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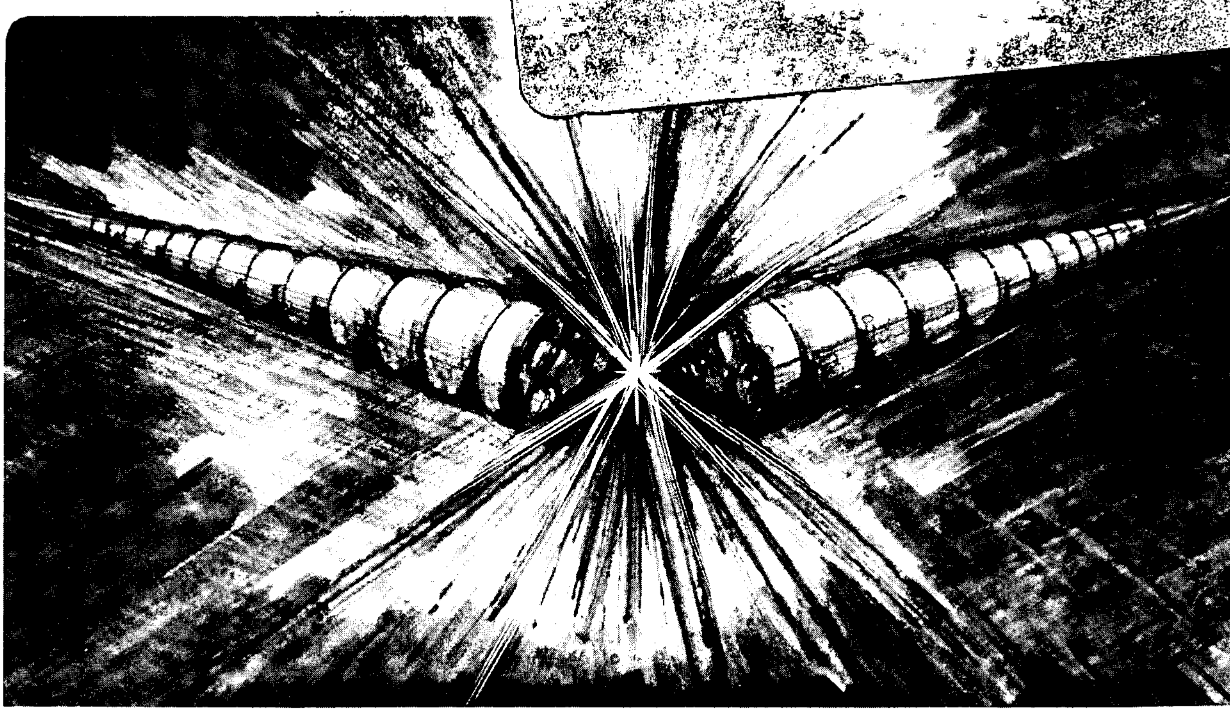

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A LIE-TRANSFORMED ACTION PRINCIPLE FOR CLASSICAL PLASMA DYNAMICS*

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ABSTRACT

The Lie transform for a single particle in a wave is imbedded in a Lagrangian action principle for self-consistent plasma dynamics. Variation of the action then yields the Vlasov equation for the oscillation-center distribution, the ray equations and amplitude transport for the wave, and the Poisson equation for the quasistatic field.

There are many situations in classical (as well as quantum) physics where phenomena occur on very different space- and time-scales. The small parameter associated with the ratio of scales then implies the existence of a reduced description for the slow dynamics in which the fast phenomena are averaged over, and appear at higher order in the slow dynamics.

The Lie transform has proved to be a successful technique[1] for systematically performing such a reduction for a particle in a given field. Beginning with the particle Hamiltonian $H(p,q;\phi)$, where ϕ represents the field, the Lie transform produces a new Hamiltonian $K(P,Q;\phi)$ for the "oscillation center," about which the particle oscillates. Whereas H depends linearly on ϕ, and generates rapid particle oscillations, the dependence of K on ϕ is typically quadratic, generating so-called "ponderomotive" effects.

It is clearly desirable to know what the reaction of the particles is on the field, i.e., to obtain a self-consistent dynamics. Before reduction, this dynamics is generally well-known. Our aim here is to show how to find the self-consistent dynamics for the reduced description.

Our method is to formulate both the original dynamics and the reduced dynamics as a Lagrangian action principle. Variations then lead automatically to self-consistent evolution equations. Further, the symmetries of the action imply conservation laws at both levels.

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In order to clarify the essentials of our method, we shall introduce simplifying assumptions along the way. In future publications, we intend to remove these simplifications and present the full picture.

The original (unreduced) dynamics is expressed in terms of the Vlasov distribution $f(p,q;t)$ and the Coulomb field $\phi(\underline{x},t)$. The former evolves by a Liouville equation:

$$\partial f/\partial t + [f, H] = 0, \tag{1}$$

with the canonical bracket on (p,q) space. The potential satisfies the Poisson equation, with the charge density determined by f.

To obtain these equations from an action principle, we introduce the particle orbit $p(z_0;t)$, $q(z_0;t)$, where z_0 represents the initial condition of a particle. Letting $f_0(z_0)$ be the initial Vlasov distribution, we adopt the Vlasov Lagrangian action:[2]

$$S_V(p,q) = \int d^6z_0\ f_0(z_0) \int (p\dot{q} - H)dt, \tag{2}$$

where $q = \partial q(z_0;t)/\partial t$, and we have suppressed summation over the three degrees of freedom of (p,q), and over the particle species. Demanding that S_V be stationary under variation of the "fields" $p(z_0,t)$ and $q(z_0,t)$ yields the Hamiltonian equations:

$$q = \partial H/\partial p, \qquad p = - \partial H/\partial q. \tag{3}$$

Then, defining the Vlasov distribution:

$$f(p',q';t) = \int dz_0^6\ f_0(z_0)\ \delta^3(p'-p(z_0;t))\ \delta^3(q'-q(z_0;t)), \tag{4}$$

we obtain the Vlasov Eq. (1).

We now add to Eq. (2) the action of the Coulomb field

$$S_C\ (\phi) = \int d^3x \int dt\ E^2/8\pi, \tag{5}$$

where $\underline{E} = - \nabla\phi(\underline{x},t)$. Variation of $S = S_V + S_C$ with respect to $\phi(\underline{x},t)$ then yields the Poisson equation:

$$- \nabla^2\phi(\underline{x},t) = 4\pi \int d^6z_0\ f_0(z_0)\ \delta H/\delta\phi(\underline{x},t). \tag{6}$$

We introduce the particle charge density:

$$\rho(\underline{x},t;\ p,q) = \delta H(p,q;\phi)/\delta\phi(\underline{x},t) \tag{7}$$

and the identity

$$d^3p\ d^3q\ f(p,q;t) = d^6z_0\ f_0(z_0) \tag{8}$$

from (4), to express the plasma charge density as

$$\rho(\underline{x},t;f) = \int d^3p \; d^3q \; f(p,q;t) \; \rho(\underline{x},t;p,q). \tag{9}$$

Thus the Poisson equation

$$-\nabla^2 \phi(\underline{x},t) = 4\pi\rho(\underline{x},t;f) \tag{10}$$

couples f to ϕ, while the Vlasov Eq. (1) couples ϕ to f.

We now restrict the field ϕ to represent a single wave in eikonal form, plus a self-consistent quasi-static background:

$$\phi(x;\varepsilon) = \phi_0(x) + \tilde{\phi}(x) \exp i\theta(x)/\varepsilon + \text{c.c.}, \tag{11}$$

where $x = (\underline{x},t)$, and the amplitude $\tilde{\phi}$, the phase θ, and the background ϕ_0 are all slowly varying functions of space-time x.

The Hamiltonian $H(p,q;\phi)$ thus generates rapid oscillations, characterized by the local wave-vector $\underline{k}(x) = -\nabla\theta/\varepsilon$ and frequency $\omega(x) = -(\partial\theta/\partial t)/\varepsilon$. For particles not in resonance with the wave, the linear dependence of H on $\tilde{\phi}$ is Lie-transformed to a quadratic dependence of K on $\tilde{\phi}$. Further, whereas H contains the rapid phase dependence of Eq. (11), K depends on θ only thorugh its derivatives $(\underline{k},\omega)$, which are themselves slowly varying.

Because the Lie transform is a canonical transformation, the phase-space Lagrangian action is form-invariant.

$$\int dt \; (p\dot{q} - H) = \int dt \; (P\dot{Q} - K) \tag{12}$$

Substituting Eq. (12) into Eq. (2), we now vary S_V with respect to the "fields" P,Q. By steps analogous to those yielding Eq. (1), we now obtain the Vlasov equation for the oscillation-center distribution F(P,Q;t):

$$\partial F/\partial t + [F,K] = 0. \tag{13}$$

Thus the evolution of F is generated by $K(P,Q;\phi_0,\tilde{\phi},\theta)$. It remains to obtain the self-consistent evolution of $\phi_0,\tilde{\phi},\theta$.

The contribution of K to S can be expressed as

$$S^K = - \int d^4x \int d^3P \; F(\underline{x},\underline{P};t) \; K(\underline{x},\underline{P};\phi_0,\tilde{\phi},k), \tag{14}$$

where we have identified $\underline{Q}$ and $\underline{x}$, and where $k(x) = (\underline{k},\omega)$. We expand K in powers of $\tilde{\phi}$:

$$K = K_0(\underline{x},\underline{P};\phi_0) + \tilde{\phi}^2(x) \; K_2(\underline{x},\underline{P};\phi_0,k), \tag{15}$$

truncating at the quadratic term. (The linear term vanishes in the absence of resonance.) The coefficient K_2, explicitly known from the Lie-transform generating function, generates the ponderomotive effects.[3] Substituting (15) into (14), we obtain $S^K = S^K_0 + S^K_2$, with

$$S^K_2 = \int d^4x\, \tilde{\Phi}^2(x)\, \chi(x), \tag{16}$$

in terms of the susceptibility $\chi(x)$, defined as

$$\chi(x) = -\int d^3P\, F(\underline{x},\underline{P};t)\, K_2(\underline{x},\underline{P};\, \phi_0(\underline{x},t),\, k(\underline{x},t)). \tag{17}$$

This definition represents the K-χ theorem, discovered some years ago.[4] The Coulomb contribution to S is, by (5) and (11),

$$S_c = \int d^4x\, (\nabla\phi_0)^2/8\pi + \int d^4x\, \underline{k}^2(x)\tilde{\Phi}^2(x)/4\pi. \tag{18}$$

We collect the quadratic terms from (16) and (18):

$$S_2 = \int d^4x\, \tilde{\Phi}^2(x)\, \varepsilon(x), \tag{19}$$

with the local dielectric function defined as

$$\varepsilon(x) = \varepsilon(x,k(x)) = \underline{k}^2(x)/4\pi + \chi(x). \tag{20}$$

Now variation of S with respect to the amplitude $\tilde{\Phi}(x)$ yields the dispersion relation

$$\varepsilon(x,k(x)) = 0. \tag{21}$$

This is solved for the phase $\theta(x)$ by Hamilton's method; i.e., the ray equations:

$$\begin{aligned} dk_\mu/d\lambda &= \partial\varepsilon(x,k)/\partial x^\mu \\ dx^\mu/d\lambda &= -\partial\varepsilon(x,k)/\partial k_\mu \end{aligned} \tag{22}$$

are integrated to obtain

$$\theta(x) = \int^x k_\mu(x')dx' \tag{23}$$

Variation of S with respect to the phase $\theta(x)$ yields the wave-action conservation law:[5]

$$\partial J^\mu/\partial x^\mu = 0, \tag{24}$$

with the action 4-vector defined as

$$J^\mu(x) = \tilde{\Phi}^2(x)\, \partial\varepsilon(x,k)/\partial k_\mu\,. \tag{25}$$

In terms of the scalar action density:

$$J(x) = \tilde{\Phi}^2(x)\ \partial\varepsilon/\partial\omega, \qquad (26)$$

(24) is the amplitude-transport equation:

$$\partial J(\underline{x},t)/\partial t + \nabla\cdot(J\underline{V}) = 0, \qquad (27)$$

where $\underline{V}(\underline{x},t)$ is the wave group velocity.

Finally, we need to vary S with respect to $\phi_0(x)$, obtaining from (18) and (14):

$$-\nabla^2\phi_0(x) = 4\pi \int d^3P\ F(\underline{x},\underline{P};t)\ \partial K/\partial\phi_0. \qquad (28)$$

The right side of (28) is the charge density, $\rho(x;F)$, expressed in terms of the oscillation-center distribution F. From (15), we see that there are two contributions: $\partial K_0/\partial\phi_0$ represents the oscillation-center charge density, while $\partial K_2/\partial\phi_0$ is a polarization contribution from the wave. The generalization of the latter is particularly important in the magnetic case, where it represents polarization drift and wave magnetization current.

We now have obtained a closed set of equations for $F, \phi_0, \tilde{\Phi}, \theta$, representing the self-consistent slow dynamics.

In future publications we shall present generalizations to the full Maxwell equations, covariant relativistic dynamics, many waves, resonant interaction, higher order effects in $\tilde{\Phi}$. We shall also discuss the conservation laws arising from the Noether symmetries.

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