WHEN DOES MAJORITY RULE SUPPLY PUBLIC GOODS EFFICIENTLY?*

Ted C. Bergstrom

University of Michigan, Ann Arbor, Michigan, USA

Abstract

H. R. Bowen showed that majority voting leads to a Pareto efficient supply of a single public good if all voters have equal tax shares and if marginal rates of substitution for the public good are symmetrically distributed in the voting population. In general however, even if preferences are identical and tax shares equal, majority voting is not Pareto efficient if income is asymmetrically distributed. Here we formalize and generalize Bowen's theorem. In the process we propose a new idea of a public goods allocation system, a "pseudo-Lindahl equilibrium". Though it is Pareto efficient for an interesting class of societies, the informational requirements for implementing pseudo-Lindahl equilibrium are considerably less stringent than those for a true Lindahl equilibrium.

I. Introduction

From Wicksell (1896), who argued for approximate unanimity, to Arrow (1951) who showed the impossibility of a thoroughly satisfactory democratic decision mechanism, one finds little support in the literature of public finance for majority rule as an efficient means of determining supplies of public goods. A notable exception is Howard R. Bowen's *The Interpretation of Voting in the Allocation of Resources* (1943). Bowen shows that if there is a single public good, if the marginal rates of substitution for that public good are symmetrically distributed in an appropriate way and if taxes are divided equally among the population then majority rule leads to an efficient output of public goods. Although this is, as far as I know, the only theorem in the economic literature which specifies conditions under which majority rule is efficient, it has received little attention.[1] The reason for this neglect is probably that the assumptions of the theorem rarely are even approximately met. For example, in most

* The idea for this article was conceived while the author was a National Fellow at the Hoover Institution in Stanford, California. Its development was aided by useful conversations with Arthur Denzau of the University of Arizona and Robert Parks of Washington University, St. Louis, M.

[1] An interesting paper which independently pursues a closely related line of thought is Barlow (1970). See also Bergstrom (1973).

political jurisdictions, wealth is not symmetrically distributed and in consequence the symmetry which Bowen's theorem demands is not likely to be realized. Furthermore, most economists would argue that where incomes differ, some sort of income tax is more "equitable" than the "head tax" which is considered in Bowen's theorem. In this paper we show that for an interesting class of economies, Bowen's ideas can be extended to demonstrate that majority voting together with an appropriate tax system leads to Pareto efficient provision of the public good. As we will show, these ideas are closely related to Lindahl's (1919) solution to the public good problem, but suggest a notion of equilibrium that is somewhat closer to being practically implementable than Lindahl equilibrium.

II. Bowen's Model

The model presented here is essentially that of Bowen. We assume that there are n individuals, a single public good, and a single private good. Preferences of each individual i are represented by differentiable strictly quasi-concave utility function $U^i(X_i, Y)$ where X_i is i's consumption of private good and Y is the amount of public good produced. Individual i has an initial wealth of W_i units of private good. Public good can be produced at a constant unit cost of c units of private good. Letting the private good be the numeraire, suppose that individual i is taxed to pay the fraction $t_i \geqslant 0$ of the total cost of the public good. Thus if Y units of public good are produced, he will have the amount $X_i = W_i - ct_i Y$ of wealth left for private consumption. Conditional on this system of taxation, his preferences over amounts of public goods are represented by the induced utility function, $\tilde{U}^i(Y) = U^i(W_i - ct_i Y, Y)$. If U^i is strictly quasi-concave in X_i and Y, then $\tilde{U}^i$ is strictly quasi-concave in Y. A strictly quasi-concave function of a single variable is single-peaked in the sense of Duncan Black (1958). Let Y_i^* be the (unique because of single peakedness) value of Y which maximizes $\tilde{U}^i(Y)$ on the interval $[0, (1/c)\sum_i W_i]$ of feasible outputs of Y. Let $\hat{Y}^*$ be the median of the Y_i^*'s. Then it follows from Black's results on single peaked preferences that $\hat{Y}^*$ is the unique amount of public good supply that is stable under pairwise majority voting. Thus an interesting case can be made for $\hat{Y}^*$ as the natural outcome of many majoritarian public decision processes.

We call the allocation of resources that results from such a process a *Bowen equilibrium*. Thus for the model under consideration, we define a Bowen equilibrium as follows.

Definition

Let $t_i \geqslant 0$ be the tax share and $W_i > 0$ be the wealth of individual i and let $\sum_i t_i = 1$. Let Y_i^* maximize $U_i(W_i - t_i cY, Y)$ and let

$$\hat{Y}^* = \underset{i \in \{1, \ldots, n\}}{\text{median}}\ Y_i^*$$

and $X_i^{**} = W_i - t_i c \hat{Y}^*$. Then the allocation $(X_1^{**}, ..., X_n^{**}, \hat{Y}^*)$ is the *Bowen equilibrium* corresponding to the tax share distribution $(t_1, ..., t_n)$.

Bowen was able to assert the following quite remarkable result:

Theorem 1 (Bowen). *In the model described above,*

$$\text{if } \underset{i \in \{1, ..., n\}}{\text{median}} \text{ M.R.S.}_i(\hat{Y}^*) = \frac{1}{n} \sum_{i=1}^{n} \text{M.R.S.}_i(\hat{Y}^*),$$

then a Bowen equilibrium in which $t_i = 1/n$ for each i is Pareto optimal.

Proof. Strict quasi-concavity of U^i implies that M.R.S.$_i(Y)$ is a monotone decreasing function of Y. Therefore, where $Y_j^* = \hat{Y}^*$ is the median of $Y_1^*, ..., Y_n^*$, it must be that

$$\text{M.R.S.}_j(\hat{Y}^*) = \underset{i \in \{1, ..., n\}}{\text{median}} \text{ M.R.S.}_i(\hat{Y}^*) = \frac{1}{n} \sum_{i=1}^{n} \text{M.R.S.}_i(\hat{Y}^*).$$

It must also be that

$$\frac{c}{n} = \text{M.R.S.}_j(\hat{Y}^*) = \frac{1}{n} \sum_{i=1}^{n} \text{M.R.S.}_i(\hat{Y}^*).$$

Therefore $c = \sum_{i=1}^{n} \text{M.R.S.}_i(\hat{Y}^*)$. But this is just the "Samuelson necessary condition" (Samuelson, 1954)) for efficient supply of public goods. As we demonstrate in the appendix, when U^i is quasi-concave, the Samuelson conditions are sufficient as well as necessary for Pareto optimality. It follows that the allocation $(X_1^{**}, ..., X_n^{**}, \hat{Y}^*)$ is Pareto optimal where $X_i^{**} = W_i - (c/n)\hat{Y}^*$ for each i. Q.E.D.

The novel assumption for the Bowen theorem is, of course, the assumption that the mean of the marginal rates of substitution at $\hat{Y}^*$ is equal to the median of these rates. The most natural way of establishing such a condition appears to be to assume that the distribution among individuals of marginal rates of substitution given the amount of public goods and the tax system is symmetric. We illustrate by examples when this is more or less likely to be the case.

Example 1: The "transferable utility" case. For each i let $U_i(X_i, Y) = X_i + a_i f(Y)$ where $f'(Y) > 0$ and $f''(Y) < 0$ for all Y. Then if $t_i = 1/n$ for each i, it must be that $c/n = a_i f'(Y_i^*)$. Therefore $Y_i^* = f'^{-1}(c/na_i)$. Since $f''(Y) < 0$ for all Y, the inverse function, $f'^{-1}(\cdot)$ is monotone decreasing. It follows that Y_i^* is an increasing function of a_i and that $\hat{Y}^* = f'^{-1}(c/n\hat{a})$ where $\hat{a} = \text{median } \{a_1, ..., a_n\}$. Also, for any i, M.R.S.$_i(\hat{Y}^*) = a_i f'(\hat{Y}^*)$. It follows that

$$\underset{i \in \{1, ..., n\}}{\text{median}} \text{ M.R.S.}_i(\hat{Y}^*) = \hat{a} f'(\hat{Y}^*)$$

and $1/n \sum_i \text{M.R.S.}_i(\hat{Y}^*) = \bar{a} f'(\hat{Y}^*)$ where $\bar{a} = 1/n \sum_{i=1}^n a$. Thus if the a_i's are symmetrically distributed so that $\hat{a} = \bar{a}$, then the condition for the Bowen theorem is satisfied.

Example 2. Log linear utility and identical wealth. The previous example had the peculiar feature that the "income elasticity of demand" for the public good is zero. Thus Y_i^* is independent of W_i. Here we consider an example in which this is not the case. For each i, let $U_i = \ln X_i + a_i \ln Y$. If $t_i = 1/n$ for each i, then a bit of computation shows that $Y_i^* = (a_i/(1+a_i))(c/n) W_i$. If $W_i = \overline{W}$ for all i, then $\hat{Y}^* = \hat{a}/(1+\hat{a})(c/n)\overline{W}$ where

$$\hat{a} = \underset{i \in \{1, \ldots, n\}}{\text{median}}\ a_i.$$

Then $\text{M.R.S.}_i(\hat{Y}^*) = a_i(X_i^*/\hat{Y}^*) = a_i(\overline{W} - n\hat{Y}^*/\hat{Y}^*) = (a_i/\hat{a})c/n$. Thus

$$\underset{i \in \{1, \ldots, n\}}{\text{median}}\ \text{M.R.S.}_i(\hat{Y}^*) = \frac{c}{n}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \text{M.R.S.}(\hat{Y}^*) = \frac{\bar{a}}{\hat{a}} \left(\frac{c}{n}\right)$$

where

$$\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i.$$

Therefore if all individuals have the same wealth, and if the parameters a are symmetrically distributed, Bowen's theorem again applies.

Example 3. Identical log linear utility functions and different wealths. Now let us consider the case where preferences are as in example 2 but the a_i's are all the same, $a_i = \bar{a}$ for all i, and the W_i's differ. Then

$$\hat{Y}_i^* = \frac{1+\bar{a}}{\bar{a}} \left(\frac{n}{c}\right) W_i$$

and

$$\hat{Y}^* = \frac{\bar{a}}{1+\bar{a}} \left(\frac{n}{c}\right) \hat{W}$$

where

$$\hat{W} = \underset{i \in \{1, \ldots, n\}}{\text{median}}\ W_i.$$

Then

$$\text{M.R.S.}_i(\hat{Y}^*) = \bar{a} \frac{X_i^{**}}{\hat{Y}^*} = \bar{a} \left(\frac{W_i - (c/n)\hat{Y}^*}{\hat{Y}^*}\right).$$

Then computation shows that

$$\underset{i\in\{1,\ldots,n\}}{\text{median}}\ \text{M.R.S.}_i(\hat{Y}^*)=\frac{c}{n}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\text{M.R.S.}_i(\hat{Y}^*)=\left[(1+\bar{a})\frac{\overline{W}}{\hat{W}}-\bar{a}\right]\left(\frac{c}{n}\right).$$

Thus the Bowen condition will be satisfied if and only if $\overline{W}=\hat{W}$, that is, median wealth equals mean wealth. If mean wealth exceeds median wealth, then

$$\frac{1}{n}\sum_{i=1}^{n}\text{M.R.S.}_i(\hat{Y}^*)>\frac{c}{n}\ \text{so that}\ \sum_{i=1}^{n}\text{M.R.S.}_i(\hat{Y}^*)>c$$

It follows from a simple application of the calculus that if this is the case, $\hat{Y}^*$ is "too small" in the sense that it would be possible to collect revenue for an increased amount of public goods in such a way that everyone's utility would be increased.

III. A Median Voter Model with a Proportional Wealth Tax and Log Linear Utilities

Since there is evidence that in most political jurisdictions, mean income exceeds median income,[1] the result of example 3 suggests that if utility is approximately log linear, financing public goods by a "head tax" would result in a Bowen equilibrium with too little public goods. Thus we might consider a Bowen equilibrium where tax shares are positively related to wealth. As it turns out, if the distribution of "tastes" in the economy is appropriately symmetric and uncorrelated with wealth, we can show that a Bowen equilibrium with a proportional wealth tax is Pareto optimal.

Let there be n individuals. Individual i has preferences represented by $U^i(X_i, Y)=\ln X_i+a_i \ln Y$ and his initial wealth is $W_i>0$. The public good is produced at constant unit cost c. Tax rates are proportional to wealth so that $t_i=W_i/\sum W_j$ is i's tax share. Where $\text{M.R.S.}_i(Y)$ is defined as

$$\frac{U_y(W_i-t_iY, Y)}{U_x(W_i-t_iY, Y)},$$

individual i determines his favorite amount Y_i^* of public good by solving the equation $\text{M.R.S.}_i(Y)=t_ic$. Solving this equation yields

$$Y_i^*=\left(\frac{a_i}{1+a_i}\right)\frac{\sum_j W_j}{c}.$$

[1] See Bergstrom (1973).

Thus differences in Y_i^* between individuals are due only to differences in a_i and not to differences in W_i. Furthermore, the larger is a_i, the larger is Y_i^*. Therefore where $\hat{Y}^*$ is the the median of the Y_i^*'s and $\hat{a}$ is the median of the a_i's,

$$\hat{Y}^* = \frac{\hat{a}}{1+\hat{a}} \frac{\sum_j W_j}{c}.$$

Then

$$\text{M.R.S.}_i(\hat{Y}^*) = a_i \frac{W_i - ct_i\hat{Y}^*}{\hat{Y}^*} = \left(\frac{a_i}{\hat{a}}\right)\left(\frac{cW_i}{\sum_j W_j}\right).$$

Therefore $\sum_i \text{M.R.S.}_i(\hat{Y}^*) = c\sum_i a_i W_i/\hat{a}\sum_i W_i$. If the a_i's are uncorrelated with the W_i's then it must be that $\sum_i a_i W_i = \bar{a}\sum_i W_i$ where $\bar{a}$ is the mean of the a_i's. But then $\sum_i \text{M.R.S.}_i(\hat{Y}^*) = c(\bar{a}/\hat{a})$. Therefore if $\bar{a} = \hat{a}$, it must be that $\sum_i \text{M.R.S.}_i(\hat{Y}^*) = c$. But this is the Samuelson condition for efficiency. Therefore where $X_i^{**} = W_i - t_i c\hat{Y}^* = W_i/(1+\hat{a})$, the Bowen equilibrium allocation $(X_1^{**}, \ldots, X_n^{**}, \hat{Y}^*)$ is Pareto optimal. The results of this discussion are summarized by Theorem 2.

Theorem 2. *Let there be n individuals where individual i has preferences represented by* $U_i = \ln X_i + a_i \ln Y$ *and wealth* W_i. *Assume that the public good Y is produced at constant unit cost c. Let* $t_i = W_i/\sum_i W_i$ *for each i. If the* a_i*'s are symmetrically distributed and uncorrelated with the* W_i*'s, then the Bowen equilibrium is Pareto optimal.*

IV. Lindahl Equilibrium, Pseudo–Lindahl Equilibrium and Bowen Equilibrium

Here we seek to extend results of the previous section to find an equally satisfactory resolution for a more general class of preferences. In particular, we would like to find a practical way of assigning tax shares so that the corresponding Bowen equilibrium is Pareto efficient. For this purpose it is useful to consider the Lindahl theory of public expenditure determination. For a simple model of the kind discussed above, a Lindahl equilibrium is defined as follows.

Definition. A *Lindahl equilibrium* is a vector of tax shares $(t_1^*, \ldots, t_n^*) \geqslant 0$ such that $\sum_i t_i^* = 1$ and an allocation vector $(X_1^*, \ldots, X_n^*, Y^*)$ such that for each i, (X_i^*, Y^*) maximizes $U^i(X_i, Y)$ subject to $X_i + t_i cY \leqslant W$. Bergstrom (1973), Foley (1970), and others have shown that Lindahl equilibrium exists and is Pareto optimal for a rich class of models. Lemmas 1 and 2 state these results for the simple model studied here.

Lemma 1. *If utility functions are quasi-concave and continuous, and* $W_i > 0$ *for each i, then there exists a Lindahl equilibrium.*

Lemma 2. *If preferences are locally non-satiated, a Lindahl equilibrium is Pareto optimal.*

If tax shares are all set at their Lindahl equilibrium levels, then there is unanimous agreement about the appropriate amount of public goods. Therefore the Lindahl equilibrium quantity of public goods is also the Bowen equilibrium corresponding to Lindahl tax shares. If the conditions of Lemmas 1 and 2 are satisfied, then Lindahl equilibrium exists and is Pareto optimal. Therefore when these conditions are satisfied, there exist assignments of tax shares such that the corresponding Bowen equilibrium is Pareto optimal.

As has often been observed in discussions of the "free rider problem", computation of a true Lindahl equilibrium for a community would require a detailed knowledge of individual preferences which not only would swamp the data processing (and equilibrium computing) capabilities of any government, but would also, in general, require the individuals to reveal accurate information about their preferences, even though no mechanism can be devised which would give selfish individuals an incentive to do so; see for example, Gibbard (1973), Groves & Ledyard (1977) and Bergstrom (1976).

Our analysis of the previous section suggests an interesting possibility for resolving this difficulty. There we showed that if preferences are all log linear and appropriately symmetric, then a Bowen equilibrium with proportional wealth taxation is Pareto optimal. It is also true that if all individual preferences were identical and representable by the "average" utility function, $U(X_i, Y) = \ln X_i + \bar{a} \ln Y$, then the Lindahl tax would be a proportional wealth tax and the Lindahl quantity of public goods would be the same as the Bowen quantity $\hat{Y}^*$ found in the previous section. This suggests that more generally we could compute Lindahl equilibrium for a hypothetical community in which preferences are "averaged", ignoring individual eccentricities of tastes that are not easily observable. Under certain circumstances, the Lindahl equilibria so computed are Pareto efficient for the actual community and may also be Bowen equilibria.

Let there be m observable types of individuals. Let n_j be the number of type j individuals. Assume that all individuals of type j have the same wealth, W_j, and that preferences of the ith individual of type j are representable by a utility function $U^i(X_i, Y, a_j^i)$ where a_j^i is a parameter of i's preferences that need not be observable to anyone other than i.

For each j, let $\bar{a}_j = (1/n_j) \sum_i a_j^i$ and consider the hypothetical community in which all type j individuals have the same utility function, $U^j(X_i, Y, \bar{a}_j)$. Then under the assumptions of Lemma 1 there will exist a Lindahl equilibrium for the hypothetical economy and in this equilibrium, tax shares of all individuals of the same type will be the same.

Definition. Let $\bar{t}_j$ be the common Lindahl share of type j's, and $\bar{Y}$ the Lindahl equilibrium quantity of public goods for the hypothetical community

described above. The allocation in which type j's consume $\bar{X}_j = W_j - \bar{t}_j c\bar{Y}$ of private good and everyone enjoys $\bar{Y}$ of the public good will be called a *pseudo-Lindahl equilibrium* for the actual community.

Theorem 3. *Let variations in preferences within types be such that for each j there exists a function $M^j(X_i, Y)$, for which*

$$\frac{U_y^j(X_i, Y, a_j^i)}{U_x^j(X_i, Y, a_j^i)} = a_j^i M^j(X_i, Y).$$

Then a pseudo-Lindahl equilibrium is Pareto optimal. If, in addition, for each type j,

$$\tilde{a}_j = \underset{i \in \{1, \ldots, n_j\}}{\text{median}}\ a_j^i \equiv \hat{a}_j,$$

then a pseudo-Lindahl equilibrium is also a Bowen equilibrium.

Proof. Let $\bar{t}_j$ for $j = 1, \ldots, m$ and $\bar{Y}$ be the pseudo-Lindahl equilibrium tax shares and quantity of public goods and let $\bar{X}_j = W_j - \bar{t}_j c\bar{Y}$. Then $\bar{t}_j c = \tilde{a}_j M_j(\bar{X}_j, \bar{Y})$ for all j. Then

$$c = \sum_{j=1}^{m} n_j \bar{t}_j c = \sum_{j=1}^{m} n_j \tilde{a}_j M_j(\bar{X}_j \bar{Y}) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} a_j^i M_j(\bar{X}_j, \bar{Y}) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \frac{U_y^j(\bar{X}_j, \bar{Y}, a_j^i)}{U_x^j(\bar{X}_j, \bar{Y}, a_j^i)}.$$

Therefore the Samuelson condition is satisfied. It follows that pseudo-equilibrium is Pareto optimal.

If $\tilde{a}_j = \hat{a}_j$ then

$$\bar{Y} = \underset{i \in \{1, \ldots, n_j\}}{\text{median}}\ Y_{j*}^i$$

where Y_{j*}^i is the quantity of public good that individual i of type j would most prefer given that his tax share is $\bar{t}_j$. If $\bar{Y}$ is the median demand for each type, it is also the median over all types. Therefore where tax shares $\bar{t}_j$ are assigned to each member of j, $\bar{Y} = \hat{Y}^*$ where

$$\hat{Y}^* = \underset{\substack{j \in \{1, \ldots, m\} \\ i \in \{1, \ldots, n_j\}}}{\text{median}}\ Y_{j*}^i.$$

It follows that $\bar{Y}$ is a Bowen equilibrium quantity. Q.E.D.

A Lindahl equilibrium, though Pareto efficient, requires unobtainable information to be implemented. A Bowen equilibrium, while practically implementable is, in general, not Pareto efficient. Theorem 3 suggests that in an interesting class of cases, a pseudo-Lindahl equilibrium is Pareto optimal and is a Bowen equilibrium. When the number of individuals of each type is large,

the informational requirements for implementing a pseudo-Lindahl equilibrium appear to be considerably less stringent than the requirements for a full Lindahl equilibrium. For example sampling procedures such as those suggested by Bergstrom (1974), Green & Laffont (1977) or Kurz (1974) could be used.

V. Extension to the Case of Many Public Goods

These results can be extended in a straightforward way to the case of several public goods. We can show that if variations in the marginal rates substitution take a multiplicative form as in Theorem 1, the multidimensional pseudo-Lindahl equilibrium is Pareto optimal. If, also, variations in preferences are symmetric, the pseudo-Lindahl equilibrium is an l dimensional Bowen equilibrium. This latter notion corresponds to the idea of a sophisticated voting equilibrium as defined by Kramer (1972).

These results are sketched more formally as follows. Let there be n consumers, one private good and l public goods. Each public good k is produced at constant unit cost c_k. Individual i has an initial endowment W_i of private good and a utility function $U^i(X_i, Y_1, ..., Y_l)$.

A Lindahl equilibrium consists of tax shares, $(T^*_{11}, ..., T^*_{nl})$ where T^*_{ik} is the share of the cost of the kth public good paid for by i, with $\sum_i T^*_{ik}=1$), and an allocation vector $(X^*_1, ..., X^*_n, Y^*_1, ..., Y^*_l)$ such that for each i, $(X^*_1, Y^*_1, ..., Y^*_l)$ maximizes U^i subject to $X^*_i+\sum_{k=1}^{l} T^*_{ik} Y^*_k = W_i$.

Let there be m types of consumers and let there be n_j consumers of type j. Let all consumers of the same type have the same wealth. Let the marginal rate of substitution of the ith consumer of type j between public good k and the public good $a^i_{jk} M_{jk}(X_i, Y_1, ..., Y_l)$. Consider the hypothetical community where for all j and k, $a^i_{jk}=\bar{a}_{jk}\equiv(1/n_j)\sum_{i=1}^{n_j} a^i_{jk}$. Solve for a Lindahl equilibrium for this hypothetical economy and call it a pseudo-equilibrium for the actual economy. Using the same kind of argument employed in Theorem 3, it can be shown that pseudo-Lindahl equilibrium is Pareto optimal.

Conditional on a specified assignment of tax shares for each public good and for each individual, tax shares t^i_k define an n dimensional Bowen equilibrium to be an allocation $(X^{**}_1, ..., X^{**}_n, Y^*_1, ..., Y^*_l)$ such that if changes in the amounts of public good are voted on one good at a time, no change will receive majority approval. Equivalently, this allocation is a Bowen equilibrium if and only if conditional on the tax shares and the quantities of the other public goods being fixed, each $\hat{Y}^*_k$ is the median of the most preferred values for Y_k.

If the values a^i_{jk} are symmetrically distributed in each group j then it follows that for each population subgroup, the pseudo-Lindahl equilibrium quantity of each public good is the median of the preferred quantities conditional on the Lindahl tax shares and the amounts of the other public goods. Therefore the pseudo-Lindahl equilibrium is also an l dimensional Bowen equilibrium.

This result is encouraging and somewhat surprising, since as Kramer (1973) has pointed out, if there are two or more public goods, even with quasi-concave preferences there are in general likely to be Condorcet cycles in pairwise voting.

Appendix

Theorem 0 (Samuelson)

Let there be n individuals, one private good and m public goods. Let preferences of individual i be represented by a differentiable function, $U^i(X_i, Y_1, ..., Y_l)$ and let the set of feasible allocations be $(X_1, ..., X_n, Y_1, ..., Y_l) \geqslant 0$ such that $\sum_{i=1}^{n} X_i + \sum_{k=1}^{l} c_k Y_k = W$. Let

$$M_k^i(X_i Y_1, ..., Y_l) = \frac{\dfrac{\partial U^i(X_i, Y_1, ..., Y_l)}{\partial Y_k}}{\dfrac{\partial U^i(X_i, Y_1, ..., Y_l)}{\partial X_i}}.$$

A necessary condition for the allocation $(\bar{X}_1, ..., \bar{X}_n, \bar{Y}_1, ..., \bar{Y}_l) > 0$ to be Pareto optimal is $\sum_{i=1}^{n} M_k^i(\bar{X}_i, \bar{Y}_1, ..., \bar{Y}_l) = c_k$ for all k. If U^i is quasi-concave for each i, this condition is also sufficient.

The proof of necessity is familiar (see Samuelson, 1954). Though the sufficient condition is widely believed to suffice, I have never seen a proof in print. The proof requires the following Lemma.

Lemma 0

Let U: $R_+^n \to R$ be a quasi-concave and differentiable and $DU(x^*) \neq 0$ where $DU(x^*)$ is the gradient of U at x^*. Then if $U(x) \geqslant U(x^*)$, $(x - x^*) DU(x^*) \geqslant 0$. If $x^* \gg 0$ and $U(x) > U(x^*)$, then $(x - x^*) DU(x^*) > 0$.

Proof of Lemma 0

Let $U(x) > U(x^*)$, $x(t) = tx + (1-t)x^*$ and $f(t) = U(x(t))$. Quasi-concavity of U implies that f is monotone increasing in t for $0 < t < 1$. It follows from a simple application of calculus that

$$\text{If } U(x) \geqslant U(x^*) \quad \text{then } (x - x^*) DU(x^*) \geqslant 0. \tag{1}$$

We wish to show that (1) holds with strict inequalities. Suppose that $U(x) > U(x^*)$ and $(x - x^*) DU(x^*) \leqslant 0$. Since $x^* \gg 0$ and $x \geqslant 0$, $x(t) \gg 0$ for $0 < t < 1$. Also $(x(t) - x^*) DU(x^*) \leqslant 0$ and $U(x(t)) > U(x^*)$. Continuity of U ensures that there exists a neighborhood, N, of $x(t)$ such that $U(y) > U(x^*)$ for all $y \in N$. Since $DU(x^*) \neq 0$, y can be chosen so that $(y - x) DU(x^*) < 0$ and $U(y) > U(x^*)$. But this contradicts (1). Lemma 0 therefore must be true. Q.E.D.

Proof of Sufficiency

Suppose $(\bar{X}_1, ..., \bar{X}_n, \bar{Y}_1, ..., \bar{Y}_l)$ satisfies the marginal conditions of the theorem and the feasibility equation and suppose $(X_1, ..., X_n, Y_1, ..., Y_l)$ is Pareto superior to $(\bar{X}_1, ..., \bar{X}_n, \bar{Y}_1, ..., \bar{Y}_l)$. The according to Lemma 0,

$$\frac{\partial U_i(\bar{X}_i, \bar{Y}_1, ..., \bar{Y}_l)}{\partial X_i}(X_i - \bar{X}_i) + \sum_k \frac{\partial U_i(\bar{X}_i, \bar{Y}_1, ..., \bar{Y}_l)}{\partial Y_k}(Y_k - \bar{Y}_{kj} \geqslant 0$$

for all i with strict inequality for some i.

Therefore $\sum_i (X_i - \bar{X}_i) + \sum_i \sum_k M_k^i(\bar{X}_i, \bar{Y}_1, ..., \bar{Y}_l)(Y_k - \bar{Y}_k) > 0$. But since $\sum_i M_k^i(\bar{X}_i, \bar{Y}_1, ..., \bar{Y}_l) = c_k$, this implies that $\sum_i (X_i - \bar{X}_i) + \sum_k c_k(Y_k - \bar{Y}_k) > 0$. Since $\sum_i \bar{X}_i + \sum_k c_k \bar{Y}_k = W$, it follows that $\sum_i X_i + \sum_k c_k Y_k > W$. Therefore the Pareto superior allocation $(\bar{X}_1, ..., \bar{X}_n, \bar{Y}_1, ..., \bar{Y}_l)$ is not feasible. It follows that $(\bar{X}_1, ..., \bar{X}_n, \bar{Y}_1, ..., \bar{Y}_l)$ is Pareto optimal. Q.E.D.

References

Arrow, K.: *Social choice and individual values*, Wiley, New York, 1951.

Barlow, R.: Efficiency aspects of local school finance. *J.P.E. 78*, 1028–1040, 1970.

Bergstrom, T.: A note on efficient taxation. *J.P.E. 81*, 1973.

Bergstrom, T.: Collective choice and the Lindahl allocation method. In *Theory and Measurement of Economic Externalities* (ed. S. A. Y. Lin). Academic Press, New York, 1976.

Bergstrom, T.: *On the Free Rider problem.* Mimeo, University of Michigan, Department of Economics.

Bergstrom, T.: Regulation of externalities. *Journal of Public Economics 5*, 131–138, 1976.

Black, D.: *The theory of committees and elections.* Cambridge University Press, Cambridge, 1958.

Bowen, H.: The interpretation of voting in the allocation of economic resources. *Quarterly Journal of Economics 58*, reprinted in *Readings in welfare economics* (ed. K. Arrow and T. Scitovsky), Homewood, 1969.

Foley, D.: Lindahl's solution and the core of an economy with collective goods. *Econometrica 38*, January 1970.

Gibbard, A.: Manipulation of voting schemes; A general result. *Econometrica 41*, 1973.

Green, J. & Laffont, J.-J.: On the revelation of preferences for public goods. *Journal of Public Economics 8*, 79–94, 1977.

Groves, T. & Ledyard, J.: Some limitations of preference revealing mechanisms. *Public Choice 37*, 1977.

Kramer, G.: Sophisticated voting over multidimensional choice spaces. *Journal of Math Soc 2*, 165–180, 1972.

Kramer, G.: On a class of equilibrium conditions for majority rule. *Econometrica 41*, 285–299, 1973.

Kurz, M.: Experimental approach to the determination of demand for public goods. *Journal of Public Economics 3*, 329–348, 1974.

Lindahl, E.: *Die Gerechtigkeit der Besteuerung*, Lund, Part I, Chapter 4, "Positive Lösung", English Translation, "Just taxation a positive solution" appears in *Classics in the theory of public finance* (ed. R. Musgrave and A. Peacock), Macmillan Co., London, 1958.

Samuelson, P. A.: The pure theory of public expenditure. *Review of Economics and Statistics 36*, 387–389, 1954.

Wicksell, K.: Ein neues Prinzip der gerechten Besteuerung, Jena 1896, English translation, "A new principle of just taxation", appears in *Classics in the theory of public finance* (ed. R. Musgrave and A. Peacock), pp. 72–118. Macmillan Co., New York, 1958.