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A COMBINATORIAL BASIS FOR THE FERMIONIC DIAGONAL COINVARIANT RING

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Abstract. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ and $\Xi_n = (\xi_1, \ldots, \xi_n)$ be two lists of n variables and consider the diagonal action of $\mathfrak{S}_n$ on the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$ generated by these variables. Jongwon Kim and Rhoades defined and studied the *fermionic diagonal coinvariant ring FDR_n* obtained from $\wedge\{\Theta_n, \Xi_n\}$ by modding out by the $\mathfrak{S}_n$-invariants with vanishing constant term. The author and Rhoades gave a basis for the maximal degree components of this ring where the action of $\mathfrak{S}_n$ could be interpreted combinatorially via noncrossing set partitions. This paper will do similarly for the entire ring, although the combinatorial interpretation will be limited to the action of $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$. The basis will be indexed by a certain class of noncrossing partitions.

Keywords. Skein relation, coinvariant algebra, noncrossing set partition, cyclic sieving

Mathematics Subject Classifications. 05E10, 05E18, 20C30

1. Introduction

This paper involves an algebraically defined $\mathfrak{S}_n$-module, and is concerned with modelling the $\mathfrak{S}_n$ action on this module via combinatorially defined objects. In particular, we will give a basis indexed by a certain type of noncrossing set partition for which the action of $\mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$ has a nice combinatorial interpretation.

The module in question was introduced by Jongwon Kim and Rhoades [KR20], and is defined as follows. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ and $\Xi_n = (\xi_1, \ldots, \xi_n)$ be two sets of n anticommuting variables, and let

$$\wedge\{\Theta_n, \Xi_n\} := \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} \tag{1.1}$$

be the exterior algebra generated by these symbols over $\mathbb{C}$. The symmetric group $\mathfrak{S}_n$ acts on this exterior algebra via a diagonal action given by

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \xi_i' := \xi_{w(i)}'. \tag{1.2}$$

for any permutation $w \in \mathfrak{S}_n$ and $1 \leqslant i \leqslant n$. Let $\wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}$ denote the subspace of $\mathfrak{S}_n$-invariants with vanishing constant term. Then the fermionic diagonal coinvariant ring is defined as

$$FDR_n := \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}\rangle. \tag{1.3}$$

The ring FDR_n is a variant of the Garsia–Haiman diagonal coinvariant ring [Hai94], which is defined analogously but with the anticommuting variables replaced with commuting ones. Several other variants involving more sets of variables or mixtures of anticommuting and commuting variables have been studied by other authors [Ber20, BRT20, DIV21, KR20, OZ20, PRR19, RW20, RW22, Swa21, SW20, Zab19, Zab20].

The ring $\wedge\{\Theta_n, \Xi_n\}$ has a bigrading given by

$$(\wedge\{\Theta_n, \Xi_n\})_{i,j} := \wedge^i\{\theta_1, \ldots, \theta_n\} \otimes \wedge^j\{\xi_1, \ldots, \xi_n\}. \tag{1.4}$$

The invariant ideal $\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}\rangle$ is homogeneous, so FDR_n inherits the bigrading. In [KR20], Kim and Rhoades calculated the frobenius image of FDR_n to be given by

$$\mathrm{Frob}(FDR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i-1,1^{i+1})} * s_{(n-j-1,1^{j+1})} \tag{1.5}$$

where $*$ denotes the Kronecker product of Schur functions. They remark that in the case when $i + j = n - 1$, the above shows that the dimension of $(FDR_n)_{n-k,k-1}$ is given by the Narayana number $\mathrm{Nar}(n, k)$. Narayana numbers count noncrossing set partitions of $[n]$ into k blocks, and in [KR22] a combinatorial basis of $(FDR_n)_{n-k,k-1}$ was given indexed by set partitions for which the $\mathfrak{S}_n$-action was given by a skein action on noncrossing partitions first described by Rhoades in [Rho17].

In this paper we will give a similar result for all bidegrees, although our results will not give a combinatorial description for the full $\mathfrak{S}_n$-action. Instead, we will focus on the subgroup of $\mathfrak{S}_n$ consisting of permutations which leave n fixed (which we will abusively refer to as $\mathfrak{S}_{n-1}$). We will define a basis of $(FDR_n)_{i,j}$ indexed by a certain class of noncrossing set partitions defined in Section 3 for which the action of $\mathfrak{S}_{n-1}$ can be described via combinatorial manipulations of the indexing partitions and use this basis to give an expression for the Frobenius image

$$\mathrm{Frob}(\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n)_{i,j}). \tag{1.6}$$

The rest of the paper is organized as follows. Section 2 will give relevant background information on set partitions, exterior algebras, and $\mathfrak{S}_n$ representation theory. Section 3 will describe an action of $\mathfrak{S}_{n-1}$ on certain set partitions and map this action into FDR_n. Section 4 will show that a restriction of this map is an isomorphism and use it to obtain a combinatorial basis of FDR_n. Section 5 will use the basis developed to calculate the bigraded $\mathfrak{S}_n$-structure of FDR_n. Section 6 will connect this basis to a cyclic sieving result of Thiel and address some avenues of further inquiry.

2. Background

2.1. Combinatorics

A *noncrossing set partition* of $[n]$ is a set partition of $[n]$ in which for any $1 \leqslant a < b < c < d \leqslant n$ if a and c are in the same block, and b and d are in the same block, then a, b, c, d are all in the same block.

An *integer partition* $\lambda \vdash n$ of length k is a sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 + \cdots + \lambda_k = n$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$. *Dominance order*, denoted by $\mu \preceq \lambda$, is a partial order on set partitions defined by $\mu \preceq \lambda$ if and only if $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$ for all i, taking μ_i or λ_i to be 0 whenever i exceeds the length of μ or λ respectively. The conjugate of an integer partition λ denoted by λ'.

2.2. Exterior Algebras

As in the introduction, we will use $\wedge\{\Theta_n, \Xi_n\}$ to denote the exterior algebra generated by the $2n$ symbols $\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n$. There is an isomorphism of graded vector spaces (see e.g. [KR20])

$$FDR_n \cong \wedge\{\theta_1, \ldots, \theta_{n-1}, \xi'_1, \ldots, \xi'_{n-1}\}/\langle \theta_1\xi'_1 + \cdots + \theta_{n-1}\xi'_{n-1}\rangle \tag{2.1}$$

given by

$$\begin{aligned} &\theta_i \to \theta_i && 1 \leqslant i \leqslant n-1 \\ &\theta_n \to -(\theta_1 + \cdots + \theta_{n-1}) && \\ &\xi_i \to \xi'_i - \frac{1}{n}(\xi'_1 + \cdots + \xi'_{n-1}) && 1 \leqslant i \leqslant n-1 \\ &\xi_n \to -\frac{1}{n}(\xi'_1 + \cdots + \xi'_{n-1}) && \end{aligned}$$

As this paper will focus on the action of $\mathfrak{S}_{n-1}$, we will extensively use this alternate formulation of FDR_n, and use $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ to denote $\wedge\{\theta_1, \ldots, \theta_{n-1}, \xi'_1, \ldots, \xi'_{n-1}\}$. The ring $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ inherits the action of $\mathfrak{S}_n$ and $\mathfrak{S}_{n-1}$ from FDR_n, and the action of $\mathfrak{S}_{n-1}$ simply permutes indices of variables.

Given subsets $S, T \subseteq [n-1]$, let θ_S, ξ'_T with $S = \{s_1 < \cdots < s_a\}$, $T = \{t_1 < \cdots < t_b\}$ denote the monomial

$$\theta_{s_1} \cdots \theta_{s_a} \cdot \xi'_{t_1} \cdots \xi'_{t_b} \tag{2.2}$$

The set $\{\theta_S \cdot \xi'_T : S, T \subseteq [n-1]\}$ is a basis of $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$. Define an inner product $\langle -, - \rangle$ on $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ by declaring this basis to be orthonormal.

An exterior algebra $\wedge\{\omega_1, \ldots, \omega_n\}$ acts on itself via *exterior differentiation*, denoted by $\odot$. The action $\odot : \wedge\{\omega_1, \ldots, \omega_n\} \times \wedge\{\omega_1, \ldots, \omega_n\} \to \wedge\{\omega_1, \ldots, \omega_n\}$ is defined by

$$\omega_i \odot (\omega_{s_1} \cdots \omega_{s_k}) = \begin{cases} (-1)^{j-1}\omega_{s_1} \cdots \widehat{\omega_{s_j}} \cdots \omega_{s_k} & i = s_j \\ 0 & i \neq s_j \text{ for all } 1 \leqslant j \leqslant k \end{cases}.$$

2.3. $\mathfrak{S}_n$-Representation Theory

Irreducible representations of $\mathfrak{S}_n$ are in one-to-one correspondence with partitions $\lambda \vdash n$. Given $\lambda \vdash n$ let S^λ denote the corresponding $\mathfrak{S}_n$-irreducible. The dimension of S^λ is given by the number of standard Young tableau of shape λ. Any finite dimensional $\mathfrak{S}_n$-module V can be decomposed uniquely as $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$ for some multiplicities c_λ. The *Frobenius image* of V is the symmetric function given by

$$\text{Frob } V := \sum_{\lambda \vdash n} c_\lambda s_\lambda \tag{2.3}$$

where s_λ is the Schur function corresponding to λ.

If V is an $\mathfrak{S}_n$-module and W is an $\mathfrak{S}_m$-module, their *induction product* $V \circ W$ is given by

$$V \circ W := \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} (V \otimes W) \tag{2.4}$$

where the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $V \otimes W$ is given by $(\sigma, \sigma') \cdot (v \otimes w) := (\sigma \cdot v) \otimes (\sigma' \cdot w)$. We have

$$\text{Frob } V \circ W = \text{Frob } V \cdot \text{Frob } W \tag{2.5}$$

so induction product of modules corresponds to multiplication of Frobenius image.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, let $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_n$ denote the Young subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{n-\lambda_k,\ldots,n\}}$. To any subgroup $X \subseteq \mathfrak{S}_n$ we associate two group algebra elements $[X]_+$ and $[X]_-$ defined by $[X]_+ = \sum_{w \in X} w$ and $[X]_- = \sum_{w \in X} \text{sign}(w) w$. We will need the following standard lemma.

Lemma 2.1. *Let $\lambda, \mu \vdash n$. Then $[\mathfrak{S}_\lambda]_+$ kills S^μ unless $\lambda \preceq \mu$ and $[\mathfrak{S}_{\lambda'}]_-$ kills S^μ unless $\mu \preceq \lambda$.*

For a more thorough treatment of this topic, see [Sag91]

2.4. Cyclic Sieving

Cyclic sieving was introduced by Reiner, Stanton and White [RSW04] as a way to express various related results about the enumeration of fixed points of a cyclic action. If X is a finite set, C is a cyclic group of order n generated by c acting on X, and $P(q)$ is a polynomial in $\mathbb{N}[q]$, then we say that

Definition 2.2. The triple $(X, C, P(q))$ exhibits the cyclic sieving phenomenon if for all non-negative integers d,

$$|\{x \in X \mid c^d \cdot x = x\}| = P(\zeta^d)$$

where ζ is a primitive n^{th} root of unity.

The polynomial $P(q)$ is often given in terms of *q-analogs*. The q-analog of a positive integer n is denoted $[n]_q$ and is defined to be $1 + q + q^2 + \cdots + q^{n-1}$. The q-analogs of $n!$, $\binom{n}{k}$ and the multinomial coefficient $\binom{n}{k_1,k_2,\ldots,k_l}$, are denoted and defined as follows:

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!},$$

$$\begin{bmatrix} n \\ k_1, k_2, \dots k_l \end{bmatrix}_q = \frac{[n]_q!}{[k_1]_q! \cdots [k_l]_q!}.$$

The *fake degree polynomial* of a representation is defined by

$$\textbf{fd}(S^\lambda) = q^{b(\lambda)} \frac{[r]_q!}{\prod_{(i,j \in \lambda)} [h(i,j)]_q}$$

where the product is over all cells (i, j) of λ, $h(i, j)$ is its hook length and $b(\lambda) = \lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots$ and the fake degree of a general representation is the sum of fake degrees of the irreducibles it contains, with multiplicity.

Cyclic sieving results are often proven via representation theory. In particular Reiner, Stanton and White [RSW04] realized the following was implied by a result of Springer's [Spr74].

Theorem 2.3 (Springer, 1974)**.** *Let V be a representation of $\mathfrak{S}_n$ with a basis X which is preserved by the long cycle, c. Let $P(q) = \textbf{fd}(V)$. Then $(X, \langle c \rangle, P(q))$ exhibits the cyclic sieving phenomenon.*

3. Set partitions and the action of $\mathfrak{S}_{n-1}$

The indexing set for our combinatorial basis will be a certain partially labelled subset $\Phi(n)$ of noncrossing set partitions of $[n]$.

Definition 3.1. Let n, k, x, t be nonnegative integers. We define the following sets of set partitions:

- Let $\Psi(n)$ denote the set of all set partitions of $[n]$ for which all blocks not containing n are size 1 or size 2, and blocks of size 1 not containing n are labelled with either a θ or a ξ'.

- Let $\Psi(n, k)$ be the set of partitions in $\Psi(n)$ which have exactly k blocks of size 2 not containing n.

- Let $\Psi(n, k, x, y)$ denote the set of partitions in $\Psi(n, k)$ which have exactly x singletons labelled θ and exactly y singletons labelled ξ'.

- Let $\Phi(n)$, $\Phi(n, k)$, and $\Phi(n, k, x, y)$ be the subsets of $\Psi(n)$, $\Psi(n, k)$, or $\Psi(n, k, x, y)$ respectively which consist only of the those set partitions which are noncrossing.

For the rest of this paper, when we refer to the singleton blocks of a partition $\pi \in \Psi(n)$, we only refer to those blocks of size 1 that do not contain n, even if the block containing n happens to be size 1. Similarly when we refer to the blocks of size two we refer to only the blocks of size two that do not contain n.

There is a natural action of $\mathfrak{S}_{n-1}$ on $\Psi(n)$, given by simply permuting elements between blocks and preserving labels of blocks. The sets $\Psi(n,k)$ and $\Psi(n,k,x,y)$ are closed under this action, but $\Phi(n)$ is not, as permuting the elements of a noncrossing permutation may introduce crossings. However, we can define an action of $\mathfrak{S}_{n-1}$ on the linearization $\mathbb{C}\Phi(n)$ by mapping $\mathbb{C}\Psi(n)$ into $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ in such a way that $\mathbb{C}\Phi(n)$ is $\mathfrak{S}_{n-1}$-invariant and pulling back the $\mathfrak{S}_{n-1}$-action.

Towards this goal, to each element $\pi \in \Psi(n)$ we will associate an element G_π of the exterior algebra $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$. To define G_π we will make use of a tool we will call *block operators*. Let B be a block of a set partition $\pi \in \Psi(n)$, i.e. B is a nonempty subset of $[n]$ that either contains n or is size at most two. Define the *block operator* $\tau_B : \wedge\{\Theta_{n-1}, \Xi'_{n-1}\} \to \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ by

$$\tau_B(f) = \begin{cases} (\prod_{i\in B\setminus\{n\}} \theta_i) \odot f & n \in B \\ \xi'_i \cdot (\theta_j \odot f) + \xi'_j \cdot (\theta_i \odot f) & n \notin B, B = \{i,j\} \\ f & B = \{i_\theta\} \\ \xi'_i \cdot (\theta_i \odot f) & B = \{i'_\xi\} \end{cases} \tag{3.1}$$

It will be important for what follows to note that block operators corresponding to blocks not containing n commute.

Lemma 3.2. *Let A and B be two nonempty subsets of $[n-1]$ of size at most two. Then τ_A and τ_B commute.*

Proof. The lemma reduces to the fact that the family of operators

$$\{\xi'_1\cdot, \ldots, \xi'_{n-1}\cdot, \theta_1\odot, \ldots, \theta_{n-1}\odot\}$$

all anticommute, and that each block operator is a degree two polynomial in these. □

Block operators also interact nicely with the action of $\mathfrak{S}_{n-1}$.

Lemma 3.3. *Let A be a subset of $[n-1]$ and let $\sigma \in \mathfrak{S}_{n-1}$. Then for any $f \in \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$*

$$\sigma \cdot \tau_A(f) = \tau_{\sigma\cdot A}(\sigma \circ f)$$

where the action of $\mathfrak{S}_n$ on subsets is given by $\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$.

We can now define G_π.

Definition 3.4. Let $\pi \in \Psi(n)$ with blocks $B_1, \ldots, B_k$ and $n \in B_k$. Then

$$G_\pi := \tau_{B_1} \cdots \tau_{B_k}(\theta_1\theta_2\cdots\theta_{n-1}). \tag{3.2}$$

We can also give a description of the G_π not involving block operators as follows.

Proposition 3.5. *Let $\pi \in \Psi(n)$. Take the product of $\theta_i\xi'_i - \theta_j\xi'_j$ for every size two block $\{i,j\}$ of π with $i<j$ in any order (they are homogeneous of degree two and thus commute). For each singleton block $\{i\}$ of π, multiply on the right by θ_i or ξ'_i according to its label in increasing order. Then G_π is equal to the result multiplied by $(-1)^{inv(\pi')}$ where π' is the word formed by listing the block containing n in decreasing order, then listing all size two blocks not containing n increasing within each block and by order of increasing minimal element, then listing all size one blocks not containing n in increasing order.*

For example, if $\pi = 1_\theta/2, 5/3, 4/6, 8/7_{\xi'}$, then

$$G_\pi = (-1)^{\mathrm{inv}(86253417)}(\theta_2\xi'_2 - \theta_5\xi'_5)(\theta_3\xi'_3 - \theta_4\xi'_4)\theta_1\xi'_7 \tag{3.3}$$

Proof. By Lemma 3.2 we can assume that all of the block operators corresponding to size two blocks appear before block operators according to singletons. Applying τ_{B_k} and any block operators corresponding to singletons to $(\theta_1\theta_2\cdots\theta_{n-1})$ removes all θ_i indexed by elements of B_k and replaces θ_i indexed by ξ'-labelled singletons with ξ'_i. Note that $\tau_{\{i,j\}}\theta_i\theta_j = \theta_i\xi'_i - \theta_j\xi'_j$, and the proof follows. □

The $\mathfrak{S}_{n-1}$ action on these G_π matches the natural $\mathfrak{S}_{n-1}$ action on $\Psi(n)$, up to sign.

Proposition 3.6. *Let $\sigma \in \mathfrak{S}_{n-1}$ and $\pi \in \Psi(n)$. Then $\sigma \circ G_\pi = \mathrm{sign}(\sigma)G_{\sigma\circ\pi}$.*

Proof. Using the block operator definition of G_π and Lemma 3.3 we have,

$$\begin{aligned}
\sigma \circ G_\pi &= \sigma \circ (\tau_{B_1}\cdots\tau_{B_k}(\theta_1\theta_2\cdots\theta_{n-1})) && (3.4)\\
&= \tau_{\sigma(B_1)}\cdots\tau_{\sigma(B_k)}(\sigma\circ(\theta_1\theta_2\cdots\theta_{n-1})) && (3.5)\\
&= \tau_{\sigma(B_1)}\cdots\tau_{\sigma(B_k)}(\mathrm{sign}(\sigma)\theta_1\theta_2\cdots\theta_{n-1}) && (3.6)\\
&= \mathrm{sign}(\sigma)G_{\sigma\circ\pi} && (3.7)
\end{aligned}$$

□

The goal of the remainder of this section is to show that $\mathrm{span}(\{G_\pi \mid \pi \in \Phi(n)\})$ is $\mathfrak{S}_{n-1}$ invariant. For this end we will need the following relations of block operators.

Lemma 3.7. *Let $a,b,c,d \in [n-1]$. Then*

$$\tau_{\{a,b\}}\tau_{\{c,d\}} + \tau_{\{a,c\}}\tau_{\{b,d\}} + \tau_{\{a,d\}}\tau_{\{b,c\}} = 0 \tag{3.8}$$

as operators on the ring $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$.

Proof. This is a straightforward calculation from the definition of τ. We have

$$\begin{aligned}
(\tau_{\{a,b\}}\tau_{\{c,d\}} + \tau_{\{a,c\}}\tau_{\{b,d\}} + \tau_{\{a,d\}}\tau_{\{b,c\}})(f) = \quad & \xi'_a\cdot(\theta_b\odot(\xi'_c\cdot(\theta_d\odot f)+\xi'_d\cdot(\theta_c\odot f)))\\
&+\xi'_b\cdot(\theta_a\odot(\xi'_c\cdot(\theta_d\odot f)+\xi'_d\cdot(\theta_c\odot f)))\\
&+\xi'_a\cdot(\theta_c\odot(\xi'_b\cdot(\theta_d\odot f)+\xi'_d\cdot(\theta_b\odot f)))\\
&+\xi'_c\cdot(\theta_a\odot(\xi'_b\cdot(\theta_d\odot f)+\xi'_d\cdot(\theta_b\odot f)))\\
&+\xi'_a\cdot(\theta_d\odot(\xi'_b\cdot(\theta_c\odot f)+\xi'_c\cdot(\theta_b\odot f)))\\
&+\xi'_d\cdot(\theta_a\odot(\xi'_b\cdot(\theta_c\odot f)+\xi'_c\cdot(\theta_b\odot f)))
\end{aligned}$$

Distributing and using the anticommutativity of operators $\xi'_i\cdot$ and $\theta_j\odot$, we have

$$\begin{aligned}(\tau_{\{a,b\}}\tau_{\{c,d\}}+\tau_{\{a,c\}}\tau_{\{b,d\}}+\tau_{\{a,d\}}\tau_{\{b,c\}})(f) = \quad & \xi'_a\cdot(\theta_b\odot(\xi'_c\cdot(\theta_d\odot f)))\\ &-\xi'_a\cdot(\theta_b\odot(\theta_c\odot(\xi'_d\cdot f)))\\ &-\theta_a\odot(\xi'_b\cdot(\xi'_c\cdot(\theta_d\odot f)))\\ &+\theta_a\odot(\xi'_b\cdot(\theta_c\odot(\xi'_d\cdot f)))\\ &-\xi_a\cdot(\xi'_b\cdot(\theta_c\odot(\theta_d\odot f)))\\ &+\xi'_a\cdot(\theta_b\odot(\theta_c\odot(\xi'_d\cdot f)))\\ &+\theta_a\odot(\xi'_b\cdot(\xi'_c\cdot(\theta_d\odot f)))\\ &-\theta_a\odot(\theta_b\odot(\xi'_c\cdot(\xi'_d\cdot f)))\\ &+\xi'_a\cdot(\xi'_b\cdot(\theta_c\odot(\theta_d\odot f)))\\ &-\xi'_a\cdot(\theta_b\odot(\xi'_c\cdot(\theta_d\odot f)))\\ &-\theta_a\odot(\xi'_b\cdot(\theta_c\odot(\xi'_d\cdot f)))\\ &+\theta_a\odot(\theta_b\odot(\xi'_c\cdot(\xi'_d\cdot f)))\end{aligned}$$

and each term appears twice with opposite signs. □

Lemma 3.8. *Let $A=\{a_1<a_2\}$ and $B\subset[n]$ be two disjoint sets with $n\in B$. Let $b_1<\cdots<b_m$ be the elements of b that lie between a_1 and a_2. Then for any $1\leqslant i\leqslant m$,*

$$\tau_A\tau_B+(-1)^{i+m}\tau_{\{a_1,b_i\}}\tau_{B+a_2-b_i}+(-1)^{i-1}\tau_{\{b_i,a_2\}}\tau_{B+a_1-b_i}=0 \tag{3.9}$$

as operators on the ring $\wedge\{\Theta_{n-1},\Xi'_{n-1}\}$.

Proof. We begin with the case where all elements of B lie between a_1 and a_2. Applying the definition of τ and anticommutativity gives

$$\begin{aligned}&(\tau_A\tau_B+(-1)^{i+m}\tau_{\{a_1,b_i\}}\tau_{B+a_2-b_i}+(-1)^{i-1}\tau_{\{b_i,a_2\}}\tau_{B+a_1-b_i})(f)\\ &\qquad=\xi_{a_1}\cdot\theta_{a_2}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad\quad+\xi_{a_2}\cdot\theta_{a_1}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad\quad+(-1)^{i+m}(-1)^{i+m-1}\xi_{a_1}\cdot\theta_{a_2}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad\quad+(-1)^{i+m}(-1)^{m-1}\xi_{b_i}\cdot\theta_{a_1}\odot\theta_{a_2}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_{i-1}}\odot\theta_{b_{i+1}}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad\quad+(-1)^{i-1}(-1)^{1}\xi_{b_i}\cdot\theta_{a_1}\odot\theta_{a_2}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_{i-1}}\odot\theta_{b_{i+1}}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad\quad+(-1)^{i-1}(-1)^{i}\xi_{a_2}\cdot\theta_{a_1}\odot\theta_{a_2}\odot\theta_{b_1}\odot\cdots\odot\theta_{b_{i-1}}\odot\theta_{b_{i+1}}\odot\cdots\odot\theta_{b_m}\odot f\\ &\qquad=0\end{aligned}$$

Signs coming from the statement and signs coming from anticommutativity have been kept separate for clarity. When B contains elements that do not lie between a_1 and a_2, all terms of the above computation will be affected uniformly, so the result holds in that case as well. □

Corollary 3.9. *Let $A = \{a_1 < a_2\} \subset [n-1]$ and $B \subset [n]$ be two disjoint sets with $n \in B$. Let $b_1 < b_2 < \cdots < b_m$ be the elements of B that lie between a_1 and a_2, and suppose at least one such element exists. Then*

$$(-1)^{m+1}\tau_A\tau_B + \tau_{\{a_1,b_1\}}\tau_{B+a_2-b_1} + \sum_{i=1}^{m-1} \tau_{\{b_i,b_{i+1}\}}\tau_{B+a_1+a_2-b_i-b_{i+1}} + \tau_{\{b_m,a_2\}}\tau_{B+a_1-b_m} = 0 \quad (3.10)$$

as operators on the ring $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$.

Proof. Apply Lemma 3.8 with $i = 1$ to obtain

$$(-1)^{m+1}\tau_A\tau_B + \tau_{\{a_1,b_1\}}\tau_{B+a_2-b_1} - (-1)^m\tau_{\{b_1,a_2\}}\tau_{B+a_1-b_1} \quad (3.11)$$

Inducting on m gives the result. □

Example 3.10. Let $n = 9$ and let $\pi \in \Psi(n)$ be the set partition $\{1_\theta/28/3479/56\}$. Corollary 3.9 lets us write G_π as a linear combination of G's for noncrossing set partitions in the following way. We have

$$G_\pi = \tau_{\{1_\theta\}}\tau_{\{56\}}\tau_{\{28\}}\tau_{\{3479\}}(\theta_1\theta_2\cdots\theta_8).$$

By Corollary 3.9 we can rewrite this as

$$G_\pi = \tau_{\{1_\theta\}}\tau_{\{56\}}(-\tau_{\{23\}}\tau_{\{4789\}} - \tau_{\{34\}}\tau_{\{2789\}} - \tau_{\{47\}}\tau_{\{2389\}} - \tau_{\{78\}}\tau_{\{2349\}})(\theta_1\theta_2\cdots\theta_8).$$

Distributing, this becomes

$$G_\pi = -G_{\{1_\theta/23/4789/56\}} - G_{\{1_\theta/2789/34/56\}} - G_{\{1_\theta/2389/47/56\}} - G_{\{1_\theta/2349/56/78\}}.$$

We can check that this agrees with Proposition 3.5. Applying it transforms the above into

$$\begin{aligned} -(\theta_2\xi'_2 - \theta_8\xi'_8)(\theta_5\xi'_5 - \theta_6\xi'_6)\theta_1 = &- (\theta_2\xi'_2 - \theta_3\xi'_3)(\theta_5\xi'_5 - \theta_6\xi'_6)\theta_1 \\ &- (\theta_3\xi'_3 - \theta_4\xi'_4)(\theta_5\xi'_5 - \theta_6\xi'_6)\theta_1 \\ &- (\theta_4\xi'_4 - \theta_7\xi'_7)(\theta_5\xi'_5 - \theta_6\xi'_6)\theta_1 \\ &- (\theta_7\xi'_7 - \theta_8\xi'_8)(\theta_5\xi'_5 - \theta_6\xi'_6)\theta_1 \end{aligned}$$

which is true as the right hand side telescopes.

Lemma 3.7, Lemma 3.8, and Corollary 3.9 are most easily understood via pictures, see Figure 3.1.

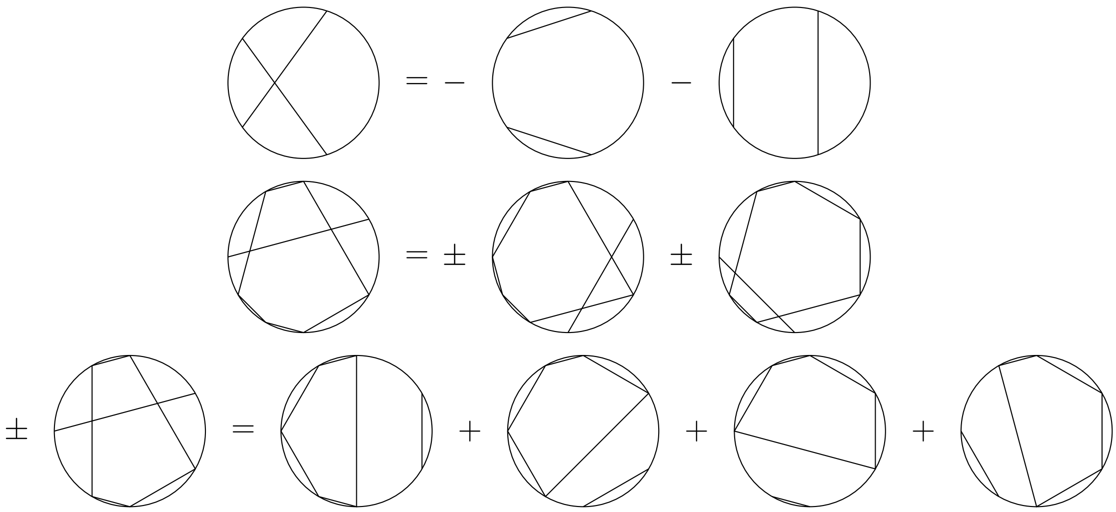

Figure 3.1: From top to bottom: Lemma 3.7, Lemma 3.8, and Corollary 3.9.

Together these lemmas allow us to demonstrate the $\mathfrak{S}_{n-1}$ invariance via a combinatorial algorithm.

Corollary 3.11. *Let $\sigma \in \mathfrak{S}_{n-1}$ and let $\pi \in \Phi(n)$. Then $\sigma \cdot G_\pi$ can be expressed as a linear combination of $\{G_\pi \mid \pi \in \Phi(n)\}$ via the following algorithm:*

1. *Apply σ to π, resulting in a set partition π' not necessarily in $\Phi(n)$.*

2. *If π' is contains any crossing two element blocks $\{a, c\}$, $\{b, d\}$, neither of which contain n, replace π' with minus the sum of the partitions obtained by replacing $\{a, c\}, \{b, d\}$ with $\{a, b\}, \{c, d\}$ and $\{a, d\}, \{b, c\}$. Repeat on each new term of the sum until all terms of the sum do not contain crossing two element blocks.*

3. *For each term of the sum obtained in step 2, replace any two element set that crosses the block containing n as described by Corollary 3.9.*

4. *Replace each partition π'' in the sum obtained from step 3 with its corresponding $G_{\pi''}$ to express $\sigma \cdot G_\pi$ as a linear combination.*

Example 3.12. Let $n = 8$ and let $\sigma \in \mathfrak{S}_{n-1}$ be the cycle (3576). Let $\pi \in \Phi(n)$ be the set partition $\{23/45/7_\theta/186\}$. An example of applying Corollary 3.11 to this situation is given in Figure 3.2.

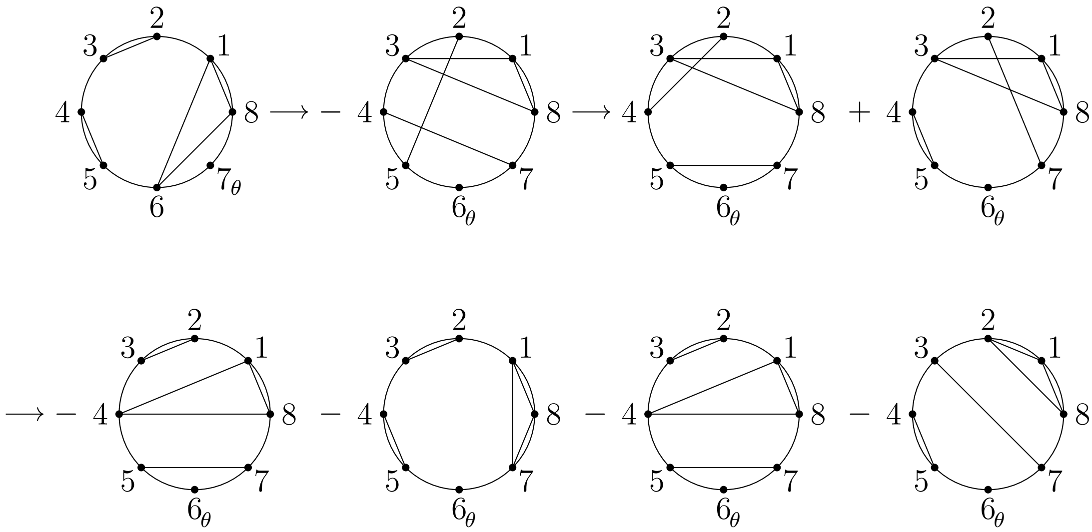

Figure 3.2: Applying Corollary 3.11.

4. A combinatorial basis

We have shown that there is a mapping of $\mathfrak{S}_{n-1}$-modules $\mathbb{C}\Psi(n) \to \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$. In this section we will show that the restriction of this mapping to $\mathbb{C}\Phi(n)$ is injective and becomes an isomorphism when composed with the quotient map $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\} \to FDR_n$, thereby proving the following.

Theorem 4.1. *The set* $\{[G_\pi] \mid \pi \in \Phi(n)\}$ *forms a basis for* FDR_n*, where* $[f]$ *denotes the equivalence class in* FDR_n *of* $f \in \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$.

Proof. We begin with a dimension count; Kim and Rhoades [KR20] gave a basis of FDR_n indexed by a set $\Pi(n)_{>0}$ of Motzkin-like lattice paths defined as follows.

Definition 4.2. Let $\Pi(n)_{>0}$ be the set of all lattice paths which

- Start at $(0,0)$
- Take steps $(1,0),(1,1)$ or $(1,-1)$
- Only touch the x-axis at $(0,0)$
- Have all $(1,0)$ steps labelled by θ or ξ'.

The two indexing sets are in bijection.

Lemma 4.3. *There is a bijection between* $\Pi(n)_{>0}$ *and* $\Phi(n)$.

Proof. Given a Motzkin path in $\Pi(n)_{>0}$, draw a horizontal line extending to the right of each up step until it first intersects the path again. Label each step after the first 1 to $n-1$. Construct a set partition by placing every up step in a block with the down step it is connected to if such a down step exists, or in the block containing n otherwise. Place every horizontal step in a singleton block with the same label. The process can be reversed, and is therefore a bijection. $\square$

The bijection is best described with a picture example as in Figure 4.1.

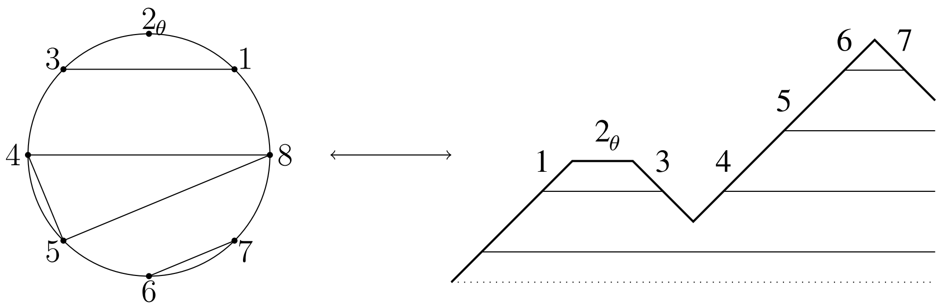

Figure 4.1: An example of Lemma 4.3.

Therefore it suffices to show that $\{[G_\pi] \mid \pi \in \Phi(n)\}$ spans. Since FDR_n is defined as a quotient, it suffices to show that together, the sets

$$\{G_\pi \mid \pi \in \Phi(n)\}$$

and

$$\beta' := \{m(\theta_1\xi'_1 + \cdots + \theta_{n-1}\xi'_{n-1}) \mid m \text{ a monomial in } \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}\}$$

span

$$\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}.$$

By Corollary 3.11, the span of $\{G_\pi \mid \pi \in \Phi(n)\}$ is equal to the span of $\beta := \{G_\pi \mid \pi \in \Psi(n)\}$, it also suffices to show spanning for the larger set. Since FDR_n has maximal bidegree $i + j = n - 1$, it suffices to show that every monomial of total degree $n - 1$ or less lies in the span of $\beta \cup \beta'$.

To show every monomial lies in the span, we will inductively show that it is possible "replace" the $(\theta_i\xi'_i - \theta_j\xi'_j)$ terms in some G_π with $\theta_i\xi'_i$ via the following lemma.

Lemma 4.4. *Let m, k, p, q be nonnegative integers with $2m + 2k + p + q \leqslant n - 1$. Given a collection of distinct indices $I = \{a_1, \ldots, a_m, b_1, \ldots, b_{2k}, c_1, \ldots, c_p, d_1, \ldots, d_q\} \subseteq [n-1]$, define the element F of $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ by*

$$F := (\theta_{a_1}\xi'_{a_1} \cdots \theta_{a_m}\xi'_{a_m})(\theta_{b_1}\xi'_{b_1} - \theta_{b_2}\xi'_{b_2}) \cdots (\theta_{b_{2k-1}}\xi'_{b_{2k-1}} - \theta_{b_{2k}}\xi'_{b_{2k}})\theta_{c_1} \cdots \theta_{c_p}\xi'_{d_1} \cdots \xi'_{d_q}.$$

Thus, I is the set of indices appearing in F. Then F is in the span of $\beta \cup \beta'$.

Proof. We proceed by induction on m. When $m = 0$, F is equal up to sign to G_π for some $\pi \in \Psi(n)$.

Let $m > 0$ and assume the result holds for $m-1$. We have

$$
\begin{aligned}
&(n-|I|)(\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_m}\xi'_{a_m})F' \\
&\qquad = (\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})\left(\sum_{i\notin I-\{a_m\}}\theta_i\xi'_i - \sum_{i\notin I}(\theta_i\xi'_i - \theta_{a_m}\xi'_{a_m})\right)F' \\
&= (\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})\left(\sum_{i\notin I-\{a_m\}}\theta_i\xi'_i\right)F' - \sum_{i\notin I}(\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})(\theta_i\xi'_i-\theta_m\xi'_m)F' \\
&= (\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})\left(\sum_{i\in[n-1]}\theta_i\xi'_i\right)F' - \sum_{i\notin I}(\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})(\theta_i\xi'_i-\theta_m\xi'_m)F'
\end{aligned}
\tag{4.1}
$$

where $F' = (\theta_{b_1}\xi'_{b_1} - \theta_{b_2}\xi'_{b_2})\cdots(\theta_{b_{2k-1}}\xi'_{b_{2k-1}} - \theta_{b_{2k}}\xi'_{b_{2k}})\theta_{c_1}\cdots\theta_{c_p}\xi'_{d_1}\cdots\xi'_{d_q}$.

The first equality holds since

$$\sum_{i\notin I-\{a_m\}}\theta_i\xi'_i - \sum_{i\notin I}(\theta_i\xi'_i - \theta_{a_m}\xi'_{a_m}) = (n-|I|)\theta_{a_m}\xi'_{a_m}.$$

The second equality distributes the sum and the third equality holds because terms in the sum

$$\sum_{i\in[n-1]}\theta_i\xi'_i$$

corresponding to $i \in I-\{a_m\}$ will contribute 0 due to θ_i, ξ'_i, or $\theta_i\xi'_i - \theta_j\xi'_j$ appearing elsewhere in the product, as $(\theta_i\xi'_i + \theta_j\xi'_j)(\theta_i\xi'_i - \theta_j\xi'_j) = 0$ for any i, j.

Then the first term

$$(\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})\left(\sum_{i\in[n-1]}\theta_i\xi'_i\right)F'$$

lies in the span of β' while the second term

$$\sum_{i\notin I}(\theta_{a_1}\xi'_{a_1}\cdots\theta_{a_{m-1}}\xi'_{a_{m-1}})(\theta_i\xi'_i-\theta_m\xi'_m)F'$$

is in the span of $\beta \cup \beta'$ by the inductive hypothesis. The result follows. □

Taking $k = 0$ in Lemma 4.4 gives any monomial of total degree at most $n-1$, so we have shown that together the sets

$$\{G_\pi \mid \pi \in \Psi(n)\}$$

and

$$\{m(\theta_1\xi'_1 + \cdots + \theta_{n-1}\xi'_{n-1}) \mid m \text{ a monomial in } \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}\}$$

span

$$\wedge\{\Theta_{n-1}, \Xi'_{n-1}\},$$

as desired.

Example 4.5. To provide an example of the inductive process by which Lemma 4.4 demonstrates spanning, let $n = 5$ and consider the monomial

$$\theta_1\xi_1'\theta_2\xi_2'$$

Let $\delta = \theta_1\xi_1' + \theta_2\xi_2' + \theta_3\xi_3' + \theta_4\xi_4'$. We have the following calculation

$$\begin{aligned}
\theta_1\xi_1'\theta_2\xi_2'\theta_6 &= \frac{1}{3}\theta_1\xi_1'((\theta_2\xi_2' - \theta_3\xi_3') + (\theta_2\xi_2' - \theta_4\xi_4') + +(\theta_2\xi_2' + \theta_3\xi_3' + \theta_4\xi_4')) \\
&= \quad \frac{1}{3}\theta_1\xi_1'(\theta_2\xi_2' - \theta_3\xi_3') \\
&\quad + \frac{1}{3}\theta_1\xi_1'(\theta_2\xi_2' - \theta_4\xi_4') \\
&\quad + \frac{1}{3}\theta_1\xi_1'(\theta_2\xi_2' + \theta_3\xi_3' + \theta_4\xi_4') \\
&= \quad \frac{1}{6}(\theta_1\xi_1' - \theta_4\xi_4')(\theta_2\xi_2' - \theta_3\xi_3') + \frac{1}{6}(\theta_1\xi_1' + \theta_4\xi_4')(\theta_2\xi_2' - \theta_3\xi_3') \\
&\quad + \frac{1}{6}(\theta_1\xi_1' - \theta_3\xi_3')(\theta_2\xi_2' - \theta_4\xi_4') + \frac{1}{6}(\theta_1\xi_1' + \theta_3\xi_3')(\theta_2\xi_2' - \theta_4\xi_4') \\
&\quad + \frac{1}{3}\theta_1\xi_1'(\theta_2\xi_2' + \theta_3\xi_3' + \theta_4\xi_4') \\
&= \frac{1}{6}G_{\{14/23\}} + \frac{1}{6}(\theta_2\xi_2' - \theta_3\xi_3')\delta + \frac{1}{6}G_{\{13/24\}} + \frac{1}{6}(\theta_2\xi_2' - \theta_4\xi_4')\delta + \frac{1}{3}\theta_1\xi_1'\delta
\end{aligned}$$

This demonstrates that the monomial $\theta_1\xi_1'\theta_2\xi_2'$ is indeed in the span of the G_π and multiples of δ. □

5. $\mathfrak{S}_{n-1}$ module structure

In this section we will describe the Frobenius image of each bigraded piece of FDR_n as an $\mathfrak{S}_{n-1}$ module. Consider the family of subspaces:

$$V(n, k, x, y) := \text{span}\{[G_\pi] \mid \pi \in \Phi(n, k, x, y)\} \subseteq FDR_n \tag{5.1}$$

These subspaces are in fact submodules of $\text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n)$, since they are closed under the action of $\mathfrak{S}_{n-1}$. To see this, note that no step of the algorithm described in Corollary 3.11 replaces a set partition with one with a different number of size two blocks, ξ'-labelled elements, or θ-labelled elements. Since $\Phi(n) = \oplus_{k,x,y}\Phi(n, k, x, y)$ the subspaces $V(n, k, x, y)$ make up all of FDR_n:

Proposition 5.1. *The i, j-graded piece of FDR_n is a direct sum of $V(n, k, x, y)$:*

$$(FDR_n)_{i,j} = \bigoplus_{\substack{k,x,y \\ k+x=i \\ k+y=j}} V(n, k, x, y)$$

Proof. From the definition of G_π it is clear that if $\pi \in \Phi(n,k,x,y)$ then G_π has bidegree $(k+x, k+y)$. The result follows. □

To determine the structure of these modules we begin with $V(n,k,0,0)$. We first need a lemma.

Lemma 5.2. *There exists a bijection from $\Phi(n,k,0,0)$ to $SYT(n-k-1,k)$, the set of standard Young tableau of shape $\lambda = (n-k-1,k)$.*

Proof. Define a function $g : \Phi(n,k,0,0) \to \binom{[n-1]}{[k]}$ by

$$g(\pi) = \{i \in [n-1] \mid i \text{ is in a block of size 2, and is the larger element in its block.}\} \tag{5.2}$$

For example, $g(14/23/78/569) = \{3,4,8\}$. Then g is injective, it is possible to recover the preimage of a set S under g by starting with the smallest i element of S, if $g(\pi) = S$, then for π to satisfy the noncrossing condition, $\{i-1, i\}$ must be a block of π. Then the next smallest element of S must be paired with the largest element smaller than it that is not already paired, and so on. This algorithm will produce a unique preimage iff S satisfies the condition that for any $k \in [n-1]$, $|S \cap [k]| \leqslant k/2$. Define another function $h : SYT(n-k-1,k) \to \binom{[n-1]}{k}$ by

$$h(T) = \{i \in [n-1] \mid i \text{ is in the second row of } T\} \tag{5.3}$$

Then h is also injective, and $S \in h(SYT(n-k-1,k))$ iff S satisfies the condition that for any $k \in [n-1]$, $|S \cap [k]| \leqslant k/2$. So the image of h and g are the same and the result follows. □

Proposition 5.3. *We have that $V(n,k,0,0) \cong_{\mathfrak{S}_{n-1}} S^{(n-k-1,k)}$.*

Proof. Let $\lambda = (n-k-1,k)$. By Theorem 4.1 and Lemma 5.2, the dimensions of the modules agree, so by Lemma 2.1 it suffices to show that $[\mathfrak{S}_\lambda]_+$ does not kill $V(n,k,0,0)$, but $[\mathfrak{S}_\mu]_+$ does kill $V(n,k,0,0)$ for all partitions $\mu \succ \lambda$.

We begin by showing that $[\mathfrak{S}_\lambda]_+$ does not kill $V(n,k,0,0)$. Let $\pi_0 \in \Phi(n,k,0,0)$ be the partition whose blocks are

$$\{n-1, n-2k\}, \{n-2, n-2k+1\}, \ldots, \{n-k, n-k-1\}, \{1,2,3,\ldots,n-2k-1,n\}.$$

By Proposition 3.5 we have

$$[\mathfrak{S}_\lambda]_+ G_{\pi_0} \;=\; \pm \sum_{\sigma \in \mathfrak{S}_\lambda} \sigma \cdot (\theta_{n-1}\xi'_{n-1} - \theta_{n-2k}\xi'_{n-2k}) \cdots (\theta_{n-k}\xi'_{n-k} - \theta_{n-k-1}\xi'_{n-k-1}) \tag{5.4}$$

Consider the coefficient of $\theta_{n-1}\xi'_{n-1}\cdots\theta_{n-k}\xi'_{n-k}$ in the above expression. Since $\sigma \in \mathfrak{S}_\lambda$, σ permutes elements in $\{n-1,\ldots,n-k\}$ among themselves, so the only contribution to this coefficient comes from

$$\sigma \cdot (\theta_{n-1}\xi'_{n-1}\cdots\theta_{n-k}\xi'_{n-k}).$$

But this does not depend on σ since degree two monomials commute. Therefore every term in the sum in equation (5.4) contributes the same sign to the coefficient of $\theta_{n-1}\xi'_{n-1}\cdots\theta_{n-k}\xi'_{n-k}$ and therefore $V(n,k,0,0)$ is not killed by $[\mathfrak{S}_\lambda]_+$.

Now let μ be any partition of $n-1$ such that $\lambda \succ \mu$, i.e. $\mu = (n-m, m-1)$ for any $m \leqslant k$. Let $\pi \in \Phi(n,k,0,0)$. Since $m-1 < k$, there must be at least two elements of i and j of $[n-m]$ in the same block in π. Then the transposition (i,j) acts on G_π via multiplication by -1, so $(1+(i,j))G_\pi = 0$. But $[\mathfrak{S}_\lambda]_+ = A(1+(i,j))$ for some symmteric group algebra element A, so indeed $[\mathfrak{S}_\lambda]_+ G_\pi = 0$, and the result follows. □

We can use $V(n,k,0,0)$ to determine the structure of $V(n,k,x,y)$ for any x, y.

Proposition 5.4. *We have that*

$$V(n,k,x,y) \cong_{\mathfrak{S}_{n-1}} \mathrm{Ind}_{\mathfrak{S}_{n-x-y-1}\otimes\mathfrak{S}_x\otimes\mathfrak{S}_y}^{\mathfrak{S}_{n-1}} S^{(n-x-y-k-1,k)} \otimes \mathrm{sign}_{\mathfrak{S}_x} \otimes \mathrm{sign}_{\mathfrak{S}_y}.$$

Proof. We can represent an element π of $\Phi(n,k,x,y)$ by the triple (X, Y, π'), where X is the set of singletons labelled by θ, Y is the set of singletons labelled by ξ', and π' is the set partition obtained by removing all singletons from π and decrementing indices. Let $G_{(X,Y,\pi')}$ denote G_π for the corresponding π. The action of a transposition (i,j) on $G_{(X,Y,\pi')}$ is then given by

$$(i,j) \circ G_{(X,Y,\pi')} = \begin{cases} -G_{(X,Y,\pi')} & \{i,j\} \subset X \text{ or } \{i,j\} \subset Y \\ G_{(X,Y,(i,j)\circ\pi')} & \{i,j\} \subset (X \cup Y)^c \\ G_{(i,j)\circ X,(i,j)\circ Y,\pi'} \text{ otherwise} & \end{cases} \tag{5.5}$$

The proposition follows from the definition of induced representation. □

Corollary 5.5. *The Frobenius image of $V(n,k,x,y)$ is given by $s_{(n-x-y-k-1,k)} s_{(1^x)} s_{(1^y)}$. The Frobenius image of $(FDR_n)_{i,j}$ is*

$$\sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} s_{(n-x-y-k-1,k)} s_{(1^x)} s_{(1^y)}$$

Proof. This follows directly from Proposition 5.4, Proposition 5.1, and equation (2.5). □

Corollary 5.6. *The bigraded Frobenius image of $Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(FDR_n)$ is given by*

$$\mathrm{grFrob}(\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(FDR_n); q, t) = (1-qt) \prod_{i=1}^{\infty} \frac{(1+x_i qz)(1+x_i tz)}{(1-x_i z)(1-x_i qtz)}\bigg|_{z^{n-1}}$$

where the operator $(\cdots)\,|_{z^{n-1}}$ extracts the coefficient of z^{n-1}.

By Proposition 5.5 we have

$$\mathrm{grFrob}(\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(FDR_n); q, t) = \sum_i \sum_j \sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} s_{(n-x-y-k-1,k)} s_{(1^x)} s_{(1^y)} q^i t^j. \tag{5.6}$$

Applying Jacobi–Trudi [Sag91] to the $s_{(n-x-y-k-1,k)}$ terms on the right gives

$$\sum_{\substack{i,j,k,x,y\\k+x=i\\k+y=j}} s_{(n-x-y-k-1,k)}s_{(1^x)}s_{(1^y)}q^it^j = \sum_{\substack{i,j,k,x,y\\k+x=i\\k+y=j}} (h_{n-x-y-k-1}h_k - h_{n-x-y-k}h_{k-1})e_xe_yq^it^j \tag{5.7}$$

and reindexing sums gives

$$\sum_{\substack{i,j,k,x,y\\k+x=i\\k+y=j}} h_{n-x-y-k-1}h_ke_xe_yq^it^j = \sum_k h_kq^kt^kz^k \sum_x e_xq^xz^x \sum_y e_yq^yz^y \sum_m h_mz^m \bigg|_{z^{n-1}} \tag{5.8}$$

and

$$\sum_{\substack{i,j,k,x,y\\k+x=i\\k+y=j}} h_{n-x-y-k}h_{k-1}e_xe_yq^it^j = \sum_k h_kq^{k+1}t^{k+1}z^k \sum_x e_xq^xz^x \sum_y e_yq^yz^y \sum_m h_mz^m \bigg|_{z^{n-1}} \tag{5.9}$$

from which the result follows.

6. Maximal bidegrees, cyclic sieving and further directions

Let X_n denote the subset of $\Phi(n)$ corresponding to bidegrees (i, j) where $i+j = n-1$, in other words,

$$X_n = \bigcup_{2k+x+y=n-1} \Phi(n, k, x, y). \tag{6.1}$$

This set consists of noncrossing set partitions set partitions of $[n]$ in which n is in a block by itself, all other blocks are size 1 or 2, and singleton blocks other than n are labelled by θ or ξ'. The set $\{G_\pi \mid \pi \in X_n\}$ is invariant (up to sign changes) under the action of the cycle $(1, 2, \dots, n-1)$, since n is in a block by itself and rotating all elements except n cannot introduce any new crossings. We therefore have the setup for a cyclic sieving result using Springer's theorem of regular elements (Theorem 2.3).

Theorem 6.1. *The triple $(X_n, C_{n-1}, q^{\binom{n}{2}}\mathbf{fd}(FDR_n)_{i+j=n-1})$ exhibits the cyclic sieving phenomenon where C_{n-1} is the cyclic group generated by $(1, 2, \dots, n-1)$.*

Proof. This follows directly from Theorem 2.3. □

Thiel [Thi16] studied a version of this cyclic action in which rotation does not introduce a sign change, while in our setup it introduces a sign when n is odd. Thiel proved the following cyclic sieving.

Theorem 6.2 (Thiel, 2016)**.** *The triple $(X_n, C_{n-1}, C_n(q))$ exhibits cyclic sieving, where C_{n-1} is the cyclic group generated by $(1, 2, \dots, n-1)$ and $C_n(q)$ is the MacMahon q-Catalan number, defined by*

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ q \end{bmatrix}_q .$$

Thiel proved his result via direct computation of $C_n(q)$ and enumeration of fixed points instead of using representation theory, so one might wonder if our basis could give an alternate algebraic proof of his result. The expression for Frobenius image given in Corollary 5.5 allows for the computation of the fake degree as

$$\mathbf{fd}((FDR_n)_{i+j=n-1}) = \sum_{\substack{k,x,y \\ 2k+x+y=n-1}} \begin{bmatrix} n-1 \\ 2k, x, y \end{bmatrix}_q C_k(q) q^{k+\binom{x}{2}+\binom{y}{2}} \tag{6.2}$$

Combining the two cyclic sieving results it must follow that $q^{\binom{n}{2}}\mathbf{fd}((FDR_n)_{i+j=n-1})$ is equivalent to $C_n(q)$ modulo $q^{n-1}-1$. We have had difficulty in determining this equivalence directly, however, so we propose the following problem:

Problem 6.3. Is there a direct computational proof that $q^{\binom{n}{2}}\mathbf{fd}((FDR_n)_{i+j=n-1})$ and $C_n(q)$ are equivalent modulo $q^n - 1$?

Such a proof would complete an alternative representation theoretic proof of Thiel's result.

In [KR22] a similar combinatorial model for the maximal bidegree components of FDR_n was developed, with a basis indexed by all noncrossing set partitions. The action of $\mathfrak{S}_n$ on that basis could be understood in terms of Skein-like relations described by Rhoades [Rho17]. Patrias, Pechenik, and Striker [PPS22] independently discovered an alternate algebraic/geometric model for the irreducible submodule of this action generated by singleton-free noncrossing set partitions as the coordinate ring of a certain algebraic variety. They associated to each partition a polynomial in this coordinate ring defined in terms of matrix minors, and showed that these polynomials satisfied the Skein relations described in [Rho17]. This suggests the following problem:

Problem 6.4. Can our basis for $S^{(n-k-1,k)}$ be realized as a set of polynomials, similarly to the methods of Patrias, Pechenik, and Striker [PPS22]?

One reason for thinking an analogous model might exist is that the relation of block operators described in Lemma 3.7 also appears in the maximal bidegree model and corresponds to a certain identity of two-by-two matrix minors in the work of Patrias, Pechenik and Striker. Mimicking their construction would therefore give a model for the submodule generated by partitions in $\Phi(n)$ for which the block containing n is at most size two, but we have as yet been unable to discover a treatment of larger blocks satisfying our other relations.

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