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A Proof of the Asymptotic Radiance Hypothesis*

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I. Introduction. The asymptotic radiance hypothesis was first formulated in the field of experimental radiative transfer dealing with the penetration of natural light into the oceans and deep lakes. It may be stated as follows: The form of the angular distribution of light (radiance, specific intensity) about a point in an optical medium approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions at the upper boundary of the medium and which depends only on the inherent optical properties of the medium. Some relatively recent references to the hypothesis may be found in the experimental papers of Whitney [1] , [2] , and Lenoble [3] . Some recent theoretical discussions for particular cases may be found in Lenoble [4] , and Poole [5] . Subsequently, the mathematical problem underlying the hypothesis took on meaning in a wider set of contexts such as astrophysical optics and neutron transport theory. The statement of the hypothesis for these contexts is essentially the same.

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In this note a proof of the hypothesis is given for a rather wide class of inhomogeneous spaces known as <u>eventually separable</u> spaces, a term which is defined in detail below. The discussion is designed so that the main results are also applicable to the astrophysical and neutron contexts. The approach used is direct in the sense that it is based on a study of the equation of transfer itself rather than first solving the equation for particular cases and then inspecting the properties of the resulting solutions. Furthermore, the quantities introduced in the study are for the most part directly observable quantities, a feature which reflects the experimental origins of the problem and which keeps sight of possible practical applications of the asymptotic radiance hypothesis. In this way the discussion complements a different approach to the problem, namely the formal-solution approach initiated by Chandrasekhar [6] and extended by Kuščer [7] in the radiative transfer context, and also considered, for example, by Davison [8] in the neutron transport context. In particular, the present discussion shows in terms of directly observable quantities that when an asymptotic radiance distribution exists in a medium, it is represented by a formal-solution distribution and is approached in a continuous way by the natural distributions in the medium. Two illustrations of this fact are given. One is based on tables compiled from theoretical calculations made in the neutron transport context, the other is drawn from an experiment which documented the light field in a natural hydrosol.

The practical consequences of the asymptotic radiance theorem are many. They take on especial utility in the field of geophysical optics. While an exhaustive discussion of these consequences is out of place here, we should observe that the classical two-flow equations of the light field [9], [10], are accurate and become exact with increasing depth whenever the hypothesis holds. This results in an enormous simplification of the standard experimental procedures dealing with the determination of the optical properties of natural hydrosols. Finally, the present method allows a means of estimating, with repect to a given preassigned criterion, the optical depth at which the asymptotic distribution has been attained.

2. Preliminary Definitions. We begin with the general equation of transfer for the radiance (specific intensity) function N on a general space $\mathsf{X} \times \Xi$ as used in geophysical optics [11]:

$$\frac{n^2(x,t)}{v(x,t)} D\left[N(x,\xi,t)/n^2(x,t)\right]/Dt = -\alpha(x,t)N(x,\xi,t) + N_*(x,\xi,t) + N_\eta(x,\xi,t) \tag{1}$$

and recast it into a form which will be most suitable for the present discussion, and which will insure the widest domain of applicability of the present results to related fields such as astrophysical optics and neutron transport theory. Here,

$$N_*(x,\xi,t) = \int_{\Xi} \sigma(x;\xi;\xi',t)\, N(x,\xi,t)\, d\Omega(\xi')$$

defines the path function N_* . σ is the volume scattering function, and Ξ is the unit sphere of direction vectors ξ . N_η is the emission function, α the volume attenuation function, n is the index of refraction function, v the speed of light function, $D[\cdot]/Dt$ the Lagrangian derivative operator on $X \times \Xi$, t denotes time, and Ω is the solid angle measure on Ξ .

The present problem is meaningful only in the steady state case, and is most immediately concerned with emission-free arbitrarily strati-fied plane-parallel media with constant index of refraction. Adopt-ing these conditions and applying them to (1), the resultant simple form is obtained:

$$-\underset{\sim}{\xi} \cdot \underset{\sim}{n} \, \frac{dN(z,\theta,\phi)}{dz} = -\alpha(z) N(z,\theta,\phi) + N_*(z,\theta,\phi) \, , \tag{2}$$

where $\underset{\sim}{n}$ is the unit outward normal to the plane-parallel medium $Z \times \Xi$, where Z is a copy of the non negative real numbers. Furthermore, $\theta = \arccos(\underset{\sim}{\xi} \cdot \underset{\sim}{n})$, so that $\underset{\sim}{\xi} \in \Xi$ may be represented as usual by a pair of angles $(\theta, \phi) \in \Xi$. Finally, we adopt the parameters

$$\mu = \underset{\sim}{\xi} \cdot \underset{\sim}{n} \, ,$$

and

$$\tau(z) = \int_0^z \alpha(z') \, dz' \, ,$$

and observe that the phase function p as used in astrophysics is related to σ by

$$p = 4\pi\,\sigma/\alpha .$$

With these notations, (2) takes the form

$$\mu \, \frac{dN(\tau,\mu,\phi)}{d\tau} = N(\tau,\mu,\phi) - N_q(\tau,\mu,\phi) , \tag{3}$$

where

$$N_q(\tau,\mu,\phi) = \frac{1}{4\pi} \int_{\Xi} p(\tau;\mu,\phi;\mu',\phi')\, N(\tau,\mu',\phi')\, d\mu' d\phi'$$

defines the equilibrium radiance function N_q . Thus (1) reduces, under the above assumptions -- which will be considered in force in the sequel -- to the standard form for the equation of transfer in plane-parallel media.

The discussion will require consideration of the scattering-order decomposition of (3):

$$\mu \, \frac{dN^j(\tau,\mu,\phi)}{d\tau} = N^j(\tau,\mu,\phi) - N_q^j(\tau,\mu,\phi) , \quad j = 1, 2, \ldots , \tag{4}$$

where

$$N_q^j(\tau,\mu,\phi) = \frac{1}{4\pi} \int_{\Xi} p(\tau;\mu,\phi;\mu',\phi')\, N^{j-1}(\tau,\mu',\phi')\, d\mu' d\phi' , \tag{5}$$

and where N^j and N_q^j are positive valued functions on $Z \times \Xi$ which refer to radiant flux which has been scattered precisely j times, so that

$$N(\tau,\mu,\phi) = \sum_{j=0}^{\infty} N^j(\tau,\mu,\phi) . \tag{6}$$

As outlined in the introduction, the present discussion employs, whenever possible, directly observable quantities. From the point of view of the experimenter, the depth dependence of the radiance distribution $N(\tau,\cdot,\cdot)$ on Ξ, $\tau \geq 0$, is most conveniently studied by means of the associated functions $K(\tau,\cdot,\cdot)$ on Ξ defined by

$$K(\tau,\mu,\phi) = \frac{-1}{N(\tau,\mu,\phi)} \frac{dN(\tau,\mu,\phi)}{d\tau} . \tag{7}$$

The present discussion will also require consideration of the function $K_q(\tau,\cdot,\cdot)$ on Ξ, $\tau \geq 0$, defined by

$$K_q(\tau,\mu,\phi) = \frac{-1}{N_q(\tau,\mu,\phi)} \frac{dN_q(\tau,\mu,\phi)}{d\tau} . \tag{8}$$

Similarly, we define

$$K^{j}(\tau,\mu,\phi) = \frac{-1}{N^{j}(\tau,\mu,\phi)} \frac{dN^{j}(\tau,\mu,\phi)}{d\tau}, \qquad j=0,1,\ldots,\ \tau>0, \tag{9}$$

and

$$K_{q}^{j}(\tau,\mu,\phi) = \frac{-1}{N_{q}^{j}(\tau,\mu,\phi)} \frac{dN_{q}^{j}(\tau,\mu,\phi)}{d\tau} \qquad j=1,2,\ldots,\ \tau\geq 0. \tag{10}$$

Finally, corresponding to

$$N_{q}^{(n)}(\tau,\mu,\phi) = \sum_{j=1}^{n} N_{q}^{j}(\tau,\mu,\phi), \tag{11}$$

we define

$$K_{q}^{(n)}(\tau,\mu,\phi) = \frac{-1}{N_{q}^{(n)}(\tau,\mu,\phi)} \frac{dN_{q}^{(n)}(\tau,\mu,\phi)}{d\tau} = \frac{\sum_{j=1}^{n} N_{q}^{j}(\tau,\mu,\phi)\, K_{q}^{j}(\tau,\mu,\phi)}{\sum_{j=1}^{n} N_{q}^{j}(\tau,\mu,\phi)}, \tag{12}$$

where, of course,

$$\lim_{n\to\infty} N_{q}^{(n)}(\tau,\mu,\phi) = N_{q}(\tau,\mu,\phi). \tag{13}$$

The equations which govern the behavior of the K-functions play a central role in what follows. The equations have the outward appearance of Riccati differential equations, and it will actually be possible to use to advantage some of the well-known properties of such differential equations.

It is easy to verify the following formula with the help of (3) and the definitions of K and K_q :

$$\frac{dK(\tau,\mu,\phi)}{d\tau} = \left[K(\tau,\mu,\phi) - K_q(\tau,\mu,\phi)\right]\left[K(\tau,\mu,\phi) + \frac{1}{\mu}\right]. \tag{14}$$

Furthermore, from (4) and the definition of K^j and K_q^j

$$\frac{dK^j(\tau,\mu,\phi)}{d\tau} = \left[K^j(\tau,\mu,\phi) - K_q^j(\tau,\mu,\phi)\right]\left[K^j(\tau,\mu,\phi) + \frac{1}{\mu}\right], \quad j = 1,2,\ldots, \quad \tau > 0. \tag{15}$$

We now can give the motivation for the adoption of the K-functions in the present approach to the asymptotic radiance problem. Suppose there is some depth τ_0 in the medium below which the functions $K(\tau,\cdot,\cdot)$, $\tau \geq \tau_0$ are constant functions on Ξ, and whose values are equal to a fixed number k_∞. Then we may write, for all $\tau \geq \tau_0$,

$$\begin{aligned} N(\tau,\mu,\phi) &= N(0,\mu,\phi)\, \exp\left\{-\int_0^{\tau_0} K(\tau',\mu,\phi)\,d\tau' - \int_{\tau_0}^{\tau} K(\tau',\mu,\phi)\,d\tau'\right\} \\ &= N(\tau_0,\mu,\phi)\, \exp\left\{-(\tau-\tau_0)\,k_\infty\right\}. \end{aligned}$$

Thus if we set

$$g(\mu,\phi) = N(\tau_0,\mu,\phi)\, exp\left\{\tau_0 k_\infty\right\}$$

we have the following equivalent way of representing the distributions:

$$N(\tau,\mu,\phi) = g(\mu,\phi)\; exp\left\{-\tau k_\infty\right\}. \tag{16}$$

Relation (16) is the starting point of the formal procedures which lead to the determination of a specific radiance distribution g on Ξ. From the point of view of the present approach, however, (16) is an incidental end rather than a means. That is, we will be concerned with the determination of a class of spaces in which the radiance distributions tend continuously to a structure of the kind summarized in (16), and thus as a matter of course, determine a class of spaces in which such a formal procedure for the asymptotic radiance distribution is meaningful.

The preceding heuristic argument leading to (16) supplies the motivation for the following definition of an asymptotic radiance distribution: An asymptotic radiance distribution is said to exist if (i) $K_\infty(\mu,\phi) \equiv \lim_{\tau\to\infty} K(\tau,\mu,\phi)$ exists for each $(\mu,\phi) \in \Xi$ and (ii) $K_\infty(\cdot,\cdot)$ is a constant function on Ξ.

It is quite possible for condition (i) of the preceding definition to hold, while condition (ii) does not hold. This state of affairs is encountered, for example, in spaces in which $\tilde{\omega}_0(\tau) = \mathcal{A}(\tau)/\alpha(\tau) = 0$ for all $\tau \geq 0 \left(\mathcal{A}(\tau) = \int_{\Xi} \sigma(\tau; \mu', \phi'; \mu, \phi) \, d\mu' d\phi' \right)$ i.e., in purely absorbing media. However, such spaces are clearly trivial from the present point of view.

3. Formulation of the Problem. In order to keep the usual operations on ρ meaningful, we will assume, as a matter of course, that ρ is piecewise continuously differentiable with respect to τ, and that ρ is continuous on $\Xi \times \Xi$ for each $\tau \geq 0$. Furthermore, we will require that the boundary radiance function $N^0(0, \cdot, \cdot)$ on Ξ_- be a non negative valued, non trivial integrable function with respect to the measure Ω. Here $\Xi_- = \{ \xi : \xi \cdot \mathbf{n} \leq 0 \} = \{ (\mu, \phi) : -1 \leq \mu \leq 0 \}$. In addition, we define Ξ_+ as the complement of Ξ_- with respect to Ξ.

A separable medium is one in which the phase function is independent of position. The term 'separable' is used to suggest the multiplicative uncoupling of position and directional dependence that σ undergoes in such spaces: $\sigma(\mathbf{x}; \xi; \xi') = \frac{1}{4\pi} \alpha(\mathbf{x}) \, \rho(\xi; \xi')$. Separable media form a class of harmlessly inhomogeneous spaces. From the point of view of the equation of transfer (3), such spaces are homogeneous. The present discussion can be carried out in a rather wide class of non-separable spaces which we will call eventually separable. We will say that a semi-infinite stratified plane parallel medium is eventually separable if, (i) the phase function ρ on $Z \times \Xi \times \Xi$ has the form $\rho(\tau; \mu, \phi; \mu', \phi') = \rho_\infty(\mu, \phi; \mu', \phi') + \phi(\tau; \mu, \phi; \mu', \phi')$

such that p_∞ is independent of τ and not identically zero on $\Xi \times \Xi$, and (ii) $\phi \rightarrow 0$ (the zero function) uniformly on $\Xi \times \Xi$, as $\tau \rightarrow \infty$.

We can now state the main result:

<u>An asymptotic radiance distribution exists in every plane-parallel medium if and only if the medium is eventually separable.</u> This statement is understood to hold in media whose equation of transfer is given by (3), and whose boundary conditions are given as above. All of the effort below will be devoted to proving the sufficiency of the eventually separable condition. Simple counterexamples show that if a space is not eventually separable, the asymptotic radiance distribution necessarily does not exist.

We close this preliminary discussion by making some observations on the K-functions which will be required below. First we observe that from (3), if $\mu = 0$, then $N(\tau,0,\phi) = N_q(\tau,0,\phi)$. Hence, for all $\tau > 0$, $K(\tau,0,\phi) = K_q(\tau,0,\phi)$. Secondly, for each $\tau > 0$, $N(\tau,\cdot,\cdot)$ is bounded away from zero, is continuous on the compact set Ξ, and hence is uniformly continuous on Ξ and Ξ_-. A similar observation holds for $N_q^j(\tau,\cdot,\cdot)$, $j = 1, 2, \ldots$. It follows that $K_q(\tau,\cdot,\cdot)$ and $K_q^j(\tau,\cdot,\cdot)$ are uniformly continuous on Ξ and Ξ_- for all $\tau > 0$ and $j = 1, 2, \ldots$. Finally, from (3) and (4) and the definitions of K and K^j,

$$K(\tau,\mu,\phi) + \frac{1}{\mu} < 0 \quad , \tag{17}$$

$$K^{j}(\tau,\mu,\phi) + \frac{1}{\mu} < 0 \;, \quad j = 0, 1, \ldots , \tag{18}$$

for all $(\mu, \phi) \in \Xi_-$ and all $\tau > 0$. Properties (17) and (18) are particularly useful in conjunction with (14) and (15). For example if $K(\tau,\mu,\phi) \lesseqgtr K_q(\tau,\mu,\phi)$, then by (14) and (17) it follows that $dK(\tau,\mu,\phi)/d\tau \gtreqless 0$, showing that in general K always tends toward the equilibrium function K_q. This is a particularly useful fact in practice. Whether or not $K \to K_q$ as $\tau \to \infty$ depends on the relative sizes of $K_q(\mu,\phi) = \lim_{\tau\to\infty} K(\tau,\mu,\phi)$. It follows directly from the properties of the Riccati equation [12], that $K \to \min\left\{K_q(\mu,\phi), -\frac{1}{\mu}\right\}$ assuming of course that $K_q(\mu,\phi)$ exists. A similar set of remarks holds for (15) and (18).

4. The Functions P, Q, and R. In order to insure that the main sequence of arguments is uninterrupted by the development of certain required auxiliary relations, these auxiliary relations are gathered here for ready reference.

The first relation needed below gives the connection between the downwelling j-scattered flux at level $\tau > 0$ and the upwelling scattered flux at level τ:

$$N^{j+1}(\tau,\mu,\phi) = \frac{1}{\mu}\int_{\Xi_-} P(\tau;\mu,\phi;\mu',\phi')\, N^{j}(\tau,\mu',\phi')\, d\mu' d\phi' \tag{19}$$

for all $(\mu,\phi) \varepsilon \Xi_+$, and where

$$P(\tau;\mu,\phi;\mu',\phi') = \frac{1}{4\pi}\int_\tau^\infty \rho(\tau';\mu,\phi;\mu',\phi')\exp\left\{-(\tau'-\tau)\left[\frac{1}{\mu}-\frac{1}{\mu'}\right]\right\}d\tau' . \quad (20)$$

If the space were separable, i.e., ρ were independent of τ (or, in the present case, $\phi \equiv 0$) then (20) reduces to

$$P_\infty(\mu,\phi;\mu',\phi') = \frac{1}{4\pi}\frac{\mu\mu'}{\mu'-\mu}\rho_\infty(\mu,\phi;\mu',\phi') , \quad (21)$$

where $(\mu,\phi) \varepsilon \Xi_+$, $(\mu',\phi') \varepsilon \Xi_-$.

We observe that for eventually separable spaces,

$$\lim_{\tau\to\infty}\frac{dP(\tau;\mu,\phi;\mu',\phi')}{d\tau} = 0 \quad (22)$$

uniformly on $\Xi_+ \times \Xi_-$, and that

$$\lim_{\tau\to\infty} P(\tau;\mu,\phi;\mu',\phi') = P_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+ \times \Xi_-$ and finally, that:

$$\lim_{\tau\to\infty} K^{i+1}(\tau,\mu,\phi) = \lim_{\tau\to\infty}\frac{\int_{\Xi_-} P(\tau;\mu,\phi;\mu',\phi')\,N^i(\tau,\mu',\phi')\,K^i(\tau,\mu',\phi')\,d\mu'd\phi'}{\int_{\Xi_-} P(\tau;\mu,\phi;\mu',\phi')\,N^i(\tau,\mu',\phi')\,d\mu'd\phi'} \quad (23)$$

for $(\mu,\phi) \varepsilon \Xi_+$.

A similar expression holds for $K_q^{j+1}(\tau,\mu,\phi)$, which follows from (5) and (10):

$$\lim_{\tau\to\infty} K_q^{j+1}(\tau,\mu,\phi) = \lim_{\tau\to\infty} \frac{\int_{\Xi} p(\tau;\mu,\phi;\mu',\phi')\, N^j(\tau,\mu',\phi')\, K^j(\tau,\mu',\phi')\, d\mu' d\phi'}{\int_{\Xi} p(\tau;\mu,\phi;\mu',\phi')\, N^j(\tau,\mu',\phi')\, d\mu' d\phi'} \quad (24)$$

The next relation required below makes use of the forms of the principles of invariance in generally non-separable media, in particular, use will be made of:

$$N(\tau,\mu,\phi) = \frac{1}{\mu}\int_{\Xi_-} R(\tau,\infty;\mu,\phi;\mu',\phi')\, N(\tau,\mu',\phi')\, d\mu' d\phi' \qquad (25)$$

where $(\mu,\phi) \in \Xi_+$ and $R(\tau,\infty;\ \cdot\ ;\ \cdot\ )$ on $\Xi_+ \times \Xi_-$ is the diffuse reflectance function associated with the subset of $Z \times \Xi$ below level $\tau \geq 0$, [13]. If the medium were separable, then for all pairs (τ_1, τ_2) of depths,

$$R(\tau_1,\infty;\mu,\phi;\mu',\phi') = R(\tau_2,\infty;\mu,\phi;\mu',\phi').$$

In the present case it is easy to verify ([13], equation I') that

$$\lim_{\tau\to\infty} \frac{dR(\tau,\infty;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

uniformly on $\Xi_+ \times \Xi_-$, and that

$$\lim_{\tau\to\infty} R(\tau,\infty;\mu,\phi;\mu',\phi') = R_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+ \times \Xi_-$, where R_∞ on $\Xi_+ \times \Xi_-$ is defined by the phase function p_∞.

Finally, the integral operator

$$\int_{\Xi_-} Q(\tau,\infty;\mu,\phi;\mu',\phi')\,[\quad]\,d\mu'd\phi' \tag{26}$$

will be used. This operator maps the function $N(\tau,\cdot,\cdot)$ on Ξ_- into the function $N_q(\tau,\cdot,\cdot)$ on Ξ_-. The kernel Q is the form:

$$Q(\tau,\infty;\mu,\phi;\mu',\phi') = \rho(\tau;\mu,\phi;\mu',\phi') + \int_{\Xi_+} \rho(\tau;\mu,\phi;\mu'',\phi'')\,R(\tau,\infty;\mu'',\phi'';\mu',\phi')\,\frac{d\mu''}{\mu''}d\phi''.$$

The operator (26) is a positive operator*. From the definition of Q, it follows ([13] , equation I') that

$$\lim_{\tau\to\infty} \frac{dQ(\tau,\infty;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

and that

$$Q_\infty(\mu,\phi;\mu',\phi') \equiv$$

$$\lim_{\tau\to\infty} Q(\tau,\infty;\mu,\phi;\mu',\phi') = \rho_\infty(\mu,\phi;\mu',\phi') + \int_{\Xi_+} \rho(\mu,\phi;\mu'',\phi'')\,R_\infty(\mu'',\phi'';\mu',\phi')\,\frac{d\mu''}{\mu''}d\phi''$$

both uniformly on $\Xi_- \times \Xi_-$.

5. The limit of $K_q(\cdot,\mu,\phi)$. We now begin the main steps of the proof. The object of this section is to show that the function $K_q(\cdot,\mu,\phi)$ on Z defined by

$$K_q(\mu,\phi) = \lim_{\tau\to\infty} K_q(\tau,\mu,\phi)$$

exists and is continuous almost everywhere on Ξ . The discussion begins with some observations on the functions $K^j, K_q^j, K_q^{(n)}$. In particular, we observe that for $(\mu,\phi)\in\Xi$ and every $\tau \geq 0$,

* For the present discussion, an operator T is said to be positive if $Tf = 0$ (the zero function) implies f is the zero function, where is a non negative valued function on Ξ_- , and the vanishing of f is taken in the sense of Lebesgue (cf., e.g., [14] , p. 25).

$$K_q^1(\tau,\mu,\phi) =$$

$$-\frac{\left\{\int_{\Xi_-} \rho(\tau;\mu,\phi;\mu',\phi')\, N^\circ(\tau,\mu',\phi')\, \frac{d\mu'}{\mu'}\, d\phi' + \int_{\Xi_-} \rho'(\tau;\mu,\phi;\mu',\phi')\, N^\circ(\tau,\mu',\phi')\, d\mu' d\phi'\right\}}{\int_{\Xi_-} \rho(\tau;\mu,\phi;\mu',\phi')\, N^\circ(\tau,\mu',\phi')\, d\mu' d\phi'}, \tag{27}$$

where ρ' denotes the derivative of ρ with respect to τ. The function $N^\circ(\tau,\cdot,\cdot)$ on Ξ_- is related to the boundary radiance distribution by

$$N^\circ(\tau,\mu,\phi) = N^\circ(0,\mu,\phi)\, e^{\tau/\mu}.$$

Hence each integrand in (27) is integrable on Ξ_-, so that $K_q^1(\tau,\mu,\phi)$ exists and is well defined for every $\tau \geq 0$ and $(\mu,\phi) \in \Xi$. Furthermore each integrand in (27) satisfies the hypothesis of Lebesgue's bounded convergence theorem, so that by (24)

$$K_q^1(\mu,\phi) = \lim_{\tau\to\infty} K_q^1(\tau,\mu,\phi) \tag{28}$$

exists for every $(\mu,\phi) \in \Xi$ and in fact $K_q^1(\cdot,\cdot)$ is continuous and therefore bounded on Ξ. The values of $K_q^1(\cdot,\cdot)$ are readily determinable for specific choices of $N^\circ(0,\cdot,\cdot)$. For example, if we adopt the standard discrete boundary radiance distribution defined by

$$N^\circ(0,\mu,\phi) = N^\circ\, \delta(\mu-\mu_0)\, \delta(\phi-\phi_0), \quad -1 \leq \mu_0 < 0,$$

then

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_0}\,, \quad (\mu,\phi) \in \Xi\,. \tag{29}$$

Slightly more generally, if

$$N^\circ(0,\mu,\phi) = \sum_{i=0}^{n} N^\circ(\mu_i)\,\delta(\mu-\mu_i)\,\delta(\phi-\phi_i)\,, \quad -1 \le \mu_i < 0\,,$$

and if

$$\mu_0 = \min\{\mu_i : i = 0, 1, \ldots, n\}$$

then

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_0}\,, \quad (\mu,\phi) \in \Xi\,. \tag{30}$$

Other simple examples of $N^\circ(0,\cdot,\cdot)$ may be given, such as step-function representations, various simple continuous functions on Ξ_- but (29) and (30) will suffice to illustrate the general procedure. In particular they help to illustrate the use of (15) which is required in the next step of the proof and which runs as follows: By means of (15) and (28), we see that for each $(\mu,\phi) \in \Xi_-$ such that

$$K_q^1(\mu,\phi) > -\frac{1}{\mu} \tag{31}$$

we have

$$\lim_{\tau\to\infty} K^1(\mu,\phi) = -\frac{1}{\mu}$$

Since $K_q^1(\cdot,\cdot)$ is bounded, the subset of μ's for which (31) holds is a relatively open subset of $[0, 1]$ excluding 0. Finally, from (15) and (28), for each $(\mu,\phi) \in \Xi_-$ such that

$$K_q^1(\mu,\phi) \leq -\frac{1}{\mu}$$

we have

$$\lim_{\tau\to\infty} K^1(\tau,\mu,\phi) = K_q^1(\mu,\phi) .$$

Hence the function $K_\infty^1(\cdot,\cdot)$ on Ξ_- defined by

$$K_\infty^1(\mu,\phi) = \lim_{\tau\to\infty} K^1(\tau,\mu,\phi)$$

exists for all $(\mu,\phi) \in \Xi_-$ and is continuous on Ξ_-. A particular illustration of a $K_\infty^1(\cdot,\cdot)$ is given by means of (29).

The main observation to make at this point is the following: in addition to being bounded on Ξ_-, the function $K_\infty^1(\cdot,\cdot)$ has the property that

$$K_\infty^1(\mu,\phi) \leq -\frac{1}{\mu} \tag{32}$$

on Ξ_-. Observe also that $K_q^1(\cdot,\cdot)$ does not generally have this property. The discussion of $K_\infty^1(\cdot,\cdot)$ is completed by showing that it exists and is continuous on Ξ_+. This is done by applying the preceding arguments to (23). As an example, one may consider (29) once again, which yields $K_\infty^1(\mu,\phi) = -\frac{1}{\mu}$. for all $(\mu,\phi) \varepsilon \Xi_+$.

We now consider the general inductive step, that is, assume that $K_\infty^j(\cdot,\cdot), j \geq 1$ is continuous on Ξ and in particular, $K_\infty^j(\mu,\phi) \leq -\frac{1}{\mu}$ for every $(\mu,\phi) \varepsilon \Xi_-$. Then by means of (24) and the previously cited convergence arguments, we find that

$$K_q^{j+1}(\mu,\phi) = \lim_{\tau\to\infty} K_q^{j+1}(\tau,\mu,\phi) \tag{34}$$

exists for every $(\mu,\phi) \varepsilon \Xi_-$, and $K_q^{j+1}(\cdot,\cdot)$ is continuous on Ξ_- ; and in particular*,

$$K_q^{j+1}(\mu,\phi) \leq -\frac{1}{\mu} \quad , \quad (\mu,\phi) \varepsilon \Xi_- . \tag{35}$$

Furthermore, by (15),

$$K_\infty^{j+1}(\mu,\phi) = \lim_{\tau\to\infty} K^{j+1}(\tau,\mu,\phi) = K_q^{j+1}(\mu,\phi) \tag{36}$$

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* Implicit use has been made of the fact that if $F(x) = [A(x)a(x) + B(x)b(x)] / [A(x) + B(x)]$, and if we have $a(x) \to a(x_0) \leq a_0$ along with $B(x) = o(A(x))$ as $x \to x_0$, then $F(x) \to a(x_0) \leq a_0$ as $x \to x_0$.

on Ξ_- . Finally, from (23) $K_\infty^{j+1}(\cdot,\cdot)$ exists and is continuous on Ξ_+ and moreover,

$$K_\infty^{j+1}(\mu,\phi) \leq -\frac{1}{\mu} \quad , \quad (\mu,\phi) \in \Xi_+ \, . \tag{37}$$

Since the induction hypothesis is true for $j=1$, the conclusions (34) - (37) then hold for all integers $j = 1, 2, \ldots$. It follows from (12) and the preceding results that

$$K_\eta^{(n)}(\mu,\phi) = \lim_{\tau\to\infty} K_\eta^{(n)}(\tau,\mu,\phi) \leq -\frac{1}{\mu} \quad , \; n = 2,3,\ldots \tag{38}$$

exists and is continuous on Ξ .

Now consider the function $g_\eta^{(n)}(\cdot,\mu,\phi)$ on Z defined for every $(\mu,\phi) \in \Xi$ by

$$g_\eta^{(n)}(\tau,\mu,\phi) = \frac{N_\eta^{(n)}(\tau,\mu,\phi)}{N_\eta(\tau,\mu,\phi)} \quad , \quad n = 1,2,\ldots \, . \tag{39}$$

Clearly $\{g_\eta^{(n)}(\cdot,\mu,\phi)\}$ is an increasing sequence of functions on Z, such that for every $(\mu,\phi) \in \Xi$,

$$\lim_{n\to\infty} g_\eta^{(n)}(\cdot,\mu,\phi) = 1 \, ,$$

the unit function on Z. From this and the definitions of $K_q^{(n)}$ and K_q, we conclude, first of all, that the sequence $\{K_q^{(n)}(\cdot,\mu,\phi)\}$ of functions converge in the mean to $K_q(\cdot,\mu,\phi)$ on Z. This in turn implies that the convergence to $K_q(\cdot,\mu,\phi)$ is almost uniform on Z for some subsequence $\{K_q^{(n_k)}(\cdot,\mu,\phi)\}$. Hence for every $\epsilon > 0$,

$$\lim_{n_k\to\infty} \lim_{\tau\to\infty} K_q^{(n_k)}(\tau,\mu,\phi) = \lim_{\tau\to\infty} \lim_{n_k\to\infty} K_q^{(n_k)}(\tau,\mu,\phi)$$
$$= \lim_{\tau\to\infty} K(\tau,\mu,\phi)$$

on $Z' = Z - Z_0$, where $\int_{Z_0} d\tau' < \epsilon$. It follows that the function $K_q(\cdot,\cdot)$ on Ξ, defined by

$$K_q(\mu,\phi) = \lim_{\tau\to\infty} K_q(\tau,\mu,\phi) = \lim_{n_k\to\infty} K_q^{(n_k)}(\mu,\phi) \leq -\frac{1}{\mu} \quad (40)$$

exists on Ξ. Finally, from (24), (15), (23), and (12), (in that order) we establish the fact that $\{K_q^{(n)}(\cdot,\cdot)\}$ is a sequence of continuous functions whose essential suprema form a non increasing sequence of real numbers. It follows that $\{K_q^{(n)}(\cdot,\cdot)\}$ converges uniformly a.e. on Ξ to $K_q(\cdot,\cdot)$, and that $K_q(\cdot,\cdot)$ is continuous on $\Xi - \Xi_0$ where Ξ_0 is a subset of Ξ such that $\Omega(\Xi_0) = 0$.

6. The limit of $K(\cdot, \mu, \phi)$. The proof is now concluded by showing that $K(\cdot, \mu, \phi)$ satisfies the definition of asymptoticity. By (14) and the result summarized in (40), we have

$$K_\infty(\mu, \phi) = \lim_{\tau \to \infty} K(\tau, \mu, \phi) = K_g(\mu, \phi) \tag{41}$$

for all $(\mu, \phi) \varepsilon\, \Xi_-$. Let (μ_1, ϕ_1) be determined by the condition

$$K_\infty(\mu_1, \phi_1) = \inf \left\{ K_\infty(\mu, \phi) \;:\; (\mu, \phi) \varepsilon\, \Xi_- \right\}$$

then set

$$g(\tau, \mu, \phi) = \frac{N(\tau, \mu, \phi)}{N(\tau, \mu_1, \phi_1)},$$

and observe that g on Ξ_- defined by

$$g(\mu, \phi) = \lim_{\tau \to \infty} g(\tau, \mu, \phi)$$

is at least bounded and measurable (hence integrable) on Ξ_-. Then, by means of the operator defined in (26), we have

$$K_g(\mu, \phi) = \frac{\int_{\Xi_-} Q_\infty(\mu, \phi; \mu', \phi')\, g(\mu', \phi')\, K_\infty(\mu', \phi')\, d\mu' d\phi'}{\int_{\Xi_-} Q_\infty(\mu, \phi; \mu', \phi')\, g(\mu', \phi')\, d\mu' d\phi'}, \tag{41}$$

for all $(\mu, \phi) \varepsilon\, \Xi_-$. In particular, (41) holds for $(\mu_1, \phi_1) \varepsilon\, \Xi_-$.

Using (41), (42) may be rewritten as:

$$\int_{\Xi_-} Q_\infty(\mu_1,\phi_1;\mu',\phi')\, g(\mu',\phi')\left[K_\infty(\mu',\phi') - K_\infty(\mu_1,\phi_1)\right] d\mu' d\phi' .$$

This operator, as that in (26), being a positive operator, requires that the non negative valued function

$$K_\infty(\cdot,\cdot) - K_\infty(\mu_1,\phi_1)$$

on Ξ_- be the zero function, i.e.,

$$K_\infty(\mu,\phi) = K_\infty(\mu_1,\phi_1) \equiv k_\infty$$

for all $(\mu,\phi) \in \Xi_-$. An application of (25) to the definitions of $K(\tau,\mu,\phi)$ and $K_\infty(\mu,\phi)$, yields the result that

$$K_\infty(\mu,\phi) = k_\infty$$

on Ξ_+, so that $K_\infty(\cdot,\cdot)$ is a constant function on Ξ. This concludes the proof. We observe finally, that, by means of the definition of N_q and (8),

$$K_q(\mu,\phi) = K_\infty(\mu,\phi) = k_\infty$$

on Ξ.

7. Notes. (i) The physical significance of k_∞ is readily determined. We observe that the scalar irradiance function h on Z defined by

$$h(\tau) = \int_{\Xi} N(\tau,\mu,\phi)\, d\mu\, d\phi$$

has, in analogy to N, a K-function defined as:

$$k(\tau) = \frac{-1}{h(\tau)} \frac{d\,h(\tau)}{d\tau} ,$$

which is represented in terms of $K(\tau,\cdot,\cdot)$ by the formula:

$$k(\tau) = \frac{\int_{\Xi} N(\tau,\mu,\phi)\, K(\tau,\mu,\phi)\, d\mu\, d\phi}{\int_{\Xi} N(\tau,\mu,\phi)\, d\mu\, d\phi} .$$

From this and the preceding results, we see that:

$$\lim_{\tau\to\infty} k(\tau) = \frac{\int_{\Xi} g(\mu,\phi)\, K_{\infty}(\mu,\phi)\, d\mu\, d\phi}{\int_{\Xi} g(\mu,\phi)\, d\mu\, d\phi} = k_{\infty} .$$

Hence k_{∞} is the limit, as $\tau\to\infty$, of $k(\cdot)$, the K-function for scalar irradiance. The function h is related to the radiant energy density function u by $h = vu$. ¶ (ii) The equation of transfer (3) may be written in terms of $K(\tau,\cdot,\cdot)$ as follows:

$$N(\tau,\mu,\phi) = \frac{N_{q}(\tau,\mu,\phi)}{1+\mu K(\tau,\mu,\phi)} ,$$

which is the canonical form of the equation of transfer for the slab geometry. The limit of the canonical form as $\tau\to\infty$ is, by the preceding results,

$$g(\mu,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} p_{\infty}(\mu,\phi;\mu',\phi')\, g(\mu',\phi')\, d\mu'\, d\phi'}{1+\mu k_{\infty}} ,$$

which is the general form used in the formal-solution procedures discussed in the introduction. The real number k_{∞} now takes on the additional significance of being an eigenvalue of a non-linear eigenvalue problem associated with the above integral equation for g on Ξ. For the kind of boundary conditions adopted in the present paper — which as we have noted before, stem from the geophysical origins of the asymptotic radiance problem -- the resultant values of k_∞ are non negative, and in fact, $0 \leq k_\infty < 1$ (cf. (17) and the def. of g). (iii) Figure 1 shows the depth dependence of $K(\cdot, \mu, \phi)$ for several directions $(\mu, \phi) \varepsilon \Xi$. The associated medium is an hypothetical separable half-space, irradiated by normally incident collimated neutron flux, in which scattering is isotropic and $\tilde{\omega}_0 = 0.9$. These plots are based on theoretical computations of $N(\tau, \mu, \phi)$ (for neutron flux) compiled in [15] . The plots show clearly that asymptoticity has been essentially attained at $\tau = 10$, for at this depth the function $K(10, \cdot, \cdot)$ is essentially constant on Ξ .

Figure 2 shows the depth dependence of $K(\cdot, \mu, \phi)$ for several directions $(\mu, \phi) \varepsilon \Xi$. The associated medium is a natural hydrosol, namely Lake Pend Oreille, Idaho, which at the time of measurement of $N(\tau, \mu, \phi)$ (for photon flux), was irradiated by light from a clear sunny sky (angle of sun from zenith was about 40°, hence the associated $\mu_0 = -0.77$); scattering was found to be highly anisotropic and $\tilde{\omega}_0(\tau)$ approached with increasing τ a constant value of about 0.7, indicating the medium was eventually separable. These plots are based on experimental determinations of $N(\tau, \mu, \phi)$ by J. E. Tyler of Scripps Institution of Oceanography [16] . All N-measurements were made at 478 millimicrons. The plots show that asymptoticity is being approached at depths $\tau = 30$. The azimuth angle ϕ has been fixed at 0°, which denotes the vertical plane through the sun. Plots for $\phi \neq 0^{\circ}$ indicate similar trends to asymptoticity for depths at $\tau = 30$. The vertical K-scale has been exaggerated (relative to that of Figure 1) in order to more clearly show the details of the transition to asymptoticity.

REFERENCES

[1] L. V. Whitney, "The Angular Distribution of Characteristic Diffuse Light in Natural Waters," J. Marine Research, 4, 122-131 (1941).

[2] L. V. Whitney, "A General Law of Diminuation of Light Intensity in Natural Waters and the Percent of Diffuse Light at Different Depths," J. Opt. Soc. Am., 31, 714-722 (1941).

[3] J. Lenoble, "Etude de la Pénétration de L'Ultraviolet dans La Mer," Ann. de Geophys., 12, 16-31 (1956).

[4] J. Lenoble, "Etude théorique de la Pénétration du Rayonnement dans les milieux Diffusants Naturels," Opt. Acta, 4, 1-11 (1957).

[5] H. H. Poole, "The Angular Distribution of Submarine Daylight in Deep Water," Sci. Proc. Roy. Dublin Soc., 24, 29-42 (1945).

[6] S. Chandrasekhar, Radiative Transfer, Clarendon Press, Oxford (1950).

[7] I. Kuščer, "Milne's Problem for Aniostropic Scattering," J. Math. and Phys., 34, 256-266 (1955).

[8] B. Davison, Neutron Transport Theory, Clarendon Press, Oxford (1957).

[9] A. Schuster, "Radiation Through a Foggy Atmosphere," Astrophys. J., 21, 1-22 (1905).

[10] R. W. Preisendorfer, "Unified Irradiance Equations," to be published.

[11] R. W. Preisendorfer, "A Mathematical Foundation for Radiative Transfer Theory," J. Math. and Mech., 6, 685-730 (1957).

[12] E. L. Ince, Ordinary Differential Equations, Dover Publications.

[13] R. W. Preisendorfer, "Functional Relations for the R and T Operators on Plane Parallel Media," Proc. Nat. Acad. Sci., 44, 323-327 (1958).

[14] E. Hopf, Mathematical Problems of Radiative Equilibrium, Cambridge Tracts in Math. and Math. Phys., 31, Camb. Univ. Press (1934).

[15] L. A. Beach and others, Comparison of Solutions to the One-Velocity Neutron Diffusion Problem, NRL Report 5052, Naval Research Laboratory, Washington , D. C. (1957).

[16] J. E. Tyler, Radiance Distribution as a Function of Depth in the Submarine Environment, SIO Ref. 58-25, Visibility Laboratory Report, Scripps Institution of Oceanography, University of California (1958).

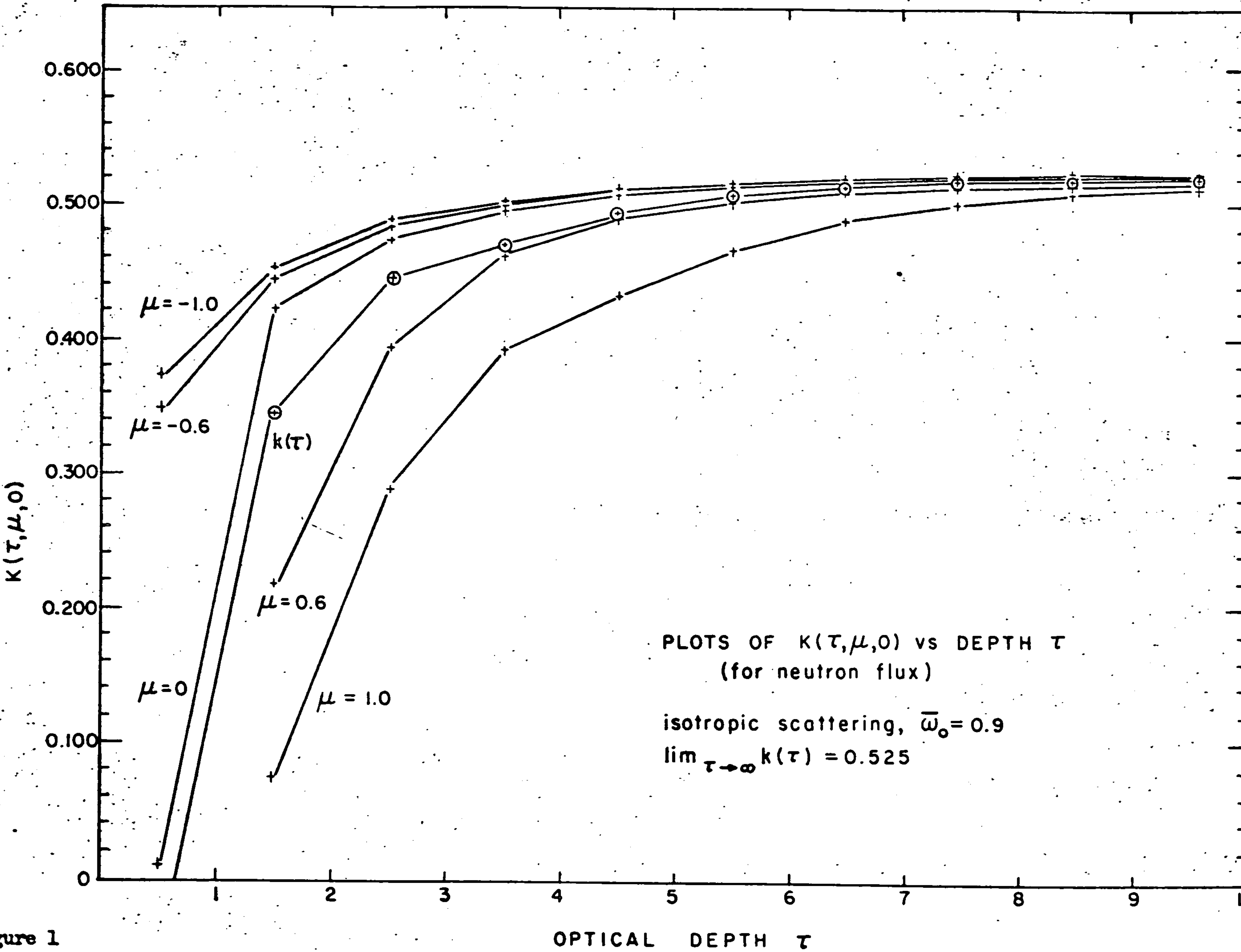

Figure 1
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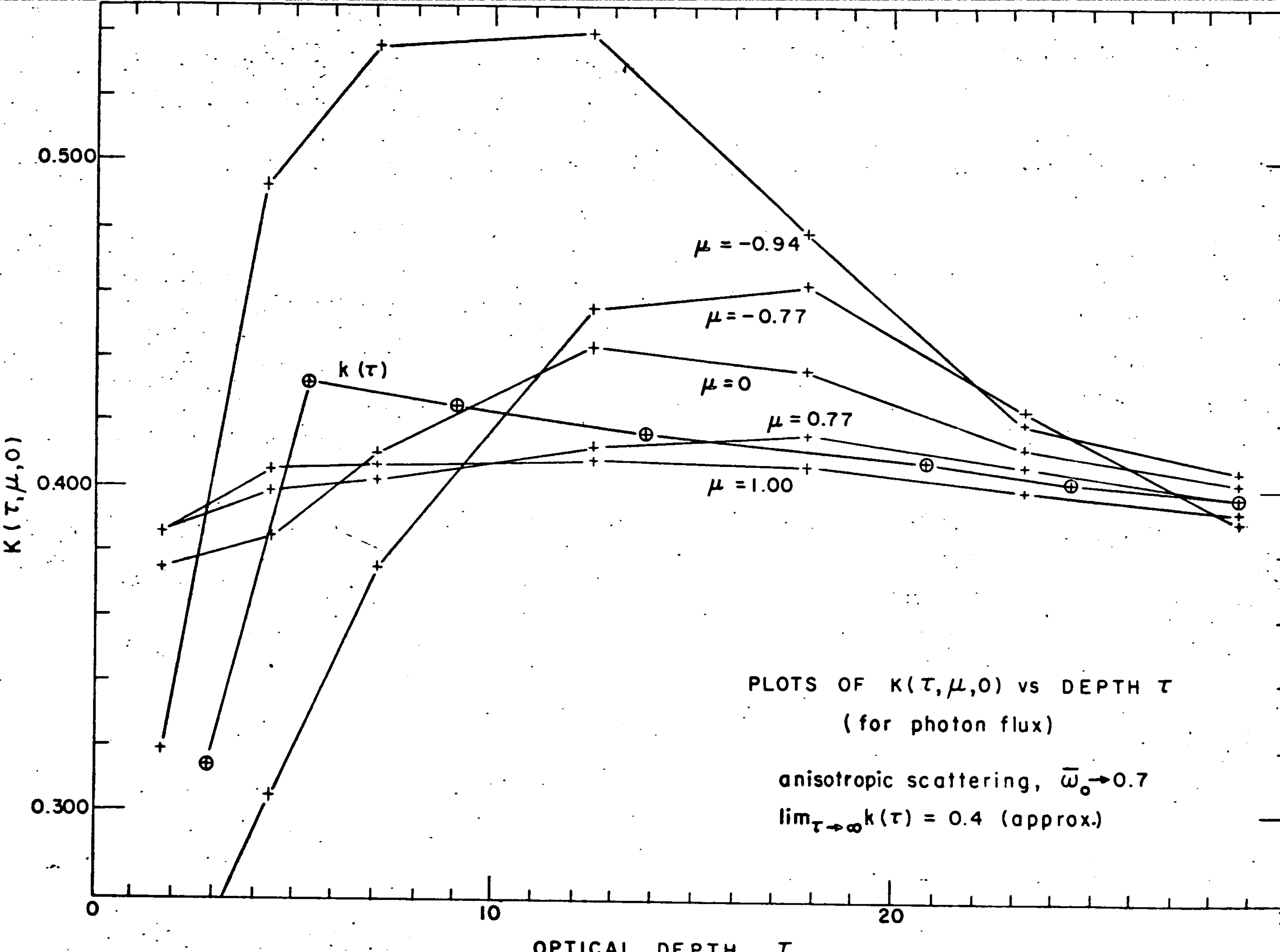

Figure 2

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