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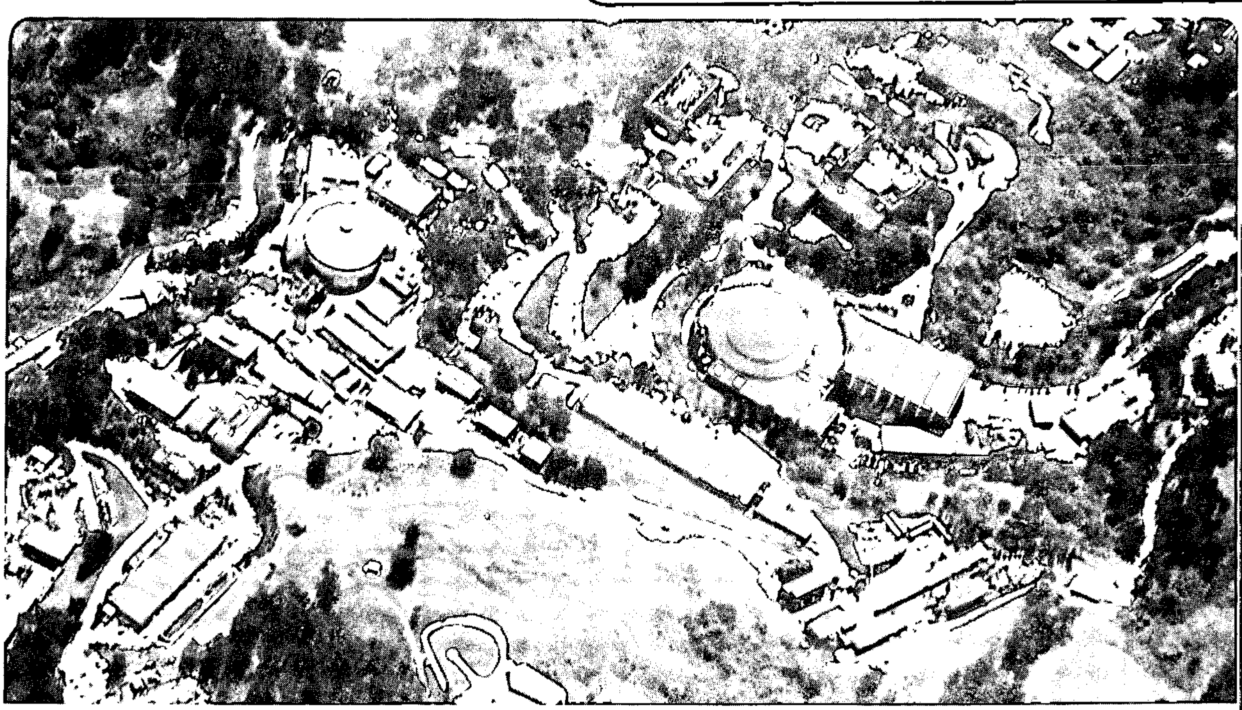

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New Symmetries of Supersymmetric Effective Lagrangians

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Abstract

We consider the structure of effective lagrangians describing the low-energy dynamics of supersymmetric theories in which a global symmetry G is spontaneously broken to a subgroup H while supersymmetry is unbroken. In accordance with the supersymmetric Goldstone theorem, these lagrangians contain Nambu–Goldstone superfields associated with a coset space $G^c/\hat{H}$, where G^c is the complexification of G and $\hat{H}$ is the largest subgroup of G^c that leaves the order parameter invariant. The lagrangian may also contain additional light matter fields. To analyze the effective lagrangian for the matter fields, we first consider the case where the effective lagrangian is obtained by integrating out heavy modes at weak coupling (but including non-perturbative effects such as instantons). We show that the superpotential of the matter fields is $\hat{H}$ invariant, which can give rise to non-trivial relations among independent H-invariants in the superpotential. We also show that the Kähler potential of the matter fields can be restricted by a remnant of $\hat{H}$ symmetry. These results are non-perturbative and have a simple group-theoretic interpretation. When we relax the weak-coupling constraint, there appear to be additional possibilities for the action of $\hat{H}$ on the matter fields, hinting that the constraints imposed by $\hat{H}$ may be even richer in strongly coupled theories.

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1. Introduction

Supersymmetry provides an elegant framework for understanding the hierarchy between the weak scale and much larger mass scales such as the grand-unification and Planck scales that are believed to play a fundamental role in nature [1]. However, there is at present no direct information about what role supersymmetry will play in the more fundamental theory that we believe lies behind the standard model of electroweak and strong interactions. Given this situation, we believe it is essential to understand the general features of supersymmetric theories as fully as possible.

In many models for physics beyond the standard model, the symmetries (approximate, exact, or gauged) that we observe are remnants of a larger symmetry that is spontaneously broken at some high energy scale Λ. Because the scale Λ is often too large to be probed directly, it is important to know what constraints this places on the physics at observable energies $E \ll \Lambda$. For non-supersymmetric theories, this question was answered in an elegant paper by Coleman, Wess, and Zumino [2]. This paper derives a useful canonical form for the most general effective lagrangian describing the low-energy physics in a model where a global symmetry G is broken spontaneously down to a subgroup H. The effective lagrangian contains fields for the Nambu–Goldstone bosons (NGB's) associated with the coset space G/H, as well as additional light "matter" fields that can be chosen to transform according to linear representations of H. The matter fields can couple to each other in the most general way allowed by H invariance, while the NGB's are derivatively coupled [3], so their interactions are suppressed by powers of E/Λ. Therefore, at sufficiently low energies, the only important interactions are those of the matter fields among themselves. Since the matter fields can interact in the most general way allowed by the unbroken group H, one can describe this result by saying that H invariance is the only remnant of the symmetry group G at energy scales small compared to Λ.

In this paper, we show that this result is modified in an interesting way in supersymmetric theories. We consider a theory with $N = 1$ supersymmetry in which a symmetry group G is spontaneously broken down to a subgroup H, while supersymmetry is left unbroken.* We consider the most general effective lagrangian describing the interactions of NGB's and their superpartners (which we collectively refer to as SNGB's), and "matter" chiral superfields. The SNGB's are described by chiral superfields living in the coset space $G^c/\hat{H}$, where G^c is the complexification of G and $\hat{H}$ is the largest subgroup of G^c that leaves the order parameter invariant; this is in agreement with the supersymmetric Goldstone theorem [4]. Clearly, $\hat{H} \supseteq H^c$, but $\hat{H}$ is in general *larger* than H^c [5][6]. (The special case where $\hat{H} = H^c$ was discussed extensively in the literature; see *e.g.*, [7].)

To analyze the matter fields, we begin by discussing the case where the effective lagrangian is obtained by integrating out heavy modes at weak coupling. Our results rely only on symmetry arguments, and are therefore valid non-perturbatively. This is important despite the fact that non-perturbative effects in weak coupling vanish faster than any power of the coupling (instanton effects, for example). This is because non-perturbative effects in supersymmetric theories can lift degeneracies that persist to all orders in perturbation theory [8][9]. Many of the non-perturbative effects in supersymmetric gauge theories discussed in the recent literature [10][11] (see also [12]) are interesting examples of this phenomenon.

For the weak-coupling case, the matter fields transform according to linear representations of the group $\hat{H}$, even though the true unbroken symmetry of the theory is H. Supersymmetry restricts the way that $\hat{H}$ is broken down to H, and our first major result is that holomorphy implies that the

* We do not treat the case where a $U(1)_R$ symmetry is spontaneously broken.

effective superpotential of the matter fields is in fact $\hat{H}$ invariant. Because $\hat{H}$ can be larger than H^c, this can lead to non-trivial relations between different H-invariants in the effective superpotential. Perhaps more surprisingly, we show that there can be a remnant of $\hat{H}$ symmetry that restricts the effective Kähler potential of the matter fields as well. We illustrate these results with simple explicit models.

We then relax the assumption of weak coupling in the fundamental theory and consider the most general effective lagrangian describing the low-energy dynamics when G is spontaneously broken to H. We are unable to classify the group action of $\hat{H}$ on the effective fields in this case: for example, there are cases where the $\hat{H}$ action cannot be made linear by redefining the effective fields. Even if $\hat{H}$ acts linearly, there are $\hat{H}$ representations for which we are unable to write kinetic terms. While it is certainly dangerous to draw any conclusions from ignorance, we note that this may be taken as a hint that the role of $\hat{H}$ may be even richer in strong-coupling theories.

This paper is organized as follows: in section 2, we consider the most general effective lagrangian that can describe the low-energy dynamics of the spontaneously broken theory. We explain the role of the groups G^c and $\hat{H}$ and give some results on the structure of these groups. In section 3, we turn our attention to the matter fields and derive a simple canonical form for the effective lagrangian describing the SNGB's and matter fields for the case where the effective theory is obtained by integrating out heavy modes at weak coupling; this section contains the main results of this paper. In section 4, we analyze the most general effective lagrangian describing spontaneous symmetry breaking. Section 5 contains our conclusions.

2. The effective lagrangian for the SNGB's

In this section, we consider the most general effective lagrangian describing the low-energy dynamics of a theory with a compact global symmetry group G spontaneously broken to a (compact) subgroup H, while supersymmetry is left unbroken. We will concentrate on the SNGB sector of the effective lagrangian in this section, leaving a detailed discussion of the matter fields for the next two sections.

2.1. The Role of G^c and $\hat{H}$

The main new feature of the supersymmetric case is that the group G^c plays an important role in restricting the low-energy couplings. G^c is the complexification of G, defined by choosing a hermitian basis of generators for G and allowing the group parameters to be complex. To understand the importance of this group, consider the underlying "fundamental" theory whose dynamics gives rise to the symmetry breaking. We assume that this theory is a $N = 1$ supersymmetric theory of chiral superfields coupled to gauge superfields. We can write the lagrangian for this theory as

$$\mathcal{L}_{\text{fund}} = \int d^2\theta\, d^2\bar{\theta}\, K(\overline{\Phi}, \Phi) + \left(\int d^2\theta\, W(\Phi) + \text{h.c.} \right), \tag{2.1}$$

where we have shown only the dependence on the chiral superfields Φ; gauge fields are also present in general, but are not written explicitly. This lagrangian is assumed to have a global symmetry G, which must of course commute with the gauge group.

The first observation is that the superpotential $W(\Phi)$ is actually invariant under G^c [13]. The reason is simply that W is a holomorphic function of Φ (that is, it is independent of $\bar{\Phi}$), and so it is invariant whether the group parameters are taken real or complex.

The Kähler potential $K(\Phi, \bar{\Phi})$ is not holomorphic, and is therefore not invariant under G^c. However, we can make the Kähler potential formally invariant under G^c by introducing "spurion" gauge field sources $\mathcal{V}$ transforming under G^c as

$$e^{\mathcal{V}} \mapsto g^{-1\dagger} e^{\mathcal{V}} g^{-1}, \qquad g \in G^c. \tag{2.2}$$

These gauge fields are not dynamical, and we will set $\mathcal{V} = 0$ at the end of the calculation.* (Differentiating with respect to components of $\mathcal{V}$ allows us to obtain information about symmetry currents and related operators, and is also useful for making contact with the "current algebra" approach to the low-energy dynamics.) We can then write the formally G^c-invariant lagrangian by replacing $\overline{\Phi}$ with

$$\overline{\Phi} e^{\mathcal{V}} \mapsto \overline{\Phi} e^{\mathcal{V}} \cdot g^{-1}, \qquad g \in G^c. \tag{2.3}$$

(If there are derivatives in K, they must be replaced by gauge-covariant derivatives constructed from $\mathcal{V}$.) The role of $\mathcal{V}$ is to keep track of how G^c is explicitly broken down to G by the Kähler terms in the fundamental lagrangian.

To understand why this is a useful thing to do, it is helpful to contrast our introduction of $e^{\mathcal{V}}$ with the more familiar case of explicit flavor symmetry breaking by current quark masses in QCD. In QCD with N_F quark flavors there is a $SU(N_F)_L \times SU(N_F)_R$ chiral symmetry that is explicitly broken by quark masses. The effects of this explicit breaking are taken into account by treating the quark mass m_q as a spurion field transforming under $SU(N_F)_L \times SU(N_F)_R$ as

$$m_q \mapsto L m_q R^\dagger, \tag{2.4}$$

which formally restores the chiral symmetry of the QCD lagrangian. This is useful if the quark masses are small (compared to $\Lambda_{\rm QCD}$), because terms proportional to many powers of m_q in the low-energy effective lagrangian below the scale $\Lambda_{\rm QCD}$ can then be neglected. If the quark masses are not small, introducing the quark mass as a spurion is not useful, since many powers of m_q can be used to write down any desired $SU(3)$-violating term with an unsuppressed coefficient.

In the supersymmetric case, the symmetry G^c is not an approximate symmetry because $e^{\mathcal{V}}$ is not small in any sense. Nevertheless, it is useful to introduce the gauge field spurion explicitly because one cannot use it to write down arbitrary G-invariant terms in the effective lagrangian. To see this, note that only terms with no (spacetime or supersymmetry) derivatives acting on $\mathcal{V}$ are non-zero when we set $\mathcal{V} = 0$, so we can restrict attention to such terms. But *functions of $\mathcal{V}$ that do not involve derivatives of $\mathcal{V}$ cannot appear in the effective superpotential*, because their transformation properties involve $g^\dagger$, which is an antichiral superfield. Therefore, *the superpotential of the effective lagrangian behaves as though the underlying theory were invariant under G^c*. Furthermore we will argue in subsection 3.4 that the dependence of the Kähler potential on $\mathcal{V}$ is restricted by the spurious gauge transformation properties of $e^{\mathcal{V}}$, and we find that *the Kähler potential of the effective lagrangian is also restricted by*

* This is analogous to the treatment of anomalies by Wess and Zumino [14]. In this case, global symmetries are enlarged to gauge symmetries by introducing spurion gauge fields, and it is required that the low-energy effective lagrangian has the same anomalous properties as the microscopic lagrangian. Even when one sets all the spurion gauge fields to zero, one is still left with a non-trivial Wess–Zumino term. In this paper we will not address the issue of the appearance of such terms in the supersymmetric effective lagrangian [15].

a remnant of G^c *symmetry*. These are the general principles behind our results; we will see them in action repeatedly below.

The group $\hat{H}$ is defined to be the largest unbroken subgroup of G^c. To be precise, we assume that G is spontaneously broken by an order parameter v that can be thought of as an element of a (reducible) representation ρ of G. We can extend ρ to a representation of G^c simply by allowing the group parameters of G to be complex. The representation matrices therefore do not depend on the complex conjugates of the group parameters, so ρ can be thought of as a holomorphic representation of G^c. The group $\hat{H}$ is then defined by

$$\hat{H} \equiv \{g \in G^c \mid \rho(g)v = v\}. \tag{2.5}$$

That is, $\hat{H}$ can be viewed as the unbroken subgroup of G^c; of course, $\hat{H}$ is broken explicitly down to H by the spurion gauge field $e^{\mathcal{V}}$. We note that $\hat{H} \supseteq H^c$, but we will see that $\hat{H}$ is in general *larger* than H^c [6]. We will describe the structure of $\hat{H}$ in more detail in subsection 2.3.

2.2. The Effective Lagrangian

We now turn to the general structure of the low-energy effective lagrangian. We begin by discussing the conditions on the effective lagrangian that encode the fact that it describes the low-energy dynamics of a theory where a global symmetry G is spontaneously broken down to a subgroup H, while supersymmetry is unbroken. First, the effective lagrangian must be supersymmetric, so we assume that it can be written in terms of chiral superfields. (Light gauge superfields can be introduced by gauging part or all of the global G symmetry. This will not be discussed here.)

Second, since the original theory (including the field $\mathcal{V}$) is invariant under G^c, there is a G^c action on the fields of the effective lagrangian that is nonlinear in general, and which we write as

$$\Phi \mapsto T(g)(\Phi), \qquad g \in G^c \tag{2.6}$$

with

$$T(g_1 g_2)(\Phi) = T(g_1)(T(g_2)(\Phi)), \qquad T(1)(\Phi) = \Phi. \tag{2.7}$$

We assume that the effective lagrangian is invariant under this transformation. The effective theory also contains the spurion gauge field $\mathcal{V}$ transforming as in eq. (2.2), which breaks G^c explicitly down to G.

Finally, we must also encode the information that the symmetry G is broken spontaneously by the order parameter v (introduced above). We want to interpret the fields in the effective lagrangian as fluctuations about the vacuum described by the order parameter v, so we demand that the target space (space of fields) in the effective lagrangian contain a special point (the origin) that is preserved by the action of the subgroup $\hat{H}$. Here we are implicitly assuming that the complex structure of the full theory is inherited by the effective theory, that is, that there are no "holomorphic anomalies" in the matching that determines the effective lagrangian. Since this matching is infrared safe, this is a reasonable assumption [16].

We now consider the most general effective lagrangian satisfying the assumptions above. We will follow closely the arguments of ref. [2]. The basic idea is to use the freedom to make field redefinitions to put the effective lagrangian into a canonical form where its physical content is manifest. Specifically, if we make a field redefinition of the form

$$\Psi = \Phi F(\Phi), \tag{2.8}$$

with $F(0) = 1$ (that is, the redefinition preserves the origin of field space), then the physics described by the the effective lagrangians written in terms of Φ and Ψ is identical.

We therefore make such a field redefinition by decomposing the target space into the orbits of the origin under G^c and the rest. Specifically, we write

$$\Phi = T(\xi)(\Psi), \tag{2.9}$$

where

$$\xi = e^{i\Pi} \in G^c/\hat{H}, \tag{2.10}$$

and Ψ are coordinates for the part of the target space that is left invariant under $\hat{H}$. We can see how the new fields ξ and Ψ transform by noting that for any $g \in G^c$

$$\Phi \mapsto T(g)(T(\xi)(\Psi)) = T(g\xi)(\Psi). \tag{2.11}$$

We then decompose

$$g\xi = \xi'(g,\xi)\hat{h}(g,\xi), \qquad \xi' \in G^c/\hat{H}, \quad \hat{h} \in \hat{H} \tag{2.12}$$

and write

$$\Phi \mapsto T(g\xi\hat{h}^{-1}(g,\xi))(T(\hat{h}(g,\xi)(\Psi))). \tag{2.13}$$

That is, the fields ξ and Ψ transform as

$$\xi \mapsto g\xi\hat{h}^{-1}(g,\xi), \tag{2.14}$$

$$\Psi \mapsto T(\hat{h}(g,\xi))(\Psi). \tag{2.15}$$

The effective lagrangian also contains the spurion gauge field transforming as in eq. (2.2).

We see that with our assumptions, the effective lagrangian automatically contains fields ξ that live in the coset space $G^c/\hat{H}$. One can check that the fields ξ couple to broken symmetry currents in the manner required by the supersymmetric version of Goldstone's theorem [4](see also [5][7]), so that we can identify them with the SNGB's. The fields Ψ are identified with light "matter" fields.

The existence of the fields $\xi \in G^c/\hat{H}$ is a direct consequence of our ability to formally promote G^c to a symmetry of the fundamental lagrangian by introducing the gauge spurion $\mathcal{V}$. Therefore, as a consistency check, we should understand why the presence of $\mathcal{V}$ in the effective lagrangian does not allow us to write a mass term for the SNGB's in a theory with no matter fields. (For example, a quark mass spurion in QCD allows us to write mass terms for the NGB's.) A mass term for the SNGB's must be a superpotential term with no derivatives (spacetime or supersymmetry). It is easy to see that the constraints of the transformation rules in eqs. (2.2) and (2.14), together with the requirement of holomorphy, imply that no such term is possible.

We will not discuss the structure of the effective lagrangian for the SNGB's in much detail, but we briefly indicate how to write an invariant kinetic term for the SNGB's. We restrict ourselves to groups $\hat{H}$ for which

$$\rho(\hat{h}^\dagger) = \rho(\hat{h})^\dagger. \tag{2.16}$$

We will see in subsection 3.4 how this condition can fail, and how to generalize the construction below to all $\hat{H}$. We then define

$$e^{\mathcal{W}} \equiv \xi^\dagger e^{\mathcal{V}} \xi \in G^c, \tag{2.17}$$

which transforms like a gauge field for the group $\hat{H}$:

$$e^{\mathcal{W}} \mapsto \hat{h}^{-1\dagger}(g,\xi)e^{\mathcal{W}}\hat{h}^{-1}(g,\xi). \tag{2.18}$$

Then we can write the kinetic term

$$\mathcal{L}_{\text{eff}} = \int d^2\theta\, d^2\bar{\theta}\, v^\dagger \rho(e^{\mathcal{W}})v, \tag{2.19}$$

where v is the order parameter in the representation ρ of G^c (see eq. (2.5)). To see that eq. (2.19) contains a kinetic term for the SNGB's, note that $\rho(\xi)v = i\rho(\Pi)v + O(\Pi^2)$, so that

$$\mathcal{L}_{\text{eff}} = \int d^2\theta\, d^2\bar{\theta}\, |\rho(\Pi)v|^2 + O(\mathcal{V}) + O(\Pi^3). \tag{2.20}$$

Note that $\rho(\Pi)v$ is linear in Π and is nonzero for all $\Pi \neq 0$ by the definition of the SNGB fields, completing the argument.* We could go on to discuss the general form of the Kähler potential for the SNGB's and the resulting low-energy theorems, but the main focus of this paper is on the matter fields, so we will leave these topics for the future.

2.3. Structure of $\hat{H}$

We now give some results on the structure of the group $\hat{H}$, and illustrate them with some simple examples. The main structure theorem is that $\hat{H}$ has the Levy decomposition

$$\hat{H} = K^c \wedge N, \tag{2.21}$$

where "$\wedge$" denotes a semidirect product (with K^c acting on N). Specifically, this means that any $\hat{h} \in \hat{H}$ can be uniquely decomposed as $\hat{h} = kn$ with $k \in K^c$, $n \in N$, and that $knk^{-1} \in N$ for any $k \in K^c$ and $n \in N$. Here, K is a compact group that can be written as a direct product of a semisimple group and an abelian group, and N is a unipotent group: that is, N is isomorphic to a group of upper-triangular matrices with 1's on the diagonal. (This is the "algebraic" version of the Levy decomposition; see *e.g.* ref. [20]. In the context of SNGB's, this result is discussed in ref. [6].) The multiplication law for the Levy factors of $\hat{H}$ is (in obvious notation)

$$\hat{h}_1\hat{h}_2 = k_1 n_1 \cdot k_2 n_2 = (k_1 k_2)\cdot(k_2^{-1} n_1 k_2 n_2) \tag{2.22}$$

where $k_1 k_2 \in K^c$ and $k_2^{-1} n_1 k_2 n_2 \in N$.

As the examples below will make clear, $H \subseteq K$, but K can be larger than H. (In fact, $K \not\subset G$ in general.) Thus, $\hat{H} \supset H$ if either $N \neq 0$ or $K \supset H$. We illustrate both of these possibilities below.

Consider first an example with $G = U(N)$ broken by an order parameter $\langle\Phi\rangle$ in the defining representation of $U(N)$. We can make a $U(N)$ transformation to put $\langle\Phi\rangle$ in the standard form

$$\langle\Phi\rangle = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.23}$$

* Similar kinetic terms to eq. (2.19) were discussed in refs. [5][6]. Note that eq. (2.19) is not of the form proposed in ref. [17]. This form of the Kähler potential can never appear in a consistent effective lagrangian describing the dynamics of SNGB fields [18][19].

and it is clear that the unbroken group is $H = U(N-1)$. The group $\hat{H}$ is given by the set of all $N \times N$ matrices of the block form

$$\hat{h} = \begin{array}{c} \\ 1 \\ N-1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1 & N-1 \end{array} \\ \begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix} \end{array}, \tag{2.24}$$

where $u \in U(N-1)^c = GL(N-1, C)$ and a is a general complex row vector with $N-1$ entries. The entries in a are allowed to be non-zero because elements of $\hat{H}$ are not required to be unitary (equivalently, the generators of $\hat{H}$ are not required to be hermitian). Therefore, in this example $K = H = U(N-1)$, and N is the group of matrices of the form

$$n = \begin{array}{c} \\ 1 \\ N-1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1 & N-1 \end{array} \\ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{array}. \tag{2.25}$$

According to the arguments given earlier in this section, there is one real scalar SNGB for each generator of $G^c/\hat{H}$. It is easy see that

$$\dim G/H = 2N - 1, \qquad \dim G^c/\hat{H} = 2N. \tag{2.26}$$

Therefore, there is one "extra" SNGB whose superpartner is a NGB. When we discuss explicit models in subsection 3.5, we will see that the SNGB's that are not NGB's can be identified with excitations along flat directions of the potential that gives rise to the spontaneous symmetry breaking.

Next, consider an example with $G = U(N)$ as before, but with two order parameters (or equivalently, an order parameter in a reducible representation of G). The order parameters are $\langle \Phi_+ \rangle$ in the defining representation, and $\langle \Phi_- \rangle$ in the complex conjugate of the defining representation. We can use $U(N)$ transformations to put the order parameters in the standard form

$$\langle \Phi_+ \rangle = \begin{pmatrix} v_+ \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \langle \Phi_- \rangle = \begin{pmatrix} v_- \\ w \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.27}$$

and it is clear that the unbroken group is $H = U(N-2)$. To see what $\hat{H}$ is in this case, note that we can use a $U(N)^c = GL(N, C)$ transformation to further simplify the order parameters: if we choose

$$g_0 = \begin{array}{c} \\ 1 \\ N-1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1 & N-1 \end{array} \\ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{array} \in U(N)^c, \qquad a^T = \begin{pmatrix} w/v_- \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.28}$$

then

$$g_0 \langle \Phi_+ \rangle = \begin{pmatrix} v_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad g_0^{-1T} \langle \Phi_- \rangle = \begin{pmatrix} v_- \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.29}$$

This shows that $\hat{H}$ is given by all matrices of the form

$$\hat{h} = g_0^{-1}\hat{k}g_0, \qquad \hat{k} = \begin{matrix} \\ 1 \\ N-1 \end{matrix}\begin{matrix} 1 \quad N-1 \\ \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \end{matrix}, \tag{2.30}$$

where $u \in U(N-1)^{\mathrm{c}}$. Therefore, in this example $N = 0$ and $K \simeq U(N-1)$. Note that $K \not\subset G$ in this case (although K is *isomorphic* to a subgroup of G). It is easy see that

$$\dim G/H = 4N - 4, \qquad \dim G^{\mathrm{c}}/\hat{H} = 2(2N-1). \tag{2.31}$$

Therefore, there are 2 "extra" SNGB's in this case. As in the previous example, when we discuss explicit models, we will see that they can be identified with excitations along flat directions of the potential.

We will see in subsection 3.5 that these symmetry breaking patterns can arise in simple toy models, and that they give rise to interesting restrictions on the low-energy effective lagrangian.

3. Matter Fields: Weak Coupling

In this section, we consider the effective lagrangian including the matter fields in the case where the effective lagrangian is obtained by integrating out heavy modes at weak coupling. The reason for making this restriction is that in this case, the group $\hat{H}$ acts linearly on the matter fields in the effective lagrangian, and we will be able to obtain a simple canonical form for the effective lagrangian: we find that the superpotential for the matter fields is invariant under $\hat{H}$, while the Kähler potential is constrained by K^{c}, both of which are in general larger than H^{c}. (As we will see in the next section, the general case is more complicated.)

3.1. Transformation of the Effective Fields

We consider a "fundamental" theory with chiral superfields Φ invariant under a global symmetry G. Because the theory is weakly coupled, the order parameter can be taken to be $\langle\Phi\rangle$. We can therefore write

$$\Phi = \rho(\xi)\left[\langle\Phi\rangle + \Psi + \Delta\right], \tag{3.1}$$

where ρ is the representation (reducible in general) of G under which Φ transforms, $\xi \in G^{\mathrm{c}}/\hat{H}$ parameterizes the excitations of Φ in the broken symmetry directions (in the generalized sense of G^{c} invariance), and Ψ and Δ are the excitations of Φ in the remaining directions. We assume that the fields Ψ remain light (relative to $\langle\Phi\rangle$), while the fields Δ get masses of order $\langle\Phi\rangle$. (For simplicity, we assume that there is a single scale set by $\langle\Phi\rangle$. The extension to the case where $\langle\Phi\rangle$ contains several different scales should be clear.) We then imagine computing an effective action containing only the light degrees of freedom by integrating out the heavy fields Δ. The fields in the resulting low-energy effective lagrangian will transform under G^{c} as

$$\xi \mapsto g\xi\hat{h}^{-1}(g,\xi), \tag{3.2}$$

$$\Psi \mapsto R(\hat{h}(g,\xi))\Psi, \tag{3.3}$$

where R is the $\hat{H}$ representation obtained by reducing the representation ρ of G^{c}. The effective lagrangian can be constructed by writing down the most general form allowed by the symmetries and then determining the coefficients by matching onto the fundamental theory. We address the first part of the problem in this section, concentrating on the matter fields Ψ.

The striking fact about eq. (3.3) is that the matter fields transform according to representations of $\hat{H}$, even though the true unbroken symmetry of the theory is only H. As discussed in subsection 2.1, the reason for this is the fact that supersymmetry constrains the way the field $\mathcal{V}$ breaks $\hat{H} \to H$.

3.2. The Effective Superpotential

As pointed out in subsection 2.1, the transformation rule for $\mathcal{V}$ in eq. (2.2) does not allow $\mathcal{V}$ to appear in the effective superpotential unless derivatives act on $\mathcal{V}$. Since such terms vanish when $\mathcal{V} = 0$, the effective superpotential is invariant under $\hat{H}$. In cases where $\hat{H}$ is larger than H^c, this leads to additional restrictions on the effective superpotential beyond those imposed by H^c invariance.

As a simple example, consider a theory with the symmetry breaking pattern of the first example in subsection 2.3 (see the discussion surrounding eqs. (2.23) and (2.24)). That is, $G = U(N)$, $H = U(N-1)$, and the order parameter is in the defining representation of $U(N)$. Suppose now that the low-energy theory contains matter fields Ψ_+ transforming according to the defining representation of $U(N)$. Under $U(N-1)$, Ψ_+ decomposes into a singlet and a defining representation, but $\hat{H}$ invariance mixes these representations, leading to restrictions on the effective superpotential. For example, if the low-energy theory also contains matter fields Ψ_- transforming according to the complex conjugate of the defining representation of $U(N)$, then we can write

$$\Psi_\pm = \begin{matrix} 1 \\ N-1 \end{matrix} \begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix}, \tag{3.4}$$

and the most general $\hat{H}$-invariant quartic terms in the effective superpotential are

$$(\Psi_+\Psi_-)^2 = (A_+A_-)^2 + 2(A_+A_-)(B_+B_-) + (B_+B_-)^2, \qquad \Psi_+\Psi_-A_-^2, \qquad A_-^4. \tag{3.5}$$

In the first term, the relative coefficients of three H invariants are fixed by $\hat{H}$ invariance. (Note that we cannot put the quartic term in this form by rescaling the fields.) Also note that the terms involving A_+ by itself are forbidden by $\hat{H}$, even though they are allowed by H. We will consider an explicit model with this structure after we have discussed the effective Kähler potential.

3.3. Structure of $\hat{H}$ Representations

In order to understand the structure of the effective Kähler potential, we need some general results about the $\hat{H}$ representations R of the matter fields. In the class of theories we are considering, the $\hat{H}$ representation R is obtained by reducing a G^c representation. To make this precise, we write (in the sense of eq. (3.1))

$$\Psi = P_\Psi[\Phi - \langle\Phi\rangle], \tag{3.6}$$

where P_Ψ is a projection operator. That is, we think of Ψ as an element of the representation space of the G^c representation ρ that is nonzero only in a subspace. Because the fields Ψ transform among themselves under $\hat{H}$, the relation between ρ and the representation R of $\hat{H}$ is

$$\rho(\hat{h})P_\Psi = R(\hat{h}) \tag{3.7}$$

for all $\hat{h} \in \hat{H}$. Here, we view R as acting on the state space of ρ, but R is non-zero only on the Ψ subspace.*

In the appendix, we show that any $\hat{H}$ representation R obtained by reducing a G^c representation as in eq. (3.7) is equivalent to a representation by matrices of the block form

$$R(kn) = \begin{pmatrix} R_1(k) & * & \cdots & * \\ 0 & R_2(k) & * & \vdots \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & R_r(k) \end{pmatrix}, \tag{3.8}$$

where $k \in K^c$ and $n \in N$ are the factors in the Levy decómposition of $\hat{H}$. That is, $R(\hat{h})$ is an upper-block-triangular matrix with representation matrices of K^c in the diagonal blocks. As explained in the appendix, this result can be thought of as a generalization of Engel's theorem for the representations of Lie algebras. To check that eq. (3.8) defines a representation of $\hat{H}$, we must use the multiplication law for the Levy factors given in eq. (2.22). As a special case of eq. (3.8), we note that any representation R_K of K^c gives a representation of $\hat{H}$, defined by $R(kn) = R_K(k)$.

3.4. The Effective Kähler Potential

We now discuss how $\mathcal{V}$ breaks $\hat{H}$ down to H in the effective Kähler potential. Our main result is that the allowed term in the Kähler potential for the matter fields are classified by K invariants (not H invariants). The best way to see this is to work in a "gauge" for $\mathcal{V}$ where the structure of the unbroken group is as simple as possible. In this language, the explicit breaking of K down to H is accomplished by the vacuum value of $\mathcal{V}$.

To make this precise, recall that the group K defined in the Levy decomposition eq. (2.21) can be larger than H when there is a G^c transformation g_0 that can simplify the order parameter. We therefore define the group

$$\hat{K} \equiv \{g_0 \hat{h} g_0^{-1} \mid \hat{h} \in \hat{H}\}. \tag{3.9}$$

The group $\hat{K}$ is *isomorphic* to $\hat{H}$, but it is a different group of matrices. Maintaining this distinction is important for understanding the construction given below.

The reason for introducing the group $\hat{K}$ is as follows: if K is larger than H, then when we choose a basis where R has the form given in eq. (3.8), we find that $R(\hat{h}^\dagger) \neq R(\hat{h})^\dagger$. (Here, we use the definition of $\dagger$ on G^c that makes G and its representations real. The subgroup $\hat{H}$ and its representations R satisfying eq. (3.7) then naturally inherit a definition of $\dagger$.) To see why this is so, consider the second example in subsection 2.3. The field Φ_+ transforms in the defining representation of $\hat{H}$, but this does not have the form of eq. (3.8). However, the defining representation of $\hat{H}$ is equivalent to the representation

$$R(\hat{h}) = g_0 \hat{h} g_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}. \tag{3.10}$$

(See eq. (2.30).) This example shows that we can find a $g_0 \in G^c$ such that

$$R(g_0^{-1} \hat{h}^\dagger g_0) = R(g_0^{-1} \hat{h} g_0)^\dagger. \tag{3.11}$$

* Strictly speaking, R defined in this way is not a representation, since it is not invertible as a linear operator on the state space of ρ. However, if we define the inverse of R on the subspace on which it acts non-trivially, we can think of R as a representation.

The reason for this is that K is larger than H only if there is a transformation $g_0 \in G^c$ that can be used to simplify the order parameter further than can be done with G transformations alone.

When K is larger than H, it is convenient to work with fields where hermitian conjugation acts in a simple way. We therefore define fields

$$\widetilde{\xi} \equiv g_0 \xi g_0^{-1} \in G^c/\hat{K}, \tag{3.12}$$

$$\widetilde{\Psi} \equiv \rho(g_0)\Psi, \tag{3.13}$$

transforming as

$$\widetilde{\xi} \mapsto \widetilde{g}\widetilde{\xi}\hat{k}^{-1}(\widetilde{g},\widetilde{\xi}), \tag{3.14}$$

$$\widetilde{\Psi} \mapsto \widetilde{R}(\hat{k}(\widetilde{g},\widetilde{\xi}))\widetilde{\Psi}, \tag{3.15}$$

where

$$\widetilde{g} \equiv g_0 g g_0^{-1} \in G^c, \qquad \hat{k}(\widetilde{g},\widetilde{\xi}) \equiv g_0 \hat{h}(g,\xi) g_0^{-1} \in \hat{K}, \tag{3.16}$$

and

$$\widetilde{R}(\hat{k}) \equiv R(g_0^{-1}\hat{k}g_0). \tag{3.17}$$

Note that with these definitions,

$$\widetilde{R}(\hat{k}^\dagger) = \widetilde{R}(\hat{k})^\dagger \tag{3.18}$$

by eq. (3.11). In order to write invariants, it is useful to work in terms of the transformed spurion gauge fields

$$e^{\widetilde{\mathcal{V}}} \equiv g_0^{-1\dagger} e^{\mathcal{V}} g_0^{-1} \mapsto \widetilde{g}^{-1\dagger} e^{\widetilde{\mathcal{V}}} \widetilde{g}^{-1}, \tag{3.19}$$

$$e^{\widetilde{\mathcal{W}}} \equiv \widetilde{\xi}^\dagger e^{\widetilde{\mathcal{V}}} \widetilde{\xi} \mapsto \hat{k}^{-1\dagger}(\widetilde{g},\widetilde{\xi}) e^{\widetilde{\mathcal{W}}} \hat{k}^{-1}(\widetilde{g},\widetilde{\xi}). \tag{3.20}$$

These redefinitions simply amount to making a transformation $g_0 \in G^c$ to twiddled fields. However, it is important to note that this is not a symmetry transformation, since

$$e^{\widetilde{\mathcal{V}}}\big|_{\mathcal{V}=0} = g_0^{-1\dagger} g_0^{-1} \neq 1 \text{ in general.} \tag{3.21}$$

The advantage of working in terms of these fields is that the most general invariants are simply the most general gauge-invariant combinations of the twiddled fields, and the explicit breaking of $\hat{H}$ down to H is *entirely* due to the vacuum value of $\widetilde{\mathcal{V}}$.

We can use the field $\widetilde{\mathcal{W}}$ to define covariant generalizations of derivative operators, such as

$$\nabla_\alpha \equiv D_\alpha + e^{-\widetilde{\mathcal{W}}} D_\alpha e^{\widetilde{\mathcal{W}}}. \tag{3.22}$$

When $\widetilde{\mathcal{V}}$ is replaced by its vacuum value, the derivative does not explicitly break $\hat{H}$, since

$$e^{-\widetilde{\mathcal{W}}} D_\alpha e^{\widetilde{\mathcal{W}}} = \widetilde{\xi}^{-1} e^{-\widetilde{\mathcal{V}}} D_\alpha e^{\widetilde{\mathcal{V}}} \widetilde{\xi} + \widetilde{\xi}^{-1} D_\alpha \widetilde{\xi}, \tag{3.23}$$

and $D_\alpha \widetilde{\mathcal{V}} = 0$ when $\mathcal{V} = 0$.

We are now ready to show that the allowed terms in the effective Kähler potential are classified by K invariants. This is done by constructing new matter fields $\widetilde{\Psi}_j$ that transform according to

$$\widetilde{\Psi}_j \mapsto \widetilde{R}_j(k(\widetilde{g},\widetilde{\xi}))\widetilde{\Psi}_j, \tag{3.24}$$

where $\widetilde{R}_j$ is the j^{th} diagonal block of the $\hat{K}$ representation $\widetilde{R}$ (see eq. (3.8)), and $k(\widetilde{g}, \widetilde{\xi}) \in K^c$ is defined by decomposing

$$\hat{k}(\widetilde{g}, \widetilde{\xi}) = k(\widetilde{g}, \widetilde{\xi})\widetilde{n}(\widetilde{g}, \widetilde{\xi}), \tag{3.25}$$

with $\widetilde{n}(\widetilde{g}, \widetilde{\xi}) \in \{g_0 n g_0^{-1} \mid n \in N\}$.

We will see that the matter fields $\widetilde{\Psi}_j$ for $j = 1, \ldots, r-1$ (where r is the total number of blocks) are not holomorphic in the original fields, so they cannot appear in the effective superpotential. However, they can appear in the Kähler potential, so any K-invariant function of $\widetilde{\Psi}_j$ is allowed in the Kähler potential. The matter fields $\widetilde{\Psi}_j$ also depend on $\widetilde{\mathcal{V}}$, and when K is larger than H, K is broken *only* by eq. (3.21). This will give rise to relations between the coefficients of different H-invariants in the Kähler potential.

We begin by defining the projection operators $P_{\geq j}$ and $P_{\leq j}$ acting on the representation space of $\widetilde{R}$:

$$P_{\leq j} = \begin{array}{c} \\ j \\ r-j \end{array}\!\!\begin{array}{c} \begin{array}{cc} j & r-j \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \qquad P_{\geq j} = \begin{array}{c} \\ j-1 \\ r-j+1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} j-1 & r-j+1 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \tag{3.26}$$

where the numbers at the border of the matrix count the number of *blocks* (see eq. (3.8)). Because $\widetilde{R}(\hat{k})$ is an upper-triangular matrix, it is easy to see that

$$P_{\geq j}\widetilde{R}(\hat{k}) = P_{\geq j}\widetilde{R}(\hat{k})P_{\geq j}, \qquad \widetilde{R}(\hat{k})P_{\leq j} = P_{\leq j}\widetilde{R}(\hat{k})P_{\leq j}, \tag{3.27}$$

where we use the abbreviation $\hat{k} \equiv \hat{k}(\widetilde{g}, \widetilde{\xi})$. Similarly, $\widetilde{R}(\hat{k})^\dagger$ is a lower-triangular matrix, and hence

$$P_{\leq j}\widetilde{R}(\hat{k})^\dagger = P_{\leq j}\widetilde{R}(\hat{k})^\dagger P_{\leq j}, \qquad \widetilde{R}(\hat{k})^\dagger P_{\geq j} = P_{\geq j}\widetilde{R}(\hat{k})^\dagger P_{\geq j}. \tag{3.28}$$

We can therefore define the fields

$$\widetilde{\Psi}_{\geq j} \equiv P_{\geq j}\widetilde{\Psi} \mapsto P_{\geq j}\widetilde{R}(\hat{k})P_{\geq j} \cdot \widetilde{\Psi}_{\geq j}. \tag{3.29}$$

That is, $\widetilde{\Psi}_{\geq j}$ transforms according to the representation $P_{\geq j}\widetilde{R}P_{\geq j}$ consisting of the last j subblocks of $\widetilde{R}$. In particular, the matter field $\widetilde{\Psi}_r \equiv P_{\geq r}\widetilde{\Psi}$ transforms according to the representation $\widetilde{R}_r$ of K.

We can isolate the "middle" blocks using the gauge field spurion. To see how to do this, note that by eq. (3.7),

$$\widetilde{R}(e^{\widetilde{\mathcal{W}}}) \equiv \rho(e^{\mathcal{W}})P_{\Psi} \mapsto \widetilde{R}(\hat{k}^{-1})^\dagger \cdot \widetilde{R}(e^{\widetilde{\mathcal{W}}}) \cdot \widetilde{R}(\hat{k}^{-1}), \tag{3.30}$$

where we have used eq. (3.18). Thus, we can define

$$S_{\leq j} \equiv P_{\leq j}\widetilde{R}(e^{\widetilde{\mathcal{W}}})P_{\leq j} \mapsto P_{\leq j}\widetilde{R}(\hat{k}^{-1})^\dagger P_{\leq j} \cdot S_{\leq j} \cdot P_{\leq j}\widetilde{R}(\hat{k}^{-1})P_{\leq j}, \tag{3.31}$$

and

$$S_{\geq j} \equiv P_{\geq j}\widetilde{R}(e^{-\widetilde{\mathcal{W}}})P_{\geq j} \mapsto P_{\geq j}\widetilde{R}(\hat{k})P_{\geq j} \cdot S_{\geq j} \cdot P_{\geq j}\widetilde{R}(\hat{k})^\dagger P_{\geq j}. \tag{3.32}$$

To see why this is useful, note that

$$S_{\geq j}^{-1}\widetilde{\Psi} \mapsto P_{\geq j}\widetilde{R}(\hat{k}^{-1})^\dagger P_{\geq j} \cdot S_{\geq j}^{-1}\widetilde{\Psi}, \tag{3.33}$$

where the inverse is defined in the obvious way on the non-zero blocks. Note that

$$S_{\geq j}^{-1}\widetilde{\Psi} = \widetilde{\Psi}_{\geq j} + O(\Pi) + O(\mathcal{V}), \tag{3.34}$$

but $S_{\geq j}^{-1}\widetilde{\Psi}$ transforms according to a *lower*-triangular representation whose first non-zero block corresponds to R_j. Therefore, we can use the projection operator $P_{\leq j}$ to construct matter fields that transform according to $\widetilde{R}_j$:

$$\widetilde{\Psi}_j \equiv P_{\leq j} S_{\geq j}^{-1} \widetilde{\Psi} \mapsto \widetilde{R}_j(k^{-1}(\widetilde{g}, \widetilde{\xi}))^{\dagger} \widetilde{\Psi}_j, \tag{3.35}$$

as claimed above. Similarly, we can write invariants involving $\widetilde{\Psi}_j^{\dagger}$ by noting that

$$\overline{\widetilde{\Psi}}_j \equiv \widetilde{\Psi}_j^{\dagger} S_{\leq j}^{-1} P_{\geq j} \mapsto \overline{\widetilde{\Psi}}_j \widetilde{R}_j(k(\widetilde{g}, \widetilde{\xi}))^{\dagger}. \tag{3.36}$$

Note that it is impossible to project the matter fields down further, for example to H. To see this, note that when $g_0 = 1$, $K = H$, and it is clear that we can only project down to K. When $g_0 \neq 1$, we can perform a "gauge transformation" to define the "twiddled" fields in which g_0 only appears in the vacuum value of the gauge field. However, the terms we write must respect the full G^{c} symmetry, and so there are no additional invariants when the gauge fields take on particular values.

3.5. Toy Models

We now give some explicit toy models that illustrate the main results obtained above, namely that the effective superpotential for the matter fields is invariant under $\hat{H}$, while the effective Kähler potential is constrained by K^{c}, both of which are in general larger than H^{c}. In order to illustrate our results, we must consider models that spontaneously break symmetries, and in addition contain "matter" fields that remain light after symmetry breaking. The models are therefore somewhat complicated, and we will discuss them in two steps: first, we construct the "symmetry breaking sector," and then we add fields to the model to get additional matter fields at low energies.

The first example has larger K than H and demonstrates that the Kähler potential is restricted by K-invariance. It has $G = U(N)$ with fields

$$\Phi_+ \sim N, \qquad \Phi_- \sim \overline{N}, \qquad \Delta \sim 1. \tag{3.37}$$

The most general renormalizable superpotential is

$$W = \frac{M}{2}\Delta^2 + \frac{g}{3}\Delta^3 + m\Phi_+\Phi_- + \lambda\Delta\Phi_+\Phi_-, \tag{3.38}$$

where we have shifted away a possible linear term in Δ. It is easy to see that there are no additional accidental symmetries. The most general vacuum of this theory is either

$$\langle\Phi_\pm\rangle = 0, \qquad \langle\Delta\rangle = 0 \text{ or } -M/g, \tag{3.39}$$

or

$$\langle\Phi_+\rangle = \begin{pmatrix} v_+ \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \langle\Phi_-\rangle = \begin{pmatrix} v_- \\ w \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.40}$$

where

$$v_+ v_- = \frac{m}{\lambda^2}\left(M - \frac{gm}{\lambda}\right), \tag{3.41}$$

but $v_\pm$ and w are otherwise arbitrary. We will study this case. (The flat directions are preserved to all orders in perturbation theory [8]; the techniques of ref. [10] show that this result is true even beyond perturbation theory.) Note that we can always take w to be real and also $v_\pm$ relatively real using G transformation. Therefore, number of flat directions in this model is that corresponding to w and v_+/v_-, both real parameters. This precisely coincides with the number of extra SNGB's as discussed in section 2.3.

The symmetry breaking structure is exactly the same as the second example in subsection 2.3 (see the discussion surrounding eqs. (2.28)–(2.30)). In this example, $\hat{H} \simeq U(N-1)^c$, so that $K = U(N-1)$ is larger than H. There are $2N-1$ massless chiral superfields that can all be identified with SNGB's. The SNGB's can be parameterized by writing

$$\Phi_+ = g_0^{-1}\tilde{\xi} g_0 \cdot [\langle\Phi_+\rangle + \cdots], \qquad \Phi_- = g_0^T \tilde{\xi}^{-1T} g_0^{-1T} \cdot [\langle\Phi_-\rangle + \cdots], \tag{3.42}$$

where g_0 is defined in eq. (2.28) and

$$\tilde{\xi} = e^{i\tilde{\Pi}} \in G^c/\hat{K}, \qquad \tilde{\Pi} = \begin{matrix} \\ 1 \\ N-1 \end{matrix} \begin{matrix} 1 \quad N-1 \\ \begin{pmatrix} \sigma & \pi_- \\ \pi_+ & 0 \end{pmatrix} \end{matrix}. \tag{3.43}$$

To get a more interesting theory, we add more fields to the theory. We write $G = SU(N) \times U(1)$ and take the fields to transform as

$$\begin{aligned} &\Phi_+ \sim (N;+1), \qquad \Phi_- \sim (\overline{N};-1), \qquad \Delta \sim (1;0), \\ &\Sigma_+ \sim (N;-1), \qquad \Sigma_- \sim (\overline{N};+1), \qquad \Gamma \sim (1;-2). \end{aligned} \tag{3.44}$$

The most general dimension-4 superpotential is now

$$\begin{aligned} W = \frac{M}{2}\Delta^2 + \frac{g}{3}\Delta^3 + m\Phi_+\Phi_- + \lambda\Delta\Phi_+\Phi_- \\ + \mu\Sigma_+\Sigma_- + \gamma\Delta\Sigma_+\Sigma_- + \beta\Gamma\Phi_+\Sigma_-. \end{aligned} \tag{3.45}$$

There is a vacuum for which $\langle\Phi_\pm\rangle$ and $\langle\Delta\rangle$ are as above and

$$\langle\Sigma_\pm\rangle = 0, \qquad \langle\Gamma\rangle = 0. \tag{3.46}$$

The light fields are now the SNGB's discussed above and the matter fields

$$\Psi_+ = \begin{matrix} 1 \\ N-1 \end{matrix} \begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = g_0 \xi^{-1} \Sigma_+, \tag{3.47}$$

$$\Psi_- = \begin{matrix} 1 \\ N-1 \end{matrix} \begin{pmatrix} 0 \\ B_- \end{pmatrix} = P g_0^{-1T} \xi^T \Sigma_-, \tag{3.48}$$

where

$$P = \begin{matrix} \\ 1 \\ N-1 \end{matrix} \begin{matrix} 1 \quad N-1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix}. \tag{3.49}$$

These fields transform as

$$A_+ \mapsto A_+, \qquad B_+ \mapsto uB_+, \qquad B_- \mapsto u^{-1T}B_-, \tag{3.50}$$

where $u \in U(N-1)$ parameterizes $\hat{k}(\widetilde{g}, \widetilde{\xi})$ as in eq. (2.30). Note that $B_\pm$ reduce under the unbroken $U(N-2)$ as a sum of a singlet and a $(N-2)$-dimensional representation. To define kinetic terms for these fields, we follow the general discussion above and define

$$\begin{aligned} S &\equiv \left[Pe^{-\widetilde{W}}P\right]^{-1} \\ &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & u^{-1\dagger} \end{pmatrix} S \begin{pmatrix} 0 & 0 \\ 0 & u^{-1} \end{pmatrix}. \end{aligned} \tag{3.51}$$

We can then write the effective Kähler potential

$$\mathcal{L}_D = \int d^2\theta\, d^2\bar{\theta}\, [\overline{A}_+ A_+ + \overline{B}_+ S B_+ + \overline{B}_- S^{-1T} B_-], \tag{3.52}$$

and the effective superpotential

$$\mathcal{L}_F = \int d^2\theta\, m_{\text{eff}} B_+ B_- + \text{h.c.} \tag{3.53}$$

Note that different $U(N-2)$ invariants are related in both the superpotential and the Kähler potential.

The second example illustrates that the $\hat{H}$ invariance relates invidual H-invariant terms in the superpotential. It has global $G = U(N) \times U(1)_R$ symmetry with fields

$$\Phi_+ \sim (N; \tfrac{1}{2}), \qquad \Phi_- \sim (\overline{N}; \tfrac{1}{2}), \tag{3.54}$$

where $U(1)_R$ is defined by

$$U(1)_R: \; \Phi_\pm(x, \theta) \mapsto e^{i\alpha/2} \Phi_\pm(x, \theta e^{i\alpha}). \tag{3.55}$$

The most general superpotential compatible with these symmetries is

$$W = G(\Phi_+ \Phi_-)^2. \tag{3.56}$$

This term can be imagined to arise from integrating out a heavy singlet chiral superfield in a renormalizable theory.* It is easy to check that there are no additional accidental symmetries. There are supersymmetric ground states for

$$\langle \Phi_+ \rangle = \begin{pmatrix} v_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \langle \Phi_- \rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_- \end{pmatrix}. \tag{3.57}$$

The potential is minimized for arbitrary $v_\pm$, so the potential has 2-dimensional space of flat directions. For simplicity, we will analyze the theory for the special case $v_- = 0$.

* For instance, $W = \lambda \Phi_+ \Phi_- \chi + M\chi^2$, where χ has quantum numbers $(1; 1)$. The superpotential including the matter fields eq. (3.65) can also be rewritten using additional singlet fields, with our results remaining unchanged. The non-renormalizable forms used in the text simplify some of the expressions.

The theory then has a symmetry breaking structure similar to that of the first example in subsection 2.3. There is an unbroken $U(1)_{R'}$ symmetry that is a combination of the original $U(1)_R$ and a broken $U(N)$ generator:

$$U(1)_{R'}: \ \Phi_\pm(x,\theta) \mapsto e^{i\alpha/2} e^{\pm i\alpha T} \Phi_\pm(x, \theta e^{i\alpha}), \tag{3.58}$$

where

$$T = \begin{matrix} \\ 1 \\ N-1 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad N-1 \end{matrix} \\ \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \in U(N). \tag{3.59}$$

Therefore, the only effect of the $U(1)_R$ symmetry is the existence of the symmetry $U(1)_{R'}$ in the effective lagrangian, and we write (for $v_- = 0$)

$$G = U(N), \qquad H = U(N-1). \tag{3.60}$$

The group $\hat{H}$ is then the same as in eq. (2.24). There are $2N$ SNGB's in this model, which are conveniently parameterized by

$$\Phi_+ = \xi^{-1} \cdot \langle \Phi_+ \rangle, \tag{3.61}$$

where

$$\xi = e^{i\Pi} \in G^c/\hat{H}, \qquad \Pi = \begin{matrix} \\ 1 \\ N-1 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad N-1 \end{matrix} \\ \begin{pmatrix} \sigma & 0 \\ \pi & 0 \end{pmatrix} \end{matrix}. \tag{3.62}$$

There is a flat direction parametrized by the real part of σ, consistent with the number of extra SNGB's as discussed in section 2.3. There are also $N-1$ light chiral matter fields

$$\Xi_- \equiv \begin{matrix} \\ 1 \\ N-1 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad N-1 \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \xi^T \Phi_- \equiv \begin{matrix} 1 \\ N-1 \end{matrix} \begin{pmatrix} 0 \\ C_- \end{pmatrix}. \tag{3.63}$$

There is no superpotential allowed for the light matter fields because of $U(1)_{R'}$ symmetry.

To get a more interesting effective lagrangian, we again add additional fields. We also impose an additional $U(1)$ symmetry, so that the full symmetry is $U(N) \times U(1) \times U(1)_R$. The fields are now

$$\begin{aligned} &\Phi_+ \sim (N; 0, \tfrac{1}{2}), \qquad \Phi_- \sim (\overline{N}; 0, \tfrac{1}{2}), \\ &\Sigma_+ \sim (N; 1, \tfrac{1}{2}), \qquad \Sigma_- \sim (\overline{N}; -1, \tfrac{1}{2}). \end{aligned} \tag{3.64}$$

The most general superpotential is

$$\begin{aligned} W = {} & G_1(\Phi_+\Phi_-)^2 + G_2(\Sigma_+\Sigma_-)^2 \\ & + G_3(\Phi_+\Phi_-)(\Sigma_+\Sigma_-) + G_4(\Phi_+\Sigma_-)(\Sigma_+\Phi_-). \end{aligned} \tag{3.65}$$

This theory has a vacuum with $\langle \Phi_\pm \rangle$ as before (we again take $v_- = 0$), and

$$\langle \Sigma_\pm \rangle = 0, \tag{3.66}$$

giving rise to the same symmetry-breaking pattern discussed above. The low-energy matter fields are the fields Ξ_- in eq. (3.63), as well as

$$\Psi_+ \equiv \xi^{-1} \cdot \Sigma_+, \qquad \Psi_- \equiv \xi^T \cdot \Sigma_-. \tag{3.67}$$

If we write

$$\Psi_{\pm} = \frac{1}{N-1}\begin{pmatrix} A_{\pm} \\ B_{\pm} \end{pmatrix}, \tag{3.68}$$

The effective superpotential is

$$W_{\text{eff}} = G_{\text{eff}}(\Psi_+\Psi_-)^2 = G_{\text{eff}}\left[(A_+A_-)^2 + 2(A_+A_-)(B_+B_-) + (B_+B_-)^2\right]. \tag{3.69}$$

Just as in the example in subsection 3.2, there are three H invariants related by $\hat{H}$. Also, terms such as A_+^4 are allowed by H as well as $U(1)_{R'}$, but are forbidden by $\hat{H}$. Terms proportional to powers of A_- are allowed by $\hat{H}$ symmetry, but are forbidden by the unbroken $U(1)$.

As described in the previous subsection, the effective Kähler potential for the model is written in terms of fields transforming under $U(N-1)$ representations. In this example, the fields are

$$B_+ \mapsto uB_+, \qquad A_- \mapsto A_-, \tag{3.70}$$

and the non-holomorphic fields

$$\begin{pmatrix} \widetilde{A}_+ \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{\mathcal{W}}\Psi_+, \qquad \begin{pmatrix} 0 \\ \widetilde{B}_- \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{-\mathcal{W}}\Psi_-, \tag{3.71}$$

transforming as

$$\widetilde{A}_+ \mapsto \widetilde{A}_+, \qquad \widetilde{B}_- \mapsto u^{-1}\widetilde{B}_-. \tag{3.72}$$

The Kähler potential for the matter fields is simply the most general $U(N-1)$ invariant function of these fields.

The final example has a matter field whose mass term is forbidden by $\hat{H}$, even though a mass is allowed by H alone. The model is a simple variation on the one just discussed: the symmetry is $SU(N) \times U(1)_R$ with "Higgs" fields

$$\Phi_+ \sim (N; \tfrac{1}{2}), \qquad \Phi_- \sim (\overline{N}; \tfrac{1}{2}), \tag{3.73}$$

and "matter" fields

$$\Sigma_+ \sim (N; \tfrac{3}{2}). \tag{3.74}$$

In addition, we impose a Z_2 symmetry under which $\Phi_\pm$ is even and Σ_+ is odd. The superpotential is then simply

$$W = G(\Phi_+\Phi_-)^2. \tag{3.75}$$

The superpotential has an accidental $U(N)$ symmetry acting only on Σ_+, but this can be broken in the Kähler potential by terms such as

$$\int d^2\theta\, d^2\bar{\theta}\;\; \overline{\Phi}_+^4 \overline{\Phi}_-^2 \Sigma_+^2, \tag{3.76}$$

where the indices are contracted in the obvious way. In the effective theory, the light matter fields are

$$\Psi_+ \equiv \xi^{-1}\Sigma_+ \equiv \begin{pmatrix} A_{\pm} \\ B_{\pm} \end{pmatrix}. \tag{3.77}$$

A mass term for A_+ is allowed by H, but forbidden by $\hat{H}$, since

$$A_+ \mapsto A_+ + a \cdot B_+, \tag{3.78}$$

where a is defined in eq. (2.24).

4. Matter Fields: General Case

In this section, we relax the assumption that the low-energy effective lagrangian arises from a weakly-coupled theory, and explore the action of the group $\hat{H}$ on the matter fields in a general effective lagrangian satisfying the assumptions stated in subsection 2.2. For the weakly-coupled case, we found that the $\hat{H}$ action on the matter fields is linear, and that the $\hat{H}$ representations that arise can be embedded in G^c representations. We will show by explicit examples that the freedom to make field redefinitions does not in general allow us to define matter fields on which $\hat{H}$ acts linearly. Furthermore, even if we restrict attention to linear $\hat{H}$ representations, we show that they cannot be embedded in G^c representations in general. This seems to make it impossible to write G^c-invariant kinetic terms. We therefore do not have a good understanding of the general effective lagrangian, and this section is mainly an attempt to quantify our ignorance.

4.1. Linearization

We first show that we can redefine the matter fields so that the action of K^c is linear. Expanding the transformation eq. (2.15) for small Ψ, we have

$$\Psi \mapsto R(\hat{h}(g,\xi))\Psi + O(\Psi^2). \tag{4.1}$$

Note that there is no Ψ-independent term on the right-hand side because $T(\hat{h})(0) = 0$. Following ref. [2], we then define

$$\Xi \equiv \int_K \omega(k)\, R(k)^{-1} T(k)(\Psi), \tag{4.2}$$

where the integral is over the compact subgroup $K \subset K^c$, and $\omega(k)$ is the invariant group measure on K. Despite the fact that the integral is defined only over K, the fields Ξ actually transform linearly under all of K^c. To see this, note that under $\ell \in K^c$,

$$\begin{aligned}\Xi \mapsto &\int_K \omega(k)\, R(k^{-1}) T(k)(T(\ell)(\Psi)) \\ &= \int_K \omega(k)\, R(k)^{-1} T(k\ell)(\Psi) \\ &= R(\ell) \int_{K\ell} \omega(k')\, R(k')^{-1} T(k')(\Psi),\end{aligned} \tag{4.3}$$

where we have changed variables to $k' = k\ell$ in the last line, so the integration is now over $K\ell \equiv \{k\ell \mid k \in K\}$. Since the group action $T(k')(\Psi)$ and the group measure are holomorphic in the group parameters, we can deform the contour back to K, and obtain*

$$\Xi \mapsto R(\ell) \cdot \Xi, \qquad \ell \in K^c. \tag{4.4}$$

The argument above relies crucially on the fact that K is compact, since the group-invariant measure is not defined for general non-compact groups. To see that this is not just a technicality,

* The measure $\omega(k)$ is the natural analytic continuation of the Haar measure on K to K^c. Note that $\omega(k)$ is closed in K^c: It is holomorphic in the group coordinates, $\bar{\partial}\omega(k) = 0$, and also the highest form in the holomorphic coordinates, $\partial\omega(k) = 0$. Therefore, one can continuously deform the integration region within K^c as long as one does not encounter singularities in the integrand.

we give an explicit example of a $\hat{H}$ group action that cannot be linearized by any redefinition of the matter fields that preserves the origin. Consider a case with $G = SU(2)$ broken by an order parameter transforming in the defining representation. In this case, we can make an $SU(2)$ transformation to put the order parameter in the form

$$\langle \Phi \rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}, \tag{4.5}$$

and we see that $H = 1$. The group $\hat{H}$ is given by the set of 2×2 matrices of the form

$$\hat{h} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

with a complex. It is easy to see that the a's add under group multiplication, so $\hat{H}$ is isomorphic to the group of translations in the complex plane. Now consider a single matter chiral superfield Ψ transforming as

$$\Psi \mapsto \frac{\Psi}{1 + a\Psi}. \tag{4.7}$$

This transformation leaves the origin invariant, and it is easily checked that it satisfies the group multiplication law. We wish to define new matter fields $\Xi(\Psi)$ that transform linearly under $\hat{H}$. These fields should transform as

$$\Xi(\Psi) \mapsto \Xi(\Psi - a\Psi^2 + O(a^2)) = \Xi + at\Xi + O(a^2), \tag{4.8}$$

where t is the "generator" in the linearized transformation. Equating the $O(a)$ terms gives the requirement

$$-\Psi^2 \frac{d\Xi}{d\Psi} = t\Xi. \tag{4.9}$$

The general solution is

$$\Xi = Ce^{t/\Psi}, \tag{4.10}$$

which does not satisfy the condition that $\Xi = 0$ when $\Psi = 0$. Thus we see that the transformation eq. (4.7) cannot be linearized.

To see what $\hat{H}$ invariants we can construct in this example, note that eq. (4.7) can be rewritten as

$$\frac{1}{\Psi} \mapsto \frac{1}{\Psi} + a. \tag{4.11}$$

Therefore, we can write $\hat{H}$ invariant terms such as

$$\int d^2\theta \, d^2\bar{\theta} \, \frac{1}{\bar{\Psi}\Psi}. \tag{4.12}$$

(This shifts by a total derivative under the transformation eq. (4.11).) This can perhaps be interpreted to give a sensible effective field theory by expanding around $\Psi = \infty$, but the resulting effective lagrangian can certainly not be interpreted as describing fluctuations about $\Psi = 0$. We would therefore be inclined to regard this effective lagrangian as "unphysical." We do not know whether the features found in this example are general to all non-linearizable group actions.

4.2. Non-embeddable Representations

We now restrict attention to effective lagrangians where the $\hat{H}$ action on the matter fields can be linearized, and make some brief comments on general $\hat{H}$ representations. We point out that there are simple $\hat{H}$ representations that cannot be embedded into G^c, and that there appears to be no way to write kinetic terms for fields transforming according to these representations.

As discussed in the Appendix, the general representations of $\hat{H}$ contain 1-dimensional representations (characters) of N. If these characters are non-trivial, then the representation cannot be embedded into a G^c representation, since elements of the subgroup $N \in G^c$ are represented by unipotent matrices in a G^c representation. We can use this fact to construct simple $\hat{H}$ representations that cannot be embedded in G^c representations.

A simple example is obtained by considering again the symmetry breaking pattern $SU(2) \to 1$ discussed in the previous subsection. Consider now a field Ψ transforming under $\hat{H}$ as

$$\Psi \mapsto e^{ta}\Psi \tag{4.13}$$

for some constant t. By the arguments in the Appendix, this representation cannot be embedded in G^c. The importance of this is that we do not know any way to couple the gauge field spurion $\mathcal{V}$ to Ψ. (In subsection 3.4, we saw that couplings of $\mathcal{V}$ are crucial for writing G^c-invariant kinetic terms for the embeddable $\hat{H}$ representations.) In the present case,

$$\overline{\Psi}\Psi \mapsto e^{ta+\bar{t}\bar{a}}\overline{\Psi}\Psi, \tag{4.14}$$

and there appears to be no way to use $\mathcal{V}$ to construct an $SU(2)^c$-invariant kinetic term.

We do not know whether there are any non-embeddable $\hat{H}$ representations for which one can write a sensible effective lagrangian. The question is an interesting one, since such matter fields would be analogs of states with fractional charge, such as dyons.

5. Conclusions

In this paper we have discussed the structure of supersymmetric effective lagrangians describing the low-energy physics in a situation where a global symmetry group G is spontaneously broken down to a subgroup H while supersymmetry remains unbroken. This effective lagrangian contains fields describing the supersymmetric Nambu–Goldstone bosons (SNGB's), as well as possible additional light "matter" fields. By introducing external "spurion" gauge fields for G, the symmetry is formally enhanced to G^c, the complexification of G. By studying the way in which this external gauge field can appear in the effective lagrangian, we have shown that the effective couplings of the matter fields are constrained by the group $\hat{H}$, the largest unbroken subgroup of G^c. The structure of $\hat{H}$ is rather non-trivial: it can be decomposed into a semidirect product $K^c \wedge N$, where K is compact and N is unipotent. K contains H, but K is larger than H in general.

We have shown how to write a manifestly supersymmetric effective lagrangian for the SNGB's, but our main results concern the matter fields. We showed that the superpotential for the matter fields is invariant under $\hat{H}$. In cases where $\hat{H}$ can be larger than H^c, the coefficients of H-invariant terms therefore obey relations imposed by $\hat{H}$ invariance. The Kähler potential for the matter fields is determined by the most general K-invariant function of the matter fields, with the explicit breaking down to H determined as a function of the order parameter. Both these results are considerably stronger than the simple H-invariance one naively expects.

The assumptions made in deriving these result are that the holomorphy of the group action is preserved in the quantum theory, and that the action of $\hat{H}$ on the matter fields can be taken to be a linear representation embedded in a G representation; both of these assumptions are valid in weakly-coupled theories. Relaxing these assumptions, we show that there are $\hat{H}$ actions on the matter fields that cannot be made linear by field redefinitions, and there are $\hat{H}$ representations for which it appears to be impossible to write a G^c-invariant kinetic term. It is not clear to us whether a physically sensible effective lagrangian can be constructed from matter fields transforming under these more general $\hat{H}$ actions.

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Appendix A. Structure of $\hat{H}$ Representations

In this appendix, we prove the structure theorem alluded to in subsection 3.2.* Given any representation R of $\hat{H}$, Engel's theorem tells us that there is a basis in which

$$R(n) = \begin{pmatrix} \lambda_1(n)S_1(n) & 0 & \cdots & 0 \\ 0 & \lambda_2(n)S_2(n) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_r(n)S_r(n) \end{pmatrix}, \tag{6.1}$$

for $n \in N$. Here, $\lambda_1, \ldots, \lambda_r$ are 1-dimensional representations (characters) of N, and $S_1, \ldots, S_r$ are unipotent matrices: that is, they are upper-triangular with 1's on the diagonal. However, if R is a representation of $\hat{H}$ obtained by reducing a representation of G^c, then elements of N are represented by matrices with $\lambda_1, \ldots, \lambda_r \equiv 1$. One way to see this is to note that the representations of G^c can be obtained by taking tensor products of fundamental representations and reducing them, and these operations preserve the property of having 1 as an eigenvalue. Therefore, every unipotent element of G^c will be represented by a unipotent matrix.

We now restrict attention to the case where the $\hat{H}$ representation is embedded in a G^c representation. In that case, we denote the state space for the representation R by V and define the subspace

$$V_1 \equiv \{v \in V \mid R(n)v = v \text{ for all } n \in N\}. \tag{6.2}$$

* We thank D. Vogan for this argument.

The considerations above tell us that $V_1 \neq 0$. It is also easy to see that V_1 is invariant under K^c, since for all $v \in V_1$ we have

$$R(n) \cdot R(k)v = R(k)R(k^{-1}nk)v = R(k)v \tag{6.3}$$

for all $k \in K^c$, $n \in N$ (because N is a normal subgroup of $\hat{H}$). This means that there is a basis for V in which the representation matrices have the block form

$$R(kn) = \begin{pmatrix} R_1(k) & * \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} R_1(k) & * \\ 0 & * \end{pmatrix}. \tag{6.4}$$

It is easy to see that the block in the lower-right corner is again a representation of $\hat{H}$, and we can apply the same argument to it. Therefore, we obtain that any embedded $\hat{H}$ representation is equivalent to a representation of the form given in eq. (3.8) in the main text.

It is interesting that a "folk theorem" in the mathematics community states that the converse of this result is also true: any $\hat{H}$ representation of the form eq. (3.8) is isomorphic to a subrepresentation of a $\hat{H}$ representation obtained by restricting a G^c representation [21].

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