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OFF-DIAGONAL ASYMPTOTIC PROPERTIES OF BERGMAN KERNELS ASSOCIATED TO ANALYTIC KÄHLER POTENTIALS

HAMID HEZARI, ZHIQIN LU, AND HANG XU

ABSTRACT. We prove a new off-diagonal asymptotic of the Bergman kernels associated to tensor powers of a positive line bundle on a compact Kähler manifold. We show that if the Kähler potential is real analytic, then the Bergman kernel accepts a complete asymptotic expansion in a neighborhood of the diagonal of shrinking size $k^{-\frac{1}{4}}$. These improve the earlier results in the subject for smooth potentials, where an expansion exists in a $k^{-\frac{1}{2}}$ neighborhood of the diagonal. We obtain our results by finding upper bounds of the form $C^m m!^2$ for the Bergman coefficients $b_m(x, \bar{y})$, which is an interesting problem on its own. We find such upper bounds using the method of [BeBeSj08]. We also show that sharpening these upper bounds would improve the rate of shrinking neighborhoods of the diagonal $x = y$ in our results. In the special case of metrics with local constant holomorphic sectional curvatures, we obtain off-diagonal asymptotic in a fixed (as $k \rightarrow \infty$) neighborhood of the diagonal, which recovers a result of Berman [Ber08] (see Remark 3.5 of [Ber08] for higher dimensions). In this case, we also find an explicit formula for the Bergman kernel mod $O(e^{-k\delta})$.

1. INTRODUCTION

Let $(L, h) \rightarrow M$ be a positive Hermitian holomorphic line bundle over a compact complex manifold of dimension n . The metric h induces the Kähler form $\omega = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(h)$ on M . For k in \mathbb{N} , let $H^0(M, L^k)$ denote the space of holomorphic sections of L^k . The Bergman projection is the orthogonal projection $\Pi_k : L^2(M, L^k) \rightarrow H^0(M, L^k)$ with respect to the natural inner product induced by the metric h^k and the volume form $\frac{\omega^n}{n!}$. The *Bergman kernel* K_k , a section of $L^k \otimes \bar{L}^k$, is the distribution kernel of Π_k . Given $p \in M$, let (V, e_L) be a local trivialization of L near p . We write $|e_L|_h^2 = e^{-\phi}$ and call ϕ a local Kähler potential. In the frame $e_L^k \otimes \bar{e}_L^k$, the Bergman kernel $K_k(x, y)$ is understood as a function on $V \times V$. We note that on the diagonal $x = y$, the function $K_k(x, x)e^{-k\phi(x)}$ is independent of the choice of the local frame, hence it is a globally defined function on M called the *Bergman function*, which is also equal to $|K_k(x, x)|_{h^k}$.

Zelditch [Ze98] and Catlin [Ca99] proved that on the diagonal $x = y$, the Bergman kernel accepts a complete asymptotic expansion of the form

$$(1.1) \quad K_k(x, x)e^{-k\phi(x)} \sim \frac{k^n}{\pi^n} \left(b_0(x, \bar{x}) + \frac{b_1(x, \bar{x})}{k} + \frac{b_2(x, \bar{x})}{k^2} + \dots \right).$$

Near the diagonal, i.e. in a $\frac{1}{\sqrt{k}}$ -neighborhood of the diagonal, one has a scaling asymptotic expansion for the Bergman kernel (see [ShZe02, MaMa07, MaMa13, LuSh15, HeKeSeXu16]). For $d(x, y) \gg \sqrt{\frac{\log k}{k}}$, where d is the Riemannian distance induced by ω , no useful asymptotics are known. However, there are off-diagonal upper bounds of Agmon type

$$(1.2) \quad |K_k(x, y)|_{h^k} \leq Ck^n e^{-c\sqrt{k}d(x, y)},$$

proved for smooth metrics by Ma-Marinescu [MaMa15]¹ In fact as shown by Christ in [Ch13b], away from the diagonal one has better decay estimates. More precisely, for any $\delta > 0$ there exist K and a function $f(k) \rightarrow \infty$ as $k \rightarrow \infty$ such that²

$$|K_k(x, y)|_{h^k} \leq C e^{-f(k)\sqrt{k \log k}}, \quad \text{for } d(x, y) > \delta \text{ and } k > K.$$

When the metric h is real analytic the much improved estimate

$$|K_k(x, y)|_{h^k} \leq C k^n e^{-ckd^2(x, y)},$$

holds³ (see estimate 6.2 in Remark 6.6 of [Ch03]). The goal of this article is to prove an asymptotic expansion in a $k^{-1/4}$ neighborhood of the diagonal in the real analytic case. In particular, we show that in the real analytic case, uniformly for all sequences x_k and y_k with $d(x_k, y_k) \leq k^{-1/4}$, we have

$$|K_k(x_k, y_k)|_{h^k} \sim \frac{k^n}{\pi^n} e^{-\frac{kD(x_k, y_k)}{2}}, \quad \text{as } k \rightarrow \infty,$$

where $D(x, y)$ is Calabi's diastasis function (1.4), which is controlled from above and below by $d^2(x, y)$. Before we state the results we must also mention that in the literature⁴ there is an *ill* off-diagonal asymptotic expansion for the Bergman kernel of the form

$$(1.3) \quad K_k(x, y) = e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} k^{-N+n} O_N(1),$$

which holds for all $d(x, y) \leq \delta$ for some $\delta > 0$. Here, $\psi(x, \bar{y})$ and $b_j(x, \bar{y})$ are holomorphic extensions of $\phi(x)$ and $b_j(x, \bar{x})$ from (1.1). However, note that this expansion is only useful when the term $e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} k^{-N+n}$ is a true remainder term, i.e. it is less than the principal term $k^n e^{k\psi(x, \bar{y})}$ in size, which holds only when $d(x, y) \leq C_N \sqrt{\frac{\log k}{k}}$. Thus, to obtain an asymptotic when $d(x, y) \gg \sqrt{\frac{\log k}{k}}$, we need a quantitative description of the term $O_N(1)$ and the coefficients b_j in (1.3) which would allow N to depend on k accordingly.

We now state our main result and its corollaries.

Theorem 1.1. *Assume that the local Kähler potential ϕ is real analytic in V . Let $\psi(x, z)$ be the holomorphic extension of $\phi(x)$ near the diagonal obtained by polarization, i.e., $\psi(x, z)$ is holomorphic and $\psi(x, \bar{x}) = \phi(x)$. Also let $b_m(x, z)$ be the holomorphic extensions of the Bergman kernel coefficients $b_m(x, \bar{x})$ in the on-diagonal expansion (1.1). Then there exist positive constants δ and C , and an open set $U \subset V$ containing p , such that for $N_0(k) = \lfloor (\frac{k}{C})^{\frac{1}{2}} \rfloor$ and uniformly for any $x, y \in U$, we have*

$$K_k(x, y) = e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k^{\frac{1}{2}}} O(1).$$

Moreover, one can differentiate this expansion. More precisely,

$$D^\alpha K_k(x, y) = D^\alpha \left(e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right) \right) + k^{|\alpha|} \alpha! e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k^{\frac{1}{2}}} O(1),$$

for any differential operator D^α of order α with respect to x and y .

¹Estimate (1.2) was first stated in [Be03], which is analogous to earlier results [Ch91, De98, Lin01] in various settings.

²In [LuSe17], similar decays are obtained in the non-compact setting under the assumption of Ricci curvature being uniformly bounded from below. See also page 64 of [Se15] for Agmon estimates in this setting.

³We were unable to find a proof of this estimate in the literature.

⁴See for example page 214 of [BeBeSj08], page 66 of [KaSc01], and page 28 of [DaLiMa06].

As a first corollary of this theorem, we get a complete asymptotic expansion in a $k^{-\frac{1}{4}}$ neighborhood of the diagonal.

Corollary 1.2. *Given the same assumptions and notations as in the above theorem, there exist positive constants C and δ , and an open set $U \subset V$ containing p , such that for all k and $N \in \mathbb{N}$, we have for all $x, y \in U$ satisfying $d(x, y) \leq \delta k^{-\frac{1}{4}}$,*

$$K_k(x, y) = e^{-k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} + \mathcal{R}_N(x, \bar{y}, k) \right),$$

where

$$|\mathcal{R}_N(x, \bar{y}, k)| \leq \frac{C^N N!^2}{k^N}.$$

As another corollary, we obtain the following off-diagonal asymptotic in terms of Calabi's diastasis [Cal53] function defined by

$$(1.4) \quad D(x, y) = \phi(x) + \phi(y) - \psi(x, \bar{y}) - \psi(y, \bar{x}).$$

We point out that near a given point $p \in M$, we have $D(x, y) = |x - y|_p^2 + O(|x - p|_p^3 + |y - p|_p^3)$, where $|z|_p^2 := \sum_{i,j=1}^n \phi_{i\bar{j}}(p) z_i \bar{z}_j$. If we use *Bochner coordinates* at p (introduced in [Bo47]), in which the Kähler potential admits the form $\phi(x) = |x|^2 + O(|x|^4)$, we have $D(x, y) = |x - y|_p^2 + O(|x - p|_p^4 + |y - p|_p^4)$.

Corollary 1.3. *Under the same assumptions and notations (and the same δ and same U) as in Theorem 1.1, we have uniformly for all $x, y \in U$ satisfying $D(x, y) \leq \frac{1}{2}\delta k^{-\frac{1}{2}}$,*

$$(1.5) \quad \frac{1}{k} \log |K_k(x, y)|_{h^k} = -\frac{D(x, y)}{2} + \frac{n \log k}{k} - \frac{n \log \pi}{k} + O\left(\frac{1}{k^2}\right).$$

The following scaling asymptotic is then immediate:

Corollary 1.4. *In Bochner coordinates at p , we have uniformly for all $u, v \in \mathbb{C}^n$ with $|u|_p$ and $|v|_p < \frac{\sqrt{\delta}}{3}$,*

$$\frac{1}{k^{\frac{1}{2}}} \log \left| K_k \left(\frac{u}{k^{\frac{1}{4}}}, \frac{v}{k^{\frac{1}{4}}} \right) \right|_{h^k} = -\frac{|u - v|_p^2}{2} + \frac{n \log k}{k^{\frac{1}{2}}} - \frac{n \log \pi}{k^{\frac{1}{2}}} + O\left(\frac{1}{k^{\frac{3}{2}}}\right).$$

One of the key ingredients in our proofs is the following estimate on the Bergman kernel coefficients $b_m(x, z)$. We emphasize again that $b_m(x, z)$ are the holomorphic extensions of the Bergman kernel coefficients $b_m(x, \bar{x})$ appearing in the on-diagonal expansion (1.1) of Zelditch [Ze98] and Catlin [Ca99].

Theorem 1.5. *Assume the Kähler potential ϕ is real analytic in some neighborhood V of p . Then, there exists a neighborhood $U \subset V$ of p , such that for any $m \in \mathbb{N}$ we have*

$$\|b_m(x, z)\|_{L^\infty(U \times U)} \leq C^m m!^2,$$

where C is a constant independent of m .

Remark 1.6. We conjecture that in the real analytic case

$$(1.6) \quad \|b_m(x, z)\|_{L^\infty(U \times U)} \leq C^m m!.$$

As we show in this paper, if this conjecture holds true, then all of the above results can be improved accordingly. In particular, the quantities $N_0(k) = (k/C)^{\frac{1}{2}}$ and $e^{-\delta k^{\frac{1}{2}}}$ in the remainder estimate

of Theorem 1.1 would be replaced by k/C and $e^{-\delta k}$, moreover Corollary 1.5 would hold for all $D(x, y) \leq \frac{\delta}{2}$. However we are unable to prove this conjecture for general real analytic Kähler metrics using our method, which is based on a recursive formula of [BeBeSj08]. In Section 6, we discuss the optimality and limitations of this method. One can also check, with a bit of more effort, that the recursive formula of [Lo04] (hence equivalently the method of [Xu12]) would also provide the same estimates as our Theorem 1.5. In addition, in Theorem 8.1, we find a parametrix representation of the Bergman kernel by means of transport equations similar to those for the wave equation, but as we discuss in Remark 8.2, this method would also give the bounds $C^m(m!)^2$. Another way to explore this conjecture is to use the method of peak sections. It would be interesting to confirm the bounds $C^m(m!)^2$ using this method. We must also mention that in [LiuLu16], it is claimed that in the analytic case

$$\|b_m(x, \bar{x})\|_{L^\infty(U)} \leq C^m,$$

however the proof contains some errors. In fact we do not expect the upper bounds C^m to be correct in general although we do not have any counterexamples. We doubt the bounds C^m because by [LuTi04] the leading term in $b_m(x, \bar{x})$ is $\frac{m}{(m+1)!} \Delta^{m-1} \rho(x)$ where ρ is the scalar curvature, so when the metric is real analytic we have $\frac{m}{(m+1)!} \Delta^{m-1} \rho(x) \approx C^m m!$. Nevertheless, as we shall see below, the bounds C^m hold trivially in the case of constant holomorphic sectional curvatures.

1.1. Metrics with local constant holomorphic sectional curvatures. As a special case, if we assume that the Kähler metric $\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi$ has constant holomorphic sectional curvatures in V , then $\phi(x)$ is analytic in V and there exists $U \subset V$ containing p such that for $x, z \in U$, $b_m(x, z)$ are all constants and vanish for $m > n = \dim M$. We now state the improved results.

Theorem 1.7. *Assume the Kähler metric ω has constant holomorphic sectional curvatures c in V . Then there exist a positive constant δ , an open set $U \subset V$ containing p , and constants b_1, \dots, b_n only dependent on n and c , such that uniformly for all $x, y \in U$, we have*

$$K_k(x, y) = e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^n \frac{b_j}{k^j} \right) + e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k} O(1).$$

Moreover, we have an explicit formula for the amplitude given by

$$1 + \sum_{j=1}^n \frac{b_j}{k^j} = \begin{cases} \frac{c^n}{k^n} \frac{\Gamma(\frac{k}{c} + n + 1)}{\Gamma(\frac{k}{c} + 1)} = \frac{c^n}{k^n} \left(\frac{k}{c} + n\right) \left(\frac{k}{c} + n - 1\right) \dots \left(\frac{k}{c} + 1\right), & \text{when } c \neq 0, \\ 1, & \text{when } c = 0. \end{cases}$$

In particular, if the Kähler metric ω has global constant holomorphic sectional curvature, we have

$$(1.7) \quad \left\| B_k(x) - \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^n \frac{b_j}{k^j} \right) \right\|_{L^\infty(M)} = e^{-\delta k} O(1),$$

where $B_k(x) := |K_k(x, x)|_{h^k} = K_k(x, x) e^{-k\phi(x)}$ is the Bergman function.

Estimate (1.7) was previously obtained by Berman [Ber08] for Riemann surfaces (see also Remark 3.5 of [Ber08] for a sketch of a proof for all dimensions).

Accordingly, we get the following corollary.

Corollary 1.8. *Under the same assumptions and notations as in the above theorem, we have uniformly for all $x, y \in U$ satisfying $D(x, y) \leq \frac{1}{2}\delta$,*

$$\frac{1}{k} \log |K_k(x, y)|_{h^k} = -\frac{D(x, y)}{2} + \frac{n \log k}{k} - \frac{n \log \pi}{k} + O\left(\frac{1}{k^2}\right).$$

There is a huge literature on Bergman kernels on compact complex manifolds. Before closing the introduction we only list some related work that were not cited above: [BoSj75, En00, Ch03, Lo04, LuTi04, MaMa08, Liu10, LiuLu15, LuZe16, Ze16]. Applications of the Bergman kernel, and the closely related Szegő kernel, can be found in [Do01], [BIShZe00], [ShZe02], [YuZh16]. The book of Ma and Marinescu [MaMa07] contains an introduction to the asymptotic expansion of the Bergman kernel and its applications. See also the book review [Ze09] for more on the applications of Bergman kernels.

1.2. Organization of the paper. In Sections 2 and 3, we follow the construction of local Bergman kernel in [BeBeSj08], but we obtain precise estimates for the error term by using the growth rate of Bergman coefficients $b_m(x, z)$ provided by Theorem 1.5. In Section 4, we give the proofs of Theorem 1.1 and Corollaries 1.2 and 1.3. The proof of Theorem 1.5 will be given in Section 5. Section 6 discusses the optimality of our bounds on Bergman coefficients. Section 7 concerns the special case of constant holomorphic sectional curvatures. Finally, in Section 8 we derive transport equations for the amplitude of the parametrix of [BeBeSj08], which is analogous to transport equations for the parametrices of the wave kernel.

2. LOCAL BERGMAN KERNELS

In [BeBeSj08], by using *good* complex contour integrals, Berman-Berndtsson-Sjöstrand constructed *local reproducing kernels* (mod $e^{-k\delta}$) for $U = B^n(0, 1) \subset \mathbb{C}^n$, which reproduce holomorphic sections in U up to $e^{-k\delta}$ error terms. These kernels are in general not holomorphic. By allowing more flexibility in choosing the amplitudes in the integral, the authors modified these local reproducing kernels to local Bergman kernels, which means that they are *holomorphic* local reproducing kernels mod $O(k^{-N})$. The global Bergman kernels are then approximated using the standard Hörmander's L^2 estimates.

Throughout this paper, we assume that ϕ is real analytic in a small open neighborhood $V \subset M$ of a given point p . Let $B^n(0, r)$ be the ball of radius r in \mathbb{C}^n . We identify p with $0 \in \mathbb{C}^n$ and V with the ball $B^n(0, 3) \subset \mathbb{C}^n$ and denote $U = B^n(0, 1)$. Let e_L be a local holomorphic frame of L over V as introduced in the introduction. For each positive integer k , we denote $H_{k\phi}(U)$ to be the inner product space of L^2 -holomorphic functions on U with respect to

$$(u, v)_{k\phi} = \int_U u\bar{v} e^{-k\phi} d\text{Vol},$$

where $d\text{Vol} = \frac{\omega^n}{n!}$ is the natural volume form induced by the Kähler form $\omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial}\phi$. So the norm of $u \in H_{k\phi}(U)$ is given by

$$\|u\|_{k\phi}^2 = \int_U |u|^2 e^{-k\phi} d\text{Vol}.$$

Let $\chi \in C_0^\infty(B^n(0, 1))$ be a smooth cut-off function such that $\chi = 1$ in $B^n(0, \frac{1}{2})$ and vanishes outside $B^n(0, \frac{3}{4})$. The following result gives a refinement of the the result of [BeBeSj08] by giving a more precise estimate for the error term when the Kähler potential is real analytic. The main ingredient of the proof is Theorem 1.5, whose proof is delayed to Section 5.

Proposition 2.1. *For each $N \in \mathbb{N}$, there exist $K_{k,x}^{(N)}(y) \in H_{k\phi}(U)$ and a positive constant C independent of N and k , such that for all $u \in H_{k\phi}(U)$ we have*

$$(2.1) \quad \forall x \in B^n(0, 1/4) : \quad u(x) = \left(\chi u, K_{k,x}^{(N)} \right)_{k\phi} + k^n e^{\frac{k\phi(x)}{2}} \mathcal{R}_{N+1}(\phi, k) \|u\|_{k\phi},$$

where

$$(2.2) \quad |\mathcal{R}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N+1)!^2}{k^{N+1}}.$$

The function $K_{k,x}^{(N)}$ is called a local Bergman kernel of order N .

Remark 2.2. In [BeBeSj08], only the qualitative estimate $\mathcal{R}_{N+1}(\phi, k) = O_N(\frac{1}{k^{N+1}})$ is given.

To prove this proposition we first need to recall the techniques of [BeBeSj08].

2.1. Review of the method of Berman-Berndtsson-Sjöstrand. The main idea is to construct the local holomorphic reproducing kernel (also called local Bergman kernel) by means of the calculus of contour pseudo-differential operators (contour Ψ DO for short) introduced by Sjöstrand [Sj82]. Before we introduce the notion of contour integrals we present some notations and definitions.

Suppose $\phi(x)$ is analytic in $V = B^n(0, 3)$. Without the loss of generality we assume that the radius of convergence of its power series in terms of x and \bar{x} is 3. By replacing $\phi(x)$ by $\phi(x) - \phi(0)$, we can assume that $\phi(0) = 0$. We then denote $\psi(x, z)$ to be the holomorphic extension of $\phi(x)$ by replacing \bar{x} with z . This procedure is called polarization. One can easily verify that $\psi(x, z)$ satisfies the formal definition of holomorphic extension, namely

- $\psi(x, z)$ is holomorphic in $B^n(0, 3) \times B^n(0, 3)$.
- $\psi(x, \bar{x}) = \phi(x)$.

Moreover, since $\phi(x)$ is real-valued, we have $\overline{\psi(x, z)} = \psi(\bar{z}, \bar{x})$. We also define

$$(2.3) \quad \theta(x, y, z) = \int_0^1 (D_x \psi)(tx + (1-t)y, z) dt,$$

where the differential operator D_x is the gradient operator defined by

$$D_x = (D_{x_1}, D_{x_2}, \dots, D_{x_n}).$$

Note that $\theta(x, x, z) = \psi_x(x, z)$. It is easy to prove that the Jacobian of the map $(x, y, z) \rightarrow (x, y, \theta)$ at $(x, y, z) = (0, 0, 0)$ is non-singular. Thus the map is actually a biholomorphic map between two neighborhoods of the origin of \mathbb{C}^{3n} . As a result, we can use (x, y, z) or (x, y, θ) as local coordinates interchangeably. Without loss of generality we can assume that $(x, y, z) \in B^n(0, 3) \times B^n(0, 3) \times B^n(0, 3)$ and $\theta \in W$, where

$$W = \theta(B^n(0, 3) \times B^n(0, 3) \times B^n(0, 3)).$$

Note that W contains the origin because by our assumption $\phi(0) = 0$.

A fundamental idea of [BeBeSj08] is to use the estimate

$$(2.4) \quad u(x) = c_n \left(\frac{k}{2\pi} \right)^n \int_{\Lambda} e^{k\theta \cdot (x-y)} u(y) \chi(y) d\theta \wedge dy + O(e^{-k\delta}) e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi}$$

which holds uniformly for $x \in B^n(0, \frac{1}{4})$, for any holomorphic function u defined on $B^n(0, 1)$. Here, $c_n = i^{-n^2}$, δ is a positive constant, and $\Lambda = \{(y, \theta) : \theta = \theta(x, y)\}$ is a *good contour*, which means that there exists $\delta > 0$ such that for any x, y in a neighborhood of the origin,

$$(2.5) \quad 2 \operatorname{Re} \theta \cdot (x - y) \leq -\delta |x - y|^2 - \phi(y) + \phi(x).$$

One can easily verify that

$$(2.6) \quad \Lambda = \{(y, \theta) : \theta = \theta(x, y, \bar{y})\},$$

with $\theta(x, y, z)$ defined by (2.3), is a good contour by observing that

$$\theta \cdot (x - y) = \psi(x, \bar{y}) - \psi(y, \bar{y}).$$

To put (2.4) into a useful perspective, one should think of the integral in (2.4) as a contour Ψ DO defined as follows. Let $a = a(x, y, \theta, k)$ be a holomorphic symbol in $B^n(0, 3) \times B^n(0, 3) \times W$, with an asymptotic expansion of the form

$$a(x, y, \theta, k) \sim a_0(x, y, \theta) + \frac{a_1(x, y, \theta)}{k} + \frac{a_2(x, y, \theta)}{k^2} + \dots .$$

For simplicity, we will suppress the dependency on k and write $a = a(x, y, \theta)$.

A Ψ DO associated to a good contour Λ and an amplitude $a(x, y, \theta)$, is an operator on $C_0^\infty(U)$ defined by

$$\text{Op}_\Lambda(a)u = c_n \left(\frac{k}{2\pi} \right)^n \int_\Lambda e^{k\theta \cdot (x-y)} a(x, y, \theta) u(y) d\theta \wedge dy.$$

Thus in this language (2.4) means that for $x \in B^n(0, 1/4)$

$$(\chi u)(x) = \text{Op}_\Lambda(1)(\chi u) + O(e^{-k\delta}) e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi}.$$

Roughly speaking this says that $\text{Op}_\Lambda(1)$ is the identity operator mod $O(e^{-k\delta})$. We define the integral kernel $K_{k,x}(y)$ of $\text{Op}_\Lambda(a)$ with respect to the inner product $(\cdot, \cdot)_{k\phi}$, by

$$\text{Op}_\Lambda(a)u = (u, K_{k,x})_{k\phi}.$$

The first observation is that the kernel $K_{k,x}(y)$ of $\text{Op}_\Lambda(1)$, associated to the contour (2.6), is not holomorphic. The idea of [BeBeSj08] is to replace $\text{Op}_\Lambda(1)$ by $\text{Op}_\Lambda(1+a)$ where $a(x, y, \theta)$ is a *negligible amplitude* and the kernel of $\text{Op}_\Lambda(1+a)$ is holomorphic. An amplitude $a(x, y, \theta)$ is negligible if

$$\text{Op}_\Lambda(a)(\chi u) = O(k^{-\infty}) e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi}.$$

To find a suitable condition for negligible amplitudes one formally writes

$$\text{Op}_\Lambda(a) = \text{Op}_\Lambda(Sa|_{x=y}),$$

where S is a standard operator that is used in microlocal analysis to turn a symbol $a(x, y, \theta)$ of a Ψ DO to a symbol of the form $\tilde{a}(x, \theta)$. The operator S is formally defined by

$$S = e^{\frac{D_\theta \cdot D_y}{k}} = \sum_{m=0}^{\infty} \frac{(D_\theta \cdot D_y)^m}{m! k^m}.$$

Then an amplitude a is negligible if $Sa|_{x=y} \sim 0$ as a formal power series. This implies that there exists a holomorphic vector field $A(x, y, \theta)$ with formal power series

$$A(x, y, \theta) \sim A_0(x, y, \theta) + \frac{A_1(x, y, \theta)}{k} + \frac{A_2(x, y, \theta)}{k^2} + \dots .$$

such that

$$(2.7) \quad Sa \sim k(x - y) \cdot SA.$$

By a straightforward calculation, it can be seen that this is equivalent to

$$(2.8) \quad a = \nabla A := D_\theta \cdot A + k(x - y) \cdot A.$$

By comparing coefficients, (2.8) is equivalent to the following relations between a_m and A_m :

$$(2.9) \quad a_m = D_\theta \cdot A_m + (x - y) \cdot A_{m+1}.$$

Here $a_m(x, y, \theta)$ are holomorphic functions and $A_m(x, y, \theta)$ are holomorphic vector fields in \mathbb{C}^n , defined on $B^n(0, 3) \times B^n(0, 3) \times W$.

Next, we observe that the integral kernel of $\text{Op}_\Lambda(1+a)$ is holomorphic if

$$(2.10) \quad 1 + a(x, y, \theta) \sim B(x, z(x, y, \theta))\Delta_0(x, y, \theta),$$

where

$$\Delta_0(x, y, \theta) = \frac{\det \psi_{yz}(y, z)}{\det \theta_z(x, y, z)} = \det \partial_\theta \psi_y(x, y, \theta),$$

and $B(x, z)$ is holomorphic and has an asymptotic expansion of the form

$$(2.11) \quad B(x, z) \sim b_0(x, z) + \frac{b_1(x, z)}{k} + \frac{b_2(x, z)}{k^2} + \dots,$$

where $b_m(x, z)$ are holomorphic. In fact, as it turns out, $b_m(x, z)$ are the holomorphic extensions of $b_m(x, \bar{x})$, the Bergman kernel coefficients of the on-diagonal asymptotic expansion of Zelditch-Catlin (1.1).

If the amplitude a is negligible, then by applying $S(\cdot)|_{x=y}$ to both sides of (2.10), we get

$$S(B(x, z(x, y, \theta))\Delta_0(x, y, \theta))|_{x=y} \sim 1.$$

From this, one gets the following recursive equations for Bergman kernel coefficients $b_m(x, z)$, which will play a key role in the proof of Theorem 1.5:

$$(2.12) \quad b_m(x, z(x, y, \theta)) = - \sum_{l=1}^m \frac{(D_y \cdot D_\theta)^l}{l!} (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta))|_{y=x}.$$

Additionally, by comparing the coefficients on both sides of (2.10), we have the following relations between a_m and b_m :

$$(2.13) \quad a_m(x, y, \theta) = \begin{cases} \Delta_0(x, y, \theta) - 1 & \text{when } m = 0, \\ b_m(x, z(x, y, \theta))\Delta_0(x, y, \theta) & \text{when } m \geq 1. \end{cases}$$

These equations will be useful in estimating a_m in terms of the bounds on b_m from Theorem 1.5.

We are now prepared to prove Proposition 2.1.

3. THE REMAINDER ESTIMATES AND THE PROOF OF PROPOSITION 2.1

Let a_m , A_m , and b_m be given by (2.13), (2.9), and (2.11). Define $a^{(N)}$, $A^{(N)}$, and $B^{(N)}$ to be the partial sums of a , A , and B up to order k^{-N} . We claim that

$$(3.1) \quad u(x) = \text{Op}_\Lambda \left(1 + a^{(N)} \right) (\chi u) + k^n \mathcal{R}_{N+1}(\phi, k) e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi},$$

where uniformly for $x \in B(0, \frac{1}{4})$ we have

$$(3.2) \quad |\mathcal{R}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N+1)!^2}{k^{N+1}},$$

and the integral kernel of $\text{Op}_\Lambda(1+a^{(N)})$ is holomorphic. The complex conjugate of this kernel is given by

$$(3.3) \quad \overline{K_{k,x}^{(N)}}(y) = \left(\frac{k}{\pi} \right)^n e^{k\psi(x, \bar{y})} \left(1 + a^{(N)}(x, y, \theta(x, y, \bar{y})) \right) \Delta_0(x, y, \theta(x, y, \bar{y}))^{-1},$$

which by the relation (2.10) is reduced to

$$\overline{K_{k,x}^{(N)}}(y) = \left(\frac{k}{\pi} \right)^n e^{k\psi(x, \bar{y})} B^{(N)}(x, \bar{y}).$$

Hence $K_{k,x}^{(N)}(y)$ is holomorphic in y because $B(x, z)$ is holomorphic.

In the light of (2.4), to prove (3.1) it suffices to show that

$$\forall x \in B(0, \frac{1}{4}) : \quad \left| \text{Op}_\Lambda \left(a^{(N)} \right) (\chi u)(x) \right| \leq \frac{C^{N+1} (N+1)!^2}{k^{N+1-n}} e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi}.$$

By definition,

$$\text{Op}_\Lambda \left(a^{(N)} \right) (\chi u)(x) = c_n \left(\frac{k}{2\pi} \right)^n \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) a^{(N)} d\theta \wedge dy.$$

We observe that by the definition of ∇ in (2.8),

$$a^{(N)} - \nabla \left(A^{(N+1)} \right) = -\frac{D_\theta \cdot A_{N+1}}{k^{N+1}}.$$

Then it is easy to see that using integration by parts (see for example the proof of Proposition 2.2 in [BeBeSj08]), we get

$$\begin{aligned} \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) a^{(N)} d\theta \wedge dy &= - \int_\Lambda d\chi \wedge u(y) e^{k\theta \cdot (x-y)} A^{(N+1)} \wedge dy \\ &\quad - \int_\Lambda e^{k\theta \cdot (x-y)} u(y) \chi(y) \frac{D_\theta \cdot A_{N+1}}{k^{N+1}} d\theta \wedge dy. \end{aligned}$$

In the first integral, we have identified the n -vector A as an $(n-1, 0)$ form defined by $A = \sum_{j=1}^n A_j \widehat{d\theta}_j$, where $\widehat{d\theta}_j$ is the wedge product of all $\{d\theta_k\}_{k \neq j}$ such that $d\theta_j \wedge \widehat{d\theta}_j = d\theta$.

We now estimate the two integrals on the right hand side of the above equality. For the first integral, since $d\chi(y) = 0$ for $y \in B^n(0, \frac{1}{2})$, $|x-y| \geq \frac{1}{4}$ for $x \in B^n(0, \frac{1}{4})$. Since θ satisfies (2.5), by changing δ to a smaller constant, the integrand of the first integral is bounded by some constant times

$$|u(y)| e^{\frac{k\phi(x)}{2} - \frac{k\phi(y)}{2} - \delta k} |A^{(N+1)}(x, y, \theta(x, y, \bar{y}))|.$$

So by using the Cauchy-Schwartz inequality, the first integral is bounded by

$$\|A^{(N+1)}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi} O(e^{-\delta k}),$$

where $W_1 \subset W$ is defined by

$$W_1 = \theta(B^n(0,1) \times B^n(0,1) \times B^n(0,1)).$$

Similarly, the integrand of the second integral is bounded by some constant times

$$|u(y)| e^{\frac{k\phi(x)}{2} - \frac{k\phi(y)}{2}} \left| \frac{(D_\theta \cdot A_{N+1})(x, y, \theta(x, y, \bar{y}))}{k^{N+1}} \right|.$$

Again using the Cauchy-Schwartz inequality, the second integral is bounded by

$$k^{-N-1} \|D_\theta \cdot A_{N+1}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} e^{\frac{k\phi(x)}{2}} \|u\|_{k\phi}.$$

Finally, (3.1) and (3.2), hence Proposition 2.1, follow quickly from the following lemmas.

Lemma 3.1. *For any $N \in \mathbb{N}$,*

$$ke^{-k\delta} \leq \left(\frac{2}{\delta} \right)^{N+2} \frac{(N+1)!}{k^{N+1}}.$$

Proof. Define $f(x) = x^{N+2} e^{-\delta x}$ on $[0, \infty)$. It is easy to see that f attains its maximum at $x = \frac{N+2}{\delta}$. Therefore, by using Stirling's formula, we have

$$f(x) \leq \left(\frac{N+2}{\delta} \right)^{N+2} e^{-(N+2)} \leq \left(\frac{1}{\delta} \right)^{N+2} (N+2)!.$$

The Lemma follows from $N+2 \leq 2^{N+2}$. □

Lemma 3.2. *We have*

$$(3.4) \quad \|A_N\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \leq C^N N!^2,$$

$$(3.5) \quad \|A^{(N)}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \leq Ck + \frac{C^N N!^2}{k^N},$$

$$\|D_\theta \cdot A_N\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \leq C^N N!^2,$$

for a constant C independent of N .

Proof. The idea is to explicitly express A_m 's in terms of b_m 's, and then use Theorem 1.5.

First, we observe that since

$$Sa \sim \sum_{i=0}^{\infty} \frac{(D_\theta \cdot D_y)^i}{i! k^i} \sum_{j=0}^{\infty} \frac{a_j}{k^j} \sim \sum_{m=0}^{\infty} \sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i a_j}{i! k^m},$$

we have

$$(Sa)_m = \sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i a_j}{i!}.$$

Using the relation (2.13), this becomes

$$(Sa)_m = \sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i (b_j \Delta_0)}{i!} \quad \text{for any } m \geq 1.$$

So by the relation between Sa and SA in (2.7), we get

$$(3.6) \quad (x-y) \cdot (SA)_{m+1} = \sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i (b_j \Delta_0)}{i!}.$$

We emphasize that a formal asymptotic expansion A satisfying the above equation is not unique, and in fact here, we choose the particular A so that $(SA)_0 = 0$ and SA satisfies

$$(SA)_{m+1}(x, y, \theta) = \int_0^1 D_y \left(\sum_{i+j=m} \frac{(D_\theta \cdot D_y)^i (b_j \Delta_0)}{i!} \right) (x, tx + (1-t)y, \theta) dt \quad \text{for } m \geq 0.$$

That why (3.6) holds for this particular A is evident by the fundamental theorem of calculus. Given that we have SA , by applying the inverse operator S^{-1} , we can obtain A as follows

$$\begin{aligned} A_m &= \sum_{i+j=m} \frac{(-D_\theta \cdot D_y)^i (SA)_j}{i!} \\ &= \sum_{\substack{i+j=m, \\ j \geq 1}} \frac{(-D_\theta \cdot D_y)^i}{i!} \int_0^1 D_y \left(\sum_{s+t=j-1} \frac{(D_\theta \cdot D_y)^s (b_t \Delta_0)}{s!} \right) (x, tx + (1-t)y, \theta) dt. \end{aligned}$$

Now, by using the Cauchy integral formula twice and Theorem 1.5, we obtain

$$\begin{aligned}
& \|A_m\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \\
& \leq \sum_{\substack{i+j=m, \\ j \geq 1}} C^{2i} i! \left\| \int_0^1 D_x \left(\sum_{s+t=j-1} \frac{(D_\theta \cdot D_y)^s (b_t \Delta_0)}{s!} \right) (x, tx + (1-t)y, \theta) dt \right\|_{L^\infty(B^n(0, \frac{3}{2}) \times B^n(0, \frac{3}{2}) \times W_2)} \\
& \leq \sum_{\substack{i+j=m, \\ j \geq 1}} C^{2i} i! \left\| D_y \left(\sum_{s+t=j-1} \frac{(D_\theta \cdot D_y)^s (b_t \Delta_0)}{s!} \right) (x, y, \theta) \right\|_{L^\infty(B^n(0, \frac{3}{2}) \times B^n(0, \frac{3}{2}) \times W_2)} \\
& \leq \sum_{\substack{i+j=m, \\ j \geq 1}} C^{2i} i! \sum_{s+t=j-1} C^{2s+1} s! \|b_t \Delta_0\|_{L^\infty(B^n(0, \frac{5}{2}) \times B^n(0, \frac{5}{2}) \times W_3)} \\
& \leq \sum_{\substack{i+j=m, \\ j \geq 1}} C^{2i} i! \sum_{s+t=j-1} C^{2s+1} s! C^t t!^2 \\
& \leq C^m m!^2.
\end{aligned}$$

In the above series of estimates $W_1 \Subset W_2 \Subset W_3 \Subset W$, where \Subset is our notation for relative compactness. Also, the constants C are renamed to be C again at each line.

Using the estimate on A_m and (2.9), we get

$$\begin{aligned}
& \|D_\theta \cdot A_m\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \\
& \leq \|a_m\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} + \|(x-y) \cdot A_{m+1}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \leq C^m m!^2,
\end{aligned}$$

for a new C .

To prove (3.5), we note that by the definition of $A^{(N)}$ and (3.4), we have

$$\|A^{(N)}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} \leq \frac{C1!^2}{k} + \frac{C^2 2!^2}{k^2} + \cdots + \frac{C^N N!^2}{k^N}.$$

To estimate the right hand side, we need to study the function $\frac{C^x (\Gamma(1+x))^2}{k^x}$. By Sterling's formula this is more or less equivalent to studying the function $\frac{C^x x^{2x}}{e^{2x} k^x}$. To minimize this function we consider its logarithm

$$f(x) = \log \frac{C^x x^{2x}}{e^{2x} k^x} = x \log C + 2x \log x - 2x - x \log k \quad \text{for } x \in (0, \infty).$$

Since

$$f'(x) = \log C + 2 \log x - \log k,$$

the only critical point is $x_0 = (\frac{k}{C})^{\frac{1}{2}}$, and the function is decreasing on the interval $(0, x_0]$ and increasing on the interval $[x_0, \infty)$. Therefore, if we take $N_0 = N_0(k) = \lceil (\frac{k}{C})^{\frac{1}{2}} \rceil$, then by using

Stirling's formula twice

$$\begin{aligned}
\sum_{m=1}^N \frac{C^m m!^2}{k^m} &\leq e^2 \sum_{m=1}^N \frac{C^m m^{2m+1}}{e^{2m} k^m} \\
&\leq e^2 \left(N_0^2 + N(N - N_0) \frac{C^N N^{2N}}{e^{2N} k^N} \right) \\
&\leq e^2 \left(\frac{k}{C} + \frac{C^N N^{2N+2}}{e^{2N} k^N} \right) \\
&\leq C' \left(k + \frac{C'^N N!^2}{k^N} \right).
\end{aligned}$$

□

4. FROM LOCAL TO GLOBAL AND THE PROOF OF THEOREM 1.1

Let $K_k(x, y)$ be the Bergman kernel of (M, L^k) . As we noted before, we also write $K_k(x, y)$ for the representation of the Bergman kernel in the local frame $e_L^k \otimes \overline{e_L^k}$ and we denote $K_{k,y}(x) := K_k(x, y)$. In the last section, we constructed the local Bergman kernel of order N , which we denoted by $K_k^{(N)}(x, y) = K_{k,y}^{(N)}(x)$. In this section, we show that $K_k(x, y)$ is equal to $K_k^{(N)}(x, y)$ up to order k^{-N} when x, y are sufficiently close to each other. Moreover, we will give a precise upper bound for the error term.

Proposition 4.1. *There exists $\delta > 0$ such that whenever $d(x, y) < \delta$, we have*

$$(4.1) \quad K_k(x, y) = K_k^{(N)}(x, y) + k^{\frac{3n}{2}} \tilde{\mathcal{R}}_{N+1}(\phi, k) e^{\frac{k\phi(x)}{2} + \frac{k\phi(y)}{2}},$$

where

$$(4.2) \quad |\tilde{\mathcal{R}}_{N+1}(\phi, k)| \leq \frac{C^{N+1} (N+1)!^2}{k^{N+1}},$$

and the constant C is independent of N, x, y , and k .

Proof. We fix $x \in M$ and assume that ϕ is analytic in $B^n(x, 3)$. Let χ be a smooth cut-off function such that

$$\chi(z) = \begin{cases} 1 & z \in B^n(x, \frac{1}{2}) \\ 0 & z \notin B^n(x, \frac{3}{4}) \end{cases}.$$

We assume $y \in B^n(x, \frac{1}{4})$. We first observe that

$$(4.3) \quad K_k(y, x) = \left(\chi K_{k,x}, K_{k,y}^{(N)} \right)_{k\phi} + \mathcal{S}_{N+1}(\phi, k) k^{\frac{3n}{2}} e^{k(\frac{\phi(x)}{2} + \frac{\phi(y)}{2})},$$

where $|\mathcal{S}_{N+1}(\phi, k)| \leq \frac{C^{N+1} (N+1)!^2}{k^{N+1}}$. This is because, by Proposition 2.1, we have

$$K_{k,x}(y) = \left(\chi K_{k,x}, K_{k,y}^{(N)} \right)_{k\phi} + k^n \mathcal{S}_{N+1}(\phi, k) e^{\frac{k\phi(y)}{2}} \|K_{k,x}\|_{k\phi},$$

and by the reproducing property of Bergman kernel, we have

$$\|K_{k,x}\|_{k\phi} e^{-k\frac{\phi(x)}{2}} \leq \left\| \|K_{k,x}\|_{L^2(M, L^k)} \right\|_{hk} = \sqrt{|K_k(x, x)|_{hk}} \leq Ck^{\frac{n}{2}}.$$

That why $|K_k(x, x)|_{hk} \leq Ck^n$ follows from the extreme property of the Bergman function and also the sub-mean value inequality. For a simple proof see for example Lemma 4.1 of [HeKeSeXu16].

Next, we define

$$(4.4) \quad u_{k,y}(z) = \chi(z)K_{k,y}^{(N)}(z) - \left(\chi K_{k,y}^{(N)}, K_{k,z} \right)_{k\phi}.$$

Our goal is to estimate $|u_{k,y}(x)|$. Since $\left(\chi K_{k,y}^{(N)}, K_{k,x} \right)_{k\phi}$ is the Bergman projection of $\chi K_{k,y}^{(N)}$, $u_{k,y}$ is the minimal L^2 solution to the equation

$$\bar{\partial}u = \bar{\partial}(\chi K_{k,y}^{(N)}).$$

So by using Hörmander's L^2 estimates [Ho66] (see [Be10] for an exposition), we have

$$\|u_{k,y}\|_{L^2}^2 \leq \frac{C}{k} \|\bar{\partial}(\chi K_{k,y}^{(N)})\|_{L^2}^2.$$

As $K_{k,y}^{(N)}(z)$ is holomorphic, we have $\bar{\partial}(\chi K_{k,y}^{(N)})(z) = \bar{\partial}\chi(z)K_{k,y}^{(N)}(z)$. We recall that by (3.3)

$$K_{k,y}^{(N)}(z) = \left(\frac{k}{\pi} \right)^n e^{k\psi(z,\bar{y})} B^{(N)}(z, \bar{y}).$$

Since $\bar{\partial}\chi(z)$ is supported in $d(z, x) \geq \frac{1}{2}$ and since $d(x, y) \leq \frac{1}{4}$, using

$$\operatorname{Re}\psi(z, \bar{y}) \leq -\delta|z - y|^2 + \frac{\phi(z)}{2} + \frac{\phi(y)}{2},$$

we get

$$\left| \bar{\partial}(\chi K_{k,y}^{(N)}(z)) \right|^2 \leq \left(\frac{k}{\pi} \right)^{2n} e^{-k\delta + k\phi(y) + k\phi(z)} \left\| B^{(N)} \right\|_{L^\infty(U \times U)}^2,$$

for a new $\delta > 0$. Therefore,

$$\begin{aligned} \left\| \bar{\partial}(\chi K_{k,y}^{(N)}) \right\|_{L^2}^2 &\leq Ck^{2n} \int_{B(x,1)} e^{-k\delta + k\phi(y)} \left\| B^{(N)} \right\|_{L^\infty(U \times U)} \frac{\omega^n}{n!} \\ &\leq Ck^{2n} e^{-k\delta + k\phi(y)} \left\| B^{(N)} \right\|_{L^\infty(U \times U)}^2, \end{aligned}$$

which implies

$$\|u_{k,y}\|_{L^2}^2 \leq Ck^{2n} e^{-k\delta + k\phi(y)} \left\| B^{(N)} \right\|_{L^\infty(U \times U)}^2.$$

Thus, using the sub-mean value inequality in $B(x, \frac{1}{\sqrt{k}})$ (with respect to the standard Euclidean volume form dV_0) as performed below

$$\begin{aligned} \|u_{k,y}\|^2 &\geq e^{-k\phi(x)} \int_{B(x, \frac{1}{\sqrt{k}})} |u_{k,y}(z)|^2 e^{-k(\phi(z) - \phi(x))} \frac{\omega^n}{n!} \\ &\geq \frac{e^{-k\phi(x)}}{C} \int_{B(x, \frac{1}{\sqrt{k}})} |u_{k,y}(z)|^2 dV_0 \\ &\geq \frac{e^{-k\phi(x)}}{Ck^n} |u_{k,y}(x)|^2, \end{aligned}$$

we get

$$|u_{k,y}(x)| \leq Ck^{\frac{3n}{2}} e^{-k\delta} e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} \left\| B^{(N)} \right\|_{L^\infty(U \times U)}.$$

We can estimate $\|B^{(N)}\|_{L^\infty(U \times U)}$ using our Theorem 1.5

$$\begin{aligned} \|B^{(N)}\|_{L^\infty(U \times U)} &\leq 1 + \frac{1}{k} \|b_1\|_{L^\infty(U \times U)} + \dots + \frac{1}{k^N} \|b_N\|_{L^\infty(U \times U)} \\ &\leq 1 + \frac{C1!^2}{k} + \frac{C^2 2!^2}{k^2} + \dots + \frac{C^N N!^2}{k^N} \\ &\leq C \left(k + \frac{C^N N!^2}{k^N} \right). \end{aligned}$$

Hence,

$$|u_{k,y}(x)| \leq C k^{\frac{3n}{2}} e^{-k\delta} e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} \left(k + \frac{C^N N!^2}{k^N} \right) \leq k^{\frac{3n}{2}} e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} \frac{C^{N+1} (N+1)!^2}{k^{N+1}}.$$

Combining this estimate with (4.3) and recalling the definition of $u_{k,y}$ in (4.4), we get the result.

We point out that we have renewed the constant C at each step, but the final constant is independent of k and N . We also note that the constant C may depend on the point x , however by a simple compactness argument one can see that each such C can be bounded by a uniform constant independent of x . \square

Now we are ready to prove Theorem 1.1 and its corollaries.

4.1. Proof of Theorem 1.1. By Proposition 4.1, we just need to show that with $N = N_0 - 1 = \lfloor \sqrt{k/C} \rfloor - 1$ we have⁵

$$k^{\frac{3n}{2}} \mathcal{R}_{N_0}(\phi, k) e^{\frac{k\phi(x)}{2} + \frac{k\phi(y)}{2}} = e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k^{\frac{1}{2}}} O(1).$$

However, by the same proposition we know that

$$|R_{N_0}(\phi, k)| \leq \frac{C^{N_0} N_0!^2}{k^{N_0}}.$$

Hence it is enough to show that

$$\frac{C^{N_0} N_0!^2}{k^{N_0}} = e^{-\delta k^{\frac{1}{2}}} O(1).$$

By Stirling's formula

$$\frac{C^{N_0} N_0!^2}{k^{N_0}} \leq C' N_0 \frac{C^{N_0} N_0^{2N_0}}{e^{2N_0} k^{N_0}} \leq C' N_0 e^{-2N_0} \leq C'' k^{\frac{1}{2}} e^{-\frac{2}{\sqrt{C}} k^{\frac{1}{2}}}.$$

Since,

$$k^{\frac{3n}{2} + \frac{1}{2}} e^{-\frac{2}{\sqrt{C}} k^{\frac{1}{2}}} \leq C''' e^{-\frac{1}{\sqrt{C}} k^{\frac{1}{2}}},$$

$\delta = \frac{1}{\sqrt{C}}$ would do the job. The estimate on the derivatives follow immediately by using the Cauchy integral formula over the boundary of a polydisc with radius $\frac{1}{k}$, since the kernels $K_k(x, y)$ and $K_k^{(N)}(x, y)$ are both holomorphic in x and \bar{y}

⁵For convenience, we use N_0 for $N_0(k) = \lfloor \sqrt{k/C} \rfloor$.

4.2. Proof of Corollary 1.2.

Proof. By Theorem 1.1, uniformly for any $x, y \in U$, we have

$$K_k(x, y) = e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{k\left(\frac{\phi(x)}{2} + \frac{\phi(y)}{2}\right)} e^{-\delta k^{\frac{1}{2}}} O(1).$$

For any given positive integer N , we rewrite the above formula as follows.

$$K_k(x, y) = e^{-k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} + \sum_{j=N}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j} + e^{\frac{k}{2}(\phi(x) + \phi(y) - 2\psi(x, \bar{y}))} e^{-\delta k^{\frac{1}{2}}} O(1) \right).$$

Our first observation is that,

$$\left| e^{\frac{k}{2}(\phi(x) + \phi(y) - 2\psi(x, \bar{y}))} e^{-\delta k^{\frac{1}{2}}} \right| = e^{\frac{k}{2}D(x, y) - \delta k^{\frac{1}{2}}} \leq e^{-\frac{3}{4}\delta k^{\frac{1}{2}}}.$$

Now we estimate the term $\sum_{j=N}^{N_0(k)-1} \frac{b_j(x, \bar{y})}{k^j}$. By Stirling's formula, we have

$$\left| \frac{b_j(x, \bar{y})}{k^j} \right| \leq \frac{C^j j!^2}{k^j} \leq C' j \frac{C^j j^{2j}}{e^{2j} k^j}.$$

Since $\frac{C^j j^{2j}}{e^{2j} k^j}$ is monotonically decreasing for $1 \leq j \leq N_0(k) - 1$ (with the help of Stirling's formula once more), we get

$$\begin{aligned} \left| \sum_{j=N}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| &\leq \frac{C^N N!}{k^N} + \sum_{j=N+1}^{N_0-1} C' j \frac{C^j j^{2j}}{e^{2j} k^j} \\ &\leq \frac{C^N N!}{k^N} + C' N_0^2 \frac{C^{N+1} (N+1)^{2(N+1)}}{e^{2(N+1)} k^{N+1}} \\ &\leq \frac{C''^N N!^2}{k^N}. \end{aligned}$$

Therefore,

$$K_k(x, y) = e^{-k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N-1} \frac{b_j(x, \bar{y})}{k^j} + \frac{C''^N N!^2}{k^N} + e^{-\frac{3}{4}\delta k^{\frac{1}{2}}} O(1) \right).$$

But by a similar argument as in Lemma 3.1,

$$e^{-\frac{3}{4}\delta k^{\frac{1}{2}}} \leq \left(\frac{4}{3\delta} \right)^{2N} \frac{(2N)!}{k^N} \leq \left(\frac{8}{3\delta} \right)^{2N} \frac{N!^2}{k^N},$$

and the result follows. \square

4.3. Proof of Corollary 1.3. By Theorem 1.1, we have

$$K_k(x, y) = e^{k\psi(x, \bar{y})} \frac{k^n}{\pi^n} \left(1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right) + e^{\frac{k\phi(x)}{2} + \frac{k\phi(y)}{2}} e^{-\delta k^{\frac{1}{2}}} O(1).$$

Recall that $D(x, y) = \phi(x) + \phi(y) - \psi(x, \bar{y}) - \psi(y, \bar{x})$. Then

$$\log |K_k(x, y)|_{h^k} = -\frac{kD(x, y)}{2} + n \log k - n \log \pi + \log \left| 1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} + e^{\frac{Q(x, y)}{2}} k^{-\delta k^{1/2}} O(1) \right|,$$

where $Q(x, y) = \phi(x) + \phi(y) - 2\psi(x, \bar{y})$. So it is sufficient to prove

$$\log \left| 1 + \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} + e^{\frac{Q(x, y)}{2} k - \delta k^{1/2}} O(1) \right| = \log \left(1 + O\left(\frac{1}{k}\right) \right).$$

To do this we note that by our assumption $D(x, y) \leq \frac{1}{2}\delta k^{-\frac{1}{2}}$, hence

$$\left| e^{\frac{Q(x, y)}{2} k - \delta k^{1/2}} \right| = e^{\frac{D(x, y)}{2} k - \delta k^{1/2}} \leq e^{-\frac{3\delta}{4} k^{1/2}}.$$

It remains to show that

$$\left| \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| = O\left(\frac{1}{k}\right).$$

By the estimates on $b_j(x, \bar{y})$ from Theorem 1.5 and Stirling's formula, we have

$$\frac{|b_j(x, \bar{y})|}{k^j} \leq \frac{C^j j!^2}{k^j} \leq C' j \frac{C^j j^{2j}}{e^{2j} k^j}.$$

As shown in Lemma 3.2, the function $f(x) = \log \frac{C^x x^{2x}}{e^{2x} k^x}$ is decreasing on the interval $(0, (\frac{k}{C})^{\frac{1}{2}}]$, thus for $j \in [2, N_0 - 1]$,

$$\frac{|b_j(x, \bar{y})|}{k^j} \leq C'(N_0 - 1) \frac{C^2 2^4}{e^4 k^2} \leq C' C^2 \frac{N_0}{k^2}.$$

Therefore,

$$\left| \sum_{j=1}^{N_0-1} \frac{b_j(x, \bar{y})}{k^j} \right| \leq \frac{C}{k} + C' C^2 \frac{N_0^2}{k^2} \leq \frac{C}{k} + C' C \frac{1}{k} = O\left(\frac{1}{k}\right).$$

5. ESTIMATES ON BERGMAN KERNEL COEFFICIENTS

As before, we assume the Kähler metric is analytic in a neighborhood V of p . We will estimate the growth rate of the Bergman kernel coefficients $b_m(x, z)$ as $m \rightarrow \infty$ for x, z in some open set $U \subset V$ containing p . Our goal is to prove Theorem 1.5.

The key ingredient for the proof is the following recursive formula⁶ on $b_m(x, z)$ established in [BeBeSj08].

$$(5.1) \quad b_m(x, z(x, x, \theta)) = - \sum_{l=1}^m \frac{(D_y \cdot D_\theta)^l}{l!} (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta))|_{y=x}.$$

We will break the proof of Theorem 1.5 into two steps. The first step is to derive from the recursive formula (5.1), a recursive inequality on $\|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)}$ for any multi-index $\xi \in (\mathbb{Z}^{\geq 0})^n$. The second step is to estimate $\|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)}$ by induction.

In the following we shall use the following standard notations for multi-indices.

- $\mathbb{1} = (1, 1, \dots, 1)$.
- $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.
- $\alpha \leq \beta$ if $\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2, \dots, \alpha_n \leq \beta_n$.
- $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.
- $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \dots \binom{\alpha_n}{\beta_n}$.
- $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$.

⁶We discussed its proof in (2.12).

- $\binom{l}{\delta_1, \delta_2, \dots, \delta_n} = \frac{l!}{\delta_1! \delta_2! \dots \delta_n!}$ for any non-negative integer l and multi-index $\delta \geq 0$ such that $|\delta| = l$.
- $D_y^\alpha = D_{y_1}^{\alpha_1} D_{y_2}^{\alpha_2} \dots D_{y_n}^{\alpha_n}$.

Lemma 5.1. *Suppose ϕ is real analytic in some neighborhood V of p . Then there exist some positive constant $C \geq 1$ and an open set $U \subset V$ containing p , such that for any non-negative integer m and multi-index $\xi \geq 0$*

$$(5.2) \quad b_{m, \xi} \leq \sum_{l=1}^m \sum_{|\delta|=l} \delta! \sum_{\alpha, \beta \leq \delta} \sum_{|\gamma| \leq |\alpha + \beta|} \sum_{\xi_0 \leq \xi} \frac{b_{m-l, \gamma + \xi_0}}{\gamma!} \frac{\xi!}{\xi_0!} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} C^{|\xi - \xi_0| + 2|\delta| + |\gamma| + 1},$$

where

$$b_{m, \xi} := \|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)}.$$

Proof. We first work on $(D_y \cdot D_\theta)^l (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta))$. We expand the operator $(D_y \cdot D_\theta)^l$ and obtain

$$(5.3) \quad \begin{aligned} & (D_y \cdot D_\theta)^l (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta)) \\ &= \sum_{|\delta|=l} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} D_y^\delta D_\theta^\delta (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta)) \\ &= \sum_{|\delta|=l} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} \sum_{\alpha, \beta \leq \delta} \binom{\delta}{\alpha} \binom{\delta}{\beta} D_y^\alpha D_\theta^\beta (b_{m-l}(x, z(x, y, \theta))) D_y^{\delta - \alpha} D_\theta^{\delta - \beta} \Delta_0 \end{aligned}$$

To compute $D_y^\alpha D_\theta^\beta (b_{m-l}(x, z(x, y, \theta)))$, or equivalently (up to a factor $\alpha! \beta!$) the Taylor coefficients of $b_{m-l}(x, z(x, y, \theta))$ at arbitrary points, we use the following more general strategy by means of formal Taylor expansions. Assume

$$\begin{aligned} f(z) &\sim \sum_{\gamma \geq 0} a_\gamma (z - z_0)^\gamma, \\ z(y, \theta) &\sim \sum_{\alpha, \beta \geq 0} a_{\alpha\beta} (y - y_0)^\alpha (\theta - \theta_0)^\beta, \end{aligned}$$

where $a_{\alpha\beta} = (a_{\alpha\beta}^1, a_{\alpha\beta}^2, \dots, a_{\alpha\beta}^n)$ is a vector and $z(y_0, \theta_0) = z_0$. Then we calculate the Taylor series of the composition $f(z(y, \theta))$ as follows.

$$\begin{aligned} f(z(y, \theta)) &\sim \sum_{\gamma \geq 0} a_\gamma \left(\sum_{\alpha, \beta \geq 0} a_{\alpha\beta} (y - y_0)^\alpha (\theta - \theta_0)^\beta - z_0 \right)^\gamma \\ &= \sum_{\gamma \geq 0} a_\gamma \left(\sum_{\alpha + \beta > 0} a_{\alpha\beta} (y - y_0)^\alpha (\theta - \theta_0)^\beta \right)^\gamma \\ &= \sum_{\gamma \geq 0} a_\gamma \left(\sum_{\alpha + \beta > 0} a_{\alpha\beta}^1 (y - y_0)^\alpha (\theta - \theta_0)^\beta \right)^{\gamma_1} \cdots \left(\sum_{\alpha + \beta > 0} a_{\alpha\beta}^n (y - y_0)^\alpha (\theta - \theta_0)^\beta \right)^{\gamma_n}. \end{aligned}$$

By taking advantage of the following set of indices for convenience,

$$I(\alpha, \beta, i) = \{ \{ \alpha_{ij} \}_{1 \leq j \leq \gamma_i}, \{ \beta_{ij} \}_{1 \leq j \leq \gamma_i} : \alpha_{ij} + \beta_{ij} > 0 \},$$

we get

$$\begin{aligned}
f(z(y, \theta)) &= \sum_{\gamma \geq 0} a_\gamma \sum_{I(\alpha, \beta, 1)} a_{\alpha_{11}\beta_{11}}^1 a_{\alpha_{12}\beta_{12}}^1 \cdots a_{\alpha_{1\gamma_1}\beta_{1\gamma_1}}^1 (y - y_0)^{\alpha_{11} + \alpha_{12} + \cdots + \alpha_{1\gamma_1}} (\theta - \theta_0)^{\beta_{11} + \beta_{12} + \cdots + \beta_{1\gamma_1}} \\
&\quad \cdots \sum_{I(\alpha, \beta, n)} a_{\alpha_{n1}\beta_{n1}}^n a_{\alpha_{n2}\beta_{n2}}^n \cdots a_{\alpha_{n\gamma_n}\beta_{n\gamma_n}}^n (y - y_0)^{\alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{n\gamma_n}} (\theta - \theta_0)^{\beta_{n1} + \beta_{n2} + \cdots + \beta_{n\gamma_n}} \\
&= a_0 + \sum_{\gamma > 0} a_\gamma \sum_{I(\alpha, \beta, 1)} \cdots \sum_{I(\alpha, \beta, n)} a_{\alpha_{11}\beta_{11}}^1 a_{\alpha_{12}\beta_{12}}^1 \cdots a_{\alpha_{1\gamma_1}\beta_{1\gamma_1}}^1 \cdots a_{\alpha_{n1}\beta_{n1}}^n a_{\alpha_{n2}\beta_{n2}}^n \cdots a_{\alpha_{n\gamma_n}\beta_{n\gamma_n}}^n \\
&\quad \cdot (y - y_0)^{\alpha_{11} + \alpha_{12} + \cdots + \alpha_{n\gamma_n}} (\theta - \theta_0)^{\beta_{11} + \beta_{12} + \cdots + \beta_{n\gamma_n}} \\
&= a_0 + \sum_{\alpha + \beta > 0} \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} a_\gamma \sum_{A(\alpha, \beta, \gamma)} \prod_{i,j} a_{\alpha_{ij}\beta_{ij}}^i (y - y_0)^\alpha (\theta - \theta_0)^\beta.
\end{aligned}$$

Here, the last sum runs over the set $A_{\alpha\beta\gamma}$ defined by⁷

$$A_{\alpha\beta\gamma} = \left\{ \left\{ \alpha_{ij} \right\}_{1 \leq i \leq n, 1 \leq j \leq \gamma_i}, \left\{ \beta_{ij} \right\}_{1 \leq i \leq n, 1 \leq j \leq \gamma_i} : \begin{array}{l} \sum_{1 \leq i \leq n, 1 \leq j \leq \gamma_i} \alpha_{ij} = \alpha, \\ \sum_{1 \leq i \leq n, 1 \leq j \leq \gamma_i} \beta_{ij} = \beta, \\ \alpha_{ij} + \beta_{ij} > 0 \end{array} \right\}.$$

Therefore, the coefficient of $(y - y_0)^\alpha (\theta - \theta_0)^\beta$ is

$$\sum_{1 \leq |\gamma| \leq |\alpha + \beta|} a_\gamma \sum_{A(\alpha, \beta, \gamma)} \prod_{i,j} a_{\alpha_{ij}\beta_{ij}}^i$$

when $\alpha + \beta > 0$, and is a_0 when $\alpha = \beta = 0$.

Applying this formula to $b_{m-l}(x, z(x, y, \theta))$ and plugging it into (5.3), we get

$$\begin{aligned}
&(D_y \cdot D_\theta)^l (b_{m-l}(x, z(x, y, \theta)) \Delta_0(x, y, \theta)) \\
&= b_{m-l}(x, z(x, y, \theta)) (D_y \cdot D_\theta)^l \Delta_0 \\
(5.4) \quad &+ \sum_{|\delta|=l} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} \sum_{\substack{\alpha, \beta \leq \delta \\ \alpha + \beta > 0}} \binom{\delta}{\alpha} \binom{\delta}{\beta} \alpha! \beta! \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} \frac{D_z^\gamma b_{m-l}(x, z)}{\gamma!} \\
&\cdot \sum_{A_{\alpha\beta\gamma}} \prod_{i,j} \frac{D_y^{\alpha_{ij}} D_\theta^{\beta_{ij}} z_i}{\alpha_{ij}! \beta_{ij}!} \cdot D_y^{\delta - \alpha} D_\theta^{\delta - \beta} \Delta_0.
\end{aligned}$$

We now substitute (5.4) into equation (5.1) and obtain

$$\begin{aligned}
&b_m(x, z(x, x, \theta)) \\
&= - \sum_{l=1}^m \frac{1}{l!} \left(b_{m-l}(x, z(x, x, \theta)) (D_y \cdot D_\theta)^l \Delta_0(x, x, \theta) + \sum_{|\delta|=l} \sum_{\substack{\alpha, \beta \leq \delta \\ \alpha + \beta > 0}} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} \binom{\delta}{\alpha} \binom{\delta}{\beta} \alpha! \beta! \right. \\
&\quad \cdot \left. \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} \frac{D_z^\gamma b_{m-l}(x, z(x, x, \theta))}{\gamma!} \sum_{A_{\alpha\beta\gamma}} \prod_{i,j} \frac{D_y^{\alpha_{ij}} D_\theta^{\beta_{ij}} z_i}{\alpha_{ij}! \beta_{ij}!} (x, x, \theta) D_y^{\delta - \alpha} D_\theta^{\delta - \beta} \Delta_0(x, x, \theta) \right).
\end{aligned}$$

⁷By our notation, α_{ij} and β_{ij} are vectors that have nothing to do with α and β .

The correspondence $(x, x, z) \leftrightarrow (x, x, \theta = \psi_x(x, z))$, turns this into

$$\begin{aligned} & b_m(x, z) \\ &= - \sum_{l=1}^m \frac{1}{l!} \left(b_{m-l}(x, z) (D_y \cdot D_\theta)^l \Delta_0(x, x, \psi_x(x, z)) + \sum_{|\delta|=l} \sum_{\substack{\alpha, \beta \leq \delta \\ \alpha + \beta > 0}} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} \binom{\delta}{\alpha} \binom{\delta}{\beta} \alpha! \beta! \right. \\ & \quad \cdot \left. \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} \frac{D_z^\gamma b_{m-l}(x, z)}{\gamma!} \sum_{A_{\alpha\beta\gamma}} \prod_{i,j} \frac{D_y^{\alpha_{ij}} D_\theta^{\beta_{ij}} z_i}{\alpha_{ij}! \beta_{ij}!} (x, x, \psi_x(x, z)) D_y^{\delta-\alpha} D_\theta^{\delta-\beta} \Delta_0(x, x, \psi_x(x, z)) \right). \end{aligned}$$

Note that in this recursive formula, the coefficients b_m depend on not only the previous coefficients b_{m-l} , but also the derivatives of b_{m-l} . Hence, we need to include $D_z^\xi b_m$ in our inductive argument. To do this we apply D_z^ξ on both sides and obtain a recursive formula for the derivatives of b_m .

(5.5)

$$\begin{aligned} D_z^\xi b_m(x, z) &= - \sum_{l=1}^m \frac{1}{l!} \sum_{\xi_0 \leq \xi} \binom{\xi}{\xi_0} \left(D_z^{\xi_0} b_{m-l}(x, z) D_z^{\xi-\xi_0} \left((D_y \cdot D_\theta)^l \Delta_0(x, x, \psi_x(x, z)) \right) \right. \\ & \quad + \sum_{|\delta|=l} \sum_{\substack{\alpha, \beta \leq \delta \\ \alpha + \beta > 0}} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} \delta!^2 \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} \frac{D_z^{\gamma + \xi_0} b_{m-l}(x, z)}{\gamma!} \\ & \quad \cdot \left. D_z^{\xi-\xi_0} \left(\sum_{A_{\alpha\beta\gamma}} \prod_{i,j} \frac{D_y^{\alpha_{ij}} D_\theta^{\beta_{ij}} z_i}{\alpha_{ij}! \beta_{ij}!} (x, x, \psi_x(x, z)) \frac{D_y^{\delta-\alpha} D_\theta^{\delta-\beta} \Delta_0}{(\delta-\alpha)! (\delta-\beta)!} (x, x, \psi_x(x, z)) \right) \right). \end{aligned}$$

As $z(x, y, \psi_x(x, z)), \Delta_0(x, y, \psi_x(x, z))$ are holomorphic, by the Cauchy integral formula, there exists a fixed neighborhood U such that for any $x, z \in U$, we have

$$\begin{aligned} & \left| D_z^{\xi-\xi_0} \left(\sum_{A_{\alpha\beta\gamma}} \prod_{i,j} \frac{D_y^{\alpha_{ij}} D_\theta^{\beta_{ij}} z_i}{\alpha_{ij}! \beta_{ij}!} (x, x, \psi_x(x, z)) \frac{D_y^{\delta-\alpha} D_\theta^{\delta-\beta} \Delta_0}{(\delta-\alpha)! (\delta-\beta)!} (x, x, \psi_x(x, z)) \right) \right| \\ & \leq \sum_{A(\alpha, \beta, \gamma)} C^{|\xi-\xi_0|+2|\delta|+|\gamma|+1} (\xi - \xi_0)! \\ & \leq \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} C^{|\xi-\xi_0|+2|\delta|+|\gamma|+1} (\xi - \xi_0)!. \end{aligned}$$

Similarly,

$$\left| D_z^{\xi-\xi_0} (D_y \cdot D_\theta)^l \Delta_0(x, x, \psi_x(x, z)) \right| \leq \sum_{|\delta|=l} \binom{l}{\delta_1, \delta_2, \dots, \delta_n} C^{|\xi-\xi_0|+2|\delta|+1} \delta!^2 (\xi - \xi_0)!.$$

Recall that

$$b_{m, \xi} = \|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)}.$$

Then (5.5) implies the following inequality

$$\begin{aligned}
b_{m,\xi} &\leq \sum_{l=1}^m \sum_{|\delta|=l} \delta! \sum_{\xi_0 \leq \xi} \binom{\xi}{\xi_0} b_{m-l,\xi_0} C^{|\xi-\xi_0|+2|\delta|+1} (\xi - \xi_0)! \\
&\quad + \sum_{l=1}^m \sum_{|\delta|=l} \delta! \sum_{\substack{\alpha, \beta \leq \delta \\ \alpha + \beta > 0}} \sum_{1 \leq |\gamma| \leq |\alpha + \beta|} \sum_{\xi_0 \leq \xi} \binom{\xi}{\xi_0} \frac{b_{m-l,\gamma+\xi_0}}{\gamma!} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} C^{|\xi-\xi_0|+2|\delta|+|\gamma|+1} (\xi - \xi_0)! \\
&\leq \sum_{l=1}^m \sum_{|\delta|=l} \delta! \sum_{\alpha, \beta \leq \delta} \sum_{|\gamma| \leq |\alpha + \beta|} \sum_{\xi_0 \leq \xi} \frac{\xi!}{\xi_0!} \frac{b_{m-l,\gamma+\xi_0}}{\gamma!} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} C^{|\xi-\xi_0|+2|\delta|+|\gamma|+1}
\end{aligned}$$

Thus Lemma 5.1 follows. Next we use this lemma to prove Theorem 1.5. \square

5.1. Proof of Theorem 1.5. For convenience we define

$$(5.6) \quad a_{m,\xi} = \frac{b_{m,\xi}}{(2m+1)! \xi!}.$$

Then by Lemma 5.1

$$(5.7) \quad a_{m,\xi} \leq \sum_{l=1}^m \sum_{|\delta|=l} \sum_{\substack{\alpha, \beta \leq \delta \\ |\gamma| \leq |\alpha + \beta| \\ \xi_0 \leq \xi}} a_{m-l,\gamma+\xi_0} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{(2^{m+1})} \frac{\delta!}{(2l)!} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} C^{|\xi-\xi_0|+2|\delta|+|\gamma|+1}.$$

Since $b_0(x, z) = 1$, we have

$$(5.8) \quad a_{0,\xi} = \begin{cases} 1 & \xi = (0, 0, \dots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

We will argue by induction on m and prove that for any integer $m \geq 0$ and multi-index $\xi \geq 0$,

$$(5.9) \quad a_{m,\xi} \leq \binom{2m + |\xi|}{|\xi|} A^m (2C)^{|\xi|},$$

where C is the same constant which appears on the right hand side of (5.7) and A is a bigger constant to be selected later. Obviously (5.8) and the fact that $C \geq 1$ imply that (5.9) holds for $m = 0$ and any $\xi \geq 0$. Assume that (5.9) holds up to $m - 1$ and we proceed to m . By (5.7) and because $\frac{\delta!}{(2l)!} \leq 1$, we have

$$\begin{aligned}
a_{m,\xi} &\leq \sum_{l=1}^m \sum_{|\delta|=l} \sum_{\alpha, \beta \leq \delta} \sum_{|\gamma| \leq |\alpha + \beta|} \sum_{\xi_0 \leq \xi} A^{m-l} \binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{(2^{m+1})} \\
&\quad \cdot \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} 2^{|\gamma + \xi_0|} C^{|\xi|+2|\gamma|+2|\delta|+1} \\
&\leq \sum_{l=1}^m \sum_{|\delta|=l} \sum_{|\gamma| \leq 2|\delta|} \sum_{\xi_0 \leq \xi} A^{m-l} \binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{(2^{m+1})} \\
&\quad \cdot \sum_{\alpha \leq \delta} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \sum_{\beta \leq \delta} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} 2^{|\gamma + \xi_0|} C^{|\xi|+7l}.
\end{aligned}$$

Due to the fact

$$\sum_{\alpha \leq \delta} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} = \sum_{\beta \leq \delta} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} = \binom{\delta + (|\gamma| + 1) \mathbb{1}}{(|\gamma| + 1) \mathbb{1}} \leq 2^{|\delta| + n|\gamma| + n},$$

it follows that

$$\begin{aligned} a_{m,\xi} &\leq \sum_{l=1}^m \sum_{|\delta|=l} \sum_{|\gamma| \leq 2|\delta|} \sum_{\xi_0 \leq \xi} A^{m-l} \binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{\binom{2m+1}{2l}} 2^{|\gamma + \xi_0| + 2|\delta| + 2n|\gamma| + 2n} C^{|\xi| + 7l} \\ &\leq A^m (2C)^{|\xi|} \sum_{l=1}^m \sum_{|\delta|=l} \sum_{|\gamma| \leq 2l} \sum_{\xi_0 \leq \xi} 2^{|\xi_0| - |\xi|} \left(\frac{2^{6n+4} C^7}{A} \right)^l \binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{\binom{2m+1}{2l}}. \end{aligned}$$

Moreover, since

$$\#\{\delta = l\} = \binom{l+n-1}{n-1} \leq 2^{l+n-1} \leq 2^{nl},$$

we have

$$(5.10) \quad a_{m,\xi} \leq A^m (2C)^{|\xi|} \sum_{l=1}^m \sum_{|\gamma| \leq 2l} \sum_{\xi_0 \leq \xi} 2^{|\xi_0| - |\xi|} \left(\frac{2^{7n+4} C^7}{A} \right)^l \binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{\gamma + \xi_0}{\xi_0} \frac{1}{\binom{2m+1}{2l}}.$$

In the next step we apply the combinatorial inequality

$$\binom{\gamma + \xi_0}{\xi_0} \leq \binom{|\gamma + \xi_0|}{|\xi_0|},$$

and the combinatorial identity

$$\binom{2m - 2l + |\gamma + \xi_0|}{|\gamma + \xi_0|} \binom{|\gamma + \xi_0|}{|\xi_0|} = \binom{2m - 2l + |\gamma + \xi_0|}{|\xi_0|} \binom{2m - 2l + |\gamma|}{2m - 2l}.$$

Observe that, since $|\gamma| \leq 2l$ and $\xi_0 \leq \xi$, we have

$$\binom{2m - 2l + |\gamma + \xi_0|}{|\xi_0|} \leq \binom{2m + |\xi|}{|\xi|}.$$

Plugging these into (5.10), we obtain

$$a_{m,\xi} \leq A^m (2C)^{|\xi|} \binom{2m + |\xi|}{|\xi|} \sum_{l=1}^m \sum_{|\gamma| \leq 2l} \sum_{\xi_0 \leq \xi} 2^{|\xi_0| - |\xi|} \left(\frac{2^{7n+4} C^7}{A} \right)^l \binom{2m - 2l + |\gamma|}{2m - 2l} \frac{1}{\binom{2m+1}{2l}}.$$

Again since

$$\#\{\gamma = k\} = \binom{k+n-1}{n-1} \leq 2^{k+n-1},$$

the sum over γ on the right hand side can be estimated as

$$\sum_{|\gamma| \leq 2l} \binom{2m - 2l + |\gamma|}{2m - 2l} = \sum_{k=0}^{2l} \sum_{|\gamma|=k} \binom{2m - 2l + k}{2m - 2l} \leq 2^{2l+n-1} \binom{2m + 1}{2m - 2l + 1}.$$

Therefore,

$$\begin{aligned} a_{m,\xi} &\leq A^m (2C)^{|\xi|} \binom{2m + |\xi|}{|\xi|} \sum_{l=1}^m \left(\frac{2^{8n+5} C^7}{A} \right)^l \sum_{\xi_0 \leq \xi} 2^{|\xi_0| - |\xi|} \\ &\leq A^m (2C)^{|\xi|} \binom{2m + |\xi|}{|\xi|} \sum_{l=1}^m 2^n \left(\frac{2^{8n+5} C^7}{A} \right)^l. \end{aligned}$$

By taking $A = 2^{9n+6}C^7$ we surely have $\sum_{l=1}^m 2^n \left(\frac{2^{8n+5}C^7}{A}\right)^l < 1$, which implies that $a_{m,\xi} \leq A^m(2C)^\xi \binom{2m+|\xi|}{|\xi|}$, hence concluding the induction. Remembering the definition (5.6) of a_m in terms of b_m , we get

$$(5.11) \quad \|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)} = (2m+1)! \xi! a_{m,\xi} \leq (64A)^{m+|\xi|} (m!)^2 \xi!,$$

So Theorem 1.5 follows by renaming $64A$ to C and taking $\xi = 0$.

6. OPTIMALITY OF THE UPPER BOUNDS ON BERGMAN COEFFICIENTS b_m

In this section we show that although it would be desirable to improve the estimate (5.11) to

$$(6.1) \quad \|D_z^\xi b_m(x, z)\|_{L^\infty(U \times U)} \leq C^{m+|\xi|} m! \xi!,$$

it is not possible to prove it simply by the recursive inequality (5.2). Here we provide an example which satisfies (5.2) while fails (6.1). For simplicity, we assume $C = 1$ in (5.2). Let us consider the worst case when equality holds in (5.2), i.e.

$$(6.2) \quad b_{m,\xi} = \sum_{l=1}^m \sum_{|\delta|=l} \delta! \sum_{\alpha, \beta \leq \delta} \sum_{|\gamma| \leq |\alpha+\beta|} \sum_{\xi_0 \leq \xi} \frac{b_{m-l, \gamma+\xi_0}}{\gamma!} \frac{\xi!}{\xi_0!} \binom{\alpha + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}} \binom{\beta + |\gamma| \mathbb{1}}{|\gamma| \mathbb{1}}.$$

One can easily check that this recursive equation uniquely defines $\{b_{m,\xi}\}$ given an initial data $\{b_{0,\xi}\}$. We shall only focus on the terms b_{m,ke_1} where $e_1 = (1, 0, \dots, 0)$ and show by induction that

$$(6.3) \quad b_{m,ke_1} \geq (2m - 2 + k)! \quad \text{for any } m \geq 1, k \geq 0.$$

First let us check this for b_{1,ke_1} . Since in our case

$$b_{0,\xi} = \begin{cases} 1 & \xi = 0 \\ 0 & \text{otherwise,} \end{cases}$$

by (6.2), we have

$$b_{1,\xi} = \sum_{|\delta|=1} \sum_{\alpha, \beta \leq \delta} \xi! \geq \xi!.$$

Therefore (6.3) holds for b_{1,ke_1} . Assume that (6.3) holds for $b_{1,ke_1}, b_{2,ke_1}, \dots, b_{m-1,ke_1}$. Then by only considering the terms with $l = |\alpha| = |\beta| = 1$ and $\gamma = 2e_1$ in (6.2), we obtain for $m \geq 2$

$$\begin{aligned} b_{m,ke_1} &\geq \sum_{|\delta|=1} \sum_{|\alpha|=|\beta|=1} \sum_{j=0}^k \frac{b_{m-1, (j+2)e_1}}{2!} \frac{k!}{j!} \binom{\alpha + 2 \cdot \mathbb{1}}{2 \cdot \mathbb{1}} \binom{\beta + 2 \cdot \mathbb{1}}{2 \cdot \mathbb{1}} \\ &\geq \sum_{j=0}^k (2m - 2 + j)! \frac{k!}{j!} \\ &\geq (2m - 2 + k)!. \end{aligned}$$

Note that if in particular we put $k = 0$ into (6.3) we get

$$b_{m,0} \geq \left(\frac{1}{4}\right)^m m!^2,$$

which shows that up to an exponential factor C^m , $m!^2$ is the best upper bound one can hope to obtain from the recursive inequality (5.2).

7. KÄHLER MANIFOLDS WITH LOCAL CONSTANT HOLOMORPHIC SECTIONAL CURVATURES

In this section, we consider Kähler manifolds with local constant holomorphic sectional curvature and prove Theorem 1.7 and Corollary 1.8. If the Kähler manifold has constant holomorphic sectional curvatures only near a point p , then we have the following properties on b_m near p .

Proposition 7.1. *Assume the Kähler manifold has constant holomorphic sectional curvature c in some neighborhood V containing p . Then there exists $U \subset V$ containing p such that for any $x, y \in U$, the Bergman kernel coefficients $b_m(x, \bar{y})$ are all constants that vanish for $m > n$, and are given by the polynomial relation*

$$\sum_{j=0}^n b_j k^{n-j} = \frac{c^n \Gamma(\frac{k}{c} + n + 1)}{\Gamma(\frac{k}{c} + 1)} = c^n \left(\frac{k}{c} + n\right) \left(\frac{k}{c} + n - 1\right) \dots \left(\frac{k}{c} + 1\right).$$

In the case $c = 0$, the right hand side is understood as the limit when $c \rightarrow 0$, which equals k^n .

To prove this proposition we first prove a lemma that gives a recursive formula for the constants b_m .

Lemma 7.2. *For all c , $b_0 = 1$ and for $m \geq 1$ we have*

$$(7.1) \quad b_m = - \sum_{l=1}^m (-c)^l \frac{(l+n-1)!}{(n-1)!} a_l b_{m-l},$$

where $\{a_l\}_{l=0}^{\infty}$ are given by the Taylor expansion

$$(7.2) \quad e^x \left(\frac{e^x - 1}{x}\right)^{n-1} = \sum_{l=0}^{\infty} a_l x^l.$$

Proof. When the Kähler manifold has constant holomorphic sectional curvature c in V containing p , it is known that (see for example equation (28) in [Bo47]), in a specific coordinate (Bochner coordinate) at p , the Kähler potential near p , say in U , can be uniquely written⁸ as

$$\phi(x) = \begin{cases} \frac{1}{c} \ln(1 + c \sum_{i=1}^n |x_i|^2) & c \neq 0, \\ \sum_{i=1}^n |x_i|^2 & c = 0, \end{cases}$$

where the case $c = 0$ can be regarded as the limiting case when $c \rightarrow 0$. We now simplify the recursive formula (5.1) on b_m with the above explicit expression for ϕ . By polarizing $\phi(x)$, we get $\psi(x, z) = \frac{1}{c} \ln(1 + cx \cdot z)$. Taking partial derivatives,

$$\psi_{x_i}(x, z) = \frac{z_i}{1 + cx \cdot z}, \quad \psi_{x_i z_j} = \frac{\delta_{ij}}{1 + cx \cdot z} - \frac{cz_i x_j}{(1 + cx \cdot z)^2}.$$

The *matrix determinant lemma* tells that if A is an invertible matrix and u and v are column vectors, then

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det A.$$

Therefore,

$$\det \psi_{xz}(x, z) = \frac{1}{(1 + cx \cdot z)^{n+1}}.$$

⁸Note that we are using a different notation from the one in [Bo47], so that $\mathbb{C}\mathbb{P}^n$ has holomorphic sectional curvature 1 instead of 2.

Recalling the definition of $\theta = \theta(x, y, z)$ in (2.3), we have

$$\theta = \frac{\ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)}{c(x - y) \cdot z} z.$$

Hence,

$$\begin{aligned} \frac{\partial \theta_i}{\partial z_j}(x, y, z) &= \frac{\ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)}{c(x - y) \cdot z} \delta_{ij} + \frac{\frac{cx_j}{1+cx \cdot z} - \frac{cy_j}{1+cy \cdot z}}{c(x - y) \cdot z} z_i \\ &\quad - \frac{\ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)}{(c(x - y) \cdot z)^2} c(x_j - y_j) z_i. \end{aligned}$$

Using the *matrix determinant lemma* again, we obtain

$$\det \theta_z = \frac{1}{(1 + cx \cdot z)(1 + cy \cdot z)} \left(\frac{\ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)}{c(x - y) \cdot z} \right)^{n-1}.$$

Therefore,

$$\begin{aligned} \Delta_0(x, y, z) &= \frac{\det \psi_{yz}(y, z)}{\det \theta_z(x, y, z)} \\ &= \frac{1 + cx \cdot z}{1 + cy \cdot z} \left(\frac{c(x - y) \cdot z}{1 + cy \cdot z} \right)^{n-1} \left(\frac{1}{\ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)} \right)^{n-1}. \end{aligned}$$

We then use the relation $c\theta \cdot (x - y) = \ln(1 + cx \cdot z) - \ln(1 + cy \cdot z)$, to change the variables from (x, y, z) to (x, y, θ) and get

$$\Delta_0(x, y, \theta) = e^{c\theta \cdot (x-y)} \left(\frac{e^{c\theta \cdot (x-y)} - 1}{c\theta \cdot (x - y)} \right)^{n-1} = \sum_{l=0}^{\infty} a_l (c\theta \cdot (x - y))^l.$$

Remember that $b_0(x, z) = 1$, so if we assume that $b_1(x, z), b_2(x, z) \cdots b_{m-1}(x, z)$ are all constants, then by (5.1) we have

$$b_m = - \sum_{l=1}^m \frac{(D_y \cdot D_\theta)^l}{l!} (b_{m-l} \Delta_0) \Big|_{y=x} = - \sum_{l=1}^m b_{m-l} \frac{(D_y \cdot D_\theta)^l \Delta_0}{l!} \Big|_{y=x}.$$

Now we calculate $\frac{(D_y \cdot D_\theta)^l \Delta_0}{l!} \Big|_{y=x}$ as follows.

$$\begin{aligned} \frac{(D_y \cdot D_\theta)^l \Delta_0}{l!} \Big|_{y=x} &= \sum_{j=0}^{\infty} a_j \frac{(D_y \cdot D_\theta)^l}{l!} (c\theta \cdot (x - y))^j \Big|_{y=x} \\ &= a_l \frac{(D_y \cdot D_\theta)^l}{l!} (c\theta \cdot (x - y))^l \Big|_{y=x} \\ &= \frac{c^l a_l}{l!} \sum_{|\xi|=l} \binom{l}{\xi_1, \xi_2, \dots, \xi_n} D_y^\xi D_\theta^\xi \sum_{|\eta|=l} \binom{l}{\eta_1, \eta_2, \dots, \eta_m} \theta^\eta (x - y)^\eta \Big|_{y=x} \\ &= \frac{c^l a_l}{l!} \sum_{|\xi|=l} \binom{l}{\xi_1, \xi_2, \dots, \xi_n}^2 \xi!^2 (-1)^{|\xi|} \\ &= (-c)^l a_l \frac{(l + n - 1)!}{(n - 1)!}. \end{aligned}$$

Therefore, $b_m = - \sum_{l=1}^m (-c)^l \frac{(l+n-1)!}{(n-1)!} a_l b_{m-l}$ is a constant and the lemma follows. \square

Proof of Proposition 7.1. By Lemma 7.2, it is obvious that if $c = 0$ then $b_m = 0$ for any $m \geq 1$. When $c \neq 0$, we define $\tilde{b}_m = \frac{b_m}{c^m}$. Then $\tilde{b}_0 = b_0 = 1$ and \tilde{b}_m 's satisfy

$$\tilde{b}_m = - \sum_{l=1}^m (-1)^l \frac{(l+n-1)!}{(n-1)!} a_l \tilde{b}_{m-l}.$$

Since \tilde{b}_m share the same initial data and the same recursive formula as the Bergman coefficients for $c = 1$, \tilde{b}_m are identical to the Bergman coefficients for $c = 1$, in particular those of $\mathbb{C}\mathbb{P}^n$. On the other hand, one can find by direct computation (see for instance, Example 1 in [Lu00]) that for $\mathbb{C}\mathbb{P}^n$, $\tilde{b}_m = 0$ for $m > n$, and

$$\sum_{j=0}^n \frac{\tilde{b}_j}{k^j} = \frac{1}{k^n} \frac{\Gamma(k+n+1)}{\Gamma(k+1)} = \frac{1}{k^n} (k+n)(k+n-1)\dots(k+1).$$

Then, using this we write

$$\sum_{j=0}^n b_j k^{n-j} = \sum_{j=0}^n c^j \tilde{b}_j k^{n-j} = k^n \sum_{j=0}^n \tilde{b}_j \left(\frac{k}{c}\right)^{-j} = \frac{c^n \Gamma(\frac{k}{c} + n + 1)}{\Gamma(\frac{k}{c} + 1)},$$

and the result follows. \square

Remark 7.3. If the Kähler manifold has global constant holomorphic sectional curvature c , then the fact that the Bergman coefficients are constants and vanish for $m > n$, can be obtained from the Hirzebruch-Riemann-Roch Theorem as we demonstrate below.

Suppose M has constant holomorphic sectional curvature. Then by Theorem 10 of [Bo47], the curvature $R_{i\bar{j}k\bar{l}}$ can be written in terms of the metric $g_{i\bar{j}}$ as

$$R_{i\bar{j}k\bar{l}} = -c (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}).$$

By Theorem 1.1 in [Lu00], each $b_m(x, \bar{x})$ is a polynomial of the curvature and its covariant derivatives at x . Therefore, each $b_m(x, \bar{x})$ is a constant function on M . Denote $B_k(x)$ to be the Bergman function on M defined by $B_k(x) = |K_k(x, x)|_{h^k}$. Then

$$\dim H^0(M, L^k) = \int_M B_k(x) \frac{\omega^n}{n!}.$$

By the Kodaira Vanishing theorem and the Hirzebruch-Riemann-Roch theorem, for sufficiently large k , we have

$$\dim H^0(M, L^k) = \chi(M, L^k) = \int_M Ch(L^k) \wedge Td(M),$$

where $Ch(L^k)$ is the Chern character and $Td(M)$ is the Todd class. Note that, since

$$\begin{aligned} & \int_M Ch(L^k) \wedge Td(M) \\ &= \int_M Td(M) + \frac{k}{\pi} \int_M \frac{\omega}{\pi} \wedge Td(M) + \left(\frac{k}{\pi}\right)^2 \int_M \frac{\omega^2}{2!} \wedge Td(M) + \dots + \left(\frac{k}{\pi}\right)^n \int_M \frac{\omega^n}{n!}, \end{aligned}$$

the quantity $\dim H^0(M, L^k)$ must be a polynomial in k . On the other hand, by the asymptotic expansion of $K_k(x, x)$ we have

$$\int_M B_k(x) \frac{\omega^n}{n!} = \left(\frac{k}{\pi}\right)^n \int_M \frac{\omega^n}{n!} \left(b_0(x, \bar{x}) + \frac{b_1(x, \bar{x})}{k} + \frac{b_2(x, \bar{x})}{k^2} + \dots \right).$$

By comparing the coefficients and knowing that $b_m(x, \bar{x})$ are constants, we conclude that $b_m(x, \bar{x})$ vanishes when $m > n$. Note that because $b_m(x, z)$ is the polarization of $b_m(x, \bar{x})$, each $b_m(x, z)$ wherever it is defined, is the same constant as $b_m(x, \bar{x})$. So $b_m(x, z) = 0$ for $m > n$.

Now that in this particular case, we have better estimates on the growth of $b_m(x, z)$, the proofs of Theorem 1.7 and Corollary 1.8 follow by using the same argument as in the previous sections. We shall only indicate the differences.

7.1. Proof of Theorem 1.7. Since $\|b_m(x, z)\|_{L^\infty(U \times U)}$ is now trivially bounded by $C^m m!$ (in fact it is bounded by a constant C) instead of $C^m m!^2$, estimates in Lemma 3.2 can be improved to

$$\begin{aligned} \|A_N\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} &\leq C^N N!, \\ \|A^{(N)}\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} &\leq Ck^2 + \frac{C^N N!}{k^N}, \\ \|D_\theta \cdot A_N\|_{L^\infty(B^n(0,1) \times B^n(0,1) \times W_1)} &\leq C^N N!. \end{aligned}$$

Accordingly, the error term estimates in Proposition 4.1 and 2.1 can be both improved to

$$|\mathcal{R}_{N+1}(\phi, k)| \leq \frac{C^{N+1}(N+1)!}{k^{N+1}}.$$

By choosing $N = \lceil \frac{k}{C} \rceil - 1$, we can minimize the error term $\mathcal{R}_{N+1}(\phi, k)$ to be $e^{-\delta k} O(1)$ and thus obtain the desired results by observing that $\lceil \frac{k}{C} \rceil - 1 > n$ for sufficiently large k .

8. TRANSPORT EQUATIONS FOR THE AMPLITUDE

In this final section, we present a parametrix for the local Bergman kernel by means of *transport equations*, similar to those for the wave and heat equations, which could be of independent interest. One can also give a proof of Theorem 1.5 using this parametrix representation.

In the following $K_k^{(N)}(x, y) = K_{k,y}^{(N)}(x)$ is the local Bergman kernel of order N , defined in Proposition 2.1. See Section 2.1 for all other necessary notations.

Theorem 8.1. *For all $N \in \mathbb{N}$, the local Bergman kernel of order N can be expressed as*

$$K_k^{(N)}(x, y) = \left(\frac{k}{\pi}\right)^n e^{k\psi(x, \bar{y})} B^{(N)}(x, \bar{y}),$$

where $B^{(N)}(x, \bar{y})$ is the truncation up to order k^{-N} of the formal expansion

$$B(x, \bar{y}) = b_0(x, \bar{y}) + \frac{b_1(x, \bar{y})}{k} + \frac{b_2(x, \bar{y})}{k^2} + \dots,$$

given by

$$B(x, \bar{y}) = \frac{1 + k(x - y) \cdot A(x, y, \theta(x, y, \bar{y})) + (D_\theta \cdot A)(x, y, \theta(x, y, \bar{y}))}{\Delta_0(x, y, \theta(x, y, \bar{y}))},$$

where

$$A(x, y, \theta) = \frac{A_1(x, y, \theta)}{k} + \frac{A_2(x, y, \theta)}{k^2} + \dots,$$

is the formal expansion whose coefficients satisfy the following transport equations

$$A_1(x, y, \theta) = - \int_0^1 (D_y \Delta_0)(x, tx + (1-t)y, \theta) dt,$$

and for $m \geq 2$

$$\begin{aligned} (8.1) \quad A_m(x, y, \theta) &= - \int_0^1 D_y \left(\Delta_0(x, y, \theta) (D_\theta \cdot A_{m-1})(x, x, \theta(x, x, z(x, y, \theta))) \right) (x, tx + (1-t)y, \theta) dt \\ &\quad + \int_0^1 D_y \left((D_\theta \cdot A_{m-1})(x, y, \theta) \right) (x, tx + (1-t)y, \theta) dt. \end{aligned}$$

Proof. Recall that the asymptotic expansion $a(x, y, \theta)$ satisfies

$$1 + a(x, y, \theta(x, y, z)) = B(x, z)\Delta_0(x, y, \theta(x, y, z)).$$

Since a is negligible, we have $(Sa)(x, x, \theta) = 0$ and hence there exists a formal expansion $C(x, y, \theta)$ such that

$$Sa(x, y, \theta) = (x - y) \cdot C(x, y, \theta).$$

We can solve for a by taking S^{-1} and obtain

$$a = (x - y) \cdot S^{-1}C + \frac{1}{k}D_\theta \cdot S^{-1}C.$$

Then we set $A = \frac{1}{k}S^{-1}C$. Obviously,

$$a(x, y, \theta) = k(x - y) \cdot A(x, y, \theta) + D_\theta \cdot A(x, y, \theta).$$

Thus $B(x, z)$ is related to A by

$$(8.2) \quad B(x, z) = \frac{1 + k(x - y) \cdot A(x, y, \theta(x, y, z)) + (D_\theta \cdot A)(x, y, \theta(x, y, z))}{\Delta_0(x, y, \theta(x, y, z))}.$$

Since $B(x, z)$ is independent of the variable y , the right hand side is unchanged when we vary y . In particular, we can put $y = x$ and get

$$\frac{1 + k(x - y) \cdot A(x, y, \theta(x, y, z)) + (D_\theta \cdot A)(x, y, \theta(x, y, z))}{\Delta_0(x, y, \theta(x, y, z))} = 1 + (D_\theta \cdot A)(x, x, \theta(x, x, z)).$$

Plugging the asymptotic expansion

$$A \sim A_0 + \frac{A_1}{k} + \frac{A_2}{k^2} + \dots,$$

and comparing the coefficients, we obtain $A_0 = 0$ and

$$(x - y) \cdot A_1(x, y, \theta) = \Delta_0(x, y, \theta) - 1.$$

By the fundamental theorem of calculus, this equation has a particular solution given by

$$A_1(x, y, \theta) = - \int_0^1 (D_y \Delta_0)(x, tx + (1 - t)y, \theta) dt.$$

For $m \geq 2$, we get

$$(x - y) \cdot A_m(x, y, \theta(x, y, z)) = \Delta_0(x, y, \theta(x, y, z))(D_\theta \cdot A_{m-1})(x, x, \theta(x, x, z)) - (D_\theta \cdot A_{m-1})(x, y, \theta(x, y, z)),$$

which by changing variables to (x, y, θ) , turns into

$$(x - y) \cdot A_m(x, y, \theta) = \Delta_0(x, y, \theta)(D_\theta \cdot A_{m-1})(x, x, \theta(x, x, z(x, y, \theta))) - (D_\theta \cdot A_{m-1})(x, y, \theta).$$

Similarly, we can see that A_m given by (8.1) is a particular solution to the above equation. \square

Remark 8.2. In the recursive formula (8.1), each A_m depends on second order derivatives of the previous term A_{m-1} . This is similar to the relations between the coefficients in Hadamard's parametrix (see [Ze12]) for the fundamental solution of wave equation. In Hadamard's parametrix, we have

$$U = U_0 + \Gamma U_1 + \dots + \Gamma^m U_m + \dots,$$

where each $U_m(x, t)$ is a smooth function defined on $W \times \mathbb{R}^+$ and W is some neighborhood on the given Riemannian manifold (M, g) . The function U_0 is smooth and non-vanishing. For $m \geq 1$,

$$(8.3) \quad U_m = - \frac{U_0}{4ms^{n+m+1}} \int_0^s U_0^{-1} s^{n+m} \square U_{m-1} ds,$$

where $\square = \frac{\partial^2}{\partial t^2} + \Delta_g$. Hadamard's result says that when (M, g) is real analytic, U_m is dominated by some C^m . But the same procedure does not work for A_m because we do not have the m

appearing in the denominator of (8.3), nor the factor s^{n+m} in the integrand, which contributes as $\frac{1}{s^{n+m+1}} \int_0^s s^{n+m} ds = \frac{1}{n+m+1}$. With these differences in mind, it seems that A_m can only be controlled by $C^m m!^2$. We obtain the same bounds for b_m by the relation between A_m and b_m in (8.2).

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