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Operators on k -tableaux and the k -Littlewood–Richardson rule for a special case

by

Sarah Elizabeth Iveson

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Mark Haiman, Chair
Assistant Professor Lauren Williams
Professor Jonathan Shewchuk

Fall 2012

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Abstract

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Professor Mark Haiman, Chair

This thesis proves a special case of the k -Littlewood–Richardson rule, which is analogous to the classical Littlewood–Richardson rule but is used in the case for k -Schur functions. The classical Littlewood–Richardson rule gives a combinatorial formula for the coefficients $c_{\mu\nu}^{\lambda}$ appearing in the expression $s_{\mu}s_{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}$ for two Schur functions multiplied together. k -Schur functions are another class of symmetric functions which were introduced by Lascoux, Morse, and Morse and are indexed by and related to k -bounded partitions. We investigate what occurs when multiplying two k -Schur functions with some restrictions. More specifically, we investigate what happens when a k -Schur function is multiplied by a k -Schur function corresponding to a partition of length two. In this restricted case we are able to provide a combinatorial description of the k -Littlewood–Richardson coefficients that appear in the expansion of the product as a sum of k -Schur functions. These k -Littlewood–Richardson coefficients can be computed in terms of the number of k -tableaux with a certain property we call k -lattice. Furthermore, we conjecture that the result holds for any k -Schur functions, even when no restrictions are imposed. The proofs presented rely on a class of operators on k -tableaux which we introduce that are similar to the crystal operators on classical tableaux, but we provide a specific example that implies they are not actually crystal operators on k -tableaux. In addition to this, we also provide numerous examples and dedicate a chapter to examples of computation for some k -Littlewood–Richardson coefficients.

To Mom.

Thank you for putting up with all the tears and late night phone calls. I could not have gotten through it all without your unconditional love and support.

Contents

Contents	ii
List of Figures	iv
List of Tables	v
1 Introduction	1
1.1 Introduction	1
1.2 Where k -Schur functions came from	2
1.3 Strategies used to prove the k -Littlewood–Richardson rule	3
2 Background	5
2.1 Partitions, tableaux, $k + 1$ -cores	5
2.2 Symmetric functions and Schur functions	9
2.3 k -tableaux and k -Schur functions	12
2.4 Properties of entries in k -tableaux	16
3 Some operators on k-tableaux	19
3.1 Definitions	19
3.2 Properties of the operators	22
3.3 The operator $s_i^{(k)} e_i^{(k)}$	26
3.4 The k -lattice property	28
4 A proof of the k-Littlewood–Richardson rule in a restricted case	30
4.1 The proof of the k -Littlewood–Richardson rule	30
4.2 An example of the bijection used for the proof	33
4.3 An explanation of the requirements on μ and problems with generalizing the proof	34
5 An alternate proof of the k-Littlewood–Richardson rule for $\mu = (a, b)$	37
5.1 The alternate proof	37
5.2 Problems with generalizing the classical case and an example	41

6	A strategy for computing k-Littlewood–Richardson coefficients	44
7	The scalar product of a dual k-Schur function and a Schur function	47
	Bibliography	50

List of Figures

2.1	Ferrers diagrams for $\lambda = (3, 3, 1, 1)$ and its conjugate, $\lambda' = (4, 2, 2)$	6
2.2	The skew diagram corresponding to $(3, 2, 2, 1)/(2, 1)$	6
2.3	The hook length for the box $(2, 1)$ in the diagram of $\lambda = (4, 2, 2, 1)$ is 4.	7
2.4	An example of a tableau of shape $(4, 2, 1)$ with weight $(3, 1, 2, 1)$	7
2.5	The 4-core $(6, 3, 2, 1)$ first showing all of the boxes with hook length greater than 4 are labeled with a dot (all other boxes have hook length less than 4), and then showing the 4-residue of each box.	8
2.6	An example of the bijection between k -bounded partitions and $k + 1$ -cores for $k = 3$ ($k + 1 = 4$).	8
2.7	Skew tableaux of shape $(5, 3, 1)/(4, 2)$ with weight $(2, 1)$. Note that the empty boxes are not really in the skew diagram, but we leave them in the diagrams for clarity on the shape of the skew tableaux. This means $K_{(5,3,1)/(4,2),(2,1)} = 3$. . .	11
2.8	Skew tableau of shape $(7, 4, 3, 3)/(4, 3, 1)$ with weight $(3, 3, 2, 1)$ and reverse reading word $w = 111232432$. Note that this skew tableau is lattice.	11
2.9	An example of a k -tableau T for $k = 4$. The k -weight of T is $(2, 3, 2, 1)$. Note that the subscripts on the entries in the tableau correspond to the residues of the boxes.	12
2.10	A k -tableau T for $k = 3$, with shape $\mathfrak{c}(2, 1, 1) = (3, 1, 1)$ and k -weight $(1, 1, 0, 1, 1)$. Recall that the subscript on the entries refers to the residue of the box. For instance, the box labeled 4_3 is meant to contain the number 4, and the residue of this entry is 3.	17
3.1	How $e_i^{(k)}$ and $f_i^{(k)}$ change the entries in a fundamental row R . The subscripts d and s refer to whether each entry is divorced or single.	25

List of Tables

5.1	All of the k -tableaux of shape $\mathbf{c}(\tau)$ with k -content $\lambda = (4, 2, 1)$ for $k = 5$ and the k -Kostka numbers $K_{\tau(4,2,1)}^{(k)}$ for all $\tau \succeq (4, 2, 1)$	42
7.1	Table of values of the scalar product $\langle \mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}, s_{\mu} \rangle$ for different partitions μ . Notice that the scalar product can be positive, negative, or zero.	48
7.2	All possible k -tableaux of shape $\mathbf{c}(3, 2, 2, 1)/\mathbf{c}(1, 1)$ for $k = 3$ with k -weight μ where μ is a partition.	49

Acknowledgments

I would like to thank my academic advisor Mark Haiman. Without his helpful conversations and guidance I would not have been able to complete this.

Thanks to my other committee members, Lauren Williams and Jonathan Shewchuk for taking the time to read this and giving very useful suggestions.

I would also like to thank the UC Berkeley Mathematics Department staff for all of the advice and moral support they gave to me during the time that I was at Berkeley.

Lastly, I would like to thank Julianna Tymoczko for getting me interested in Combinatorics and also for all of the encouragement that she has given me.

Chapter 1

Introduction

1.1 Introduction

k -Schur functions were introduced by Lapointe, Lascoux, and Morse in [3], first developed to assist in studying Macdonald polynomials. These symmetric functions are believed to play the same fundamental combinatorial role in the symmetric function subspace $\Lambda^k = \mathbb{Z}[h_1, \dots, h_k]$ as Schur functions play in Λ , the space of all symmetric functions. In particular, they are believed to satisfy a k -Pieri rule, and a k -Littlewood–Richardson rule, both of which would be analogous to the case of classical Schur functions. There are approximately six definitions of k -Schur functions, all of which are conjectured to be equivalent. For some of the definitions, the k -Pieri rule and k -Littlewood–Richardson rule are clear or have been proven, and for others they are conjectured to hold.

In this thesis, we investigate some properties of the k -Schur functions from a combinatorial viewpoint. In particular, we derive a formula for some special cases of the k -Littlewood–Richardson coefficients, $c_{\mu\nu}^{\lambda,k}$ which appear in the expansion when we multiply two k -Schur functions.

$$s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu\nu}^{\lambda,k} s_{\lambda}^{(k)} \quad (1.1)$$

Lam proved in [2] that the coefficients $c_{\mu\nu}^{\lambda,k}$ appearing in the above expression are all non-negative integers, using a geometric strategy. In this paper we investigate the possibility of a combinatorial interpretation of the coefficients being positive. More specifically, we try to find a set of combinatorial objects which can be used to count the coefficients, and in a special case we are able to get such a description which is presented in the theorem below. In addition, we also conjecture in Chapter 4 that the combinatorial description of the k -Littlewood–Richardson coefficients, presented in the theorem below, holds in general for any k -Schur functions.

Theorem 1.1. Let μ be a k -bounded partition with length 2, meaning $\mu = (a, b)$, and let

λ be any k -bounded partition. Then the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$ appearing in the expansion

$$s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu\nu}^{\lambda,k} s_{\lambda}^{(k)}$$

is equal to the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are k -lattice.

In Chapter 5 we are also able to define a recursive formula for the k -Littlewood–Richardson coefficients which could potentially prove useful in some cases. More specifically, we prove the following proposition.

Proposition 1.2. Let λ , μ , and ν be k -bounded partitions. Then we have the following formula for the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$:

$$c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)} - \sum_{\tau \triangleright \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} \quad (1.2)$$

where $K_{\lambda/\nu,\mu}^{(k)}$ is the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight μ , and $K_{\tau\mu}^{(k)}$ is the number of k -tableaux of shape τ and k -weight μ .

This proposition then allows us to conclude the same result as in Chapter 4 where we give a combinatorial description of the k -Littlewood–Richardson coefficients.

1.2 Where k -Schur functions came from

As stated previously, k -Schur functions arose from a study of Macdonald polynomials. Macdonald polynomials can be decomposed as a sum of Schur functions as

$$H_{\mu}[X; q, t] = \sum_{\lambda} K_{\lambda\mu}(q, t) s_{\lambda}[X] \quad (1.3)$$

where $H_{\mu}[X; q, t]$ is a Macdonald polynomial, obtained from a modification of the Macdonald integral form $J_{\mu}[X; q, t]$, so $H_{\mu}[X; q, t] = J_{\mu}[X/(1-t); q, t]$ and the coefficients $K_{\lambda\mu}(q, t)$ are the q, t -Kostka polynomials.

Lapointe, Lascoux, and Morse in [3] developed a family of symmetric functions denoted by $A_{\lambda}^{(k)}[X; t]_{\lambda_1 \leq k}$ which are called atoms, and are indexed by k -bounded partitions λ . These are believed to form a basis of the space spanned by Macdonald polynomials $H_{\mu}[X; q, t]$ which are indexed by k -bounded partitions μ . This then gives a refinement of the decomposition given in Equation (1.3), in particular we have the following:

$$H_{\mu}[X; q, t] = \sum_{k\text{-bounded } \nu} K_{\nu\mu}^{(k)}(q, t) A_{\nu}^{(k)}[X; t]$$

$$A_\nu^{(k)}[X; t] = \sum_{\lambda} \pi_{\lambda\nu}(t) s_{\lambda}[X]$$

where the coefficients are believed to satisfy some positivity properties. It was proven in [8] and [6] that $K_{\nu\mu}^{(k)}(q, t) \in \mathbb{N}[q, t]$ and $\pi_{\lambda\nu}(t) \in \mathbb{N}[t]$ for $k = 2$, and it is believed that this is true for all k .

In the original definition of k -Schur functions given in [3], the k -Schur functions are just defined to be the restriction of the atoms to the case $t = 1$, meaning the k -Schur function s_{λ} is defined as

$$s_{\lambda}[X] = A_{\lambda}^{(k)}[X; 1]$$

for λ being a k -bounded partition. This definition of k -Schur functions was very combinatorial in nature, and was useful for computing many examples of atoms and k -Schur functions, but was not as good for proving properties of the k -Schur functions.

The definition of k -Schur functions that we will be using was developed by Lapointe and Morse in [7]. The idea behind this definition is that the k -Kostka numbers, $K_{\lambda\mu}^{(k)}$, which we will define more precisely in Chapter 2, satisfy a triangularity property. In particular, for k -bounded λ and μ , $K_{\mu\lambda}^{(k)} = 0$ unless $\mu \succeq \lambda$, and $K_{\lambda\lambda}^{(k)} = 1$. Therefore, the inverse of the matrix $\|K_{\mu\lambda}^{(k)}\|$ exists, and is denoted by $\|\overline{K}^{(k)}\|$. Then the k -Schur functions can be defined as $s_{\lambda}^{(k)} = \sum_{\mu \succeq \lambda} \overline{K}_{\lambda\mu}^{(k)} h_{\mu}$, where λ and μ are of course k -bounded partitions.

This definition from [7] was shown by Lapointe and Morse in [4] to satisfy a k -Pieri rule which is analogous to the classical Pieri rule for Schur functions. This k -Pieri rule will be fundamental in our proofs of the k -Littlewood–Richardson rule for some special cases.

1.3 Strategies used to prove the k -Littlewood–Richardson rule

As stated previously, the proofs in this thesis rely heavily on the k -Pieri rule. Thus it should be made clear that the results presented in this thesis apply only to definitions of k -Schur functions that have been proven to satisfy the k -Pieri rule, in particular for the definition of k -Schur functions presented by Lapointe and Morse in [7]. If, as conjectured, the proposed definitions of k -Schur functions are equivalent, then the results presented apply to all of them.

We use two approaches to prove our results about the k -Littlewood–Richardson rule. The first proof, presented in Chapter 4, works for a slightly more restricted case than the

second proof which is given in Chapter 5. Both of these proofs rely on some operators on k -tableaux that are defined in Chapter 3.

The three operators defined in Chapter 3 are used throughout this thesis. The notation for and description of these operators is very similar to that of the usual crystal operators on classical Schur functions, but the operators we use are not in fact crystal operators on k -tableaux. We elaborate more on this in Chapter 7 giving an example that leads us to conclude that there are in fact no crystal operators for k -tableaux, meaning that our operators defined in Chapter 3 are not crystal operators. Nonetheless, they are useful for proving the theorems in this thesis. The operators act on the set of k -tableau by changing the k -weight of a given tableau. We also describe the property for a k -tableau to be k -lattice, which is used for the combinatorial description of the k -Littlewood–Richardson coefficients.

In Chapter 4, to prove our results about the k -Littlewood–Richardson rule we use a strategy based on a proof of the classical Littlewood–Richardson rule given by Remmel and Shimozono in [11]. For our proof using k -Schur functions though, we require that the μ appearing in Equation (1.1) has hook length at most k , meaning $s_\mu^{(k)} = s_\mu$, so the k -Schur function is actually just a Schur function. Additionally, we require that $\mu = (a, b)$. We provide an explanation for why this was necessary for our proof at the end of Chapter 4. We then can use the determinantal expansion for s_μ to express it in terms of homogeneous symmetric functions, h_r . This then allows us to use the k -Pieri rule on the product $s_\lambda^{(k)} s_\mu$. We then will have an expression for the k -Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu, k}$ as a sum of the number of k -tableaux with certain weight and shapes, some counted as positive and some counted as negative. A bijection using the operators on k -tableaux defined in Chapter 3 then gives the desired result for the case of $\mu = (\mu_1, \mu_2)$.

The second strategy we use for the proof of the k -Littlewood–Richardson rule is given in Chapter 5. We give a somewhat more direct proof after first proving a recursive formula for k -Littlewood–Richardson coefficients that works for any μ . The proof of the k -Littlewood–Richardson rule does not require the restriction on the hook length of μ , but it still does require that μ have length at most 2, that is $\mu = (a, b)$. We get a result for a formula of the k -Littlewood–Richardson coefficients in terms of other k -Littlewood–Richardson coefficients and some k -Kostka numbers, which holds for any μ . Then for the case of $\mu = (a, b)$, we can refine the result to give a combinatorial formula for k -Littlewood–Richardson coefficients in terms of certain skew k -tableaux that have a property we call “ k -lattice” which is analogous to the property of being lattice for semistandard Young tableaux. This proof in Chapter 5 relies on the expansion of h_λ in terms of k -Schur functions, and the k -Pieri rule. Chapter 6 provides some examples for how the k -Littlewood–Richardson coefficients can actually be computed using the results presented.

Chapter 2

Background

2.1 Partitions, tableaux, $k + 1$ -cores

In this chapter we review basic combinatorial definitions and then some properties of $k + 1$ -cores, k -tableaux and k -tableaux, much of which can be found in [4] and [7]. But before we can begin working with k -tableaux and k -Schur functions, we will review some basic combinatorial concepts which are the basis for the definitions and concepts presented later in this section.

Before we can review the definition of k -tableaux, we first recall some basic properties of partitions and the definition and properties of $k + 1$ -cores, in particular, we review the bijection between $k + 1$ -cores and k -bounded partitions, which will be used later on.

A *partition* is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_m)$. A composition $\alpha = (\alpha_1, \dots, \alpha_m)$ is a sequence of nonnegative integers, not necessarily decreasing. The *length* of the partition, $l(\lambda)$, is the number of parts of the partition, which would be m for $\lambda = (\lambda_1, \dots, \lambda_m)$. For a partition of length m , $|\lambda|$ is defined as $\lambda_1 + \dots + \lambda_m$, and if $|\lambda| = n$ we say that λ is a partition of n . Every partition can be visualized by a *Ferrers diagram* which is a collection of boxes that are right and bottom aligned, with λ_i boxes in the i^{th} row. In this thesis we will be using the convention that the rows go from bottom-to-top. The box (i, j) in a Ferrers diagram denotes the box in the i^{th} row (counted from bottom-to-top) and j^{th} column (counted from left-to-right). Given a partition λ , the *conjugate* of λ is obtained by reflecting the Ferrers diagram for λ over the line $y = x$. So the number of boxes in the i^{th} row of the conjugate is the number of boxes in the i^{th} column of λ (where columns are ordered from left-to-right). The conjugate partition is denoted λ' .

Example 2.1. Let $\lambda = (3, 3, 1, 1)$. Then the Ferrers diagram for this partition, seen in Figure 2.1, has 3 boxes in the first (bottom) row, 3 boxes in the second row, and 1 box each in the third and fourth (top) rows. With the same $\lambda = (3, 3, 1, 1)$, if we reflect the Fer-

ers diagram across the diagonal, we will get a Ferrers diagram with 4 boxes in the bottom row, then 2 boxes in the second and third rows. This diagram corresponds to the partition $\lambda' = (4, 2, 2)$ which can also be seen in Figure 2.1.

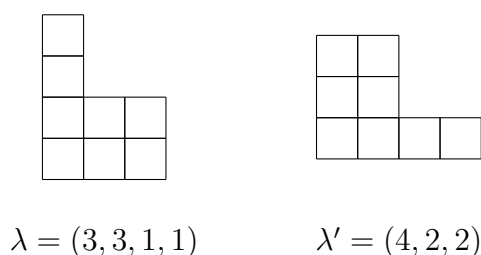


Figure 2.1: Ferrers diagrams for $\lambda = (3, 3, 1, 1)$ and its conjugate, $\lambda' = (4, 2, 2)$.

We say that $\lambda = (\lambda_1, \dots, \lambda_m)$ is a k -bounded partition if $\lambda_i \leq k$. The set of all k bounded partitions is denoted \mathcal{P}^k , so $\lambda \in \mathcal{P}^k$ means λ is a k -bounded partition. A composition $\alpha = (\alpha_1, \dots, \alpha_m)$ is k -bounded if $\alpha_i \leq k$ for all i .

In the previous examples, $\lambda = (3, 3, 1, 1)$ is a 3-bounded partition, since every part has length at most 3, but $\lambda' = (4, 2, 2)$ is not a 3-bounded partition, since the first part of λ' is 4, which is greater than 3.

For two partitions μ and λ , we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i . For partitions $\mu \subseteq \lambda$, we can obtain a *skew diagram* λ/μ which is the collection of boxes in the diagram of λ which are not in the diagram of μ . For example, consider the partitions $(2, 1) \subseteq (3, 2, 2, 1)$. The skew diagram corresponding to $(3, 2, 2, 1)/(2, 1)$ can be seen in Figure 2.2.

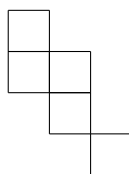


Figure 2.2: The skew diagram corresponding to $(3, 2, 2, 1)/(2, 1)$.

There is a partial ordering on partitions of n , called the *dominance order*. With this partial ordering, if λ and μ are two partitions of n (meaning $|\lambda| = |\mu| = n$) we say that $\lambda \supseteq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i . In this thesis, if we use the notation $\lambda \triangleright \mu$ this means that λ is strictly larger than μ with respect to the dominance order.

For an example of dominance order, let $\lambda = (3, 2, 2, 1)$, then under the dominance order, $(4, 4) \supseteq \lambda$, and $(3, 3, 1, 1) \supseteq \lambda$, but $\mu = (3, 1, 1, 1, 1)$ is not larger than λ with respect to the dominance order, since the sum of the first two terms of μ is $3 + 1$ which is less than the sum of the first two terms of λ , $3 + 2$.

For any box in a Ferrers diagram, the *hook length* of that box is the number of boxes above it plus the number of boxes to the right of it, including the box itself. To see an example of this, in Figure 2.3, for the box $(2, 1)$, which is indicated with a dot, the hook length is 4 since there are two boxes above it, one to the right of it, and the box itself.

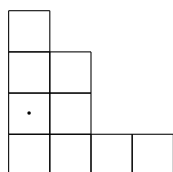


Figure 2.3: The hook length for the box $(2, 1)$ in the diagram of $\lambda = (4, 2, 2, 1)$ is 4.

When we refer to the hook length of a partition, this is the hook length of the box $(1, 1)$ (the bottom-left corner box) in the diagram. So for the partition $\lambda = (4, 2, 2, 1)$, we would say the hook length of the partition, $h(\lambda)$ is 7. Note that for any partition, the hook length is $h(\lambda) = \lambda_1 + l(\lambda) - 1$, that is, the number of boxes in the bottom row plus the number of rows, minus one.

Given a Ferrers diagram corresponding to a partition λ , fill the diagram with the numbers $1, 2, \dots$, according to the rule that numbers are strictly increasing going up columns and weakly increasing from left to right in the rows. We call this filled in diagram a *semi-standard Young tableau* (in this thesis we will often just use the word tableau) of shape λ . If there are α_i boxes filled with the number i for all i , then we say that the tableau has weight $\alpha = (\alpha_1, \alpha_2, \dots)$ where α is a composition.

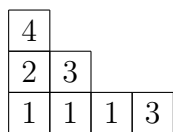


Figure 2.4: An example of a tableau of shape $(4, 2, 1)$ with weight $(3, 1, 2, 1)$.

A $k + 1$ -core is a partition that has no hooks of length $k + 1$. Given a $k + 1$ -core, the $k + 1$ -residue of the square (i, j) is $j - i \pmod{k + 1}$. In Figure 2.5 we see an example of a

$k + 1$ -core with the $k + 1$ residues for each square labeled.

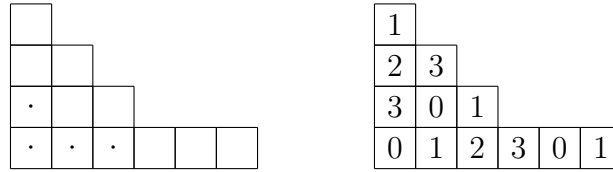


Figure 2.5: The 4-core $(6, 3, 2, 1)$ first showing all of the boxes with hook length greater than 4 are labeled with a dot (all other boxes have hook length less than 4), and then showing the 4-residue of each box.

There is a bijection between k -bounded partitions and $k + 1$ -cores, which was given by Morse and Lascoux in [7]. We give a quick explanation of the bijection here, as it will be necessary to use for proofs presented later.

Given a k -bounded partition λ , first place the top row in the Ferrers diagram for λ , then take the next row down and slide it to the right as much as necessary so that there are no $k + 1$ hooks in the diagram, but do not slide it any further than necessary. Then do the same thing with the next row down and so on. After placing the bottom row the same way, we will have the unique skew diagram of shape τ/ν that contains λ_i boxes in row i , and has no $k + 1$ hooks, but for all the boxes of τ that are in ν , the hook length is greater than $k + 1$. Then τ is the $k + 1$ -core associated to λ . For this bijection, we let λ be a k -bounded partition, and $\mathbf{c}(\lambda)$ be the $k + 1$ -core associated to it. For the other direction of the bijection it is easy to see we take a $k + 1$ -core τ , then remove all of the boxes with hook length greater than $k + 1$, then slide all of the rows in that skew diagram to the left to get a diagram $\mathbf{c}^{-1}(\tau)$ which will be the diagram for a k -bounded partition. This bijection is illustrated in the Figure 2.6.

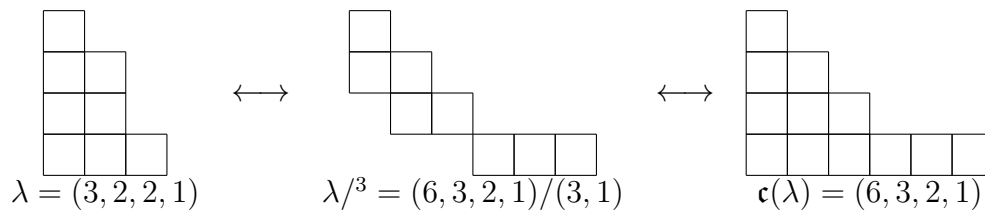


Figure 2.6: An example of the bijection between k -bounded partitions and $k + 1$ -cores for $k = 3$ ($k + 1 = 4$).

Note that for any k -bounded partition λ , $\mathbf{c}(\lambda)$ will be the $k + 1$ -core which corresponds to λ under the bijection described above. We will use this notation throughout this thesis.

For a k -bounded partition λ , the k -conjugate of λ is denoted λ^{ω_k} and is defined as $\lambda^{\omega_k} = \mathbf{c}^{-1}(\mathbf{c}(\lambda)')$. So with the partition $\lambda = (3, 2, 2, 1)$ and $k = 3$ from Figure 2.6, $\mathbf{c}(\lambda) = (6, 3, 2, 1)$, then the normal conjugate diagram of $\mathbf{c}(\lambda)$ is $\mathbf{c}(\lambda)' = (4, 3, 2, 1, 1, 1)$, and the k -conjugate of λ is $\lambda^{\omega_k} = \mathbf{c}^{-1}(\mathbf{c}(\lambda)') = (2, 2, 1, 1, 1, 1)$.

2.2 Symmetric functions and Schur functions

Let Λ denote the ring of symmetric functions over \mathbb{Q} . The elements of this ring are formal power series

$$f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}$$

where α ranges over all compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ of n , $c_{\alpha} \in \mathbb{Q}$, and $f(x_{\omega(1)}, x_{\omega(2)}, \dots) = f(x_1, x_2, \dots)$ for every permutation ω of the positive integers. There are many bases of Λ , but the bases that we will be most concerned with are the complete homogeneous symmetric functions and Schur functions.

Let r be a positive integer, then we can define the symmetric function h_r as

$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$

For any partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the *complete homogeneous symmetric function* h_{λ} can be defined in terms of all of the h_{λ_i}

$$h_{\lambda} = h_{\lambda_1} h_{\lambda_2} \cdots$$

Note that for any positive integer n , we can take the subring of Λ consisting of symmetric functions on the variables x_1, x_2, \dots, x_n . This subring will be denoted Λ_n , and consists of symmetric functions on the variables x_1, x_2, \dots, x_n .

Consider the symmetric functions on 3 variables, Λ_3 . Then $h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$, and $h_1 = x_1 + x_2 + x_3$. Then we can see that

$$h_{(2,1)} = h_2 h_1 = x_1^2 + x_2^2 + x_3^2 + 2(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2) + 3x_1 x_2 x_3.$$

Another basis for symmetric functions are the monomial symmetric functions. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. Then we can define the *monomial symmetric function* m_{λ} by

$$m_{\lambda} = \sum_{\alpha} x^{\alpha}$$

where $\alpha = (\alpha_1, \alpha_2, \dots)$ ranges over all distinct permutations of the entries in λ , and $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

The Schur functions also are a basis for the ring of symmetric functions, and they are indexed by partitions λ , and we denote them by s_λ . The Schur functions have many nice properties, some of which generalize nicely to the case of k -Schur functions.

Let λ be a partition of n and α be a composition of n , and let $K_{\lambda\alpha}$ be the number of semistandard Young tableaux of shape λ with weight α . $K_{\lambda\alpha}$ is called a *Kostka number*. The Kostka numbers also satisfy a triangularity property. Specifically, if μ and λ are partitions, then $K_{\lambda\mu} = 0$ unless $\mu \supseteq \lambda$ (the dominance order), and $K_{\lambda\lambda} = 1$ whenever $\mu = \lambda$. We will soon introduce the k -Kostka numbers, which satisfy a similar triangularity property. The Kostka numbers give many nice relationships between the different bases of symmetric functions. In particular, they satisfy the relationship $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$. In addition, they also appear as the coefficients in the expansion of a homogeneous symmetric function in terms of Schur functions, $h_\lambda = \sum_\mu K_{\lambda\mu} s_\mu$. There is a formula for Schur functions which is analogous to this, which is a consequence of the definition that we use for k -Schur functions.

The Schur functions also have another nice property, the Pieri rule for Schur functions, which gives a combinatorial description of how to multiply a complete homogeneous symmetric function by a Schur function, and express the product in terms of Schur functions.

Let r be a positive integer, and let ν be a partition. Then the *Pieri rule* states that $h_r s_\nu = \sum_\lambda s_\lambda$, where λ runs over all partitions that can be obtained from ν by adding a horizontal strip with r elements. When we say that λ is obtained by adding a horizontal strip with r elements to ν , we mean that we can add r boxes to the Ferrers diagram for ν , which are all in different columns (so no two boxes that we add are in the same column), and the resulting Ferrers diagram corresponds to the partition λ .

Let μ and ν be partitions. Then using the Pieri rule we can get a formula for multiplying a Schur function by a homogeneous symmetric function. This rule states that $h_\mu s_\nu = \sum_\lambda K_{\lambda/\nu, \mu} s_\lambda$ where $K_{\lambda/\nu, \mu}$ is called a skew Kostka number, and is given by the number of semistandard Young tableaux of shape λ/ν and weight μ .

Example 2.2. Let $\mu = (2, 1)$ and $\nu = (4, 2)$. To compute $h_\mu s_\nu$, we can use the formula above, so $h_{(2,1)} s_{(4,2)} = \sum_\lambda K_{\lambda/(4,2), (2,1)} s_\lambda$ where $K_{\lambda/(4,2), (2,1)}$ is the number of skew tableaux of shape $\lambda/(4, 2)$ and weight $(2, 1)$. For $\lambda = (5, 3, 1)$, there are three skew tableaux of shape $(5, 3, 1)/(4, 2)$ with weight $(2, 1)$ seen in Figure 2.7. This means that $K_{(5,3,1)/(4,2), (2,1)} = 3$ is the coefficient of $s_{(5,3,1)}$ in the expansion of $h_{(2,1)} s_{(4,2)}$ in terms of Schur functions.

Lastly, the Littlewood–Richardson rule for Schur functions is an important rule which we attempt to generalize to k -Schur functions, and successfully do in some special cases. For classical Schur functions, this rule gives a combinatorial description of how to multiply two

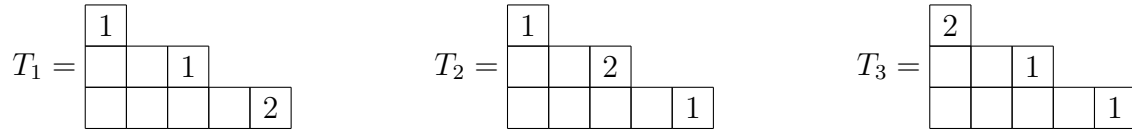


Figure 2.7: Skew tableaux of shape $(5, 3, 1)/(4, 2)$ with weight $(2, 1)$. Note that the empty boxes are not really in the skew diagram, but we leave them in the diagrams for clarity on the shape of the skew tableaux. This means $K_{(5,3,1)/(4,2),(2,1)} = 3$

Schur functions, and express their product in terms of other Schur functions.

Let μ and ν be partitions. Then $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$, where the $c_{\mu\nu}^\lambda$ are known as the Littlewood–Richardson coefficients, and they are equal to the number of skew tableaux of shape λ/ν with content μ that are lattice. Recall that a tableau T is lattice if when we take the reverse reading word $w = w_1 w_2 \cdots w_m$ of a tableau by reading from right to left across rows starting with the bottom row and then moving up, then for $j = 1, \dots, n$, the subword $w_1 w_2 \dots w_j$ contains at least as many i 's as $i + 1$'s for all $i = 1, \dots, n - 1$ (where T is filled with the numbers $1, \dots, n$). For example, the skew tableaux in Figure 2.8 has reverse reading word $w = 111232432$, and is not lattice.

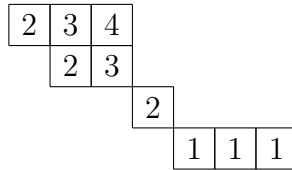


Figure 2.8: Skew tableau of shape $(7, 4, 3, 3)/(4, 3, 1)$ with weight $(3, 3, 2, 1)$ and reverse reading word $w = 111232432$. Note that this skew tableau is lattice.

Example 2.3. Using the Littlewood–Richardson rule, if we want to compute the expansion for $s_{(2,1)} s_{(4,2)}$ in terms of Schur functions, we can do so for each λ by finding all of the skew tableau of shape $\lambda/(4, 2)$ with weight $(2, 1)$ that are lattice, and this number will be the Littlewood–Richardson coefficient $c_{(2,1)(4,2)}^\lambda$ appearing in the expansion

$$s_{(2,1)} s_{(4,2)} = \sum_\lambda c_{(2,1)(4,2)}^\lambda s_\lambda.$$

Of the three skew tableaux of shape $(5, 3, 1)/(4, 2)$ and weight $(2, 1)$ seen in Figure 2.7 only T_2 and T_3 are lattice. The reverse reading word of T_1 is 211 which is not lattice since there are initially more 2's than there are 1's. But for 121 and 112, the reverse reading words of T_2 and T_3 , respectively, at any point in the word there are at least as many 1's previous to that

point as there are 2's, so they are lattice. Thus $c_{(2,1)(4,2)}^{(5,3,1)} = 2$. A similar computation for the other Littlewood–Richardson coefficients (which are the coefficients in the above equation) gives the expansion

$$s_{(2,1)}s_{(4,2)} = s_{(6,3)} + s_{(6,2,1)} + s_{(5,4)} + 2s_{(5,3,1)} + s_{(5,2,2)} + s_{(5,2,1)} + s_{(4,4,1)} + s_{(4,3,2)} + s_{(4,3,1,1)} + s_{(4,2,2,1)}.$$

2.3 k -tableaux and k -Schur functions

All of the definitions presented in this section can be found explained in further detail in [3], [7], and [4]. We begin by reviewing the definition of k -tableaux presented by Lapointe and Morse [7] and presenting some examples, and then discussing the definition for k -Schur functions introduced by Lapointe and Morse [4] that is used throughout this thesis.

Definition 2.4 (Lapointe, Morse [7]). Let γ be a $k + 1$ -core, let m be the number of k -bounded hooks in γ , and let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of m . Then a k -tableau with shape γ and k -weight α is a filling of γ with the numbers $1, 2, \dots, r$, in which rows are weakly increasing and columns are strictly increasing and there are exactly α_i distinct residues for the cells occupied by the letter i .

4 ₁					
3 ₂	3 ₃				
2 ₃	2 ₀	4 ₁			
1 ₀	1 ₁	2 ₂	2 ₃	2 ₀	4 ₁

Figure 2.9: An example of a k -tableau T for $k = 4$. The k -weight of T is $(2, 3, 2, 1)$. Note that the subscripts on the entries in the tableau correspond to the residues of the boxes.

The operators we introduce in Chapter 3 which act on skew k -tableau are used to prove the k -Littlewood–Richardson rule in Chapters 4 and 5. This definition of k -tableau is extended by Lapointe and Morse to skew k -tableau as follows.

Definition 2.5 (Lapointe, Morse [4]). Let $\delta \subseteq \gamma$ be $k + 1$ -cores, with m_1 k -bounded hooks in δ and m_2 k -bounded hooks in γ . Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of $m_2 - m_1$. Then a *skew k -tableau* with shape γ/δ and k -weight α is a filling of a γ/δ with the numbers $1, 2, \dots, r$ in which rows are weakly increasing and columns are strictly increasing and there are exactly α_i distinct residues for the cells occupied by the letter i .

Lapointe and Morse [7] proved a number of facts about k -tableaux. In particular they proved that there is a characterization equivalent to the definition of k -tableaux which will allow us to construct k -tableaux more easily. We present their result below in Theorem 2.6.

Theorem 2.6 (Lapointe, Morse [7]). Let T be a semistandard tableaux filled with the numbers $1, \dots, n$ whose shape is a $k + 1$ -core. Furthermore, for $i = 1, \dots, n$, let S_i be the part of T containing entries $1, \dots, i$. Then T is a k -tableaux if and only if each S_i can be constructed from S_{i-1} by adding a sequence of boxes containing i to S_{i-1} according to the rule that whenever we add a box of residue r , we add every box of residue r that still gives a diagram whose shape is a partition.

To construct a k -tableaux, one way is of course to start with a $k + 1$ -core and fill all of the boxes according to the rules. But sometimes we will only know that we want a k -tableaux with a given k -weight $\mu = (\mu_1, \dots, \mu_m)$. By Theorem 2.6, we can construct a k -tableau by first adding boxes labeling them with the number 1 and keeping track of $k + 1$ residues. Whenever we add a box with a given residue though, we must add every possible box of the same residue while still having a shape that is a Ferrers diagram. Add enough boxes so there are exactly m_1 residues for the boxes occupied by the number 1. Next, continue by adding boxes containing 2's and keeping track of $k + 1$ residues, and again whenever we add a box with a given residue, we must add every box of the same residue that still gives us a Ferrers diagram. Continue until there are m_2 residues for all the boxes occupied by the number 2. Continue this process and we will eventually have a k tableaux of k -weight μ . This process works to construct both k -tableaux or skew k -tableaux, except for the skew case we first start drawing boxes outside of the shape $\mathfrak{c}(\nu)$.

Example 2.7. Suppose we want to construct a k -tableaux of k -weight $\mu = (2, 2, 1)$ for $k = 3$. We begin by adding boxes containing 1's. There is only one way to do this so we get the following diagram

$$\begin{array}{cc} 3 & 0 \\ \boxed{1_0} & \boxed{1_1} \end{array} \quad 2 \quad 3$$

Notice that outside the boxes containing 1's we write the residue of the corresponding cells. Now, to add two residues containing 2's, we have some choices. We will place the first one in the cell of residue 2, and then if we place a 2 in the cell of residue 3, we must place a 2 in both of the cells with residue 3.

$$\begin{array}{cccc} & 2 & & \\ \boxed{2_3} & & 0 & \\ \boxed{1_0} & \boxed{1_1} & \boxed{2_2} & \boxed{2_3} \end{array} \quad 0$$

To add one residue containing a 3, we have the option of placing the 3 in the cell of residue 2, or else in both of the cells of residue 0 (again, we have to add numbers in every

cell of a given residue that will still give us a Ferrers diagram as the shape). If we choose to place the 3 in the cell of residue 0, our final k -tableau is the following k -tableau of shape $\mathfrak{c}(3, 2) = (5, 2)$ (the k -bounded partition is $(3, 2)$ which corresponds to the $k + 1$ -core $(5, 2)$ for $k = 3$ and $k + 1 = 4$).

2 ₃	3 ₀			
1 ₀	1 ₁	2 ₂	2 ₃	3 ₀

In addition to the characterization provided in Theorem 2.6, Lapointe and Morse [7] also showed there is another characterization for k -tableaux, which is presented in the following theorem.

Theorem 2.8 (Lapointe, Morse [7]). Let T be a semistandard tableaux filled with the numbers $1, \dots, n$ whose shape is a $k + 1$ -core. Furthermore, for $i = 1, \dots, n$, let S_i be the part of T containing entries $1, \dots, i$, and let P_i be the partition corresponding to the shape of S_i . Then T is a k -tableaux if and only if the shape of P_i/P_{i-1} is a horizontal strip and the shape of $P_i^{\omega_k}/P_{i-1}^{\omega_k}$ is a vertical strip for each i . Recall that $P_i^{\omega_k}$ denotes the k -conjugate as defined in Chapter 2.

The characterization of k -tableaux provided in Theorems 2.6 and 2.8 are used frequently throughout the remainder of this thesis. Note that if T is a k -tableau, then the the S_i as defined in Theorems 2.6 and 2.8 are k -tableaux.

As mentioned previously, there are a number of definitions for k -Schur functions, all of which are conjectured to be equivalent. For this thesis we will be using the definition given by Lapointe and Morse [7]. All of definitions and facts presented below can be found explained in greater detail in [7] and [4].

Recall that the Kostka number $K_{\mu\alpha}$ is equal to the number of tableaux of shape μ with weight α . This was generalized to k -tableaux by Lapointe and Morse [7], so the k -Kostka number $K_{\mu\alpha}^{(k)}$ is the number of k -tableaux of shape $\mathfrak{c}(\mu)$ (the $k + 1$ -core corresponding to μ) and k -weight α . Lapointe and Morse [4] proved the following property of k -Kostka numbers which we present as a theorem. This property is used implicitly in this thesis.

Theorem 2.9 (Lapointe, Morse [4]). Let λ and μ be two k -bounded partitions with $|\mu| = |\lambda|$. Then if α is any composition obtained by rearranging the parts of μ , then

$$K_{\lambda\alpha}^{(k)} = K_{\lambda\mu}^{(k)}.$$

It was shown in [7] that the k -Kostka numbers satisfy a triangularity property similar to that of the Kostka numbers, in particular, for μ and λ partitions, $K_{\mu\lambda}^{(k)} = 0$ whenever $\mu \not\leq \lambda$, and $K_{\mu\mu}^{(k)} = 1$. Because of this triangularity property, the inverse of the matrix $\|K_{\mu\lambda}^{(k)}\|_{\lambda, \mu \in \mathcal{P}^k}$

exists and we denote it by $\|K^{(k)}\|^{-1} = \|\overline{K}^{(k)}\|$. The k -Schur functions were defined in terms of this inverted system as

$$s_\lambda^{(k)} = \sum_{\mu \supseteq \lambda} \overline{K}_{\mu\lambda}^{(k)} h_\mu$$

where λ is a k -bounded partition and the sum runs over all k -bounded $\mu \supseteq \lambda$. Note that the k -Schur functions are also symmetric functions. Additionally, if $\lambda \in \mathcal{P}^k$ and $h(\lambda) \leq k$, then $s_\lambda^{(k)} = s_\lambda$, meaning if a partition has hook length at most k , then its corresponding k -Schur function is actually just a classical Schur function [4]. In addition, this definition of k -Schur functions implies that

$$h_\lambda = \sum_{\mu} K_{\lambda\mu}^{(k)} s_\mu^{(k)}.$$

Example 2.10. For $k = 4$, $\lambda = (2, 2, 1, 1)$ is a k -bounded partition, and the k -Schur function $s_{(2,2,1,1)}^{(3)}$ can be expanded in terms of Schur functions,

$$s_{(2,2,1,1)}^{(4)} = s_{(2,2,1,1)} + s_{(3,2,1)}.$$

For $k = 4$, $\lambda = (4, 2)$ is also a k -bounded partition, and the k -Schur function $s_{(4,2)}^{(3)}$ can also be expanded in terms of Schur functions,

$$s_{(4,2)}^{(4)} = s_{(4,2)} + s_{(5,1)} + s_{(6)}.$$

For $k = 4$ $\lambda = (3, 1)$ is k -bounded partition and also has hook length 4, so $h(\lambda) \leq k$ meaning the k -Schur function is equal to a Schur function,

$$s_{(3,1)}^{(4)} = s_{(3,1)}.$$

As a side note, generally wherever we are using k -tableaux or k -Schur functions, we are assuming k is fixed, and all of our partitions will be k -bounded unless otherwise noted.

This definition of k -Schur functions implies that the set of all k -Schur functions $\{s_\lambda^{(k)}\}_{\lambda_1 \leq k}$ forms a basis of $\Lambda^k = \mathbb{Z}[h_1, \dots, h_k]$, the space of all k -bounded symmetric functions, which is a subspace of Λ . Lapointe and Morse proved [4] that this definition implies a k -Pieri rule for multiplying a k -Schur function by h_l where $l \leq k$, which we present below.

Theorem 2.11 (Lapointe, Morse [4]). For $\nu \in \mathcal{P}^k$ and $l \leq k$,

$$h_l s_\nu^{(k)} = \sum_{\mu \in H_{\nu,l}^{(k)}} s_\mu^{(k)}$$

where $H_{\nu,l}^{(k)} = \{\mu \in \mathcal{P}^k \mid \mu/\nu \text{ is a horizontal } l \text{ strip and } \mu^{\omega_k}/\nu^{\omega_k} \text{ is a vertical } l \text{ strip}\}$.

The skew k -Kostka number $K_{\lambda/\nu,\mu}^{(k)}$ is the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ . Lapointe and Morse [4] go further to prove the following corollary of the previous theorem (the k -Pieri rule), which gives a combinatorial description for the coefficients of the k -Schur functions which appear when multiplying a homogeneous symmetric function h_λ by a k -Schur function.

Corollary 2.12 (Lapointe, Morse [4]). For any k -bounded partitions ν and μ ,

$$h_\mu s_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^k} K_{\lambda/\nu,\mu}^{(k)} s_\lambda^{(k)}$$

where $K_{\lambda/\nu,\mu}^{(k)}$ is the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ .

Example 2.13. $h_{(2,1)} s_{(4,2)}^{(4)} = s_{(4,4,1)}^{(4)} + 2s_{(4,3,2)}^{(4)} + s_{(4,3,1,1)}^{(4)} + s_{(4,2,2,1)}^{(4)}$. In this example, the coefficient in front of $s_{(4,3,2)}^{(4)}$ is $K_{(4,3,2)/(4,2),(2,1)}^{(4)}$ which is the number of skew 4-tableaux of shape $\mathbf{c}(4,3,2)/\mathbf{c}(4,2) = (7,3,2)/(6,2)$ with 4-weight $(2,1)$. These 2 skew 4-tableaux are shown below. (Note that the boxes left empty are not really in the skew diagram, but they are included in the figure to clarify the shape of the skew diagram).



Lam proved in [2], using a geometric strategy, that when two k -Schur functions $s_\mu^{(k)}$ and $s_\nu^{(k)}$ are multiplied, the k -Littlewood–Richardson coefficients $c_{\mu\nu}^{\lambda,k}$ appearing in the expansion

$$s_\mu^{(k)} s_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^k} c_{\mu\nu}^{\lambda,k} s_\lambda^{(k)}$$

are all nonnegative integers. It would also be nice to also obtain a combinatorial description for them which generalizes the combinatorial description of classical Littlewood–Richardson coefficients. In Chapters 4 and 5 we give a combinatorial description of $c_{\mu\nu}^{\lambda,k}$ for specific partitions μ , which does in fact generalize the classical Littlewood–Richardson rule, and at the end of Chapter 4 we conjecture that the combinatorial description does in fact hold even when no restrictions are imposed on μ .

2.4 Properties of entries in k -tableaux

In this section we present the concepts of entries of k -tableaux being married, divorced, and single. These concepts are of great importance for the operators we define in Chapter 3. The terminology and concepts were all developed by Lapointe and Morse in [4].

Remark 2.14. Whenever we use the terms married, divorced, or single when referencing entires i or $i + 1$ of a k -tableau, we will be referencing the relation of i to $i + 1$ or vice versa. In other words we always have two consecutive entires (such as 2 and 3 or 5 and 6) that we are discussing. So if we say an entry 2 is married, this could mean the 2 is married to a 1 or the 2 is married to a 3, but our intentions regarding which we mean will always be clear.

Let T be a k -tableau. When a box of residue c is occupied by an i we say that T contains an i_c . For example, in the tableau seen in Figure 2.10 there is a 1_0 , 2_1 , 4_3 , and 5_2 .

$$T = \begin{array}{|c|c|c|} \hline 5_2 & & \\ \hline 4_3 & & \\ \hline 1_0 & 2_1 & 5_2 \\ \hline \end{array}$$

Figure 2.10: A k -tableau T for $k = 3$, with shape $\mathbf{c}(2, 1, 1) = (3, 1, 1)$ and k -weight $(1, 1, 0, 1, 1)$. Recall that the subscript on the entries refers to the residue of the box. For instance, the box labeled 4_3 is meant to contain the number 4, and the residue of this entry is 3.

Fix i and $i + 1$ to be consecutive entries in a tableau T . We say that two entries i and $i + 1$ are *married* if they occur in the same column (so that the $i + 1$ is immediately above the i). If we say that an entry i is married it will be implied that it is married to an $i + 1$ and vice versa. An entry containing an i (or $i + 1$) is *divorced* if it is not married but has the same residue as a married i (or $i + 1$). If an entry containing an i or an $i + 1$ is not married or divorced, then we say that it is *single*. In addition, when we say that an entry is *unmarried*, we mean that it is not married and is therefore either divorced or single.

In Figure 2.10 for boxes containing 1 and 2, both the 1_0 and 2_1 are single. For boxes containing 4 and 5, the 4_3 in the second row is married to the 5_2 in the third row (counting rows from bottom to top), and the 5_2 in the bottom row is divorced.

For a row r in a tableau T (where we count the rows from bottom to top starting with the bottom row being 1), we let $\text{URes}_r(i, i + 1)$ be the set of all residues occupied by an unmarried i or $i + 1$ in row r . In our example from Figure 2.10, $\text{URes}_1(4, 5) = \{2\}$, $\text{URes}_2(4, 5) = \emptyset$, and $\text{URes}_1(1, 2) = \{0, 1\}$. In [4] it is shown that there is an equivalence relation on rows in a k -tableau T , which proves that whenever $\text{URes}_m(i, i + 1) \cap \text{URes}_r(i, i + 1) \neq \emptyset$ for two rows where $r < m$, then $\text{URes}_m(i, i + 1) \subseteq \text{URes}_r(i, i + 1)$. This allowed the definition of a *fundamental row* which is a row m such that $\text{URes}_m(i, i + 1) \not\subseteq \text{URes}_r(i, i + 1)$ for any r where $r < m$ (so the unmarried residues of i and $i + 1$ in the fundamental row are not contained in the set of unmarried residues of i and $i + 1$ for any higher row).

In our example again from Figure 2.10, concerning 4 and 5, there is one fundamental row, which is the bottom row (row 1), and concerning 3 and 4, the only fundamental row is row 2 (there are no boxes containing a 3, so the 4_3 is single).

We now present a more in-depth example of all of the concepts presented in this section.

Example 2.15. We present here an example showing fundamental rows for two different k -tableaux, and mention the single, married, or divorced status of some entries in them. Consider first the following k -tableaux T for $k = 5$.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 5_2 & 5_3 & & & & & & & & & \\ \hline 3_3 & 4_4 & & & & & & & & & \\ \hline 2_4 & 2_0 & 4_1 & 5_2 & 5_3 & & & & & & \\ \hline 1_0 & 1_1 & 2_2 & 3_3 & 4_4 & 4_0 & 4_1 & 5_2 & 5_3 & & \\ \hline \end{array}$$

In T , if we consider the fundamental rows for $4/5$, the only fundamental row is the bottom row, R_1 . In this row we have single entries 4_0 , 4_1 , and 5_2 , and divorced entry 5_3 . The 4_4 is married. Notice that all of the other single or divorced 4's or 5's that appear in T have the same residue as one of the 4's or 5's in the fundamental row R_1 .

Next we will consider a skew k -tableaux S , and the fundamental rows in S . The concepts of fundamental rows work the same for skew k -tableaux as they do for k -tableaux.

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 5_4 & & & & & & & & & & \\ \hline 4_0 & 5_1 & & & & & & & & & \\ \hline 2_1 & 4_2 & & & & & & & & & \\ \hline 1_2 & 3_3 & 5_4 & & & & & & & & \\ \hline & 2_4 & 4_0 & 5_1 & & & & & & & \\ \hline & & 1_0 & 2_1 & 4_2 & 4_3 & 5_4 & & & & \\ \hline & & & 1_2 & 2_3 & 2_4 & 4_0 & 5_1 & & & \\ \hline \end{array}$$

In S , there are two fundamental rows for $4/5$, the bottom two rows: R_1 and R_2 . In R_1 we have a married 4_0 and a divorced 5_1 . In R_2 we have married entries 4_2 and 5_4 and a single 4_3 . In this example, any 4's or 5's appearing above the two fundamental rows are actually married.

Chapter 3

Some operators on k -tableaux

3.1 Definitions

In this section we define and describe properties of three operators on the set of k -tableaux and skew k -tableaux. It should be noted here that while other operators on classical tableaux (e_i , f_i , and s_i) are in fact crystal operators, the operators we introduce ($e_i^{(k)}$, $f_i^{(k)}$, and $s_i^{(k)}$) are not crystal operators, but they have some nice properties similar to crystal operators, and they work to help prove the special case of the k -Littlewood–Richardson rule in later chapters. The similar notation that we use is merely to help clarify.

First, following Lascoux and Schützenberger [9], we recall three operators on words e_i , f_i and s_i . Let w be a word on the letters $1, \dots, n$, and let i be a positive integer such that $1 \leq i \leq n - 1$. Next, take the subword of w consisting of only the letters i and $i + 1$. Replace each i with a right parenthesis, and each $i + 1$ with a left parenthesis. Pair the parentheses according to the normal rules, and consider the unpaired ones. These will correspond to a subword of the form $i^p(i + 1)^q$. The operators e_i , f_i and s_i replace the subword $i^p(i + 1)^q$ with a similar word according to the following rules.

- For e_i , replace $i^p(i + 1)^q$ by $i^{p+1}(i + 1)^{q-1}$. Note that we can only apply e_i if $q > 0$, meaning that u has an $i/(i + 1)$ -unpaired letter $i + 1$.
- For f_i , replace $i^p(i + 1)^q$ by $i^{p-1}(i + 1)^{q+1}$. Note that we can only apply f_i if $p > 0$, meaning that u has an $i/(i + 1)$ -unpaired letter i .
- For s_i , replace $i^p(i + 1)^q$ by $i^q(i + 1)^p$.

For an example of these operators, let $w = 3212423224131$. We will show how the operators e_2 , f_2 and s_2 act on this word so we will be acting on the number of 2's and 3's in the word. First we take the subword of w consisting of only the 2's and 3's. Call this subword u , so $u = 32223223$. If we replace the 2's and 3's with right and left parentheses, we get

$u = ()))())$. If we pair the parentheses according to the usual rules, we see that the third, fourth, and last two are unpaired. We will replace those with the numbers that they correspond to, so $u = ()22()23$. The operators will change the unpaired numbers left in u . e_2 will send the the unpaired $2223 = 2^33^1$ to $2222 = 2^43^0$. Then $s_2u = ()22()22 = 32223222$, and by replacing this back into w in place of u , we get $s_2w = \mathbf{3212423224121} = 3212423224121$. The bold is just to emphasize the subword s_2u that we replaced in w . Now $f_2u = ()22()33 = 32223233$, so $f_2w = \mathbf{3212423234131} = 3212423234131$, and $s_2u = ()23()33 = 32233233$, so $s_2w = \mathbf{3212433234131} = 3212433234131$.

It is now time to define a particular subword of a k -tableau or skew k -tableau which we call the *fundamental $i/(i+1)$ subword*.

Definition 3.1. Let T be a k -tableau or skew k -tableau filled with $1, 2, \dots, n$, and let i be a positive integer, $1 \leq i \leq n-1$. Let R_1, R_2, \dots, R_m be the fundamental rows of T (for the chosen $i/(i+1)$), ordered from top to bottom. Ignore all of the married and divorced entries of i and $i+1$, and just take the word consisting of single i and $i+1$ entries reading from left to right across R_1 , then R_2 , and so on, ending with R_m . We call this word the *fundamental $i/(i+1)$ subword*.

Definition 3.2. Let T be a k -tableau or skew k -tableau filled with $1, 2, \dots, n$, and let i be a positive integer, $1 \leq i \leq n-1$. Let R_1, R_2, \dots, R_m be the fundamental rows of T (for the chosen $i/(i+1)$), ordered from top to bottom. Let w be the fundamental $i/(i+1)$ subword which consists of the letters i and $i+1$. Then perform the parentheses pairing after replacing each i with a right parenthesis and each $i+1$ with a left parenthesis. Take the subword u which consists of all of the unpaired i and $i+1$ entries. We call this subword the *fundamental $i/(i+1)$ -unpaired subword* of T .

When we say that a tableau T has an $i/(i+1)$ -unpaired i (resp. $i+1$), this just means that in the fundamental $i/(i+1)$ -unpaired subword of T , there is at least one i (resp. $i+1$).

The operators $e_i^{(k)}$, $f_i^{(k)}$, and $s_i^{(k)}$ will be operators defined on k -tableaux and skew k -tableaux as follows. First let T be a k -tableau or skew k -tableau. Take the fundamental $i/(i+1)$ -unpaired word of T and call this u . For $e_i^{(k)}$ replace u by $e_i u$ and put it back into T in place of u . For $f_i^{(k)}$ and $s_i^{(k)}$, do the same except replace u by $f_i u$ and $s_i u$, respectively. Once we do this, we need to make a few adjustments so that we still have a k -tableau or skew k -tableau as follows

1. In each of the fundamental rows, if there are any entries containing a lying to the right of an entry we relabeled with $i+1$, relabel it with $i+1$, and if there is any entry containing $i+1$ lying to the left of an entry that we relabeled with i , replace it with i .

3_4														
2_5	3_6													
1_6	1_7	4_8												
		2_0	3_1	3_2	4_3	4_4								
			1_2	1_3	2_4	2_5	2_6	3_7	4_8					
						1_6	1_7	1_8	2_0	3_1	3_2	4_3	4_4	

There are no divorced entries containing a 2 to the right of the 3 that we changed, so we only need to change the 3_6 in a higher row to 2_6 , so we get the following:

$$f_2^{(8)}T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 3_4 & & & & & & & & & & & & & & & \\ \hline 2_5 & 2_6 & & & & & & & & & & & & & & \\ \hline 1_6 & 1_7 & 4_8 & & & & & & & & & & & & & \\ \hline & & 2_0 & 3_1 & 3_2 & 4_3 & 4_4 & & & & & & & & & \\ \hline & & & 1_2 & 1_3 & 3_4 & 2_5 & 2_6 & 3_7 & 4_8 & & & & & & \\ \hline & & & & & & & 1_6 & 1_7 & 1_8 & 2_0 & 3_1 & 3_2 & 4_3 & 4_4 & \\ \hline \end{array}$$

Finally, for $s_2^{(8)}T$, we change $u = 2_43_63_13_2$ to $2_42_62_13_2$, and then perform the same corrections of changing divorced entries as needed and replacing entries in higher rows, and we get the following:

$$s_2^{(8)}T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 3_4 & & & & & & & & & & & & & & & \\ \hline 2_5 & 2_6 & & & & & & & & & & & & & & \\ \hline 1_6 & 1_7 & 4_8 & & & & & & & & & & & & & \\ \hline & & 2_0 & 2_1 & 3_2 & 4_3 & 4_4 & & & & & & & & & \\ \hline & & & 1_2 & 1_3 & 2_4 & 2_5 & 2_6 & 3_7 & 4_8 & & & & & & \\ \hline & & & & & & & 1_6 & 1_7 & 1_8 & 2_0 & 2_1 & 3_2 & 4_3 & 4_4 & \\ \hline \end{array}$$

3.2 Properties of the operators

There remain a few facts to prove about the operators. First of all that the tableau we get after performing any of the above operations is actually a k -tableau or skew k -tableau. To do this we need to show that $e_i^{(k)}T$, $f_i^{(k)}T$ and $s_i^{(k)}T$ are all column-strict and weakly increasing in rows, and then show that the k -weight changes appropriately. This is sufficient to prove that they are k -tableau or skew k -tableau because if a k -tableau or skew k -tableau is column strict, weakly increasing in rows, and has k -weight α where α is a composition of the number of k -bounded hooks in the given tableau then it is clearly a k -tableau or skew k -tableau by definition. $e_i^{(k)}T$, $f_i^{(k)}T$ and $s_i^{(k)}T$ are all weakly increasing in rows by the definition of the operators, so it suffices to show that they are column strict and have the appropriate k -weight. For simplicity, in the rest of this section when we say k -tableau we

mean it to imply k -tableau or skew k -tableau. The proofs given all apply to both cases.

Before we begin our proofs we recall a fact given by Lapointe and Morse [4] which is necessary for many of the proofs in this chapter.

Remark 3.4. (Lapointe, Morse [4]). Given a sequence of i and $i + 1$ entries in a row of a k -tableau, the married i 's all lie at the beginning of the sequence, and the married $i + 1$'s all lie at the end of the sequence. This is easy to see, since any unmarried i has an entry larger than an i above it, so anything to the right of it must also be unmarried. Similarly, any unmarried $i + 1$ has an entry smaller than i below it, so anything to the right of it is unmarried.

We also will need the following property, also proven by Lapointe and Morse [4] which we restate here.

Property 3.5. (Lapointe, Morse [4])

- Given an unmarried $(i + 1)_c$ (that is an $i + 1$ occupying a box with residue c) in a k -tableau T , any $i_c \in T$ is married and lies weakly higher than the highest unmarried $(i + 1)_c$. Furthermore, i_c occurs in T if and only if there is a divorced $(i + 1)_{c-1}$ left-adjacent to the unmarried $(i + 1)_c$.
- Given an unmarried i_c (that is an i occupying a box with residue c) in a k -tableau T , any $(i + 1)_c \in T$ is married and lies strictly higher than the highest unmarried i_c . Furthermore, $(i + 1)_c$ occurs in T if and only if there is a divorced $(i + 1)_{c+1}$ right-adjacent to the unmarried i_c .

We can now continue with the proofs about our operators $e_i^{(k)}$, $f_i^{(k)}$, and $s_i^{(k)}$.

Proposition 3.6. If T is a k -tableau that has at least one $i/(i + 1)$ -unpaired $i + 1$ (meaning $e_i^{(k)}T$ is defined) and k -weight $\alpha = (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$, then $e_i^{(k)}T$ is a k -tableau of the same shape as T , with k -weight $(\alpha_1, \dots, \alpha_i + 1, \alpha_{i+1} - 1, \dots, \alpha_n)$.

Proof. The proof of this is similar to and follows a proof of a similar fact for a different operator on k -tableau given by Lapointe and Morse in [4].

To prove that $e_i^{(k)}$ is a k -tableau, it suffices to show that it is column strict and has the appropriate k -weight, as explained previously.

We begin by showing that $e_i^{(k)}$ is column strict. First note that in $e_i^{(k)}T$, the only entries which were changed from T are unmarried $i + 1$'s. In any given row of T , if we change an

unmarried $i + 1$ to an i , since T is column strict, the entry above $i + 1$ was strictly greater than $i + 1$ (and also greater than the i that we replaced the $i + 1$ with), and since the $i + 1$ was unmarried, any entry below the $i + 1$ would have to be strictly less than i . Thus when we replace the $i + 1$ with an i , the entry below is still less. This shows that changing unmarried $i + 1$ entries to i does not affect column strictness, so $e_i^{(k)}T$ is column strict.

Now we need only show that the k -weight changes appropriately for $e_i^{(k)}T$. Let α_i and α_{i+1} denote the number of residues occupied in T by the letters i and $i + 1$, respectively, and β_i and β_{i+1} denote the number of residues occupied in $e_i^{(k)}T$ by the letters i and $i + 1$. Then let α_i^m and α_i^s denote the number of residues occupied in T by married i and single i entries, respectively. We use similar notation for married and single $i + 1$ entries in T , and for married and single i and $i + 1$ entries in $e_i^{(k)}T$. Then $\alpha_i = \alpha_i^m + \alpha_i^s$ and $\alpha_{i+1} = \alpha_{i+1}^m + \alpha_{i+1}^s$ for the entries in T and $\beta_i = \beta_i^m + \beta_i^s$ and $\beta_{i+1} = \beta_{i+1}^m + \beta_{i+1}^s$ for the entries in $e_i^{(k)}T$. But since there is a one-to-one correspondence between married i and $i + 1$ entries in T , and similarly in $e_i^{(k)}T$, we have $\alpha_i^m = \alpha_{i+1}^m$ and $\beta_i^m = \beta_{i+1}^m$. Also because any married i or $i + 1$ entry in T is still married in $e_i^{(k)}T$, and any unmarried entry that we changed from T to $e_i^{(k)}T$ is still unmarried, we have $\alpha_i^m = \alpha_{i+1}^m = \beta_i^m = \beta_{i+1}^m$. Therefore to show that the weight changes appropriately, we need only show that $\beta_i^s = \alpha_i^s + 1$ and $\beta_{i+1}^s = \alpha_{i+1}^s$. In other words, we need to show that $e_i^{(k)}T$ has exactly one more single i than does T , and has exactly one less single $i + 1$ than T .

We only change one fundamental row under the operator $e_i^{(k)}$, let us call this row R . Then in all other fundamental rows of T the number of single i and $i + 1$ entries stay the same, so the only possible change must occur in row R . Let c be the residue of the single $i + 1$ in T which gets changed to an i in $e_i^{(k)}T$. This single $i + 1$ that gets changed must be the leftmost single $i + 1$ in row R , and because married $i + 1$ entries appear to the right of single and divorced $i + 1$ entries, any $i + 1$ entry to the left of the one with residue c must be divorced. When we perform $e_i^{(k)}$, we change the $(i + 1)_c$ to an entry i_c . The entry i_c is either single or divorced. By Property 3.5, since T contained an unmarried $(i + 1)_c$, any $i_c \in T$ is married and occurs if and only if there is a divorced $(i + 1)_{c-1}$ left-adjacent to the unmarried $(i + 1)_c$. So if the $(i + 1)_c$ in T is changed to a single i_c in $e_i^{(k)}$, then this means there was no married i_c which occurred in T (otherwise the i_c in row R would be divorced), and hence by Property 3.5, there is no divorced $(i + 1)_{c-1}$ left-adjacent to the unmarried $(i + 1)_c$ in row R , so there are no entries to the left of the $(i + 1)_c$ in row R that get changed to an i . On the other hand, if the $(i + 1)_c$ in T is changed to a divorced i_c , then this means there was a divorced $(i + 1)_{c-1}$ left-adjacent to the $(i + 1)_c$ in row R in T . Then by the definition of $e_i^{(k)}$, we must also change the $(i + 1)_{c-1}$ in row R to an i_{c-1} . Then this i_{c-1} is either single or divorced, and if we use the same argument repeatedly, we can see that the entries that are changed in T are exactly a string of divorced $i + 1$ entries which end with the single $(i + 1)_c$ at the right, and these are changed into a string of i entries, where the leftmost one is single in $e_i^{(k)}T$, and all others are divorced. Therefore when we perform $e_i^{(k)}$ on T we lose one single

$i + 1$, but gain one single i , as desired. □

Proposition 3.7. Let T be a k -tableau and let $u = i^p(i + 1)^q$ be the fundamental $i/(i + 1)$ -unpaired subword of T . If $s_i^{(k)}u$, $e_i^{(k)}u$, and $f_i^{(k)}u$, are the fundamental $i/(i + 1)$ -unpaired subwords of $s_i^{(k)}T$, $e_i^{(k)}T$, and $f_i^{(k)}T$, respectively, then we have the following

- $s_i^{(k)}u = i^q(i + 1)^p$.
- $e_i^{(k)}u = i^{p+1}(i + 1)^{q-1}$ for $q > 1$.
- $f_i^{(k)}u = i^{p-1}(i + 1)^{q+1}$ for $p > 1$.

Proof. First, we prove the proposition for $e_i^{(k)}$. When we perform $e_i^{(k)}$ on T we are only changing one single $i + 1$ to an i in a fundamental row R , and then changing any divorced $i + 1$'s which are to the left of the single $i + 1$ that we changed in row R (and then changing corresponding entries in higher rows). Then as explained in the proof of Proposition 3.6, in row R we are changing a string of divorced $i + 1$'s followed by a single $i + 1$ to a string with a single i at the left, followed by divorced i entries.

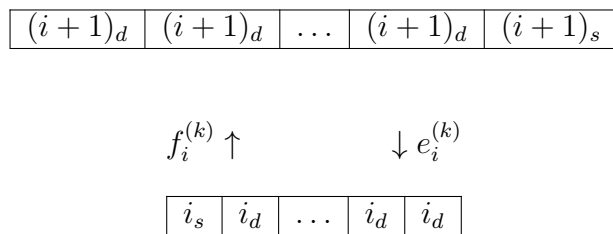


Figure 3.1: How $e_i^{(k)}$ and $f_i^{(k)}$ change the entries in a fundamental row R . The subscripts d and s refer to whether each entry is divorced or single.

So we gain a single i and lose a single $i + 1$ and it will be in the same relative position in the fundamental $i/(i + 1)$ -unpaired subword, and all other single i and $i + 1$'s will remain unchanged. So $e_i^{(k)}u = i^{p+1}(i + 1)^{q-1}$.

Similar reasoning proves the case for $f_i^{(k)}u$.

All that we have left is to consider the case of $s_i^{(k)}u$. But for this word, for each of the $i + 1$ we change to an i or i we change to an $i + 1$, the same thing happens as when we do $e_i^{(k)}$ or $f_i^{(k)}$, so the fundamental $i/(i + 1)$ -unpaired subword will just change by swapping the number of unpaired i and $i + 1$'s, so $s_i^{(k)}u = i^q(i + 1)^p$. □

One more important fact about the operators that we will need is the fact that $f_i^{(k)}$ and $e_i^{(k)}$ are inverses.

Proposition 3.8. The operators $e_i^{(k)}$ and $f_i^{(k)}$ are inverses. That is, if T is a k -tableau on which $e_i^{(k)}$ is defined, then $f_i^{(k)}e_i^{(k)}T = T$. Likewise, if T is a k -tableau on which $f_i^{(k)}$ is defined, then $e_i^{(k)}f_i^{(k)}T = T$.

Proof. First, let T be a k -tableaux on which $e_i^{(k)}$ is defined, and we will consider $f_i^{(k)}e_i^{(k)}T$. Notice from the proof of Proposition 3.7 that the only thing $e_i^{(k)}$ changes in T is a string of divorced $i + 1$ followed by a single $i + 1$ in a fundamental row R of T , which it changes into a string consisting of a single i followed by divorced i , and corresponding entries in higher rows of T . This idea can be seen illustrated in Figure 3.2. But the location of the single $i + 1$ in the fundamental $i/(i + 1)$ -unpaired subword of T that we change is the same relative location as the corresponding single i that it was changed into, in the fundamental $i/(i + 1)$ -unpaired subword of $e_i^{(k)}T$. Since it was the leftmost unpaired $i + 1$ in the fundamental $i/(i + 1)$ -unpaired subword of T , it will be correspond to the rightmost unpaired i in the fundamental $i/(i + 1)$ -unpaired subword of $e_i^{(k)}T$. Hence when we perform $f_i^{(k)}$, that unpaired i will get changed back to an $i + 1$, and the entire string of entries that was changed in row R will get changed back to what they were originally, along with the corresponding entries in higher rows. We can visualize this by thinking that all $e_i^{(k)}$ and $f_i^{(k)}$ do are change a string of entries in fundamental row R back and forth as in Figure 3.2, and the corresponding entries in higher rows. So $f_i^{(k)}e_i^{(k)}T = T$. The argument that $e_i^{(k)}f_i^{(k)}T = T$ is essentially the same. \square

3.3 The operator $s_i^{(k)}e_i^{(k)}$

The operator $s_i^{(k)}e_i^{(k)}$ has some nice properties, and will be used in the bijection for our proof of the k -Littlewood–Richardson rule in Chapter 4.

Lemma 3.9. Let T be a k -tableau or skew k -tableau filled with the letters $1, \dots, n$ and k -weight $w(\mu + \rho) - \rho$ where $\mu = (\mu_1, \dots, \mu_n)$ is a partition, $w \in S_n$ is a permutation, and $\rho = (n - 1, \dots, 1, 0)$. Then if there is at least one $i/(i + 1)$ -unpaired $i + 1$ in T , $s_i^{(k)}e_i^{(k)}T$ is defined, and $s_i^{(k)}e_i^{(k)}T \neq T$.

Proof. The k -weight of T is $w(\mu + \rho) - \rho = w(\mu_1 + n - 1, \dots, \mu_i + n - i, \mu_{i+1} + n - (i + 1), \dots, \mu_n + 0) - (n - 1, \dots, n - i, n - i + 1, \dots, 0)$. In particular, T has $\mu_{w_i} + n - w_i - (n - i)$ $k + 1$ -residues filled with i , and $\mu_{w_{i+1}} + n - w_{i+1} - (n - (i + 1))$ $k + 1$ -residues filled with $i + 1$. But then $s_i^{(k)}e_i^{(k)}T$ has $\mu_{w_{i+1}} + n - w_{i+1} - (n - (i + 1)) - 1$ $k + 1$ -residues filled with i , and $\mu_{w_i} + n - w_i - (n - i) + 1$ $k + 1$ -residues filled with $i + 1$. If T and $s_i^{(k)}e_i^{(k)}T$ were the same k -tableau, then they would have the same number of $k + 1$ -residues filled with i , so we would have

$$\mu_{w_i} + n - w_i - (n - i) = \mu_{w_{i+1}} + n - w_{i+1} - (n - (i + 1)) - 1.$$

But then simplifying this expression, we get $\mu_{w_{i+1}} - \mu_{w_i} = w_{i+1} - w_i$. But there are two possible cases here.

- If $w_{i+1} > w_i$, then $\mu_{w_{i+1}} \leq \mu_{w_i}$, meaning $\mu_{w_{i+1}} - \mu_{w_i} \leq 0 < w_{i+1} - w_i$.
- If if $w_{i+1} < w_i$, then $\mu_{w_{i+1}} \geq \mu_{w_i}$, meaning $\mu_{w_{i+1}} - \mu_{w_i} \geq 0 > w_{i+1} - w_i$.

Either of these cases contradicts the above equality ($\mu_{w_{i+1}} - \mu_{w_i} = w_{i+1} - w_i$), meaning T and $s_i^{(k)} e_i^{(k)} T$ have different numbers of $k + 1$ -residues occupied by i , and thus they cannot be the same k -tableau. □

Lemma 3.10. Let u be the fundamental $i/(i + 1)$ -unpaired subword of T , so $u = i^p(i + 1)^q$. If $q \geq p$, then $s_i^{(k)} T = (e_i^{(k)})^{q-p} T$, otherwise if $q < p$, then $s_i^{(k)} T = (f_i^{(k)})^{q-p} T$.

Proof. First, suppose that $q \geq p$. When we apply $e_i^{(k)}$ to a k -tableau T , the rightmost $i/(i + 1)$ -unpaired $i + 1$ in the fundamental $i/(i + 1)$ -unpaired subword of T is changed to an i , but all of the $i + 1$'s that appeared to the right of the one we changed in the fundamental $i/(i + 1)$ -unpaired subword are exactly the same as the $i/(i + 1)$ -unpaired $i + 1$'s in the fundamental $i/(i + 1)$ -unpaired subword of $e_i^{(k)} T$. Performing the corrections from the definition of $e_i^{(k)}$ (which result in only changing some divorced $i + 1$'s to i 's) will not change the positions of the $i/(i + 1)$ -unpaired $i + 1$'s in the k -tableau. This means that if we apply $e_i^{(k)}$ to T a total of $q - p$ times, the resulting k -tableau will be identical to the one we would get by first changing the $q - p$ leftmost $i + 1$'s in u to i entries, and then perform the correcting parts of the operators (changing any divorced $i + 1$ in the fundamental rows to i as needed, and changing unmarried $i + 1$'s in rows above the fundamental rows as needed), which is exactly $s_i^{(k)} T$ by definition. A similar argument using the $f_i^{(k)}$ operators proves the remaining case. □

Using Lemma 3.10 and Proposition 3.8, we will be able to prove the following corollary which is necessary for our proof in Chapter 4.

Corollary 3.11. The operator $s_i^{(k)} e_i^{(k)}$ is an involution on the set of k -tableau of a given shape that have at least one $i/(i + 1)$ -unpaired $i + 1$.

Proof. First of all, let $u = i^p(i + 1)^q$ be the fundamental $i/(i + 1)$ -unpaired subword of T . Since there is at least one $i/(i + 1)$ -unpaired $i + 1$, $q > 0$ and $e_i^{(k)} T$ is defined. If we pay attention to what happens to the fundamental $i/(i + 1)$ -unpaired subword of T when we apply $e_i^{(k)}$ and then $s_i^{(k)}$, we see that

$$u = i^p(i + 1)^q \xrightarrow{e_i^{(k)}} i^{p+1}(i + 1)^{q-1} \xrightarrow{s_i^{(k)}} i^{q-1}(i + 1)^{p-1}.$$

By Lemma 3.10, since $s_i^{(k)}$ is the same as applying $e_i^{(k)}$ or $f_i^{(k)}$ some number of times, and since $e_i^{(k)}$ and $f_i^{(k)}$ are inverses by Proposition 3.8, this means for our specific k -tableau T we can express $s_i^{(k)}e_i^{(k)}T$ as $(e_i^{(k)})^\alpha T$ (where if $\alpha < 0$ we interpret this as meaning $s_i^{(k)}e_i^{(k)}T = (f_i^{(k)})^{-\alpha}T$). To find exactly what α is, we notice that the word $u = i^p(i+1)^q$ must get sent to the word $i^{q-1}(i+1)^{p-1}$, so $\alpha = q - p - 1$. Let $T' = s_i^{(k)}e_i^{(k)}T$ and consider what happens to the word $u' = i^{q-1}(i+1)^{p-1}$ of T' when we perform $s_i^{(k)}e_i^{(k)}$ a second time.

$$i^{q-1}(i+1)^{p-1} \xrightarrow{e_i^{(k)}} i^q(i+1)^p \xrightarrow{s_i^{(k)}} i^p(i+1)^q.$$

By the same reasoning as before, $s_i^{(k)}e_i^{(k)}T' = (e_i^{(k)})^\beta T'$, and in order for the word to change appropriately, we need $\beta = -q + p + 1 = -\alpha$. But then

$$s_i^{(k)}e_i^{(k)}T' = (e_i^{(k)})^{-\alpha}T' = (e_i^{(k)})^{-\alpha}(e_i^{(k)})^\alpha T = T.$$

□

3.4 The k -lattice property

The last section of this chapter introduces a property of a k -tableau T which we call the k -lattice property. We define the property and give a few examples which demonstrate when the k -lattice property holds and when it does not.

Definition 3.12. Let T be a k -tableau filled with the numbers $1, \dots, n$. If T has no $i/(i+1)$ -unpaired $i+1$ for $i = 1, \dots, n-1$, then we say that T is k -lattice.

Remark 3.13. A k -tableau T is k -lattice if and only if it does not admit any $e_i^{(k)}$ operator. This follows from the definition of $e_i^{(k)}$. Note that this is the same as in the classical case (with “no k ”) where a tableau T is lattice if and only if it does not admit any e_i operator.

Recall that the property for a semistandard Young tableaux T filled with the numbers $1, \dots, n$ to be lattice is that when we take the reverse reading word of the tableau, $w = w_1w_2 \cdots w_m$, then in any left factor $w_1w_2 \cdots w_j$ the number of i 's is at least as great as the number of $i+1$'s (for $i = 1, \dots, n$) [12]. It is easy to see that if we take k to be sufficiently large, then the k -lattice definition becomes the traditional lattice definition for a tableau. The k -lattice property is of importance in this thesis because it is used in giving a combinatorial description of the k -Littlewood–Richardson coefficients.

Remark 3.14. For the proofs presented in Chapters 4 and 5, we will only be filling k -tableaux with the numbers 1 and 2. In this case the k -lattice property will just be that there is no $1/2$ -unpaired 2 in a tableau T .

We now present two examples illustrating the k -lattice property.

Example 3.15. Let $k = 9$ and let T be the following 9-tableau:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2_6 & 3_7 & 3_8 & & & & & & \\ \hline 1_7 & 1_8 & 1_9 & & & & & & \\ \hline & & & 2_1 & 2_2 & & & & \\ \hline & & & & & & & & \\ \hline & & & & & & 1_6 & 1_7 & 1_8 & 1_9 \\ \hline \end{array}$$

If we take the word consisting of all single 1's and 2's in fundamental rows, reading from right to left, top to bottom, we get $2_1 2_2 1_6 1_8 1_9$, and after pairing them using parenthesis rules, we get the fundamental 1/2-unpaired subword consisting of just 1_9 . So there is no 1/2-unpaired 2. Looking at single 2's and 3's in fundamental rows we get the word $2_6 3_7 3_8 2_1 2_2$, and after parenthesis pairing we are left with the fundamental 2/3-unpaired subword 2_6 . This means there is also no 2/3-unpaired 3, so T is k -lattice.

Note that if only one i violates the k -lattice property, then the tableau will not be k -lattice. This can be seen in the following example.

Example 3.16. Let $k = 9$ and let T be the following 9-tableau:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 3_6 & 3_7 & 3_8 & & & & & & \\ \hline 1_7 & 2_8 & 2_9 & & & & & & \\ \hline & & & 1_1 & 1_2 & & & & \\ \hline & & & & & & & & \\ \hline & & & & & & 3_6 & 3_7 & 3_8 & 3_9 \\ \hline \end{array}$$

In T , the fundamental 1/2-unpaired subword will consist of 1_7 , so there is no 1/2-unpaired 2, so the 1's and 2's do not violate the lattice property. But if we consider the 2's and 3's, the fundamental 2/3-unpaired subword is $3_6 3_9$, so there is a 2/3-unpaired 3, which violates the k -lattice property. Hence T is not k -lattice.

Chapter 4

A proof of the k -Littlewood–Richardson rule in a restricted case

4.1 The proof of the k -Littlewood–Richardson rule

In this section we follow the methods used by Remmel and Shimozono in [11] to prove the k -Littlewood–Richardson rule in the restricted case, which we restate after first reviewing the definition of k -Littlewood–Richardson coefficients.

Definition 4.1. Let μ , ν , and λ be any k -bounded partitions. Then the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$ is the coefficient of $s_{\lambda}^{(k)}$ appearing in the expansion of the product of the two k -Schur functions $s_{\mu}^{(k)}$ and $s_{\nu}^{(k)}$

$$s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu\nu}^{\lambda,k} s_{\lambda}^{(k)}$$

where the sum is taken over all k -bounded partitions λ .

Theorem 4.2 (The k -Littlewood–Richardson rule for a special case). Fix an integer $k > 0$ and let μ , ν , and λ be k -bounded partitions, and require that the hook length of μ is at most k ($h(\mu) \leq k$). Furthermore, require that $l(\mu) \leq 2$. Then the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$ can be described combinatorially as the number of skew k -tableaux of shape λ/ν with content μ that are k -lattice.

We begin by setting up some notation and recalling some facts that will be used. Recall the determinantal expansion for Schur functions, $s_{\mu} = \det(h_{\mu_i+i-j})_{1 \leq i,j \leq n}$. Expanding out the determinant we see $s_{\mu} = \sum_{\omega \in S_n} (-1)^{\omega} \prod_{i=1}^n h_{\mu_{\omega(i)}+\omega(i)-i} = \sum_{\omega \in S_n} (-1)^{\omega} h_{\omega(\mu+\rho)-\rho}$ where ρ

is the partition $(n-1, n-2, \dots, 1, 0)$. For the rest of this chapter, we will take n to be equal to $l(\mu)$. Also, for simplicity throughout this chapter, when we use say that T is a k -tableau, we mean that T is either a k -tableau or skew k -tableau.

Let μ be a k -bounded partition with hook length at most k , meaning $h(\mu) \leq k$, so then $s_\mu^{(k)} = s_\mu$ [7]. We do not yet require that $l(\mu) \leq 2$. Then for any other k -bounded partition ν ,

$$s_\mu^{(k)} s_\nu^{(k)} = s_\mu s_\nu^{(k)} = \sum_{\omega \in S_n} (-1)^\omega h_{\omega(\mu+\rho)-\rho} s_\nu^{(k)}. \quad (4.1)$$

We will apply a consequence of the k -Pieri rule (given in Corollary 4.4) to the terms in the previous equation, but first we need the following lemma to be sure that $\omega(\mu + \rho) - \rho$ is k -bounded (otherwise we cannot use Corollary 4.4).

Lemma 4.3. If μ is a partition with hook length at most k , meaning $h(\mu) \leq k$, then $\omega(\mu + \rho) - \rho$ is a composition with all parts having length at most k for any $\omega \in S_n$. This means that for $i = 1, \dots, n$, $\mu_{\omega(i)} + \rho_{\omega(i)} - \rho(i) \leq k$.

Proof. For any $i = 1, \dots, n$, $\mu_{\omega(i)} + \rho_{\omega(i)} - \rho(i) = \mu_{\omega(i)} + n - \omega(i) - (n - i) = \mu_{\omega(i)} - \omega(i) + i$. The largest that this term can possibly be is if $\omega(i) = 1$ and $i = n$. So $\mu_{\omega(i)} + \rho_{\omega(i)} - \rho(i) \leq \mu_1 - 1 + n = h(\mu) \leq k$. □

We need to use a consequence of the k -Pieri rule which was derived by Lapointe and Morse [4]. This is a formula which gives a combinatorial rule for the coefficients when we multiply a homogeneous symmetric function and a k -Schur function which we previously stated in Chapter 2 but review again here.

Corollary 4.4 (Lapointe, Morse [4]). For any k -bounded partitions ν and τ ,

$$h_\tau s_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^k} K_{\lambda/\nu, \tau}^{(k)} s_\lambda^{(k)} \quad (4.2)$$

where $K_{\lambda/\nu, \tau}^{(k)}$ is the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight λ , and \mathcal{P}^k denotes the set of all k -bounded partitions.

Suppose that we fix k and let ν and λ be any k -bounded partitions, and $\mu = (a, b)$ be a k -bounded partition of length 2, with $h(\lambda) \leq k$ (this is our restriction on μ for this chapter). Then combining Equations (4.1) and (4.2), we have the following

$$s_\mu^{(k)} s_\nu^{(k)} = s_\mu s_\nu^{(k)} = \sum_{\omega \in S_n} \sum_{\lambda \in \mathcal{P}^k} (-1)^\omega K_{\lambda/\nu, \omega(\mu+\rho)-\rho}^{(k)} s_\lambda^{(k)}. \quad (4.3)$$

Lemma 4.5. Let ν and λ be any k -bounded partitions, and μ be a k -bounded partition with $l(\mu) \leq 2$ and $h(\mu) \leq k$. Then the k -Littlewood–Richardson coefficient appearing in the expansion $s_\mu^{(k)} s_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^k} c_{\mu\nu}^{\lambda,k} s_\lambda^{(k)}$ can be expressed as

$$c_{\mu\nu}^{\lambda,k} = \sum_{\omega \in S_n} (-1)^\omega K_{\lambda/\nu, \omega(\mu+\rho)-\rho}^{(k)}$$

where $K_{\lambda/\nu, \omega(\mu+\rho)-\rho}^{(k)}$ is the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight $\omega(\mu+\rho)-\rho$.

Proof. This formula can be obtained easily from Equation (4.3) by rearranging terms slightly. \square

We are ready to prove Theorem 4.2. To prove the k -Littlewood–Richardson rule in our special case and give a combinatorial description of the k -Littlewood–Richardson coefficients, we will construct a bijection that cancels terms on the right hand side of the equation in the above Lemma.

Proof of Theorem 4.2 (The k -Littlewood–Richardson rule for a special case). First, for $l(\mu) = 1$, we take $n = 1$, and S_1 consists of just the identity permutation. Then the expression from Lemma 4.5 becomes $c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu, \mu}^{(k)}$ where $K_{\lambda/\nu, \mu}^{(k)}$ is the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight μ . But since $l(\mu) = 1$, the k -tableaux are just filled with 1's, so they will all be k -lattice. Therefore $c_{\mu\nu}^{\lambda,k}$ is equal to the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight μ which are k -lattice when $l(\mu) = 1$. It remains to prove the case when $l(\mu) = 2$.

For the case where $l(\mu) = 2$, we take $n = 2$ by convention, and S_2 consists of two permutations, $\sigma = (12)$ and the identity, 1. There are thus two possibilities for $\omega(\mu+\rho)-\rho$.

- If $\omega = 1$, then $\omega(\mu+\rho)-\rho = (a, b)$.
- If $\omega = \sigma$, then $\omega(\mu+\rho)-\rho = \sigma((a, b) + (1, 0)) - (1, 0) = \sigma((a+1, b)) - (1, 0) = (b, a+1) - (1, 0) = (b-1, a+1)$.

We can simplify the expression from Lemma 4.5 slightly, since for our case $\mu = (a, b)$ and $S_n = S_2$.

$$c_{(a,b)\nu}^{\lambda,k} = \sum_{\omega \in S_2} (-1)^\omega K_{\lambda/\nu, \omega((a,b)+(1,0))-(1,0)}^{(k)} = K_{\lambda/\nu, (a,b)}^{(k)} - K_{\lambda/\nu, (b-1, a+1)}^{(k)},$$

where $K_{\lambda/\nu, (a,b)}^{(k)}$ is the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight (a, b) , and $K_{\lambda/\nu, (b-1, a+1)}^{(k)}$ is the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight $(b-1, a+1)$.

For the bijection, let T be a skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight (a, b) that is not k -lattice. Then T has at least one $1/2$ - k -unpaired 2. Then $s_1^{(k)} e_1^{(k)} T$ is defined and is

a skew k -tableaux of the same shape, with k -weight $(b - 1, a + 1)$ (this is because $e_1^{(k)}T$ has k -weight $(a + 1, b - 1)$) so then $s_1^{(k)}e_1^{(k)}T$ has k -weight $(b - 1, a + 1)$. For the other direction of the bijection, given any skew k -tableau T' of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ of k -weight $(b - 1, a + 1)$, then we must have at least one $1/2$ - k -unpaired 2, since $b \leq a$, meaning $b - 1 < a + 1$, so there are more k -residues occupied with 2 than with 1 in T' . This means $s_1^{(k)}e_1^{(k)}(T')$ is defined, and it will be a skew k -tableau of the same shape, with k -weight (a, b) . By Corollary 3.11 in Chapter 3, $s_1^{(k)}e_1^{(k)}$ is an involution, which gives our bijection between skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight (a, b) that are not k -lattice, and skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight $(b - 1, a + 1)$. Therefore in the above expression, we get cancellation between these terms and are left with $c_{(a,b)\nu}^{\lambda,k}$ being equal to the number of k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight (a, b) that are k -lattice. □

4.2 An example of the bijection used for the proof

To see a short example illustrating the bijection used in the proof of Theorem 4.2, let $k = 3$, $\mu = (2, 1)$ and $\nu = (3, 2)$ and $\lambda = (3, 3, 2)$. As before $n = l(\mu) = 2$, so in the expression for the k -Littlewood–Richardson coefficient from Corollary 4.5, we have

$$c_{(2,1)(3,2)}^{(3,3,2),3} = \sum_{\omega \in S_2} (-1)^\omega K_{(3,3,2)/(3,2), \omega((2,1)+(1,0))-(1,0)}^{(3)} = K_{(3,3,2)/(3,2),(2,1)}^{(3)} - K_{(3,3,2)/(3,2),(0,3)}^{(3)}.$$

If we compute all of the k -tableaux of shape $\mathbf{c}(3, 3, 2)/\mathbf{c}(3, 2) = (5, 3, 2)/(3, 2)$ with k -weight $(2, 1)$, we get the following two k -tableau:

$$T_1 = \begin{array}{|c|c|c|c|c|} \hline 1_3 & 1_4 & & & \\ \hline & & 2_1 & & \\ \hline & & & 1_3 & 1_4 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|} \hline 1_3 & 2_4 & & & \\ \hline & & 1_1 & & \\ \hline & & & 1_3 & 2_4 \\ \hline \end{array}$$

Computing all of the k -tableaux of shape $\mathbf{c}(3, 3, 2)/\mathbf{c}(3, 2) = (5, 3, 2)/(3, 2)$ with k -weight $(0, 3)$, there is only one, which we call T_3 .

$$T_3 = \begin{array}{|c|c|c|c|c|} \hline 2_3 & 2_4 & & & \\ \hline & & 2_1 & & \\ \hline & & & 2_3 & 2_4 \\ \hline \end{array}$$

For this example, the bijection in the proof of Theorem 4.2 is a bijection between the k -tableau of shape $\mathbf{c}(3, 3, 2)/\mathbf{c}(3, 2)$ and k -weight $(1, 2)$ which are not k -lattice, and the k -tableau of shape $\mathbf{c}(3, 3, 2)/\mathbf{c}(3, 2)$ and k -weight $(0, 3)$, with the operator $s_1^{(3)}e_1^{(3)}$ being used to provide the bijection in either direction. First note that of the two k -tableaux with k -weight

$(1, 2)$, T_1 is k -lattice, and T_2 is not k -lattice. This is because when we take the $1/2$ fundamental subword of T_1 , we get $2_1 1_3 1_4$, then after we perform the parenthesis matching, we are left with the fundamental $1/2$ -unpaired subword being 1_4 . There are no unpaired 2's, so T_1 is k -lattice. On the other hand, the $1/2$ fundamental subword of T_2 , we get $1_1 1_3 2_4$, then even after we perform the parenthesis matching, we are left with the fundamental $1/2$ -unpaired subword $1_1 1_3 2_4$. There is an unpaired 2, so T_2 is not k -lattice.

For the bijection, notice that $e_1^{(3)} T_2$ is the following tableau.

$$e_1^{(3)} T_2 = \begin{array}{|c|c|c|c|} \hline 1_3 & 1_4 & & \\ \hline & & 1_1 & \\ \hline & & & 1_3 & 1_4 \\ \hline \end{array}$$

Then $s_1^{(3)} e_1^{(3)} T_2 = T_3$, so with the bijection the k -tableaux T_2 and T_3 cancel out, and we are left with only one k -tableau T_1 contributing to the k -Littlewood–Richardson coefficient, meaning that $c_{(2,1)(3,2)}^{(3,3,2),3} = 1$.

4.3 An explanation of the requirements on μ and problems with generalizing the proof

In the proofs of the k -Littlewood–Richardson rule in both Chapters 4 and 5 we use the restriction that μ has length 2 ($\mu = (a, b)$). The expression for the k -Littlewood–Richardson coefficients from Lemma 4.5, $c_{\mu\nu}^{\lambda,k} = \sum_{\omega \in \mathcal{S}_n} (-1)^\omega K_{\lambda/\nu, \omega(\mu+\rho)}^{(k)}$, actually still holds for any μ with $h(\mu) \leq k$, even if μ has length greater than 2. It is possible then that there may be some sign reversing involution to generalize our proof for μ of any length. In [11], for the proof of the classical Littlewood–Richardson rule, Remmel and Shimozono used a similar involution where if T was a tableau, and i was the smallest number $1 \leq i \leq n-1$ that violated the lattice condition on T , then the involution sent T to $s_i e_i T$. Then in $s_i e_i T$, i is still the smallest number violating the lattice condition. This does not exactly generalize to the case of k -tableau though, for the following reason which we present as a remark.

Remark 4.6. If T is a k -tableau, and i is the smallest number $1 \leq i \leq n-1$ that violates the k -lattice condition, meaning that there is an $i/(i+1)$ -unpaired $i+1$, but there is no $j/(j+1)$ -unpaired j for any $j < i$, then when we perform any of the operators $e_i^{(k)}$, $s_i^{(k)}$, or $f_i^{(k)}$, i may not be the smallest number violating the k -lattice condition anymore.

The following example presents a case where the problem mentioned above occurs.

Example 4.7. Let T be the following k -tableau for $k = 5$.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & 3_3 & & \\ \hline & & & 1_4 & 3_5 & 3_0 \\ \hline & & & & & \\ \hline & & & & & 3_3 & 3_4 & 3_5 & 3_0 & 3_1 \\ \hline \end{array}$$

Then T is a 5-tableau of shape $\mathbf{c}(5, 3, 3, 1)/\mathbf{c}(3, 3)$ and 5-weight $(1, 0, 5)$. Note that the fundamental 2/3-unpaired subword of T is $3_3 3_4 3_5 3_0 3_1$, and the fundamental 1/2-unpaired subword of T is 1_4 . Thus 2 is the smallest i such that $i/i + 1$ violates the k -lattice condition. We will now compute $e_2^{(5)}T$ and $s_2^{(5)}e_2^{(5)}T$.

$$e_2^{(5)}T = \begin{array}{|c|c|c|c|c|c|} \hline & & & 2_3 & & \\ \hline & & & 1_4 & 3_5 & 3_0 \\ \hline & & & & & \\ \hline & & & & & 2_3 & 3_4 & 3_5 & 3_0 & 3_1 \\ \hline \end{array} \qquad s_2^{(5)}e_2^{(5)}T = \begin{array}{|c|c|c|c|c|c|} \hline & & & 2_3 & & \\ \hline & & & 1_4 & 2_5 & 2_0 \\ \hline & & & & & \\ \hline & & & & & 2_3 & 2_4 & 2_5 & 2_0 & 3_1 \\ \hline \end{array}$$

The fundamental 1/2-unpaired and 2/3-unpaired subwords of $e_2^{(5)}T$ are \emptyset (the empty word) and $2_3 3_4 3_5 3_0 3_1$, respectively, so 2 is still the smallest i such that $i/(i + 1)$ violates the k -lattice condition. The fundamental 1/2-unpaired and 2/3-unpaired subwords of $s_2^{(5)}e_2^{(5)}T$ are $2_4 2_5 2_0$ and $2_3 2_4 2_5 2_0 3_1$, respectively, so we can see that although 2/3 still violates the k -lattice condition, now 1/2 does as well since there are 1/2-unpaired 2's. So even though 2 was the smallest i such that $i/(i + 1)$ violates the k -lattice condition in T , 1 is the smallest such number in $s_2^{(5)}e_2^{(5)}T$.

Despite this, it still appears in examples that have been computed that our k -lattice definition for k -Littlewood–Richardson coefficients may work. For instance, consider the following example where $\mu = (3, 2, 1)$, $\nu = (3, 3)$, and $k = 5$ in which the above problem was observed.

Example 4.8. Let $\mu = (3, 2, 1)$, $\nu = (3, 3)$ and $k = 5$. Then it was computed that $s_{(3,2,1)}^{(5)}s_{(3,3)}^{(5)} = s_{(3,3,3,2,1)}^{(5)} + s_{(4,3,2,2,1)}^{(5)} + s_{(4,4,2,1,1)}^{(5)}$, which means the k -Littlewood–Richardson coefficients are $c_{(3,2,1)(3,3)}^{\lambda,5} = 1$ when $\lambda = (3, 3, 3, 2, 1)$, $(4, 3, 2, 2, 1)$, or $(4, 4, 2, 1, 1)$, and $c_{(3,2,1)(3,3)}^{\lambda,5} = 0$ for all other λ . If we compute all 5-tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(3, 3)$ and 5-weight $(3, 2, 1)$ for any λ , we find that exactly 3 are k -lattice.

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline & & & 3_2 & & \\ \hline & & & 2_3 & 2_4 & \\ \hline & & & 1_4 & 1_5 & 1_0 \\ \hline & & & & & 3_2 \\ \hline & & & & & 2_3 & 2_4 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline & & & 3_2 & & \\ \hline & & & 2_3 & 2_4 & \\ \hline & & & 1_4 & 1_5 & \\ \hline & & & & & 3_2 \\ \hline & & & & & 1_3 & 1_4 & 1_5 \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline & & & 3_2 & & \\ \hline & & & 2_3 & & \\ \hline & & & 1_4 & 1_5 & \\ \hline & & & & & 2_2 & 2_3 \\ \hline & & & & & 1_3 & 1_4 & 1_5 \\ \hline \end{array}$$

These k -tableaux are of shapes $\mathfrak{c}(3, 3, 3, 2, 1)/\mathfrak{c}(3, 3)$ (for T_1), $\mathfrak{c}(4, 3, 2, 2, 1)/\mathfrak{c}(3, 3)$ (for T_2), and $\mathfrak{c}(4, 4, 2, 1, 1)/\mathfrak{c}(3, 3)$ (for T_3). Notice that the shapes correspond to the λ for which $c_{(3,2,1)(3,3)}^{\lambda,5} = 1$, which leads us to conclude that in at least some other cases, the k -lattice condition seems to work, even without taking the restriction on μ .

While computing a large number of other examples, it does appear that the definition of k -lattice is correct (in at least all of the cases that were computed) for determining the k -Littlewood–Richardson coefficients $c_{\mu\nu}^{\lambda,k}$ in terms of k -lattice k -tableaux of shape $\mathfrak{c}(\lambda)/\mathfrak{c}(\nu)$ and k -weight μ , which leads us to the following conjecture.

Conjecture 4.9. Fix an integer $k > 0$ and let μ , ν , and λ be k -bounded partitions (with no restrictions on μ). Then the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$ is conjectured to be equal to the number of skew k -tableaux of shape λ/ν with content μ that are k -lattice.

The reason why this would hold in general is not clear, but it is possible that the techniques presented in the Chapter 5 may help lead to a proof since some of the results presented there do not require imposing any restrictions on μ .

Chapter 5

An alternate proof of the k -Littlewood–Richardson rule for $\mu = (a, b)$

5.1 The alternate proof

We will present an alternate proof here which slightly generalizes the result found in Chapter 4. For the first few things we prove here, we do not require any restriction on the partitions we use other than the fact that they are k -bounded. We will specifically mention when we need to restrict to the case $\mu = (a, b)$. First of all, recall that the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda, k}$ is the coefficient of $s_{\lambda}^{(k)}$ appearing in the expansion of $s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu\nu}^{\lambda, k} s_{\lambda}^{(k)}$. We will also be using a specific notation convention stated in the following remark.

Remark 5.1. Given two partitions μ and τ , we say $\tau \supseteq \mu$ if τ is larger than or equal to μ with respect to the dominance order which we reviewed in Chapter 2. When we use the notation $\tau \triangleright \mu$ this means that τ is *strictly* larger than μ with respect to the dominance order, so in particular $\tau \neq \mu$. Also, when we use the notation $\tau \supseteq \mu$ or $\tau \triangleright \mu$, where μ is a k -bounded partition, we require τ to be k -bounded as well. If τ is not k -bounded, then the k -Kostka numbers which appear are all zero [7], so those terms for τ not k -bounded would not appear anyway.

Proposition 5.2. Let λ , μ , and ν be k -bounded partitions. Then we have the following formula for the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda, k}$

$$c_{\mu\nu}^{\lambda, k} = K_{\lambda/\nu, \mu}^{(k)} - \sum_{\tau \triangleright \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda, k} \quad (5.1)$$

where $K_{\lambda/\nu, \mu}^{(k)}$ is the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and k -weight μ , and $K_{\tau\mu}^{(k)}$ is the number of k -tableaux of shape $\mathbf{c}(\tau)$ and k -weight μ .

Note that Proposition 5.2 holds for any k -bounded partitions λ , μ , and ν . We only make a restriction on μ further on.

Proof. Recall that by the definition of k -Schur functions that we are using [7], we have

$$h_\mu = \sum_{\tau \supseteq \mu} K_{\tau\mu}^{(k)} s_\tau^{(k)}$$

where λ is a k -bounded partition. If we multiply the above equation by $s_\nu^{(k)}$, where ν is a k -bounded partition, we will have

$$h_\mu s_\nu^{(k)} = \sum_{\tau \supseteq \mu} K_{\tau\mu}^{(k)} s_\tau^{(k)} s_\nu^{(k)}.$$

Using the fact that $s_\tau^{(k)} s_\nu^{(k)} = \sum_\lambda c_{\tau\nu}^{\lambda,k} s_\lambda^{(k)}$, and substituting this into the above equation and then simplifying, we get

$$h_\mu s_\nu^{(k)} = \sum_\lambda \left(\sum_{\tau \supseteq \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} \right) s_\lambda^{(k)}. \quad (5.2)$$

Alternately, we have the formula proven by Lapointe and Morse [4] using the Pieri rule for k -Schur functions

$$h_\mu s_\nu^{(k)} = \sum_\lambda K_{\lambda/\nu,\mu}^{(k)} s_\lambda^{(k)}. \quad (5.3)$$

If we set Equations (5.2) and (5.3) equal to each other, we get

$$\sum_\lambda \left(\sum_{\tau \supseteq \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} \right) s_\lambda^{(k)} = \sum_\lambda K_{\lambda/\nu,\mu}^{(k)} s_\lambda^{(k)}.$$

The coefficients of $s_\lambda^{(k)}$ on each side of this must be equal since the k -Schur functions form a basis. Thus we get

$$\sum_{\tau \supseteq \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)}.$$

Now we are almost at our desired result, but this is not completely simplified. It was shown in [7] that the k -Kostka numbers $K_{\tau\mu}^{(k)}$ satisfy a triangularity property similar to the one for the Kostka numbers, so this means that in particular $K_{\mu\mu}^{(k)} = 1$, which can be used to can simplify the above expression a bit.

$$\begin{aligned} K_{\mu\mu}^{(k)} c_{\mu\nu}^{\lambda,k} + \sum_{\tau \supset \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} &= K_{\lambda/\nu,\mu}^{(k)} \\ c_{\mu\nu}^{\lambda,k} + \sum_{\tau \supset \mu} K_{\tau\mu}^{(k)} c_{\tau\nu}^{\lambda,k} &= K_{\lambda/\nu,\mu}^{(k)} \end{aligned}$$

The last expression is of course our desired result. □

From this result, we can derive some specific combinatorial formulas for the k -Littlewood–Richardson coefficients in a more specific case. Specifically we can get expressions for any partition of the form $\mu = (a, b)$ where μ is k -bounded and $a \geq b$ and a combinatorial description of the coefficients, which implies their positivity in our specific case.

Theorem 5.3. Let μ be a k -bounded partition of the form $\mu = (a, b)$, and let λ and ν be any k -bounded partitions. Then

$$c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)} - K_{\lambda/\nu,\mu+(1,-1)}^{(k)}, \quad (5.4)$$

where $c_{\mu\nu}^{\lambda,k}$ is the k -Littlewood–Richardson coefficient appearing in the expression $s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu\nu}^{\lambda,k} s_{\lambda}^{(k)}$, and $K_{\lambda/\nu,\mu}^{(k)}$ and $K_{\lambda/\nu,\mu+(1,-1)}^{(k)}$ are the number of k -tableaux of shape $\mathfrak{c}(\lambda)/\mathfrak{c}(\nu)$ and k -weights μ and $\mu + (1, -1)$, respectively.

Note that this means if $\mu = (k, b)$, meaning the first part of μ is k , then the second term in the expansion $K_{\lambda/\nu,\mu+(1,-1)}^{(k)}$ is zero, since there are no k -tableaux of weight $(k+1, b-1)$, so in this case $c_{(k,b)\nu}^{\lambda,k} = K_{\lambda/\nu,(k,b)}^{(k)}$. Before we prove Theorem 5.3 we state and prove a Lemma that will be used in the proof.

Lemma 5.4. Let $\mu = (a, b)$ be a k -bounded partition, and let $0 \leq t \leq k - a$, then $\tau = (a + t, b - t)$ is a k -bounded partition and $K_{\tau\mu}^{(k)} = 1$.

Proof. First of all, since $0 \leq t \leq k - a$, $\tau = (a + t, b - t)$ is a k -bounded partition since $a + t \leq a + k - a = k$. $K_{\tau\mu}^{(k)}$ is the number of k -tableaux of shape $\mathfrak{c}(\tau)$ and k -weight μ . But given any diagram of shape $\mathfrak{c}(\tau)$, in order to get a k -tableau with k -weight μ , we must fill it with a 1's and b 2's. But in order to do this, we must put all of the 1s in the bottom row, and then fill the remaining boxes with 2s. Thus there is only one possible way to do this, so $K_{\tau\mu}^{(k)} = 1$. \square

Now we can continue to the proof of Theorem 5.3.

Proof of Theorem 5.3. First of all, note that if $\tau \supseteq \mu$ where $\mu = (a, b)$ is k -bounded, then the only possibilities for τ are $\tau = (a + t, b - t)$ where $0 \leq t \leq k - a$. This means that $K_{\tau\mu}^{(k)} = 1$ for all $\tau \supseteq \mu$ by Lemma 5.4. Using this fact in Equation (5.1), we get

$$c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)} - \sum_{\tau \supseteq \mu} c_{\tau\nu}^{\lambda,k}. \quad (5.5)$$

We will now proceed to prove the theorem by induction on a .

First of all, the largest that a can possibly be is $a = k$. Then there are no k -bounded partitions larger than $\mu = (k, b)$ with respect to the dominance order, so from Equation

(5.5), we get $c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)}$ since there are no $\tau \triangleright \mu$.

Next, suppose that Equation (5.4) holds for $a = k, k-1, \dots, k-(s-1)$. We will prove that it holds for $a = k-s$. If $\mu = (k-s, b)$ then the possibilities for τ if τ is k -bounded and $\tau \triangleright \mu$ are $\tau = (k-(s-i), b-i)$ where $i = 1, \dots, s$. Then using Equation (5.5),

$$c_{(k-s,b)\nu}^{\lambda,k} = K_{\lambda/\nu,(k-s,b)}^{(k)} - \sum_{\tau \triangleright (k-s,b)} c_{\tau\nu}^{\lambda,k},$$

but by assumption, for any $\tau \triangleright (k-s, b)$, $c_{\tau\nu}^{\lambda,k} = K_{\lambda/\nu,\tau}^{(k)} - K_{\lambda/\nu,\tau+(1,-1)}^{(k)}$, so we can substitute this in and get

$$c_{(k-s,b)\nu}^{\lambda,k} = K_{\lambda/\nu,(k-s,b)}^{(k)} - \sum_{\tau \triangleright (k-s,b)} (K_{\lambda/\nu,\tau}^{(k)} - K_{\lambda/\nu,\tau+(1,-1)}^{(k)}).$$

But since $\tau = (k-(s-i), b-i)$ where $i = 1, \dots, s$,

$$c_{(k-s,b)\nu}^{\lambda,k} = K_{\lambda/\nu,(k-s,b)}^{(k)} - \sum_{i=1}^s (K_{\lambda/\nu,(k-(s-i),b-i)}^{(k)} - K_{\lambda/\nu,(k-(s-i)+1,b-i-1)}^{(k)}).$$

Notice that when $i = s$, $K_{\lambda/\nu,(k-(s-i)+1,b-i-1)}^{(k)} = K_{\lambda/\nu,(k+1,b-s-1)}^{(k)} = 0$ since there are no k -tableaux of k -weight $(k+1, b-s-1)$. Next we split up this sum and simplify a bit

$$\begin{aligned} c_{(k-s,b)\nu}^{\lambda,k} &= K_{\lambda/\nu,(k-s,b)}^{(k)} - \sum_{i=1}^s K_{\lambda/\nu,(k-(s-i),b-i)}^{(k)} + \sum_{i=1}^{s-1} K_{\lambda/\nu,(k-(s-i)+1,b-i-1)}^{(k)} \\ c_{(k-s,b)\nu}^{\lambda,k} &= K_{\lambda/\nu,(k-s,b)}^{(k)} - \sum_{i=1}^s K_{\lambda/\nu,(k-(s-i),b-i)}^{(k)} + \sum_{i=2}^s K_{\lambda/\nu,(k-(s-i),b-i)}^{(k)} \end{aligned}$$

In this expression, all the terms for $i = 2, \dots, s$ in the left sum cancel with the corresponding terms in the right sum, and we are left with just the term for $i = 1$.

$$c_{(k-s,b)\nu}^{\lambda,k} = K_{\lambda/\nu,(k-s,b)}^{(k)} - K_{\lambda/\nu,(k-(s-1),b-1)}^{(k)} = K_{\lambda/\nu,(k-s,b)}^{(k)} - K_{\lambda/\nu,(k-s+1,b-1)}^{(k)}$$

By induction, since this holds for $a = k-s$, we have our desired result. \square

Corollary 5.5. In this case where $\mu = (a, b)$ is a k -bounded partition, and λ and ν are also k -bounded partitions, the k -Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda,k}$ is equal to the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are k -lattice.

Proof. We will use the expression for $c_{\mu\nu}^{\lambda,k}$ from Theorem 5.3 to construct a bijection between skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are not k -lattice and skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight $\mu + (1, -1)$. This gives cancellation that leaves

us with the desired result.

Let T be a skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that is not k -lattice. Then T has at least one $1/2$ - k -unpaired 2. Then $e_1^{(k)}T$ is defined and is a skew k -tableaux of the same shape, with k -weight $\mu + (1, -1)$. For the other direction of the bijection, given any skew k -tableau T' of k -weight $\mu + (1, -1) = (a + 1, b - 1)$, then we must have at least one $1/2$ - k -unpaired 1, since $a \geq b$, meaning $a + 1 > b - 1$ so there are more k -residues occupied with 1 than with 2 in T' . This means $f_1^{(k)}(T')$ is defined, and it will be a skew k -tableau of the same shape, with k -weight μ .

This is in fact a bijection between skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are not k -lattice and skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight $\mu + (1, -1)$, because $e_1^{(k)}$ and $f_1^{(k)}$ are inverses by Proposition 3.8. This means that in Equation (5.4),

$$c_{\mu\nu}^{\lambda,k} = K_{\lambda/\nu,\mu}^{(k)} - K_{\lambda/\nu,\mu+(1,-1)}^{(k)},$$

all of the skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are not k -lattice totally cancel with all of the skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight $\mu + (1, -1)$, so we are left with $c_{\mu\nu}^{\lambda,k}$ being equal to the number of skew k -tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ with k -weight μ that are k -lattice. \square

5.2 Problems with generalizing the classical case and an example

For classical Schur functions, in the case where λ and ν are any partitions, and $\mu = (a, b)$, a similar formula holds. It can be proven in a similar way to the k -Schur case, and also can be derived from the k -Schur case by taking k to be sufficiently large. So in the classical case, we have the formula

$$K_{\lambda/\nu,\mu} = \sum_{\tau \supseteq \mu} K_{\tau\mu} c_{\tau\nu}^{\lambda}$$

where $K_{\lambda/\nu,\mu}$ is the number of skew tableaux of shape λ/ν and weight μ , $K_{\tau\mu}$ is the number of tableaux of shape τ and weight μ , and $c_{\tau\nu}^{\lambda}$ is the usual Littlewood–Richardson coefficient. When $\mu = (a, b)$, this gives the formula

$$c_{(a,b)\nu}^{\lambda} = K_{\lambda/\nu,(a,b)} - K_{\lambda/\nu,(a+1,b-1)}.$$

We can see that in the case of $\mu = (a, b)$ the k -Schur Littlewood–Richardson coefficient has a formula that is nearly identical to the classical case with the exception that we use k -Kostka numbers instead of the classical Kostka numbers. This leads to the question of whether we can just generalize the formulas from the classical case for partitions μ which are not of the form $\mu = (a, b)$ to the k -Schur case by just adding in a k everywhere. The

answer unfortunately appears to be no, we cannot. We will illustrate the fact that it does not always generalize with the following example.

Example 5.6. Let $\mu = (4, 2, 1)$ and $k = 5$. In order to compute the k -Littlewood–Richardson coefficient $c_{\tau(4,2,1)}^{\lambda,5}$ we need to use Proposition 5.2; Theorem 5.3 does not apply here since $l(\mu) > 2$. The only possibilities for k -bounded $\tau \supseteq \mu$ are $\tau = (4, 2, 1), (4, 3), (5, 1, 1)$, or $(5, 2)$. We do not need to consider τ that are not k -bounded, since $K_{\tau\mu}^{(k)} = 0$ when τ is not k -bounded, since there are no k -tableaux of shape $\mathbf{c}(\tau)$ for non- k -bounded τ .

We must first compute the k -Kostka numbers $K_{\tau,(4,2,1)}^{(5)}$ which are the number of k -tableaux of shape $\mathbf{c}(\tau)$ and k -weight $(4, 2, 1)$ for $k = 5$. We have illustrated all of these k -tableaux for $\tau \supseteq (4, 2, 1)$ in the following table.

$\mu = (4, 2, 1)$ $K_{(4,2,1)(4,2,1)}^{(5)} = 1$ <table style="margin: auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">3₄</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">2₅</td><td style="border: 1px solid black; padding: 2px;">2₀</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1₀</td><td style="border: 1px solid black; padding: 2px;">1₁</td><td style="border: 1px solid black; padding: 2px;">1₂</td><td style="border: 1px solid black; padding: 2px;">1₃</td></tr> </table>	3 ₄	2 ₅	2 ₀	1 ₀	1 ₁	1 ₂	1 ₃	$\mu = (4, 3)$ $K_{(4,3)(4,2,1)}^{(5)} = 1$ <table style="margin: auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">2₅</td><td style="border: 1px solid black; padding: 2px;">2₀</td><td style="border: 1px solid black; padding: 2px;">3₁</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1₀</td><td style="border: 1px solid black; padding: 2px;">1₁</td><td style="border: 1px solid black; padding: 2px;">1₂</td><td style="border: 1px solid black; padding: 2px;">1₃</td></tr> </table>	2 ₅	2 ₀	3 ₁	1 ₀	1 ₁	1 ₂	1 ₃			
3 ₄																		
2 ₅	2 ₀																	
1 ₀	1 ₁	1 ₂	1 ₃															
2 ₅	2 ₀	3 ₁																
1 ₀	1 ₁	1 ₂	1 ₃															
$\mu = (5, 1, 1)$ $K_{(5,1,1)(4,2,1)}^{(5)} = 1$ <table style="margin: auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">3₄</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">2₅</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1₀</td><td style="border: 1px solid black; padding: 2px;">1₁</td><td style="border: 1px solid black; padding: 2px;">1₂</td><td style="border: 1px solid black; padding: 2px;">1₃</td><td style="border: 1px solid black; padding: 2px;">2₄</td><td style="border: 1px solid black; padding: 2px;">2₅</td></tr> </table>	3 ₄	2 ₅	1 ₀	1 ₁	1 ₂	1 ₃	2 ₄	2 ₅	$\mu = (5, 2)$ $K_{(5,2)(4,2,1)}^{(5)} = 1$ <table style="margin: auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">2₅</td><td style="border: 1px solid black; padding: 2px;">3₀</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1₀</td><td style="border: 1px solid black; padding: 2px;">1₁</td><td style="border: 1px solid black; padding: 2px;">1₂</td><td style="border: 1px solid black; padding: 2px;">1₃</td><td style="border: 1px solid black; padding: 2px;">2₄</td><td style="border: 1px solid black; padding: 2px;">2₅</td><td style="border: 1px solid black; padding: 2px;">3₀</td></tr> </table>	2 ₅	3 ₀	1 ₀	1 ₁	1 ₂	1 ₃	2 ₄	2 ₅	3 ₀
3 ₄																		
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1 ₀	1 ₁	1 ₂	1 ₃	2 ₄	2 ₅													
2 ₅	3 ₀																	
1 ₀	1 ₁	1 ₂	1 ₃	2 ₄	2 ₅	3 ₀												

Table 5.1: All of the k -tableaux of shape $\mathbf{c}(\tau)$ with k -content $\lambda = (4, 2, 1)$ for $k = 5$ and the k -Kostka numbers $K_{\tau(4,2,1)}^{(k)}$ for all $\tau \supseteq (4, 2, 1)$.

We can see that the k -Kostka number $K_{\tau,(4,2,1)}^{(k)} = 1$ for all k -bounded $\tau \supseteq (4, 2, 1)$. Plugging this into Equation (5.1), we get the formula

$$c_{(4,2,1)\nu}^{\lambda,5} = K_{\lambda/\nu,(4,2,1)}^{(5)} - \sum_{\tau \triangleright (4,2,1)} c_{\tau\nu}^{\lambda,5}. \tag{5.6}$$

We must compute the k -Littlewood–Richardson coefficients for $\mu \triangleright (4, 2, 1)$. For $\mu = (5, 2)$ and $\mu = (4, 3)$ we can use the formula from Proposition 5.2, and we get

$$c_{(5,2)\nu}^{\lambda,5} = K_{\lambda/\nu,(5,2)}^{(5)}.$$

$$c_{(4,3)\nu}^{\lambda,5} = K_{\lambda/\nu,(4,3)}^{(5)} - K_{\lambda/\nu,(5,2)}^{(5)}.$$

For $\mu = (5, 1, 1)$, the calculation takes a bit more effort, but we can compute it using the same strategy we are using for this example.

$$c_{(5,1,1)\nu}^{\lambda,5} = K_{\lambda/\nu,(5,1,1)}^{(5)} - K_{\lambda/\nu,(5,2)}^{(5)}.$$

Next, we substitute these into Equation (5.6).

$$c_{(4,2,1)\nu}^{\lambda,5} = K_{\lambda/\nu,(4,2,1)}^{(5)} - (K_{\lambda/\nu,(4,3)}^{(5)} - K_{\lambda/\nu,(5,2)}^{(5)}) - (K_{\lambda/\nu,(5,1,1)}^{(5)} - K_{\lambda/\nu,(5,2)}^{(5)}) - K_{\lambda/\nu,(5,2)}^{(5)}.$$

$$c_{(4,2,1)\nu}^{\lambda,5} = K_{\lambda/\nu,(4,2,1)}^{(5)} - K_{\lambda/\nu,(4,3)}^{(5)} + K_{\lambda/\nu,(5,2)}^{(5)} - K_{\lambda/\nu,(5,1,1)}^{(5)}.$$

We just computed the k -Littlewood–Richardson coefficient $c_{(4,2,1)\nu}^{\lambda,5}$, where our partition $(4, 2, 1)$ is not of the form $\mu = (a, b)$. If we use the analogous strategy to compute the classical Littlewood–Richardson coefficient $c_{(4,2,1),\nu}^{\lambda}$, we get

$$c_{(4,2,1)\nu}^{\lambda} = K_{\lambda/\nu,(4,2,1)} - K_{\lambda/\nu,(4,3)} + K_{\lambda/\nu,(6,1)} - K_{\lambda/\nu,(5,1,1)}.$$

We can see from this, that while the formulas are similar, it does not appear that it is as simple as just turning the Kostka numbers to k -Kostka numbers to go from the classical Littlewood–Richardson coefficients to the k -Littlewood–Richardson coefficients, since the classical case contains the term $K_{\tau/\nu,(6,1)}$, while the $k = 5$ case contains the term $K_{\tau/\nu,(5,2)}^{(5)}$. So while we have nice a nice formula that generalizes the classical case for $\mu = (a, b)$ it does not appear that there is an analogous generalization for other μ .

Chapter 6

A strategy for computing k -Littlewood–Richardson coefficients

The purpose of this chapter is to compute some examples and discuss how one may go about finding k -Littlewood–Richardson coefficients. In practice, we may want to find all of the k -Littlewood–Richardson coefficients $c_{\mu\nu}^{\lambda,k}$ for a given μ , ν , and k , where λ can be any k -bounded partition. We first give an example using this strategy.

Example 6.1. Let $\mu = (2, 1)$, $\nu = (3, 2)$ and $k = 4$. We will construct k -tableaux of shape $\mathfrak{c}(\lambda)/\mathfrak{c}(\nu)$ and k -weight $\mu = (2, 1)$ (so 2 residues filled with 1's, and 1 residue filled with a 2). We begin by first drawing $\mathfrak{c}(\nu) = (3, 2)$ taking note of the residues of empty cells where we can start adding numbers. We first will fill 2 residues with 1's, and then after that fill another residue with a 2.

3	4		
		1	
			3 4

First we need to fill two residues with 1's, and our options are either to use the residues 3 and 4, or else 1 and 3. We first consider the case of placing 1's in boxes with residues 3 and 4. We then get the following k -tableau:

	2					
1 ₃	1 ₄					
		1				
			1 ₃	1 ₄		0

There are three residues for boxes in which we can place a 2, so we get the following three k -tableaux with weight $(2, 1)$.

$$T_1 = \begin{array}{|c|c|c|c|c|} \hline & 2_2 & & & \\ \hline 1_3 & 1_4 & & & \\ \hline & & & & \\ \hline & & & 1_3 & 1_4 \\ \hline \end{array}
 \quad
 T_2 = \begin{array}{|c|c|c|c|c|} \hline 1_3 & 1_4 & & & \\ \hline & & 2_1 & & \\ \hline & & & 1_3 & 1_4 \\ \hline \end{array}
 \quad
 T_3 = \begin{array}{|c|c|c|c|c|} \hline 1_3 & 1_4 & & & \\ \hline & & & & \\ \hline & & & 1_3 & 1_4 & 2_0 \\ \hline \end{array}$$

We could have, on the other hand, placed the 1's in boxes of residues 1 and 3. In this case, we would have the following k -tableau:

$$\begin{array}{|c|c|c|c|c|} \hline & 2 & & & \\ \hline 1_3 & 4 & & & \\ \hline & & 1_1 & 2 & \\ \hline & & & 1_3 & 4 \\ \hline \end{array}$$

There are now two residues for boxes in which we can place our one 2, either residue 2 or residue 4. Thus we get the following two k -tableaux with k -weight $(2, 1)$:

$$T_4 = \begin{array}{|c|c|c|c|} \hline 2_2 & & & \\ \hline 1_3 & & & \\ \hline & 1_1 & 2_2 & \\ \hline & & & 1_3 \\ \hline \end{array}
 \quad
 T_5 = \begin{array}{|c|c|c|c|} \hline 1_3 & 2_4 & & \\ \hline & & 1_1 & \\ \hline & & & 1_3 & 2_4 \\ \hline \end{array}$$

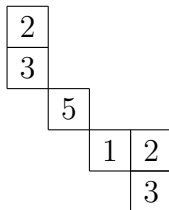
So there are five 4-tableaux of shape $\mathbf{c}(\lambda)/\mathbf{c}(3, 2)$ and 4-weight $(2, 1)$ where λ can be any 4-bounded partition. Of these, $T_1, T_2,$ and T_4 (of shapes $\mathbf{c}(3, 2, 2, 1)/\mathbf{c}(3, 2), \mathbf{c}(3, 3, 2)/\mathbf{c}(3, 2),$ and $\mathbf{c}(3, 3, 1, 1)/\mathbf{c}(3, 2),$ respectively) are k -lattice, and T_3 and T_5 are not. Therefore we get the following for the k -Littlewood–Richardson coefficients in this case.

$$c_{(2,1)(3,2)}^{\lambda,4} = 1 \text{ for } \lambda = (3, 2, 2, 1), (3, 3, 2), \text{ or } (3, 3, 1, 1).$$

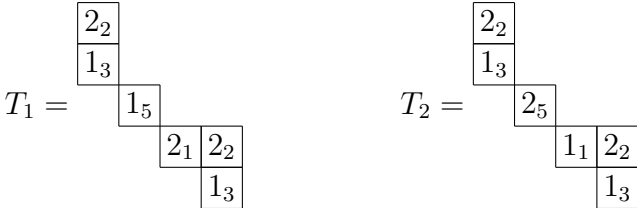
$$c_{(2,1)(3,2)}^{\lambda,4} = 0 \text{ for all other } \lambda.$$

Our other strategy for computing k -Littlewood–Richardson coefficients is if we have a specific λ that we would like to compute it for. In this case, we first find the shape $\mathbf{c}(\lambda)/\mathbf{c}(\nu)$ and the residues of each box in the shape, and then determine how we can fill in the boxes with k -weight μ .

Example 6.2. Let $k = 5, \mu = (2, 2), \nu = (3, 2, 1),$ and $\lambda = (3, 3, 2, 1, 1).$ Then if we want to compute $c_{(2,2)(3,2,1)}^{(3,3,2,1,1),5},$ we first determine the shape of $\mathbf{c}(3, 3, 2, 1, 1)/\mathbf{c}(3, 2, 1) = (4, 4, 2, 1, 1)/(3, 2, 1)$ and the $k + 1$ -residues of each box in the skew diagram.



It turns out that with these residues there are exactly two ways to label 2 residues with the number 2, and 2 residues with the number 1.



Of these, T_1 is not k -lattice, since the fundamental 1/2-unpaired subword of T_1 is $1_5 2_1$, but T_2 is k -lattice, since the fundamental 1/2-unpaired subword of T_2 is \emptyset (so there are no 1/2-unpaired 2's, meaning it is k -lattice). Therefore, we see that $c_{(2,2)(3,2,1)}^{(3,3,2,1,1),5} = 1$, since there is exactly one k -lattice k -tableaux of shape $\mathbf{c}(3, 3, 2, 1, 1)/\mathbf{c}(3, 2, 1)$, with k -weight $\mu = (2, 2)$ for $k = 5$.

Chapter 7

The scalar product of a dual k -Schur function and a Schur function

We will be considering the scalar product of a Schur function with a certain symmetric function related to k -Schur functions, the dual skew k -Schur function. The dual skew k -Schur function $\mathfrak{S}_{\nu/\mu}^{(k)}$ was defined by Lapointe and Morse [5]. We start off by reviewing their definition. Note that $\mathfrak{S}_{\lambda/\nu}^{(k)}$ is a symmetric function.

Definition 7.1 (Lapointe, Morse [5]). The dual skew k -Schur function $\mathfrak{S}_{\lambda/\nu}^{(k)}$ is defined as

$$\mathfrak{S}_{\lambda/\nu}^{(k)} = \sum_T x^{k\text{-weight}(T)}$$

where the sum is over all skew k -tableaux of shape $\mathfrak{c}(\lambda)/\mathfrak{c}(\nu)$.

We first recall the usual scalar product on symmetric functions $\langle f, g \rangle$ where f and g are symmetric functions, and some facts about it, which can be found in [10], and [12]. Recall that Schur functions form an orthonormal basis of the space of symmetric functions, meaning that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, where $\delta_{\mu\mu} = 1$ and $\delta_{\lambda\mu} = 0$ for $\lambda \neq \mu$. It is also worth noting that Lapointe and Morse showed that $\langle \mathfrak{S}_{\lambda/\nu}^{(k)}, s_\mu^{(k)} \rangle = c_{\mu\nu}^{\lambda,k}$ in [5].

In this chapter though, we will be concerned with taking the scalar product of a dual k -Schur function and a classical Schur function. We will give an example of a dual k -Schur function $\mathfrak{S}_{\lambda/\nu}^{(k)}$, that when we take the inner product of it with certain Schur functions, we are able to get negative integer results. The reason that this example is of interest to us is in relation to whether or not there exist crystal operators on k -tableaux. If there were crystal operators, then the dual k -Schur functions would be Schur-positive, meaning $\langle \mathfrak{S}_{\lambda/\nu}^{(k)}, s_\mu^{(k)} \rangle \geq 0$ for all k -bounded λ and ν , and all partitions μ [1]. But in this chapter we present an example that proves the dual k -Schur functions are in fact not Schur-positive.

Let $\lambda = (3, 2, 2, 1)$, $\nu = (1, 1)$ and $k = 3$. We will be considering the dual k -Schur function $\mathfrak{S}_{\lambda/\nu}^{(k)} = \mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}$. It is not too difficult of a computation to find all skew k -tableaux of shape $\mathfrak{c}(3, 2, 2, 1)/\mathfrak{c}(1, 1) = (6, 3, 2, 1)/(1, 1)$ and k -weight μ for all partitions μ . First of all, for there to be a k -tableaux of that shape and k -weight μ , the length of μ must be at least 3, and μ must also be a partition of 6. So the only possibilities we must consider for μ are $(3, 2, 1)$, $(3, 1, 1)$, $(2, 2, 2)$, $(2, 2, 1, 1)$, $(2, 1, 1, 1, 1)$, and $(1, 1, 1, 1, 1, 1)$. We have listed all of the possibilities for those partitions which can be seen in Table 7.2 which can be found at the end of this chapter. To compute $\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)} = \sum_T x^{k\text{-weight}(T)}$ it is enough that we have computed all of the skew k -tableaux where the weight is a partition μ , as these will correspond to the monomial symmetric functions m_μ that show up in the expansion for $\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}$. More specifically, the number of skew k -tableaux of k -weight μ will be the coefficient of m_μ in the expansion. So we have the following:

$$\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)} = m_{(3,2,1)} + 2m_{(3,1,1,1)} + m_{(2,2,2)} + 2m_{(2,2,1,1)} + 3m_{(2,1,1,1,1)} + 4m_{(1,1,1,1,1,1)}.$$

It turns out that this expansion in terms of Schur functions is the following:

$$\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)} = s_{(3,2,1)} - s_{(2,2,2)} - s_{(2,2,1,1)} + 2s_{(1,1,1,1,1,1)}.$$

This expansion of $\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}$ in terms of Schur functions demonstrates that the dual k -Schur functions are not Schur-positive, since the coefficients of $s_{(2,2,2)}$ and $s_{(2,2,1,1)}$ are negative. In particular, we also have the following values of the scalar product between $\mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}$ and s_μ illustrated in Table 7.1.

μ	$\langle \mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}, s_\mu \rangle$
$(3, 2, 1)$	1
$(2, 2, 2)$	-1
$(2, 2, 1, 1)$	-1
$(1, 1, 1, 1, 1, 1)$	2
All other partitions μ	0

Table 7.1: Table of values of the scalar product $\langle \mathfrak{S}_{(3,2,2,1)/(1,1)}^{(3)}, s_\mu \rangle$ for different partitions μ . Notice that the scalar product can be positive, negative, or zero.

$\mu = (3, 2, 1)$			
$\begin{array}{c} 2_1 \\ 1_2 \ 3_3 \\ \quad 2_0 \ 2_1 \\ \quad\quad 1_1 \ 1_2 \ 1_3 \ 2_0 \ 2_1 \end{array}$			
$\mu = (3, 1, 1, 1)$			
$\begin{array}{c} 4_1 \\ 1_2 \ 3_3 \\ \quad 2_0 \ 4_1 \\ \quad\quad 1_1 \ 1_2 \ 1_3 \ 2_0 \ 4_1 \end{array}$	$\begin{array}{c} 3_1 \\ 1_2 \ 4_3 \\ \quad 2_0 \ 3_1 \\ \quad\quad 1_1 \ 1_2 \ 1_3 \ 2_0 \ 3_1 \end{array}$		
$\mu = (2, 2, 2)$			
$\begin{array}{c} 3_1 \\ 1_2 \ 3_3 \\ \quad 2_0 \ 3_1 \\ \quad\quad 1_1 \ 1_2 \ 2_3 \ 2_0 \ 3_1 \end{array}$			
$\mu = (2, 2, 1, 1)$			
$\begin{array}{c} 4_1 \\ 1_2 \ 3_3 \\ \quad 2_0 \ 4_1 \\ \quad\quad 1_1 \ 1_2 \ 2_3 \ 2_0 \ 4_1 \end{array}$	$\begin{array}{c} 3_1 \\ 1_2 \ 4_3 \\ \quad 2_0 \ 3_1 \\ \quad\quad 1_1 \ 1_2 \ 2_3 \ 2_0 \ 3_1 \end{array}$		
$\mu = (2, 1, 1, 1, 1)$			
$\begin{array}{c} 5_1 \\ 1_2 \ 4_3 \\ \quad 3_0 \ 5_1 \\ \quad\quad 1_1 \ 1_2 \ 2_3 \ 2_0 \ 5_1 \end{array}$	$\begin{array}{c} 4_1 \\ 1_2 \ 5_3 \\ \quad 3_0 \ 4_1 \\ \quad\quad 1_1 \ 1_2 \ 2_3 \ 3_0 \ 4_1 \end{array}$	$\begin{array}{c} 5_1 \\ 1_2 \ 3_3 \\ \quad 2_0 \ 5_1 \\ \quad\quad 1_1 \ 1_2 \ 3_3 \ 4_0 \ 5_1 \end{array}$	
$\mu = (1, 1, 1, 1, 1, 1)$			
$\begin{array}{c} 5_1 \\ 2_2 \ 6_3 \\ \quad 4_0 \ 5_1 \\ \quad\quad 1_1 \ 2_2 \ 3_3 \ 4_0 \ 5_1 \end{array}$	$\begin{array}{c} 6_1 \\ 2_2 \ 5_3 \\ \quad 4_0 \ 6_1 \\ \quad\quad 1_1 \ 2_2 \ 3_3 \ 4_0 \ 6_1 \end{array}$	$\begin{array}{c} 6_1 \\ 2_2 \ 4_3 \\ \quad 3_0 \ 6_1 \\ \quad\quad 1_1 \ 2_2 \ 4_3 \ 5_0 \ 6_1 \end{array}$	$\begin{array}{c} 6_1 \\ 3_2 \ 4_3 \\ \quad 2_0 \ 6_1 \\ \quad\quad 1_1 \ 3_2 \ 4_3 \ 5_0 \ 6_1 \end{array}$

Table 7.2: All possible k -tableaux of shape $\mathfrak{c}(3, 2, 2, 1)/\mathfrak{c}(1, 1)$ for $k = 3$ with k -weight μ where μ is a partition.

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