  $t \in W$  because  $(P, \mathcal{L}_P)$  is a projective plane and  $W \in \mathcal{L}_P$ . So  $s$  belongs to the block  $B_t \in \mathcal{B}$  joining  $w$  and  $t$  in  $\mathcal{H}(C|R)$ . Now  $B_t$  and  $B$  are Baer sublines with respect to  $C|R$ , and  $\{w, s, t\} \subseteq B_t \cap B$  yields  $B_t = B$ . □

#### 4. Projective spaces of higher dimension

Let  $C|R$  be a quadratic field extension, and let  $V$  be a vector space of (possibly infinite) dimension greater than three over  $C$ . Consider a nontrivial linear space  $(Q, \mathcal{B})$  with  $Q \subseteq \text{PG}(V)$ , such that  $Q$  spans  $\text{PG}(V)$ , and such that each member of  $\mathcal{B}$  is a Baer subline of  $\text{PG}(V)$  with respect to  $C|R$ . Assume that the mapping  $\lambda|_{\mathcal{B}}$  is injective.

**Definition 4.1.** For any set  $X$  of points of  $\text{PG}(V)$ , let  $\mathcal{L}_X$  denote the set of intersections of  $X$  with secants; i.e. with lines of  $\text{PG}(V)$  that contain more than one point of  $X$ . A *flat* of  $X$  is the intersection of  $X$  with a plane of  $\text{PG}(V)$  containing at least one triangle of points in  $X$ .

**Theorem 4.2.** *Suppose there exists a plane  $\pi$  of  $\text{PG}(V)$  such that the flat  $\pi_Q := Q \cap \pi$  is a Baer subplane of  $\pi$ . Then  $(Q, \mathcal{B})$  is a Baer subspace of  $\text{PG}(V)$ .*

*Proof.* It is obvious that the graph with vertices the flats of  $Q$ , adjacent when they share a block, is connected. Hence, to show that every flat is a Baer subplane, it suffices to show this for flats sharing a block with  $\pi_Q$ . Let  $\alpha_Q = Q \cap \alpha$  be such a flat, where  $\alpha$  is a plane of  $\text{PG}(V)$ . Select  $p \in \alpha_Q \setminus \pi_Q$ . For each block  $B$  in  $\pi_Q$ , we denote by  $\beta_B$  the flat obtained by intersecting  $Q$  with the plane spanned by  $\{p\} \cup B$ . Aiming at a contradiction, by Theorem C, we assume that  $\alpha_Q$  is a generalized Hermitian unital.

**Claim.** *For at most one block  $A$  of  $\pi_Q$ , the flat  $\beta_A$  is the point set of a Baer subplane.*

Suppose for a contradiction that  $\beta_{B_1}$  and  $\beta_{B_2}$  are Baer subplanes, with  $B_1 \neq B_2$  two blocks of  $\pi_Q$ . As  $\pi_Q$  is a Baer subplane, the blocks  $B_1$  and  $B_2$  meet in a point  $b \in \pi_Q$ . Let  $\Sigma$  be the solid of  $\text{PG}(V)$  spanned by  $\pi$  and  $\alpha$ , and let  $B$  be the block of  $(Q, \mathcal{B})$  containing  $p$  and  $b$ . There is a unique Baer subspace  $\Sigma_B$  (with respect to  $C|R$ ) of  $\Sigma$  containing all points of  $\pi_Q \cup B$ . For  $i \in \{1, 2\}$ ,  $\Sigma_B$  clearly contains  $\beta_{B_i}$ , because a Baer subplane (with respect to  $C|R$ ) is determined

by any two of its blocks. Now pick  $p_i \in B_i \setminus \{b\}$ . For  $i \in \{1, 2\}$ , the intersection of  $\Sigma_B$  and the line through  $p$  and  $p_i$  then is a block (i.e., belongs to  $\mathcal{B}$ ).

Lemma 3.2 implies that  $Q$  contains the intersection of  $\Sigma_B$  with  $\langle p, p_1, p_2 \rangle$ . It now follows easily that  $\Sigma_B \subseteq Q$ . This implies that  $\alpha_Q$  contains a Baer subplane, contradicting Lemma 3.3. Thus the Claim is established (under the assumption that the flat  $\alpha_Q$  is not a Baer subplane).

Pick any point  $u$  of  $\pi_Q \setminus \alpha_Q$ . By the Claim, we find two blocks  $C_1$  and  $C_2$  in  $\pi_Q$  with  $u \in C_1 \cap C_2$  such that the flats  $\beta_{C_1}$  and  $\beta_{C_2}$  are generalized Hermitian unitals. Let  $C$  be the block of  $(Q, \mathcal{B})$  through  $p$  and  $u$ . Select any point  $x \in \lambda(C) \setminus C$ . Let  $\Sigma_x$  be the Baer subspace (with respect to  $C|R$ ) of  $\Sigma$  defined by  $\pi_Q$  and the Baer subline (with respect to  $C|R$ ) containing  $\{p, x, u\}$ .

By Lemma 3.3, there exists a point  $u_i \in C_i$  such that  $\Sigma_x$  contains the block  $D_i$  of  $(Q, \mathcal{B})$  through  $p$  and  $u_i$ . But then the flat determined by  $\{p, u_1, u_2\}$  is a Baer subplane, by Lemma 3.2. Varying  $x$  in  $\lambda(C) \setminus C$ , we obtain at least  $|R|$  such Baer subplanes, contradicting the Claim. This contradiction leaves only the possibility that each flat is a Baer subplane.

Now let  $(M, \mathcal{L}_M)$  be a Baer subspace of  $\text{PG}(V)$  with respect to  $C|R$ , maximal among the Baer subspaces contained in  $Q$ . Aiming at a contradiction, we assume that there exists a point  $x \in Q \setminus M$ . Let  $K$  be the subspace of  $\text{PG}(V)$  spanned by  $\{x\} \cup M$ . For any point  $y \in M$  and any block  $B \in \mathcal{B}$  through  $y$  in  $M$ , the flat  $\beta_B := Q \cap \langle x, B \rangle$  is a Baer subplane. Let  $A$  be the block joining  $x$  and  $y$ , and let  $K'$  be the Baer subspace of  $K$  generated by  $M \cup A$ . Then  $K'$  contains  $\beta_B$ , and  $K'$  is spanned (as a projective space over  $R$ ) by  $M \cup \{x\}$ . Moreover, we have  $A \subseteq Q$  because  $A$  is a block of  $\beta_B$ . Varying  $B$  through  $y$ , we see that  $K' \cap Q$  contains every block joining  $x$  to a point in  $M$ . As every line in  $K'$  meets the hyperplane  $M$ , we obtain  $K' \subseteq Q$ , contradicting maximality of  $M$ .  $\square$

## Hermitian linear spaces in projective space

**Definition 4.3.** A *Hermitian linear space* in  $\text{PG}(V)$  (with respect to  $C|R$ ) is a set  $Q$  of points of  $\text{PG}(V)$  such that each member of  $\mathcal{L}_Q$  is a Baer subline of  $\text{PG}(V)$  with respect to  $C|R$ , at least one flat of  $Q$  is not a Baer subplane, and  $\text{PG}(V)$  is generated by  $Q$ .

Note that  $(Q, \mathcal{L}_Q)$  is in fact a linear space, as the name suggests.

As there exists a flat, the dimension of  $V$  is at least 3 if there exists a Hermitian linear space in  $\text{PG}(V)$ . The mapping  $\lambda|_{\mathcal{L}_Q} : \mathcal{L}_Q \rightarrow \mathcal{L} : B \mapsto \lambda(B) = \langle B \rangle$  is injective by definition of  $\mathcal{L}_Q$ . From Theorem 4.2 we know that every flat is a Baer subplane if there exists at least one such flat. We conclude from Theorem C that every flat in a Hermitian linear space in  $\text{PG}(V)$  is isomorphic to the generalized Hermitian unital  $\mathcal{U}(C|R)$ , in its standard embedding into the plane inducing the flat. We shall use this observation without further reference. It implies that, if  $\dim V = 3$ , a Hermitian linear space is a generalized Hermitian unital. Hence we are only interested in the case  $\dim V \geq 4$ .

**Lemma 4.4.** *Let  $Q$  be a Hermitian linear space in  $\text{PG}(V)$ . If  $\dim V \geq 4$  then no plane of  $\text{PG}(V)$  contains exactly one member of  $\mathcal{L}_Q$ .*

*Proof.* Aiming at a contradiction, suppose that some plane  $\pi$  contains exactly one member  $B$  of  $\mathcal{B}$ .

Assume  $\dim V = 4$  first. Pick an arbitrary point  $b \in B$ . It is easy to find a plane  $\alpha$  containing  $b$ , not containing  $B$ , and intersecting  $Q$  in a generalized Hermitian unital  $\mathcal{U}$ . Let  $b' \in B \setminus \{b\}$ . Let  $B'$  be a block of  $\mathcal{U}$  containing  $b$ . The plane  $\langle B', B \rangle$  intersects  $Q$  in a generalized Hermitian unital, the tangent line at  $b'$  intersects  $\alpha$  in a point  $x \in \lambda(B') \setminus Q$ . Let  $K \neq \alpha \cap \langle x, B \rangle$  be a line in  $\alpha$  containing  $x$  and intersecting  $\mathcal{U}$  in a block  $D$ . Then  $B \not\subseteq \langle b', D \rangle$ . Now the plane  $\langle b', D \rangle$  contains two lines (namely,  $\langle b', x \rangle$  and  $\langle b', D \rangle \cap \pi$ ) through  $b'$  that intersect  $Q$  in exactly one point, namely  $b'$ . But  $\langle b', D \rangle \cap Q$  is a generalized Hermitian unital, which has only one tangent at  $b'$ . This contradiction shows that the lemma is true if  $\dim V = 4$ .

Now assume  $\dim V > 4$ . Pick a point  $x \in \pi \setminus B$ . Since  $Q$  generates  $\text{PG}(V)$ , we find a finite set  $F$  of points of  $Q$  such that the subspace  $\langle F \rangle$  contains  $x$ . Then  $U := \langle F, B \rangle$  is a finite-dimensional subspace of  $\text{PG}(V)$  spanned by  $U \cap Q$ . Let  $W$  be a subspace of  $U$  maximal with respect to the properties  $\pi \subseteq W$  and  $W \neq \langle W \cap Q \rangle$  (such a subspace exists since  $\pi$  satisfies the given properties and  $\dim U$  is finite). Since  $W \neq U$ , there exists a subspace  $W' \supseteq W$  with  $\dim W' = 1 + \dim W$ . By maximality of  $W$  we have  $\langle W' \cap Q \rangle = W'$ . Hence we can pick points  $p \in Q \setminus W$  and  $q \in Q \setminus (\langle W \cap Q, p \rangle \cup W)$ . Then  $A := \langle p, q, B \rangle$  is a projective 3-space with  $\langle A \cap Q \rangle = A$ . The plane generated by  $B$  and  $\langle p, q \rangle \cap W$  is contained in  $A$  but intersects  $Q$  in just  $B$ . This contradicts the previous paragraph. This contradiction completes the proof of the lemma.  $\square$

**Lemma 4.5.** *Let  $Q$  be a Hermitian linear space in  $\text{PG}(V)$ . For each point  $p \in Q$ , the union  $T_p$  of the set of lines  $L$  with  $L \cap Q = \{p\}$  is a hyperplane.*

*Proof.* We first show that  $T_p$  is a subspace. Let  $u, v \in T_p \setminus \{p\}$  be two distinct points. Then  $\langle p, u \rangle$  and  $\langle p, v \rangle$  both intersect  $Q$  in only  $p$ . We may assume that  $\langle p, u \rangle \neq \langle p, v \rangle$ . Then the plane  $\langle p, u, v \rangle$  does not intersect  $Q$  in a generalized Hermitian unital, since there would be two tangent lines at  $p$  to that generalized Hermitian unital. Using Lemma 4.4 we infer  $\langle p, u, v \rangle \cap Q = \{p\}$ . Thus  $\langle p, u, v \rangle \subseteq T_p$ , and  $T_p$  is a subspace.

It remains to show that each line intersects  $T_p$  nontrivially. Let  $L$  be an arbitrary line. We may assume  $p \notin L$ . Lemma 4.4 yields that the plane  $\langle p, L \rangle$  intersects  $Q$  in either exactly  $\{p\}$  (and then  $L \subseteq T_p$ ), or a generalized Hermitian unital (and then  $L$  intersects the tangent line  $M \subseteq T_p$  at  $p$  to that generalized Hermitian unital nontrivially).  $\square$

The following Proposition establishes Theorem A.

**Proposition 4.6.** *If  $C$  is finite then either  $(Q, \mathcal{B})$  is a Baer subspace (and  $\text{PG}(V)$  has arbitrary dimension), or a Hermitian unital (and  $\text{PG}(V)$  is a plane).*

*Proof.* The existence of tangent hyperplanes (as established by Lemma 4.5) rules out the finite case for  $\dim V \geq 4$ . Indeed, let  $\mathcal{U}$  be any flat of  $Q$  and  $p \in Q \setminus \mathcal{U}$ . A simple counting argument yields that every line of the plane  $\langle \mathcal{U} \rangle$  meets  $\mathcal{U}$  in at least one point. This leads to a contradiction with the definition of  $T_p$  and Lemma 4.5.  $\square$

**Lemma 4.7.** *Let  $Q$  be a Hermitian linear space in  $\text{PG}(V)$ , with respect to a (not necessarily separable) quadratic field extension  $C|R$ . Then every full subspace  $S$  of  $(Q, \mathcal{B})$  spanning  $\text{PG}(V)$  coincides with  $Q$ .*

*Proof.* Every point  $p \in Q$  is contained in the span  $\langle F \rangle$  of a finite subset  $F$  of  $S$ . Clearly  $\langle F \rangle \cap Q$  is a Hermitian linear space in  $\langle F \rangle$ , and the full subspace  $\langle F \rangle \cap S$  of  $\langle F \rangle \cap Q$  spans  $\langle F \rangle$  because it contains  $F$ . So it suffices to consider the case where  $\text{PG}(V) = \langle F \rangle$  (so  $d := \dim V$  is finite). Without loss of generality, we may assume that  $F$  consists of exactly  $d$  points.

The existence of a Hermitian linear space in  $\text{PG}(V)$  implies  $d \geq 3$ . If  $d = 3$  then  $(Q, \mathcal{L}_Q)$  is a (generalized) Hermitian unital in the projective plane over  $C$ , and has been treated in Lemma 3.1.

Now we proceed by induction on  $d$ , starting with  $d = 4$ . Consider an arbitrary point  $x \in Q$ , and pick  $p \in F$ . The three points in  $F \setminus \{p\}$  span a plane  $\pi$  meeting  $Q$  in a flat. Let  $L$  be the line joining  $x$  with  $p$ , and let  $z$  be the point where  $L$  meets  $\pi$ . We find a secant through  $z$  if either  $C|R$  is separable or  $C|R$  is inseparable but  $z$  is not the nucleus of the flat. In these cases, the plane generated by  $p$  and that secant defines a flat completely contained in  $S$ , and  $x \in S$  follows.

There remains the case where  $C|R$  is inseparable, and  $z$  is the nucleus of the flat in  $\pi$ . Then all points of  $Q \setminus L$  belong to  $S$ , and  $Q = \langle Q \setminus L \rangle = S$ .

Now consider  $d > 4$ . By assumption  $F$  consists of  $d$  points spanning  $\text{PG}(V)$ . Let  $p$  be one of them and let the projective subspace  $\Pi$  be spanned by the  $d - 1$  points in  $F \setminus \{p\}$ . Set  $\mathcal{U} = \Pi \cap Q$ , then  $\mathcal{U} \subseteq \Pi \cap S \subseteq S$  by the induction hypothesis. Let  $x \in Q \setminus \Pi$  be arbitrary, and let  $z$  be the point where the line  $L = \langle p, x \rangle$  meets  $\Pi$ . If there exists a secant of  $\mathcal{U}$  through  $z$  (in particular, if the extension  $C|R$  is separable) then the plane spanned by  $p$  and that secant intersects  $Q$  in a flat and Lemma 3.1 implies  $x \in S$ .

It remains to consider the case where  $C|R$  is inseparable, and every line through  $z$  and a point of  $\mathcal{U}$  meets  $\mathcal{U}$  in just that point. Take any block  $B$  of  $\mathcal{U}$ . Then  $\langle B, z, p \rangle$  generates a projective subspace of projective dimension 3, and the induction hypothesis (for  $d = 4$ ) yields  $p \in S$ .  $\square$

### Constructing a polarity in the separable case

We now assume that  $Q$  is a Hermitian linear space in  $\text{PG}(V)$  with respect to a *separable* quadratic extension  $C|R$ .

**Lemma 4.8.** *Every point of  $\text{PG}(V)$  is contained in a secant.*

*Proof.* Let  $x \in \text{PG}(V)$  be arbitrary and select a block  $B \in \mathcal{L}_Q$ . If  $x$  is contained in the line  $\lambda(B)$  spanned by  $B$ , then we are done. So suppose  $\langle x, B \rangle$  is a plane, which then induces a Hermitian unital by Lemma 4.4. Clearly  $x$  is contained in a secant of the Hermitian unital since the extension  $C|R$  is separable.  $\square$

**Lemma 4.9.** *Consider the Hermitian unital  $(U, \mathcal{B})$  in the projective plane  $\text{PG}(2, C)$ , and let  $x \mapsto x^\perp$  be the (unique) polarity such that  $U$  is the set of absolute points of that polarity. On each secant  $L$  of the unital, the mapping  $L \rightarrow L: x \mapsto x^\perp \cap L$  coincides with the unique Baer involution (with respect to  $C|R$ ) fixing each point of the block  $L \cap U$  (and no other). If  $L$  is a tangent, the same geometric construction yields a constant map (mapping each point to the unique point in  $L \cap U$ ).  $\square$*

**Lemma 4.10.** *Let  $x \in \text{PG}(V) \setminus Q$  be arbitrary. For each secant line  $L$  through  $x$  define the point  $x_L$  as the image of the unique Baer involution (with respect to  $C|R$ ) on  $L$  fixing  $Q \cap L$  pointwise. For each tangent line  $L$  through  $x$ , define  $x_L$  to be the unique point in  $L \cap Q$ . Then the set  $S_x$  of all such points  $x_L$  generates a hyperplane  $T_x$ .*

*Proof.* Let  $S'_x \subseteq S_x$  be the set of points  $x_L$  with  $L$  a secant. We claim that  $\{x\} \cup S'_x$  spans  $\text{PG}(V)$ . Indeed, let  $K$  be any line through  $x$  and let  $L$  be a secant through  $x$ . By Lemma 4.4, the plane  $\langle K, L \rangle$  induces a Hermitian unital in  $Q$ , and  $x$  is on a second secant  $M$  of that unital. Now  $K$  intersects  $\langle S'_x \rangle$  in a point of  $\langle x_M, x_L \rangle$ , and that point is distinct from  $x$ . Consequently,  $K \subseteq \langle x, S'_x \rangle$  and the claim follows.

So there exists a set  $E \subseteq S'_x$  such that  $\{x\} \cup E$  is a basis of  $\text{PG}(V)$ . Then  $E$  generates a hyperplane  $H$ . Let  $Q_H$  be the set of points of  $Q$  on the secants  $L$  through  $x$  such that  $x_L \in H$ . For each such secant  $L$ , the block  $L \cap Q$  generates  $x$  and  $x_L$ . This implies that  $\{x\} \cup E \subseteq \langle x, H \rangle$ . Hence  $\langle Q_H \rangle = \langle x, H \rangle = \text{PG}(V)$ .

Now let  $P \subseteq Q$  be the set of points  $y \in Q$  such that  $x_{\langle x, y \rangle} \in H$ . Clearly  $Q_H \subseteq P$ . For any two points  $u, v$  of  $P$  such that  $x, u, v$  are not collinear, the plane  $\pi := \langle x, u, v \rangle$  intersects  $Q$  in a Hermitian unital. If  $L \subseteq \pi$  is a secant through  $x$  then the point  $x_L$  is contained in the line  $M$  of  $\pi$  spanned by  $x_{\langle x, u \rangle}$  and  $x_{\langle x, v \rangle}$  because that line is the polar of  $x$  with respect to the polarity corresponding to the Hermitian unital in  $\pi$  (see Lemma 4.9). If  $L \subseteq \pi$  is a tangent through  $x$  then  $x_L$  lies on  $M$ , anyway.

We obtain that each point  $w$  on the block  $\langle u, v \rangle \cap Q$  satisfies  $x_{\langle x, w \rangle} \in H$ . Hence  $P$  is a full subspace of  $Q$ . As  $x$  lies on at least two secants, we have  $x \in \langle P \rangle$  and thus  $\langle P \rangle = \langle x, S'_x \rangle = \text{PG}(V)$ . Lemma 4.7 implies  $P = Q$ . So  $S_x \subseteq H$ , and  $T_x := H = \langle S_x \rangle$ . The proof is complete.  $\square$

**Theorem D.** *Let  $Q$  be a Hermitian linear space in  $\text{PG}(V)$ , with respect to a separable quadratic field extension  $C|R$ . Then there exists a (non-degenerate) polarity of  $\text{PG}(V)$  such that  $Q$  consists of the absolute points of that polarity. That polarity is represented by a non-degenerate Hermitian form with respect to the involution generating  $\text{Gal}(C|R)$ .*

*Proof.* For  $x \in Q$ , we have constructed the tangent hyperplane  $T_x$  in Lemma 4.5. For  $x \notin Q$ , we use the hyperplane  $T_x$  as defined in Lemma 4.10. We claim that the mapping  $x \mapsto T_x$  is a (non-degenerate) polarity (as defined by Tits [Tit86, 8.3.2, p. 128]). It suffices to prove that the correspondence  $x \in T_y$  is symmetric, that is,  $x \in T_y$  if and only if  $y \in T_x$ . By considering a plane containing  $x$  and a secant through  $y$ , this reduces to the case of a plane, where the correspondence holds by Lemma 4.9.

The restriction of the polarity to any plane intersecting  $Q$  in a flat is represented by a Hermitian form (with respect to the involution generating  $\text{Gal}(C|R)$ ). Thus the polarity is represented by a Hermitian form with respect to the same involution (again, see [Tit86, 8.3.2, p. 128]).  $\square$

Theorem D, together with the planar case of Theorem C, yields Theorem B.

**Open Problems.** Various questions suggest further research.

- a. Do there exist linear spaces embedded in the projective plane  $\text{PG}(2, C)$  such that the point set generates the plane, every block is a Baer subline and Baer sublines with respect to distinct subfields of  $C$  occur as blocks?
- b. Is there an analogue for Theorem B in the inseparable case?
- c. If one omits the injectivity of  $\lambda$  in the conditions of Theorem C, are there additional examples besides the ones mentioned after the statement of Theorem C in the introduction?

## References

- [BE08] Susan G. Barwick and Gary Ebert. *Unitals in projective planes*. Springer Monographs in Mathematics. Springer, New York, 2008. doi:10.1007/978-0-387-76366-8.
- [FK83] Giorgio Faina and Gábor Korchmáros. A graphic characterization of Hermitian curves. In *Combinatorics '81 (Rome, 1981)*, volume 18 of *Ann. Discrete Math.*, pages 335–342. North-Holland, Amsterdam-New York, 1983. doi:10.1016/S0304-0208(08)73314-0.
- [GSV19] Theo Grundhöfer, Markus Johannes Stroppel, and Hendrik Van Maldeghem. Embeddings of hermitian unitals into pappian projective planes. *Aequationes Math.*, 93(5):927–953, 2019. doi:10.1007/s00010-019-00652-x.
- [GSV21] Theo Grundhöfer, Markus Johannes Stroppel, and Hendrik Van Maldeghem. Embeddings of unitals such that each block is a subline. *Australas. J. Combin.*, 79:295–301, 2021. URL: [https://ajc.maths.uq.edu.au/pdf/79/ajc\\_v79\\_p295.pdf](https://ajc.maths.uq.edu.au/pdf/79/ajc_v79_p295.pdf).
- [LP82] Christiane Lefèvre-Percsy. Characterization of Hermitian curves. *Arch. Math. (Basel)*, 39(5):476–480, 1982. doi:10.1007/BF01899549.
- [O’N72] Michael E. O’Nan. Automorphisms of unitary block designs. *J. Algebra*, 20:495–511, 1972. doi:10.1016/0021-8693(72)90070-1.
- [Tit86] Jacques Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer, Berlin, 2nd edition, 1986. 2nd corrected printing.