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Moduli of Continuity, Gauss Curvature Flow and Ricci Solitons

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by

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Professor Kenneth Intriligator
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TABLE OF CONTENTS

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PUBLICATIONS

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Xiaolong Li, Kui Wang, "Nonparametric Hypersurfaces Moving by Powers of Gauss Curvature", arxiv:1606.01287[math.AP], 2016.

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ABSTRACT OF THE DISSERTATION

Moduli of Continuity, Gauss Curvature Flow and Ricci Solitons

by

Xiaolong Li

Doctor of Philosophy in Mathematics

University of California, San Diego, 2017

Professor Lei Ni, Chair
Professor Bennett Chow, Co-Chair

This thesis is a summary of the work accomplished by the author and his coauthors in geometric analysis during his Ph.D. studies. It consists of four parts.

The first and the second parts are the estimates of modulus of continuity for viscosity solutions of nonlinear partial differential equations in domains in Euclidean spaces and on manifolds. The main results generalize B. Andrews and J. Clutterbuck's modulus of continuity estimates for smooth solutions to viscosity

solutions. The main ingredients of the proofs are the parabolic maximum principle for semicontinuous functions, its generalized version on manifolds, and the multi-point estimates method.

The third part studies asymptotic behavior of nonparametric hypersurfaces of dimension n moving by α powers of its Gaussian Curvature with $\alpha > 1/n$. Our work generalizes the results for $\alpha = 1$ obtained by V. Oliker to all $\alpha > 1/n$. Although we are using similar ideas, the proof is quite technical.

In the fourth part, we study classification of shrinking gradient Ricci solitons. Our main result asserts that any four-dimensional complete gradient shrinking Ricci soliton with positive isotropic curvature is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. This gives a clean classification result removing the earlier additional assumptions in by L. Ni and N. Wallach. This also generalizes a result of Perelman on three-dimensional gradient shrinking Ricci solitons to dimension four. The result has important consequences in studying Ricci flow on four-dimensional manifolds.

Chapter 1

Introduction and Main Results

1.1 Moduli of Continuity for Viscosity Solutions

In Chapter 2, we investigate the moduli of continuity for viscosity solutions of a wide class of nonsingular quasilinear evolution equations and also for the level set mean curvature flow, which is an example of singular degenerate equations. We prove that the modulus of continuity is a viscosity subsolution of some one dimensional equation. This work extends B. Andrews' recent result on moduli of continuity for smooth spatially periodic solutions. Chapter 2 is a reprint of the author's paper [44] with slight modifications. The main theorem in [44] states that

Theorem 1.1.1. *Let* $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ *be a continuous periodic viscosity solution of*

$$u_t = a^{ij}(Du, t)D_i D_j u + b(Du, t) \tag{1.1}$$

where $A(p,t) = (a^{ij}(p,t))$ *is positive semi-definite. Assume there exists a continuous function* $\alpha : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ *with*

$$0 < \alpha(R,t) \le R^2 \inf_{|p|=R,(v\cdot p)\neq 0} \frac{v^T A(p,t) v}{(v\cdot p)^2}. \tag{1.2}$$

Then the modulus of continuity $\omega(s,t) = \sup\left\{\frac{u(y,t)-u(x,t)}{2} \,\middle|\, \frac{|y-x|}{2} = s\right\}$ *of* u *is a viscosity subsolution of the one dimensional equation*

$$\varphi_t = \alpha(|\varphi'|,t)\varphi''. \tag{1.3}$$

These results obtained in [44] can be used to obtain gradient estimates and eigenvalue estimates. It is also of great significance to control the growth of unbounded solutions in studying Cauchy problems on the whole space.

1.2 Moduli of Continuity for Viscosity Solutions on Manifolds

The main results of Chapter 3 are generalizations of the results in Chapter 2 to viscosity solutions on manifolds. This is joint work with Dr. Kui Wang [47].

Theorem 1.2.1. *Let* $u : M \times [0,T) \to \mathbb{R}$ *be a viscosity solution of*

$$u_t = \left[\alpha(|Du|,t)\frac{D_iuD_ju}{|Du|^2} + \beta(|Du|,t)\left(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\right)\right] D_iD_ju + b(|Du|,t). \tag{1.4}$$

on a closed manifold M *and denote by* D *the diameter of* M*. Assume further that* $Ric_g \ge (n-1)\kappa g$*. Then the modulus of continuity* $w : [0, \frac{D}{2}] \times [0,T) \to \mathbb{R}$ *of* u

satisfies

$$w_t \le \alpha(w', t)w'' + (n-1)\frac{c_\kappa'(s)}{c_\kappa(s)}\beta(w', t)w' \tag{1.5}$$

in the viscosity sense provided ω *is increasing in* s, *and where* $c_\kappa(s)$ *is defined by* $c_\kappa'' + \kappa c_\kappa = 0, c_\kappa(0) = 1, c_\kappa'(0) = 0.$

In [47], we also proved the viscosity version generalizations of Theorem 4, 5, and 6 in [3] and Theorem 1.2 in [8].

1.3 Nonparametric Hypersurfaces Moving by Powers of Gauss Curvature

Chapter 4 presents the joint work with Dr. Kui Wang on the asymptotic behavior of nonparametric hypersurfaces of dimension n moving by α powers of Gauss Curvature with $\alpha > 1/n$. Our main result in [46] generalizes the result for $\alpha = 1$ obtained by V. Oliker in [59] to all $\alpha > 1/n$. Let u be a solution of

$$\begin{aligned} &u_t = \frac{[\det(u_{ij})]^\alpha}{(1+|\nabla u|^2)^{\alpha\beta}} \text{ in } \Omega \times (0,\infty), \\ &u(x,t) = 0 \text{ in } \partial\Omega \times (0,\infty), \\ &u(x,t) \text{ is strictly convex for each } t \ge 0, \end{aligned} \tag{1.6}$$

where $\alpha \in (\frac{1}{n}, \infty)$ and $\beta \ge 0$ are constants and Ω is a strictly convex domain in $\mathbb{R}^n$. When $\beta = \frac{n+2-\frac{1}{\alpha}}{2}$, the normal speed of the point $(x, u(x,t))$ is equal to α powers of the Guass curvature of the graph. We prove that, the solution

$u(x,t)$ asymptotically picks up the symmetry of the domain Ω. More precisely, u becomes radially symmetric regardless of the initial shape if Ω is a ball. If Ω is centrally symmetric, then u asymptotically becomes centrally symmetric as well. The strategy we use is similar to that of V. Oliker[59], but the proof is technically much more difficult.

Our first result establishes the existence of self-similar solutions to the following equation:

$$\begin{aligned} &u_t = M^\alpha(u) \text{ in } \Omega \times (0,\infty), \\ &u(x,t) = 0 \text{ in } \partial\Omega \times (0,\infty), \\ &u(x,t) \text{ is strictly convex for each } t \geq 0. \end{aligned} \tag{1.7}$$

Theorem 1.3.1. *Let Ω be a bounded strictly convex domain with smooth boundary $\partial\Omega$. Then problem* (1.7) *admits a self-similar solution in $\overline{\Omega} \times (0,\infty)$ given by*

$$u(x,t) = (1+t)^{\frac{1}{1-n\alpha}}\psi(x), \tag{1.8}$$

where ψ is the unique solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the equation

$$\begin{aligned} &M(\psi) = \left(\frac{-\psi}{|1-n\alpha|}\right)^{\frac{1}{\alpha}} \text{ in } \Omega,\ \ \psi = 0 \text{ on } \partial\Omega, \\ &\psi \text{ is stictly convex and } \psi < 0 \text{ in } \Omega, \end{aligned} \tag{1.9}$$

and $\sup_\Omega |\psi(x)|$ admits an estimate depending only on n, α and the domain Ω. Furthermore, if $\tilde{u}(x,t) = \varphi(t)\tilde{\psi}(x)$ is an arbitrary self-similar solution of (4.2)*,* *then there exists a unique* $c > 0$ *such that* $\tilde{\psi}(x) = c\psi(x)$ *and*

$$\tilde{u}(x,t) = u(x,t)\left\{\frac{1+t}{[c\varphi(0)]^{1-n\alpha}+t}\right\}^{\frac{1}{n\alpha-1}}. \tag{1.10}$$

The main theorem concerning the asymptotic behavior of the solution is the following:

Theorem 1.3.2. *Let* $u(x,t) \in C^2(\overline{\Omega} \times (0,\infty))$ *be a solution of the problem*

$$\begin{aligned} &u_t = \frac{M^\alpha(u)}{(1+|\nabla u|^2)^{\alpha\beta}} \ \textit{in } \Omega \times (0,\infty),\\ &u(x,t) = 0 \ \textit{in } \partial\Omega \times (0,\infty), \\ &u(x,t) \ \textit{is strictly convex for each } t \geq 0, \end{aligned} \tag{1.11}$$

where $\alpha > 1/n$ *and* $\beta \geq 0$ *are constants. If* $\beta = 0$*, then there exists positive constant* C_1 *depending only on dimension* n*,* α*,* Ω *and* $u(x,0)$*, such that for all* $t \geq 0$*,*

$$\sup_{\Omega}\left|(1+t)^{\frac{1}{n\alpha-1}}u(x,t) - \psi(x)\right| \leq \frac{C_1}{1+t}, \tag{1.12}$$

If $\beta > 0$*, then*

$$\left[\frac{C_2}{1+t} + G^{\frac{1}{1-n\alpha}} - 1\right]\psi \leq (1+t)^{\frac{1}{n\alpha-1}}u(x,t) - \psi(x) \leq \frac{-C_3\psi}{1+t}, \tag{1.13}$$

where C_2 *and* C_3 *are positive constants depending only on dimension* n*,* α*,* Ω*,* $u(x,0)$ *and*

$$G = \inf_{\Omega}\left(1+|\nabla u(x,0)|^2\right)^{-\alpha\beta}.$$

Moreover,

$$\lim_{t\to\infty}(1+t)^{\frac{1}{n\alpha-1}}u(x,t) = \psi(x) \ \textit{uniformly on } \overline{\Omega}. \tag{1.14}$$

We have gradient estimates for solutions of (4.9).

Corollary 1.3.3. *Suppose the same conditions as in Theorem 4.2.2 holds. Then for all $t \geq 0$,*

$$\sup_{\Omega} |\nabla u(x,t)| \leq G^{\frac{1}{1-n\alpha}} \sup_{\partial\Omega} \psi_\nu(x)(C_4 + t)^{\frac{1}{1-n\alpha}}$$

where ψ_ν is the derivative in the direction of the outward unit normal to $\partial\Omega$, and C_4 depends only on $u(x,0)$.

An interesting geometric consequence of Theorem 4.2.2 is the following:

Theorem 1.3.4. *If Ω is a ball in $\mathbb{R}^n$ and $u(x,t) \in C^2(\overline{\Omega} \times (0,\infty))$ is a solution of (4.9). Then*

$$(1+t)^{\frac{1}{n\alpha-1}} u(x,t) \to \psi(|x|) \textit{ uniformly on } \overline{\Omega} \textit{ as } t \to \infty.$$

This theorem implies that, $u(x,t)$ asymptotically becomes radially symmetric regardless of the initial shape. More generally, if Ω is centrally symmetric, then

$$(1+t)^{\frac{1}{n\alpha-1}} u(x,t) \to \psi(x) \text{ uniformly on } \overline{\Omega} \text{ as } t \to \infty,$$

where $\psi(x) = \psi(-x)$. The proof of Theorem 4.2.4 is the same as in [59, Section 6] and we omit it here.

1.4 Four-dimensional Shrinking Solitons with Positive Isotropic Curvature

Chapter 5 studies classification of shrinking gradient Ricci solitons. We first review some background information about shrinking gradient Ricci solitons.

The main theorem in [45], joint with Lei Ni and Kui Wang, asserts that

Theorem 1.4.1. *Any four-dimensional complete gradient shrinking Ricci soliton with positive isotropic curvature is either a quotient of S^4 or a quotient of $S^3 \times \mathbb{R}$.*

This gives a clean classification result removing the earlier additional assumptions in [57] by L. Ni and N. Wallach. An immediate corollary is that any four-dimensional gradient shrinking soliton with positive curvature operator must be isometric to S^4. This generalizes a result of Perelman on three-dimensional gradient shrinking solitons to dimension four. This result has important consequences in studying Ricci flow on four-dimensional manifolds.

Chapter 2

Estimates of Modulus of Continuity for Viscosity Solutions

In this paper, we investigate the moduli of continuity for viscosity solutions of a wide class of nonsingular quasilinear evolution equations and also for the level set mean curvature flow, which is an example of singular degenerate equations. We prove that the modulus of continuity is a viscosity subsolution of some one dimensional equation. This work extends B. Andrews' recent result on moduli of continuity for smooth spatially periodic solutions.

2.1 Introduction

Given a function $u : \mathbb{R}^n \to \mathbb{R}$, any function $f : [0, \infty) \to \mathbb{R}_+$ satisfying

$$|u(y) - u(x)| \leq 2f\left(\frac{|y-x|}{2}\right)$$

for all x and y is called a modulus of continuity of u. The (optimal) modulus of continuity ω of u is defined by

$$\omega(s) = \sup\left\{\frac{u(y) - u(x)}{2}\middle|\frac{|y-x|}{2} = s\right\}.$$

The estimate of modulus of continuity has been studied by B. Andrews and J. Clutterbuck in several papers [4] [5]. B. Andrews and J. Clutterbuck [6], B. Andrews and L. Ni [8] and L. Ni [55] have also studied the modulus of continuity for heat equations on manifolds.

More precisely, B. Andrews and J. Clutterbuck considered the following quasilinear evolution equation

$$u_t = a^{ij}(Du, t)D_iD_ju + b(Du, t) \tag{2.1}$$

where $A(p, t) = (a^{ij}(p, t))$ is positive semi-definite. Under the assumption that there exists a continuous function $\alpha : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ with

$$0 < \alpha(R, t) \leq R^2 \inf_{|p|=R, (v\cdot p)\neq 0} \frac{v^T A(p, t)v}{(v \cdot p)^2}, \tag{2.2}$$

They showed [5, Theorem 3.1] that the modulus of continuity of a regular periodic solution to (2.1) is a viscosity subsolution of the one dimensional equation

$$\phi_t = \alpha(|\phi'|, t)\phi''. \tag{2.3}$$

Note that their result is applicable to any anisotropic mean curvature flow and can be used to obtain gradient estimate and thus existence and uniqueness of (2.1).

The first result of this paper is that the same holds for viscosity solutions of (2.1) when (2.2) holds and $a^{ij}, b : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ are continuous functions.

Theorem 2.1.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a continuous periodic viscosity solution of* (2.1)*. Then the modulus of continuity $\omega(s,t) = \sup\left\{ \frac{u(y,t)-u(x,t)}{2} \,\middle|\, \frac{|y-x|}{2} = s \right\}$ of u is a viscosity subsolution of the one dimensional equation* (2.3)*.*

We also study the modulus of continuity for singular evolution equations. As summarized in a recent survey [3] by B. Andrews, for the isotropic flows of the form

$$u_t = \left[a(|Du|) \frac{D_i u D_j u}{|Du|^2} + b(|Du|) \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2} \right) \right] D_i D_j u, \tag{2.4}$$

the modulus of continuity of a spatially periodic smooth solution of (2.4) is a viscosity subsolution of the corresponding one-dimensional heat equation $\omega_t = a(\omega')\omega''$. Note that equation (2.4) covers the classical heat equation, the graphical mean curvature flow and the p-Laplace heat equation with suitable choices of a and b. When (2.4) is nonsingular, it is covered by (2.1). When it is singular, it has to be treated differently since there are various definitions for viscosity solutions of singular equations. We will focus on the particular case $a = 0$ and $b = 1$, which corresponds to the level set mean curvature flow:

$$u_t = \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2} \right) D_i D_j u. \tag{2.5}$$

Equation (2.5) was studied by L. Evans and J. Spruck in [31]. They gave a definition of viscosity solution and proved that for an initial data g that is continuous and constant on $\mathbb{R}^n \cap \{|x| \geq S\}$, there exists a unique viscosity solution u that is continuous and constant on $\mathbb{R}^n \cap \{|x| \geq R\}$, with R depending only on S. We will recall their definition in Section 2.

We prove the following theorem:

Theorem 2.1.2. *Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a viscosity solution of (2.5) with continuous initial data g that is a constant on $\mathbb{R}^n \cap \{|x| \geq S\}$. Then the modulus of continuity $\omega(s,t) = \sup \left\{ \frac{u(y,t)-u(x,t)}{2} \middle| \frac{|y-x|}{2} = s \right\}$ of u is a viscosity subsolution of $\omega_t = \max\{0, \frac{1}{4}(\omega'' + |\omega''|)\}$ on $(0, \infty) \times (0, \infty)$.*

As an immediate consequence, we have that any concave modulus of continuity for the initial data is preserved by the level set mean curvature flow.

Corollary 2.1.3. *Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a viscosity solution of (2.5) with continuous initial data g that is a constant on $\mathbb{R}^n \cap \{|x| \geq S\}$. Assume ϕ is nonnegative, concave and satisfies $|g(y) - g(x)| \leq 2\phi\left(\frac{|y-x|}{2}\right)$ for all x, y, then*

$$|u(y,t) - u(x,t)| \leq 2\phi\left(\frac{|y-x|}{2}\right)$$

for all x, y and $t \geq 0$.

Proof of Corollary 2.1.3. Since the function $\phi_\epsilon = \phi + \epsilon e^t$ satisfies

$$\partial_t \phi_\epsilon > 0 = \max\{0, \frac{1}{4}(\phi'' + |\phi''|)\},$$

so it cannot touch ω from above by Theorem 2.1.2. $\square$

2.2 Definitions of Viscosity Solutions

We give definition of a viscosity solution for the general equation

$$u_t + F(x, t, u, \nabla u, \nabla^2 u) = 0 \tag{2.6}$$

assuming $F : \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \to \mathbb{R}$ is continuous and degenerate elliptic. Let O be an open subset of $\Omega \times (0, T)$. We write $z = (x, t)$ and $z_0 = (x_0, t_0)$.

The following notations are useful:

$$\mathrm{USC}(O) = \{u : O \to \mathbb{R} |\ u \text{ is upper semicontinuous }\},$$

$$\mathrm{LSC}(O) = \{u : O \to \mathbb{R} |\ u \text{ is lower semicontinuous }\},$$

Definition 2.2.1. *(i) A function $u \in USC(O)$ is a viscosity subsolution of* (2.6) *in O if for any $\phi \in C^\infty(O)$ such that $u - \phi$ has a local maximum at $z_0 \in O$, then*

$$\phi_t(z_0) + F(z_0, u(z_0), \nabla\phi(z_0), \nabla^2\phi(z_0)) \le 0.$$

(ii) A function $u \in LSC(O)$ is a viscosity supersolution of (2.6) *in O if for any $\phi \in C^\infty(O)$ such that $u - \phi$ has a local minimum at $z_0 \in O$, then*

$$\phi_t(z_0) + F(z_0, u(z_0), \nabla\phi(z_0), \nabla^2\phi(z_0)) \ge 0.$$

(iii) A viscosity solution of (2.6) *in O is defined to be a continuous function that is both a viscosity subsolution and a viscosity supersolution of* (2.6) *in O.*

We have an equivalent definition in terms of parabolic semijets. Assume $u \in \mathrm{USC}(O)$ and $z_0 \in O$. The parabolic superjet of u at z_0, denoted by $\mathcal{P}^{2,+}u(z_0)$, is defined by

$$\mathcal{P}^{2,+}u(z_0) = \{(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} | u(z) \le u(z_0) + \tau(t - t_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T X (x - x_0) + o(|x - x_0|^2 + |t - t_0|) \text{ as } z \to z_0\}.$$

The parabolic subjet of $u \in \mathrm{LSC}(O)$ at z_0, denoted by $\mathcal{P}^{2,-}u(z_0)$, is defined by

$$\mathcal{P}^{2,-}u(z_0) = -\mathcal{P}^{2,+}(-u)(z_0).$$

Definition 2.2.2. *(i) A function $u \in USC(O)$ is a viscosity subsolution of* (2.6) *in O if for all $(x, t) \in O$ and $(\tau, p, X) \in \mathcal{P}^{2,+}u(x, t)$,*

$$\tau + F(z, u(z), p, X) \le 0.$$

(ii) A function $u \in LSC(O)$ is a viscosity supersolution of (2.6) *in O if for all $(x, t) \in O$ and $(\tau, p, X) \in \mathcal{P}^{2,-}u(x, t)$,*

$$\tau + F(z, u(z), p, X) \ge 0.$$

Remark 2.2.3. *In the above definitions, since F is continuous, we can replace $\mathcal{P}^{2,+}u(z_0)$ and $\mathcal{P}^{2,-}u(z_0)$ by $\overline{\mathcal{P}}^{2,+}u(z_0)$ and $\overline{\mathcal{P}}^{2,-}u(z_0)$ respectively, where the closures are defined by*

$$\begin{aligned} \overline{\mathcal{P}}^{2,+}u(z_0) = \{&(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} | \textit{ there is a sequence } (z_j, \tau_j, p_j, X_j) \\ &\textit{such that } (\tau_j, p_j, X_j) \in \mathcal{P}^{2,+}u(z_j) \\ &\textit{and } (z_j, u(z_j), \tau_j, p_j, X_j) \to (z_0, u(z_0), \tau, p, X) \textit{ as } j \to \infty\}. \\ \overline{\mathcal{P}}^{2,-}u(z_0) = -&\overline{\mathcal{P}}^{2,+}(-u)(z_0). \end{aligned}$$

For singular equations, there are different ways to define viscosity solutions. For the level set mean curvature flow (2.5), we use the definition given by L. Evans and J. Spruck in [31], where viscosity solutions are called weak solutions.

Definition 2.2.4. *A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of* (2.5) *if for any $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then we have*

$$\begin{cases} \phi_t \le \left(\delta_{ij} - \frac{D_i\phi D_j\phi}{|D\phi|^2}\right) D_iD_j\phi \ \ \textit{at}\ (x_0, t_0) \\ \textit{if}\ D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$\begin{cases} \phi_t \le (\delta_{ij} - \eta_i\eta_j)\, D_iD_j\phi \ \ \textit{at}\ (x_0, t_0) \\ \textit{for some}\ \eta \in \mathbb{R}^n\ \textit{with}\ |\eta| \le 1,\ \ \textit{if}\ D\phi(x_0, t_0) = 0 \end{cases}$$

Definition 2.2.5. *A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of* (2.5) *if for any $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $u - \phi$ has a local minimum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then we have*

$$\begin{cases} \phi_t \ge \left(\delta_{ij} - \frac{D_i\phi D_j\phi}{|D\phi|^2}\right) D_iD_j\phi \ \ \textit{at}\ (x_0, t_0) \\ \textit{if}\ D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$\begin{cases} \phi_t \ge (\delta_{ij} - \eta_i\eta_j)\, D_iD_j\phi \ \ \textit{at}\ (x_0, t_0) \\ \textit{for some}\ \eta \in \mathbb{R}^n\ \textit{with}\ |\eta| \le 1,\ \ \textit{if}\ D\phi(x_0, t_0) = 0 \end{cases}$$

Definition 2.2.6. *A continuous and bounded function* $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ *is a viscosity solution of* (2.5) *provided* u *is both a viscosity subsolution and a viscosity supersolution.*

We also have alternative definitions in terms of parabolic semijets.

Definition 2.2.7. *A continuous and bounded function* $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ *is a viscosity subsolution of* (2.5) *if for all* $(x, t) \in \mathbb{R}^n \times (0, \infty)$ *and* $(\tau, p, X) \in \mathcal{P}^{2,+}u(x, t)$,

$$\tau \leq \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) X_{ij} \text{ if } p \neq 0$$

and

$$\tau \leq (\delta_{ij} - \eta_i \eta_j) X_{ij} \text{ for some } |\eta| \leq 1, \text{ if } p = 0.$$

Definition 2.2.8. *A continuous and bounded function* $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ *is a viscosity supersolution of* (2.5) *if for all* $(x, t) \in \mathbb{R}^n \times (0, \infty)$ *and* $(\tau, p, X) \in \mathcal{P}^{2,-}u(x, t)$,

$$\tau \geq \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) X_{ij} \text{ if } p \neq 0$$

and

$$\tau \geq (\delta_{ij} - \eta_i \eta_j) X_{ij} \text{ for some } |\eta| \leq 1, \text{ if } p = 0.$$

Remark 2.2.9. *One can replace* $\mathcal{P}^{2,+}u(z_0)$ *and* $\mathcal{P}^{2,-}u(z_0)$ *by* $\overline{\mathcal{P}}^{2,+}u(z_0)$ *and* $\overline{\mathcal{P}}^{2,-}u(z_0)$ *respectively in the above definitions for the reason of continuity.*

2.3 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. We must show that if ϕ is a smooth function such that $\omega - \phi$ has a local maximum at (s_0, t_0) for $s_0 > 0$ and $t_0 > 0$, then at (s_0, t_0)

$$\phi_t \leq \alpha(|\phi'|, t)\phi''.$$

Since u is continuous and periodic, there exist points x_0 and y_0 with $|y_0 - x_0| = 2s_0$ attaining the supremum,

$$\omega(s_0, t_0) = \frac{u(y_0, t_0) - u(x_0, t_0)}{2}.$$

Define

$$Z(x, y, t) := u(y, t) - u(x, t) - 2\phi\left(\frac{|y - x|}{2}, t\right).$$

In view of the definition of ω, we obtain that

$$Z(x, y, t) \leq Z(x_0, y_0, t_0)$$

for all $|y - x|$ close to $2s_0$ and t close to t_0. Thus Z has a local maximum at (x_0, y_0, t_0). Since Z is continuous, by the parabolic version maximum principle for semicontinuous functions [29, Theorem 8.3], for any $\lambda > 0$, there exist $X, Y \in S^{n\times n}$ such that

$$(b_1, 2D_y\phi(s_0, t_0), X) \in \overline{\mathcal{P}}^{2,+}u(y_0, t_0),$$

$$(-b_2, -2D_x\phi(s_0, t_0), Y) \in \overline{\mathcal{P}}^{2,-}u(x_0, t_0),$$

$$b_1 + b_2 = 2\phi_t(s_0, t_0),$$

$$-\left(\lambda^{-1}+\|M\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M+\lambda M^2, \tag{2.7}$$

where

$$M = 2D^2\phi = 2\begin{pmatrix} D_y^2\phi & D_{y,x}^2\phi \\ D_{x,y}^2\phi & D_x^2\phi \end{pmatrix} = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $B = 2D_y^2\phi(s_0,t_0)$.

To simplify, we choose an orthonormal basis of $\mathbb{R}^n$ with $e_n = \frac{y-x}{|y-x|}$, then

$$2D_y\phi(s_0,t_0) = -2D_x\phi(s_0,t_0) = \phi'(s_0,t_0)e_n.$$

$$B = \begin{pmatrix} \frac{\phi'}{2s_0} & & & \\ & \ddots & & \\ & & \frac{\phi'}{2s_0} & \\ & & & \frac{1}{2}\phi'' \end{pmatrix}.$$

Since u is both a subsolution and a supersolution of (2.1), we have

$$b_1 \leq tr(A(\phi' e_n)X) + b(\phi' e_n)$$

$$-b_2 \geq tr(A(\phi' e_n)Y) + b(\phi' e_n)$$

By choosing a symmetric matrix C such that $\begin{pmatrix} A & C \\ C & A \end{pmatrix} \geq 0$, we obtain using (3.17)

$$2\phi_t(s_0,t_0) = b_1 + b_2 \leq tr\left(A(\phi' e_n)(X-Y)\right) = tr \begin{pmatrix} A & C \\ C & A \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}$$

$$\leq tr \begin{pmatrix} A & C \\ C & A \end{pmatrix} \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \lambda tr \begin{pmatrix} A & C \\ C^T & A \end{pmatrix} \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}^2$$

$$= 2tr\left((A-C)B\right) + 4\lambda tr\left((A-C)B^2\right)$$

Taking $C = A - 2\alpha(|\phi'|)e_n \otimes e_n$, it's easy to verify $\begin{pmatrix} A & C \\ C & A \end{pmatrix} \geq 0$ due to (2.2). Thus at (s_0, t_0)

$$\phi_t \leq \alpha(|\phi'|)\phi'' + \lambda\alpha(|\phi'|)(\phi'')^2 \tag{2.8}$$

Since $\lambda > 0$ is arbitrary, we get

$$\phi_t \leq \alpha(|\phi'|)\phi''$$

at (s_0, t_0). $\square$

2.4 Proof of Theorem 2.1.2

Proof of Theorem 2.1.2. Suppose that ϕ is a smooth function such that $\omega - \phi$ has a strict local maximum at (s_0, t_0) with $s_0 > 0$ and $t_0 > 0$. As in the proof of

Theorem 2.1.1, we arrive at that the function

$$Z(x,y,t) := u(y,t) - u(x,t) - 2\phi\left(\frac{|y-x|}{2}, t\right).$$

has a local maximum at (x_0, y_0, t_0). Again we use an orthonormal frame with $e_n = \frac{y-x}{|y-x|}$. The maximum principle for semicontinuous functions [29, Theorem 8.3] implies that for any $\lambda > 0$, there exist $X, Y \in S^{n\times n}$ such that

$$(b_1, \phi'(s_0,t_0)e_n, X) \in \overline{\mathcal{P}}^{2,+}u(y_0,t_0)$$

$$(-b_2, \phi'(s_0,t_0)e_n, Y) \in \overline{\mathcal{P}}^{2,-}u(x_0,t_0)$$

$$b_1 + b_2 = 2\phi_t(s_0,t_0)$$

$$-\left(\lambda^{-1} + \|M\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2, \tag{2.9}$$

where

$$M = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with

$$B = 2D_y^2\phi(s_0,t_0) = \begin{pmatrix} \frac{\phi'}{2s_0} & & & \\ & \ddots & & \\ & & \frac{\phi'}{2s_0} & \\ & & & \frac{1}{2}\phi'' \end{pmatrix}.$$

For any vector $p \in \mathbb{R}^n$, we have

$$p^T X p - p^T Y p = (p,p)^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} (p,p) \leq (p,p)^T(M + \lambda M^2)(p,p) = 0$$

Therefore $X \leq Y$. For simiplicity, we denote $A(p) = I - \frac{p \otimes p}{|p|^2}$. If $\phi'(s_0, t_0) \neq 0$, then by the definition of viscosity solution of (2.5),

$$b_1 \leq tr(A(\phi' e_n)X),$$

$$-b_2 \geq tr(A(\phi' e_n)Y).$$

Using the fact $X \leq Y$, we get

$$\phi_t = \frac{1}{2}(b_1 + b_2) \leq \frac{1}{2} tr\left(A(\phi' e_n)(X - Y)\right) \leq 0.$$

If $\phi'(s_0, t_0) = 0$, then it follows from the definition of a viscosity solutions that for some ξ, η with $|\xi|, |\eta| \leq 1$,

$$b_1 \leq tr(A(\xi)X),$$

$$-b_2 \geq tr(A(\eta)Y).$$

In view of (2.9), we have $X \leq B + 2\lambda B^2$ and $-Y \leq B + 2\lambda B^2$. Thus

$$tr((A(\xi)X)) \leq tr\left((A(\xi)(B + 2\lambda B^2)\right) = \frac{1}{2}(1 - \xi_n^2)\left((\phi'' + \lambda(\phi'')^2\right),$$

$$tr((A(\eta)Y)) \geq tr\left((A(\eta)(-B - 2\lambda B^2)\right) = -\frac{1}{2}(1 - \eta_n^2)\left((\phi'' + \lambda(\phi'')^2\right).$$

Therefore,

$$2\phi_t = b_1 + b_2 \leq tr(A(\xi)X) - tr(A(\eta)Y) \leq \left(1 - \frac{\xi_n^2 + \eta_n^2}{2}\right)\left(\phi'' + \lambda(\phi'')^2\right).$$

Letting $\lambda \to 0$ yields

$$\phi_t \leq \frac{1}{4}\left(\phi'' + |\phi''|\right).$$

$\square$

The results of Chapter 2 are my own work, published in Proceedings of the American Mathematical Society in 2016.

Chapter 3

Estimates of Modulus of Continuity for Viscosity Solutions on Manifolds

3.1 Preliminaries

We establish the estimates of modulus of continuity for viscosity solutions of nonlinear evolution equations on manifolds, extending previous work of B. Andrews and J. Clutterbuck for regular solutions on manifolds [6] and the first author's recent work for viscosity solutions in Euclidean spaces [44].

3.2 Introduction

In this paper, we study the moduli of continuity for viscosity solutions to nonlinear evolution equations on manifolds. Let (M, g) be a compact Riemannian manifold. Recall that given a continuous function $u : M \to \mathbb{R}$, the *optimal modulus of continuity* w of u can be defined by

$$w(s) = \sup\left\{\frac{u(y)-u(x)}{2} : \quad d(x,y) = 2s\right\},$$

where d is the induced distance function on (M, g).

We will mainly consider the following isotropic flow:

$$u_t = \left[\alpha(|Du|,t)\frac{D_iuD_ju}{|Du|^2} + \beta(|Du|,t)\left(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\right)\right] D_iD_ju + b(|Du|,t). \quad (3.1)$$

We make the assumptions that equation (4.3) is nonsingular, i.e., the right hand side of (4.3) is a continuous function on $\mathbb{R}_+ \times \mathbb{R}^n \times S^{n\times n}$, where $S^{n\times n}$ is the set of $n \times n$ symmetric matrices, and that α, β are nonnegative functions.

For domains in Euclidean spaces, B. Andrews and J. Clutterbuck [5] proved that the modulus of continuity for a regular solution of (4.3) is a viscosity subsolution of the one-dimensional equation $\phi_t = \alpha(\phi', t)\phi''$. Recently this was shown by the first author [44] to be true for viscosity solutions as well. On manifolds the estimates of modulus of continuity for regular solutions have been investigated by B. Andrews and J. Clutterbuck [6], B. Andrews and L. Ni[8] and L. Ni[55]. More precisely, if the Ricci curvature of the manifold has a lower bound: $\mathrm{Ric}_g \geq (n-1)\kappa g$, then the modulus of continuity of a regular solution of (4.3) satisfies

$$w_t \leq \alpha(w', t)w'' + (n-1)\frac{c_\kappa'(s)}{c_\kappa(s)}\beta(w', t)w' \tag{3.2}$$

in the viscosity sense, where $c_\kappa(s)$ is defined by $c_\kappa'' + \kappa c_\kappa = 0, c_\kappa(0) = 1, c_\kappa'(0) = 0$. The main goal of this paper is to show that various modulus of continuity estimates remain valid for viscosity solutions on manifolds as well. We would like to mention some important examples of such equations:

(i) If we take $\alpha = 1, \beta = 1$ and $b = 0$, then equation (4.3) reduces to the heat equation:

$$u_t = \Delta u;$$

(ii) If we take $\alpha = \frac{1}{1+|Du|^2}, \beta = 1$ and $b = 0$, then equation (4.3) reduces to the graphical mean curvature flow:

$$u_t = \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}\right) D_i D_j u;$$

(iii) If we take $\alpha = (p-1)|Du|^{p-2}, \beta = |Du|^{p-2}$ and $b = 0$, then equation (4.3) reduces to the p-Laplacian equation with $p > 2$:

$$\operatorname{div}\left(|Du|^{p-2} Du\right) = 0.$$

The paper is organized as follows: In Section 2, we recall the definitions of viscosity solutions on manifolds and state the parabolic maximum principle for semicontinuous functions on manifolds, which is the main technical tool we use in this paper. The main proof is given in Section 3. In Section 4, we prove height dependent gradient bounds, which is useful to derive gradient estimates for nonlinear equations. A generalization of Section 3 to Bakry-Emery manifolds is done in Section 5. In Section 6, we treat Neumann and Dirichlet boundary value problems and establish the estimates of modulus of continuity.

3.3 Preliminaries

3.3.1 Definition of Viscosity Solutions on manifolds

Let M be a Riemannian manifold. The following notations are useful:

$$\mathrm{USC}(M \times (0,T)) = \{u : M \times (0,T) \to \mathbb{R} |\ u \text{ is upper semicontinuous } \},$$

$$\mathrm{LSC}(M \times (0,T)) = \{u : M \times (0,T) \to \mathbb{R} |\ u \text{ is lower semicontinuous } \}.$$

We first introduce the notion of parabolic semijets on manifolds. We write $z = (x,t)$ and $z_0 = (x_0, t_0)$.

Definition 3.3.1. *For a function $u \in USC(M \times (0,T))$, we define the parabolic second order superjet of u at a point $z_0 \in M \times (0,T)$ by*

$$\mathcal{P}^{2,+}u(z_0) := \{(\varphi_t(z_0), D\varphi(z_0), D^2\varphi(z_0)) : \varphi \in C^{2,1}(M \times (0,T)),$$
$$\textit{such that } u - \varphi \textit{ attains a local maximum at } z_0\}.$$

For $u \in LSC(M \times (0,T))$, the parabolic second order subjet of u at $z_0 \in M \times (0,T)$ is defined by

$$\mathcal{P}^{2,-}u(z_0) := -\mathcal{P}^{2,+}(-u)(z_0).$$

We also define the closures of $\mathcal{P}^{2,+}u(z_0)$ and $\mathcal{P}^{2,-}u(z_0)$ by

$$\begin{aligned}\overline{\mathcal{P}}^{2,+}u(z_0) = \{(\tau, p, X) \in \mathbb{R} \times T_{x_0}M \times Sym^2(T^*_{x_0}M) | \text{ there is a sequence}\\ (z_j, \tau_j, p_j, X_j) \text{ such that } (\tau_j, p_j, X_j) \in \mathcal{P}^{2,+}u(z_j)\\ \text{and } (z_j, u(z_j), \tau_j, p_j, X_j) \to (z_0, u(z_0), \tau, p, X) \text{ as } j \to \infty\};\\ \overline{\mathcal{P}}^{2,-}u(z_0) = -\overline{\mathcal{P}}^{2,+}(-u)(z_0).\end{aligned}$$

Now we give the definition of a viscosity solution for the general equation

$$u_t + F(x, t, u, Du, D^2u) = 0 \tag{3.3}$$

on M. Assume $F \in C(M \times [0,T] \times \mathbb{R} \times T_{x_0}M \times Sym^2(T^*_{x_0}M))$ is proper, i.e.

$$F(x, t, r, p, X) \leq F(x, t, s, p, Y) \text{ whenever } r \leq s, Y \leq X.$$

Definition 3.3.2. *(i) A function $u \in USC(M \times (0,T))$ is a viscosity subsolution of* (3.3) *if for all $z \in M \times (0,T)$ and $(\tau, p, X) \in \mathcal{P}^{2,+}u(z)$,*

$$\tau + F(z, u(z), p, X) \leq 0.$$

(ii) A function $u \in LSC(M \times (0,T))$ is a viscosity supersolution of (3.3) *if for all $z \in M \times (0,T)$ and $(\tau, p, X) \in \mathcal{P}^{2,-}u(z)$,*

$$\tau + F(z, u(z), p, X) \geq 0.$$

(iii) A viscosity solution of (3.3) *is defined to be a continuous function that is both a viscosity subsolution and a viscosity supersolution of* (3.3).

3.3.2 Parabolic Maximum Principle for Semicontinuous Functions on Manifolds

The main technical tool we use is the parabolic version maximum principle for semicontinous functions on manifolds, which is a restatement of [29, Theorem 8.3], for Riemannian manifolds. One can also find it in [40, Section 2.2] or [10, Theorem 3.8].

Theorem 3.3.3. *Let $M_1^{N_1}, \cdots, M_k^{N_k}$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Let $u_i \in USC((0,T) \times \Omega_i)$, and φ defined on $(0,T) \times \Omega_1 \times \cdots \times \Omega_k$ such that φ is continuously differentiable in t and twice continuously differentiable in $(x_1, \cdots x_k) \in \Omega_1 \times \cdots \times \Omega_k$. Suppose that $\hat{t} \in (0,T), \hat{x}_i \in \Omega_i$ for $i = 1, \cdots, k$ and the function*

$$\omega(t, x_1, \cdots, x_k) := u_1(t, x_1) + \cdots + u_k(t, x_k) - \varphi(t, x_1, \cdots, x_k)$$

attains a maximum at $(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k)$ on $(0,T) \times \Omega_1 \times \cdots \times \Omega_k$. Assume further that there is an $r > 0$ such that for every $\eta > 0$ there is a $C > 0$ such that for $i = 1, \cdots, k$

$$b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \overline{\mathcal{P}}^{2,+} u_i(t, x_i),$$

$$d(x_i, \hat{x}_i) + |t - \hat{t}| \leq r \text{ and } |u_i(t, x_i)| + |q_i| + \|X_i\| \leq \eta.$$

Then for each $\lambda > 0$, there are $X_i \in Sym^2(T^*_{\hat{x}_i} M_i)$ such that

$$(b_i, D_{x_i}\varphi(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k), X_i) \in \overline{\mathcal{P}}^{2,+} u_i(\hat{t}, \hat{x}_i),$$

$$-\left(\frac{1}{\lambda} + \|M\|\right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq M + \lambda M^2,$$

$$b_1 + \cdots + b_k = \varphi_t(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k),$$

where $M = D^2\varphi(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k)$.

3.4 Modulus of continuity estimates on manifolds

For any given constant κ, let

$$c_\kappa(t) = \begin{cases} \cos\sqrt{\kappa}t, & \kappa > 0, \\ 1, & \kappa = 0, \\ \cosh\sqrt{|\kappa|}t, & \kappa < 0. \end{cases}$$

Note that $c_\kappa(s)$ satisfies $c_\kappa'' + \kappa c_\kappa = 0, c_\kappa(0) = 1, c_\kappa'(0) = 0$. The following theorem is a generalization to viscosity solutions of Theorem 1 in [6].

Theorem 3.4.1. *Let $u : M \times [0,T) \to \mathbb{R}$ be a viscosity solution of (4.3) on a closed manifold M and denote by D the diameter of M. Assume further that*

$Ric_g \geq (n-1)\kappa g$. Then the modulus of continuity $w : [0, \frac{D}{2}] \times [0, T) \to \mathbb{R}$ of u satisfies

$$w_t \leq \alpha(w', t)w'' + (n-1)\frac{c_\kappa'(s)}{c_\kappa(s)}\beta(w', t)w' \tag{3.4}$$

in the viscosity sense, provided ω is increasing in s.

Proof. From the definition of viscosity solution, it suffices to show the following

> *For any given (s_0, t_0), a small neighborhood U of s_0, $\epsilon_0 > 0$, and any smooth function ϕ lying above w for $U \times (t_0 - \epsilon_0, t_0 + \epsilon_0)$ with equality at (s_0, t_0), then*
>
> $$\phi_t \leq \alpha(\phi', t_0)\phi'' + (n-1)\frac{c_\kappa'}{c_\kappa}\beta(\phi', t_0)\phi' \tag{3.5}$$
>
> *holds at (s_0, t_0).*

Let ϕ be a smooth function lying above w for $U \times (t_0 - \epsilon_0, t_0 + \epsilon_0)$ with equality at (s_0, t_0). The assumption that ω is increasing in s implies $\phi'(s_0, t_0) \geq 0$. Since M is compact, there exist x_0 and y_0 in M with $d(x_0, y_0) = 2s_0$ such that

$$u(y_0, t_0) - u(x_0, t_0) = 2w(s_0, t_0) = 2\phi(s_0, t_0).$$

Then it follows that

$$u(y, t) - u(x, t) - 2\phi(\frac{d(x, y)}{2}, t)$$

attains a local maximum at (x_0, y_0, t_0). Note that the distance function d may not be smooth at (x_0, y_0), so one cannot apply the maximum principle for semicontinuous functions on manifolds directly. To overcome this, we replace d by a smooth function ρ, which is defined as follows. Let U_{x_0} and U_{y_0} be small neighborhoods of x_0 and y_0 respectively. Let $\gamma_0 : [0,1] \to M$ be a minimizing geodesic joining x_0 and y_0 with $|\gamma_0'| = 2s_0$. We choose Fermi coordinates $\{e_i(s)\}$ $(i = 1, 2, \cdots, n)$ along γ_0 with $e_n(s) = \gamma_0'(s)$ for $s \in [0,1]$. For $1 \leq i \leq n-1$, we define $V_i(s)$ along $\gamma_0(s)$ by

$$V_i(s) = \frac{c_\kappa\left((2s-1)s_0\right)}{c_\kappa(s_0)} e_i(s),$$

and set $V_n(s) = e_n(s)$. We then define a smooth function $\rho(x,y)$ in $U_{x_0} \times U_{y_0}$ to be the length of the curve $\exp_{\gamma_0(s)}\left(\sum_{i=1}^n \left((1-s)a_i(x) + sb_i(y)\right) V_i(s)\right)$ $(s \in [0,1])$, where $a_i(x)$ and $b_i(x)$ are so defined that

$$x = \exp_{x_0}\left(\sum_{i=1}^n a_i(x) e_i(0)\right), \qquad y = \exp_{y_0}\left(\sum_{i=1}^n b_i(y) e_i(1)\right).$$

From the definition of $\rho(x,y)$, we see $d(x,y) \leq \rho(x,y)$ and with equality at (x_0, y_0). We write $\psi(x,y,t) = 2\phi(\frac{\rho(x,y)}{2}, t)$. Then the function

$$Z(y,x,t) := u(y,t) - u(x,t) - \psi(x,y,t)$$

has a local maximum at (x_0, y_0, t_0). Now we can apply the parabolic version maximum principle for semicontinuous functions on manifolds to conclude that for each $\lambda > 0$, there exist symmetric tensors X, Y such that

$$(b_1, D_y\psi(x_0,y_0,t_0), X) \in \overline{\mathcal{P}}^{2,+} u(y_0,t_0),$$

$$(-b_2, -D_x\psi(x_0,y_0,t_0), Y) \in \overline{\mathcal{P}}^{2,-} u(x_0,t_0),$$

$$b_1 + b_2 = \psi_t(x_0,y_0,t_0) = 2\phi_t(s_0,t_0),$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2, \tag{3.6}$$

where $M = D^2\psi(x_0, y_0, t_0)$.

The first derivative of ψ yields

$$D_y\psi(x_0, y_0, t_0) = \phi'(s_0, t_0)\frac{\gamma_0'(1)}{2s_0}, \tag{3.7}$$

and

$$D_x\psi(x_0, y_0, t_0) = -\phi'(s_0, t_0)\frac{\gamma_0'(0)}{2s_0}. \tag{3.8}$$

Since u is both a subsolution and a supersolution of (4.3), we have

$$b_1 \leq \operatorname{tr}(AX) + b(|\phi'|, t_0),$$

and

$$-b_2 \geq \operatorname{tr}(AY) + b(|\phi'|, t_0),$$

where

$$A = \begin{pmatrix} \beta(|\phi'|, t_0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta(|\phi'|, t_0) & 0 \\ 0 & \cdots & 0 & \alpha(|\phi'|, t_0) \end{pmatrix}.$$

Set

$$C = \begin{pmatrix} \beta(|\phi'|, t_0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta(|\phi'|, t_0) & 0 \\ 0 & \cdots & 0 & -\alpha(|\phi'|, t_0) \end{pmatrix},$$

and simple calculation shows $\begin{pmatrix} A & C \\ C & A \end{pmatrix} \geq 0$. Then we obtain that

$$\begin{aligned} 2\phi_t(s_0, t_0) &= b_1 + b_2 \leq \operatorname{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\right] \\ &\leq \operatorname{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M\right] + \lambda \operatorname{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M^2\right]. \end{aligned}$$

We easily get

$$\begin{aligned} \operatorname{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M\right] &= \alpha(|\phi'|, t_0) D^2\psi\left((e_n(1), -e_n(0)), (e_n(1), -e_n(0))\right) \\ &\quad + \beta(|\phi'|, t_0) \sum_{i=1}^{n-1} D^2\psi\left((e_i(1), e_i(0)), (e_i(1), e_i(0))\right) \end{aligned} \tag{3.9}$$

It remains to estimate the terms involving second derivatives of ψ. The estimate is analogous to [6, Theorem 3]. For $1 \leq i \leq n-1$, we choose the variation vector fields $V_i(s) = \frac{c_\kappa((2s-1)s_0)}{c_\kappa(s_0)} e_i(s)$ along $\gamma_0(s)$, then the first variation formulas gives

$$\left.\frac{d}{dv}\right|_{v=0} |\gamma_v| = \frac{1}{2s_0} g(\gamma', V_i)|_0^1 = 0,$$

and the second variation formula gives

$$\left.\frac{d^2}{dv^2}\right|_{v=0} |\gamma_v| = \frac{1}{2s_0} g(\gamma', \nabla_{V_i} V_i)\ |_0^1 + \frac{1}{2s_0}\int_0^1 |(\nabla_{\gamma'} V_i)^\perp|^2 - \langle R(\gamma', V_i)\gamma', V_i\rangle\, ds. \tag{3.10}$$

By the way of variation, we can also require $\nabla_{V_i} V_i = 0$ for $s \in [0,1]$. Therefore direct calculation gives

$$\frac{1}{2s_0}\int_0^1 |(\nabla_{\gamma'} V_i)^{\perp}|^2 ds = 2s_0 \int_0^1 \left(\frac{c'_\kappa\left((2s-1)s_0\right)}{c_\kappa(s_0)}\right)^2 ds = \int_{-s_0}^{s_0} \left(\frac{c'_\kappa(x)}{c_\kappa(s_0)}\right)^2 dx.$$

Using the integration by parts, the definition of c_κ and equation $c''_\kappa + \kappa c_\kappa = 0$, we have

$$\int_{-s_0}^{s_0} \left(\frac{c'_\kappa(x)}{c_\kappa(s_0)}\right)^2 dx = 2\frac{c'_\kappa(s_0)}{c_\kappa(s_0)} + \int_{-s_0}^{s_0} \kappa \left(\frac{c_\kappa(x)}{c_\kappa(s_0)}\right)^2 dx.$$

Combining with (3.10), we see

$$\left.\frac{d^2}{dv^2}\right|_{v=0} |\gamma_v| = 2\frac{c'_\kappa(s_0)}{c_\kappa(s_0)} + \int_{-s_0}^{s_0} \left(\frac{c_\kappa(x)}{c_\kappa(s_0)}\right)^2 (\kappa - \langle R(e_n,e_i)e_n, e_i\rangle) dx.$$

Then we conclude that

$$\begin{aligned}
&\frac{1}{2}D^2\psi\left((e_i(1),e_i(0)),(e_i(1),e_i(0))\right)\\
= &\left.\frac{d^2}{dv^2}\right|_{v=0} \phi\left(\frac{1}{2}\rho\left(\exp_{x_0} ve_i(0), \exp_{y_0} ve_i(1)\right), t_0\right)\\
= &\left.\frac{d^2}{dv^2}\right|_{v=0} \phi\left(\frac{1}{2}L\left[\left(\exp_{\gamma_0(s)}\left((1-s)v + sv\right)V_i(s)\right)\right], t_0\right)\\
= &\left.\frac{d^2}{dv^2}\right|_{v=0} \phi\left(\frac{1}{2}L\left[\left(\exp_{\gamma_0(s)} vV_i(s)\right)\right], t_0\right)\\
= &\left.\frac{d^2}{dv^2}\right|_{v=0} \phi(\frac{|\gamma_v|}{2}, t_0)\\
= &\ \phi'(s_0,t_0)\left(\frac{c'_\kappa(s_0)}{c_\kappa(s_0)} + \frac{1}{2}\int_{-s_0}^{s_0}\left(\frac{c_\kappa(x)}{c_\kappa(s_0)}\right)^2 (\kappa - \langle R(e_n,e_i)e_n,e_i\rangle)dx\right), \quad (3.11)
\end{aligned}$$

from which, the following holds

$$
\begin{aligned}
& \sum_{i=1}^{n-1} \frac{1}{2} D^2\psi\left((e_i(1), e_i(0)), (e_i(1), e_0(0))\right) \\
= \; & (n-1)\phi'(s_0, t_0)\left(\frac{c_\kappa'(s_0)}{c_\kappa(s_0)} + \frac{1}{2}\int_{-s_0}^{s_0}\left(\frac{c_\kappa(x)}{c_\kappa(s_0)}\right)^2 \left((n-1)\kappa - \mathrm{Ric}(e_n, e_n)\right) dx\right) \\
\leq \; & (n-1)\frac{c_\kappa'(s_0)}{c_\kappa(s_0)}\phi'(s_0, t_0),
\end{aligned} \tag{3.12}
$$

where we used the curvature assumption.

It follows easily from the variation along e_n that

$$
\frac{1}{2} D^2\psi\left((e_n(1), -e_n(0)), (e_n(1), -e_n(0))\right) = \phi''(s_0, t_0). \tag{3.13}
$$

Thus we conclude from (3.9), (3.12) and (3.13) that

$$
\mathrm{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M\right] \leq 2\beta(\phi', t_0)(n-1)\frac{c_\kappa'(s_0)}{c_\kappa(s_0)}\phi'(s_0, t_0) + 2\alpha(\phi', t_0)\phi''(s_0, t_0). \tag{3.14}
$$

Since $\lambda > 0$ is arbitrary, (4.11) comes true from (3.9) and (3.14). Hence we complete the proof. $\square$

As an immediate corollary, we have the following Ricci flow version, which generalizes Theorem 4 in [3].

Theorem 3.4.2 (Ricci flow version)**.** *Let M^n be a closed Riemannian manifold, and $g(t)$ a family of time-dependent metrics on M satisfying $\frac{\partial g}{\partial t} \geq -2Ric$, and let $u : M \times [0, T) \to \mathbb{R}$ be a viscosity solution of (4.3). Then the modulus of continuity*

$w : [0, \frac{D}{2}] \times [0, T) \to \mathbb{R}$ *of* u *satisfies*

$$w_t \leq \alpha(w', t)w''$$

in the viscosity sense, provided ω *is increasing in* s.

Proof. As before, we consider a smooth function ϕ which lies above the modulus of continuity w and is equal at (s_0, t_0). Then via maximum principle as before, it holds that

$$2\phi_t(s_0, t_0) - \int_{-s_0}^{s_0} \mathrm{Ric}(e_n(s), e_n(s))\, ds \leq b_1 + b_2 \leq \mathrm{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M\right].$$

On the other hand, choosing the variation fields $V_i(s) = e_i(s)$ yields

$$\mathrm{tr}\left[\begin{pmatrix} A & C \\ C & A \end{pmatrix} M\right] \leq 2\alpha(\phi', t_0)\phi''(s_0, t_0) - \int_{-s_0}^{s_0} \mathrm{Ric}(e_n(s), e_n(s))\, ds,$$

completing the proof. Here we used inequality in (3.12) with $c_\kappa = 1$ and (3.13). $\square$

3.5 Height-dependent gradient bounds

In this section, we obtain height-dependent gradient bounds for viscosity solutions, generalizing Theorem 6 in [3].

Theorem 3.5.1. *Let* (M^n, g) *be a closed Riemannian manifold with diameter* D *and Ricci curvature satisfying* $Ric_g \geq 0$, *and suppose* $u : M \times [0, T) \to \mathbb{R}$ *is a viscosity solution of an equation of the form*

$$\frac{\partial u}{\partial t} = \left[\alpha(|Du|, u, t)\frac{D_i u D_j u}{|Du|^2} + \beta(t)\left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right)\right] D_i D_j u. \tag{3.15}$$

Let $\varphi : [0, D] \times [0, T) \to \mathbb{R}$ *be a solution of*

$$\varphi_t = \alpha(\varphi', \varphi, t)\varphi'',$$

with Neumann boundary condition, which is increasing in the first variable, such that the range of $u(\cdot, 0)$ *is contained in* $[\varphi(0, 0), \varphi(D, 0)]$. *Let* $\Psi(s, t)$ *be given by inverting* φ *for each* t, *and assume that for all* x *and* y *in* M,

$$\Psi(u(y, 0), 0) - \Psi(u(x, 0), 0) - d(x, y) \leq 0.$$

Then

$$\Psi(u(y, t), t) - \Psi(u(x, t), t) - d(x, y) \leq 0.$$

for all $x, y \in M$ *and* $t \in [0, T)$.

We begin with a lemma about the behavior of parabolic semijets when composed with an increasing function.

Lemma 3.5.1. *Let* u *be a continuous function. Let* $\varphi : \mathbb{R} \times [0, T) \to \mathbb{R}$ *be a* $C^{2,1}$ *function with* $\varphi' \geq 0$. *Let* $\Psi : \mathbb{R} \times [0, T) \to \mathbb{R}$ *be such that*

$$\Psi(\varphi(u(y, t), t), t) = u(y, t)$$

$$\varphi(\Psi(u(y, t), t), t) = u(y, t)$$

(i) Suppose $(\tau, p, X) \in \mathcal{P}^{2,+}\Psi(u(y_0, t_0), t_0)$, *then*

$$(\varphi_t + \varphi'\tau, \varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{P}^{2,+}u(y_0, t_0),$$

where all derivatives of φ *are evaluated at* $(\Psi(u(y_0,t_0)),t_0)$.

(ii) Suppose $(\tau,p,X) \in \mathcal{P}^{2,-}\Psi(u(y_0,t_0),t_0)$, *then*

$$(\varphi_t + \varphi'\tau, \varphi' p, \varphi'' p\otimes p + \varphi' X) \in \mathcal{P}^{2,-}u(y_0,t_0),$$

where all derivatives of φ *are evaluated at* $(\Psi(u(y_0,t_0)),t_0)$.

(iii) The same holds if one replaces the parabolic semijets by the their closures.

Proof. The lemma is an easy consequence of the following characterization of the semijets.

$$\mathcal{P}^{2,+}u(y_0,t_0) = \{\big(\varphi_t(y_0,t_0), D\varphi(y_0,t_0), D^2\varphi(y_0,t_0)\big) \mid \varphi \in C^{2,1} \text{ and } u-\varphi \text{ has a local maximum at } (y_0,t_0)\}$$

$$\mathcal{P}^{2,-}u(y_0,t_0) = \{\big(\varphi_t(y_0,t_0), D\varphi(y_0,t_0), D^2\varphi(y_0,t_0)\big) \mid \varphi \in C^{2,1} \text{ and } u-\varphi \text{ has a local mimimum at } (y_0,t_0)\}.$$

For (i), Suppose $(\tau,p,X) \in \mathcal{P}^{2,+}\Psi(u(y_0,t_0),t_0)$. Let h be a $C^{2,1}$ function such that $\Psi(u(y,t),t) - h(y,t)$ has a local maximum at (y_0,t_0) and $(h_t, Dh, D^2h)(y_0,t_0) = (\tau,p,X)$. Since φ is increasing, we have $u(y,t)-\varphi(h(y,t),t) = \varphi(\Psi(u(y,t),t),t) - \varphi(h(y,t),t)$ has a local maximum at (y_0,t_0). Then it follows that

$$(\varphi_t + \varphi'\tau, \varphi' p, \varphi'' p\otimes p + \varphi' X) \in \mathcal{P}^{2,+}u(y_0,t_0).$$

For (ii), Suppose $(\tau,p,X) \in \mathcal{P}^{2,-}\Psi(u(y_0,t_0),t_0)$. Let h be a $C^{2,1}$ function such that $\Psi(u(y,t),t)-h(y,t)$ has a local minimum at (y_0,t_0) and $(h_t, Dh, D^2h)(y_0,t_0) =$

(τ, p, X). Since φ is increasing, we have $u(y,t) - \varphi(h(y,t),t) = \varphi(\Psi(u(y,t),t),t) - \varphi(h(y,t),t)$ has a local minimum at (y_0, t_0). Then it follows that

$$(\varphi_t + \varphi'\tau, \varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{P}^{2,-}u(y_0, t_0).$$

(iii) then follows from approximation. □

Proof of Theorem 3.5.1. The theorem is valid if we show that for any $\epsilon > 0$,

$$Z^\epsilon(x,y,t) := \Psi(u(y,t),t) - \Psi(u(x,t),t) - d(x,y) - \frac{\epsilon}{T-t} \leq 0. \tag{3.16}$$

To prove inequality (3.16), it suffices to show Z^ϵ can not attain the maximum in $M \times M \times (0,T)$. Assume by contradiction that there exist $t_0 \in (0,T)$, x_0 and y_0 in M at which the function Z^ϵ attains its maximum. Take ρ defined as before. Then the function

$$\Psi(u(y,t),t) - \Psi(u(x,t),t) - \rho(x,y) - \frac{\epsilon}{T-t}$$

has a local maximum at (x_0, y_0, t_0). If $\epsilon > 0$, then we necessarily have $x_0 \neq y_0$. By the parabolic maximum principle for semicontinuous functions on manifolds, for any $\lambda > 0$, there exist X, Y satisfying

$$(b_1, D_y\rho(x_0,y_0), X) \in \overline{\mathcal{P}}^{2,+}\Psi(u(y_0,t_0),t_0),$$

$$(-b_2, -D_x\rho(x_0,y_0), Y) \in \overline{\mathcal{P}}^{2,-}\Psi(u(x_0,t_0),t_0),$$

$$b_1 + b_2 = \frac{\epsilon}{(T-t_0)^2},$$

$$-\left(\lambda^{-1}+\|M\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M+\lambda M^2, \tag{3.17}$$

where $M = D^2\rho(x_0, y_0)$.

By Lemma 3.5.1, we have

$$(b_1\varphi'(z_{y_0}, t_0) + \varphi_t(z_{y_0}, t_0), \varphi'(z_{y_0}, t_0)D_y\rho(s_0, t_0),$$

$$\varphi'(z_{y_0}, t_0)X + \varphi''(z_{y_0}, t_0)e_n(1) \otimes e_n(1)) \in \overline{\mathcal{P}}^{2,+}u(y_0, t_0),$$

and

$$(-b_2\varphi'(z_{x_0}, t_0) + \varphi_t(z_{x_0}, t_0), -\varphi'(z_{x_0}, t_0)D_x\rho(s_0, t_0),$$

$$\varphi'(z_{x_0}, t_0)Y + \varphi''(z_{x_0}, t_0)e_n(0) \otimes e_n(0)) \in \overline{\mathcal{P}}^{2,-}u(x_0, t_0),$$

where $z_{x_0} = \Psi(u(x_0, t_0), t_0)$, $z_{y_0} = \Psi(u(y_0, t_0), t_0)$ and $e_n(s)$ is as defined in Section 2.

Since u is both a subsolution and a supersolution of (3.15), we have

$$b_1\varphi'(z_{y_0}, t_0) + \varphi_t(z_{y_0}, t_0) \leq \operatorname{tr}\left(\varphi'(z_{y_0}, t_0)A_1X + \varphi''(z_{y_0}, t_0)A_1e_n(1) \otimes e_n(1)\right),$$

and

$$-b_2\varphi'(z_{x_0}, t_0) + \varphi_t(z_{x_0}, t_0) \geq \operatorname{tr}(\varphi'(z_{x_0}, t_0)A_2Y + \varphi''(z_{x_0}, t_0)A_2e_n(0) \otimes e_n(0)),$$

where

$$A_1 = \begin{pmatrix} \beta(t_0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta(t_0) & 0 \\ 0 & \cdots & 0 & \alpha(|\varphi'(z_{y_0}, t_0)|, \varphi(z_{y_0}, t_0), t_0) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \beta(t_0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta(t_0) & 0 \\ 0 & \cdots & 0 & \alpha(|\varphi'(z_{x_0}, t_0)|, \varphi(z_{x_0}, t_0), t_0) \end{pmatrix}.$$

Set

$$C = \begin{pmatrix} \beta(t_0) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta(t_0) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

and simple calculation shows $\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} \geq 0$. Then we obtain that

$$\begin{aligned}
\frac{\epsilon}{(T-t_0)^2} &= b_1 + b_2 \leq \frac{\mathrm{tr}(\varphi'(z_{y_0},t_0)A_1X) - \varphi_t(z_{y_0},t_0)}{\varphi'(z_{y_0},t_0)} \\
&\quad + \frac{\mathrm{tr}(A_1 e_n(1)\otimes e_n(1))\varphi''(z_{y_0},t_0)}{\varphi'(z_{y_0},t_0)} \\
&\quad + \frac{\mathrm{tr}(-\varphi'(z_{x_0},t_0)A_2Y) + \varphi_t(z_{x_0},t_0)}{\varphi'(z_{x_0},t_0)} \\
&\quad - \frac{\mathrm{tr}(A_2 e_n(0)\otimes e_n(0))\varphi''(z_{x_0},t_0)}{\varphi'(z_{x_0},t_0)} \\
&\quad + \lambda\,\mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} M^2\right] \\
&\leq \mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\right] + \lambda\,\mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} M^2\right] \\
&\quad + \frac{\varphi_t(z_{x_0},t_0) - \alpha(\varphi'(z_{x_0},t_0),\varphi(z_{x_0},t_0),t_0)\varphi''(z_{x_0},t_0)}{\varphi'(z_{x_0},t_0)} \\
&\quad - \frac{\varphi_t(z_{y_0},t_0) - \alpha(\varphi'(z_{y_0},t_0),\varphi(z_{y_0},t_0),t_0)\varphi''(z_{y_0},t_0)}{\varphi'(z_{y_0},t_0)} \\
&= \mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} M\right] + \lambda\,\mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} M^2\right].
\end{aligned}$$

Using (3.12) with $\kappa = 0$, we obtain

$$\mathrm{tr}\left[\begin{pmatrix} A_1 & C \\ C & A_2 \end{pmatrix} M\right] \leq 0.$$

We have arrived at $\epsilon \leq 0$ by letting $\lambda \to 0$, which is a contradiction. Therefore (3.16) is true, hence completing the proof. $\square$

3.6 Estimates on Bakry-Emery Manifolds

We prove a generalization to viscosity solutions of Theorem 1.2 in [8].

Theorem 3.6.1. *Let M be a closed Riemannian manifold satisfying*

$$Ric_{ij} + f_{ij} \geq a g_{ij}$$

for some $a \in \mathbb{R}$. Denote $D = diam(M)$. Let u be a viscosity solution of

$$u_t = \Delta_f u$$

with the operator $\Delta_f := \Delta - \langle \nabla(\cdot), \nabla f \rangle$. Then the modulus of continuity $\omega : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ of u is a viscosity subsolution of

$$\omega_t = \omega'' - as\omega',$$

provided ω is increasing in s.

Proof. The idea is the same as before. Let ϕ be a smooth function lying above w for $U \times (t_0 - \epsilon_0, t_0 + \epsilon_0)$ with equality at (s_0, t_0). Then it follows that the function

$$Z(y, x, t) := u(y, t) - u(x, t) - \psi(x, y, t)$$

has a local maximum at (x_0, y_0, t_0), where $\psi(x, y, t) = \phi\left(\frac{\rho(x,y)}{2}, t\right)$ as before. Now we can apply the parabolic version maximum principle for semicontinuous functions on manifolds to conclude that for each $\lambda > 0$, there exist symmetric tensors X, Y such that

$$(b_1, D_y\psi(x_0, y_0, t_0), X) \in \overline{\mathcal{P}}^{2,+} u(y_0, t_0),$$

$$(-b_2, -D_x\psi(x_0, y_0, t_0), Y) \in \overline{\mathcal{P}}^{2,-}u(x_0, t_0),$$

$$b_1 + b_2 = \psi_t(x_0, y_0, t_0) = 2\phi_t(s_0, t_0),$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2, \tag{3.18}$$

where $M = D^2\psi(x_0, y_0, t_0)$.

The first derivative of ψ yields

$$D_y\psi(x_0, y_0, t_0) = \phi'(s_0, t_0)\frac{\gamma_0'(1)}{2s_0} = \phi'(s_0, t_0)e_n(1), \tag{3.19}$$

and

$$D_x\psi(x_0, y_0, t_0) = -\phi'(s_0, t_0)\frac{\gamma_0'(0)}{2s_0} = \phi'(s_0, t_0)e_n(0). \tag{3.20}$$

Since u is a viscosity solution, we have

$$b_1 \leq \mathrm{tr}(X) - \phi'\langle \nabla f(y_0), e_n(1)\rangle,$$

$$-b_2 \geq \mathrm{tr}(Y) - \phi'\langle \nabla f(x_0), e_n(0)\rangle.$$

Therefore

$$\begin{aligned} 2\phi_t(s_0, t_0) = b_1 + b_2 &\leq \mathrm{tr}(X) - \mathrm{tr}(Y) - \phi'\langle \nabla f(y_0), e_n(1)\rangle + \phi'\langle \nabla f(x_0), e_n(0)\rangle \\ &\leq \mathrm{tr}(M) - \phi'\langle \nabla f(y_0), e_n(1)\rangle + \phi'\langle \nabla f(x_0), e_n(0)\rangle. \end{aligned}$$

We estimate that

$$
\begin{aligned}
\operatorname{tr}(M) = &\sum_{i=1}^{n-1} D^2\phi\left((e_i(1), e_i(0)), (e_i(1), e_i(0))\right) \\
&+ D^2\phi\left((e_n(1), -e_n(0)), (e_n(1), -e_n(0))\right) \\
&\leq \phi'' - \phi' \int_{-s_0}^{s_0} \operatorname{Ric}(e_n(s), e_n(s)) ds \\
&\leq \phi'' + \phi' \int_{-s_0}^{s_0} \nabla\nabla f(e_n, e_n) - ag\langle e_n, e_n\rangle ds \\
&= \phi'' + 2as_0\phi' + \phi'\langle \nabla f(y_0), e_n(1)\rangle + \phi'\langle \nabla f(x_0), e_n(0)\rangle,
\end{aligned}
$$

where we have used (3.11) with $\kappa = 0$. Hence at (s_0, t_0),

$$\phi_t \leq \phi'' - as_0\phi'$$

holds, proving the theorem. $\square$

Next we give the evolutionary analogue of the above theorem, which is a generalization of Theorem 5 in [3].

Theorem 3.6.2. *Let M^n be a closed manifold with time-dependent metrics and smooth function $f(\cdot, t)$. Suppose that*

$$g_t \geq -2\left(Ric_{ij} + f_{ij}\right) + 2ag_{ij}.$$

Let u be a viscosity solution of the drift-Laplacian heat flow

$$u_t = \Delta_f u.$$

Then the modulus of continuity $\omega : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ of u is a viscosity subsolution of

$$\omega_t = \omega'' - as\omega'$$

provided ω is increasing in s.

Proof. The proof is immediate by combining

$$\phi_t - \int_{-s_0}^{s_0} \big(\mathrm{Ric}(e_n, e_n) + \nabla^2 f(e_n, e_n) - ag\langle e_n, e_n\rangle\big)\ ds \leq b_1 + b_2,$$

$$b_1 + b_2 \leq \mathrm{tr}(M) - \phi'\langle \nabla f(y_0), e_n(1)\rangle + \phi'\langle \nabla f(x_0), e_n(0)\rangle,$$

and

$$\mathrm{tr}(M) \leq \phi'' - \phi' \int_{-s_0}^{s_0} \mathrm{Ric}(e_n(s), e_n(s))\ ds.$$

□

Remark 3.6.3. *The same proof works for the more general equation: If u is a viscosity solutions of*

$$u_t = \left[\alpha(|Du|, t)\frac{D_i u D_j u}{|Du|^2} + \beta(|Du|, t)\left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right)\right] D_i D_j u$$
$$+ b(|Du|, t) + \langle \nabla f, \nabla u\rangle.$$

Then the modulus of continuity ω of u satisfies

$$\omega_t \leq \alpha(\omega')\omega'' - as\omega'$$

in the viscosity sense, provided ω is increasing in s.

3.7 Boundary Value Problems

3.7.1 Definition of Viscosity Solution for Boundary Value Problems

We recall the definition of viscosity solutions to boundary value problems from [29, Section 7]. Let Ω be an open subset of $\mathbb{R}^n$ and $T > 0$. For brevity we write $z = (x, t)$. Consider the boundary problem of the form

$$u_t + F(z, u, Du, D^2u) = 0 \text{ in } \Omega \times (0, T), \tag{3.21}$$

$$B(z, u, Du, D^2u) = 0 \text{ on } \partial\Omega \times (0, T).$$

Assume $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n})$ and $B \in C(\partial\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n})$ are both proper.

Definition 3.7.1. *A function $u \in USC(\overline{\Omega} \times (0, T))$ is a viscosity subsolution of* (3.21) *if*

$$\tau + F(z, u(z), p, X) \leq 0 \text{ for } z \in \Omega \times (0, T), (\tau, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} u(z),$$

and

$$\min\{\tau + F(z, u(z), p, X), B(z, u(z), p, X)\} \leq 0$$

$$\text{for } z \in \partial\Omega \times (0, T), (\tau, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} u(z).$$

Similarly, $u \in LSC(\overline{\Omega} \times (0,T))$ is a viscosity supersolution of (3.21) *if*

$$\tau + F(z, u(z), p, X) \geq 0 \text{ for } z \in \Omega \times (0,T), (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}u(z),$$

and

$$\max\{\tau + F(z, u(z), p, X), B(z, u(z), p, X)\} \geq 0$$
$$\text{for } z \in \partial\Omega \times (0,T), (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}u(z).$$

Finally, u is a viscosity solution of (3.21) *if it is both a viscosity subsolution and a viscosity supersolution of* (3.21).

3.7.2 Neumann problem

We consider the following quasilinear evolution equations:

$$\frac{\partial u}{\partial t} = a^{ij}(Du, t)D_iD_ju + b(Du, t) \text{ in } \Omega \times (0,T),$$
$$\langle Du(x,t), n(x)\rangle = 0 \text{ on } \partial\Omega \times (0,T),$$

where $A(p,t) = (a^{ij}(p,t))$ is positive semi-definite and $n(x)$ is the exterior unit normal vector at x. As in [5], we assume that there exists a continuous function $\alpha : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ with

$$0 < \alpha(R,t) \leq R^2 \inf_{|p|=R,(v\cdot p)\neq 0} \frac{v^T A(p,t)v}{(v\cdot p)^2}, \tag{3.22}$$

The following theorem is a generalization of Theorem 4.1 in [5] to viscosity solutions.

Theorem 3.7.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded and convex domain. Let u be a viscosity solution of the Neumann problem. Then the modulus of continuity ω is a viscosity subsolution of the one dimensional equation $\omega_t = \alpha(|\omega'|, t)\omega''$, provided that ω is increasing in s.*

Proof. We must show that if ϕ is a smooth function such that $\omega - \phi$ has a local maximum at (s_0, t_0) for $s_0 > 0$ and $t_0 > 0$, then at (s_0, t_0)

$$\phi_t \leq \alpha(|\phi'|, t)\phi''.$$

As before, we consider the function

$$Z(y, x, t) = u(y, t) - u(x, t) - 2\phi(\frac{|y - x|}{2}, t)$$

and arrive at that there exists (x_0, y_0, t_0) with $|x_0 - y_0| = 2s_0$ such that Z attains a local maximum at (x_0, y_0, t_0). Now replacing ϕ by

$$\varphi(s, t) = \phi(s, t) + (s - s_0)^4 + (t - t_0)^4$$

if necessary, we may assume Z has a strict local maximum at (x_0, y_0, t_0).

If $(x_0, y_0) \in \Omega \times \Omega$, then the same argument as in [44] would prove the theorem. For the case $(x_0, y_0) \in \partial(\Omega \times \Omega)$, the strategy is to produce approximations u^ϵ, u_ϵ such that $u^\epsilon \to u$, $u_\epsilon \to u$ uniformly as $\epsilon \to 0$ and u^ϵ, u_ϵ are a supersolution and a subsolution of some modified equation, for which we have the same inequalities no matter the maximum point lies in $\Omega \times \Omega$ or on $\partial(\Omega \times \Omega)$.

Fix a point $z_0 \in \Omega$. Let $\delta = d(z_0, \partial\Omega) > 0$. Define $v(x) = \frac{1}{2}(x - z_0)^2$. Then $Dv(x) = x - z_0$ and $D^2 v(x) = I$. Moreover for any $x \in \partial\Omega$,

$$\langle Dv(x), n(x)\rangle = \langle x - z_0, n(x)\rangle \geq d(z_0, \partial\Omega) = \delta.$$

Define $u_\epsilon(x) = u(x) - \epsilon v(x)$ and $u^\epsilon(x) = u(x) + \epsilon v(x)$. Then we have the following: u_ϵ is a viscosity subsolution of

$$\frac{\partial u}{\partial t} = a^{ij}(Du + \epsilon Dv, t) D_i D_j u + b(Du + \epsilon Dv, t) + \epsilon \operatorname{tr}(a^{ij}(Du + \epsilon Dv, t)) \text{ in } \Omega, \tag{3.23}$$

$$\langle Du(x,t), n(x)\rangle + \epsilon \langle Dv(x), n(x)\rangle = 0 \text{ on } \partial\Omega, \tag{3.24}$$

and u^ϵ is a viscosity supersolution of

$$\frac{\partial u}{\partial t} = a^{ij}(Du - \epsilon Dv, t) D_i D_j u + b(Du - \epsilon Dv, t) - \epsilon \operatorname{tr}(a^{ij}(Du - \epsilon Dv, t)) \text{ in } \Omega, \tag{3.25}$$

$$\langle Du(x,t), n(x)\rangle - \epsilon \langle Dv(x), n(x)\rangle = 0 \text{ on } \partial\Omega. \tag{3.26}$$

We complete the proof by considering the following approximation of the function Z:

$$Z_\epsilon(x, y, t) = u_\epsilon(y, t) - u^\epsilon(x, t) - 2\phi\left(\frac{|y - x|}{2}, t\right).$$

Then Z_ϵ has a local max at $(x_\epsilon, y_\epsilon, t_\epsilon)$ with $(x_\epsilon, y_\epsilon, t_\epsilon) \to (x_0, y_0, t_0)$ and $s_\epsilon = |y_\epsilon - x_\epsilon|/2 \to s_0$. As usual, choose $e_n = \frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|}$ and the maximum principle for semicontinuous functions [29, Theorem 8.3] gives $b_{1,\epsilon}, b_{2,\epsilon} \in \mathbb{R}$ and $X_\epsilon, Y_\epsilon \in S^{n\times n}$ for any $\lambda > 0$,

$$(b_{1,\epsilon}, \phi'(s_\epsilon, t_\epsilon)e_n, X_\epsilon) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} u_\epsilon(y_\epsilon, t_\epsilon),$$

$$(-b_{2,\epsilon}, \phi'(s_\epsilon, t_\epsilon)e_n, Y_\epsilon) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)} u^\epsilon(x_\epsilon, t_\epsilon),$$

$$b_{1,\epsilon} + b_{2,\epsilon} = 2\phi_t(s_\epsilon, t_\epsilon),$$

$$-\left(\lambda^{-1} + \|M\|\right) I \leq \begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix} \leq M + \lambda M^2,$$

where $M = 2D^2\phi(s_\epsilon, t_\epsilon)$. By the definition of viscosity solution for boundary problem, we have if $y_\epsilon \in \Omega$, then at $(x_\epsilon, y_\epsilon, t_\epsilon)$,

$$b_{1,\epsilon} \leq tr(A(\phi' e_n + \epsilon Dv)X_\epsilon) - b(\phi' e_n + \epsilon Dv) + \epsilon\,\mathrm{tr}(A(Du + \epsilon Dv)), \tag{3.27}$$

and if $y_\epsilon \in \partial\Omega$, then at $(x_\epsilon, y_\epsilon, t_\epsilon)$

$$\begin{aligned} \min \quad &\{b_{1,\epsilon} - tr(A(\phi' e_n + \epsilon Dv)X_\epsilon) - b(\phi' e_n + \epsilon Dv) - \epsilon\,\mathrm{tr}(A(Du + \epsilon Dv)), \\ &\phi'\langle e_n, n(y_\epsilon)\rangle + \epsilon\langle Dv(y_\epsilon), n(y_\epsilon)\rangle\} \leq 0, \end{aligned}$$

However, since Ω is convex and $\phi' \geq 0$, $\phi'\langle e_n, n(y_\epsilon)\rangle + \epsilon\langle Dv(y_\epsilon), n(y_\epsilon)\rangle \geq \epsilon\delta$. Thus equation (3.27) is valid no matter y_ϵ lies in Ω or on $\partial\Omega$. Similarly, if $x_\epsilon \in \Omega$, then at $(x_\epsilon, y_\epsilon, t_\epsilon)$

$$-b_{2,\epsilon} \geq tr(A(\phi' e_n - \epsilon Dv)Y_\epsilon) - b(\phi' e_n - \epsilon Dv) - \epsilon\,\mathrm{tr}(A(\phi' e_n - \epsilon Dv)), \tag{3.28}$$

and if $x_\epsilon \in \partial\Omega$, then at $(x_\epsilon, y_\epsilon, t_\epsilon)$

$$\max \quad \{-b_{2,\epsilon} - tr(A(\phi' e_n - \epsilon Dv)Y_\epsilon) - b(\phi' e_n) + \epsilon\, \mathrm{tr}(A(\phi' e_n - \epsilon Dv)),$$
$$\phi'\langle e_n, n(x_\epsilon)\rangle - \epsilon\langle Dv(x_\epsilon), n(x_\epsilon)\rangle\} \geq 0.$$

Observe that $\phi'\langle e_n, n(x_\epsilon)\rangle - \epsilon\langle Dv(x_\epsilon), n(x_\epsilon)\rangle \leq -\epsilon\delta < 0$, because Ω is convex and $\phi' \geq 0$. Therefore, equation (3.28) is valid no matter x_ϵ lies in Ω or on $\partial\Omega$. By passing to subsequences if necessary, we have $b_{1,\epsilon} \to b_1$, $b_{2,\epsilon} \to b_2$, $X_\epsilon \to X$ and $Y_\epsilon \to Y$ as $\epsilon \to 0$. The limits satisfies

$$b_1 + b_2 = 2\phi_t(s_0, t_0),$$
$$-\left(\lambda^{-1} + \|M\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2,$$

Letting $\epsilon \to 0$ in (3.27) and (3.28), we obtain at (y_0, t_0) (x_0, t_0)

$$b_1 \leq tr(A(\phi' e_n)X) - b(\phi' e_n),$$

$$-b_2 \geq tr(A(\phi' e_n)Y) - b(\phi' e_n),$$

The rest of the proof is the same as the proof of Theorem 1.1 in [44]. □

Remark 3.7.3. *The same modulus of continuity estimate holds on manifolds for Neumann boundary problem. The argument is the same as in Section 3.*

3.7.3 Dirichlet Problem

We consider the following quasilinear evolution equations:

$$\frac{\partial u}{\partial t} = a^{ij}(Du, t) D_i D_j u \text{ in } \Omega \times (0, T),$$

$$u(x, t) = 0 \text{ on } \partial\Omega \times (0, T).$$

Where $A(p, t) = (a^{ij}(p, t))$ is positive semi-definite and $n(x)$ is the exterior unit normal vector at x. As in [5], we assume that there exists a continuous function $\alpha : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ with

$$0 < \alpha(R, t) \leq R^2 \inf_{|p|=R, (v \cdot p) \neq 0} \frac{v^T A(p, t) v}{(v \cdot p)^2}.$$

For Dirichlet problem, we cannot formulate the same theorem as for Neumann problem. B. Andrews' proof for regular solutions assumes concavity of the modulus of continuity to rule out the case that maximum can occur on the boundary. Instead, we prove the following theorem, which is a generalization of Theorem 4.2 in [5] to viscosity solutions.

Theorem 3.7.4. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded and convex domain. Let u be a continuous viscosity solution of the Dirichlet problem. Let φ_0 be a modulus of continuity of $u(\cdot, 0)$. Suppose φ is increasing and concave in the first variable and satisfies*

$$\varphi_t \geq \alpha(|\varphi'|, t)\varphi'',$$

and $\varphi(z, t) \geq \varphi_0(z)$ for all $z \geq 0$. Then $\varphi(s, t)$ is a modulus of continuity for $u(s, t)$ for all $t > 0$.

Proof. The proof is exactly as the proof of Theorem 4.2 in [5] except one replaces the usual comparison principle by the comparison principle for viscosity solutions . □

The results of Chapter 3 are joint work with Dr. Kui Wang, published in Journal of Geometric Analysis in 2017.

Chapter 4

Nonparametric Hypersurfaces Moving by Powers of Gauss Curvature

4.1 Introduction

In this chapter, we study asymptotic behavior of nonparametric hypersurfaces moving by α powers of Gauss curvature with $\alpha > 1/n$. Our work generalizes the results of V. Oliker [59] for $\alpha = 1$.

Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. We consider a solution of the following initial boundary problem

$$
\begin{aligned}
&u_t = \frac{[\det(u_{ij})]^\alpha}{(1+|\nabla u|^2)^{\alpha\beta}} \text{ in } \Omega \times (0,\infty),\\
&u(x,t) = 0 \text{ in } \partial\Omega \times (0,\infty), \qquad (4.1)\\
&u(x,t) \text{ is strictly convex for each } t \geq 0,
\end{aligned}
$$

where $\alpha > 1/n$ and $\beta \geq 0$ are constants and

$$u_t := \frac{\partial u}{\partial t},\ u_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j},\ \nabla u := \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

Equation (4.1) describes the graphs $(x, u(x,t)), (x,t) \in \overline{\Omega} \times [0,\infty)$ evolving in $\mathbb{R}^{n+1}$ with relative boundaries $(x, u(x,t))|_{\partial\Omega}$ remain fixed. When $\beta = \frac{n+2-\frac{1}{\alpha}}{2}$, the normal speed of the point $(x, u(x,t))$ is equal to α powers of the Guass curvature of the graph. Such parabolic Monge-Ampère equations have been studied by many authors in recent years. See, for instance, [38][30]. On the other hand, in the parametric setting, flow by Gauss curvature or its powers have received considerable interests, see [64][24][25][1] [2] [33] [7] and the references therein.

V. Oliker considered (4.1) with $\alpha = 1$ in [59]. He analyzed the asymptotic behavior of smooth convex solutions of (4.1). It turned out that solutions with different β all have the same asymptotic behavior. Moreover, if Ω is centrally symmetric or rotationally symmetric, then the solution $u(x,t)$ asymptotically becomes centrally symmetric or rotational symmetric, regardless of its initial shape.

The goal of this work is to generalize V. Oliker's results in [59] to any power $\alpha > 1/n$. We investigate the asymptotic behavior of a smooth convex solution of (4.1) and show that, by comparing with self-similar solutions of (4.1) with $\beta = 0$, the solution $u(x,t)$ asymptotically converges to the solution of the following nonlinear elliptic problem:

$$[\det(\psi_{ij})]^{\alpha} = \frac{1}{1-n\alpha}\psi \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega,$$

$$\psi \text{ is stictly convex and } \psi < 0 \text{ in } \Omega.$$

Furthermore, our estimate implies geometric properties of the flow by α powers of the Gauss curvature. For instance, the asymptotic behavior of $u(x,t)$ reflects the symmetries of Ω. More precisely, if Ω is centrally or rotationally symmetric, then the solution $u(x,t)$ asymptotically becomes centrally or rotational symmetric, regardless of its initial shape, and we also give sharp estimates on the rate of this process.

Throughout out the paper, we denote by M the Monge-Ampère operator $M(u) := \det(u_{ij})$ and $M^{\alpha}(u) := [\det(u_{ij})]^{\alpha}$.

4.2 Main Results

Consider the following initial boundary problem:

$$\begin{aligned} &u_t = M^{\alpha}(u) \text{ in } \Omega \times (0,\infty), \\ &u(x,t) = 0 \text{ in } \partial\Omega \times (0,\infty), \\ &u(x,t) \text{ is strictly convex for each } t \geq 0. \end{aligned} \tag{4.2}$$

We seek for self-similar solutions of (4.2) of the form

$$u(x,t) = \varphi(t)\psi(x), \tag{4.3}$$

where $\varphi(t) \in C^\infty([0,\infty))$ and $\psi(x) = C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$. By convexity of $u(x,0) = \varphi(0)\psi(x)$, we have either $\varphi(0) < 0$ and $\psi(x) > 0$ in Ω and concave or $\varphi(0) > 0$ and $\psi(x) < 0$ in Ω and convex. Since both cases are equivalent for our purpose, we always deal with the latter one. Substituting (4.3) into (4.2) yields

$$\frac{\varphi(t)}{\varphi^{n\alpha}} = \frac{M^\alpha(\psi)}{\psi} = \lambda = \text{constant.}$$

Noting that $\psi(x) < 0$ and convex in Ω, we get $\lambda \le 0$ and

$$\varphi(t) = \left(\varphi(0)^{1-n\alpha} - (n\alpha - 1)\lambda t\right)^{\frac{1}{1-n\alpha}}, \tag{4.4}$$

$$M(\psi) = (\lambda\psi)^{\frac{1}{\alpha}} \text{ in } \Omega \text{ and } \psi = 0 \text{ on } \partial\Omega. \tag{4.5}$$

An easy argument shows that $\lambda = 0$ implies $u(x,t) \equiv 0$. Thus we only consider the case $\lambda < 0$. By scaling, it suffices to consider one negative value of λ and thus we fix $\lambda = \frac{1}{1-n\alpha} < 0$ for convenience. The following result establishes the existence of self-similar solutions to (4.2).

Theorem 4.2.1. *Let Ω be a bounded strictly convex domain with smooth boundary $\partial\Omega$. Then problem* (4.2) *admits a self-similar solution in $\overline{\Omega} \times (0,\infty)$ given by*

$$u(x,t) = (1+t)^{\frac{1}{1-n\alpha}}\psi(x), \tag{4.6}$$

where ψ *is the unique solution in* $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ *of the equation*

$$M(\psi) = \left(\frac{-\psi}{|1 - n\alpha|} \right)^{\frac{1}{\alpha}} \text{ in } \Omega, \ \psi = 0 \text{ on } \partial\Omega, \tag{4.7}$$

$$\psi \text{ is stictly convex and } \psi < 0 \text{ in } \Omega,$$

and $\sup_\Omega |\psi(x)|$ *admits an estimate depending only on* n, α *and the domain* Ω. *Furthermore, if* $\tilde{u}(x,t) = \varphi(t)\tilde{\psi}(x)$ *is an arbitrary self-similar solution of* (4.2), *then there exists a unique* $c > 0$ *such that* $\tilde{\psi}(x) = c\psi(x)$ *and*

$$\tilde{u}(x,t) = u(x,t) \left\{ \frac{1+t}{[c\varphi(0)]^{1-n\alpha} + t} \right\}^{\frac{1}{n\alpha - 1}}. \tag{4.8}$$

The main theorem concerning the asymptotic behavior of the solution is the following:

Theorem 4.2.2. *Let* $u(x,t) \in C^2(\overline{\Omega} \times (0,\infty))$ *be a solution of the problem*

$$u_t = \frac{M^\alpha(u)}{(1 + |\nabla u|^2)^{\alpha\beta}} \text{ in } \Omega \times (0,\infty),$$

$$u(x,t) = 0 \text{ in } \partial\Omega \times (0,\infty), \tag{4.9}$$

$$u(x,t) \text{ is strictly convex for each } t \geq 0,$$

where $\alpha > 1/n$ *and* $\beta \geq 0$ *are constants. If* $\beta = 0$, *then there exists positive constant* C_1 *depending only on dimension* n, α, Ω *and* $u(x,0)$, *such that for all* $t \geq 0$,

$$\sup_\Omega \left| (1+t)^{\frac{1}{n\alpha - 1}} u(x,t) - \psi(x) \right| \leq \frac{C_1}{1+t}, \tag{4.10}$$

If $\beta > 0$, *then*

$$\left[\frac{C_2}{1+t} + G^{\frac{1}{1-n\alpha}} - 1 \right] \psi \leq (1+t)^{\frac{1}{n\alpha - 1}} u(x,t) - \psi(x) \leq \frac{-C_3\psi}{1+t}, \tag{4.11}$$

where C_2 and C_3 are positive constants depending only on dimension n, α, Ω, $u(x,0)$ and

$$G = \inf_{\Omega} \left(1 + |\nabla u(x,0)|^2\right)^{-\alpha\beta}.$$

Moreover,

$$\lim_{t\to\infty} (1+t)^{\frac{1}{n\alpha-1}} u(x,t) = \psi(x) \text{ uniformly on } \overline{\Omega}. \tag{4.12}$$

We have gradient estimates for solutions of (4.9).

Corollary 4.2.3. *Suppose the same conditions as in Theorem 4.2.2 holds. Then for all $t \geq 0$,*

$$\sup_{\Omega} |\nabla u(x,t)| \leq G^{\frac{1}{1-n\alpha}} \sup_{\partial\Omega} \psi_\nu(x)(C_4 + t)^{\frac{1}{1-n\alpha}}$$

where ψ_ν is the derivative in the direction of the outward unit normal to $\partial\Omega$, and C_4 depends only on $u(x,0)$.

An interesting geometric consequence of Theorem 4.2.2 is the following:

Theorem 4.2.4. *If Ω is a ball in $\mathbb{R}^n$ and $u(x,t) \in C^2(\overline{\Omega} \times (0,\infty))$ is a solution of (4.9). Then*

$$(1+t)^{\frac{1}{n\alpha-1}} u(x,t) \to \psi(|x|) \text{ uniformly on } \overline{\Omega} \text{ as } t \to \infty.$$

This theorem implies that, $u(x,t)$ asymptotically becomes radially symmetric regardless of the initial shape. More generally, if Ω is centrally symmetric, then

$$(1+t)^{\frac{1}{n\alpha-1}} u(x,t) \to \psi(x) \text{ uniformly on } \overline{\Omega} \text{ as } t \to \infty,$$

where $\psi(x) = \psi(-x)$. The proof of Theorem 4.2.4 is the same as in [59, Section 6] and we omit it here.

4.3 Proof of Theorem 4.2.1

The existence of solution to (4.7) is claimed in [65] without a detailed proof. Thus we include a detailed proof here for the sake of completeness. V. Oliker [59] proved that (4.7) has a unique solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ when $\alpha = 1$. The proof we present here is a generalization of his proof for all $\alpha > 1/n$.

We use the same strategy as in [59] to prove the existence of a solution to (4.2). The idea is to consider the family of boundary problems

$$\begin{aligned} &M(\psi) = (\lambda(\psi - \delta))^{\frac{1}{\alpha}} \text{ in } \Omega, \\ &\psi = 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.13}$$

where $\delta \in (0, \delta_0)$ for some fixed $\delta_0 > 0$. Then we show that for any $\delta \in (0, \delta_0)$ there exists a smooth solution ψ_δ of (4.13) and then proceed to give a solution to (4.7) by letting $\delta \to 0$.

A smooth solution ψ_δ of (4.13) can be obtained by applying Theorem 7.1 in [14] if we are able to verify the conditions needed. Clearly, the function $f(x, \psi) = (\lambda(\psi - \delta))^{\frac{1}{\alpha}} > 0$ on $\overline{\Omega} \times (-\infty, 0]$. Then it remains to check there exists a convex subsolution $\underline{\psi} \in C^2(\overline{\Omega})$ such that

$$M(\underline{\psi}) \geq (\lambda(\underline{\psi} - \delta))^{\frac{1}{\alpha}} \text{ in } \Omega \tag{4.14}$$

$$\underline{\psi} = 0 \text{ on } \partial\Omega.$$

This can be done using the construction of P. L. Lions[48]. Since Ω is strictly convex, $\partial\Omega$ can be written as a level hypersurface $\rho = 0$ for a strictly convex smooth function ρ satisfying $\rho < 0$ in Ω and $|\nabla\rho| \neq 0$ on $\partial\Omega$. Take

$$\underline{\psi} = \rho + k(e^\rho - 1),$$

where k is a positive constant to be chosen later. Then

$$\underline{\psi}_{ij} = \rho_{ij} + ke^\rho(\rho_{ij} + \rho_i\rho_j).$$

Since $(\rho_{ij}) \geq c\,\mathrm{I}$ for some constant $c > 0$, one computes using $\det(\mathrm{I} + aa^T) = 1 + a^Ta$ that

$$M(\underline{\psi}_{ij}) \geq (ke^\rho)^n c^n \left[1 + \frac{|\nabla\rho|^2}{c}\right].$$

Therefore, in view of $\alpha > 1/n$, we may choose k large enough, independent of $\delta \in (0, \delta_0)$ so that

$$(ke^\rho)^n c^n \left[1 + \frac{|\nabla\rho|^2}{c}\right] \geq (\lambda(\rho + k(e^\rho - 1) - \delta_0))^{\frac{1}{\alpha}} \geq \left(\lambda(\underline{\psi} - \delta)\right)^{\frac{1}{\alpha}}.$$

Now we can apply Theorem 7.1 in [14] and obtain a strictly convex smooth solution ψ_δ to (4.13) and $\psi_\delta \geq \underline{\psi}$. Next, we show that the family of functions $\{\psi_\delta \mid \delta \in (0, \delta_0)\}$ contains a subsequence that converges to a solution of (4.7). To do this, we establish up to C^3 interior estimates independent of δ.

4.3.1 C^0 estimates

We start with the following C^0 estimate.

Lemma 4.3.1.

$$0 < \kappa \leq \sup_{\Omega} |\psi_\delta| \leq \sup_{\Omega} |\underline{\psi}|. \tag{4.15}$$

Before heading to the proof, we need to recall some properties of the Monge-Ampere operator M. The proofs of these properties can be found in [34]. Given a function $u : \Omega \to \mathbb{R}$, the normal mapping of u, is the set function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$\partial u(x_0) = \{p : u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in \Omega\}.$$

For any $E \subset \Omega$, we define

$$\partial u(E) = \bigcup_{x \in \Omega} \partial u(x).$$

If $u \in C(\Omega)$, then the class

$$\mathcal{S} = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable } \}$$

is a Borel σ-algebra and the set function $Mu : \mathcal{S} \to \overline{\mathbb{R}}$ defined by

$$Mu(E) = |\partial u(E)|$$

is a measure, finite on compact sets, that is called the Monge-Ampere measure associated with the function u. If $u \in C^2(\Omega)$ is a convex function, then the Monge-Ampere measure Mu associated with u satisfies

$$Mu(E) = \int_E \det(u_{ij})\, dx$$

for all Borel sets $E \subset \Omega$.

Given a Borel measure ν on Ω. A convex function $u \in C(\Omega)$ is called a generalized solution to the Monge-Ampere equation

$$\det D^2 u = \nu$$

if the Monge-Ampere measure Mu associated with u equals ν. In particular, smooth convex solutions are also generalized solutions. Let g be a nonnegative integrable function defined on $\overline{\Omega}$. Then

$$\nu(E) = \int_E g(x)dx$$

defines a Borel measure on $\overline{\Omega}$. It is shown in [11, Section 11.5] that if for all x sufficiently close to $\partial\Omega$,

$$g(x) \leq a\left(d(x, \partial\Omega)\right)^s, \; a = \text{const} \geq 0, s \geq 0, \tag{4.16}$$

$$\mu(\Omega) < \infty. \tag{4.17}$$

then there exists a unique convex function $u \in C(\overline{\Omega})$ such that $Mu = \nu$ as measures and $u = 0$ on $\partial\Omega$. Thus on the set of measures, generated as above, and satisfying (4.16), (4.17), a solution operator A is well-defined. Moreover, if $g_1(x) \leq g_2(x)$ on $\overline{\Omega}$, then

$$Ag_1(x) \geq Ag_2(x) \text{ for all } x \in \overline{\Omega}, \tag{4.18}$$

$$A(Cg) = C^{\frac{1}{n}} A(g) \text{ for any constant } C > 0. \tag{4.19}$$

Now we are ready to prove Lemma 4.3.1.

Proof of Lemma 4.3.1. Since ψ_δ is convex, its maximum is achieved on $\partial\Omega$ and we immediately get a uniform estimate

$$\sup_\Omega |\psi_\delta| \le \sup_\Omega |\underline{\psi}|.$$

The lower bound is not easy and we omit the subscript δ for simplicity. Since ψ is convex and vanishes on $\partial\Omega$, it achieves its minimum at some interior point $\bar{x} \in \Omega$. Consider the cone K with vertex $(\bar{x}, \psi(\bar{x}))$ and base $\partial\Omega$ and let $\theta(x)$ be the convex function whose graph is K. Clearly, $\theta(x) \ge \psi(x)$ for all $x \in \overline{\Omega}$. For any $x \in \Omega$, there is a line segment originating from $\bar{x}$ and going through x until it reaches boundary at a point $l(x)$. Then by considering similar triangles in the two-plane through the segment and the point, we have

$$\theta(x) \le \frac{\psi(\bar{x})d(x, l(x))}{d(\bar{x}, l(x))} \le \psi(\bar{x})\frac{d(x, \partial\Omega)}{\text{diam}\Omega}.$$

For sufficient small $\epsilon > 0$, define $E = \{x \in \Omega : d(x, \partial\Omega) \ge \epsilon \ \text{diam}\Omega\}$. It's easy to see

$$\psi(x) \le \frac{\chi_E \psi(\bar{x})d(x, \partial\Omega)}{\text{diam}\Omega} \le \epsilon\chi_E\psi(\bar{x})$$

for all $x \in \overline{\Omega}$. Let $g(x) = (\epsilon\lambda\chi_E\psi(\bar{x}))^{\frac{1}{\alpha}}$, then g satisfies the conditions (4.16), (4.17) and therefore there is a convex function $u \in C(\Omega)$ such that for any Borel subset $F \subset \Omega$,

$$Mu(F) = (\epsilon\lambda\psi(\bar{x}))^{\frac{1}{\alpha}} \int_F \chi_E dx.$$

Since $(\lambda(\psi - \delta))^{\frac{1}{\alpha}} \ge (\lambda\epsilon\chi_E\psi(\bar{x}))^{\frac{1}{\alpha}}$, then by (4.18) and (4.19),

$$u(x) = (\epsilon\lambda\psi(\bar{x}))^{\frac{1}{n\alpha}}(A\chi)(x) \ge \psi(x) \ge \psi(\bar{x}).$$

It then follows that

$$|\psi(\bar{x})| \geq \left(|\epsilon\lambda|^{\frac{1}{n\alpha-1}} \sup_{\Omega} |(A\chi)(x)|^{\frac{n\alpha}{n\alpha-1}} \right) := \kappa.$$

Noting that $\sup_{\Omega} |(A\chi)(x)| > 0$, we obtain

$$\sup_{\Omega} |\psi| = |\psi(\bar{x})| \geq \kappa > 0.$$

$\square$

4.3.2 C^1 estimates

The following lemma establishes the C^1 estimate.

Lemma 4.3.2.

$$\sup_{\Omega} |\nabla \psi_\delta| \leq \sup_{\partial\Omega} |\underline{\psi}_\nu|. \tag{4.20}$$

Proof. We omit the subscript δ as in previous lemma. By convexity of ψ, $|\nabla\psi|$ attains its maximum on $\partial\Omega$. Since $\psi = 0$ on $\partial\Omega$, it suffices to estimate the normal derivative $|\psi_\nu|$, where ν denotes the outward unit normal vector field on $\partial\Omega$. Note that $\psi \geq \underline{\psi}$ and they agree on $\partial\Omega$, it follows that $\psi_\nu \leq \underline{\psi}_\nu$ on $\partial\Omega$. On the other hand, since ψ is convex, it's easy to see that $\psi_\nu \geq 0$. Thus the desired estimate is established. $\square$

4.3.3 C^2 estimates

Now let's turn to the C^2 interior estimate.

Lemma 4.3.3. *There exists positive constant $\bar{z}$, independent of δ, such that for any direction γ we have*

$$0 \leq \psi_{\gamma\gamma} \leq \frac{\bar{z}\ diam\Omega}{d(x, \partial\Omega)\kappa}. \tag{4.21}$$

Before giving a proof the above lemma, we present here the calculation for estimating the second derivatives of a more general Monge-Ampere equation:

$$M(\psi) = f(x, \psi, \nabla\psi), \tag{4.22}$$

assuming that f is positive and later we restrict to equation (4.13). Although the basic calculation appeared in [58], we give some details here for the sake of completeness. Again we omit the subscript δ for brevity. Consider the function

$$z = -\psi \exp\left(\frac{c\psi_\gamma^2}{2}\right)\psi_{\gamma\gamma}, \tag{4.23}$$

where ψ_γ is the derivative in the direction γ, $\psi_{\gamma\gamma}$ is the second derivative in the same direction and c is a constant to be chosen later. Clearly, the function z attains its maximum at some interior point $q \in \Omega$. Without loss of generality, we may assume that q is the origin and choose an orthonormal basis that diagonalizes ψ_{ij}. Differentiating $\log z$ in the i direction yields

$$\frac{z_i}{z} = \frac{\psi_i}{\psi} + c\psi_\gamma\psi_{\gamma i} + \frac{\psi_{\gamma\gamma i}}{\psi_{\gamma\gamma}}. \tag{4.24}$$

When $\gamma \neq i$, this reduces to

$$\frac{\psi_i^2}{\psi^2} = \frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}^2}. \tag{4.25}$$

Differentiating one more time in the i direction gives,

$$\frac{z_{ii}}{z}-\frac{z_i^2}{z^2}=\frac{\psi_{ii}}{\psi}-\frac{\psi_i^2}{\psi^2}+c\psi_{\gamma i}^2+c\psi_\gamma\psi_{\gamma ii}+\frac{\psi_{\gamma\gamma ii}}{\psi_{\gamma\gamma}}-\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}^2}. \tag{4.26}$$

Taking into account that $\psi_{\gamma i}=0$ when $\gamma\neq i$ and $z_i=0$ at q, we get after multiplying (4.26) by $\frac{\psi_{\gamma\gamma}}{\psi_{ii}}$ and summing over i,

$$\begin{aligned}\sum_i\frac{z_{ii}\psi_{\gamma\gamma}}{z\psi_{ii}}=&n\frac{\psi_{\gamma\gamma}}{\psi}-\frac{\psi_\gamma^2}{\psi^2}-\sum_{i\neq\gamma}\frac{\psi_i^2}{\psi^2}\frac{\psi_{\gamma\gamma}}{\psi_{ii}}+c\psi_{\gamma\gamma}^2+c\psi_\gamma\psi_{\gamma\gamma}\sum_i\frac{\psi_{\gamma ii}}{\psi_{ii}}\\&+\sum_i\frac{\psi_{\gamma\gamma ii}}{\psi_{ii}}-\sum_i\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}}.\end{aligned}\tag{4.27}$$

Substituting (4.25) in (4.27) gives

$$\begin{aligned}\sum_i\frac{z_{ii}\psi_{\gamma\gamma}}{z\psi_{ii}}=&n\frac{\psi_{\gamma\gamma}}{\psi}-\frac{\psi_\gamma^2}{\psi^2}-\sum_{i\neq\gamma}\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}}+c\psi_{\gamma\gamma}^2+c\psi_\gamma\psi_{\gamma\gamma}\sum_i\frac{\psi_{\gamma ii}}{\psi_{ii}}\\&+\sum_i\frac{\psi_{\gamma\gamma ii}}{\psi_{ii}}-\sum_i\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}}.\end{aligned}\tag{4.28}$$

By differentiating (4.22) in γ direction once and twice respectively, we obtain

$$\sum_i\frac{\psi_{ii\gamma}}{\psi_{ii}}=(\log f)_\gamma \tag{4.29}$$

$$\sum_i\frac{\psi_{ii\gamma\gamma}}{\psi_{ii}}-\sum_{i,j}\frac{\psi_{ij\gamma}^2}{\psi_{ii}\psi_{jj}}=(\log f)_{\gamma\gamma}. \tag{4.30}$$

Subtracting (4.30) from (4.28) yields

$$\begin{aligned}&n\frac{\psi_{\gamma\gamma}}{\psi}-\frac{\psi_\gamma^2}{\psi^2}-\sum_{i\neq\gamma}\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}}+c\psi_{\gamma\gamma}^2+c\psi_\gamma\psi_{\gamma\gamma}\sum_i\frac{\psi_{\gamma ii}}{\psi_{ii}}-\sum_i\frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}}\\&+\sum_{i,j}\frac{\psi_{ij\gamma}^2}{\psi_{ii}\psi_{jj}}=\sum_i\frac{z_{ii}\psi_{\gamma\gamma}}{z\psi_{ii}}-(\log f)_{\gamma\gamma}.\end{aligned}\tag{4.31}$$

Noting that

$$\sum_{i,j} \frac{\psi_{ij\gamma}^2}{\psi_{ii}\psi_{jj}} - \sum_{i} \frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}} - \sum_{i\neq\gamma} \frac{\psi_{\gamma\gamma i}^2}{\psi_{\gamma\gamma}\psi_{ii}} \geq 0, \tag{4.32}$$

we obtain from (4.31) that

$$c\psi_{\gamma\gamma}^2 + n\frac{\psi_{\gamma\gamma}}{\psi} - \frac{\psi_\gamma^2}{\psi^2} + c\psi_\gamma\psi_{\gamma\gamma}(\log f)_\gamma + (\log f)_{\gamma\gamma} \leq 0. \tag{4.33}$$

where we also used $z_{ii} \leq 0$ at q.

Proof of Lemma 4.3.3. Now we return to equation (4.13) and plug in $f(x, \psi, \nabla\psi) = (\lambda(\psi - \delta))^{\frac{1}{\alpha}}$ and $c = 1$ to get

$$\psi_{\gamma\gamma}^2 + \left(\frac{n}{\psi} + \frac{\psi_\gamma^2 + 1}{\alpha(\psi - \delta)}\right)\psi_{\gamma\gamma} - \left(\frac{1}{\psi^2} + \frac{1}{\alpha(\psi - \delta)^2}\right)\psi_\gamma^2 \leq 0. \tag{4.34}$$

Multiplying (4.34) by $\psi^2 \exp(\psi_\gamma^2)$ implies,

$$z^2 - \exp\left(\frac{\psi_\gamma^2}{2}\right)\left(n + \frac{\psi(\psi_\gamma^2 + 1)}{\alpha(\psi - \delta)}\right)z - \exp(\psi_\gamma^2)\psi_\gamma^2\left(1 + \frac{\psi^2}{\alpha(\psi - \delta)^2}\right) \leq 0. \tag{4.35}$$

The fact that the coefficients in (4.35) are uniformly bounded then follows from $0 \leq \frac{\psi}{\psi - \delta} \leq 1$ and $|\nabla\psi|$ is uniformly bounded. Hence $0 \leq z \leq \bar{z}$ for $\bar{z}$ independent of δ. It then follows that $0 \leq \psi_{\gamma\gamma} \leq \frac{\bar{z}}{|\psi|}$. Convexity of ψ also implies

$$\frac{|\psi(x)|}{d(x, \partial\Omega)} \geq \frac{\max|\psi(x)|}{\mathrm{diam}\Omega} \tag{4.36}$$

Combining this inequality with (4.20) and (4.35) implies

$$0 \leq \psi_{\gamma\gamma} \leq \frac{\bar{z}\ \mathrm{diam}\Omega}{d(x, \partial\Omega)\kappa}. \tag{4.37}$$

Then by rotating the coordinates axes with origin at q, any mixed second derivative can be written in terms of second derivatives of ψ in the directions of the principle axes of the second differential of ψ. Since the direction γ was arbitrary, thus we have proved the C^2 estimate. $\square$

4.3.4 C^3 estimates

Now we establish the interior estimates for third derivatives. The idea, due to E. Calabi [15], is to consider the following function on Ω,

$$w = \psi^{kl}\psi^{pq}\psi^{rs}\psi_{kpr}\psi_{lqs}. \tag{4.38}$$

This expression measures the square of the third derivatives in term of the Riemannian metric $g = \psi_{ij}dx^i dx^j$. The Laplace operator with respect to this metric g is given by

$$\Delta w = \psi^{ij}w_{ij},$$

where (ψ^{ij}) is the inverse of (ψ_{ij}). The C^3 estimate is then obtained by computing the differential inequality satisfied by w and applying the maximum principle. Note that we have omitted the subscript δ.

For any $x \in \Omega$, we can choose an orthonormal frame that diagonalizes (ψ_{ij}) at x. It is shown in [14, Section 3] that if ψ is a solution of $M(\psi) = f(x, \psi, \nabla\psi)$, then at x,

$$\begin{aligned}\Delta w =& 2\frac{1}{\psi_{ii}\psi_{kk}\psi_{pp}\psi_{rr}}\left(\psi_{kpri}-\frac{1}{\psi_{ll}}\left(\psi_{kli}\psi_{plr}+\psi_{pli}\psi_{klr}+\psi_{rli}\psi_{kpl}\right)\right)^2\\ &-\frac{1}{2}\frac{1}{\psi_{ii}\psi_{kk}\psi_{pp}\psi_{rr}}\left|\frac{1}{\psi_{ll}}\left(\psi_{kli}\psi_{plr}+\psi_{pli}\psi_{klr}+\psi_{rli}\psi_{kpl}\right)\right|^2\\ &+2A+6B-\frac{3\psi_{kpr}\psi_{lpr}}{\psi_{kk}\psi_{pp}\psi_{rr}\psi_{ll}}\left(\frac{\psi_{kab}\psi_{lab}}{\psi_{aa}\psi_{bb}}+(\log f)_{kl}\right)+\frac{2\psi_{kpr}}{\psi_{kk}\psi_{pp}\psi_{rr}}(\log f)_{kpr},\end{aligned}$$

where

$$A=\frac{\psi_{kpr}\psi_{lqr}\psi_{kli}\psi_{pqi}}{\psi_{kk}\psi_{pp}\psi_{rr}\psi_{ll}\psi_{ii}\psi_{qq}},\ B=\frac{\psi_{kpr}\psi_{lpr}\psi_{kai}\psi_{lai}}{\psi_{kk}\psi_{pp}\psi_{rr}\psi_{ll}\psi_{aa}\psi_{ii}}.$$

The following estimate is also proved in [14]:

$$\begin{aligned}&-\frac{1}{2}\frac{1}{\psi_{ii}\psi_{kk}\psi_{pp}\psi_{rr}}\left|\frac{1}{\psi_{ll}}\left(\psi_{kli}\psi_{plr}+\psi_{pli}\psi_{klr}+\psi_{rli}\psi_{kpl}\right)\right|^2\\ &+2A+6B-\frac{3\psi_{kpr}\psi_{lpr}\psi_{kab}\psi_{lab}}{\psi_{kk}\psi_{pp}\psi_{rr}\psi_{ll}\psi_{aa}\psi_{bb}}\\ &\geq\frac{1}{2}B\geq\frac{w^2}{4n}.\end{aligned}$$

We substitute into $f(x,\psi,\nabla\psi)=(\lambda(\psi-\delta))^{\frac{1}{\alpha}}$ and obtain that

$$\Delta w\geq\frac{w^2}{4n}-\frac{w}{\alpha(\psi-\delta)}-\sqrt{E}\frac{w+6\sqrt{nw}}{\alpha(\psi-\delta)^2}-\sqrt{E^3}\frac{4\sqrt{w}}{\alpha(\psi-\delta)^3}\geq\frac{w^2}{4n}-Q(w),\quad (4.39)$$

using $\psi-\delta<0$, where $E=\frac{\psi_k^2}{\psi_{kk}}$ and $Q(w)=\sqrt{E}\frac{w+6\sqrt{nw}}{\alpha(\psi-\delta)^2}$.

We use an argument similar to Lemma 3.1 in [14] to estimate w at any interior point O. We take O to be the origin, whose distance to $\partial\Omega$ is $2R$. We make use of the function $\zeta=R^2-|x|^2$ in $B_R(0)$ and $\zeta=0$ for all $|x|\geq R$. Set

$\tau = \zeta^2 w$, then for all $|x| \leq R$,we have

$$\frac{(\zeta^{-2}\tau)^2}{4n} \leq \Delta w + Q(w) \tag{4.40}$$
$$= \zeta^{-2}\Delta\tau + 2\langle\nabla(\zeta^{-2}), \nabla\tau\rangle + \tau\psi^{ij}\left(4\zeta^{-3}\delta_{ij} + 24\zeta^{-4}\bar{x}_i\bar{x}_j\right) + Q(w).$$

At a point $\bar{x} \in B_R(0)$, where the function $\tau = \zeta^2 w$ attains its maximum on $B_R(0)$, $\nabla\tau = 0$ and $\Delta\tau \leq 0$ and so at $\bar{x}$,

$$\frac{\tau^2}{4n} \leq \tau\psi^{ij}\left(4\zeta\delta_{ij} + 24\bar{x}_i\bar{x}_j\right) + \zeta^4 Q(w) \leq C\tau\,\mathrm{tr}(\psi^{ij}) + \zeta^4 Q(w), \tag{4.41}$$

where C depends only on the diameter of Ω. Next we estimate $\zeta^4 Q(w)$. Note that (4.57) and (4.36) imply

$$\frac{1}{(\psi-\delta)^2} \leq \left(\frac{\mathrm{diam}\Omega}{\kappa\, d(x,\partial\Omega)}\right)^2. \tag{4.42}$$

It follows from (4.20), (4.37) and $\det(\psi_{ij}) = (\lambda(\psi-\delta))^{\frac{1}{\alpha}}$ that

$$\mathrm{tr}(\psi^{ij}) = \sum_i \psi^{ii} = \sum_i \frac{\Pi_{j\neq i}\psi_{jj}}{(\lambda(\psi-\delta))^{\frac{1}{\alpha}}} \leq \frac{\tilde{C}_1}{(d(x,\partial\Omega))^{n-1+\frac{1}{\alpha}}}, \tag{4.43}$$

$$E = \sum_k \frac{\psi_k^2}{\psi_{kk}} \leq \frac{\tilde{C}_2}{(d(x,\partial\Omega))^{n-1+\frac{1}{\alpha}}}, \tag{4.44}$$

where $\tilde{C}_i$ are constants depending only on $n, \kappa, \alpha, \mathrm{diam}\Omega, \sup_\Omega|\psi|$ and $\sup_\Omega|\nabla\psi|$. Therefore, if $w \geq 1$,

$$\zeta^4 Q(w) \leq \frac{\tilde{C}_3\tau}{(d(x,\partial\Omega))^{\frac{1}{2}(n-1+\frac{1}{\alpha})+2}}, \tag{4.45}$$

where $\tilde{C}_3$ is a constant depending only on $n, \kappa, \alpha, \mathrm{diam}\Omega, \sup_\Omega|\psi|$ and $\sup_\Omega|\nabla\psi|$. Inserting these estimates in (4.41), we get

$$\tau(0) \leq \tau(\bar{x}) \leq \tilde{C}\left(\frac{1}{d(x,\partial\Omega)^{n-1+\frac{1}{\alpha}}} + \frac{1}{d(x,\partial\Omega)^{\frac{1}{2}(n-1+\frac{1}{\alpha})+2}}\right), \tag{4.46}$$

where $\tilde{C}$ is a constant depending only on $n, \kappa, \lambda, \alpha, \mathrm{diam}\Omega, \sup_\Omega |\psi|$ and $\sup_\Omega |\nabla\psi|$. Since $d(\bar{x}, \partial\Omega) \geq R$,

$$w(0) \leq \tilde{C}\left(\frac{1}{R^{n-1+\frac{1}{\alpha}+2}} + \frac{1}{R^{\frac{1}{2}(n-1+\frac{1}{\alpha})+4}}\right). \tag{4.47}$$

In view of (4.37), we see that

$$\frac{1}{\psi^{ii}} \geq \tilde{C} d(0, \partial\Omega). \tag{4.48}$$

This and the preceding inequality imply that the third derivatives $|\psi_{ijk}|$ are uniformly bounded on compact subsets of Ω.

4.3.5 Passing to a subsequence

Finally, we can pass to a subsequence and obtain a solution of (4.13). From the uniform bounds of the norms $\|\psi_\delta\|_{C^3(\Omega)}$ and $\|\psi_\delta\|_{C^1(\overline{\Omega})}$, it follows that the sequence ψ_δ contains a subsequence converging to a strictly convex function $\psi \in C^{2,\gamma}(\Omega) \cap \mathrm{Lip}(\Omega)$ for any $\gamma \in (0,1)$. Obviously the function ψ satisfies (4.13) and $\psi = 0$ on $\partial\Omega$. Standard theory on nonlinear elliptic PDE's then implies $\psi \in C^\infty(\Omega) \cap \mathrm{Lip}(\Omega)$.

4.3.6 Proof of Theorem 4.2.1

Proof. Direct calculation shows $u(x,t) = (1+t)^{\frac{1}{1-n\alpha}}\psi(x)$ solves (4.2) with initial data $u_0(x) = \psi(x)$. Next we prove $\sup_\Omega |\psi(x)|$ depends only on n, α and Ω. Since ψ is strictly convex and vanishes on $\partial\Omega$, there exists a point $\bar{x} \in \Omega$ such that

$\sup_\Omega |\psi| = |\psi(\bar{x})|$. Consider a cone K generated by the linear segments joining the vertex $(\bar{x}, \psi(\bar{x}))$ with points on $\partial\Omega$. Denote $\theta(x), x \in \overline{\Omega}$, the function whose graph is K. Obviously, $\theta \geq \psi$ in Ω and $\theta = \psi = 0$ on $\partial\Omega$. Then by [34, Lemma 1.4.1] $M\theta(\Omega) \leq M\psi(\Omega)$, where Mu denotes the Monge-Ampere measure associated with the function u(see [34, Theorem 1.1.13]). Since ψ is C^∞ and convex on Ω,

$$M\psi(\Omega) = \int_\Omega M(\psi) = \int_\Omega (\lambda\psi)^{\frac{1}{\alpha}} \leq |\lambda|^{\frac{1}{\alpha}} |\psi(\bar{x})|^{\frac{1}{\alpha}} |\Omega|. \tag{4.49}$$

On the other hand, the Aleksandrov-Bakelman-Pucci maximum principle (see, for instance, [34, Theorem 1.4.5]) says $M\theta(\Omega) \geq \omega_n |\psi(\bar{x})|^n (\mathrm{diam}\Omega)^{-n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus

$$\sup_\Omega |\psi(x)| = |\psi(\bar{x})| \leq \left(\frac{|\lambda|^{\frac{1}{\alpha}} |\Omega| (\mathrm{diam}\Omega)^n}{\omega_n} \right)^{\frac{\alpha}{n\alpha - 1}}. \tag{4.50}$$

Finally, the proof of (4.8) parallels that in [59, Section 4.3]. $\square$

4.4 Completion of the proof of Theorem 4.2.1

4.4.1 Uniqueness of ψ

We then prove the uniqueness of ψ. Let ψ_1, ψ_2 be two convex solutions of (4.7), i.e.

$$M(\psi_i) = (\lambda\psi_i)^{\frac{1}{\alpha}} \text{ in } \Omega \text{ and } \psi_i = 0 \text{ on } \partial\Omega, i = 1, 2. \tag{4.51}$$

Suppose $\psi_1 \neq \psi_2$, we may assume there exist $x \in \Omega$ such that $\psi_1(x) > \psi_2(x)$. Let $R \in (0, 1)$ be such that $R\psi_2 \geq \psi_1$ for all $x \in \Omega$ and at some point $x_0 \in \Omega$,

$R\psi_2(x_0) = \psi_1(x_0)$. Then

$$M(R^{\frac{1}{n\alpha}}\psi_2) = R^{\frac{1}{\alpha}}M(\psi_2) = R^{\frac{1}{\alpha}}(\lambda\psi_2)^{\frac{1}{\alpha}} \leq (\lambda\psi_1)^{\frac{1}{\alpha}} = M(\psi_1).$$

By monotonicity of solution operator, $R^{\frac{1}{n\alpha}}\psi_2 \geq \psi_1$ in Ω. In particular, $R^{\frac{1}{n\alpha}}\psi_2(x_0) \geq \psi_1(x_0)$. On the other hand, in view of $R < 1$ and $\psi_2 < 0$ in Ω, $R^{\frac{1}{n\alpha}}\psi_2(x_0) \leq R\psi_2(x_0)$. We arrived at a contradiction and hence $\psi_1 \equiv \psi_2$.

4.4.2 Estimate for $\sup_\Omega |\psi(x)|$

The following lemma says $\sup_\Omega |\psi(x)|$ admits an estimate depending only on n, α, and Ω.

Lemma 4.4.1.

$$\sup_{\Omega} |\psi(x)| \leq C(n, \Omega, \alpha). \tag{4.52}$$

Proof. Since ψ is strictly convex and vanishes on $\partial\Omega$, there exists a point $\bar{x} \in \Omega$ such that $\sup_\Omega |\psi| = |\psi(\bar{x})|$. Consider a cone K generated by the linear segments joining the vertex $(\bar{x}, \psi(\bar{x}))$ with points on $\partial\Omega$. Denote $\theta(x), x \in \overline{\Omega}$, the function whose graph is K. Obviously, $\theta \geq \psi$ in Ω and $\theta = \psi$ on $\partial\Omega$. Then $M\theta(\Omega) \leq M\psi(\Omega)$, where Mu denotes the Monge-Ampere measure associated with the function u.

$$M\psi(\Omega) = \int_\Omega M(\psi) = \int_\Omega (\lambda\psi)^{\frac{1}{\alpha}} \leq |\lambda|^{\frac{1}{\alpha}}|\psi(\bar{x})|^{\frac{1}{\alpha}}|\Omega|. \tag{4.53}$$

On the other hand, it is shown [34, Theorem 1.4.5] that $M\theta(\Omega) \geq \omega_n|\psi(\bar{x})|^n(\mathrm{diam}\Omega)^{-n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus

$$\sup_{\Omega} |\psi(x)| = |\psi(\bar{x})| \leq \left(\frac{|\lambda|^{\frac{1}{\alpha}} |\Omega| (\mathrm{diam}\Omega)^n}{\omega_n} \right)^{\frac{\alpha}{n\alpha-1}}. \tag{4.54}$$

$\square$

4.4.3 Uniqueness of Self-similar Solutions

We prove equation (4.8) here, by modifying the argument used in [59]. Let $\tilde{u}(x,t) = \varphi(t)\tilde{\psi}(x)$ be a self-similar solution of (4.7) and $\varphi(t) \in C^\infty([0,\infty))$ and $\tilde{\psi}(x) \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$ strictly convex. Then φ and $\tilde{\psi}$ satisfy

$$\varphi(t) = \left(\varphi(0)^{1-n\alpha} - (n\alpha - 1)\tilde{\lambda} t \right)^{\frac{1}{1-n\alpha}}, \tag{4.55}$$

$$M(\tilde{\psi}) = (\tilde{\lambda}\tilde{\psi})^{\frac{1}{\alpha}} \text{ in } \Omega \text{ and } \tilde{\psi} = 0 \text{ on } \partial\Omega. \tag{4.56}$$

If we choose $\tilde{c}$ so that $\tilde{c}^{n\alpha-1}\tilde{\lambda} = \frac{1}{1-n\alpha}$, Then

$$M^\alpha(\tilde{c}\tilde{\psi}) = \tilde{c}^{n\alpha} M^\alpha(\tilde{\psi}) = \tilde{c}^{n\alpha}\tilde{\lambda}\tilde{\psi} = \tilde{c}^{n\alpha-1}\tilde{\lambda} \left(\tilde{c}\tilde{\psi} \right) = \frac{1}{1-n\alpha}\tilde{\psi}.$$

By uniqueness of solution to (4.7), we must have $\tilde{c}\tilde{\psi} = \psi$. Then direct calculation shows

$$\tilde{u}(x,t) = \varphi(t)\tilde{\psi}(x) = \tilde{c}^{-1}\varphi(t)\psi(x) = u(x,t) \left(\frac{1+t}{(c\varphi(0))^{1-n\alpha} + t} \right)^{\frac{1}{n\alpha-1}},$$

where

$$c := \tilde{c}^{-1} = \left(\tilde{\lambda}(1-n\alpha) \right)^{\frac{1}{n\alpha-1}}.$$

We end this section by briefly discussing what happens when $\alpha \leq 1/n$.

Remark 4.4.1. *When $\alpha = 1/n$, it was shown by P. L. Lions[48] that*

$$M(\psi) = \mu(-\psi)^n \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \tag{4.57}$$

admits a unique solution pair (μ, ψ) in the sense that if (ν, ϕ), where ν is positive and ϕ is convex, solves (4.57), then we must have $\mu = \nu$ and ϕ is a constant multiple of ψ. The number μ is called the first (in fact the only) eigenvalue of the Monge-Ampère operator M, and the corresponding (normalized) eigenfunction is in $C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$. The asymptotic behavior for $\alpha = 1/n$ remains interesting and open.

Remark 4.4.2. *When $0 < \alpha < 1/n$, K. Tso[65, Theorem E] showed that (4.7) admits a convex solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$. The uniqueness, however, is not known. In this case, the reader will see easily from the comparison with self-similar supersolutions in Section 4 that smooth convex solutions of (4.9) must vanish at finite time.*

4.5 Proof of Theorem 4.2.2

In this section, we determine the asymptotic behavior of u by comparing with self-similar solutions of (4.2). A direct generalization of the proof given by V. Oliker in [59] works for $\alpha \geq 2/n$. New estimates are introduced in the following lemma to take care of the case $1/n < \alpha < 2/n$.

Lemma 4.5.1. *Let* $F : (0, S) \times [0, \infty) \to (0, \infty), S < \infty$ *be defined by*

$$F(s,t) = \left(\frac{1+t}{s+t}\right)^{\frac{1}{n\alpha-1}} \equiv \left(1 + \frac{1-s}{s+t}\right)^{\frac{1}{n\alpha-1}}. \tag{4.58}$$

Then we have for all $t \geq 0$*,*

$$F(s,t) \leq 1 + \frac{1}{n\alpha - 1}\frac{1-s}{s(1+t)}, \qquad \text{if } s \leq 1, \alpha \geq 2/n; \tag{4.59}$$

$$F(s,t) \leq 1 + \frac{1}{n\alpha - 1}\left(\frac{1}{s}\right)^{\frac{1}{n\alpha-1}}\frac{1-s}{1+t}, \qquad \text{if } s \leq 1, \alpha \leq 2/n; \tag{4.60}$$

$$F(s,t) \geq 1 - \frac{s-1}{1+t}, \qquad \text{if } s \geq 1, \alpha \geq 2/n; \tag{4.61}$$

$$F(s,t) \geq 1 - \frac{1}{n\alpha - 1}\frac{s-1}{1+t} \qquad \text{if } s \geq 1, \alpha \leq 2/n. \tag{4.62}$$

Proof. This lemma follows from elementary calculus. When $\alpha \geq 2/n$, $\gamma := \frac{1}{n\alpha-1} \leq 1$. Then (4.59) follows from $(1+x)^\gamma \leq 1 + \gamma x$ for all $x \geq 0$ and (4.61) follows from $x^\gamma \geq x$ for all $0 \leq x \leq 1$. When $\alpha \leq 2/n$, $\gamma := \frac{1}{n\alpha-1} \geq 1$. Now (4.60) is a consequence of $(1+x)^\gamma \leq 1 + \gamma(1+a)^{\gamma-1}x$ for all $0 \leq x \leq a$ and (4.62) is a consequence of $(1+x)^\gamma \geq 1 + \gamma x$ for all $-1 < x \leq 0$. □

Proof of Theorem 4.2.2. First of all, a uniform estimate of $|\nabla u(x,t)|$ is obtained similarly as in [59]. For any $t \geq 0$,

$$\sup_{\overline{\Omega}} |\nabla u(x,t)| \leq \sup_{\partial\Omega} |\nabla u(x,t)| = \sup_{\partial\Omega} |u_\nu(x,t)| \leq \sup_{\partial\Omega} |u_\nu(x,0)|. \tag{4.63}$$

Self-similar subsolution and supersolution are then constructed as follows: Let

$$G = \inf_{\Omega} \left(1 + |\nabla u(x,0)|^2\right)^{-\alpha\beta}.$$

Clearly we have $0 < G \leq 1$. It follows from (4.63) that

$$GM^{\alpha}(u) \leq \left(1 + |\nabla u(x,t)|^2\right)^{-\alpha\beta} M^{\alpha}(u) = u_t \text{ in } \Omega \times (0, \infty). \tag{4.64}$$

Put $\underline{u}(x,t) = G^{\frac{1}{1-n\alpha}}\underline{\varphi}(t)\psi(x)$ and $\overline{u}(x,t) = \overline{\varphi}(t)\psi(x)$, where ψ is the solution of (4.7) and

$$\underline{\varphi}(t) = \left(\underline{\varphi}(0)^{1-n\alpha} + t\right)^{\frac{1}{1-n\alpha}},$$

$$\overline{\varphi}(t) = \left(\overline{\varphi}(0)^{1-n\alpha} + t\right)^{\frac{1}{1-n\alpha}}.$$

Then $\underline{u}$ and $\overline{u}$ satisfy $\underline{u}_t = GM^{\alpha}(\underline{u})$ and $\overline{u}_t = M^{\alpha}(\overline{u})$ in $\Omega \times (0, \infty)$, respectively. Finally we define $\tilde{u}(x,t) = \underline{u}(x,t) - u(x,t)$ and it satisfies

$$\tilde{u}_t = GM^{\alpha}(\underline{u}) - \left(1 + |\nabla u(x,t)|^2\right)^{-\alpha\beta} M^{\alpha}(u) \leq GM^{\alpha}(\underline{u}) - GM^{\alpha}(u) \text{ in } \Omega \times (0, \infty). \tag{4.65}$$

Observe that the operator $L(\tilde{u}) = M^{\alpha}(\underline{u}) - M^{\alpha}(u)$ is elliptic since

$$L(\tilde{u}) = \sum_{ij} \left(\int_0^1 \alpha \det(u_{\tau ij})^{\alpha-1} \mathrm{cof}(u_{\tau ij}) d\tau \right) \tilde{u}_{ij},$$

where $u_\tau(x,t) = \tau\underline{u}(x,t) + (1-\tau)u(x,t)$ is strictly convex and the cofactor matrix $\mathrm{cof}(u_{\tau ij})$ is positive definite on any compact subset of $\Omega \times (0, T]$ for any $T < \infty$. Next we choose $\underline{\varphi}(0)$ and $\bar{\varphi}(0)$ so that $\underline{\varphi}(0)\psi(x) \leq u(x,0) \leq \bar{\varphi}(0)\psi(x)$ on Ω. Then

$$\tilde{u}(x,0) \leq 0 \text{ in } \overline{\Omega} \text{ and } \tilde{u}(x,t) = 0 \text{ in } \partial\Omega \times [0, \infty), \tag{4.66}$$

and we can then apply the classical maximum principle to conclude that $\tilde{u}(x,t) = \underline{u}(x,t) - u(x,t) \leq 0$ on $\overline{\Omega} \times [0, \infty)$. Consequently,

$$\left\{ (1+t)^{\frac{1}{n\alpha-1}} \left(G(\underline{\varphi}(0)^{1-n\alpha} + t) \right)^{\frac{1}{1-n\alpha}} - 1 \right\} \psi(x) \leq (1+t)^{\frac{1}{n\alpha-1}} u(x,t) - \psi(x). \tag{4.67}$$

Similarly, one derives that $u(x,t) \le \bar{u}(x,t)$, namely,

$$(1+t)^{\frac{1}{n\alpha-1}}u(x,t)-\psi(x) \le \left\{(1+t)^{\frac{1}{n\alpha-1}}(\overline{\varphi}(0)^{1-n\alpha}+t)^{\frac{1}{1-n\alpha}}-1\right\}\psi(x) \tag{4.68}$$

Without loss of generality we may assume $\underline{\varphi}(0) \ge 1$ and $\bar{\varphi}(0) \le 1$. Thus by Lemma 4.5.1,

$$F(\underline{\varphi}(0)^{1-n\alpha},t) \le 1 + C_2/(1+t)$$

$$F(\overline{\varphi}(0)^{1-n\alpha},t) \ge 1 - C_3/(1+t),$$

where C_2, C_3 depend on n, α and $u_0(x)$. Combining now (4.67) and (4.68), we arrive at that for all $t \ge 0$ and $x \in \overline{\Omega}$,

$$\left[\frac{C_2}{1+t}+G^{\frac{1}{1-n\alpha}}-1\right]\psi \le (1+t)^{\frac{1}{n\alpha-1}}u(x,t)-\psi \le \frac{-C_3\psi}{1+t}, \tag{4.69}$$

If $\beta = 0$, then $G = 1$ and (4.69) implies (4.10) with $C_1 = \max\{C_2, C_3\}\sup_\Omega|\psi|$. If $\beta > 0$, one needs to estimate $|\nabla u(x,t)|$ more carefully as V. Oliker did[59, Pages 255-256]. Take an increasing sequence $t_m \to \infty$ and let $G_m = \inf_\Omega(1 + |\nabla u(x,t_m)|^2)^{-\alpha\beta}$. The same argument as in deriving (4.69) yields for all $t \ge t_m$ and $x \in \overline{\Omega}$,

$$\left[\frac{c_m}{1+t}+G_m^{\frac{-1}{n\alpha-1}}-1\right]\psi \le (1+t)^{\frac{1}{n\alpha-1}}u(x,t)-\psi \le \frac{-C_3\psi}{1+t}. \tag{4.70}$$

where $c_m = \left(1-\underline{\varphi}(t_m)^{1-n\alpha}\right)\underline{\varphi}(t_m)^{n\alpha}/(n\alpha-1) < \infty$ uniformly in m due to (4.67) . The same argument as in [59] allows one to let $t_m \to \infty$ and deduce (4.12), hence completing the proof of Theorem 4.2.2. $\square$

Remark 4.5.1. *Similarly to [9] one sees the sharpness of the estimate* (4.70) *by considering the function* $u(x,t) = (s+t)^{\frac{1}{n\alpha-1}}\psi(x)$ *for any* $s > 0$.

Remark 4.5.2. *Corollary 4.2.3 with* $C_4 = \underline{\varphi}(0)^{1-n\alpha}$ *follows from* $\underline{u}(x,t) \leq u(x,t)$, *namely,*

$$G^{\frac{1}{1-n\alpha}}(\underline{\varphi}(0)^{1-n\alpha} + t)^{\frac{1}{1-n\alpha}}\psi(x) \leq u(x,t).$$

Parts of Chapter 4 are joint work with Dr. Kui Wang, which was submitted to a journal and currently under review.

Chapter 5

Classification of Shrinking Gradient Ricci Solitons

5.1 Introduction

Over the last three decades, Ricci Flow has emerged as an indispensable tool for addressing classical problems in differential geometry, topology, and geometric analysis. It has been an extremely active field since the work of R. Hamilton for Ricci flow[35] in 1982 to prove that every closed 3-manifold with positive Ricci curvature is diffeomorphic to a 3-spherical form. Many important and difficult problems were solved using geometric flows, such as the Poincaré conjecture and geometrization conjecture by G. Perelman [60] [61] [62], the solution to the conjecture of H. Rauch and R. Hamilton by C. Böhm and B. Wilking [12] and its consequences, and the quarter-pinched differentiable sphere theorem by S. Brendle and R. Schoen[13].

One of the most important aspects of understanding the effects of Ricci flow in geometry and topology is to understand the formation of singularities in Ricci flow. Gradient Ricci solitons arise naturally in the study of the singularity analysis of the Ricci flow [4, 15]. This is our main motivation for studying and classifying gradient Ricci solitons, although they can also be viewed as generalizations of Einstein metrics. There are three types of gradient Ricci solitons and we will focus on shrinking gradient Ricci solitons as they model Type-I singularities.

A gradient Ricci soliton is a triple (M, g, f), a complete Riemannian manifold with a smooth potential function satisfying

$$\mathrm{Ric} + \nabla\nabla f = \frac{\lambda}{2} g. \tag{5.1}$$

By normalizing the metric, one can always assume that $\lambda = 0, 1$ or -1. When $\lambda = 1$, the solitons is said to be shrinking. When $\lambda = 0$, it is called steady and when $\lambda = -1$, it is called expanding.

The main theme of this chapter is to study classification of shrinking gradient Ricci solitons. In section 2, we give some examples shrinking gradient Ricci solitons. Section 3 collects basic properties of gradient shrinking Ricci solitons, including basic equations satisfied by them, the growth estimate of the potential function, volume growth estimate, integral and pointwise bounds on the curvature, and the evolution equations of some natural geometric quantities that will play an important role in later sections. Section 4 presents a Liouville type theorem for the weighted Laplacian operator Δ_f on complete (not necessarily compact) manifolds which generalizes the maximum principle for elliptic equations. Since it will be frequently used in proving many classification results, we also include a proof of it here. In section 5, we give classification of gradient shrinking Ricci solitons in dimension three.In section 6, we present the main result of this chapter, which is the classification of four-dimensional shrinking gradient Ricci solitons with positive or nonnegative isotropic curvature. This is joint work with L. Ni and K. Wang [45]. The proof is divided into three steps to better illustrate the ideas behind the proof. In section 7, we collect other known classification results on shrinking gradient Ricci solitons up to date. Their proofs can be easily found in the literature.

At last, we remark that a lot of work has been done on understanding steady and expanding gradient Ricci solitons as well.

5.2 Examples of Shrinking Gradient Ricci Solitons

In this section, we give some examples of shrinking gradient Ricci solitons. It is essential to keep the known examples in mind when it comes to classification results.

Example 5.2.1. *The Gaussian shrinking soliton* $(\mathbb{R}^n, g_E, f)$, *where* g_E *is the standard metric on* $\mathbb{R}^n$ *and* $f(x) = \frac{1}{4}|x|^2$. *Although the geometry of a Gaussian shrinking soliton is trivial, its role in the analysis of both gradient Ricci solitons and Ricci flow is nontrivial.*

Example 5.2.2. *Any Einstein Manifold* M^n *with positive Ricci curvature normalized so that* $\mathrm{Ric} = \frac{1}{2}g$, *is a shrinking gradient Ricci soliton with the potential function* $f = 0$. *Such examples include the round spheres* $\mathbb{S}^n$ *and its quotients.*

Example 5.2.3. *Product solitons. If* $(M_1^{n_1}, g_1, f_1)$ *and* $(M_2^{n_2}, g_2, f_2)$ *are shrinking gradient Ricci solitons, then* $(M_1 \times M_2, g_1 \times g_2, f_1 \circ p_1 + f_2 \circ p_2)$, *where* $p_i : M_1 \times M_2 \to M_i$ *for* $i = 1, 2$ *are the projections, is a shrinking gradient Ricci soliton. Some particular examples of this type are the generalized cylinders* $\mathbb{S}^k \times \mathbb{R}^{n-k}$.

Example 5.2.4. *Quotient solitons. Suppose that* (M, g, f) *is a shrinking gradient Ricci soliton and* Γ *is a discrete group of isometries of* g *acting freely and properly discontinuously and such that* $f \circ \gamma$ *for every* $\gamma \in \Gamma$. *Let* g_0 *denote the quotient Riemannian metric on* M/Γ *and* f_0 *denote the unique smooth function on* M/Γ *satisfying* $f_0 \circ \pi = f$, *where* $\pi : M \to M/\Gamma$ *is the quotient map. Then* $(M/\Gamma, g_0, f_0)$ *is a shrinking gradient Ricci soliton.*

The above examples are"standard". In dimension greater or equal to 4, there are non-standard examples.

Example 5.2.5. *The first example of compact non-Einstein shrinking gradient*

Ricci soliton was found by H.D. Cao [16] and N. Koiso [42] independently. In particular, in real dimension four, there is a shrinking Kähler-Ricci soliton structure on $\mathbb{C}P^2\sharp(-\mathbb{C}P^2)$.

Example 5.2.6. *M. Feldman, T. Ilmanen and D. Knopf [32] discovered the first example of noncompact non-Einstein shrinking gradient Ricci soliton. Their examples are a family of Kähler-Ricci solitons with* $U(n)$ *symmetry and a cone-like end at infinity on the twisted line bundle over* $\mathbb{C}P^{n-1}$.

We note that there are other examples besides the above mentioned ones.

5.3 Properties of Shrinking Gradient Ricci Solitons

In this section, we collect some results on gradient shrinking Ricci solitons that will be used in this paper. The results below are valid in all dimensions $n \geq 2$.

After normalizing the potential function f via translating, we have the following identities first discovered by R. Hamilton [36]:

Lemma 5.3.1.

$$S + \Delta f = \frac{n}{2},$$

$$S + |\nabla f|^2 = f,$$

where S *denotes the scalar curvature of* M.

Regarding the growth of the potential function f and the volume of geodesic balls, H.D. Cao and D. Zhou [18] showed that

Lemma 5.3.2. *Let (M, g) be a complete gradient shrinking Ricci soliton and $p \in M$. Then there are positive constants c_1, c_2 and C such that*

$$\frac{1}{4}\left(d(x,p) - c_1\right)_+^2 \leq f(x) \leq \frac{1}{4}\left(d(x,p) + c_2\right)^2,$$

$$\mathrm{Vol}(B_p(r)) \leq C r^n.$$

We have the following bounds for the scalar curvature due to B.L. Chen [22]. In fact, he proved that any complete ancient solution of the Ricci flow has nonnegative scalar curvature. In dimension three, he proved that any complete ancient solutions of the Ricci flow must have nonnegative sectional curvature.

Lemma 5.3.3. *Let (M, g) be a complete gradient shrinking Ricci soliton. Then we have*

$$S \geq 0,$$

where S denotes the scalar curvature of M.

A consequence of the strong maximum principle is that if $S(p) = 0$ for some $p \in M$, then M is isometric to the Gaussian shrinking gradient Ricci soliton. An improvement of the scalar curvature lower bound is obtained by B. Chow, P. Lu and B. Yang [27].

Lemma 5.3.4. *Let (M, g) be a complete non-flat gradient shrinking Ricci soliton. Then for any given point $p \in M$, there exists a constant $C > 0$ such that $S(x)d(x,p)^2 \geq C^{-1}$ wherever $d(x,p) \geq C$, where S denotes the scalar curvature of M.*

Obtaining curvature upper bounds turns out to be one of the most challenging questions in the study of Ricci solitons, although the most optimistic conjecture is that all shrinking Ricci solitons have bounded Riemann curvature tensor. O. Munteanu and N. Sesum [50] were able to obtain the following integral bound for the Ricci curvature. A proof of this lemma is included below since it plays an important role in the proofs of many classification results.

Lemma 5.3.5. *Let (M, g) be a complete gradient shrinking Ricci soliton. Then for any $\lambda > 0$, we have*

$$\int_M |\operatorname{Ric}|^2 e^{-\lambda f} < \infty.$$

Proof. For any smooth function ϕ with compact support, we have

$$\begin{aligned}
& \int_M |\operatorname{Ric}|^2 e^{-\lambda f} \phi^2 \\
= \; & \int_M R_{ij} \left(\frac{1}{2} g_{ij} - \nabla_i \nabla_j f \right) e^{-\lambda f} \phi^2 \\
= \; & \frac{1}{2} \int_M S e^{-\lambda f} \phi^2 + \int_M \nabla_i f \nabla_j (R_{ij} e^{-\lambda f} \phi^2) \\
= \; & \frac{1}{2} \int_M S e^{-\lambda f} \phi^2 + (1-\lambda) \int_M R_{ij} \nabla_i f \nabla_j f e^{-\lambda f} \phi^2 + \int_M R_{ij} \nabla_i f \nabla_j (\phi^2) e^{-\lambda f} \\
\leq \; & \frac{1}{2} \int_M S e^{-\lambda f} \phi^2 + \frac{1}{4} \int_M |\operatorname{Ric}|^2 e^{-\lambda f} \phi^2 + |1-\lambda| \int_M |\nabla f|^4 e^{-\lambda f} \phi^2 \\
& + \frac{1}{4} \int_M |\operatorname{Ric}|^2 e^{-\lambda f} \phi^2 + 4 \int_M |\nabla f|^2 e^{-\lambda f} |\nabla \phi|^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality twice in the last inequality. Now note that from Lemma 5.3.1 and Lemma 5.3.2, we know that $\int_M Se^{-\lambda f} < \infty$ and $\int_M |\nabla f|^4 e^{-\lambda f} < \infty$ for any $\lambda > 0$. Thus $\int_M |\operatorname{Ric}|^2 e^{-\lambda f} < \infty$ for any $\lambda > 0$. $\square$

When studying classification results, it is natural to look for quantities that satisfy nice PDEs and try to apply the maximum principle. We have the following evolution equations. See, for instance, [56] for their proofs.

Proposition 5.3.1. *Let $(M, g(t))$ be a solution to the Ricci flow. Assume that $S > 0$. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) = \frac{4P_0}{S^3} - \frac{2}{S^4}|S\nabla_p R_{ij} - \nabla_p S R_{ij}|^2 + \left\langle \nabla \frac{|\operatorname{Ric}|^2}{S^2}, \nabla \log S^2 \right\rangle, \tag{5.2}$$

where

$$P_0 = SR_{ijkl}R_{ik}R_{jl} - |\operatorname{Ric}|^4. \tag{5.3}$$

Proposition 5.3.2. *Let $(M, g(t))$ be a solution to the Ricci flow. Assume that $S > 0$. Then*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|R_{ijkl}|^2}{S^2}\right) =& \frac{4P}{S^3} - \frac{2}{S^4}|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 \\ &+ \left\langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2 \right\rangle,\end{aligned}$$

where P is defined by

$$P = 4S\langle R^2 + R^\sharp, R\rangle - |\operatorname{Ric}|^2 |R_{ijkl}|^2.$$

The above evolution equations under Ricci flow can be applied to shrinking gradient Ricci soitons. In fact, gradient Ricci solitons are self-similar solutions of Ricci flow and can be interpreted as fixed points of the Ricci flow in the space of metrics modulo the actions of diffeomorphisms and scalings. A canonical form for the associated time-dependent version of a gradient Ricci soliton is given in [26, Theorem 4.1]. In particular, on a gradient Ricci soliton, every scaling invariant smooth function h satisfies

$$\frac{\partial h}{\partial t} = \langle \nabla f, \nabla h \rangle.$$

Thus the heat operator $\frac{\partial}{\partial t} - \Delta$ is the same as $-\Delta_f$. As one can see from the above equation, the key to obtain the classification results is to show that $P_0 \leq 0$ or $P \leq 0$. Then one can apply a Liouville type theorem in the next section to conclude that we indeed must have $P_0 = 0$ or $P = 0$. This will give enough information to obtain the classification result. When we have no assumptions on the curvature growth, we need to justify carefully that the Liouville theorem is applicable when the manifold is noncompact. That is where we need the integral bound on Ricci curvature in Lemma 5.3.5.

5.4 A Liouville Theorem

In this section, we prove the following Liouville Theorem, which turns out to be useful in studying Ricci Solitons since the operator Δ_h defined by $\Delta_h u :=$

$\Delta u - \langle \nabla h, \nabla u\rangle$ arises naturally on Ricci solitons.

Theorem 5.4.1. *Let (M, g, h) be a Riemannian manifold with a smooth function h satisfying $\int_M e^{-h} d\mu_g < \infty$. Let u be a locally Lipschitz function in $L^2(e^{-h} d\mu_g)$ which is bounded from below and satisfies*

$$\Delta_h u \geq 0$$

in the sense of distribution, where $\Delta_h u := \Delta u - \langle \nabla h, \nabla u\rangle$. Then u is a constant.

Proof. By adding a constant to u, we may assume u is nonnegative. Let η be a smooth cut-off function with support in $\{x \in M : d(x, x_0) \leq r\}$ and $|\nabla \eta| \leq C/r$. Multiplying the above inequality by $u\eta^2 e^{-h}$ and integrating over M yields

$$0 \leq \int_M u\eta^2 \Delta u \ e^{-h} \leq -\int_M |\nabla u|^2 \eta^2 e^{-h} - \int_M 2\eta u \langle \nabla u, \nabla \eta\rangle u e^{-h},$$

where we used integration by parts. Note that

$$\begin{aligned} -\int_M 2u\eta\langle \nabla u, \nabla \phi\rangle e^{-h} &\leq \frac{1}{2}\int_M |\nabla u|^2 \eta^2 e^{-h} + 2\int_M |\nabla \eta|^2 u^2 e^{-h} \\ &\leq \frac{1}{2}\int_M |\nabla u|^2 \eta^2 e^{-h} + \frac{C}{r^2}\int_M u^2 e^{-h}. \end{aligned}$$

Combining the above two inequalities together, we obtain

$$0 \leq \int_M u\eta^2 \Delta u \ e^{-h} \leq -\frac{1}{2}\int_M |\nabla u|^2 \eta^2 e^{-h} + \frac{C}{r^2}\int_M u^2 e^{-h}.$$

Therefore, we know that $|\nabla u| = 0$ in the sense of distribution by letting $r \to \infty$. It follows that u must be a constant since it is locally Lipschitz. $\square$

Remark 5.4.2. *The above Liouville theorem also holds if the inequality $\Delta_f u \geq 0$ is assumed to hold in the sense of viscosity. This is because the inequality then also holds in the sense of distribution. See [41].*

Remark 5.4.3. *The above Liouville theorem also holds if the inequality $\Delta_f u \geq 0$ is assumed to hold in the sense of barriers. See P. Petersen and W. Wylie [63, Theorem 4.2].*

5.5 Classification of Shrinking Solitons in Dimension Three

Theorem 5.5.1. *Let (M^3, g, f) be a complete gradient shrinking Ricci soliton. Then M is either isometric to $\mathbb{R}^3$ or it is a quotient of $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$.*

This classification result is due to Perelman[62] in the compact case. L. Ni and N. Wallach provided an alternative proof in [56] which also works in the complete case, but they had to assume that M has nonnegative Ricci curvature to ensure certain integrals are finite. By appealing to the Ricci integral bound in Lemma 5.3.5, one can easily remove the nonnegative Ricci curvature assumption. Alternative proof are also obtained by other groups.

Proof. In dimension 3, Proposition 5.3.1 was due to Hamilton [35] and under an orthonormal frame that diagonalizes the Ricci tensor, the quantity P_0 can be written as

$$P_0 = -\frac{1}{8}\left((\mu+\nu-\lambda)^2(\mu-\nu)^2 + (\lambda+\nu-\mu)^2(\lambda-\nu)^2 + (\lambda+\mu-\nu)^2(\lambda-\mu)^2\right) \tag{5.4}$$

where λ, μ and ν are eigenvalues of Ricci tensor. Thus one clearly has $P \leq 0$. In order to apply Theorem 5.4.1, we need to consider the function $u = \frac{|\operatorname{Ric}|}{S}$, instead fo the function $\frac{|\operatorname{Ric}|^2}{S^2}$. A direct calculation using Kato's inequality shows that

$$\Delta_h u \geq 0$$

with $h = f - \log S^2$. By Lemma 5.3.5, the function $u \in L^2(e^{-h} d\mu_g)$ and satisfies all assumptions in Theorem 5.4.1 and therefore we can conclude that u is a constant. Thus u^2 also being a constant implies that $P_0 = 0$ and $|S\nabla_p R_{ij} - \nabla_p S R_{ij}|^2 = 0$. The theorem then follows easily as in [56]. $\square$

Remark 5.5.2. *A similar argument was used by G. Huisken [39] and T.Colding and W. Minicozzi [28] in the classification of mean convex shrinking solitons of mean curvature flow in* $\mathbb{R}^{n+1}$.

5.6 Classification of Four-dimensional Shrinking Solitons with Positive Isotropic Curvature

In four dimensions, it becomes very difficult to obtain a classification result without any curvature assumptions because of the existence of non-standard examples. Also, the present of the Weyl tensor makes the analysis much harder than in dimensions two and three. Under the assumption of positive isotropic curvature and nonnegative curvature operator together with some reasonable curvature growth conditions, L. Ni and N. Wallach [57] managed to prove the following result.

Theorem 5.6.1. *Let (M, g, f) be a four-dimensional complete gradient shrinking soliton with positive isotropic curvature and nonnegative curvature operator. Moreover, assume that the components of the curvature operator satisfy*

$$\frac{B_3^2}{(A_1 + A_2)(C_1 + C_2)}(x) \leq \exp(a(r(x) + 1)),$$

and

$$|R_{ijkl}(x)| \leq \exp(br(x) + 1)),$$

for some positive constants a and b, where $r(x)$ is the distance to a fixed point. Then M is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^2 \times \mathbb{R}$.

Later, A. Naber [53] was able to classify four-dimensional shrinking gradient Ricci solitons with bounded and nonnegative curvature operator.

Theorem 5.6.2. *Let (M, g, f) be a four-dimensional complete gradient shrinking soliton with bounded nonnegative curvature operator, then its universal cover must be one of the following spaces $\mathbb{R}^4$, $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$ and $\mathbb{S}^3 \times \mathbb{R}$.*

It is then natural to ask that whether the three assumptions except positive isotropic curvature in the classification theorem of L. Ni and N. Wallach can be removed. The positive isotropic curvature condition was first introduced by M. Micalleff and J. Moore [49] in applying the index computation of harmonic spheres to the study of the topology of manifolds. The Ricci flow on four-manifolds with positive isotropic curvature was studied by R. Hamilton [37]. This condition was proven to be invariant under Ricci flow in dimension four by R. Hamilton and in high dimensions by S. Brendle and R. Schoen [13] and H. Nguyen [54]. It is hence then interesting to understand the solitons under the positive isotropic curvature condition. Since the classification theorem of L. Ni and N. Wallach, there have been much progresses in understanding the general shrinking solitons [43] [52] and particularly the four-dimensional ones [51]. In particular, a classification result was obtained in [52] for solitons with nonnegative curvature operator for all dimensions. The main result of this chapter is the following classification result on shrinking solitons with positive isotropic curvature by removing all the additional assumptions in [57]. The result was obtained in [45] by L. Ni, K. Wang and the author.

Theorem 5.6.3. *Any four-dimensional complete gradient shrinking Ricci soliton with positive isotropic curvature is either a quotient of* $\mathbb{S}^4$ *or a quotient of* $\mathbb{S}^3 \times \mathbb{R}$.

The strong maximum principle together with the classification of positive case implies the following corollary for the solitons with nonnegative isotropic curvature.

Corollary 5.6.4. *If* (M, g, f) *is a complete gradient shrinking soliton with non-*

negative isotropic curvature then its universal cover must be one of the following spaces $\mathbb{R}^4$, $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$ *and* $\mathbb{S}^3 \times \mathbb{R}$.

This corollary improves the result of A. Naber [53] because nonnegative curvature operator implies nonnegative isotropic curvature. The proof of our classification result is not short, and we divide it into three steps to present.

5.6.1 Step 1: $B^t B = b^2 \,\mathrm{id}$

It is well known that, in dimension four, the curvature operator R can be written as

$$R = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

according to the natural splitting $\wedge^2(\mathbb{R}^4) = \wedge_+ \oplus \wedge_-$, where $\wedge_+$ and $\wedge_-$ are the self-dual and anti-self-dual parts respectively. It is easy to see that A and C are symmetric. Denote $A_1 \leq A_2 \leq A_3$ and $C_1 \leq C_2 \leq C_3$ the eigenvalues of A and C, respectively. Also let $B_1 \leq B_2 \leq B_3$ be the singular eigenvalues of B. Note that a direct consequence of the first Bianchi identity is that $\mathrm{tr}(A) = \mathrm{tr}(C) = \frac{S}{4}$, where S is the scalar curvature. In [57], L. Ni and N. Wallach computed that $\overset{\circ}{\mathrm{Ric}}$ can be expressed in terms of components of B, where $\overset{\circ}{\mathrm{Ric}}$ is the traceless part of the Ricci tensor. In particular, $4\|B\|^2 = |\overset{\circ}{\mathrm{Ric}}|^2$ and $\sum_1^4 \lambda_i^3 = 24 \det B$, where λ_i's are the eigenvalues of $\overset{\circ}{\mathrm{Ric}}$. The first step is to show that on a four-dimensional shrinking gradient Ricci soliton, it holds that $B^t B$ is a multiple of the identity.

Proposition 5.6.1. *Let M be a four-dimensional shrinking gradient Ricci soliton with positive isotropic curvature. Then $BB^t = b^2\,\mathrm{id}$.*

The idea to prove this identity is to find quantities satisfying nice differential equations or inequalities and apply Liouville type theorem. Before giving the proof, we recall certain partial differential inequalities satisfied by components of the curvature operator under Ricci flow. The inequalities below were observed in [57, Proposition 3.1] and they play a significant role in the classification results.

Proposition 5.6.2. *Let $(M, g(t))$ be a solution to the Ricci flow, then we have the following differential inequalities*

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(A_1 + A_2) &\geq A_1^2 + A_2^2 + 2(A_1 + A_2)A_3 + B_1^2 + B_2^2, \\
\left(\frac{\partial}{\partial t} - \Delta\right)(C_1 + C_2) &\geq C_1^2 + C_2^2 + 2(C_1 + C_2)C_3 + B_1^2 + B_2^2, \\
\left(\frac{\partial}{\partial t} - \Delta\right)B_3 &\leq A_3B_3 + C_3B_3 + 2B_1B_2
\end{aligned}$$

in the distributional sense.

For brevity, we introduce the same notations as in [57]: $\psi_1 = A_1 + A_2$, $\psi_2 = C_1 + C_2$, $\varphi = B_3$ and

$$\begin{aligned}
-E = & -\frac{4B_1(B_3 - B_2)}{B_3} - \frac{(A_1 - B_1)^2 + (A_2 - B_2)^2 + 2A_2(B_2 - B_1)}{A_1 + A_2} \\
& -\frac{(C_1 - B_1)^2 + (C_2 - B_2)^2 + 2C_2(B_2 - B_1)}{C_1 + C_2}.
\end{aligned}$$

It's clear that $-E \leq 0$ with equality holds only if $A_1 = C_1 = B_1 = B_2 = A_2 = C_2 = B_3$. In particular, $E = 0$ implies $BB^t = b^2\,\mathrm{id}$ for some b. Also notice that M

has positive isotropic curvature amounts to $\psi_1 > 0$ and $\psi_2 > 0$. Then we have the following proposition.

Proposition 5.6.3. *The following differential inequality holds in the sense of distribution:*

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} \quad\le\quad & -\frac{1}{2}\frac{\varphi}{\sqrt{\psi_1\psi_2}}E-\frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}} \\ & +\left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle.\end{aligned}\tag{5.5}$$

Proof. Straightforward calculations yield

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} \quad=\quad & \frac{\left(\frac{\partial}{\partial t}-\Delta\right)\varphi}{\sqrt{\psi_1\psi_2}}-\frac{1}{2}\frac{\varphi\psi_1\left(\frac{\partial}{\partial t}-\Delta\right)\psi_2+\varphi\psi_2\left(\frac{\partial}{\partial t}-\Delta\right)\psi_1}{(\psi_1\psi_2)^{\frac{3}{2}}} \\ & -\frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}}+\left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle\end{aligned}$$

Substituting the differential inequalities in Proposition 5.6.3 into the above equation gives, after some cancelations, that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} \quad\le\quad & -\frac{1}{2}\frac{\varphi}{\sqrt{\psi_1\psi_2}}E-\frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}} \\ & +\left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle.\end{aligned}$$

This finishes the proof. □

Now we are ready to prove Proposition 5.6.1.

Proof of Proposition 5.6.1. Let $u=\frac{\varphi}{\sqrt{\psi_1\psi_2}}$. Note that on the shrinking gradient Ricci soliton,

$$\frac{\partial u}{\partial t}=\langle\nabla f,\nabla u\rangle.$$

Proposition 5.6.3 then implies that u satisfies

$$\Delta_h u \geq \frac{1}{2} E u \geq 0$$

in the sense of distribution with $h = e^{-f+\log(\psi_1\psi_2)}$.

Since $4\varphi^2 = 4B_3^2 \leq 4\|B\|^2 = |\overset{\circ}{\mathrm{Ric}}|^2 = |\operatorname{Ric}|^2 - \frac{S^2}{4}$, we obtain, in view of Lemma 5.3.1, Lemma 5.3.2 and Lemma 5.3.5,

$$\int_M u^2 e^{-h} = \int_M \varphi^2 e^{-f} \leq \frac{1}{4} \int_M \left(|\operatorname{Ric}|^2 - \frac{S^2}{4} \right) e^{-f} < \infty.$$

This verifies that $u \in L^2(e^{-h} d\mu_g)$ and we can apply Theorem 5.4.1 to conclude that u must be constant. Therefore, we must have either $\varphi = B_3 = 0$ or $E = 0$. It then follows that $BB^t = b^2 \operatorname{id}$. □

5.6.2 Step 2: $P \leq 0$

Recall from Proposition 5.3.2 that if $S \neq 0$, then

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \left(\frac{|R_{ijkl}|^2}{S^2} \right) =& \frac{4P}{S^3} - \frac{2}{S^4} |S \nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 \\ &+ \left\langle \nabla \left(\frac{|R_{ijkl}|^2}{S^2} \right), \nabla \log S^2 \right\rangle, \end{aligned}$$

where P is defined by

$$P = 4S \langle R^2 + R^\sharp, R \rangle - |\operatorname{Ric}|^2 |R_{ijkl}|^2.$$

Proposition 5.6.4. *Let M be a four-dimensional gradient shrinking Ricci soliton with positive isotropic curvature. Then $P \leq 0$.*

Proof. In dimension four, P can be expressed in terms of A, B and C. Let $\overset{\circ}{A}$ and $\overset{\circ}{C}$ be the traceless parts of A and C, respectively. By choosing suitable basis of $\wedge_+$ and $\wedge_-$, we may diagonalize $\overset{\circ}{A}$ and $\overset{\circ}{C}$ such that

$$A = \begin{pmatrix} \frac{S}{12}+a_1 & 0 & 0 \\ 0 & \frac{S}{12}+a_2 & 0 \\ 0 & 0 & \frac{S}{12}+a_3 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{S}{12}+c_1 & 0 & 0 \\ 0 & \frac{S}{12}+c_2 & 0 \\ 0 & 0 & \frac{S}{12}+c_3 \end{pmatrix}.$$

Then P can be written as (See [57])

$$\begin{aligned} P &= -S^2\left(\frac{1}{6}\sum_1^4 \lambda_i^2 + \sum_1^3 a_i^2 + \sum_1^3 c_i^2\right) \\ &\quad +4S\left(\sum_1^3 (a_i^3+c_i^3) + 6a_1a_2a_3 + 6c_1c_2c_3 - \frac{1}{2}\sum_1^4 \lambda_i^3\right) \\ &\quad +12S\left(a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + c_1\tilde{b}_1^2 + c_2\tilde{b}_2^2 + c_3\tilde{b}_3^2\right) \\ &\quad -2\left(\sum_1^4 \lambda_i^2\right)^2 - 4\left(\sum_1^4 \lambda_i^2\right)\left(\sum_1^3 (a_i^2+c_i^2)\right), \end{aligned}$$

where $\sum_1^3 a_i = \sum_1^3 c_i = \sum_1^4 \lambda_i = 0$, $b_i^2 = \sum_{j=1}^3 B_{ij}^2$ and $\tilde{b}_i^2 = \sum_{j=1}^3 B_{ji}^2$.

After plugging into $BB^t = b^2\,\mathrm{id}$, P has a much simpler expression:

$$\begin{aligned} P &= -S^2\left(\sum_1^3 a_i^2 + \sum_1^3 c_i^2\right) + 12S\left(\sum_1^3 (a_i^3+c_i^3)\right) \\ &\quad -2b^2\left(S+12b\right)^2 - 48b^2\left(\sum_1^3 a_i^2 + \sum_1^3 c_i^2\right) \\ &\le -S\left(S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3\right) - S\left(S\sum_1^3 c_i^2 - 12\sum_1^3 c_i^3\right), \end{aligned}$$

where we have used the following

$$
\begin{aligned}
&3b^2 = \sum_1^3 b_i^2 = \sum_1^3 \tilde{b}_i^2 = \frac{1}{4}\sum_1^4 \lambda_i^2,\\
&\sum_1^3 a_i^3 + 6a_1a_2a_3 = 3\sum_1^3 a_i^3,\\
&\sum_1^4 \lambda_i^3 = 24\det B = 24b^3.
\end{aligned}
$$

In order to prove $P \leq 0$, it suffices to show that

$$S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \geq 0.$$

With suitable choices of the orthonormal basis for $\wedge_+$, we can assume that $A_i = \frac{S}{12} + a_i$.Note that we have the constraints $\sum_1^3 a_i = 0$ and $A_i + A_j = \frac{S}{6} + a_i + a_j > 0$ for $i \neq j$ because M has positive isotropic curvature. By the change of variables $x_i = 1 - \frac{6}{S}a_i$, the constraints become $\sum_1^3 x_i = 3$ and $x_i > 0$ for $1 \leq i \leq 3$, and the objective function becomes

$$
\begin{aligned}
F(x_1, x_2, x_3) \quad &:= \quad S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 = \frac{S^3}{36}\left(\sum_1^3 (1-x_i)^2 - 2(1-x_i)^3\right)\\
&= \quad \frac{S^3}{36}\left(2\sum_1^3 x_i^3 - 5\sum_1^3 x_i^2 + 9\right).
\end{aligned}
$$

Using Lagrange multipliers, we find two critical points $Z = (1,1,1)$ and $W = \left(\frac{1}{3}, \frac{4}{3}, \frac{4}{3}\right)$ with $F(Z) = 0$ and $F(W) = \frac{S^3}{162}$. On the boundary, we have $x_i = 0$. Since F is symmetric, we can assume without loss of generality that $x_1 = 0$. Then we have, using $x_3 = 3 - x_2$, that

$$F(x_1, x_2, x_3) = \frac{S^3}{36}\left(2(x_2^3 + (3-x_2)^3) - 5(x_2^2 + (3-x_2)^2) + 9\right) = \frac{S^3}{18}(2x_2-3)^2 \geq 0.$$

Therefore, under the constraints $\sum_1^3 a_i = 0$ and $\frac{S}{6} + a_i + a_j \geq 0$ for $i \neq j$,

$$F(x_1, x_2, x_3) = S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \geq 0.$$

The terms involving c_i's can be handled similarly. Hence $P \leq 0$. □

5.6.3 The Proof of Theorem 5.6.3

Proof. Recall that if $S \neq 0$, then

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|R_{ijkl}|^2}{S^2}\right) =& \frac{4P}{S^3} - \frac{2}{S^4}|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 \\ &+ \left\langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2 \right\rangle,\end{aligned} \tag{5.6}$$

Trying to apply Theorem 5.4.1 here would require a stronger integral bound of the Ricci curvature than the one we actually have in Lemma 5.3.5. To overcome this difficulty, we adopt a similar idea that was used to prove $BB^t = b^2\,\mathrm{id}$. We consider, instead, the function $u = \frac{|R_{ijkl}|}{S}$ and $T = \frac{R_{ijkl}}{S}$. A direct calculation shows that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)u &= \frac{2P}{uS^3} + \langle \nabla u, \nabla \log S^2\rangle + \frac{|\nabla u|^2 - |\nabla T|^2}{u} \\ &\leq \frac{2P}{uS^3} + \langle \nabla u, \nabla \log S^2\rangle,\end{aligned}$$

where we have used Kato's inequality in the last line. On a shrinking gradient Ricci soliton, it then holds that

$$\Delta_h u \geq \frac{-2P}{uS^3} \geq 0,$$

where $h = f - \log S^2$. We need to verify that $u \in L^2(e^{-h}d\mu_g)$. The integral $\int_M u^2 e^{-h} = \int_M |R_{ijkl}|^2 e^{-f}$ is finite in view of Lemma 5.3.5, since if M has positive isotropic curvature, then the components of curvature operator A, B and C can be estimated by

$$\begin{aligned}
&-\frac{S}{4} \leq A_1 \leq A_2 \leq A_3 \leq \frac{S}{4},\\
&-\frac{S}{4} \leq C_1 \leq C_2 \leq C_3 \leq \frac{S}{4},\\
&4\|B\| \leq |\overset{\circ}{\mathrm{Ric}}|.
\end{aligned}$$

Therefore we know that u is a positive constant and $P = 0$. Then it follows from (5.6) that $|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 = 0$. Theorem 5.6.3 then follows from the proof of the main theorem in [56]. $\square$

5.7 Some Other Classification Results

In this section, we collect other known classification results on shrinking gradient Ricci solitons, but we will not give their proofs in this thesis. In higher dimensions, the problem of classifying shrinking gradient Ricci solitons becomes difficult because of the present of Weyl curvature tensor. Besides assuming the Weyl curvature tensor vanishes as in the locally conformally flat case, one can also obtain classification results by imposing vanishing conditions on the derivatives of Weyl tensor.

5.7.1 Classification Results in Dimension Four

First of all, there are several classification results in four dimensions.

Theorem 5.7.1. *Any four-dimensional complete gradient shrinking Ricci soliton which is half locally conformally flat, is a finite quotient of* $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^3 \times \mathbb{R}$ *or* $\mathbb{R}^4$.

This is proved by X. Chen and Y. Wang [23].

Theorem 5.7.2. *Any four-dimensional complete gradient shrinking Ricci soliton having half harmonic Weyl tensor is either Einstein or a finite quotient of* $\mathbb{R}^4$, $\mathbb{S}^2 \times \mathbb{R}^2$ *or* $\mathbb{S}^3 \times \mathbb{R}$.

This is due to J.Y. Wu, P. Wu and W. Wylie [66].

Recall for any n-dimensional Riemannian manifolds (M, g) with $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla_i \nabla_k W_{ijkl} + \frac{1}{n-2} R_{ik} W_{ijkl}. \tag{5.7}$$

For Bach-flat gradient shrinking solitons, H.D. Cao and Q. Chen [17] obtained the following classification results.

Theorem 5.7.3. *Any four-dimensional complete gradient shrinking Ricci soliton which is Bach flat is either Einstein or locally conformally flat.*

5.7.2 Classification Results in Higher Dimensions

Below are some classification results in high dimensions. The first result is the classification of gradient shrinking solitons which are locally conformally flat.

Theorem 5.7.4. *Let (M^n, g, f), $n \geq 4$, be a complete gradient shrinking Ricci soliton. If (M, g) is locally conformally flat, then M is either isometric to $\mathbb{R}^n$ or it is a quotient of $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

This theorem was proved by L. Ni and N. Wallach [56], assuming M has nonnegative Ricci curvature. Alternative proofs are given by X. Cao, B. Wang and Z. Zhang [19] requiring only Ricci curvature bounded from below, and by P. Petersen and W. Wylie [63] requiring only an Ricci curvature integral bound. In [67], it was shown by Z. H. Zhang that any gradient shrinking soliton vanishing Weyl tensor must have nonnegative curvature operator, thus having nonnegative Ricci curvature, which by any of the results mentioned above proves the classification result. The assumptions on Ricci curvature can also be removed by using Lemma 5.3.5.

Generalizing all previous results concerning locally conformally flat gradient shrinking solitons, G. Catino [21] showed the same classification result under a pinching condition on the Weyl tensor.

Theorem 5.7.5. *Any n-dimensional complete gradient shrinking Ricci soliton satisfying*

$$|W|S \leq \sqrt{\frac{2(n-1)}{n-2}} \left(|\mathring{\mathrm{Ric}}| - \frac{S}{\sqrt{n(n-1)}} \right)^2 \tag{5.8}$$

must be a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$.

To state the rest results, we introduce the following definition for brevity.

Definition 5.7.6. *A gradient Ricci soliton is said to be rigid if it is isometric to a quotient of* $N \times \mathbb{R}^k$*, where* N *is an Einstein manifold and* $f = \frac{\lambda}{2}|x|^2$ *on the Euclidean factor.*

Theorem 5.7.7. *Any* n*-dimensional complete gradient shrinking Ricci soliton having harmonic Weyl curvature tensor is rigid.*

This is proved by M. Fernandez-Lopez and E. Garcia-Rio with one additional assumption, which was removed by O. Munteanu and N. Sesum in [50].

Theorem 5.7.8. *Any* n*-dimensional* $(n \geq 5)$ *complete gradient shrinking Ricci soliton which is Bach flat is either Einstein, or a quotient of* $\mathbb{R}^n$*, or a finite quotient of* $N^{n-1} \times \mathbb{R}$*, where* N *is an Einstein manifold of positive scalar curvature.*

Very recently, G. Catino, P. Mastrolia and D. Monticelli [20] are able to classify gradient shrinking solitons satisfying a fourth-order vanishing condition, improving previously known results.

Theorem 5.7.9. *Any* n*-dimensional* $(n \geq 4)$ *complete gradient shrinking Ricci soliton with* $div^4(W) := \nabla_j \nabla_k \nabla_l \nabla_i W_{ijkl} = 0$ *is either Einstein, or isometric to a finite quotient of* $N^{n-k} \times \mathbb{R}^k$ *with* $k > 0$*, where* N *is an Einstein manifold.*

Lastly, gradient shrinking Ricci solitons with nonnegative curvature operator have been classified by O. Munteanu and J.P. Wang [52].

Theorem 5.7.10. *Let* (M, g, f) *be an* n*-dimensional gradient shrinking Ricci soliton with nonnegative curvature operator. Then* (M, g) *must be a quotient of the* *sphere* $\mathbb{S}^n$*, or* $\mathbb{R}^n$*, or the product* $\mathbb{R}^k \times \mathbb{S}^{n-k}$ *with* $1 \leq k \leq n-2$.

The main result, Theorem 5.6.3 of Chapter 5, is joint work with Professor Lei Ni and Dr. Kui Wang. The paper is published in International Mathematical Research Notices in 2016.

Bibliography

[1] Ben Andrews. Gauss curvature flow: the fate of the rolling stones. *Invent. Math.*, 138(1):151–161, 1999.

[2] Ben Andrews. Motion of hypersurfaces by Gauss curvature. *Pacific J. Math.*, 195(1):1–34, 2000.

[3] Ben Andrews. Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations. In *Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations*, volume 19 of *Surv. Differ. Geom.*, pages 1–47. Int. Press, Somerville, MA, 2015.

[4] Ben Andrews and Julie Clutterbuck. Lipschitz bounds for solutions of quasilinear parabolic equations in one space variable. *J. Differential Equations*, 246(11):4268–4283, 2009.

[5] Ben Andrews and Julie Clutterbuck. Time-interior gradient estimates for quasilinear parabolic equations. *Indiana Univ. Math. J.*, 58(1):351–380, 2009.

[6] Ben Andrews and Julie Clutterbuck. Sharp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue. *Anal. PDE*, 6(5):1013–1024, 2013.

[7] Ben Andrews, Pengfei Guan, and Lei Ni. Flow by powers of the Gauss curvature. *Adv. Math.*, 299:174–201, 2016.

[8] Ben Andrews and Lei Ni. Eigenvalue comparison on Bakry-Emery manifolds. *Comm. Partial Differential Equations*, 37(11):2081–2092, 2012.

[9] D. G. Aronson and L. A. Peletier. Large time behaviour of solutions of the porous medium equation in bounded domains. *J. Differential Equations*, 39(3):378–412, 1981.

[10] D. Azagra, M. Jiménez-Sevilla, and F. Macià. Generalized motion of level sets by functions of their curvatures on Riemannian manifolds. *Calc. Var. Partial Differential Equations*, 33(2):133–167, 2008.

[11] Ilya J. Bakelman. *Convex analysis and nonlinear geometric elliptic equations*. Springer-Verlag, Berlin, 1994. With an obituary for the author by William Rundell, Edited by Steven D. Taliaferro.

[12] Christoph Böhm and Burkhard Wilking. Manifolds with positive curvature operators are space forms. *Ann. of Math. (2)*, 167(3):1079–1097, 2008.

[13] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[14] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3):369–402, 1984.

[15] Eugenio Calabi. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.*, 5:105–126, 1958.

[16] Huai-Dong Cao. Existence of gradient Kähler-Ricci solitons. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pages 1–16. A K Peters, Wellesley, MA, 1996.

[17] Huai-Dong Cao and Qiang Chen. On Bach-flat gradient shrinking Ricci solitons. *Duke Math. J.*, 162(6):1149–1169, 2013.

[18] Huai-Dong Cao and Detang Zhou. On complete gradient shrinking Ricci solitons. *J. Differential Geom.*, 85(2):175–185, 2010.

[19] Xiaodong Cao, Biao Wang, and Zhou Zhang. On locally conformally flat gradient shrinking Ricci solitons. *Commun. Contemp. Math.*, 13(2):269–282, 2011.

[20] G. Catino, P. Mastrolia, and D.D. Monticelli. Gradient ricci solitons with vanishing conditions on weyl. *Journal de Mathmatiques Pures et Appliques*, pages –, 2017.

[21] Giovanni Catino. Complete gradient shrinking Ricci solitons with pinched curvature. *Math. Ann.*, 355(2):629–635, 2013.

[22] Bing-Long Chen. Strong uniqueness of the Ricci flow. *J. Differential Geom.*, 82(2):363–382, 2009.

[23] Xiuxiong Chen and Yuanqi Wang. On four-dimensional anti-self-dual gradient Ricci solitons. *J. Geom. Anal.*, 25(2):1335–1343, 2015.

[24] Bennett Chow. Deforming convex hypersurfaces by the nth root of the Gaussian curvature. *J. Differential Geom.*, 22(1):117–138, 1985.

[25] Bennett Chow. On Harnack's inequality and entropy for the Gaussian curvature flow. *Comm. Pure Appl. Math.*, 44(4):469–483, 1991.

[26] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[27] Bennett Chow, Peng Lu, and Bo Yang. Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons. *C. R. Math. Acad. Sci. Paris*, 349(23-24):1265–1267, 2011.

[28] Tobias H. Colding and William P. Minicozzi, II. Generic mean curvature flow I: generic singularities. *Ann. of Math. (2)*, 175(2):755–833, 2012.

[29] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[30] Panagiota Daskalopoulos and Ovidiu Savin. $C^{1,\alpha}$ regularity of solutions to parabolic Monge-Ampére equations. *Amer. J. Math.*, 134(4):1051–1087, 2012.

[31] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[32] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.*, 65(2):169–209, 2003.

[33] Pengfei Guan and Lei Ni. Entropy and a convergence theorem for gauss curvature flow in high dimension. *J. Euro. Math. Soc. to appear*, 2016.

[34] Cristian E. Gutiérrez. *The Monge-Ampère equation*, volume 44 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.

[35] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[36] Richard S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Int. Press, Cambridge, MA, 1995.

[37] Richard S. Hamilton. Four-manifolds with positive isotropic curvature. *Comm. Anal. Geom.*, 5(1):1–92, 1997.

[38] Qingbo Huang and Guozhen Lu. On a priori $C^{1,\alpha}$ and $W^{2,p}$ estimates for a parabolic Monge-Ampère equation in the Gauss curvature flows. *Amer. J. Math.*, 128(2):453–480, 2006.

[39] Gerhard Huisken. Local and global behaviour of hypersurfaces moving by mean curvature. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 175–191. Amer. Math. Soc., Providence, RI, 1993.

[40] Tom Ilmanen. Generalized flow of sets by mean curvature on a manifold. *Indiana Univ. Math. J.*, 41(3):671–705, 1992.

[41] H. Ishii and P.-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differential Equations*, 83(1):26–78, 1990.

[42] Norihito Koiso. On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In *Recent topics in differential and analytic geometry*, volume 18 of *Adv. Stud. Pure Math.*, pages 327–337. Academic Press, Boston, MA, 1990.

[43] Brett Kotschwar and Lu Wang. Rigidity of asymptotically conical shrinking gradient Ricci solitons. *J. Differential Geom.*, 100(1):55–108, 2015.

[44] Xiaolong Li. Moduli of continuity for viscosity solutions. *Proc. Amer. Math. Soc.*, 144(4):1717–1724, 2016.

[45] Xiaolong Li, Lei Ni, and Kui Wang. Four-dimentional gradient shrinking solitons with positive isotropic curvature. *Int. Math. Res. Not. IMRN*, (00):1–11, 2016.

[46] Xiaolong Li and Kui Wang. Nonparametric hypersurfaces moving by powers of gauss curvature. *submitted*.

[47] Xiaolong Li and Kui Wang. Moduli of continuity for viscosity solutions on manifolds. *J. Geom. Anal.*, 27(1):557–576, 2017.

[48] Pierre-Louis Lions. Sur les équations de Monge-Ampère. I. *Manuscripta Math.*, 41(1-3):1–43, 1983.

[49] Mario J. Micallef and John Douglas Moore. Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. *Ann. of Math. (2)*, 127(1):199–227, 1988.

[50] Ovidiu Munteanu and Natasa Sesum. On gradient Ricci solitons. *J. Geom. Anal.*, 23(2):539–561, 2013.

[51] Ovidiu Munteanu and Jiaping Wang. Geometry of shrinking Ricci solitons. *Compos. Math.*, 151(12):2273–2300, 2015.

[52] Ovidiu Munteanu and Jiaping Wang. Positively curved shrinking ricci solitons are compact. *J. Differential Geom., to appear*, 2017.

[53] Aaron Naber. Noncompact shrinking four solitons with nonnegative curvature. *J. Reine Angew. Math.*, 645:125–153, 2010.

[54] Huy T. Nguyen. Isotropic curvature and the Ricci flow. *Int. Math. Res. Not. IMRN*, (3):536–558, 2010.

[55] Lei Ni. Estimates on the modulus of expansion for vector fields solving nonlinear equations. *J. Math. Pures Appl. (9)*, 99(1):1–16, 2013.

[56] Lei Ni and Nolan Wallach. On a classification of gradient shrinking solitons. *Math. Res. Lett.*, 15(5):941–955, 2008.

[57] Lei Ni and Nolan Wallach. On four-dimensional gradient shrinking solitons. *Int. Math. Res. Not. IMRN*, (4):Art. ID rnm152, 13, 2008.

[58] V. I. Oliker. Hypersurfaces in $\mathbf{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampère type. *Comm. Partial Differential Equations*, 9(8):807–838, 1984.

[59] Vladimir Oliker. Evolution of nonparametric surfaces with speed depending on curvature. I. The Gauss curvature case. *Indiana Univ. Math. J.*, 40(1):237–258, 1991.

[60] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv:math/0211159.*

[61] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *arXiv:math/0307245.*

[62] Grisha Perelman. Ricci flow with surgery on three manifolds. *arXiv:math/0303109.*

[63] Peter Petersen and William Wylie. On the classification of gradient Ricci solitons. *Geom. Topol.*, 14(4):2277–2300, 2010.

[64] Kaising Tso. Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.*, 38(6):867–882, 1985.

[65] Kaising Tso. On a real Monge-Ampère functional. *Invent. Math.*, 101(2):425–448, 1990.

[66] Jia-yong Wu, Peng Wu, and William Wylie. Gradient shrinking ricci solitons with half harmonic weyl curvature are rigid. *arXiv:math.DG/1410.7303.*

[67] Zhu-Hong Zhang. Gradient shrinking solitons with vanishing Weyl tensor. *Pacific J. Math.*, 242(1):189–200, 2009.