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Publication Date

1951-08-27

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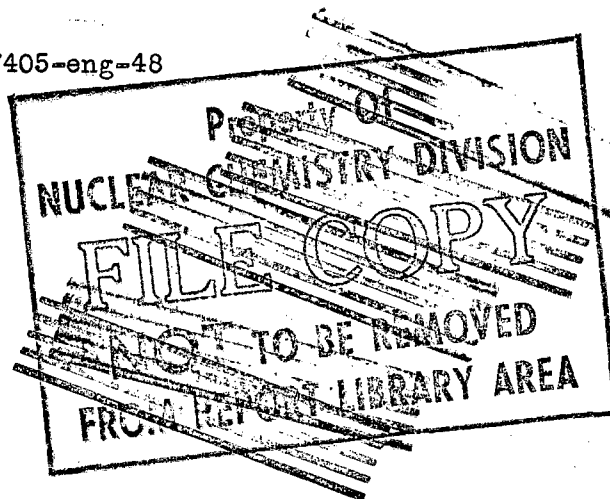
ORBITS IN A PROPOSED SPHERICAL ELECTROSTATIC SPECTROMETER

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I. Introduction

It is proposed by F. L. Reynolds to construct an ion energy spectrometer out of a spherical condenser. Following is a theoretical discussion of the orbits in and focussing properties of such an instrument,

The idea is to place source and detector between the two spherical metal shells which are at a potential difference from each other. There will be a central, inverse square electric field in the annulus which will focus ions from the source into the detector if they do not deviate too far from circular orbits, and if the detector is about 180 degrees removed from the source. The energy of the ions is then measured by adjusting the potential difference between the condenser plates until the ions are caught by the detector. The chief merit of this device is the relatively large solid angle available for ions which can be focussed into the detector, compared to more usual apparatus which will only accept ions moving in or near to a given plane.

II. Theory

1. The Field

We suppose the spherical shells have radii R_1 and R_2 and are at potentials V_1 and V_2. Then the potential between the shells relative to ground is

$$V_g(r) = V_1 - k(\frac{1}{r} - \frac{1}{R_1})$$

where

$$k = (V_2 - V_1)/(\frac{1}{R_1} - \frac{1}{R_2}) \qquad (1a)$$
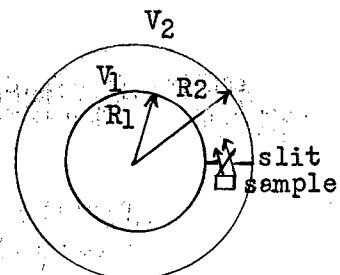
Fig. 1

If the zero of potential is taken at infinity than the potential and electric fields are

$$V(r) = -\frac{k}{r} \qquad , \quad \vec{E}(r) = -\frac{k}{r^3} \, \vec{r}$$

2. Orbits in the Field

Suppose the sample ejecting the ions is at potential corresponding to radius a; i.e.,

$$V_s = V_1 - k(\frac{1}{a} - \frac{1}{R_1})$$

This sample will probably be located behind a circular slit which will actually define the source. It is supposed that the field at the slit is undisturbed. One defines

   r = radius of ion

   T = kinetic energy

   U = T + rest energy

   E = U + potential energy with zero at infinity

   Ze = charge of ion

   Subscript 0 refers to reference ion

   Subscript 1 refers to ion as it leaves sample

   Subscript 2 refers to ion just as it passes through the slit.

By energy conservation, one has

$$E_1 = U_1 - \frac{Zek}{a} = E_2 = U_2 - \frac{Zek}{r_2} = E = U - \frac{Zek}{r} \tag{1}$$

$$= \frac{mc^2}{\sqrt{1 - \beta^2}} - \frac{Zek}{r}$$

The ions will move in planes. Their motion is described in coordinates r, Θ, ϕ, where the slit is in the yz plane and the z axis is taken so that the ion is at the z axis when it passes through the slit. Then the ion moves in a plane defined by ϕ = const., since it never experiences a force out of this plane.
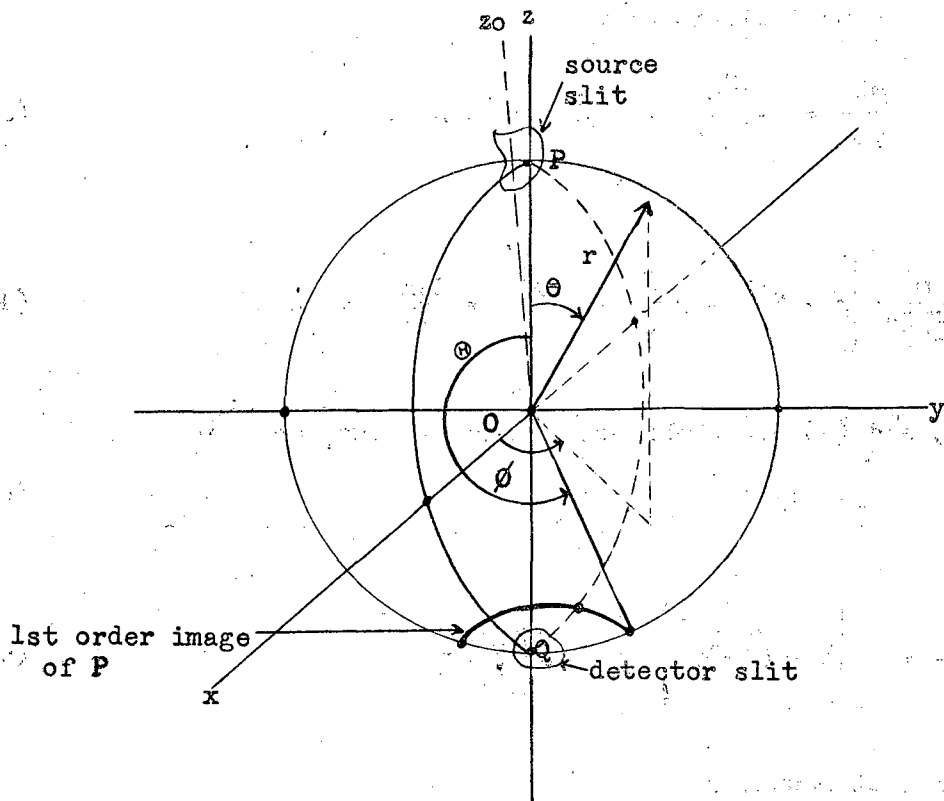

Fig. 2

The Lagrangian equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$q_i = r, \theta, \phi \tag{2}$$

$$L = - mc^2 \sqrt{1 - \beta^2} + \frac{Zek}{r}$$

In the spherical coordinates taken as described above

$$\beta^2 = \frac{1}{c^2} (\dot{r}^2 + r^2 \dot{\theta}^2)$$

so from (2) $\left[\text{dots indicate } d/dt\right]$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{mr^2 \dot{\theta}}{\sqrt{1-\beta^2}} = \ell = \text{const.} \tag{3}$$

$$\frac{m}{\sqrt{1-\beta^2}} \left[\ddot{r} - 2\frac{\dot{r}^2}{r} - \dot{r} \frac{\ddot{\theta}}{\dot{\theta}} - r\dot{\theta}^2 \right] + \frac{Zek}{r^2} = 0 \tag{4}$$

If (1), (3), and (4) are combined, eliminate time, and set

$$u = 1/r \tag{5}$$

then

$$\frac{d^2 u}{d\theta^2} + \left[1 - \left(\frac{Zek}{\ell c}\right)^2 \right] u = \frac{ZekE}{\ell^2 c^2} \tag{6}$$

whose solution obviously is

$$u = \frac{1}{r} = \frac{\frac{ZekE}{\ell^2 c^2}}{1 - \left(\frac{Zek}{\ell c}\right)^2} \left[1 + \epsilon \cos \sqrt{1 - \left(\frac{Zek}{\ell c}\right)^2} (\theta - \theta') \right] \tag{7}$$

where ϵ and θ' are determined by the initial conditions. This is the

expression for a precessing ellipse, since r is periodic in $\Delta\theta = \dfrac{2\pi}{\sqrt{1-\left(\dfrac{Zek}{\ell c}\right)^2}}$.

By setting

$$\psi = \sqrt{1-\left(\frac{Zek}{\ell c}\right)^2}\;(\theta-\theta') \tag{8}$$

Equation (7) is written in the standard forms for a conic section;

$$r = \frac{a'(1-\epsilon^2)}{1+\epsilon\cos\psi} \tag{7'}$$

Here a' is the semimajor axis and ϵ the eccentricity of the ellipse. Comparing (7) and (7') it is seen that

$$a'(1-\epsilon^2) = \frac{Zek}{E}\left[\left(\frac{\ell c}{Zek}\right)^2 - 1\right] \tag{9}$$

Now suppose, as shown in Fig. 3, the ion traverses the slit at point P, at radius $r = r_2$, energy E and is going at an angle α to the sphere about O of radius r_2. The orbit may be expressed entirely in terms of $E, r_2,$ and α.

In the first place the angular momentum $\vec{\ell}$ has magnitude
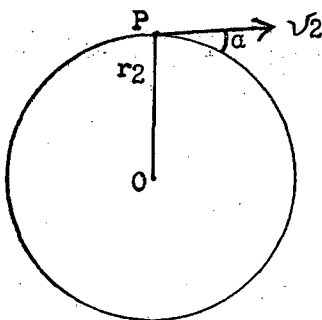
Fig. 3

$$\ell = \frac{mr^2\dot\theta}{\sqrt{1-\beta^2}} = \frac{m\,r_2\,v_2\,\cos\alpha}{\sqrt{1-\beta_2^2}}$$

so $\dfrac{m}{\sqrt{1-\beta^2}} = \dfrac{\ell}{r^2\dot\theta}$

and from (1)

$$E = \frac{\ell c^2}{r^2\dot\theta} - \frac{Zek}{r}\,,\quad \dot\theta = \frac{\ell c^2}{Er^2+Zekr}$$

Putting $1 = d/d\theta$, $\beta^2 = \dfrac{1}{c^2}(\dot r^2 + r^2\dot\theta^2) = \dfrac{\dot\theta^2}{c^2}(r'^2 + r^2)$

$$= \frac{\ell^2 c^2\,(r'^2+r^2)}{(Er^2+Zekr)^2}$$

Putting this back into (1) one obtains

$$E = \frac{mc^2}{\sqrt{1 - \frac{\ell^2 c^2 (r'^2 + r^2)}{(Er^2 + Zekr)^2}}} - \frac{Zek}{r} \tag{10}$$

The turning points (points where r is a maximum or minimum) are points where $r' = 0$. If in (10) one sets $r' = 0$ and solves for r, the turning points are obtained which turn out in this way to be

$$r_t = \frac{EK \pm \sqrt{E_r^2 K^2 + (E^2 - E_r^2)\ell^2 c^2}}{E_r^2 - E^2} \tag{11}$$

where $\quad K = Zek$

$$E_r = mc^2$$

The sum of the two turning points is equal to the major axis of the ellipse, $2a'$, whence

$$a' = \frac{EK}{E_r^2 - E^2} \tag{12}$$

Note that the major axis depends only on the energy. From (9) and (12) one gets the eccentricity

$$\in = \sqrt{\left(\frac{E_r}{E}\right)^2 + \left(\frac{\ell c}{K}\right)^2 - \left(\frac{E_r}{E} \frac{\ell c}{K}\right)^2} \quad , \tag{13}$$

where the energy E has the value E_0 when $a' = a$.

Now

$$\ell = \frac{mc^2}{\sqrt{1 - \beta^2}} \frac{r_2 \beta_2}{c} \cos \alpha \tag{14}$$

If one sets

$$T = T_o(1 + \lambda)$$

$$r_2 = a(1 + \rho)$$

$$F = T_o/E_o \tag{15}$$

$$A = \frac{E_r{}^2 - E_o{}^2}{E_o{}^2}$$

Then from (1), (8), (12), (13), (14), and (15) it can be shown by tedious but straightforward algebra that to 2nd order in λ, ρ, and α:

$$\epsilon = \sqrt{(\rho - \frac{A + 2}{A} F\lambda)^2 + (1 + A)\alpha^2} \tag{13'}$$

$$a' = a \left[1 + \frac{2 + A}{A} F\lambda + \frac{3A + 4}{A^2} F^2\lambda^2 + \dots \right] \tag{12'}$$

$$\psi = \frac{1}{\sqrt{1 + A}} \left[1 + F\lambda - \frac{[(2+A)F\lambda - A\rho]^2}{2A(1 + A)} - \frac{A}{2}\alpha^2 + \dots \right] \Theta \tag{8'}$$

From (1) and (12) it may be shown that

$$F = \frac{\sqrt{1 + \sigma^2} - 1 + \sigma}{\sqrt{1 + \sigma^2} - \sigma}$$

$$A = \frac{\sigma \left[\sqrt{1 + \sigma^2} - 2\sigma \right]}{1 - \sigma \left[\sqrt{1 + \sigma^2} - 2\sigma \right]} \tag{15'}$$

where

$$\sigma = \frac{Zek}{2aE_r} = \frac{Ze\,\Delta V}{(R_1 + R_2)(\frac{1}{R_1} - \frac{1}{R_2})}$$

$$= \frac{T}{E_r} \quad \text{nonrelativistically}$$

σ is the ratio of half the potential energy to the rest energy.

It may now be shown that a first order focus exists at the point for which ψ has increased by π from the source. From (7') and (12')

$$\frac{r(\psi_o) + r(\psi_o + \pi)}{2a} = \frac{(1 + \frac{2+A}{A} F\lambda + \frac{3A+4}{A^2} F^2\lambda^2)(1 - \epsilon^2)}{1 - \epsilon^2 \cos^2 \psi_o}$$

which is of the form $1 + g(\lambda) + h(\lambda,\rho)a^2 + \ldots$

Thus to 2nd order in a a perfect focus exists for $\Delta\psi = \pi$. From (8') this means where

$$\Delta\theta = \frac{\sqrt{1 + A} \quad \pi}{1 + F\lambda - \frac{[(2+A) F\lambda - A\rho]^2}{2A(1+A)} + \ldots} \equiv \bigoplus$$

Since ϕ in Fig. 2 can have any value from 0 to π this first order focus will be a semicircular ring at $\theta = \bigoplus$. One could use a detecting wire at this place except that $\bigoplus$ will vary with the energy of source used, so the instrument would get rather complicated. However, if σ is fairly small as well as the annular space between the spheres, a sufficiently good focus can be obtained at the single point Q ($\Delta\theta = \pi$). In fact from (7'), (8'), (13'), and (15') it can be shown that

$$\frac{PQ}{2a} = \frac{r(\theta_o) + r(\theta_o + \pi)}{2a} = 1 + 2\lambda + 6\sigma\lambda + 4\lambda^2 + \frac{\pi}{4}\sigma a - a^2 + \ldots \qquad (16)$$

$$+ \text{3rd order terms in } \sigma, \lambda, \rho, a$$

(16) measures the distance PQ where Q is the intersection of the orbit with the diameter through P, the point at which the ion traverses the source slit.

III. Spectrometer Design

The points to be kept in mind in designing the instrument are the desirability of reasonably small dimensions, good resolution, high intensity,

and large range of measurable energies. However, most of these good features conflict with each other. Probably the most fundamental physical limitation is the electric field which can be sustained in the gap without breakdown.

Now of course the energies will actually be measured by changing the voltage difference until the ions are focussed into the detector slit. It is easy to show from (1) and (12) that

$$\frac{T}{E_r} = \sqrt{1 + \sigma^2} - 1 + \sigma \tag{17}$$

where σ is given by (15'), and is proportional to the voltage difference between the electrodes, ΔV. The instrument should be designed for the maximum energy which one wants to observe. In order to make it as small as possible the field in the gap should be as large as possible, perhaps as large as the sparking field. Let the maximum field be B and the maximum energy (kinetic + rest) be $\bar{U}$. If $u = \dfrac{\bar{U}}{E_r} = \dfrac{1}{\sqrt{1 - \bar{\beta}^2}}$

$$B = \frac{\bar{k}}{R_1^2}$$

$$\frac{mc^2 \bar{\beta}^2}{a\sqrt{1 - \bar{\beta}^2}} = \frac{Ze\bar{k}}{a^2}$$

$$a = \frac{R_1 + R_2}{2}$$

Combining these with (1a) one gets

$$\frac{R_1}{R_2}(R_2 - R_1) = \frac{\Delta V}{B}$$

$$\frac{R_2^2 - R_1^2}{R_1 R_2} = \frac{2Ze\,\Delta V}{E_r}\frac{u}{u^2 - 1}$$

If these are solved for R_1 and R_2 one obtains

$$R_1 = c + b + \sqrt{c^2 + b^2}$$

$$R_2 = c + b + \sqrt{c^2 + b^2} + \frac{2b^2}{c} + 2\frac{b}{c}\sqrt{c^2 + b^2} \qquad (18)$$

$$a = c + b + \sqrt{c^2 + b^2} + \frac{b}{c}(b + \sqrt{b^2 + c^2})$$

where

$$c = \frac{E_r}{2ZeB}\frac{u^2 - 1}{u}$$

$$b = \frac{\Delta V}{2B}$$

It is seen that b, or ΔV controls the separation of the electrodes, which in turn defines the angular spread at the source seen by the detector. In fact from (7')

$$r_{max} = \frac{a(1 - \epsilon^2)}{1 - \epsilon} = a(1 + \epsilon) = a(1 + \alpha_{max}\sqrt{1 + A})$$

$$r_{max} - a = a\sqrt{1 + A} \qquad \alpha_{max} = \frac{R_2 - R_1}{2}$$

$$\Delta\alpha = 2\alpha_{max} = \frac{R_2 - R_1}{a\sqrt{1 + A}}$$

$$\alpha_{max} = \frac{\frac{b}{c}(b + \sqrt{b^2 + c^2})}{\sqrt{1 + A}\left[b + c + \sqrt{b^2 + c^2} + \frac{b}{c}(b + \sqrt{b^2 + c^2})\right]} \qquad (19)$$

From (16') the spread from the image center is

$$S = 2a(2\lambda + 6\sigma\lambda + 4\lambda^2 + \frac{\pi}{4}\sigma\alpha - \alpha^2 + \ldots)$$

so the dispersion

$$\frac{\partial S}{\partial \lambda} = 4a(1 + 3\sigma) \qquad (20)$$

The resolution is

$$\delta\lambda = \frac{\delta S}{4a(1 + 3\sigma)}$$

where δS is the total uncertainty in S. This is the sum of the uncertainty in the source slit, the detector slit size and the uncertainty due to the angular spread. Let

h_s = radial width of source slit

h_d = radial width of detector slit

h_{scatt} = r.m.s. deviation due to gas scattering

$$\delta S = h_s + h_d - 2a\alpha_{max} (\frac{\pi}{4} \sigma - \alpha_{max}) + h_{scatt}.$$

$$\delta\lambda = \frac{h_s + h_d + h_{scatt} + 2a(\frac{\pi}{4} \sigma \alpha_{max} + \alpha^2_{max})}{4a(1 + 3\sigma)} \qquad (21)$$

Numerical example

8 Mev α-particles

Let B be the sparking potential $\sim$ 30kv/cm = 3×10^4 v/cm

$E_r = 3727 \times 10^6$ ev

$T = 8 \times 10^6$ ev

$$\frac{u^2 - 1}{u} = \frac{(u - 1)(u + 1)}{u} \sim 2\frac{T}{E_r}$$

$$c \cong \frac{E_r}{2ZeB} \cdot \frac{2T}{E_r} = \frac{T}{ZeB} = \frac{8 \times 10^6}{2 \times 3 \times 10^4} \sim 133 \text{ cm}$$

a_{min} = 2c = 266 cm $\sim$ 100 in. $\sim$ 9 ft. or <u>18 ft. diameter</u>.

Also $\sigma \sim \frac{T}{2E_r} = \frac{8}{2 \times 3727} \sim 0.001$

b and hence $R_1 - R_2$ and α_{max} may be chosen at will.