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Spreading of the Free Boundary of an Ideal Fluid in a Vacuum

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Abstract

The diameter of a region occupied by an ideal fluid surrounded by vacuum will be shown to grow linearly in time provided the pressure is positive and there are no singularities. A family of explicit spherically symmetric, self-similar global solutions is constructed which illustrate the result in the compressible case.

Introduction

Consider an ideal fluid occupying a bounded region in space surrounded by vacuum. The boundary of this region is free to move with the fluid. We shall show under the physically reasonable assumptions of positive density and pressure that the diameter of this region grows at least linearly in time as long as the fluid motion remains C^1. The linear spreading rate is illustrated with an explicit family of global spherically symmetric, self-simialr compressible solutions.

The method of proof is based on simple identities for integral averages as introduced in [1] and further explored in [2], [3], [4]. In the earlier works [2], [3], [1], the goal was to establish formation of singularities in classical solutions in the entire space, but the relationship between regularity and propagation of wave fronts plays a major role. For classical solutions in the entire space, the maximum propagation speed of a front into a constant state is *a priori* determined by the background state, whereas in the case of a free boundary, the speed at which the disturbance penetrates the vacuum will, in general, depend on the unknown solution. Although there is a certain amount of flexibility, the method will be applied in two situations: compressible ideal gasses and incompressible ideal fluids, see Theorems 1 and 2.

The local existence of classical solutions to the initial free boundary value problem for ideal fluids has been much-studied in the last decade. The full water wave problem for incompressible, irrotational flow with gravity was considered in 2d and 3d, see [5] and [6], respectively. Estimates for the incompressible case without gravity were initiated in [7], well-posedness for the linearized problem was established in [8], and well-posedness for the nonlinear problem was subsequently proven in [9], see also [10], [11]. The well-posedness for compressible liquids (positive fluid density on the free boundary) was established in [12], see also [13], [14]. Finally, local existence for compressible gasses (vanishing density on the free boundary) was recently shown in [15], [16], see also [17]. A key role in these works is played by a vacuum boundary condition. For liquids, the normal derivative of the pressure is negative on the free boundary. This implies that the pressure is positive near the boundary, which is consistent with condition (6) in Theorem 2. For compressible fluids, the normal derivative of the square of the sound speed must be negative. This is consistent with our result in Theorem 1, as well as with the example presented in Theorem 3.

General problem

Suppose that for times $0 \leq t \leq T$ a three-dimensional fluid occupies a bounded open region Ω_t, with C^1 boundary $\partial\Omega_t$. Define the space-time region

$$\mathcal{C}_T = \{(t,x) : x \in \Omega_t,\ 0 < t < T\},$$

and assume that its lateral (free) boundary

$$\mathcal{B}_T = \{(t,x) : x \in \partial\Omega_t,\ ; 0 \leq t < T\},$$

is C^1 with unit outward normal $\eta(t,x) \in \mathbb{R}^4$, for $(t,x) \in \mathcal{B}_T$.

The motion of an ideal fluid is modeled by the Euler equations involving the density ρ, velocity u, and the pressure p. These are

$$D_t\rho + \rho\nabla \cdot u = 0, \qquad \text{in} \quad \mathcal{C}_T, \tag{1a}$$

$$\rho D_t u + \nabla p = 0, \qquad \text{in} \quad \mathcal{C}_T, \tag{1b}$$

where $D_t = \partial_t + u \cdot \nabla$ is the usual material time derivative.

Assume that Ω_t moves with the fluid flow, so that

$$(1,u) \cdot \eta = 0, \qquad \text{on} \quad \mathcal{B}_T. \tag{2a}$$

We assume that the fluid is surrounded by vacuum, whence the normal stress should also vanish on the free boundary. In the case of fluids, the appropriate boundary condition is

$$p = 0, \qquad \text{on} \quad \mathcal{B}_T. \tag{2b}$$

Of course, the system (1a), (1b) is underdetermined, and further assumptions are necessary. We shall consider two cases: ideal gasses and incompressible ideal fluids.

The compressible case

In the compressible case, the pressure is expressed as a function of other state variables, in this instance the density and the internal energy ε. The system (1a), (1b) is supplemented by an evolution equation for ε and an equation of state.

$$D_t\rho + \rho\nabla\cdot u = 0, \qquad \text{in} \quad \mathcal{C}_T, \tag{3a}$$

$$\rho D_t u + \nabla[p(\rho,\varepsilon)] = 0, \qquad \text{in} \quad \mathcal{C}_T, \tag{3b}$$

$$D_t\varepsilon + (\gamma-1)\varepsilon\nabla\cdot u = 0, \qquad \text{in} \quad \mathcal{C}_T. \tag{3c}$$

Here we shall employ the state equation of an ideal gas

$$p(\rho,\varepsilon) = (\gamma-1)\rho\varepsilon, \quad \gamma > 1, \tag{3d}$$

in which γ is the adiabatic index.

To give a concise statement of the results, we define the following quantities:

$$M(t) = \int_{\Omega_t} \rho\, dx \qquad \text{(total mass)} \tag{4a}$$

$$\bar{x}(t) = M(t)^{-1}\int_{\Omega_t} \rho x\, dx \qquad \text{(center of mass)} \tag{4b}$$

$$\bar{u}(t) = M(t)^{-1}\int_{\Omega_t} \rho u\, dx \qquad \text{(average velocity)} \tag{4c}$$

$$X(t) = \int_{\Omega_t} \tfrac{1}{2}\rho|x-\bar{x}|^2\, dx \qquad \text{(moment of inertia)} \tag{4d}$$

$$Y(t) = \int_{\Omega_t} \rho\langle x-\bar{x}, u-\bar{u}\rangle\, dx \qquad \text{(average radial momentum)} \tag{4e}$$

$$E(t) = \int_{\Omega_t} \rho\left[\tfrac{1}{2}|u-\bar{u}|^2 + \varepsilon\right]\, dx \qquad \text{(total energy)} \tag{4f}$$

Our first result gives conditions under which compressible solutions spread.

Theorem 1. *Let $\rho,\varepsilon \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ be a solution of* (3a)–(3c), (2a), (2b) *with $\rho(0,\cdot) > 0$ and $\varepsilon(0,\cdot) \geq 0$ in Ω_0. Then*

$$[\operatorname{diam}\Omega_t]^2 \geq [\underline{\sigma}E(0)t^2 + Y(0)t + X(0)]/M(0), \quad 0 \leq t < T,$$

with $\underline{\sigma} = \min\{1, \frac{3}{2}(\gamma-1)\}$.

Remark 1. If the initial data are equal to smooth functions on $\mathbb{R}^3$ with support in Ω_0, then there can be no spreading and a singularity must develop in finite time, see [3], for example.

Remark 2. A version of this result can be formulated for elastic solids, but the assumptions do not seem to allow for physically meaningful constitutive relations.

The incompressible case

For incompressible motion, the density remains constant $\rho = \bar{\rho} > 0$, and (1a), (1a) reduce to

$$\nabla \cdot u = 0, \qquad \text{in} \quad \mathcal{C}_T, \tag{5a}$$

$$\bar{\rho} D_t u + \nabla p = 0, \qquad \text{in} \quad \mathcal{C}_T. \tag{5b}$$

The quantities (4a)–(4e) are defined using $\rho = \bar{\rho}$. Define the energy

$$E_K(t) = \int_{\Omega_t} \tfrac{1}{2} \bar{\rho} |u - \bar{u}|^2 \, dx.$$

Theorem 2. *Let* $p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ *be a solution of* (5a), (5b), (2a), (2b). *If* $\bar{\rho} > 0$ *and*

$$\int_{\Omega_t} p \, dx > 0, \quad 0 \le t < T, \tag{6}$$

then

$$[\operatorname{diam} \Omega_t]^2 \ge [E_K(0) t^2 + Y(0) t + X(0)]/M(0), \quad \textit{for} \quad 0 \le t < T.$$

In the irrotational case, assumption (6) is unnecessary.

Corollary 1. *Let* $\bar{\rho} > 0$, $p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ *be a solution of* (5a), (5b)(2a), (2b). *If* u *is irrotational, then*

$$[\operatorname{diam} \Omega_t]^2 \ge [E(0) t^2 + Y(0) t + X(0)]/M(0), \quad \textit{for} \quad 0 \le t < T.$$

Analysis

Both theorems rely on the following identites.

Lemma 1. *Let* $\rho, p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ *be a solution of* (1a), (1b), (2a), (2b). *Then*

$$M(t) = M(0), \tag{7a}$$

$$\bar{x}(t) = \bar{x}(0) + \bar{u}(0) t, \tag{7b}$$

$$\bar{u}(t) = \bar{u}(0), \tag{7c}$$

$$X'(t) = Y(t), \tag{7d}$$

$$Y'(t) = \int_{\Omega_t} [\rho |u - \bar{u}|^2 + 3p] \, dx. \tag{7e}$$

Proof. Let $\rho, p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ be a solution of (1a), (1b), (2a), (2b).

For $y \in \overline{\Omega_0}$, define a flow $x(t, y)$ by

$$\frac{d}{dt} x(t, y) = u(t, x(t, y)), \quad x(0, y) = y.$$

Thanks to the boundary condition (2a) and the fact that $u \in C^1(\overline{\mathcal{C}_T})$, $x(t, y)$ is defined on $[0, T) \times \overline{\Omega_0}$ and $y \mapsto x(t, y)$ is a C^1 diffeomorphism of $\overline{\Omega_0}$ onto $\overline{\Omega_t}$, $0 \leq t < T$. It follows that $J(t, y) = \det D_y x(t, y)$ satisfies

$$D_t J(t, y) = \nabla \cdot u(t, x(t, y))\ J(t, y), \quad J(0, y) = 1. \tag{8a}$$

By (1a), (8a) we obtain

$$D_t[\rho(t, x(t, y)) J(t, y)] = 0, \quad \text{for} \quad (t, y) \in (0, T) \times \Omega_0,$$

and since

$$\rho(t, x(t, y)), J(t, y) \in C^0([0, T] \times \overline{\Omega_0}),$$

we see that

$$\rho(t, x(t, y)) J(t, y) = \rho(0, y), \quad \text{for} \quad y \in \overline{\Omega_0}. \tag{8b}$$

The identities (7a)-(7e) are a consequence of the transport theorem, which says that if $\Omega_t = \{x(t, y) \in \mathbb{R}^3 : y \in \Omega_0\}$ and $f \in C^0(\overline{\mathcal{C}_T})$, $D_t f \in C^0(\overline{\mathcal{C}_T})$, then

$$\frac{d}{dt} \int_{\Omega_t} \rho f(t, x)\, dx = \int_{\Omega_t} \rho D_t f(t, x)\, dx. \tag{9}$$

This is seen by changing the integral to material coordinates y and using (8b):

$$\begin{aligned} \int_{\Omega_t} \rho f(t, x)\, dx &= \int_{\Omega_0} \rho f(t, x(t, y))\ J(t, y)\, dy \\ &= \int_{\Omega_0} \rho(0, y) f(t, x(t, y))\, dy. \end{aligned} \tag{10}$$

Thus, if $f \in C^0(\overline{\mathcal{C}_T})$, $D_t f \in C^0(\overline{\mathcal{C}_T})$, $V \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$ and $g \in C^0(\overline{\mathcal{C}_T})$ satisfy the transport equation

$$\rho D_t f + \nabla \cdot V = g, \quad \text{with} \quad (0, V) \cdot \eta = 0 \quad \text{on} \quad \mathcal{B}_T,$$

then

$$\frac{d}{dt} \int_{\Omega_t} \rho f(t, x)\, dx = \int_{\Omega_t} g(t, x)\, dx.$$

This fact is now used together with the pdes (1a), (1b) and the boundary condition (2b) to compute the derivatives of the five quantities defined in (4a)-(4e).

Since

$$M'(t) = \int_{\Omega_t} \rho D_t(1) dx = 0,$$

we see that (7a) holds.

Using this and (2b), we may apply the transport theorem

$$\bar{u}'(t) = M(0)^{-1} \int_{\Omega_t} \rho D_t u \, dx = -M(0)^{-1} \int_{\Omega_t} \nabla p \, dx = 0,$$

which gives (7c).

Next, we obtain (7b) from

$$\bar{x}'(t) = M(0)^{-1} \int_{\Omega_t} \rho D_t x \, dx = M(0)^{-1} \int_{\Omega_t} \rho u \, dx = \bar{u}(t) = \bar{u}(0).$$

The remaining two identities (7d), (7e) result from the following transport equations,

$$\rho D_t \tfrac{1}{2}|x - \bar{x}|^2 = \rho \langle x - \bar{x}, u - \bar{u} \rangle$$
$$\rho D_t \langle x - \bar{x}, u - \bar{u} \rangle + \nabla \cdot p(x - \bar{x}) = \rho |u - \bar{u}|^2 + 3p.$$

Now, for solutions of the compressible system, we show that the moment of interia $X(t)$ is sandwiched between a pair of quadratic functions. In the special case of a monoatomic gas, $\gamma = 5/3$, $X(t)$ is equal to a quadratic.

Lemma 2. *Let* $\rho, \varepsilon \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ *be a solution of* (3a)–(3c), (2a), (2b) *with* $\rho(0, \cdot) > 0$ *and* $\varepsilon(0, \cdot) \geq 0$ *in* Ω_0. *Then*

$$X(t) \geq \underline{\sigma} E(0) t^2 + Y(0) t + X(0), \quad 0 \leq t < T, \tag{11a}$$

with $\underline{\sigma} = \min\{1, \frac{3}{2}(\gamma - 1)\}$, *and*

$$X(t) \leq \overline{\sigma} E(0) t^2 + Y(0) t + X(0), \tag{11b}$$

with $\overline{\sigma} = \max\{1, \frac{3}{2}(\gamma - 1)\}$.

Proof. Let $\rho, \varepsilon \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ be a solution of (3a)–(3c), (2a), (2b).

It follows from (8b) that

$$\rho(0, \cdot) > 0 \quad \text{in} \quad \Omega_0 > 0 \quad \text{implies that} \quad \rho > 0 \quad \text{in} \quad \mathcal{C}_T. \tag{12a}$$

In addition to (8b), we also have using (8a), (3c) that

$$\varepsilon(t, x(t, y)) J(t, y)^{\gamma - 1} = \varepsilon(0, y), \quad \text{for} \quad y \in \overline{\Omega_0}, \tag{12b}$$

and from (12b), we see that that

$$\varepsilon(0, \cdot) \geq 0 \quad \text{in} \quad \Omega_0 > 0 \quad \text{implies that} \quad \varepsilon \geq 0 \quad \text{in} \quad \mathcal{C}_T. \tag{12c}$$

Using (3b), (3c), (3d), and (7c), it is straightforward to verify that

$$\rho D_t [\tfrac{1}{2}|u - \bar{u}|^2 + \varepsilon] + \nabla \cdot p(u - \bar{u}) = 0$$

holds. It follows from (3c) that $D_t\varepsilon \in C^0(\overline{\mathcal{C}_T})$. We may thus apply the transport theorem (9) with the boundary condition (2a) to conclude that the total energy is conserved:

$$E(t) = \int_{\Omega_t} \rho[\tfrac{1}{2}|u-\bar{u}|^2 + \varepsilon]\; dx = E(0). \tag{13}$$

By (7d), (7e), (13), (3d), we may write

$$X''(t) = Y'(t) = 2E(0) + (3\gamma - 5)\int_{\Omega_t} \rho\varepsilon\; dx$$

and also

$$X''(t) = Y'(t) = 3(\gamma-1)E(0) + \tfrac{1}{2}(5-3\gamma)\int_{\Omega_t} \rho|u-\bar{u}|^2\; dx.$$

Therefore, using (12a) and (12c) (when $\gamma > 5/3$), we obtain the lower bound

$$X''(t) \geq 2\underline{\sigma}E(0), \quad \underline{\sigma} = \min\{1, \tfrac{3}{2}(\gamma-1)\}.$$

Similarly, using (12a) and (12c) (when $\gamma < 5/3$), we obtain the upper bound

$$X''(t) \leq 2\overline{\sigma}E(0), \quad \overline{\sigma} = \max\{1, \tfrac{3}{2}(\gamma-1)\}.$$

The lemma follows by integration.

Proof of Theorem 1. We shall verify shortly that

$$\sup_{x\in\Omega_t} |x-\bar{x}| \leq \text{diam}\;\Omega_t. \tag{14}$$

Given (11a), (14), the following estimate now completes the argument:

$$\begin{aligned} X(t) &= \int_{\Omega_t} \rho|x-\bar{x}|^2\; dx \\ &\leq \sup_{x\in\Omega_t} |x-\bar{x}|^2 \int_{\Omega_t} \rho\; dx \\ &\leq [\text{diam}\;\Omega_t]^2 M(0). \end{aligned}$$

To finish the proof, we demonstrate (14). Suppose that $x' \in \Omega_t$ and $x' \neq \bar{x}(t) \equiv \bar{x}$. Define the unit vector

$$\omega = (x'-\bar{x})/|x'-\bar{x}|.$$

By (7a), (7b), we have

$$\int_{\Omega_t} \rho\langle x-\bar{x}, \omega\rangle\; dx = \langle \int_{\Omega_t} \rho(x-\bar{x})\; dx, \omega\rangle = \langle 0, \omega\rangle = 0.$$

Thus, by (12a), both of the domains

$$\Omega_t^+ = \Omega_t \cap \{x : \langle x-\bar{x}, \omega\rangle > 0\} \quad \text{and} \quad \Omega_t^- = \Omega_t \cap \{x : \langle x-\bar{x}, \omega\rangle < 0\}$$

have positive measure. By the definition of ω, it follows that $x' \in \Omega_t^+$. Choose any $x'' \in \Omega_t^-$. By virtue of the fact that $\bar{x}$ is the point on the plane $\langle x-\bar{x}, \omega\rangle = 0$ closest to x', we have

$$|x' - \bar{x}| < |x' - x''| \leq \sup_{x,y\in\Omega_t} |x - y| = \operatorname{diam} \Omega_t.$$

Now take the supremum over all $x' \in \Omega_t$ to get (14).

Remark 3. It is possible that $X'(0) = Y(0) < 0$, in which case the function $X(t)$ is decreasing, initially. Since $X(t)$ remains positive, as long as the solution is C^1, an upper bound for $X(t)$ by a function which vanishes in finite time would signal the development of a singularity and possible collapse of Ω_t. Notice that when $Y(0) < 0$, the quadratic upper bound in (11b) has a minimum value of

$$\frac{4\overline{\sigma}E(0)X(0) - [Y(0)]^2}{4\overline{\sigma}E(0)}$$

at the positive time

$$t = -Y(0)/[2\overline{\sigma}E(0)].$$

However, this minimum value is positive, by the Cauchy-Schwarz inequality:

$$[Y(0)]^2 \leq \int_{\Omega_0} \rho(0)|x|^2\, dx \int_{\Omega_0} \rho|u|^2(0)\, dx < 4X(0)E(0) \leq 4\overline{\sigma}X(0)E(0).$$

One could force this minimum to be negative when $\gamma \geq 5/3$, by assuming the physically implausible condition $\varepsilon(0,\cdot) \leq 0$.

Proof of Theorem 2. Let $\bar{\rho} > 0$, $p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$ be a solution of (5a), (5b), (2a), (2b). By Lemma 1 and assumption (6), we have

$$X''(t) = \int_{\Omega_t} [\bar{\rho}|u - \bar{u}|^2 + 3p]\, dx \geq 2E_K(t).$$

Using equations (5a), (5b), (7c), we see that

$$D_t \tfrac{1}{2}\bar{\rho}|u - \bar{u}|^2 + \nabla \cdot p(u - \bar{u}) = 0,$$

so by the transport theorem, we have that $E_K(t)$ is conserved. The remainder of the proof is the same as for Theorem 1.

Proof of Corollary 1. Since u is irrotational, there exists $\phi \in C^2(\overline{\mathcal{C}_T})$ such that $u = \nabla\phi$. Taking the divergence of (5a), we find that p is a weak solution of

$$-\Delta p = |\nabla\phi|^2.$$

Since $p \in C^0(\overline{\mathcal{C}_T}) \cap C^1(\mathcal{C}_T)$, $u \in C^1(\overline{\mathcal{C}_T})$, the maximum principle implies that $p \geq 0$ in $\mathcal{C}_T$ so that (6) holds and we may apply Theorem 2.

Example

To conclude, we construct an explicit family of global spherically symmetric, self-similar, compressible solutions which illustrate Theorem 1. An analogous construction was performed in [18] for isentropic flows with damping, although there the goal was to establish time asymptotics of solutions.

In the next two lemmas, we characterize the initial data for our example.

Lemma 3. *Let*

$$\mathcal{Y} = \{f \in C^0[0,1] \cap C^1[0,1) : f(s) > 0,\ s \in [0,1),\ f'(0) = f(1) = 0\}.$$

Choose a function $\bar{\rho}_0 \in \mathcal{Y}$ *such that*

$$0 < \lim_{s \to 1^-} (1-s)^{-\delta} \bar{\rho}_0(s) < \infty, \quad \textit{for some} \quad \delta > 0. \tag{15}$$

If

$$\bar{\varepsilon}_0(s) = \frac{\lambda \int_s^1 \varsigma \bar{\rho}_0(\varsigma)\, d\varsigma}{\bar{\rho}_0(s)}, \quad \lambda > 0, \tag{16}$$

then $\bar{\varepsilon}_0 \in \mathcal{Y}$*, and*

$$\lim_{s \to 1^-} (1-s)^{-1} \bar{\varepsilon}_0(s) = \lambda (1+\delta)^{-1}. \tag{17}$$

Proof. The boundary behavior (17) follows from (16) and (15). By (17), we get $\bar{\varepsilon}_0(1) = 0$, and then it is clear that $\bar{\varepsilon}_0 \in \mathcal{Y}$.

Lemma 4. *Fix* $\lambda > 0$*. Define the spherically symmetric functions*

$$\rho_0(x) = \bar{\rho}_0(|x|), \quad \varepsilon_0(x) = \bar{\varepsilon}_0(|x|), \tag{18}$$

in which $\bar{\rho}_0$, $\bar{\varepsilon}_0$ *are given in Lemma 3. Let* $\Omega_0 = \{|y| < 1\}$*. Then the functions* ρ_0, ε_0 *are nonnegative, they belong to* $C^0(\overline{\Omega_0}) \cap C^1(\Omega_0)$*, they vanish on* $\partial\Omega_0$*, and* ε_0 *satisfies the vacuum boundary condition*

$$\lim_{|x| \to 1^-} (1-|x|)^{-1} \varepsilon_0(x) = \lambda/(1+\delta). \tag{19}$$

Proof. A spherically symmetric function $f(x) = \bar{f}(|x|)$ belongs to $C^0(\overline{\Omega_0}) \cap C^1(\Omega_0)$ if and only if $\bar{f}$ belongs to $C^0[0,1] \cap C^1[0,1)$ and $\bar{f}'(0) = 0$.

The next theorem presents the construction of our explicit solutions.

Theorem 3. *Define* $\Omega_0 = \{|y| < 1\}$*. For any* $\alpha \in \mathbb{R}$*, let* $u_0(x) = \alpha\, x$*. Given* $\lambda > 0$*, let* ρ_0, ε_0 *be the spherically symmetric functions defined in Lemmas 3 and 4. The initial value problem* (3a), (3b), (3c), (2a), (2b) *has a global, spherically symmetric, self-similar solution, and the fluid domain* Ω_t *is a ball which satisfies*

$$\lim_{t \to \infty} t^{-1} \operatorname{diam} \Omega_t = 2 \left[\alpha^2 + 2\lambda/3\right]^{1/2}.$$

PROOF. We start with a motion on Ω_0 of the form

$$x(t,y) = A(t)y, \quad A(0) = 1, \quad A'(0) = \alpha, \tag{20a}$$

where the scalar function $A > 0$ will be determined below. The fluid domain remains a ball:

$$\Omega_t = \{|x| < A(t)\}. \tag{20b}$$

Note that the Jacobian of the motion is

$$J(t,y) = \det D_y x(t,y) = A(t)^3. \tag{20c}$$

The material velocity associated to the motion (20a) is

$$u(t, x(t,y)) = \frac{d}{dt} x(t,y) = A'(t)y, \tag{21a}$$

which when converted to spatial coordinates yields

$$u(t,x) = A(t)^{-1}A'(t)x. \tag{21b}$$

Thus, we have that

$$u(0,x) = u_0(x) = \alpha\, x. \tag{21c}$$

Next, we impose our choices for the spherically symmetric initial density ρ_0 and internal energy ε_0. According to (8b), (12b), (20c), and (18), we must take

$$\rho(t,x) = A(t)^{-3}\rho(0, x/A(t)) = A(t)^{-3}\rho_0(x/A(t)) = A(t)^{-3}\bar{\rho}_0(r/A(t)), \tag{22a}$$

and

$$\begin{aligned} \varepsilon(t,x) &= A(t)^{-3(\gamma-1)}\varepsilon(0, x/A(t)) \\ &= A(t)^{-3(\gamma-1)}\varepsilon_0(x/A(t)) = A(t)^{-3(\gamma-1)}\bar{\varepsilon}_0(r/A(t)), \end{aligned} \tag{22b}$$

where $r = |x|$. Using (3d), (22a), (22b), we have that

$$p(t,x) = A(t)^{-3\gamma}\bar{p}_0(r/A(t)), \quad \bar{p}_0 = (\gamma - 1)\bar{\rho}_0\bar{\varepsilon}_0. \tag{22c}$$

With these purely kinematic observations in hand, we now focus on satisfying (3b). We compute from (21a)

$$D_t u(t,x) = \frac{d}{dt}[u(t, x(t,y))]|_{y=x/A(t)} = A(t)^{-1}A''(t)x, \tag{23a}$$

and from (22c)

$$\nabla p(t,x) = A(t)^{-3\gamma-1}\bar{p}_0'(r/A(t))\frac{x}{r}, \quad r = |x|. \tag{23b}$$

Thus, by (22a), (23a), (23b), we see that (3b) will hold provided that

$$\bar{\rho}_0(s)A(t)^{-4}A''(t) + s^{-1}\bar{p}_0'(s)A(t)^{-3\gamma-2} = 0, \tag{23c}$$

where $s = r/A(t)$. Finally, separating variables, we conclude that (23c) is satisfied if

$$A(t)^{3\gamma-2}A''(t) = -[s\bar{\rho}_0(s)]^{-1}\bar{p}_0'(s) = \text{Constant}, \tag{23d}$$

for all $0 \le s \le 1$ and $0 \le t < T$. From our definitions (16), (22c), we find that (23d) holds with the constant $(\gamma-1)\lambda$ on the right.

Now consider the resulting ODE for the radius $A(t)$:

$$A''(t) = (\gamma-1)\lambda A(t)^{-3\gamma+2}, \quad A(0) = 1, \quad A'(0) = \alpha. \tag{24a}$$

This equation has the Hamiltonian

$$H(v,w) = \frac{1}{2}w^2 + \frac{\lambda}{3}v^{-3(\gamma-1)}.$$

which is conserved along solution trajectories, and so

$$\begin{aligned}\frac{1}{2}[A'(t)]^2 + \frac{\lambda}{3}[A(t)]^{-3(\gamma-1)} &= H(A(t), A'(t)) \\ &= H(A(0), A'(0)) = \frac{1}{2}\alpha^2 + \frac{\lambda}{3}.\end{aligned} \tag{24b}$$

Since $\lambda > 0$, it follows that the ODE (24a) has a *globally defined* positive solution $A(t)$. Because $A(t) > 0$, equation (24a) says that $A'(t)$ is strictly increasing. Therefore, thanks to the conservation law (24b), we obtain

$$A'(t) \nearrow L \equiv [2H(A(0), A'(0)]^{1/2} = [\alpha^2 + 2\lambda/3]^{1/2} \quad \text{and} \quad A(t) \to \infty, \quad \text{as} \quad t \to \infty.$$

Given any $\mu > 0$, there exists a time t_μ such that

$$L - \mu < A'(t) < L, \quad \text{for all} \quad t \ge t_\mu.$$

After an integration, this in turn yields the estimate

$$(L-\mu)(1 - t_\mu/t) + A(t_\mu)/t < A(t)/t < L(1 - t_\mu/t) + A(t_\mu)/t,$$

for $t \ge t_\mu$, and thus,

$$L - \mu \le \liminf_{t\to\infty} A(t)/t \le \limsup_{t\to\infty} A(t)/t \le L.$$

Since $\mu > 0$ is arbitrary, we obtain

$$\lim_{t\to\infty} A(t)/t = L.$$

This proves the result, since $\operatorname{diam} \Omega_t = 2A(t)$, by (20b).

Remark 4. This example yields an asymptotic linear growth rate for the diameter of the fluid domain. Thus, in general the order of lower bound of Theorem 1 is sharp. The asymptotic spreading rate is constant along each level curve of the Hamiltonian. By choosing α and λ appropriately, we can achieve any spreading rate $0 < L < \infty$.

Remark 5. If $\alpha < 0$, then the domain Ω_t contracts for some finite time interval, reaching a minimum diameter $0 < \underline{A} < 1$ determined by the equation $H(\underline{A}, 0) = H(1, \alpha)$, before it begins to spread.

Remark 6. The sign of the parameter λ has a dramatic effect on the behavior of the solution. If $\lambda = 0$ and $\alpha < 0$ or if $\lambda < 0$, then the radius $A(t)$ collapses to zero in finite time. However, this corresponds to the physically dubious situation of negative pressure and internal energy.

Remark 7. The boundary behavior (19) is consistent with the vacuum boundary condition, a key assumption for local well-posedness in [15], [16], [14].

Remark 8. In the isentropic case, $p = \rho^\gamma$, i.e. $\varepsilon = \frac{1}{(\gamma-1)}\rho^{\gamma-1}$, the definition (16) yields

$$\bar{\rho}_0(s)/\bar{\rho}_0(0) = [1 - s^2]^{1/(\gamma-1)},$$

as in [18].

Remark 9. The solution of Theorem 3 can be related to the quantities defined previously. For example, we have

$$A(t) = \left[\frac{X(t)}{X(0)}\right]^{1/2}, \quad A'(t) = \frac{Y(t)}{2[X(t)X(0)]^{1/2}}, \quad H(A(t), A'(t)) = \frac{E(0)}{2X(0)}.$$

Thus, by Lemma 2, we see that $A(t)^2$ is sandwiched between quadratic functions, and in the case of a monoatomic gas, $\gamma = 5/3$, $A(t)^2$ is quadratic. Also, Theorem 3 translates to

$$\lim_{t\to\infty} \frac{\operatorname{diam} \Omega_t}{t} = \lim_{t\to\infty} \frac{2}{t}\left[\frac{X(t)}{X(0)}\right]^{1/2} = 2\left[\frac{E(0)}{X(0)}\right]^{1/2}.$$

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