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Issler, Paulo Floriano

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Essays on consumption cycles and corporate finance

by

Paulo Floriano Issler

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Business Administration

in the

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of the

University of California, Berkeley

Committee in charge:

Professor Nancy E. Wallace, Chair
Professor Nicolae B. Gârleanu
Professor Demian G. Pouzo

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Essays on consumption cycles and corporate finance

Abstract

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This dissertation consists of two chapters that concern with the consumption cycle and corporate finance. The first chapter analyzes the role of durability in characterizing the consumption cycle. There is strong empirical evidence demonstrating that decreases in residential investments and durable expenditures are early indicators of economic downturns. Analogously, once the economy goes into recession, early increases in residential investments and durable expenditures signal economic recoveries. So far, little work has been done detailing the mechanisms explaining these important empirical stylized facts. In this article, I develop a general equilibrium asset pricing production model that includes durability and substitutability between perishable and durable service consumption. Results indicate that large shocks in the productivity of the capital accumulation process and a high elasticity of intertemporal substitution are both needed to create the correct timing of changes in durable expenditures and nondurable consumption characterized in the data. The study also uses this general equilibrium model as a framework to make predictions about the term structure of forward contracts settled on a national housing price index. Such work will create a foundation for further developing this important derivatives market.

The second chapter analyzes the link between debt maturity and the term spread. This chapter is co-authored with Pratish Anilkumar Patel. Evidence shows that a firm's debt maturity and term spread are intricately linked. Firms issue short term debt when the term spread is significantly positive and they increase maturity as the term spread decreases. The current literature explains this link with market frictions such as agency problems, asymmetric information, and liquidity risk. We explain the link between debt maturity and term spread using the trade-off theory of capital structure. When the term spread is small or even negative, transaction costs of debt rollover outweigh bankruptcy costs. Therefore, the firm optimally chooses to increase debt maturity. On the other hand, when the term spread is significantly positive, bankruptcy costs outweigh transaction costs of debt rollover. Therefore shorter debt maturity is optimal as it minimizes the chance of bankruptcy. In addition, we contribute to the current discussion in the literature concerning the speed of adjustments

of capital structure, finding that firms are active in adjusting their capital structure. The model is consistent with a variety of stylized facts concerning debt maturity.

Dedication

I dedicate this dissertation to my wife Carla Di Castro. I deeply thank her for her extra patience and understanding during the most difficult moments over the past six years. Without her love and support this research would never have been possible.

Contents

List of Figures

List of Tables

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Chapter 1

Durability and the consumption cycle

1.1   Introduction

Since World War II, nine out of a total of eleven recessions in the U.S. have been preceded by steep declines in housing investments, making housing starts the best forward-looking indicator of economic cycles. This can be seen clearly in Figure 1.1 below, which shows the smoothed percentage change in consumption, expenditures and residential investment for the household sector, as well as investments in structures from the business sector for the 1981-Q3 National Bureau of Economic Research (NBER) recession.

Note the remarkable negative percentage change in residential investment characterizing steep declines in new home construction two quarters before the recession starts. Note also that, to a lesser extent, household expenditures in durable goods such as cars and appliances follow the same pattern. Percentage change in business structures goes into negative territory only after 1981-Q3. This shows that the declines in residential investment preceding recessions are not due to phenomena occurring in the construction industry. Once into the recession, demand softens on nondurable consumption and services, and stays low until the end of the recession period. Finally, note that before recession ends in 1982-Q4 there is a markedly strong recovery in residential investment and durables expenditures.

Appendix 3.1.1 presents the same type of graph for all eleven post-WWII recessions. With the exception of the 1953-Q1 Korean War and the 2001-Q1 dot-com recessions, all others show a similar pattern. These findings are not new. Leamer (2007) provides a detailed empirical analysis of the relevance of residential investment and durable expenditures for predicting economic fluctuations. His article, however, provides little work on framing an economic model that explains the mechanisms leading to such patterns.

In this paper, I construct a rational expectation consumption-based asset pricing model consistent with the stylized facts above. The model establishes a framework for explaining
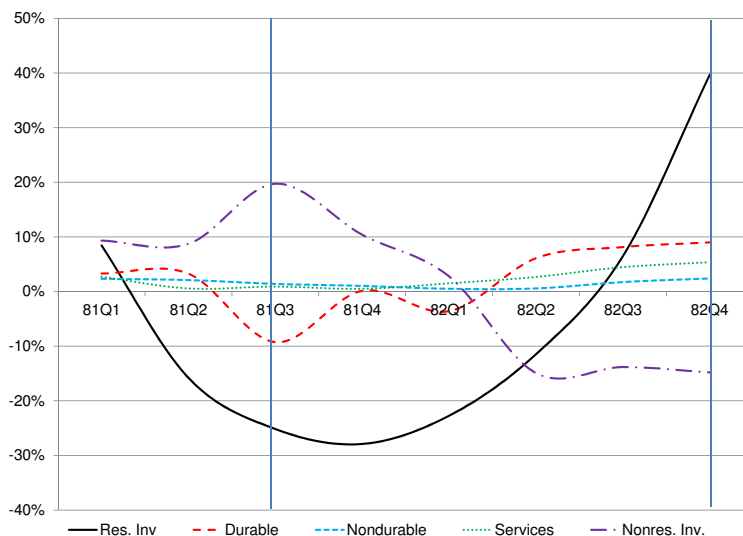
Figure 1.1: National Income and Product Accounts' (NIPA's) smoothed percentage change in residential investment, durable expenditures, nondurable consumption, services, and non residential investment for the 1981-Q3 NBER recession ending in 1982-Q4.

how the sustainable flow of services provided by the housing stock (the durability aspect of consumption) and the substitutability between durable and nondurable consumption interact to create the timing of events laid out by the empirical evidence. In addition, in the spirit of Routledge, Seppi, and Spatt (2000) and Casassus, Collin-Dufresne, and Routledge (2005), I use the general equilibrium model as a structural framework to make predictions about the term structure of forward contracts settled on a national housing price index. The general equilibrium model creates a foundation for investigating this important but underdeveloped derivatives market.

Substitutability creates an interesting dynamics as the economy transitions from recovery to boom and from downturn to recession. In the early stages of an economic downturn, households are able to cut durable and housing expenditures deeply, while maintaining high levels of services from their current durable stocks. This allows them to reallocate their income to avoid significant cuts in nondurable consumption. Conversely, in recoveries consumers face a depleted stock of housing and other durables. A shift to a regime of high capital productivity induces a demand for durables and a significant positive change in durables expenditures, thus marking the beginning of a new economic cycle.

To fully capture the interaction between producers and consumers and the feedback of their optimal decisions, I model the economy in general equilibrium. On the consumer's side of the economy, I model individuals with standard preferences over durable and nondurable consumption. I assume complete markets. Individuals invest their wealth in financial securities which enable them to fully hedge the risks embedded into their optimal consumption plans. Individuals also hold the stock of capital which is used as input flows by producing firms.

On the producing side of the economy, I assume two production sectors — durable producers and nondurable producers. Each sector has an infinite number of firms using capital supplied by individuals to produce goods for final consumption. More particularly, capital is a factor input for producing two types of goods: a nondurable (perishable) consumption good, and a finished durable good (housing), which ultimately provides a continuous stream of services to consumers. The model also implements a third production technology in which capital is employed to produce capital that would be available for producing durable and perishable goods for subsequent periods. More specifically, uncertainty in the economy is implemented by a Wiener process driving the law of motion for the capital accumulation process and a 2-state Markov chain governing changes in its production efficiency and volatility.

I make the simplified assumption that the durable stock is represented by the sum of the aggregate housing stock plus other durable goods such as automobiles, furniture, and appliances. As a result, unless stated otherwise I use the terms *housing* and *durables* interchangeably throughout the article. Moreover, since the economic value of the housing stock is larger than the stock of other types of durables, I set the depreciation rate of the durable stock to a value matching that of the housing stock.

With the proper set of parameters, the model is able to produce the correct timing of events characterizing the consumption cycle. In particular, the leading indicator aspect of residential investment and durables expenditures for recessions and recoveries can only be reproduced with large shocks in the productivity of the capital accumulation process, as implemented by the Markov chain. Small continuous changes in the productivity of capital driven just by the Wiener process can only generate continuous changes in the durable expenditure process. This cannot explain large variations in residential investment marking incoming recessions and recoveries.

Furthermore, in order to replicate the stylized facts, the model must be parameterized with a large elasticity of intertemporal substitution (EIS). When the economy is modeled with low EIS values the results are contrary to the observed facts – a shock in the efficiency of the capital accumulation process is immediately followed by a sizable discontinuous change in perishable consumption, with a small, negligible change in durables expenditures.

In spite of its single Wiener process, the general equilibrium model is able to produce a

wide variety of shapes for the term structure of forward housing prices. Furthermore, the resulting dynamics of the modeled housing forward price curve supports the idea that, similar to the interest rate market, the slope of the term structure of the housing market would be a good predictor of upcoming shifts in the business cycle.

There is a vast literature studying the role of durability on asset pricing. The effects of incorporating the consumption of durable services in the representative agent's utility index has been analyzed in Hindy and Huang (1993). They look into the agent's optimal consumption and investment decisions considering that households derive utility solely from the consumption of durable services. In their formulation, purchases of durable goods are irreversible. The representative household cannot scale down the level of durable services by selling durable assets. This happens either because there is no market for such transactions, or because the agent would have to pay a very high transaction cost to adjust the durable stock. This situation might be viewed as a limiting case of Grossman and Laroque (1990)'s model by specifying infinite selling costs. With these frictions, adjustments in the durable stock are sporadic and happen in gulps. Though groundbreaking, these papers focus on the impact of durability on the consumption and investment decisions for an individual agent in the economy. As highlighted by Marshall and Parekh (1999), the modeled frictions impose severe restrictions for analyzing the economy at an aggregate level. Furthermore, they model the economy in partial equilibrium with exogenous durable prices. As a result, their framework is not able to provide any insights about how the housing markets respond to changes in the economic environment.

Yogo (2006) explores the effects of durability in a frictionless environment. He models the consumer's problem with the representative agent deriving utility from nondurable goods and a flow of durable services obtained from durable stock holdings. The representative household's intertemporal utility has a recursive form, as in Epstein and Zin (1989) and Weil (1989). The study demonstrates that non-separable within-time preferences on the consumption of perishable goods and durable services can explain the predictability of stock price returns in both time-series and cross-sectional samples, when the intratemporal elasticity of substitution between both types of goods is higher than the intertempotal elasticity of substitution. By including durability, Yogo (2006)'s framework is able to provide an explanation for the value premium in the U.S. equity market. Unfortunately, in his analysis durable consumption is restricted to NIPA's durable good expenditures and consequently his results might change significantly when residential investment is included in the analysis.

Though motivated by rather different reasons, my modeling approach is to some extent related to that of Gomes, Kogan, and Yogo (2009). Their article extend Yogo (2006)'s by modeling a general equilibrium with firms producing durables and nondurable goods. In their model setup, there is an inelastic supply of capital provided by households to nondurable and durable producers. By contrast, my framework models the supply of capital in the economy as evolving over time. Also, since the focus of my study is on characterizing the

production and consumption behavior across business cycles, as opposed to differences in equity returns, I model consumers' preferences more parsimoniously by employing a simpler CRRA specification over an intratemporal Cobb-Douglas utility function. Finally, by using a fixed supply of capital, Gomes, Kogan, and Yogo (2009)'s setup cannot provide insights about how durable expenditures respond to sudden and large changes in the productivity of capital accumulation — a critical model component for explaining the timing of consumption and expenditures, as shown in the NIPA data. They are mute about the empirical findings described in Leamer (2007)'s study.

I organize the paper in the following way. In Section 2, I describe the general structure of the model, its assumptions, and the details of the market players. In Section 3, I derive the economic equilibrium including the main steps to derive the base case model, which is characterized by consumers' preference implemented with a logarithm utility. Section 4 explores the implementation of the numerical approach taken to solve the model, including the choice of model parameters. In Section 5, I explore the economic implications of the model results, focusing on the timing of events that characterize the consumption cycle and the resulting forward markets for housing. Section 6 shows potential generalizations of the model. Section 7 contains my concluding remarks.

1.2 Model

I develop a model of general equilibrium with producing firms using capital as the sole input for the production processes. In the spirit of Cox, Ingersoll, and Ross (1985) and Casassus, Collin-Dufresne, and Routledge (2005), I consider a capital growth model. However, instead of assuming that labor is not necessary in production, I make the assumption that, at time t, the capital stock in the economy, named K_t, is a fixed-proportion blend of both physical and human capital.

1.2.1 Production of goods

Capital is employed as a factor for producing two types of goods: a nondurable (perishable) consumption good N_t, which is taken as the numeraire for the economy, and a durable good E_t, which is used to provide a continuous stream of services to consumers. Production of nondurable goods Y_{Nt} employs a linear technology. It takes ψ units of capital to produce one unit of nondurable good. Imposing the market clearing condition gives

$$Y_{Nt} = N_t = \psi K_{Nt}. \tag{1.1}$$

Production of durable goods Y_{Dt} employs a non-linear technology. A capital flow K_{Dt} is employed as the single input for the production process. Moreover, the efficiency of the process depends on the level of the durable stock in the economy, or equivalently on the service level of the durable stock Z_t, as well as on an efficiency factor α. The durable good production flow, which with market clearing is equal to the durable expenditures E_t, is defined as

$$Y_{Dt} = E_t = \alpha K_{Dt}^{\eta} Z_t^{1-\eta}. \tag{1.2}$$

The convex functional form for durable production generates interior solutions for the flow of durable expenditures. The efficiency dependence on the durable stock Z is a mathematical choice to establish a production technology that is homogeneous of degree one on K_{Dt} and Z. This choice is critical for reducing the state space and simplifying the model solution as shown in subsection 1.3.3.

1.2.2 Capital

The third production technology relates to the capital accumulation process. New capital is produced by means of a linear technology using capital itself as the only factor for production. The stock of capital in the economy evolves according to the following law of motion

$$dK_t = (\mu_i K_t - K_{Nt} - K_{Dt}) \, dt + \sigma_{iK} K_t d\omega_t. \tag{1.3}$$

All agents in the economy construct their information set exclusively from the history of a continuous time standard Brownian motion $\omega \equiv \{\omega_t, t \in [0, \infty)\}$, and an independent Poisson process representing changes in the productivity regime of the capital accumulation process. More specifically, the Poisson process drives a 2-state Markov chain with regimes (states) labeled as h for the "high productivity" of capital regime and ℓ for the "low productivity" of capital regime. In this framework, regime i switches into regime $j \neq i$ at the first jump time of the Poisson process with intensity λ_j, i.e, λ_ℓ is the intensity of the Poisson process when switching from the high productivity regime to the low productivity regime.

The parameters μ_i and σ_{ik} in equation (1.3) define the productivity and the volatility of capital reinvestment for the productivity regime i, respectively.

Note that a change in regime does not impact the level of capital stock in the economy. Immediately after a switch in regime, the capital stock remains the same. The impact, however, is in the rate in which capital grows in the economy, or equivalently, the productivity of capital accumulation μ_i, as well as in the volatility of the capital accumulation σ_{ik}.

The economic motivation for implementing a Markov chain follows from the fact that large "shocks" are at the heart of business cycles, as documented by Fuhrer and Schuh (1998). They point out that modern macroeconomic theory identifies different causal factors behind recessions: technology shocks, energy price shocks, actions taken by monetary policymakers, and international disturbances. In my modeling approach, I do not attempt to endogenously identify each individual causal channel. I assume that, all these events ultimately create a significant and abrupt negative change in the productivity of capital accumulation prior to an economic downturn. In my general equilibrium model, capital productivity shocks are exogenously defined. After occurring, they propagate to producers and consumers, as these agents respond to negative shocks by adjusting their supply and demand functions to lower levels.

As a result, the model does not provide any insights about which individual channel leads to an economic downturn. For instance, Bernanke et al. (1997) and Hamilton (1985) attribute rising oil prices, induced by disruptions in supply, as the main cause for the 1973-Q4, 1980-Q1, 1981-Q3, and 1990-Q3 recessions. In the model framework, all these events translates into a shift in regime from a high to a low productivity of capital accumulation as implemented by the the Markov-Chain. Similarly, the model is not designed to explain the remarkable escalation in house prices that preceded the most recent recession in 2007-Q4. From the model point of view, the main source of inefficiencies in the capital accumulation process results from the significant reduction in the supply of capital by financial intermediaries, triggered by a significant shortfall in capital reserves after the collapse of house prices.

1.2.3 Households

Households in the economy are infinitely lived. At each point in time $t \in [0, \infty)$ they continuously purchase two types of goods: a nondurable (perishable) good N_t, and a durable good E_t. The nondurable good deteriorates immediately and becomes worthless if not used for consumption.

The flow of purchases of durable goods E_t, or durable expenditures, is added to the household's durable stock Z_t. A unit of durable expenditure starts providing a flow of services to consumers immediately after its purchase, and keeps providing a flow of services into the

future with its level depreciating at a constant rate δ. As common in the literature, e.g., Grossman and Laroque (1990), Hindy and Huang (1993), and Yogo (2006), I assume that the service flow from finished durable goods at time t is a linear function of the stock level of durables. More precisely, the following expression defines the relation between the service level of the durable stock and the history of durable expenditures

$$Z_t = Z_0 e^{-\delta t} + \int_0^t e^{-\delta(t-s)} E_s ds. \tag{1.4}$$

I emphasize that the durable service level Z_t is proportional to the aggregate stock of durable goods. In addition, production of any additional unit of durable good is irreversible since this same unit cannot be converted back into capital. More specifically, in the context of the housing markets, once a new home is built it might change hands between households, but it will always count as part of the existing stock.

Also, the equation above establishes a clear distinction between durable expenditures and durable consumption. Durable expenditures E_t equates to additions to the durable stock. Durable consumption equates to the level of services Z_t provided by the durable stock. Finally, this distinction does not exist in the case of nondurable goods. Due to its perishable nature, nondurable consumption and nondurable expenditures must be the same at any point in time.

Applying the first derivative to equation (1.4) provides the following law of motion for the aggregate durable service level in the economy

$$dZ_t = \left(\alpha K_{Dt}^{\eta} Z_t^{1-\eta} - \delta Z_t \right) dt.$$

1.2.4 Markets

I assume complete markets. In the modeled economy, households can invest their wealth in three long-lived securities traded continuously in a frictionless market: a riskless money-market account valued B_t and two risky securities valued S_{1t} and S_{2t} that provide hedges for both the standard Brownian motion $\omega \equiv \{\omega_t, t \in [0, \infty)\}$ and the Poisson counting process $n \equiv \{n_t, t \in [0, \infty)\}$. I assume that the value of these securities evolve according to the following processes

$$\frac{dB_t}{B_t} = r_t dt,$$

$$\frac{dS_{1t}}{S_{1t}} = \mu_{1t} dt + \sigma_{1t} d\omega_t + \nu_{1t} dn_t,$$

$$\frac{dS_{2t}}{S_{2t}} = \mu_{2t} dt + \sigma_{2t} d\omega_t + \nu_{2t} dn_t$$

respectively, where r_t represents the known rate of return of the riskless security at time t, μ_{1t} and μ_{2t} specify the drift parameters of the risky securities, and σ_{1t}, σ_{2t}, ν_{1t}, and ν_{2t} characterize the volatility parameters for the standard Brownian motion and Poisson counting processes, respectively.

Any feasible plan for nondurable consumption $N \equiv \{N_t, t \in [0, \infty)\}$ and durable expenditures $E \equiv \{E_t, t \in [0, \infty)\}$ can be established from a unique investment strategy in the securities above.

1.2.5 Household's preferences

The intraperiod utility is specified by a time-separable logarithm utility function on both the consumption of nondurable goods N and the service level provided by the durable stock Z

$$u(N, Z) = \ln\left(N^\beta Z^{1-\beta}\right) = \beta \ln N + (1 - \beta) \ln Z. \tag{1.5}$$

The utility specification above has the feature that the agent's felicity at time t depends on a combination of both nondurable consumption and the service flow provided by her holdings of the durable stock. Nondurable consumption contributes with weight β and durable service consumption contributes with weight $(1 - \beta)$. The Cobb-Douglas functional form above can be seen as a limiting case of a constant elasticity of substitution (CES) aggregator utility specification with the elasticity of substitution parameter taken to the value one.

I emphasize that the main objective of this paper is to understand the timing of events characterizing economic cycles. In this sense, it is expected that the elasticity of intertemporal substitution (EIS) plays an important role in defining the correct timing of changes in the optimal nondurable consumption and durable expenditures as the economy cycles. With this in mind, I first solve the general equilibrium economy with the logarithm preference as a base case. As is well known in the literature, this type of preference is a special case

of a more general specification in which consumers have preferences defined by a constant relative risk aversion (CRRA) utility function. In the CRRA setting the EIS equates to the reciprocal of the coefficient of relative risk aversion, named γ, and the logarithm specification is attained as a limiting case where γ converges to one.

I also note, that in spite of its dual role, the preference parameter γ should actually be thought to express the trade-offs of intertemporal consumption, or its role governing the EIS. The role of γ as the coefficient of relative risk aversion could in future studies be disentangled from the EIS by applying an even more general preference representation, such as the stochastic differential utility as defined in Duffie and Epstein (1992).

I assume that an infinite number of identical households exist in the economy. The equilibrium price of durable goods is not influenced by durable expenditures of a single individual household. Agents are price takers in this economy. To model the durable (housing) price dynamics at a macroeconomic level, I abstract from several details of firms' operations and the fact that firms might finance their operations by issuing both equity and debt. As a result, I use the simplifying assumption that the representative firm is constrained by financing its operation through equity issued to households.

1.3 Equilibrium

In this section, I provide the derivation of the equilibrium in the economy for both capital productivity regimes. First, I characterize the Hamilton-Jacobi-Bellman equation. This yields a system of partial differential equations along the optimal durable consumption and nondurable expenditures paths. Homogeneity of the production technologies allows me to reduce the state space. This results in a model formulation for a value function scaled for the size of the economy, i.e, an economy where the total stock, as defined by the sum of capital and durable stock, is equal to a value of one.

1.3.1 The Hamilton-Jacobi-Bellman equation

The general equilibrium model is solved from a social planner's perspective. For a given state of the economy defined by the productivity regime $i \in \{h, \ell\}$, the level of the capital stock K, and the durable service level Z, at every point in time t, the planner maximizes the representative household lifetime utility by choosing the corresponding capital flows employed in the production technologies Y_{Nt} and Y_{Dt}. The social planner solves

$$\sup_{\{K_{Ns}, K_{Ds}\}} E\left[\int_t^\infty e^{-\rho s} u(N_s, Z_s) ds\right].$$

The value function for the social planner takes the form

$$J_i(K, Z, t) = \sup_{\{K_{Ns}, K_{Ds}\}} E_t\left[\int_t^\infty e^{-\rho s} u(N_s, Z_s) ds\right]$$

$$subject\ to$$

$$dK_t = (\mu_i K_t - K_{Nt} - K_{Dt})\, dt + \sigma_{iK} K_t d\omega_t$$

$$dZ_t = \left(\alpha K_{Dt}^\eta Z_t^{1-\eta} - \delta Z_t\right) dt$$

$$N_t = \psi K_{Nt}, \tag{1.6}$$

where ρ denotes the rate of time preferences for the representative agent.

Building the Hamilton-Jacobi-Bellman (HJB) for the planner's problem involves, first, adding an integral term taken along the optimal path to both sides of the objective function above

$$M_i(K, Z, t) = \int_0^t e^{-\rho s} u(N_s^*, Z_s^*) ds + J_i(K, Z, t) = \sup_{\{K_{Ns}, K_{Ds}\}} E_t\left[\int_0^\infty e^{-\rho s} u(N_s, Z_s) ds\right].$$

This new modified process $M_i(K, Z, t)$ is a martingale since the expression on the right-hand-side does not depend on t. Applying the generalized Ito's lemma to assess $E_t[dM_i(K, Z, t)] = 0$ results in the following HJB equation for the logarithm preference case

$$0 = \sup_{\{K_{iN}, K_{iD}\}} \left\{ e^{-\rho t} \left(\beta \ln \psi + \beta \ln K_{iN} + (1-\beta) \ln Z\right) + J_{it} + J_{iK} \left(\mu_i K - K_{iN} - K_{iD}\right) \right.$$

$$\left. + \frac{1}{2} J_{iKK} \sigma_{iK}^2 K^2 + J_{iZ} \left(\alpha K_{iD}^\eta Z^{1-\eta} - \delta Z\right) + \lambda_j \left(J_j - J_i\right) \right\}. \tag{1.7}$$

1.3.2   Optimal nondurable consumption and durable expenditures

First order conditions for the controls K_{iN} and K_{iD} yield the optimal decisions for nondurable consumption and durable expenditures at time t. Applying first partial derivatives to equation (1.7) yields

$$K_{iN}^* = e^{-\rho t} \beta J_{iK}^{-1}, \tag{1.8}$$

$$K_{iD}^* = Z \left(\alpha \eta \frac{J_{iZ}}{J_{iK}} \right)^{\frac{1}{1-\eta}}. \tag{1.9}$$

Optimal durable expenditures results in

$$\alpha K_{iD}^{*\eta} Z^{1-\eta} = \alpha Z \left(\alpha \eta \frac{J_{iZ}}{J_{iK}} \right)^{\frac{\eta}{1-\eta}}. \tag{1.10}$$

1.3.3   Reducing the state space

All production technologies are homogeneous of degree one. This formulation, together with the logarithm utility, the homogeneity of degree one of the intraperiod consumption, and the linearity of the constraint equations, allows the reduction of the state space. Consequently, I define a new state variable y representing the ratio of the stock of capital to the sum of the durable service level and the capital stock

$$y = \frac{K}{K + Z}. \tag{1.11}$$

This new variable and the capital accumulation regime indicator $i \in \{h, \ell\}$ fully characterize the state of the economy. Also note that y is bounded in the interval $(0, 1)$. For a fixed durable service level $Z > 0$, as the stock of capital $K \to 0$, $y \to 0$. When $K \to \infty$, $y \to 1$. Analogously, for a fixed stock of capital $K > 0$, as the durable service level $Z \to 0$, $y \to 1$. When $Z \to \infty$, $y \to 0$.

I seek a candidate solution for the HJB equation of the form

$$J_i(K, Z, t) = e^{-\rho t} \left(A_i \ln (K + Z) + g_i(y) \right), \tag{1.12}$$

with A_i and $g_i(y)$ representing a constant term and a scaled value function for each productivity regime respectively.

I interpret the candidate value function in the following way. The term $e^{-\rho t}$ simply accounts for the household's impatience for later consumption. The term $\ln(K + Z)$ accounts for the "size" of the economy since it relates to the total stock of the economy. From this perspective, two distinct economic states (K_1, Z_1) and (K_2, Z_2) with the same sum of capital stock and durable service level $K_1 + Z_1 = K_2 + Z_2$ produce the same value for this term. In particular, when the total stock adds to one, $K + Z = 1$, this term vanishes and the value function equates to the discounted value of $g_i(y)$. Consequently, the function $g_i(y)$ is interpreted as the value function scaled by the size of the economy.

In fact, the term $g_i(y)$ has a more complex interpretation. First, the indicator i adjusts the value function for different productivity regimes of the capital accumulation process. Second, as explained below, for a fixed regime i the function $g_i(y)$ adjusts the value function by taking into account both the irreversibility and the varying efficiency of the durable production process.

At one extreme, when $y \to 0$, the economy has a low capital stock K relative to its size $K + Z$. In this range, for a constant size of the economy $K + Z$, the representative agent would be better off if she could rebalance her consumption pattern by consuming more non-durable goods at the expense of a lower durable service level. However, she cannot convert durable stock back into capital stock. As a consequence, the value function is lower relative to one with a more balanced nondurable and durable service consumption.

At another extreme, where $y \to 1$, the economy has a high capital stock K relative to its size $K + Z$. In this range the representative agent is starving for durable services. As before, considering a constant size of the economy $K + Z$, the agent would be better off if she could employ the abundant capital stock into the durable production process and increase her durable service level. However, durable production efficiency is very low in this range, and as a result the agent will keep suffering from low durable consumption until durable production efficiency increases. As in the case for the state variable $y \to 0$ described above, the value function is lower relative to one with a more balanced nondurable and durable service consumption.

Applying the candidate solution to the Hamilton-Jacobi-Bellman equation (1.7) yields a one-dimensional ordinary differential equation (ODE) for each productivity regime $g_h(y)$ and $g_\ell(y)$. Appendix 3.1.2 contains the details for deriving this system of ODEs in its com-

pact form

$$
\begin{aligned}
0 = & -\beta\left(1 - \ln\left(\psi\beta\right)\right) - \beta\ln\left(M_{iK}(y)\right) + \left(1 - \beta\right)\ln\left(1 - y\right) - \left(\rho + \lambda_j\right)g_i(y) + \lambda_j g_j(y) \\
& + \mu_i y M_{iK}(y) - \left(\alpha\eta\right)^{\frac{1}{1-\eta}} M_{iZ}(y)^{\frac{1}{1-\eta}}\left(1 - y\right) M_{iK}(y)^{\frac{-\eta}{1-\eta}} + \frac{1}{2}\sigma_{iK}^2 y^2 M_{iKK}(y) \\
& + \alpha\left(\alpha\eta\right)^{\frac{\eta}{1-\eta}} M_{iZ}(y)^{\frac{1}{1-\eta}}\left(1 - y\right) M_{iK}(y)^{\frac{-\eta}{1-\eta}} - \delta\left(1 - y\right) M_{iZ}(y),
\end{aligned}
\tag{1.13}
$$

where I use the following notation for the marginal values of the capital stock and the durable service level for a unit size economy $(K + Z = 1)$ at time zero

$$
M_{iK}(y) \equiv \phi + g_i'(y)\left(1 - y\right),
\tag{1.14}
$$

$$
M_{iZ}(y) \equiv \phi - g_i'(y)y,
\tag{1.15}
$$

and

$$
M_{iKK}(y) \equiv g_i''(y)\left(1 - y\right)^2 - 2g_i'(y)\left(1 - y\right) - \phi
\tag{1.16}
$$

for the convexity term. The derivation in appendix 3.1.2 also shows that for the candidate solution to be valid, the constant terms A_h and A_ℓ must both equate to the value $\phi = 1/\rho$.

To the best of my knowledge the system of ODEs in equation (1.13) does not have a closed-form solution. I obtain the the value function by employing numerical procedures. More particularly, the discretization scheme for the state space is guided by the dynamics of the state variable y.

1.3.4 The dynamics of the state variable

The dynamics of the state variable y in terms of the scaled value function $g_i(y)$ and its derivatives is obtained from the application of Ito's lemma

$$
dy = \mu_y dt + \sigma_y d\omega_t,
\tag{1.17}
$$

where

$$
\begin{aligned}
\mu_y = & \mu_i y\left(1 - y\right) + \delta y\left(1 - y\right) - \sigma_{iK}^2 y^2\left(1 - y\right) - \beta\frac{1 - y}{M_{iK}(y)} \\
& - \left(\alpha\eta\right)^{\frac{1}{1-\eta}}\left(1 - y\right)^2\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{1}{1-\eta}} - \alpha\left(\alpha\eta\right)^{\frac{\eta}{1-\eta}} y\left(1 - y\right)\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{1}{1-\eta}},
\end{aligned}
\tag{1.18}
$$

$$
\sigma_y = \sigma_K y\left(1 - y\right).
\tag{1.19}
$$

By construction, the state variable y increases when the capital stock K increases relative to the durable service level Z, and decreases otherwise. Each term contributing to the drift of y has an economic interpretation. The first term has a positive sign since it relates to increases in the capital stock from the capital accumulation production process. The second term is also positive since it expresses the effects of depreciation in the durable service level (a decrease in Z yields to an increase in y). The negative third term refers to the concavity of the scaled value function. The fourth term relates to the flow of capital employed in the nondurable production processes. Its negative sign expresses the fact that the stock of capital diminishes as capital outflows into production. The fifth term is analogous to the forth term: it refers to the capital flow for the durable production process. The last term expresses the effects of the output of the durable production process on the state variable. The negative sign indicates that increases in the durable service level yield to decreases in the state variable.

Note that the volatility of the state variable depends on the state variable itself. Since the ratio of capital stock K to the total stock $(K + Z)$ must be bounded between zero and one, the volatility σ_y must go to zero as y approaches these bounds. Also, the state variable volatility does not depend on the value function $g_i(y)$ or any of its derivatives. It has just a simple dependence on y. This last aspect will drive the choice of lattice for numerically solving the system of ODEs.

Driven by shocks in the capital accumulation process in equation (1.3), the economy evolves along its optimal path and the state variable y tends to mean revert. When the durable service level is large relative to the capital stock, i.e. y is low, the representative agent decreases her demand for durables. Production capital employed for creating durable goods is small and depreciation takes its toll. In this situation the absolute value of the fifth drift term is small and depreciation given in the third term dominates the sixth term. In this case y tends to increase. In contrast, when the durable stock is small compared to the capital stock, the representative agent is starving for durable service. A larger part of the production capital is used for the durable production process. In this case, the absolute value of the fifth term is large and the sixth term dominates the third term. In this case, y tends to decrease.

1.4 Model solution

1.4.1 Numerical approach

I employ a numerical scheme for solving the system of ODEs in equation (1.13). First, I assume that the efficiency parameter α for the durable production function in equation (1.2)

is dependent on the ratio variable y according to

$$\alpha(y) = \theta\left(1 - y\right). \tag{1.20}$$

The motivation for this specification is two-fold:

a) The $(1 - y)$ functional form expresses the stacking order in which durable producers allocate productive resources. A downward drift of the state variable y indicates a relatively low durable expenditures. This implies that when y decreases overall efficiency increases, since producers retain the more efficient resources and relinquish the less efficient ones. Conversely, an upward drift in the state variable increases the demand for additional production resources, which induces firms to employ less efficient capital at the margin. The ultimate effect is that overall productivity decreases as y increases.

b) This approach yields a well-behaved scaled value function $g_i(y)$ in the limiting case where $y \to 1$. In this extreme range of the state variable the representative consumer holds very little durable stock relative to capital stock and, for a constant parameter α, the optimal choice is to employ lumpy amounts of capital to create lumpy durable expenditures as in Grossman and Laroque (1990). This lumpy discontinuous use of capital creates numerical instabilities in this far range of the state variable. The specification in equation (1.20) reinforces a lower productivity in the vicinity $y \approx 1$ and guarantees stable solutions for a wide range of the parameters θ and η.[1]

Second, I establish a lower bound $\hat{J}_i(K, Z, t)$ for the value function $J_i(K, Z, t)$ in equation (1.6) for the logarithm preference case. I evaluate the lower bound by considering a non-optimal feasible strategy where the social planner commits to zero durable expenditures $(E_t = 0)$, and allocates capital exclusively into the capital accumulation and nondurable production processes. In this setting the planner simply lets the current durable stock depreciate at a rate δ. Equation (1.21) below defines the expression for the value function with this suboptimal strategy. Appendix 3.1.3 presents the details of its derivation.

$$\hat{J}_i(K, Z, t) = \frac{e^{-\rho t}}{\rho}\left[\ln\left(K + Z\right) + \beta\left(\ln y + \frac{\rho}{\beta}C_i\right) + (1 - \beta)\left(\ln\left(1 - y\right) - \frac{\delta}{\rho}\right)\right], \tag{1.21}$$

---

[1] The state variable dependence in equation (1.20) makes the economic interpretation of the parameter θ more difficult. This drawback, however, is compensated by a tractable and stable numerical procedure.

where

$$C_i = \frac{\beta}{\rho^2} \left[\frac{(\lambda_i + \rho)\,\mu_i + \lambda_j \mu_j}{\rho + \lambda_i + \lambda_j} - \rho\,(1 - \ln\rho) - \frac{1}{2} \frac{(\lambda_i + \rho)\,\sigma_{iK}^2 + \lambda_j \sigma_{jK}^2}{\rho + \lambda_i + \lambda_j} \right].$$

Moreover, this expression for the lower bound provides a limiting case for the value function when the durable stock is very large compared to the capital stock, which characterizes the behavior of the value function when $y \to 0$. This expression also gives the solution for the value function when $y \to 1$ for the efficiency specification in equation (1.20). In these extremes the value function $J_i(K, Z, t)$ converges to $\hat{J}_i(K, Z, t)$. Formally,

$$\lim_{y \to \{0,1\}} J_i(K, Z, t) = \frac{e^{-\rho t}}{\rho} \left[\ln(K + Z) + \beta \left(\ln y + \frac{\rho}{\beta} C_i \right) + (1 - \beta) \left(\ln(1 - y) - \frac{\delta}{\rho} \right) \right], \tag{1.22}$$

or equivalently

$$\lim_{y \to \{0,1\}} g_i(y) \to \frac{\beta}{\rho} \left(\ln y + \frac{\rho}{\beta} C_i \right) + \frac{1 - \beta}{\rho} \left(\ln(1 - y) - \frac{\delta}{\rho} \right). \tag{1.23}$$

This limiting expression for $g_i(y)$ motivates recasting the ODE in equation (1.13) by applying the following transformation for the scaled value function

$$g_i(y) = \phi \left[\beta \ln y + (1 - \beta) \ln(1 - y) + f_i(y) \right]. \tag{1.24}$$

This new formulation has the advantage of establishing a convergence value for $f_i(y)$ in the limits where $y \to \{0, 1\}$, which yields numerically stable procedures. Appendix 3.1.4 provides the steps for deriving the new system of ODEs in terms of the transformed value function $f_i(y)$. Appendix 3.1.5 defines the steps for numerically solving the ODE system for $f_i(y)$, and gives a sketch of the employed algorithm.

1.4.2   The model with CRRA preferences

The steps taken to solve the economy with consumer preferences defined with a constant EIS $1/\gamma > 0$ parallel those of the logarithm case. More specifically, with CRRA intraperiod preferences

$$u(N, Z) = \frac{1}{1 - \gamma} \left[N^\beta Z^{1-\beta} \right]^{1-\gamma},$$

the major changes in the derivation relate to the functional form of the candidate value function and the first-order condition on the capital employed in the nondurable production process. More specifically,

$$J(K, Z, t) = \frac{1}{1 - \gamma} (K + Z)^{1-\gamma} g_i(y),$$

$$\psi K_N^* = \left[\frac{J_{iK}}{\psi \beta} \right]^{\frac{1}{\beta(1-\gamma)-1}} Z^{\frac{(1-\beta)(1-\gamma)}{\beta(\gamma-1)-1}}.$$

The numerical approach for solving the system of ODEs, however, differs depending on the magnitude of γ. For γ in the interval $(0, 1)$ the scaled value function should converge to zero at the extremes $y = 0$ and $y = 1$. In this situation, there is no need to further transform $g_i(y)$. Conversely, for values of $\gamma > 1$ the numerical approach is analogous to the logarithm case after applying the following transformation to the scaled value function

$$g_i(y) = y^{\beta(1-\gamma)} (1 - y)^{(1-\beta)(1-\gamma)} f_i(y). \tag{1.25}$$

1.4.3 Model parameterization

By numerically solving the transformed value function $f_i(y)$, I am able to characterize the scaled value function $g_i(y)$ by simply employing the transformation equations (1.24) and (1.25). I note that the function $g_i(y)$ and its partial derivatives are the key elements for deriving optimal patterns for nondurable consumption and durable expenditures, as well as the drift and volatility of the state variable. In addition, as shown later in subsection 1.5.4, the function $g_i(y)$ is also the main ingredient for establishing the state price deflator and durable prices for the economy. In summary, the solution for $f_i(y)$ is central for deriving all results.

The choice of parameters for the model is also central for the solution, answering why and under which conditions residential investment is a leading indicator for changes in economic cycles, as well as the behavior of durable (housing) prices (spot and forwards). Table 1.1 reports the parameters used for the base case model specified with logarithm preferences.

Parameter	Symbol	Value
Durable (housing) depreciation rate	δ	1.00%
Preferences		
Discount factor	ρ	1.00%
Elasticity of intertemporal substitution	-	1.00
Relative risk aversion	-	1.00
Utility weight on durable goods	β	0.46
Capital Accumulation Technology		
High regime productivity	μ_h	5.50%
Low regime productivity	μ_l	-2.50%
High regime volatility	σ_{K_h}	2.4%
Low regime volatility	σ_{K_l}	2.4%
Durable & Nondurable Production Technology		
Durable elasticity of variable input	η	0.75
Durable production efficiency	θ	0.80
Nondurable production efficiency	ψ	7.00
Markov Chain Transition Rates		
High to low intensity	λ_l	0.20
Low to high intensity	λ_h	1.10

Table 1.1: Parameters for the base case model – logarithm preferences.

Wilhelmsson (2008) estimates different values for the rate of depreciation of houses. His results indicate depreciation rates varying between 0.42% to 1.10% depending on house age and the level of maintenance. In the model, I set the depreciation rate to 1.00%. This choice assumes that without any investments in maintenance a typical home will lose about 10% of its value over a period of 10 years. As common in the literature, I set the rate of time preference to 1.00%. The logarithm preference for the representative consumer implies a unit value for the elaticity of intertemporal substitution and the coefficient of relative risk aversion. I set the preference weights between the consumption of perishable goods and durable services to 0.46, which is close to the value 0.50 chosen by Gomes, Kogan, and Yogo (2009).

The productivity of the capital accumulation process for the high (5.5%) and low (−2.5%) states is set to match the average real return on the market portfolio, which is about 4.25% (see for instance Campbell, Lo, and MacKinlay (1997)). The volatility of capital accumulation for the high and low production regimes are set to the same value of 2.4%. With this choice, the model generates a volatility of nondurable consumption which matches NIPA's historical volatility of nondurable consumption plus services of 2.6%. This relatively low value for the volatility of capital accumulation simply reflects the difficulties of standard models, with logarithm and constant relative risk aversion preferences, to simultaneously

match a high volatility for the market portfolio and a low volatility for the perishable consumption process. Appendix 3.1.6 provides the details for deriving expressions for the drift and volatility of the perishable consumption process in terms of the scaled value function $g_i(y)$ and its derivatives.

The parameter η describing the curvature of the durable production process is set equal to 0.75, which is similar to Gomes, Kogan, and Yogo (2009). The parameters for durable and nondurable production efficiencies, θ and ψ respectively, are adjusted to produce a modeled perishable consumption that is on average about 4.2 times the modeled durable expenditures. This choice reflects the average historical ratio of NIPA's nondurable expenditures plus services to NIPA's durable expenditures plus residential investment.

Contraction		Duration in Months	
Beginning	End	Contraction	Expansion
Nov-1948	Oct-1949	11	37
Jul-1953	May-1954	10	45
Aug-1957	Apr-1958	8	39
Apr-1960	Feb-1961	10	24
Dec-1969	Nov-1970	11	106
Nov-1973	Mar-1975	16	36
Jan-1980	Jul-1980	6	58
Jul-1981	Nov-1982	16	12
Jul-1990	Mar-1991	8	92
Mar-2001	Nov-2001	8	120
Dec-2007	Jun-2009	18	73
Average (months)		11	58
Average (years)		0.9	4.9

Table 1.2: Number of months of contractions (duration of the recession), and number of months of expansions before the beginning of the recession for each recession period after World War II.

Finally, from the post-World War II NBER data listed in Table 1.2, I set the intensities λ_l and λ_h, governing the change in productivity regimes, to produce business cycles with an average contraction (recession) period of 0.9 years and an average expansion (boom) period of 4.9 years.

1.5 Results

1.5.1 State variable dynamics

I start by characterizing the probability density function (pdf) of the state variable. This gives a better understanding of the valid range and frequency within which the state variable varies as both the Poisson and Wiener shocks move the economy over time. Figure 1.2 displays the empirical marginal pdf of the state variable y. Using equation (1.17) and the model parameters shown in Table 1.1, a simulation tracks the ending values and the ending regimes of y after generating $30,000$ paths of the state variable over a period of approximately 500 years.
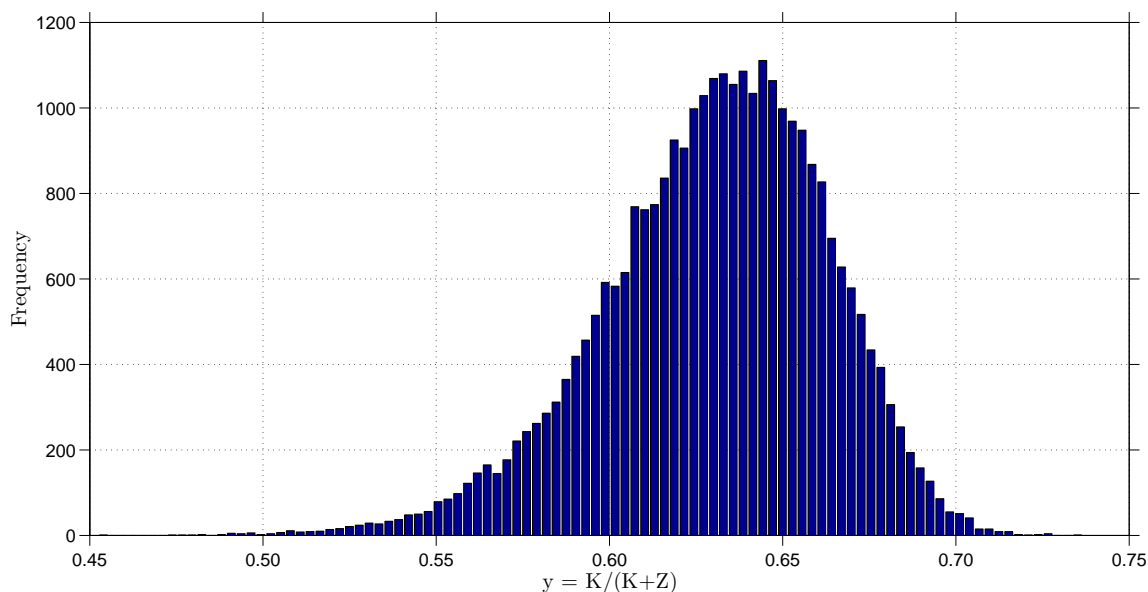
Figure 1.2: Unconditional probability density function of the state variable y for the logarithm preference case, with model parameters defined in Table 1.1. The figure results from simulating the state variable dynamics in equation (1.17) for $30,000$ paths for a period of approximately 500 years.

For simplicity, I omit the figures for the empirical distributions of y conditional on the productivity regimes since they show similar patterns to those of the unconditional pdf. Table 1.3 displays the statistics of the first two moments of the empirical unconditional and conditional distributions, as well as the ranges of the state variable for a two-tailed confidence level of 99%.

Distribution	Count	Mean	Std.	Range (99% confidence)	
				Min	Max
Unconditional	30,000	0.6303	0.0324	0.5306	0.6982
High Regime	25,385	0.6322	0.0318	0.5335	0.6982
Low Regime	4,615	0.6201	0.0342	0.5139	0.6909

Table 1.3: Statistics of the state variable y for the unconditional distribution, the distribution under the regime of high productivity of capital, and the distribution under the regime of low productivity of capital.

Figure 1.3 plots the drift of the state variable, as indicated in equation (1.18), for the model parameters defined in Table 1.1. Although the figure plots the drift for the entire range of the state variable, the distribution results shown in Figure 1.2 and Table 1.3 indicate that, in fact, the state variable takes value in a much tighter range. For convenience, the figure includes three vertical lines identifying the respective leftmost and rightmost values of y for a 99% confidence level for each regime, as well as its average. This indicates that within its range of interest, the state variable takes a positive drift for a high productivity of capital accumulation, and a negative drift for a low productivity of capital accumulation. The changing sign of the drift makes evident why the ratio variable y tends to mean-revert. Furthermore, note that if the economy stays in a high productivity regime for a long period of time, the state variable is expected to increase due to the positive drift. However, as y increases, the magnitude of the drift decreases.[2] An analogous analysis applies when the economy is in a low productivity state.

The mean-reverting behavior of the state variable has important economic implications. It means that the law of motion of the logarithm of the capital stock and the logarithm of the durable stock processes are cointegrated. In addition, the results from this general equilibrium model indicate that the ratio variable y has the potential to predict asset returns in the economy. In an empirical study, Lettau and Ludvigson (2001) find that the logarithm of the ratio of consumption to aggregate wealth, named *cay*, is a good predictor of asset returns. Their empirical findings also show that consumption, asset holdings, and labor income (proxying for human capital) are cointegrated. Deviations from a shared trend indicate expected future returns on the market portfolio. Analogously to *cay*, my model state variable y (the ratio of capital stock to durable stock plus capital stock) provides predictability to asset returns. This predictive power is also in line with Gomes, Kogan, and Yogo (2009)'s article. Since the supply of capital is constant in their general equilibrium framework, their key forecasting variable is simply the ratio of net durable expenditures to the stock of durables.

Figure 1.3 also shows how the the durable efficiency dependence on the state variable, as

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[2]In fact, the drift becomes slightly negative as the state variable approaches its rightmost 99% confidence value.
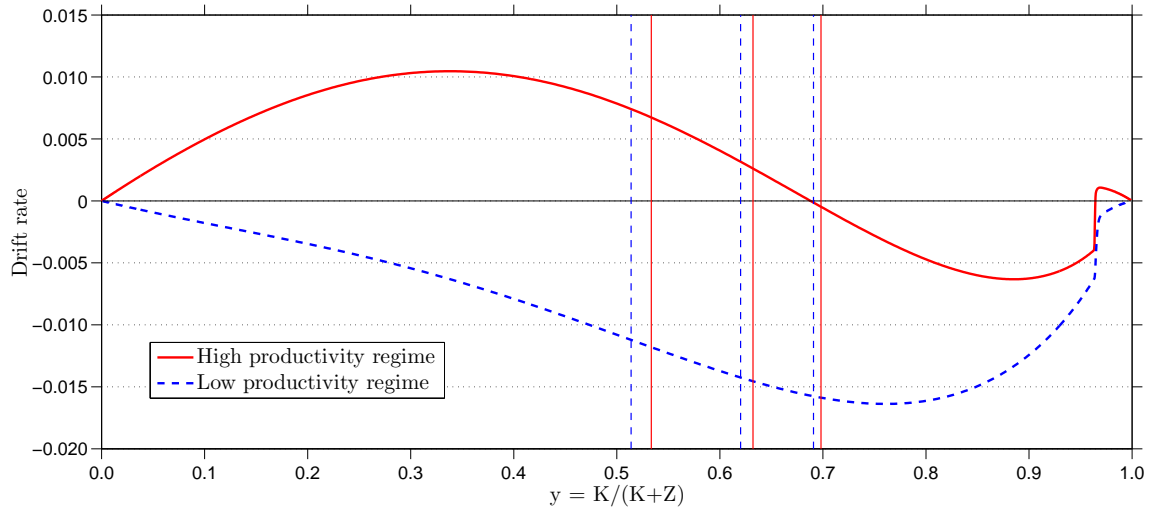
Figure 1.3: Drift rate of the state variable y for the base case model (logarithm preferences) as defined in equation (1.18). The solid line shows the drift of y when the the economy has a high productivity of capital accumulation. The dotted line shows the drift rate for a low productivity of capital accumulation. The respective solid and dotted vertical lines indicate, from left to right, the minimum, mean, and maximum values of the conditional distributions as listed in Table 1.3.

defined in equation (1.20), makes the model tractable for values of y in the vicinity of one. In this region the drift converges to zero, showing that the model predicts no gulp durable expenditures when the representative consumer has an extremely low durable stock and an abundant capital stock. I emphasize that the range in which the durable efficiency decaying functional form is heavily influencing the solution is far from the range of interest of the state variable. Still, it is illustrative to analyze the behavior of the drift function for $y \approx 0.97$. In this region, the weighted average drift quickly changes from negative to positive as y increases beyond 0.97.[3] Economically, this expresses the decision of the representative consumer to stop adding to the durable stock. In this region of the state variable, the productivity of the durable technology is extremely low, making any durable expenditures prohibitive. The representative agent simply lets the existing stock depreciate, and allocates capital only for the production of perishable goods.

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[3]This is because the positive drift of the high productivity regime contributes much more to the weighted average than the negative drift of the low productivity regime.

1.5.2 Timing of durable expenditures and nondurable consumption

Figure 1.4 plots nondurable consumption and durable expenditures for both productivity regimes over the 99% confidence interval of the state variable. The left and right vertical lines mark the average values of y for the low and high productivity regimes respectively, as defined in Table 1.3.
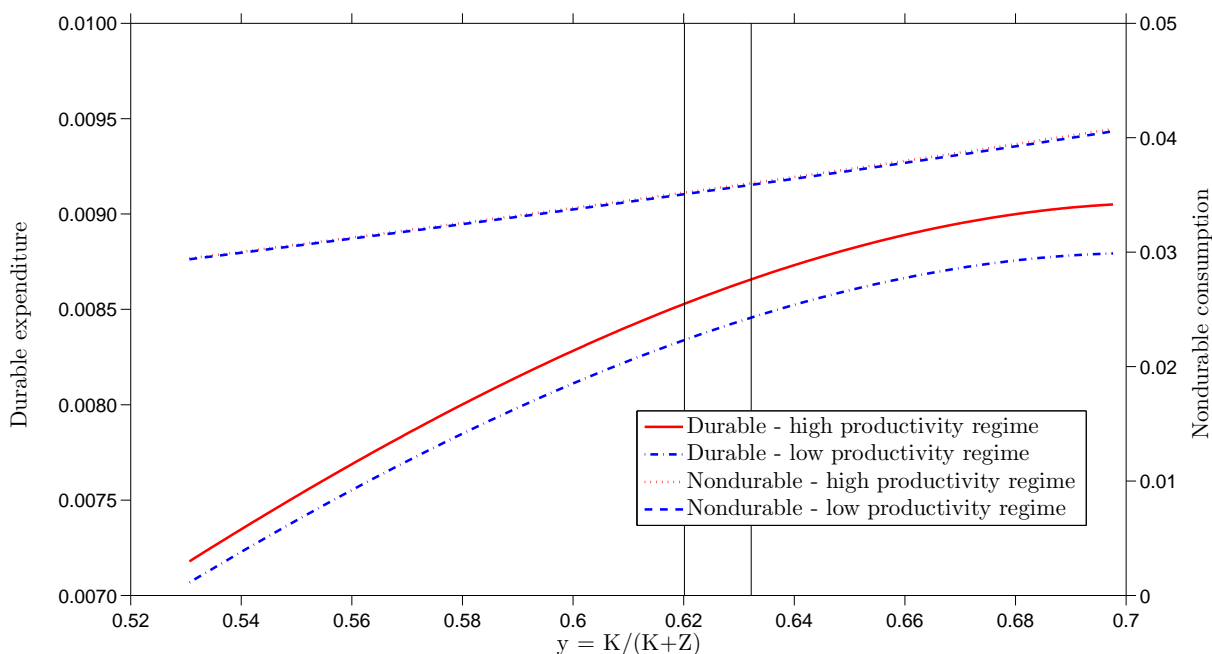
Figure 1.4: Scaled durable expenditures and nondurable consumption for the high and low productivity of capital regimes, over the 99% confidence interval of the state variable for the logarithm case. The right and left vertical lines indicate the averages of the state variable y for the high and low productivity regimes (see Table 1.3), respectively. The nondurable consumption plots for high and low regimes are almost identical and fall on top of each other.

This figure highlights the significant change in durable expenditures as the economy switches between states. This is in clear contrast with the much smaller change in nondurable consumption. When the state of the economy switches from a low to a high productivity state, the relative change in the scaled durable expenditures amounts to an average of 2.17%, compared to a 0.37% change in nondurable consumption.

Furthermore, the jump is slightly greater when the economy moves from boom to recession. This happens because when the economy stays in a high productivity state (boom) for a longer period of time, the state variable y tends to take values around its conditional mean

of 0.6322 (see Table 1.3). At this point, the gap in durable expenditures is clearly higher than the gap when y is at its conditional low productivity average of 0.6201.

The economic interpretation is that right after a negative shock in the productivity of capital accumulation, the immediate reaction of the representative consumer is to decelerate the rate of durable expenditures in favor of maintaining a high level of perishable consumption. This abrupt change in durable expenditures occurs immediately after the realization of the shock. The reverse pattern occurs after a positive shock in the productivity of capital. In this case, the economic interpretation is that consumers accelerate the rate of durable expenditures to replenish a depleted durable stock. More precisely, a negative (positive) shock in durable expenditures, immediately after a negative (positive) shock in the productivity of capital accumulation, is what signals that further declines (rises) in consumption and expenditures are expected ahead.

Figure 1.5 is a zoomed version of Figure 1.3. It gives a more detailed view of the drift of the state variable for the 99% confidence range. When the economy is booming long enough, the state variable tends to move around its conditional mean (marked by the vertical line on the right). Right after the economy switches regimes, there is a marked discontinuity in the flow of durable expenditures (as shown in Figure 1.4). The state variable y remains the same, however, since the jump in capital productivity does not immediately impact any component of the state variable y (the capital stock and the durable stock). In fact, the effect of a downward shock in capital productivity is a flip of the sign of the drift of the state variable.

Economically, after a negative productivity shock the capital stock grows at a negative pace and, in response, consumers slow down their nondurable consumption and further decrease their durable expenditures. Eventually, the economy goes into recession. A positive shock in capital productivity signals that the end of the recession is near. In response, durable expenditures jump upwards (nondurable consumption follows the pattern, but with a negligible change). As Figure 1.5 indicates, right after the positive shock, the sign of the drift of the state variable flips to a large positive value. Durable expenditures further increase after the jump, and perishable consumption starts to increase. Eventually the economy moves out of recession. As the economy fully recovers, the drift rate of y slows down. This completes the economic cycle.

The key point is that a jump in durable expenditures occurs only when there is a sizable shock in the productivity of the capital accumulation process. Small continuous changes in productivity, as frequently implemented with a persistent and slow moving variable, would yield continuous changes in the expenditure process. Without the sizable shock in the productivity of capital (from 5.5% to -2.5%), the model cannot produce quick and large variations in durable expenditures and residential investment marking incoming recessions and recoveries.[4]

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[4]As an alternative, one could implement an heteroscedastic mean-reverting process for the capital pro-
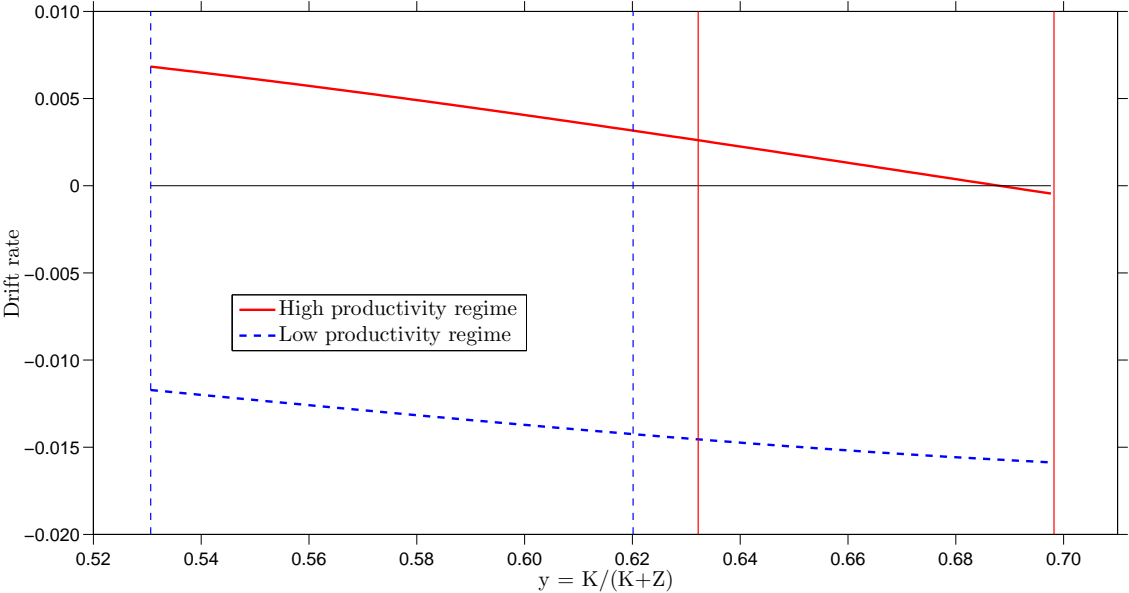
Figure 1.5: Drift of the state variable under regime switching. This figure is a zoomed version of Figure 1.3 for the 99% confidence interval of the state variable. The left and right dashed vertical lines correspond to the leftmost 99% confidence level and the conditional average of y for the low productivity regime. The left and right solid vertical lines correspond to the conditional average and rightmost 99% confidence level of y for the high productivity regime.
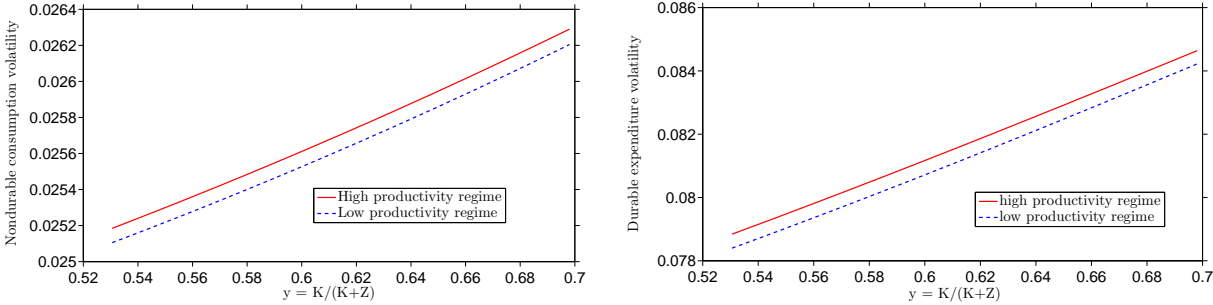
Figure 1.6: Nondurable consumption volatility and durable expenditures volatility for the 99% confidence interval of the state variable y.

Figure 1.6 depicts the derived model's volatility of nondurable consumption and durable expenditures as defined in equation (3.33) of Appendix 3.1.6. The model generates a very

ductivity in order to produce the same type of sizable changes as implemented by the Markov chain.

flat volatility for the nondurable consumption process for both productivity regimes within the range of interest of the state variable. The nondurable volatility is increasing in the state variable, and ranges from 2.51% to 2.63%. The same flat shape is also present in the case of durable expenditures volatility. The nondurable volatility also increasing in the state variable, and ranges from 7.82% to 8.35%. More interesting, though, the model predicts that the average volatility of durable expenditures is more than 3 times higher than the volatility of nondurable consumption. This result is in line with the observed data for the post-WWII period when the NIPA's volatility of nondurable consumption plus services, is 2.6%, while the NIPA's volatility of residential investment plus durable expenditures amounts to 12.3%.

1.5.3 EIS and the timing of durable expenditures and nondurable consumption

The results for the timing of durable expenditures and nondurable consumption in the previous subsection relate to a model parameterized with a high EIS. As already noted, with CRRA preferences, the curvature parameter γ governs both the EIS and the relative risk aversion. Since this paper focuses on the timing pattern of durable expenditures and nondurable consumption, I focus on the EIS aspect of the parameter γ in the subsequent analysis.

I derive the optimal durable expenditures and nondurable consumption by solving the ODE system for CRRA case, employing the same techniques used for solving the logarithm preference case. The parameters governing the production processes were modified to reproduce the same results for the levels of nondurable consumption and durable expenditures of the logarithm utility case. Table 1.4 shows the parameterization for the CRRA case.

Figure 1.7 is analogous to Figure 1.4. It shows optimal nondurable consumption and durable expenditures for the CRRA utility specification and for an EIS value of 0.50. These plotted values result from solving the ODE system for the CRRA case by employing the same techniques used for solving the logarithm preference case. The parameters θ and ψ for the production processes, however, were modified to produce the same results for the levels of consumption and expenditures of the logarithm utility case.

In this figure changes in durable expenditures and nondurable consumption, immediately after the productivity shock, are opposite to those of the logarithm preference case (EIS = 1). Note that the response of durable expenditures to shocks in capital productivity is smaller than in the logarithm case. When the state of the economy switches from a low to high productivity state, the relative change in scaled durable expenditures amount to an average of 0.76%, compared to 2.17% for the case of logarithm preferences. The average change in nondurable consumption is 2.37% for the CRRA case compared to an average of 0.370% for the logarithm utility case. Contrary to empirical evidence, the low EIS case shows that it is

Parameter	Symbol	Value
Durable (housing) depreciation rate	δ	1.00%
Preferences		
Discount factor	ρ	1.00%
Elasticity of intertemporal substitution	$1/\gamma$	0.50
Relative risk aversion	γ	2.00
Utility weight on durable goods	β	0.46
Capital Accumulation Technology		
High regime productivity	μ_h	5.50%
Low regime productivity	μ_l	-2.50%
High regime volatility	σ_{K_h}	2.4%
Low regime volatility	σ_{K_l}	2.4%
Durable & Nondurable Production Technology		
Durable elasticity of variable input	η	0.75
Durable production efficiency	θ	0.99
Nondurable production efficiency	ψ	3.59
Markov Chain Transition Rates		
High to low intensity	λ_l	0.20
Low to high intensity	λ_h	1.10

Table 1.4: Parameters for the CRRA model. By changing in the the durable and nondurable production efficiency parameters θ and ψ, the CRRA model generates the same level of nondurable consumption and durable expenditures with logarithm preferences.

nondurable consumption, as opposed to durable expenditures, that responds to shocks in the productivity of capital accumulation. These important results establish the case for setting economic models with high EIS.

1.5.4   Housing prices – spot and forward markets

The equilibrium house price p_t (durable price in the model) for the modeled economy is simply the ratio of the marginal value of durable service to the marginal value of the perishable good, or simply

$$p_{it} = \psi \frac{J_{iZ}}{J_{iK}} = \psi \frac{M_{iZ}(y)}{M_{iK(y)}}. \tag{1.26}$$

The marginal value of durable services reflects the lifetime utility gain from increasing the durable stock by one unit. This marginal value is then divided by the marginal value of
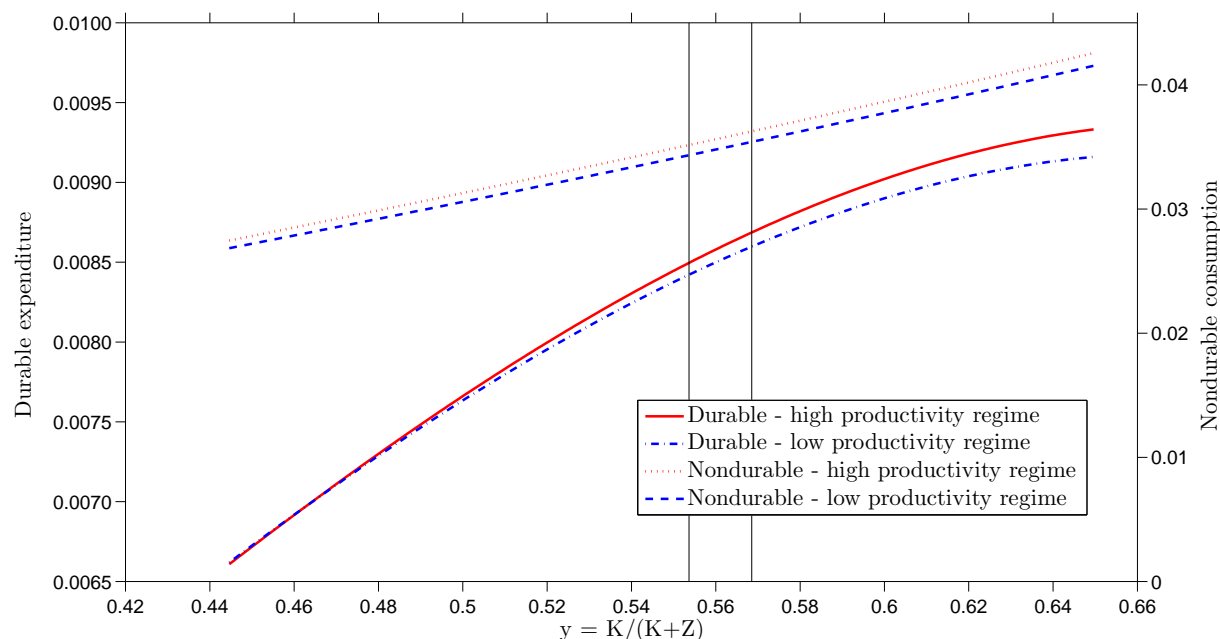
Figure 1.7: Scaled durable expenditures and nondurable consumption for the high and low productivity of capital regimes, over the 99% confidence interval of the state variable for the CRRA case ($\gamma = 2$, EIS $= 0.5$). The right and left vertical lines indicate the average of the state variable y for the high and low productivity regimes.

the perishable (the numeraire in the economy) to derive the durable price. For the modeled economy, the durable price equates to the value of a nationwide house price index.

Figure 1.8 displays durable prices as a function of the state variable y at the 99% confidence interval for both regimes. The graph shows that house prices are an increasing monotone function of y. This follows from the fact that, for a fixed level of capital stock, the higher the state variable, the more the representative agent is starving for durable services, i.e., the more she values an additional unit of durable good relative to an additional unit of perishable good. Note that the house price index is almost the same for the high and low productivity regimes. The plots are almost on top of each other. This means that jumps in capital productivity yield small changes in the house price index. In particular, for the 99% confidence interval of the state variable y the average price jump is 0.72%.

The figure also shows that the model's house price index ranges in the interval $[7.0, 16.0]$. This wide range indicates that in case of extreme events, when the state variable y moves from its highest to its lowest value on the 99% confidence level, the house price index drops by more than 50%. Interestingly, this large change comes close to the house price drops reported for the most recent housing markets crisis. The S&P/Case-Shiller Home Price Indices track
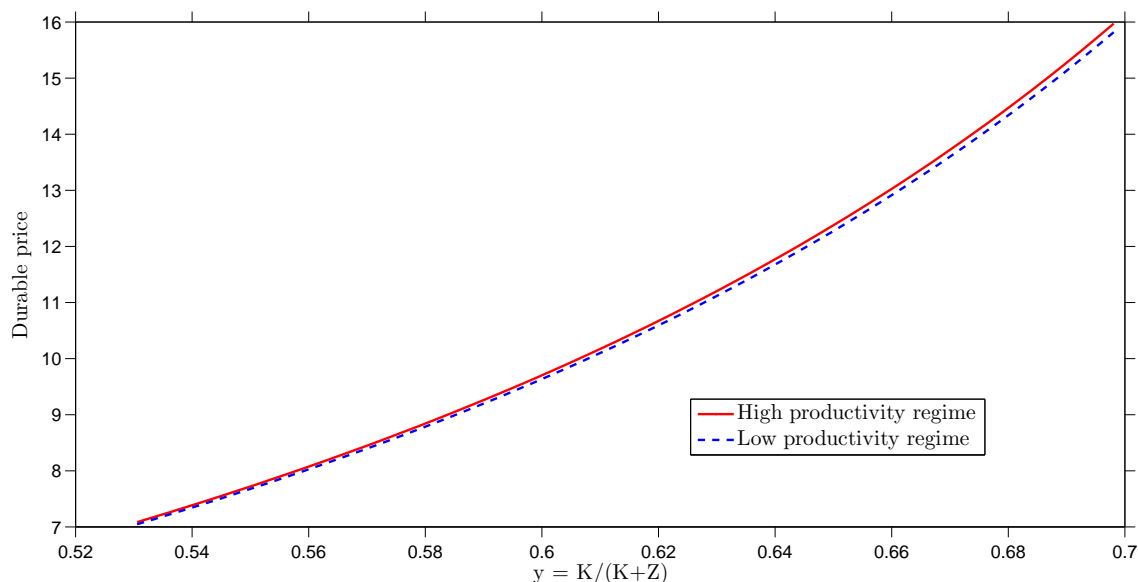
Figure 1.8: Model's house price index (durable price) as a function of the state variable y for the 99% confidence interval.

changes in the value of residential real estate both nationally and in 20 metropolitan regions. Table 1.5 lists percentage drops in the house price index for the national and metropolitan regions. The time series for the national Composite-20 index, for instance, shows an index level of 206.65 in April 2006, at its peak just before the NBER 2008-Q4 recession. This index drops to minimum of 136.77 in January 2012, representing a 34% drop. The table shows that, as predicted in the model, some metropolitan indices suffered declines of more than 50%, such as AZ-Pheonix, FL-Miami, and NV-Las Vegas.

The model can make predictions about the term structure of forward prices. The choice of forward contracts, as opposed to futures, is a natural one. The implementation of a futures contract would demand clauses for optional physical delivery by the seller – clauses similar to those used in commodity markets. Physical delivery, however, would be very difficult to manage given the wide spectrum of housing quality and the indivisibility of a dwelling unit, not to mention different regions in which the delivery units could be located. Forward contracts, in contrast, are typically settled over benchmark indices such as the S&P/Case-Shiller. These indices are constructed from periodic market surveys that can reference either a wide region such as the whole nation, or subregions such as metropolitan areas.

Following its definition, a forward contract involves no exchange of cash at the time of its inception. The time-t forward price $F(t, T)$ for a contract maturing at time T has the following expression

	Maximum		Minimum		
	Level	Month	Level	Month	Drop (%)
AZ-Phoenix	228.07	May-06	99.24	Aug-11	56%
CA-Los Angeles	273.10	Apr-06	159.92	May-09	41%
CA-San Diego	251.71	Mar-06	145.46	May-09	42%
CA-San Francisco	219.27	Mar-06	120.11	May-09	45%
CO-Denver	139.36	Mar-06	123.38	Sep-11	11%
DC-Washington	252.90	Mar-06	169.53	Jan-04	33%
FL-Miami	280.04	May-06	136.77	Nov-11	51%
FL-Tampa	239.05	May-06	124.92	Nov-11	48%
GA-Atlanta	136.11	Apr-07	85.59	Mar-12	37%
IL-Chicago	171.53	Mar-07	108.03	Mar-12	37%
MA-Boston	180.79	Nov-05	149.17	Apr-09	17%
MI-Detroit	127.92	Mar-06	67.15	Apr-11	48%
MN-Minneapolis	173.89	Apr-06	110.97	Mar-11	36%
NC-Charlotte	133.85	Aug-07	110.03	Nov-11	18%
NV-Las Vegas	235.74	Apr-06	90.20	Jan-12	62%
NY-New York	216.61	May-06	160.67	Mar-12	26%
OH-Cleveland	123.40	Jan-06	97.24	Feb-12	21%
OR-Portland	185.44	Apr-07	123.14	Jan-04	34%
TX-Dallas	125.67	Apr-07	113.81	May-11	9%
WA-Seattle	190.58	May-07	125.58	Jan-04	34%
Composite-10	226.91	Apr-06	149.46	Feb-12	34%
Composite-20	206.65	Apr-06	136.77	Jan-12	34%

Table 1.5: S&P/Case-Shiller Home Price Indices, representing the value of residential real estate both nationally and in 20 metropolitan regions. For each region, it shows the peak levels preceding the NBER 2008-Q4 recession, the minimum level after the recession, and the percentage drop in the price index.

$$F_i(t,T) = \frac{E_{it}^{\mathbb{Q}}\left[e^{-\int_t^T r_{ks}ds}p_{kT}\right]}{E_{it}^{\mathbb{Q}}\left[e^{-\int_t^T r_{ks}ds}\right]} = \frac{E_{it}\left[\zeta_{kT}p_{kT}\right]}{\zeta_{it}}, \qquad i,k \in \{h,\ell\}, \qquad (1.27)$$

where r_{ks} is the time-s value of the riskless rate, p_{kT} is value of the housing price index at maturity, $\mathbb{Q}$ refers to the risk-neutral measure, and ζ_{ks} is the state price deflator.

With the assumption of complete markets, the state price deflator ζ_{it} is unique. In this modeled production economy it equates to the marginal value of capital (see Cox, Ingersoll, and Ross (1985)). More precisely,

$$\zeta_{it} = J_{iK}. \qquad (1.28)$$

Appendix 3.1.6 provides the details for deriving the stochastic behavior of the housing price

index and the state price deflator (SPD) as an expression of the scaled value function and its derivatives. Following these results, I implement a numerical approach for characterizing the term structure of forward prices in equation (1.27). More specifically, I extend the simulation algorithm used for deriving the empirical probability density function of the state variable to create random paths for both the state price deflator and the house price index. The term structure of forward prices is simply the average of the product of the state price deflator and the spot house price index at each maturity date.
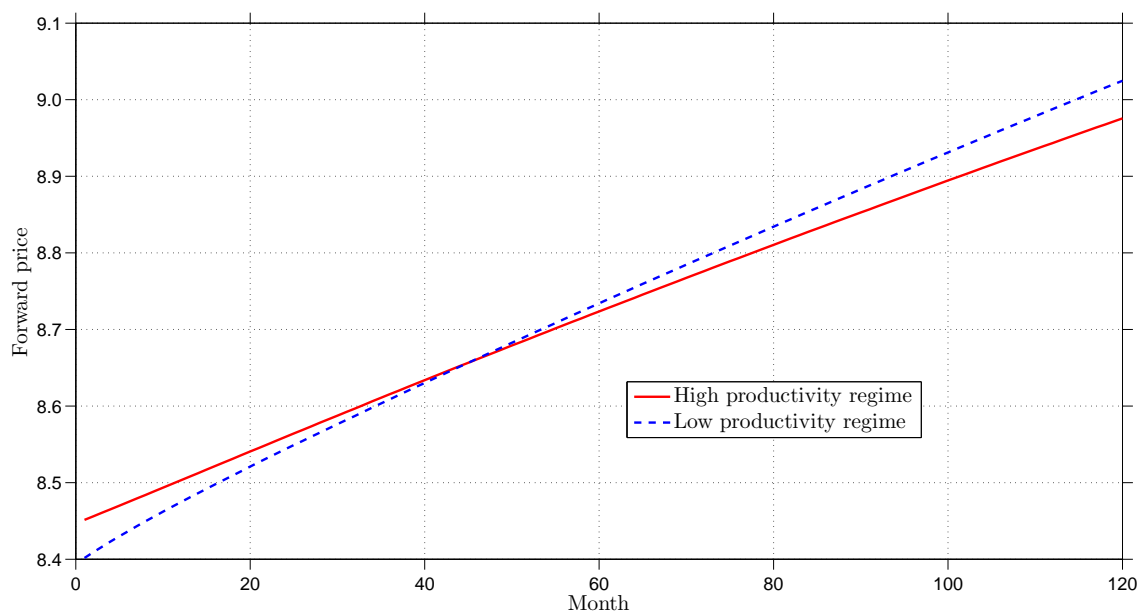
Figure 1.9: Term structure of forward house prices for both productivity regimes when the state variable y is 0.57.

Figure 1.9 displays the term structure of forward housing prices for both productivity regimes with a relatively low starting value of the state variable ($y_0 = 0.57$). Note that both curves are in contango (upward sloping), with the low regime forward curve posing a more pronounced slope. At first, this result might seem counterintuitive. When the economy is in the low productivity state, the drift of the state variable is negative, as indicated in Figure 1.5. This in turn implies that the house price index drifts to lower levels, since the house price is a positive monotone function of the state variable as depicted in Figure 1.8. Indeed, a plot of the average spot housing price p_t in Figure 1.10 shows exactly this mechanism.

A plot of the average SPD in Figure 1.11 is rather revealing. Recall that the SPD reflects the level of marginal utility of consumption, which is much higher when the economy is in recession. In this case, the increasing trend in the SPD dominates the decreasing trend in housing prices, explaining the reason for the more pronounced contango of the low produc-
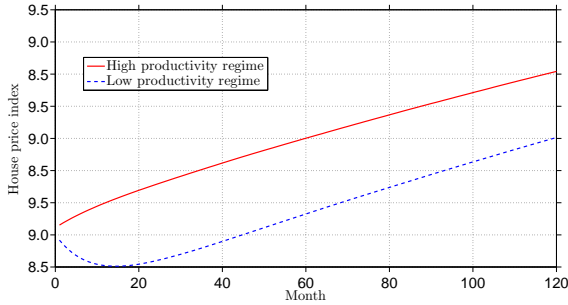
Figure 1.10: Expected house price index for $y_0 = 0.57$.
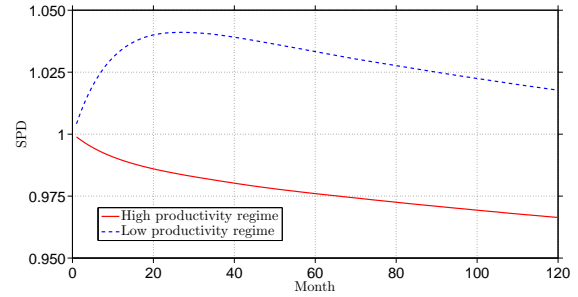
Figure 1.11: State price deflator (SPD)for $y_0 = 0.57$.

tivity state forward curve.

Figures 1.12 and 1.13 show plots of the forward curve for two initial states of the ratio variable at $y_0 = 0.66$ and $y_0 = 0.68$, respectively. These pictures characterize the slope dependence of the forward curve on the state variable. For relatively lower values of y the general shape is in contango, moving to flat when the state variable is near its unconditional average, and being in backwardation for higher values of y.
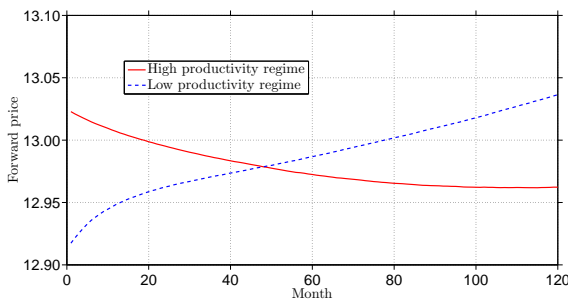
Figure 1.12: Term structure of forward house prices for both productivity regimes with $y_0 = 0.66$.
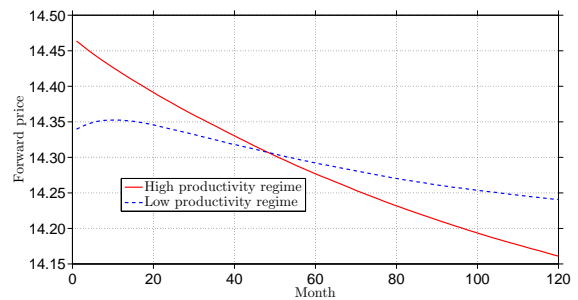
Figure 1.13: Term structure of forward house prices for both productivity regimes with $y_0 = 0.68$.

Also interesting is the hump-shaped result for the lower productivity regime when $y_0 = 0.68$. This situation occurs when the economy is in recession and the ratio of capital stock to durable stock is relatively high. It also highlights that in spite of having a single Wiener process driving the economy, the general equilibrium model is able to produce a rich variety of shapes for the forward curve.

Common to the three analyzed cases, a shift in the productivity regime results in a twist of the forward curve. This aspect is in accordance with the typical dynamics of the term struc-

ture of commodity prices and the interest rates resulting from a principal component analysis as demonstrated in Tolmasky and Hindanov (2002) and Cochrane and Piazzesi (2005). More importantly, the general equilibrium model has the ability to shed some light on the linkages between macroeconomic factors and empirical data.

1.6 Extensions

The present study can be extended in several potential areas. There is much more to explore empirically. At its current stage, the model show that with a reasonable choice of parameters it has the potential to explain key empirical facts about consumption and expenditures as the economy cycles. Clearly, the next step is to enhance the calibration of the model parameters. This effort, however, should be taken in parallel with modifications to the current structure of the general equilibrium model.

Another possible extension involves exploring the performance of a suitable measure of the ratio of capital stock to total stock (durable plus capital) for predicting future asset price returns. This study would employ an analogous approach to that used in Lettau and Ludvigson (2001).

With respect to the modeling framework, I see a list of enhancements that might prove fruitful. The first relates to relaxing the current structure for the durable production function to accommodate different specifications for the efficiency parameter. Though the current version is founded by economic arguments, other more general structures allowing for a constant efficiency or non-linear functional forms would certainly provide more realism and consequently generate a better model calibration. Along these lines, the model could adopt a more flexible intraperiod preference. The Cobb-Douglas structure pins down the elasticity of substitution between durable and nondurable consumption. A constant elasticity of substitution (CES) specification relaxes this constraint and provides more realism for modeling the agent's ability to substitute between nondurable and durable consumption.

The second enhancement refers to extending the durable production environment by introducing inventories. A realistic approach involves creating an intermediary step in the housing production process. This two-stage process relates to housing production in the sense that home developers keep units of developed land in inventory before actually engaging in the process of building the final structure and selling finished units to the market. More specifically, the model would accommodate a first production stage in which an intermediary good (developed land) is created and held in inventory. In a second stage, units of the intermediary good are taken out from inventory and serve as input for producing the finished housing unit.

Given the relevance of the EIS in defining the proper timing of consumption and expenditures, the third improvement relates to disentangling this parameter from the coefficient of relative risk aversion. This would require the implementation of a recursive type of utility representation in the fashion of Epstein and Zin (1989) for discrete time models or Duffie and Epstein (1992) for continuous time models.

Finally, the current model can be extended for studying the differences between rental prices and purchase prices in the housing markets when the economy cycles. The empirical evidence is that these two variables are co-integrated and that rental prices accelerate ahead of purchase prices during economic recoveries.

1.7   Conclusion

Empirical studies show a distinct sequence of events characterizing economic transitions into recessions and recoveries. In particular, NIPA's durable expenditures and residential investment are key economic variables signaling these transitions. In this study, I build a general equilibrium production model showing that large shocks in the productivity of capital accumulation, marking transitions to recessions and booms, are necessary to explain the cycles of nondurable consumption and durable expenditures shown in the data. In addition, the model is able to reproduce the leading aspect of residential investment and durable expenditures only when the consumer preference is set with a high elasticity of intertemporal substitution.

Finally, the paper proposes adopting the general equilibrium framework as a structural model for analyzing the dynamics of the forward market for a national house price index.

Chapter 2

Debt maturity and term spread

2.1 Introduction

Along with optimal leverage and seniority, firms also choose the time to maturity of their debt issuance. Overwhelming evidence (see Section 2.1) suggests that firms choose debt maturity based on the the term spread where term spread is defined as the difference between the 10-year Treasure note yield and the 90-day Treasury bill yield. Specifically, firms issue short term debt when the term spread is significantly positive and increase maturity as the term spread decreases. That is, debt maturity and term spread are inversely related. In this paper, we provide a theoretical explanation of how term spread affects optimal debt maturity using the trade-off theory of capital structure.

Our explanation is in contrast with the current theoretical literature that relies on the existence of either informational asymmetry, lack of liquidity, or agency conflicts. For instance, Flannery (1986) and Diamond (1991) show that good quality firms issue short-term debt as a signalling device to separate themselves from poor quality firms when there is asymmetric information between debt investors and firm managers. Milbradt and He (2012) create a model framework in which liquidity in the secondary market plays a central role in determining the optimal debt maturity of firms facing debt rollover risk. Myers (1977) and Johnson (2003) show that firms issue short-term debt to overcome the problem of under-investment created by agency conflicts. In our model, firms face the following trade-off. On one hand, debt issuance provides tax benefits to the firms; on the other hand, debt issuance is also accompanied by higher bankruptcy and transaction costs.

We develop a dynamic capital structure model with stochastic interest rates to highlight that optimal debt maturity mainly depends on the trade-off between bankruptcy costs and transaction costs of debt rollover. A significantly positive term spread induces a higher risk of financial distress because the firm's expected growth rate is lower than the long run equilibrium growth rate. Therefore, bankruptcy costs outweigh transaction costs, and firms

optimally reduce debt maturity in response. Conversely, a flat or negative term spread reduces the risk of financial distress, and firms optimally increase their debt maturity in response. In other words, as the economy cycles, the resulting term structure creates natural incentives for firms to shift their debt maturity.

In our framework, firms choose debt maturity to maximize the total firm value. Specifically, there are no agency problems between debt holders and equity holders. The firm's managers and debt investors are perfectly informed about all the relevant variables that characterize the firm and the economy. Furthermore, there is no liquidity risk in our setup. At time zero, the firm issues a T-year coupon bond after paying transaction costs related to debt issuance. If the firm has not gone bankrupt in T years, the firm rolls over its debt by issuing a new T-year coupon bond at time T, after paying transaction costs. If at the end of the second T-year period the firm is still solvent, it issues another T-year coupon bond. This process goes on indefinitely as long as the firm is solvent. If the firm goes bankrupt, debt holders take over the firm's operations after paying bankruptcy costs.

We extend the existing modeling literature by incorporating two additional stylized facts:

1. **Target leverage ratio** — According to the survey results of Graham and Harvey (2001), 44% of CFOs report having a strict or somewhat strict target leverage ratio. 37% claim to have a flexible target leverage ratio. Remarkably, only 19% of the CFOs claim that they have neither a target ratio nor a target range. Lemmon, Roberts, and Zender (2008) reinforce the evidence of target leverage ratios. They show that capital structures are remarkably stable over time; and that firms with high (low) leverage maintain relatively high (low) leverage for over twenty years, independent of being public or private. Frank and Goyal (2003), Leary and Roberts (2005), Flannery and Rangan (2006), and Huang and Ritter (2009) also corroborate that managers adjust their capital structure towards a specific target.

2. **Lumpy debt maturity** — Choi, Hackbarth, and Zechner (2011) provide empirical evidence confirming that the debt maturity structure of firms is lumpy and not granular. That is, debt maturity tends to be concentrated, as opposed to being scattered across different points in time. For example, if debt maturities are distributed uniformly in an interval $[\underline{T}, \overline{T}]$, then the maturity structure is granular. In our setup, firms issue debt with maturity T at every rollover date, characterizing a lumpy maturity structure.

The short rate follows a mean-reverting process as in Vasicek (1977). This approach allows us to examine the impact of the dynamics of the term spread. The term spread is a state variable that proxies economic conditions. As the term spread cycles, so does the economy. In the risk neutral measure, the drift of the firm value is the short rate. Therefore, when the term spread is relatively high, i.e., when the short rate is significantly lower than the

long rate, the firm growth rate is initially low and accelerates towards its long run equilibrium. This indicates that economic recovery is ahead, as would be the case at the end of a recession. On the other hand, when the term spread is relatively low, the firm growth rate is initially high and decelerates towards its long run equilibrium. This indicates a grim economic future, as would be the case at the beginning of a recession.

The economic intuition about the link between term spread and debt maturity is the following. When the term spread is significantly positive, the short rate is significantly below the long run equilibrium rate. Even though the short rate increases on average in the future, it is expected to remain below the long rate for a significant time. With the prospect of low growth ahead, the firm's probability of default is high. Therefore, the firm chooses to decrease maturity at the expense of paying higher transaction costs related to debt rollover. Conversely, when the term spread is negative, the short rate is greater than the long rate. In this case, the probability of default is low because the firm is expected to grow at higher than normal rates. Therefore, the firm chooses to increase maturity to minimize the transaction costs related to debt rollover.

Our paper contributes to the existing literature in three different ways. First, it adds to the corporate finance literature that deals with the speed of adjustment in the firm's capital structure. There is a debate in the literature about how frequently firms adjust their capital structure. Strebulaev (2007) employs a trade-off theory of capital structure model to show that firms adjust their capital structure infrequently. That is, managers optimally choose to be inactive in the process of maximizing firm value. Conversely, Welch (2012) uses empirical evidence to argue that managers are actually quite active in adjusting their capital structure. Our results indicate that the optimal maturity of firms varies between 1-3 years for a wide range of parameters. This level of activity is in line with Welch (2012). In our framework, firms incur transaction costs during debt rollover, as in Strebulaev (2007)'s. The main difference, however, concerns the choice of debt. In Strebulaev (2007)'s model, firms either increase or decrease leverage by issuing perpetual debt. In our model, firms issue a finite T-maturity debt to reach a target leverage ratio. Consequently, firms are forced to readjust their capital structure every T periods in our model while small transaction costs lead to large waiting times (inactivity) in Strebulaev (2007)'s model.

Second, our paper adds to the literature concerning debt maturity and systematic risk, as measured in our model by the term spread. Chen, Xu, and Yang (2012) explain this link in a setting with liquidity risk, whereas in our framework, markets are perfectly liquid. Lastly, our paper is closely related to Ju and Ou-Yang (2006), who explain the effect of stochastic interest rates on leverage, debt maturity, and credit spreads. To achieve a closed form solution, Ju and Ou-Yang (2006) assume that debt is not issued at par during debt rollover. An artifact of this assumption is that debt maturity is independent of the short rate, which is inconsistent with the empirical evidence. We enhance their model by assuming that debt is issued at par and that firms adjust their capital structure toward a target

leverage ratio. These features allow our model to produce results that are consistent with the empirical data, showing that debt maturity is inversely related to the term spread.

Our theoretical predictions match the empirical findings of Barclay and Smith (1995) and Julio, Kim, and Weisbach (2008). The first prediction concerns leverage. Julio, Kim, and Weisbach (2008) find that there is no substantial difference in leverage between firms that issue short-term and long-term debt. In our model, this result is mechanical as the firms re-balance their capital structure towards a target leverage ratio. The second prediction concerns volatility. Barclay and Smith (1995) and Julio, Kim, and Weisbach (2008) find that more volatile firms issue shorter term debt. They attribute this empirical finding as a validation of the agency theory of Myers (1977). Our setup matches this finding naturally. When the firm value is more volatile, big changes in the firm value are more likely. Therefore, the firm optimally decides to rebalance its capital structure more frequently as its probability of default is higher. To summarize, we find that the long run interest rate, the volatility of the interest rate process, the correlation between the short rate and the firm value, and the volatility of the firm's value are all important parameters for determining the optimal debt maturity.

The remainder of this article is organized as follows. Section II presents motivating empirical evidence that shows the inverse relationship between debt maturity and term spread. Section III presents the model setup. In order to get an intuition for the link between interest rates and default risk, we analyze the value of a risky zero coupon bond in Section IV. Section V derives the leveraged firm value using the trade-off theory of capital structure. Section VI presents the quantitative analysis. Section VII summarizes the article and makes concluding remarks.

2.2 Empirical evidence on debt maturity and term spread

We divide this section into two parts. First, we analyze the time series data of aggregate debt maturity using data from the Flow of Funds Accounts. Specifically, this part shows that in aggregate, firms reduce debt maturity when the term spread is significantly positive and they increase maturity as the term spread decreases. We also show that aggregate debt maturity is high prior to the beginning of a recession and it decreases by the end of a recession. Second, we review cross-sectional evidence of debt maturity and term spread from the corporate finance literature.

2.2.1   Analysis of aggregate debt maturity and term spread

The dark solid line in Figure 2.1 shows the cyclical component of the share of the long term debt[1] calculated by applying the Hodrick-Prescott filter.[2] The data pertains to debt issued by non-financial corporate firms. The maturity is classified as "long term" if it is greater than one year. The shaded bands indicate recessions as designated by NBER. The link between the cyclical share of long term debt and macroeconomic conditions is clear. During recessions, the long term debt share appears to dip below the trend. For example, during the first quarter of 2008, at the start of the past recession, the share of the long term debt was 3% below the trend. During the third quarter of 2009, the first quarter after the end of the past recession, the share of the long term debt was only 0.04% below the trend.
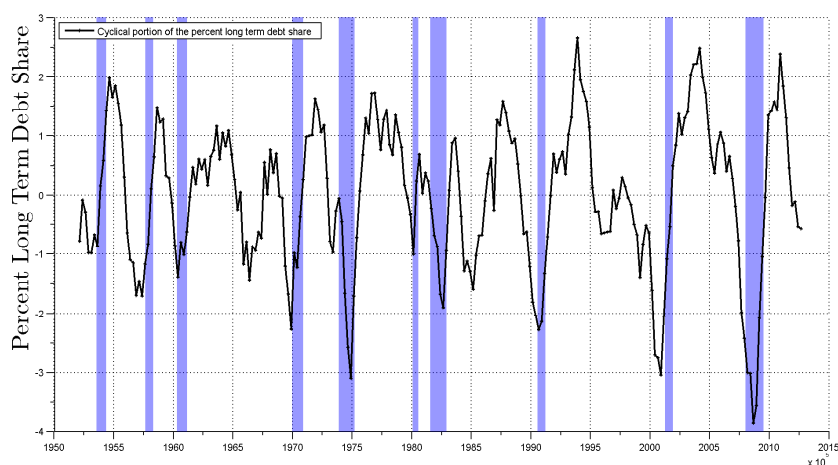
Figure 2.1: Long term debt share of non-financial corporate business. The dark line is the cyclical component of the long term debt share calculated via the Hodrick-Prescott filter. The data for the long term debt share is from the Fed Funds flow database (series L.102). The shaded bands in gray are the NBER recession dates.

Table 2.1 contains descriptive statistics of the share of long term debt since the first quarter of 1952. The share of the long term debt is 1.31% below the trend during recessions, while the share is 0.23% above the trend during non-recessionary times. Chen, Xu, and Yang (2012) and Julio, Kim, and Weisbach (2008) corroborate the results above by showing that debt maturity decreases during recessions.

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[1]The data for the long term debt share is from the Fed Funds flow database (series L.102). The data spans from first quarter of 1952 to third quarter of 2012 — a total of 243 quarters. Additionally, the data spans 10 recessions — a total of 36 quarters.

[2]For completeness, we plot the trend portion of the share of the long term debt in subsection 3.2.1 of the Appendix.
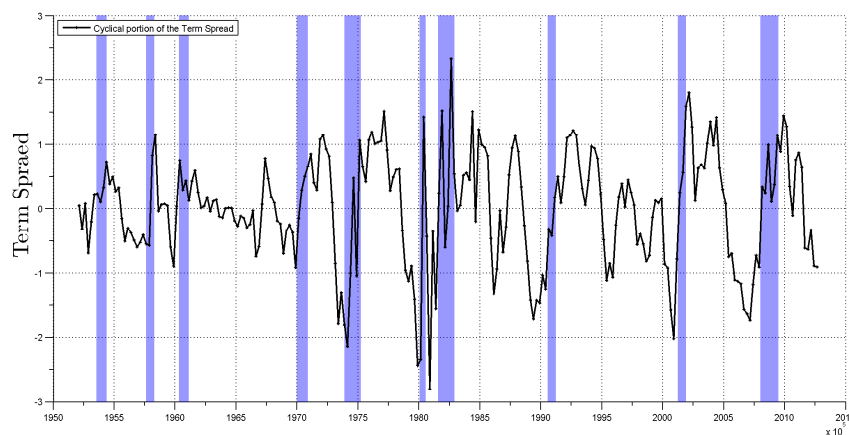
Figure 2.2: Cyclical portion of the term Spread (difference between 10-year Treasure note yield and the 3-month Treasury bill). The data is taken from Global Financial database. The shaded bands in gray are the NBER recession dates.

Event	Mean	Std. Dev	Min	Max
Recession	-1.31	1.09	0.57	-3.86
No Recession	0.23	1.10	2.67	-3.03

Table 2.1: Descriptve statistics of the percent long term debt share.

We now focus on the relationship between term spread and the state of the economy. The dark solid line in Figure 2.2 shows the cyclical portion of the term spread. A negative term spread is often a harbinger of a recession. Estrella and Hardouvelis (1991) shows that there is a marked dip in the term spread roughly six quarters prior to a recession. Figure 2.2, for instance, shows that the term spread first became negative in the third quarter of 2006, approximately five quarters before the beginning of the past recession. The dynamics of the term spread are even more evident from the descriptive statistics in Table 2.2. Three quarters prior to a recession, the term spread is flat, averaging 0.05% over the past ten recessions. Then it increases to an average of 0.93% at the beginning of a recession, and rises to its apex of 2.30% by the end of the recession. Finally, three quarters after a recession, the term spread begins to decrease, reaching an average of 2.10%.

From both figures and descriptive statistics tables, it is evident that debt maturity and term spread are intimately linked. We now review cross-sectional evidence from the corporate finance literature.

Event	Mean	Std. Dev
Term spread three quarters prior to a recession	0.05%	0.72%
Term spread at the beginning of a recession	0.93%	1.21%
Term spread at the end of a recession	2.30%	0.86%
Term spread three quarters after the end of a recession	2.10%	0.95%

Table 2.2: Dynamics of the term spread.

2.2.2 Cross-sectional evidence linking debt maturity and the term spread

Empirical evidence finds that managers time their borrowing activity. They use their beliefs about future interest rate movements to lower the cost of funds. A prominent example is the CFO survey conducted by Graham and Harvey (2001). Based on their results, CFOs say that they issue short-term debt when "short-term rates are low compared to long-term rates," or when "we are waiting for long-term rates to come down."

The fact that managers are speculating is further reinforced by Faulkender (2005)'s empirical study of financial policies for firms in the chemical industry between 1994 and 1999. Faulkender finds that firms that issue floating rate debt do not swap floating interest payments for fixed interest payments. Firms seem to be amplifying their interest rate risk as opposed to reducing it.

Using data from corporate bond issuances, Barclay and Smith (1995), Guedes and Opler (1996), Faulkender and Petersen (2006) and Julio, Kim, and Weisbach (2008) also find that firms issue shorter-term debt when the term spread is significantly positive. This result is robust to the addition of other firm-specific variables such as credit ratings, book to market ratio, and stock return volatility.

In our paper, we provide a natural explanation of why firms vary debt maturity with term spread. Based on our model, changes in the term structure resulting from economy cycles create natural incentives for firms to shift their debt maturity. The explanation does not rely on either agency conflicts or asymmetric information. This is also consistent with Graham and Harvey (2001), who show evidence that the CFOs are not concerned with either agency conflicts or asymmetric information when issuing debt. Finally, they show that CFOs are concerned with transaction costs associated with debt rollover, which supports our theoretical approach.

2.3   Model setup

In this section, we modify the setup of Leland and Toft (1996) in three ways. First, we relax the assumption of constant interest rates by assuming that the short rate process follows a mean reverting process. This assumption implies that the term spread is stochastic which in turn allows us to link debt maturity and term spread.

Second, for expositional clarity, we assume an exogenous default boundary as opposed to an endogenous default boundary. The assumption of exogenous versus endogenous boundary is important when one is concerned with agency conflicts. For example, the debt holder's incentives to default are obviously different from equity holder's incentives. We show in Section IV that the choice of an exogenous default boundary produces the same qualitative results for credit spreads as other structural credit models with an endogenous default boundary.

Third, we assume that our default boundary, which is in the spirit of Black and Cox (1976), is stochastic and not constant. This approach may seem counterproductive, but we show that our parametric form of default boundary allows us to evaluate the model in closed form. Particularly, we show with our parametric form that distance to default is directly related to the short rate. That is, when the short rate is high, firms are less likely to default. This feature forms the basis of our model.

We perform our analysis in partial equilibrium and we assume complete markets. This allows us to perform our analysis directly under the risk-neutral measure. The details of the assumptions are discussed below.

2.3.1   Environment

ASSUMPTION 1. *(Interest rate dynamics)* *Let r_t denote the short-term riskless interest rate. The dynamics of r_t are given by*

$$dr_t = \beta\left(\alpha - r_t\right)dt + \sigma_r\,d\mathbb{W}_{rt}, \tag{2.1}$$

where β, α and σ_r are constants and $\mathbb{W}_{rt}$ is a standard Wiener process under the risk-neutral measure.

The dynamics of r_t are drawn from the term structure model of Vasicek (1977). Let $\Lambda(r_t, t, T)$ be the time t price of a zero coupon bond with maturity T. Standard calculations yield the following expression for the value of a riskless bond:

$$\Lambda(r_t, t, T) = e^{A(t,T) - B(t,T)\,r_t}, \tag{2.2}$$

where

$$B(t,T) = \frac{1 - e^{\beta(T-t)}}{\beta}; \quad A(t,T) = (\alpha - \frac{\sigma_r^2}{2\beta})\left[B(t;T) - (T-t)\right] - \frac{\sigma_r^2 B(t;T)^2}{4\beta}.$$

We define the term spread as the difference in the yield of a ten-year zero coupon Treasury bond and a 3-month zero coupon Treasury bill. Mathematically, the term spread $TermSpread(r_0, T)$ is

$$TermSpread(r_0, T) = -\frac{\ln \Lambda(r_0, 0, 10)}{10} - (-\frac{\ln \Lambda(r_0, 0, 0.25)}{0.25}).$$

Loosely speaking, the term spread is well approximated as

$$TermSpread(r_0, T) \approx \alpha - r_0.$$

ASSUMPTION 2. (Firm dynamics) *Let V_t designate the market value of the firm's unleveraged assets before tax. The dynamics of V_t are given by*

$$\frac{dV_t}{V_t} = (r_t - y)\, dt + \sigma_V\, d\mathbb{W}_{vt}, \tag{2.3}$$

where y and σ_V are constants, $\mathbb{W}_{vt}$ is also a standard Wiener process. The instant correlation between $d\mathbb{W}_{vt}$ and $d\mathbb{W}_{rt}$ is $\rho\, dt$.[3]

The firm pays a constant fraction y of its unleveraged assets to its equity holders as dividends. If the firm does not issue any debt, equity holders are entitled to a fraction $(1 - \theta)$ of the dividends generated from the firm's unleveraged assets. In this case the after-tax unleveraged value of the firm is $(1 - \theta)V_t$ where θ is its marginal corporate tax rate.

ASSUMPTION 3. (Default threshold dynamics) *Following Black and Cox (1976), we assume there is an exogenous threshold value V_D at which the firm defaults on its debt. The threshold $V_D(t)$ is given by*

---

[3]Technically, uncertainty is described by two dependent Brownian motions, $\{\mathbb{W}_{vt}, \mathbb{W}_{rt}\}$ for $t \geq 0$ defined on a complete probability space $(\Omega, \mathbf{F}, \mathbb{Q})$ where $\mathbf{F} = \mathcal{F}_{\{t \geq 0\}}$ is the augmented filtration generated by $\{\mathbb{W}_{vt}, \mathbb{W}_{rt}\}$.

$$V_D(r_t, t, T) = P\Lambda(r_t, t, T)\, e^{y(T-t)}\, / \,(1-\theta) \tag{2.4}$$

where P is the face value (principal) of debt. The debt holder receives $1-\gamma$ times the after-tax unleveraged firm value upon default.

Note that the threshold has the desired property that at the maturity date T, its value is equal to the face value of debt. Therefore, as long as the after-tax market value of assets $(1-\theta)\,V_T$ remains higher than $(1-\theta)\,V_D$, debt holders will be compensated fully. At any time prior to T, the threshold value indicates that the after-tax market value of assets should remain higher than the present value of the principal. This expression for the threshold is in the spirit of that used by Black and Cox (1976) after adjusting for stochastic interest rates and dividend yield. This definition of financial distress is consistent with covenants referring to a violation of minimum net worth or working-capital requirements as implemented in Kim, Ramaswamy, and Sundaresan (1993).

Most notably, the threshold value is exogenous and not endogenous, as in Leland (1994) or Leland and Toft (1996). The magnitude of the difference in credit spreads between assuming an exogenous versus endogenous bankruptcy boundary is not significant, as highlighted in Fong (2006).

Note that the default threshold V_D depends on r_t and hence it is stochastic. At first glance, imposing such a structure might seem counterproductive. However, the study of first passage time of default is made simpler by studying the dynamics of

$$X_t \equiv \log\left[\frac{V_t}{V_{D_t}}\right],$$

which measures distance to default. By Ito's Lemma, the dynamics of X_t are

$$dX_t = \left[\frac{\sigma_p^2(t;T)}{2} - \frac{\sigma_V^2}{2}\right] dt + \sigma_V d\mathbb{W}_{vt} + \sigma_p(t,T)d\mathbb{W}_{rt}. \tag{2.5}$$

where $\sigma_p(t,T) \equiv \sigma_r B(t;T)$. Note that both the drift and volatility of X_t simplify to a deterministic expression. This is crucial for obtaining a closed form expression of the first passage time until default, as shown in a later section.

Figure 2.3 shows the relationship between the distance to default measured by X and the short rate r_0 for different times to maturity T. For a given time to maturity, the distance to default X is an always increasing function of the short rate. This is because the higher the short rate, the higher the drift of the firm value. Therefore, for a given bond with principal P_0, there is a lower chance that the firm value will breach the default threshold. The sensitivity of the distance to default X and short rate r_0 increases with time to maturity. This

is evident from the graph in which the dashed line representing a maturity of 3 years has a higher slope than the solid line representing a maturity of 1 year.
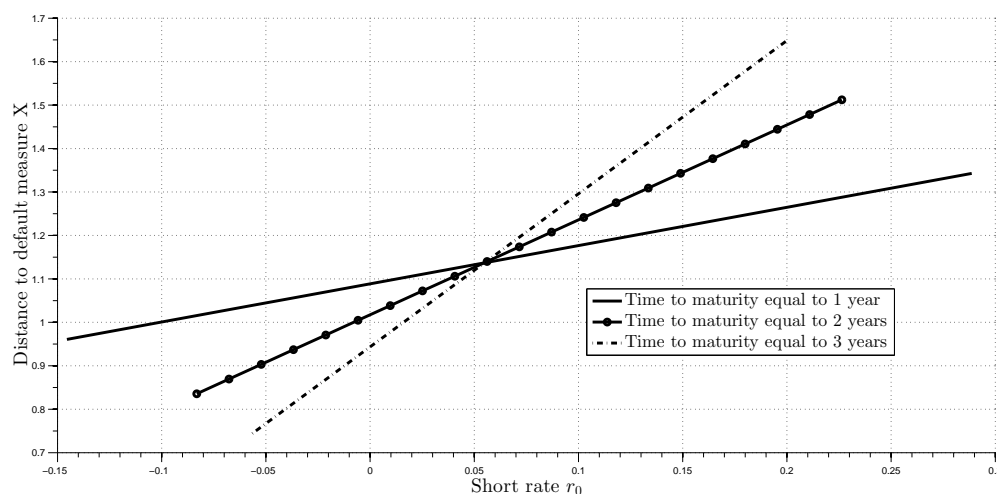
Figure 2.3: Relationship between the distance to measure X and the short rate r_0 for different time to maturity T.

The fact that X increases with the short rate is crucial for our results. A high short rate means that the term spread is low. Consider the decision making process of a firm when the term spread is low. Suppose the manager chooses a maturity T_1 of 1 year. From Figure 2.3, it is clear that the measure of distance to default is high, which means that probability of bankruptcy is low. In this case, the manager can afford to increase maturity to $T_2 > T_1$ to minimize transaction costs associated with debt rollover. Conversely, if the short rate is low so that the term spread is high, the probability of default is high. Therefore, the manager will optimally decrease maturity to minimize bankruptcy costs.

ASSUMPTION 4. *(Debt rollover dynamics)* *The firm adjusts its capital structure every T years. At time zero, the firm issues a T-year coupon bond with principal P_0 and coupon rate c_0. The firm chooses principal P_0 to achieve an exogenously specified target leverage ratio. If the firm does not default in T years, it issues another T-year coupon bond at time T. This process continues indefinitely as long as the firm is solvent. The firm incurs transaction costs equal to ϕ times the value of debt at every debt issuance.*

Most models of trade-off theory of capital structure assume either (i) perpetual debt (based

on Leland (1994))[4] or (ii) static debt (based on Leland and Toft (1996)). However, these types of specifications are clearly not suitable for analyzing optimal debt maturity. The assumption of finite maturity debt is critical for analyzing optimal debt maturity.

ASSUMPTION 5. *(Market value of debt) Debt is issued at par. Specifically, at every debt issuance nT where $n \in \{0, 1, 2, ...\}$, conditional upon not defaulting prior to nT, the market value of debt, L_{nT}, is equal to the face value of the bond, i.e. $L_{nT} = P_{nT}$.*

Assumption 5 is standard in the trade-off theory of capital structure literature.

ASSUMPTION 6. *(Specific target leverage ratio) At every debt issuance nT where $n \in \{0, 1, 2, ...\}$, conditional upon not defaulting prior to nT, the manager adjusts the capital structure toward a specific target leverage ratio ζ, where*

$$\zeta \equiv \frac{P_n}{(1 - \theta) V_{nT}} = \frac{P_0}{(1 - \theta) V_0}.$$

This assumption is in the spirit of the stylized facts identified by Graham and Harvey (2001) and Lemmon, Roberts, and Zender (2008), which states that managers adjust their capital structure toward a specific target leverage ratio. Note that in our setup, leverage is not a choice variable. That is, firms only choose maturity to maximize the firm value. In this manner, we differ from the existing literature on the trade-off theory of capital structure which focuses primarily on maximizing firm value by choosing the leverage ratio.

In order to better understand the implications of Assumption 6, it is useful to show the firm's debt issuances and default dynamics with a hypothetical sample path as in Figure 2.4. In this example, we set the time to maturity T to four years and the target leverage ratio to 64%. Therefore, at time zero, the firm issues a bond with a face value of \$42.00. The black line shows the sample path of the unleveraged firm value $\{V_t\}$ prior to default. The red line depicts the default threshold $\{V_D\}_t$. From equation (2.4), the dynamics of the default threshold depend on the dynamics of the interest rate $\{r_t\}$. Since the volatility of the interest rates is low, the fluctuations in the dynamics of the default threshold are smaller than the fluctuations in the firm value.

---

[4]Models that allow for capital structure adjustment also assume perpetual debt as in Goldstein, Ju, and Leland (2001) and Strebulaev (2007).

Note that the firm value remains above the default threshold for the first four years. In this first debt issuance, the firm does not default and it readjusts its capital structure to the specified target leverage ratio. Furthermore, since the asset value increases at the end of the fourth year, the firm rolls its debt by issuing a bond with a higher face value of $80.70. A little before year six, the firm value breaches the default threshold. At this point, the firm declares bankruptcy, and debt holders lose a fraction γ of the after-tax unleveraged firm value due to bankruptcy costs. The gray line shows the hypothetical firm value after debt holders take over the operations of the firm.
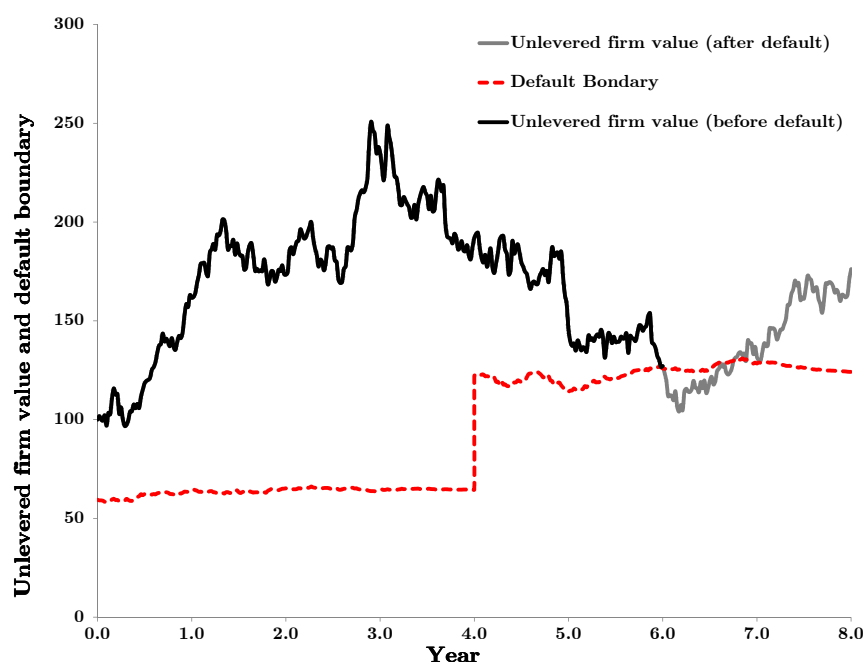
Figure 2.4: Sample path of the firm value and default threshold. The simulation shows a sample path in which 1) the firm does not default at the end of the 4th year (the time to maturity), 2) it defaults after it re-adjusts its capital structure in the 6th year. Furthermore, the firm rebalances, so that the log ratio $\ln \frac{V_0}{P_0}$ is the same at every re-adjustment date. The dark black and gray lines show the unleveraged firm values. The gray line shows the entire sample path of the firm value even though firm defaults in the 6th year. The red line shows the stochastic default threshold.

To summarize, our model set up is an enhanced version of Leland and Toft (1996) with the following modifications: stochastic interest rates, exogenous and stochastic default threshold, and exogenous target leverage ratio. Prior to analyzing the trade-off between tax benefits and the sum of bankruptcy costs and transaction costs related to debt rollover, it is useful to analyze the value of a simpler security — a zero risky coupon bond. We use the

change of numeraire technique to derive a closed-form expression for the value of such a security. This technique will be used again in later section to derive the optimal debt maturity.

2.4 Valuation of a risky zero coupon bond

In this section, we value a hypothetical risky zero coupon bond to show three features of the model. First and most importantly, we show that the credit spreads are directly related to the term spread. That is, when the term spread is low, so that the short rate is high, the probability of default is low. Therefore, the firm can afford to increase the maturity to minimize transaction costs of debt rollover. Second, we show that our model can reproduce credit spreads that are broadly consistent with other structural models of capital structure. The third motivation of this section is technical in nature. We show that the technique of change of numeraire allows us to get a closed form expression for the value of a risky zero coupon bond.

2.4.1 Setup

Let $D_{\text{zero}}(t, T, r_t; X_0)$ denote the price of a risky zero coupon bond with maturity date T at time $t \leq T$. The payoff on this contingent claim is \$1 if default does not occur during the life of the bond, and \$$(1 - \gamma)$ otherwise. This payoff function is expressed as

$$1 - \gamma \, \mathbb{I}(\text{Default happens prior to } T).$$

where $\mathbb{I}$ is an indicator function that takes the value one if V_t reaches V_{D_t} during the life of the bond, and zero otherwise. Since both V_t and V_{D_t} are stochastic, it is prudent to work with their ratio. From the definition of $X_t = \ln \frac{V_t}{V_{D_t}}$, default takes place when X_t reaches zero from above. Formally, $\mathbb{I}$ takes the value of one if $\tau \leq T$, where

$$\tau \doteq \inf\{t \geq 0 : X_t \leq 0\},$$

and zero otherwise. Therefore, the value of the risky zero bond is

$$
\begin{aligned}
D_{\text{zero}}(t, T, r_t; X_0) &= \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \{1 - \gamma \, \mathbb{I}(\tau \leq T)\} \right] \\
&= \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times 1 \right] - \gamma \, \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \mathbb{I}(\tau \leq T) \right] \quad (2.6)
\end{aligned}
$$

The first term represents the present value of one dollar upon no default. This expression is

simply the value of a default-free zero coupon bond $\Lambda(r_t, t, T)$. The second term represents the loss given default. The expectation depends on the sample path of both interest rates and firm values; hence it is difficult to derive an expressin in closed form upon first glance. In the fortunate case in which the interest rate process and firm values are independent, the second term can be written as a product of two expectations, i.e.

$$
\begin{aligned}
\mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \mathbb{I}(\tau \leq T) \right] &= \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \right] \times \mathbb{E}_t \left[\mathbb{I}(\tau \leq T) \right] \\
&= \Lambda(r_t, t, T) \times \Pr(\tau \leq T).
\end{aligned}
$$

Given the dynamics of X_t in equation 2.5, the first passage time probability can be calculated in closed form by using the Kolmogorov backward equation. However, it is hard to justify independence between the firm value and the interest rate process under the risk-neutral measure.[5] The change of measure technique, which is introduced in the Appendix (subsection 3.2.3), allows us to write the second term on the right hand side in equation 2.6 as a product of two expectations without assuming independence.

In subsection 3.2.3 of the Appendix, we show that

$$
\mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \mathbb{I}(\tau \leq T) \right] = \Lambda(r_t, t, T) \times \mathbb{E}_t^{\mathbb{T}} \left[\mathbb{I}(\tau \leq T) \right] = \Lambda(r_t, t, T) \times \Pr^{\mathbb{T}}(\tau \leq T),
$$

where $\mathbb{E}_t^{\mathbb{T}}$ is calculated using a new measure $\mathbb{Q}^{\mathbb{T}}$. Mathematically, the expectation above is the cumulative distribution function of the first passage time evaluated under the new $\mathbb{Q}^{\mathbb{T}}$ measure. The firm value as normalized by the price of a default free bond is a martingale under the new measure. In other words, under the $\mathbb{Q}^{\mathbb{T}}$ measure, a T maturity default-free zero coupon bond is used as the numeraire. By comparison, the money market account is used as the numeraire under the risk neutral measure. After deriving the dynamics of $\{X_t\}$ under the new $\mathbb{Q}^{\mathbb{T}}$ measure, we express the distribution of the first passage time in closed form using the Kolmogorov backward equation.

To summarize, the value of a zero coupon bond at time $t = 0$ is

$$
D_{\text{zero}}(0, T, r_0; X_0) = \Lambda(r_0, 0, T)(1 - \gamma \, G(T, T, X_0)),
$$

where

$$
G(t, T, X_0) = \frac{1}{\Lambda(r_0, 0, T)} \mathbb{E}_t \left[e^{-\int_0^T r_u \, du} \times \mathbb{I}(\tau \leq t) \right].
$$

---

[5]In the risk neutral measure, the drift of V_t is the $r_t - y$ and hence they cannot be independent.

The exact expression for $G(.)$ is given in equation (3.45) in the Appendix.

2.4.2 Credit spread and the shape of the term structure

Given the explicit solution for risky zero coupon bond, we can solve for the credit spread, which is defined as the difference between the yields of a risky and a riskless bond with the same maturity. Figure 2.5 graphs the term structure of credit spreads for a low leveraged firm. For this example, we set the principal P_0 to \$20.00 and the after-tax firm value to \$65.00 = \$100.00 × (1 − 0.35). The figure graphs the credit spreads for different levels of the short rate r_0. We choose the parameters α, β, and σ_r governing the short rate stochastic process to closely match the observed moments given in Ju and Ou-Yang (2006). The term structure of credit spreads are monotonically increasing as a function of maturity. This result aligns well with the empirical evidence found by Sarig and Warga (1989), who suggest that the term structure of credit spreads increases with maturity for bonds with high credit ratings.

Figure 2.6 graphs the term structure of credit spreads for highly leveraged firms where we set the principal P_0 to \$50.00, while keeping the same value for the other parameters. The term structure of credit spreads is hump shaped, which is also consistent with Sarig and Warga (1989).

The credit spread is directly proportional to the term spread. From both graphs, the credit spreads are higher when the term spread is higher, a pattern which holds true for both highly leveraged and low leveraged firms. Lastly, the concavity of the term structure of credit spreads for intermediate maturities is also dependent on and directly proportional to the term spread.

We conclude this section by pointing out that our default mechanism generates credit spreads consistent with other well-established structural default models. In the next section, we develop a model in which firms optimally choose maturity to maximize the firm value.

2.5 Leveraged firm value

In this section, we derive the firm value taking into account the trade-off between tax benefits and the sum of bankruptcy costs and transaction costs. We divide this section into two parts. In the first part, we derive closed form expressions for the present value of tax benefits, bankruptcy costs and transaction costs for one bond issuance. In the second part, using a fixed point argument, we solve for the firm value considering infinite debt issuances.

Figure 2.5: Credit spreads of low leveraged firms for different shapes of the term structures. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $P_0 = 50$ and $r_0 = 0.01, 0.07, 0.13$ for upward sloping, flat and downward sloping term structures. Credit spread is the implied yield of the bond minus the short rate.

2.5.1 One time debt issuance

Assume that the firm issues a T maturity bond so that the leverage ratio is equal to the target level ζ. This means that the firm issues a bond with principal P_0 to satisfy Assumption 6.

2.5.1.1 Expression for transaction costs

From Assumptions 4 and 5, the transaction costs for issuing debt is given by

$$tc(r_0, \zeta, T, V_0) = \phi \times L_0 = \phi \times P_0 = V_0 \times ntc(\zeta) \tag{2.7}$$

where

$$ntc(\zeta) \equiv \phi \times \zeta \times (1 - \theta).$$

The function $ntc(\zeta)$ can be interpreted as the transaction costs per unit of unleveraged firm value, so it is the normalized transaction costs. Equation (2.7) shows that transaction costs is a function of the target leverage ratio and, more importantly, it is a linear function of the

Figure 2.6: Credit spreads of highly leveraged firms for different shapes of the term structures. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $P_0 = 20$ and $r_0 = 0.01, 0.07, 0.13$ for upward sloping, flat and downward sloping term structures. Credit spread is the implied yield of the bond minus the short rate.

unleveraged firm value V_0.

2.5.1.2 Expression for bankruptcy costs

When default occurs, a fraction γ of the unleveraged firm value is lost in bankruptcy procedures. Formally, the value of bankruptcy costs is

$$bc(r_0, \zeta, T, V_0) = \mathbb{E}_0 \left[\int_0^T ds\, \gamma\, V_D(r_s, s, T)\, \delta(s - \tau)\, e^{-\int_0^\tau r_u du} \right],$$

where $\delta(.)$ is the dirac-delta function. Consider the term inside the integral. Suppose default takes place at some time $\tau = s \in [0, T]$ so that the function δ takes a value of 1 at that moment. The bondholders will recover the present value of γV_τ. At default, the firm value is equal to the default threshold, i.e. $V_{\tau=s} = V_D(r_s, s, T)$. Therefore, the term in the integral is the present value of the loss suffered by the bondholders in the event default takes place at time τ. The integral represents the loss considering the fact that default can take place at any

time between 0 and T. In the Appendix, we show that the expression for bankruptcy costs is

$$bc(r_0, \zeta, T, V_0) = V_0 \times nbc(r_0, \zeta, T) \tag{2.8}$$

where

$$nbc(r_0, \zeta, T) \;=\; \zeta\,\gamma\,\Lambda(r_0, 0, T)\,\left[G(T; T, X_0) + \widehat{G}(T; T, X_0)\right],$$

and

$$\widehat{G}(T; T, X_0) = y \int_0^T ds\, e^{y(T-s)}\, G(s; T, X_0).$$

The function $nbc(r_0, \zeta, T)$ can be interpreted as the value of bankruptcy costs per unit of unleveraged firm value or the normalized value of bankruptcy costs. Note that the vlaue of bankruptcy costs is a linear function of the unleveraged firm value V_0. Also note that X_0 is implicitly a function of the short rate r_0.

2.5.1.3 Expression for tax benefits of debt issuance

Out of the gross coupon payment C, a fraction θC is deducted to pay corporate taxes. Furthermore, the firm only enjoy tax benefits if it remains solvent. Formally, the expression for tax benefits for one debt issuance is

$$tb(r_0, \zeta, T, V_0) = \mathbb{E}_0\left[\int_0^T ds\, \mathbb{I}_{s<\tau}\,\theta C\, e^{-\int_0^s r_u du}\right].$$

The integral represents the present value of tax benefits considering the fact that default can take place at any time between 0 and T. Suppose default takes place at some time $\tau \leq T$ so that $\mathbb{I}$ takes a value of one in the interval $[0, \tau]$ and zero otherwise. The interval $[0, \tau]$ represents the times in which the firm was solvent.

We can evaluate the expression for tax benefits indirectly by using Assumption 5. The market value of debt is given by

$$
\begin{aligned}
L_0 \;=\;& \mathbb{E}_0\left[\int_0^T Ce^{-\int_0^s r(u)du}\mathbb{I}_{s<\tau}ds\right] \\
+\;& \mathbb{E}_0\left[P_0\mathbb{I}_{\tau>T}e^{-\int_0^T r_u du}\right] \\
+\;& \mathbb{E}_0\left[(1-\theta)(1-\gamma)\int_0^T V_D(r_s, s, T)\delta(s-\tau)e^{-\int_0^\tau r_u du}ds\right].
\end{aligned}
$$

The value of debt is composed of three parts: (i) present value of the flow of coupon payments prior to maturity while the firm remains solvent; (ii) present value of principal payment at time T conditional upon not defaulting prior to T and (iii) the present value of the recovery amount conditional upon defaulting at any time before T.

In subsection 3.2.5 of the Appendix, we show that the expression for tax benefits is

$$tb(r_0, \zeta, T, V_0) = V_0 \times ntb(r_0, \zeta, T) \tag{2.9}$$

where

$$ntb(r_0, \zeta, T) \equiv \theta(1-\theta)\,\zeta\left[1 - \Lambda(r_0, 0, T)(1 - G(T, T, X_0)) - (1 - \gamma)\{G(T, T, X_0) + \widehat{G}(T, T, X_0)\}\right].$$

The function $ntb(r_0, \zeta, T)$ can be interpreted as tax benefits per unit of unleveraged firm value, so it is the normalized tax benefits. Note that $tb(r_0, \zeta, T, V_0)$ is a linear function of the unleveraged firm value V_0.

At this point, we can also back solve for the value of the coupon rate C. Mathematically,

$$C = \frac{tb(r_0, \zeta, T, V_0)}{\theta\, \mathbb{E}_0\left[\int_0^T e^{-\int_0^s r(u)du}\, \mathbb{I}_{s<\tau}\, ds\right]} = \frac{tb(r_0, \zeta, T, V_0)}{\theta\, \tilde{G}(T, T, X_0)}, \tag{2.10}$$

where

$$\tilde{G}(T, T, X_0) = \int_0^T ds\, \Lambda(r_0, 0, s)\, (1 - G(s, s, X_0)).$$

The details of the derivation are given in subsection 3.2.6 of the Appendix.

2.5.2 Infinite debt issuances

2.5.2.1 Markov-Chain approximation of $\{r_t\}$

In order analyze the firm value with infinite debt issuances, it is useful to approximate the continuous time interest rate process $\{r_t\}$ by a Markov-Chain. It is well known that the Vasicek interest rate process can be expressed as an AR(1) process. We closely follow the approach outlined in Tauchen (1986), who discusses accuracy of approximating an AR(1) process with a Markov-Chain.

First note that the conditional expectation and variance of the short rate process in equation (2.1) are given by

$$r_{s|t}^{\text{mean}} \equiv \mathbb{E}_t\left[r_s\right] = \alpha + \left(r_t - \alpha\right) e^{-\beta(s-t)} \qquad \text{for } s \geq t;$$

and

$$r_{s|t}^{\text{var}} \equiv \mathbb{V}_t\left[r_s\right] = \frac{\sigma_r^2}{\alpha}\left(1 - e^{-2\beta(s-t)}\right) \qquad \text{for } s \geq t.$$

Let $\{\tilde{r}_t\}$ denote the discrete valued process that approximates the continuous valued process $\{r_t\}$. Let $\overline{r_1} < \overline{r_2} < \overline{r_3}, \ldots, \overline{r_M}$ denote the values that $\tilde{r}_t$ may take on. A method of selecting the values $\overline{r_1}$ and $\overline{r_M}$ is to let the absolute value of the difference between $\overline{r_1}$ $(\overline{r_M})$ and $r_{s|t}^{\text{mean}}$ be a multiple m of the conditional variance $r_{s|t}^{\text{var}}$. Mathematically,

$$\overline{r_1} = r_{s|t}^{\text{mean}} - m \times r_{s|t}^{\text{var}}; \qquad \overline{r_M} = r_{s|t}^{\text{mean}} + m \times r_{s|t}^{\text{var}}.$$

Let the remaining $\overline{r_k}$s be equispaced in the interval $[\overline{r_1}, \overline{r_M}]$ and denote $\triangle \overline{r} = \overline{r_j} - \overline{r_{j-1}}$ where $j \in \{2, 3, \ldots, M\}$.

We set one period in the Markov-Chain to be T years. The probability of making a transition from node $\overline{r_j}$ to node $\overline{r_k}$ in T years is calculated as follows. For each node j and for all $n \in \{0, 1, \ldots\}$

$$
\begin{aligned}
\pi_{jk} &= \Pr[\tilde{r}_{(n+1)T} = \overline{r_k} \,|\, \tilde{r}_{nT} = \overline{r_j}] \\[1em]
&= N\left[\frac{\overline{r_k} - r_{T|0}^{\text{mean}} + \triangle\overline{r}/2}{\sqrt{r_{T|0}^{\text{var}}}}\right] - N\left[\frac{\overline{r_k} - r_{T|0}^{\text{mean}} - \triangle\overline{r}/2}{\sqrt{r_{T|0}^{\text{var}}}}\right] \qquad \text{if } k \in \{2, 3, \ldots, M-1\} \\[1em]
&= N\left[\frac{\overline{r_k} - r_{T|0}^{\text{mean}} + \triangle\overline{r}/2}{\sqrt{r_{T|0}^{\text{var}}}}\right] \qquad \text{if } k = 1 \\[1em]
&= 1 - N\left[\frac{\overline{r_k} - r_{T|0}^{\text{mean}} + \triangle\overline{r}/2}{\sqrt{r_{T|0}^{\text{var}}}}\right] \qquad \text{if } k = M.
\end{aligned}
$$

Intuitively, the approximation works for the following reason. As the number of nodes M increases, the conditional distribution of $\tilde{r}_{(n+1)T}|\tilde{r}_{nT} = \overline{r_j}$ will closely approximate that of $r_{(n+1)T}|r_{nT} = \overline{r_j}$ in the sense of weak convergence.

With the discrete approximation of r_t in place, tax benefits, bankruptcy costs, and transaction costs can be written as

$$
\overline{\mathbf{tb}(\bar{\mathbf{r}}, \zeta, \mathbf{T}, \mathbf{V_0})} = V_0 \times \left[\overline{ntb_1}, \overline{ntb_2}, \ldots, \overline{ntb_M} \right]' \qquad \text{with} \quad \overline{ntb_j} \doteq ntb(r_j, \zeta, T),
$$

$$
\overline{\mathbf{bc}(\bar{\mathbf{r}}, \zeta, \mathbf{T}, \mathbf{V_0})} = V_0 \times \left[\overline{nbc_1}, \overline{nbc_2}, \ldots, \overline{nbc_M} \right]' \qquad \text{with} \quad \overline{nbc_j} \doteq nbc(r_j, \zeta, T),
$$

$$
\overline{\mathbf{tc}(\zeta, \mathbf{V_0})} = V_0 \times \left[\overline{ntc_1}, \overline{ntc_2}, \ldots, \overline{ntc_M} \right]' \qquad \text{with} \quad \overline{ntc_j} \doteq ntc(\zeta).
$$

Note that $\overline{\mathbf{tb}(\bar{\mathbf{r}}, \zeta, \mathbf{T}, \mathbf{V_0})}, \overline{\mathbf{bc}(\bar{\mathbf{r}}, \zeta, \mathbf{T}, \mathbf{V_0})}, \overline{\mathbf{tc}(\zeta, \mathbf{V_0})}$ is a $M \times 1$ vector.

2.5.2.2 Scalability

Assume for the moment that the horizon is finite so that the firm can only issue debt for $N - 1$ periods. Economically, the firm exogenously dies at time NT. The present value of tax benefits at time $(N - 1)T$ are

$$
\overline{TB_{j;N-1}} \doteq \overline{TB_{N-1}(r_j, \zeta, T, V_{(N-1)T})} = V_{(N-1)T} \times ntb_j.
$$

The following derivation shows that tax benefits at time $(N-2)T$ is linear in $V_{(N-2)T}$. Define τ_{N-2} as the first passage time when the unleveraged firm value breaches the default threshold after firm issues debt at time $(N-2)T$. Using backward induction, the present value of tax benefits at time $(N-2)T$ is

$$
\begin{aligned}
\overline{TB_{j;N-2}} &\doteq \overline{TB_{N-2}(r_j, \zeta, T, V_{(N-2)T})} \\
&= V_{(N-2)T} \times ntb_j + \sum_{k=1}^{M} \pi_{jk} \mathbb{E}_{N-2} \left[\overline{TB_{j;N-1}} \mathbb{I}_{\tau_{N-2} > T} \, e^{\int_{(N-2)T}^{(N-1)T} -r_u \, du} | r_{(N-1)T} = r_k \right] \quad (2.11)
\end{aligned}
$$

The first term of equation (2.11) is the present value of tax benefits from the debt issued at $(N - 2)T$. The second term of equation (2.11) is the present value of tax benefits from the subsequent debt issuance at $(N - 1)T$ conditional upon surviving until $(N - 1)T$. In subsection 3.2.7 of the Appendix, we show that

$$
\overline{TB_{j;N-2}} = V_{N-2} \, e^{-yT} \, ntb_k \, H_{jk} \qquad (2.12)
$$

where

$$H_{jk} \doteq H(r_j, r_k, T) = \mathbb{E}_{N-2} \left[e^{\int_{(N-2)T}^{(N-1)T} -0.5\sigma_v^2 du + \int_{(N-2)T}^{(N-1)T} \sigma_v dW_{vu} du} \mathbb{I}_{\tau_{N-2} > T} \big| r_{(N-1)T} = r_k \right].$$

We solve for H_{jk} by numerical simulation. In subsection 3.2.8 of the Appendix, we also give a closed form expression that provides a very good approximation of H_{jk}. Equation (2.12) says that the present value of tax benefits at time $(N-2)T$ is a linear function of the unleveraged firm value $V_{(N-2)T}$. By induction, the present value of tax benefits at time zero is linear in V_0. Mathematically, it means that tax benefits at time zero can be written as

$$\overline{TB_{j;0}} = V_0 \times ftb_j$$

where ftb_j is some function independent of the firm value V_0. The function ftb_j is the normalized present value of tax benefits per unit of unleveraged firm value. By letting $N \to \infty$, we apply a fixed point argument to $\overline{TB_{j;0}}$, yielding

$$\overline{TB_{j;0}} = \overline{tb_j} + \sum_{k=1}^{M} \pi_{jk} H_{jk} \overline{TB_{k;0}}.$$

In matrix notation, we have that

$$\begin{pmatrix} \overline{TB_{1;0}} \\ \overline{TB_{2;0}} \\ \vdots \\ \overline{TB_{M;0}} \end{pmatrix} = \begin{pmatrix} \overline{tb_1} \\ \overline{tb_2} \\ \vdots \\ \overline{tb_M} \end{pmatrix} + \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1M} \\ \psi_{21} & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{M1} & \cdots & \cdots & \psi_{MM} \end{pmatrix} \begin{pmatrix} \overline{TB_{1;0}} \\ \overline{TB_{2;0}} \\ \vdots \\ \overline{TB_{M;0}} \end{pmatrix}$$

where $\psi_{jk} = \pi_{jk} H_{jk}$. Define $\mathcal{I}$ as the identity matrix with M dimensions, and Ψ as a matrix with elements ψ_{jk}, then we have that

$$ftb_j = \frac{ntb_j}{\mathcal{I} - e^{-yt}\Psi}.$$

Analogously, the normalized transaction costs and bankruptcy costs are

$$ftc_j = \frac{ntc_j}{\mathcal{I} - e^{-yt}\Psi}; \qquad fbc_j = \frac{nbc_j}{\mathcal{I} - e^{-yt}\Psi}.$$

Finally, the total leveraged firm value equals the after tax unleveraged firm value, plus the value of tax benefits, less the value of bankruptcy and transaction costs. The firm value for a given $\bar{r}_j$ is given by

$$\overline{TV_{j;0}} \;=\; V_0 \times [(1-\theta) \;+\; ftb_j \;-\; fbc_j \;-\; ftc_j]\,. \tag{2.13}$$

The firm chooses maturity T to maximize the total firm value in equation (2.13).

2.6 Quantitative analysis

Before we analyze the optimal debt maturity, it is useful to analyze the leveraged firm value, tax benefits, bankruptcy costs and transaction costs for an arbitrary maturity. In Figures 2.7, 2.8 and 2.9, we plot the normalized value of each component as a function of maturity for positive, negative and zero values of the term spread, respectively. For example, bankruptcy costs of $0.1 means that bankruptcys costs are 10 cents per dollar of unleveraged firm value. These graphs show the trade-off faced by the firm when choosing optimal debt maturity. We analyze each component sequentially.

First, consider the plot for transaction costs, as shown in the dark solid line with square markers in the three figures. As expected, transaction costs decrease with maturity. This result is mechanical: as debt maturity increases, the need to roll over debt decreases, which in turn reduces transaction costs. The dependence of transaction costs on the term spread is more interesting. From the graphs, it is evident that transaction costs are almost independent of the term spread, as indicated by the small magnitude of the slope of these lines. From equation (2.7), the value of a one period transaction costs is not a function of the short rate. However, from (2.13), it is clear that transaction costs depends on the sample path of short rate $\{r_t\}$. It turns out that for choice of parameters within the range of economic interest, the dependence between transaction costs and the short rate is insignificant.

Second, consider the plot for bankruptcy costs, as shown in the dotted solid line in Figures 2.7-2.9. For each level of the term spread, bankruptcy costs approach zero as time to maturity decreases to zero. This result is expected, since in our model interest rates and firm value are driven by continuous Wiener processes. Therefore, the probability of bankruptcy smoothly reaches zero as the time to maturity decreases to zero (Duffie and Lando (2001)). In addition, observe that for a given maturity, bankruptcy costs are lowest when the term spread is negative and are highest when the term spread is positive. The intuition for this result also follows directly from Figure 2.3. When the term spread is positive, that is when the short rate is low, the chance of breaching the default threshold is high.

Third, consider the plot for tax benefits, as shown in the dark solid line with round markers in in Figures 2.7-2.9. Note that the slope of tax benefits is almost zero in all figures, and the level of tax benefits is almost the same for different term spreads. In addition to the slope, note that the level of tax benefits is the same across different term spreads. From the

figures, it is clear that tax benefits will play a minor role when firms optimize debt maturity. This evidence is consistent with Graham and Harvey (2001) whose survey results point out that CFOs do not consider tax benefits while choosing debt maturity.

Figure 2.7: Relationship between debt maturity and (i) the total firm value (ii) tax benefits (iii) bankruptcy costs, and (iv) transaction costs. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $P_0 = 40$, $\sigma_v = 20\%$, $\phi = 2\%$, and $r_0 = 2.16\%$. Note that the term spread is significantly positive since the short rate r_0 is significantly lower than the long rate α.

Lastly, with all the components in place, we analyze the effect of different parameters on the firm value.

2.6.1 Effect of term spread on firm value

Consider firm values, as shown in the dark solid line in Figures 2.7, 2.8 and 2.9. The firm value is highest when the term spread is negative and is lowest when the term spread is positive. This is also consistent with empirical evidence. Firm values are high prior to the beginning of a recession when the term spread is negative. Conversely, firm values are the lowest at the end of a recession when the term spread is positive. This result matches the literature concerning the predictability of equity returns as surveyed by Cochrane (2011).

Figure 2.8: Relationship between debt maturity and (i) the total firm value (ii) tax benefits (iii) bankruptcy costs, and (iv) transaction costs. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $P_0 = 40$, $\sigma_v = 20\%$, $\phi = 2\%$, and $r_0 = 12.16\%$. Note that the term spread is significantly negative since the short rate r_0 is significantly greater than the long rate α.

2.6.2   Effect of term spread on optimal debt maturity

Figure 2.10 plots the optimal maturity as a function of the term spread for different values of firm leverage. The solid line plots the optimal maturity for highly leveraged firms; the solid line with round markers shows the optimal maturity for medium leveraged firms; and the dotted dashed line shows the optimal maturity for low leveraged firms. From the slightly decreasing nature of the plots, it is clear that optimal maturity is a decreasing function of the term spread. This result is consistent with the empirical findings of Barclay and Smith (1995), Guedes and Opler (1996), Ozkan (2000), Graham and Harvey (2001), Faulkender and Petersen (2006), Julio, Kim, and Weisbach (2008), and Chen, Xu, and Yang (2012).

2.6.3   Effect of leverage on optimal debt maturity

From Figure 2.10, it is also clear that optimal debt maturity is inversely related to the leverage ratio. For example, for a given term spread, the optimal maturity for highly leveraged firms is lower than the optimal maturity for low leveraged firms. That is, highly leveraged firms choose lower debt maturity than low leveraged firms. This is also consistent with the evidence in Barclay and Smith (1995), and Julio, Kim, and Weisbach (2008).
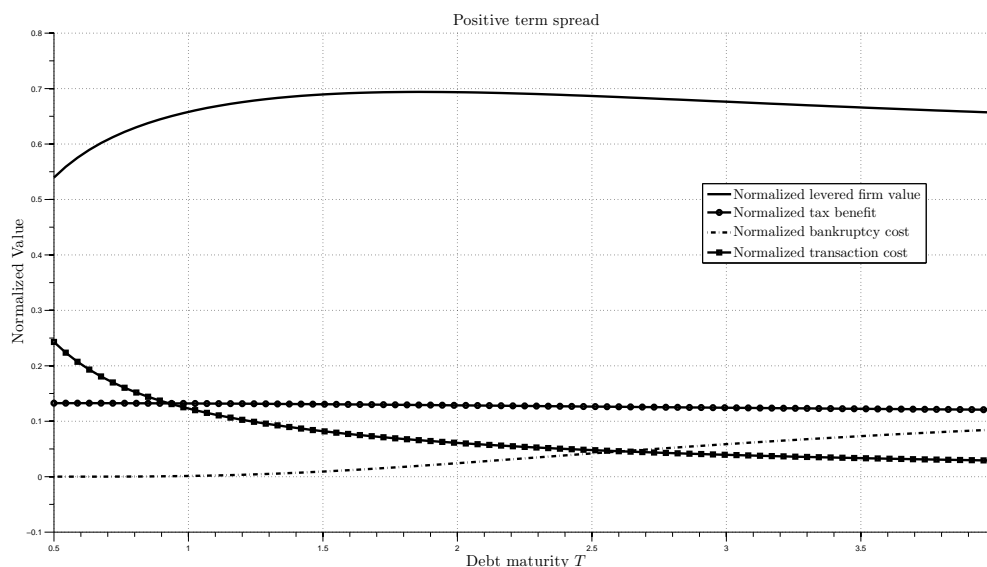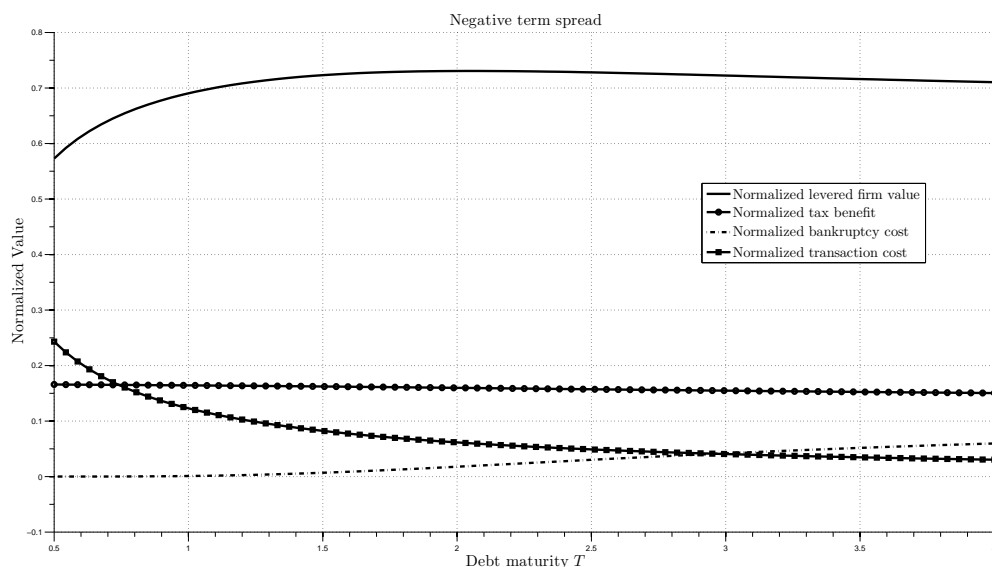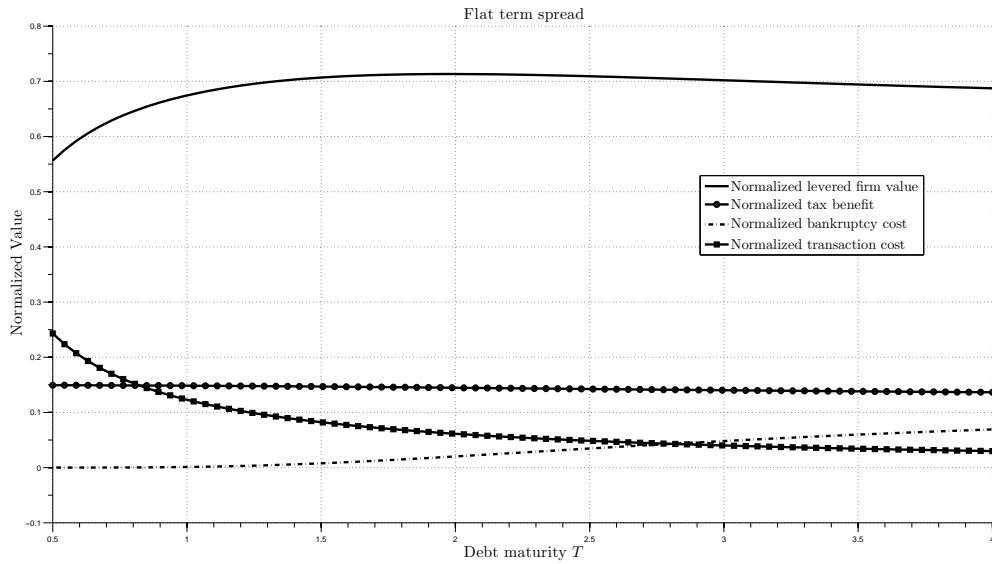
Figure 2.9: Relationship between debt maturity and (i) the total firm value (ii) tax benefits (iii) bankruptcy costs, and (iv) transaction costs. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $P_0 = 40$, $\sigma_v = 20\%$, $\phi = 2\%$, and $r_0 = 7.16\%$. Note that the term spread is approximately zero since the short rate r_0 is equal to the long rate α.
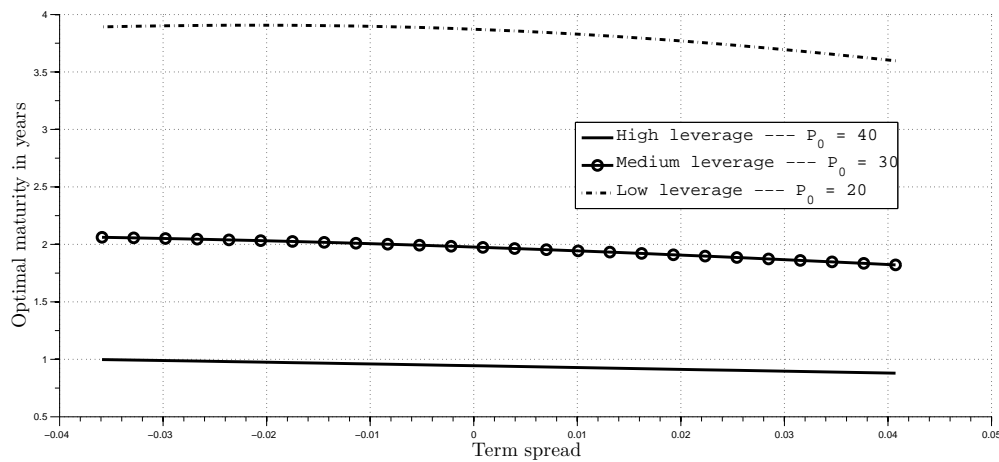
Figure 2.10: Optimal maturity as a function of the term spread for different leverage ratios. The parameters are as follows: $\beta = 0.261$, $V_0 = 100$, $\alpha = 0.0716$, $\gamma = 0.5$, $\theta = 0.35$, $y = 0.05$, $\sigma_v = 20\%$, $\phi = 2\%$, and $r_0 = 7.16\%$.

2.6.4 Effect of transaction costs, firm volatility and correlation on optimal debt maturity

Table 2.3 summarizes the effect of a small positive change in either transaction costs parameter ϕ, firm volatility σ_v or correlation ρ on the various components of the firm value. Not surprisingly, an increase in the transaction costs parameter ϕ lowers the firm value and increases optimal debt maturity. Less obvious is the comparative statics with firm volatility. An increase in firm volatility σ_v increases the probability of bankruptcy for a given leverage ratio, which increases bankruptcy costs. An increase in the probability of bankruptcy also lowers the chance of future debt issuances, which in turn lowers both tax benefits and transaction costs. However, the increase in bankruptcy costs outweighs the decrease in transaction costs. Therefore, firms optimally choose to decrease maturity. The same intuition holds for the comparative statics for correlation ρ.

Change in variable	Tax benefit	Bankruptcy cost	Transaction cost	Firm value	Optimal maturity
Transaction costs parameter ϕ	0	0	+	-	+
Firm volatility σ_v	-	+	-	-	-
Correlation ρ	-	+	-	-	-

Table 2.3: Comparative statics.

2.6.5 Discussion of the speed of adjustment toward the target leverage ratio

A primary assumption of our model is that firms adjust their capital structure toward a target leverage ratio. For a wide variety of parameters, we show that firms adjust their capital structure every 1-3 years. Specifically, our model results indicate that highly leveraged firms rollover their debt every year, while low leveraged firms roll over their debt every 2-3 years. Regardless of the leverage ratio, firms are active in adjusting their capital structure. This result is in contrast with Strebulaev (2007), who shows that firms are inactive in adjusting their capital structure most of the time.

The fact that in our results firms are active in adjusting their capital structure is consistent with Faulkender et al. (2011), who show that overleveraged firms close more than 70% of the gap between actual and target leverage ratio upon realizing positive cashflows. Even firms with near zero cash flow realization close the gap between actual and target leverage ratio by 25%. Flannery and Rangan (2006) show that a typical firm closes about one-third of the gap between actual and target leverage ratio each year. Marcus (1983), Jalilvand and Harris (1984), Leary (2002), Leary and Roberts (2005) and Welch (2012) also show that managers

are active in adjusting their capital structure.

We conclude this section by highlighting the following observation. The present literature overwhelmingly attributes the fact that firms issue short term debt during bad times to the validation of either agency theory, such as in Myers (1977), asymmetric information, such as in Diamond (1991) and Flannery and Rangan (2006), or liquidity risk, such as in Chen, Xu, and Yang (2012). Our results show that this empirical evidence is also consistent with the trade-off theory of capital structure.

2.7 Conclusion

In this paper, we explain the link between debt maturity and term spread using the trade-off theory of capital structure. Specifically, we show that:

1. Firms issue shorter term debt when term spread is positive, and increase maturity as term spread decreases.

2. Firms are incredibly active in adjusting their capital structure.

3. Highly leveraged firms issue shorter term debt compared with low leveraged firms.

4. High volatility firms issue shorter term debt compared with low volatility firms.

5. Firms choose maturity by balancing bankruptcy costs and debt rollover costs. Tax benefits play a minor role in debt maturity choice.

We develop a model of optimal maturity structure with a Vasicek (1977) interest rate process. Valuation formulas are obtained in semi-closed form. We use a novel fixed-point argument with stochastic interest rates to obtain the value of total tax benefits, bankruptcy costs, and transaction costs for a dynamic model with an infinite number of debt issuances.

We can extend the model in three major directions. First, we can endogenize the cash-flows generated by the firm. This allows us to express Tobin's Q as a function of the term spread. Second, we can add one more factor for interest rates to decompose debt maturity as a function of both term spread and level of interest rates. Lastly, we can incorporate asset substitution and other agency related frictions by endogenizing the default boundary as in Leland (1994).

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Chapter 3

Appendix

3.1 First chapter

3.1.1 Recessions and the consumption cycle

The following graphs show average percentage changes in consumption/expenditures and real-estate investments (residential and nonresidential) for the 11 NBER recessions dating from 1947-Q2 to 2012-Q2. Average percentage changes are calculated as 3-period rolling averages centered at the period. In each graph, the left vertical line marks the beginning of the recession period. The right vertical line marks its end.
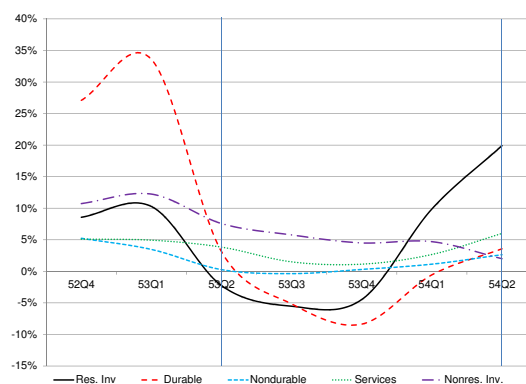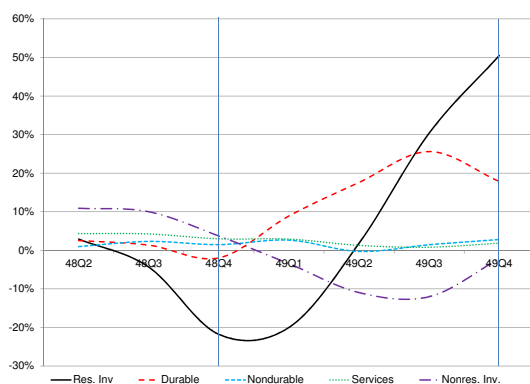
Figure 3.1: NBER recession 1948-Q4.

Figure 3.2: NBER recession 1953-Q2.
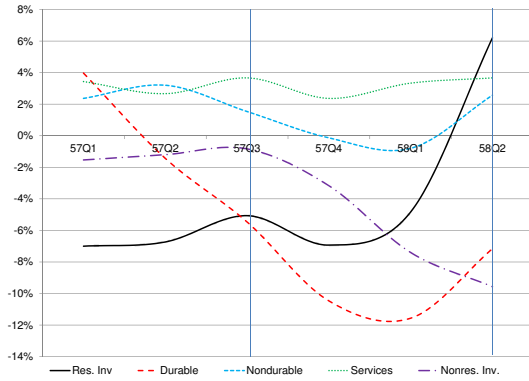
Figure 3.3: NBER recession 1957-Q3.
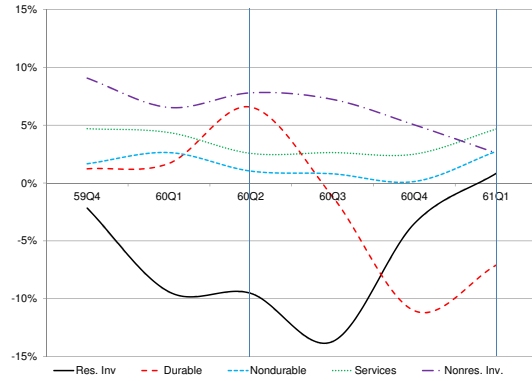
Figure 3.4: NBER recession 1960-Q2.
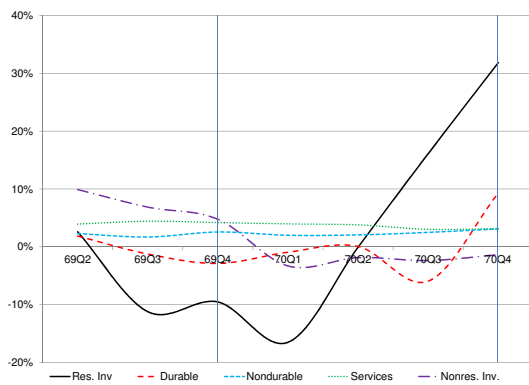
Figure 3.5: NBER recession 1969-Q4.
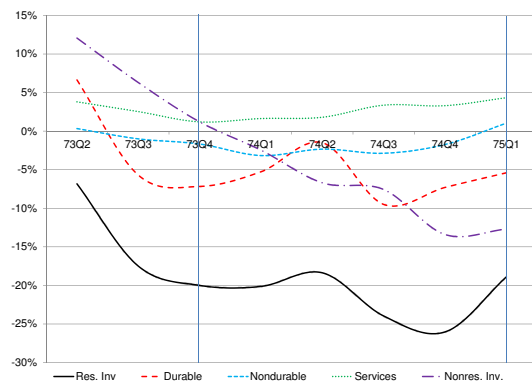
Figure 3.6: NBER recession 1973-Q4.
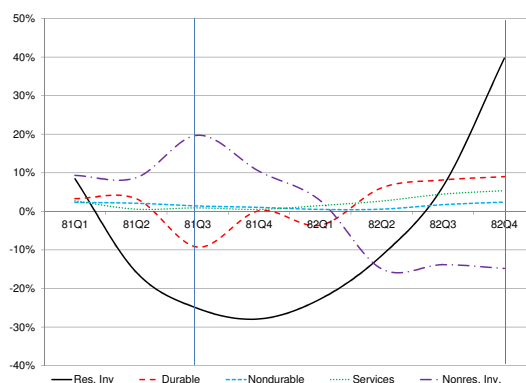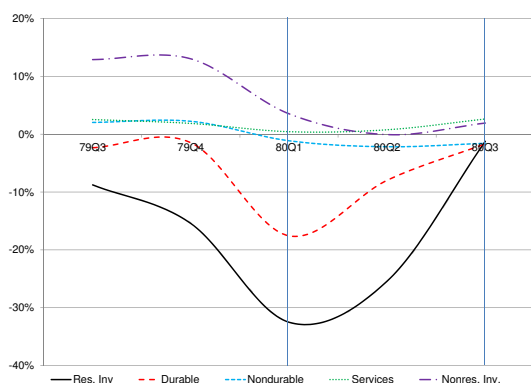
Figure 3.7: NBER recession 1980-Q1.   Figure 3.8: NBER recession 1981-Q3.
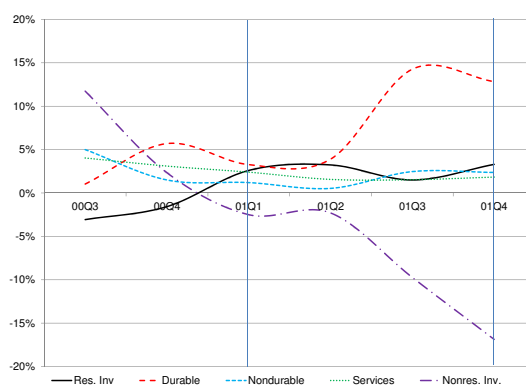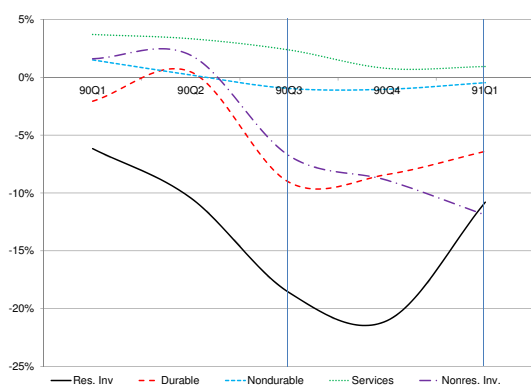
Figure 3.9: NBER recession 1990-Q3.   Figure 3.10: NBER recession 2001-Q1.
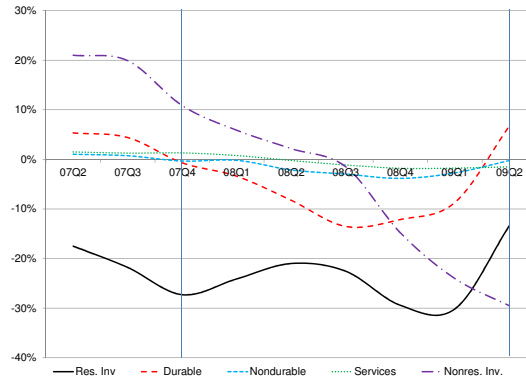
Figure 3.11: NBER recession 2007-Q4.

3.1.2 System of ODEs for the scaled value function

The planner's problem is characterized by Hamilton-Jacobi-Bellman equation (1.7). Reducing the state space involves establishing a candidate solution of the form

$$J_i(K, Z, t) = e^{-\rho t} \left(A_i \ln \left(K + Z \right) + g_i(y) \right), \tag{3.1}$$

with $y = K/(K + Z)$ representing the ratio of capital stock to total stock of the economy. After recognizing

$$\frac{Z}{K + Z} = 1 - \frac{K}{K + Z} = (1 - y),$$
$$\frac{\partial y}{\partial K} = \frac{1}{K + Z} \left(1 - \frac{K}{K + Z} \right) = \frac{(1 - y)}{(K + Z)},$$
$$\frac{\partial y}{\partial Z} = \frac{-K}{(K + Z)^2} = \frac{-y}{(K + Z)},$$

the partial derivatives of this candidate function take the following forms

$$J_{it} = -\rho e^{-\rho t} \left[A_i \ln \left(K + Z \right) + g_i(y) \right], \tag{3.2}$$

$$J_{iK} = \frac{e^{-\rho t}}{(K + Z)} \left[A_i + g_i'(y) \left(1 - y \right) \right], \tag{3.3}$$

$$J_{iKK} = \frac{e^{-\rho t}}{(K + Z)^2} \left[g_i''(y) \left(1 - y \right)^2 - 2g_i'(y) \left(1 - y \right) - A_i \right] \tag{3.4}$$

$$J_{iZ} = \frac{e^{-\rho t}}{(K + Z)} \left[A_i - g_i'(y)y \right], \tag{3.5}$$

where for clarity, I use a shorter notation with $dg_i(y)/dy = g_i'(y)$, $d^2 g_i(y)/dy^2 = g_i''(y)$.

I then express the optimal controls, the optimal durable production function, and the intraperiod utility function in terms of $g_i(y)$ and its derivatives. Use equation (1.8) and the candidate value for J_{iK} in equation (3.3) to derive

$$K_{iN}^* = \beta \frac{(K + Z)}{A_i + g_i'(y) \left(1 - y \right)}, \tag{3.6}$$

$$\ln K_{iN}^* = \ln \beta + \ln \left(K + Z \right) - \ln \left[A_i + g_i'(y) \left(1 - y \right) \right]. \tag{3.7}$$

Dividing equation (3.5) by equation (3.3) gives

$$\frac{J_{iZ}}{J_{iK}} = \frac{A_i - g_i'(y)y}{A_i + g_i'(y) \left(1 - y \right)}. \tag{3.8}$$

Plug equation (3.8) into equation (1.9). The optimal capital for the durable production process becomes

$$K_{iD}^* = Z \left[\alpha \eta \frac{A_i - g_i'(y)y}{A_i + g_i'(y) \left(1 - y \right)} \right]^{\frac{1}{1-\eta}}. \tag{3.9}$$

The optimal production flow of durable goods in equation (1.10) gives

$$\alpha K_{iD}^{*\eta} Z^{1-\eta} = \alpha Z \left[\alpha \eta \frac{A_i - g_i'(y)y}{A_i + g_i'(y) \left(1 - y \right)} \right]^{\frac{\eta}{1-\eta}}. \tag{3.10}$$

Use equation (3.7), and that $\ln Z = \ln (K + Z) + \ln (1 - y)$, to express the intraperiod utility term as

$$
\begin{aligned}
\beta \ln K_{iN}^* + (1 - \beta) \ln Z =& \beta \left(\ln \beta + \ln (K + Z) - \ln \left[A_i + g_i'(y) (1 - y) \right] \right) \\
&+ (1 - \beta) \left(\ln (K + Z) + \ln (1 - y) \right) \\
=& \beta \ln \beta - \beta \ln \left[A_i + g_i'(y) (1 - y) \right] + (1 - \beta) \ln (1 - y) + \ln (K + Z).
\end{aligned}
$$
(3.11)

I now plug into the HJB equation (1.7) the partial derivatives of the candidate solution, the optimal controls, and the intraperiod utility. After recognizing that $J_{it} = -\rho J_i$, I express the HJB equation as

$$
\begin{aligned}
0 =& e^{-\rho t} \left(\beta \ln \psi + \beta \ln K_{iN}^* + (1 - \beta) \ln Z \right) - (\rho + \lambda_j) J_i + \lambda_j J_j + J_{iK} \left(\mu_i K - K_{iN}^* - K_{iD}^* \right) \\
&+ J_{iZ} \left(\alpha K_{iD}^{*\eta} Z^{1-\eta} - \delta Z \right) + \frac{1}{2} J_{iKK} \sigma_{iK}^2 K^2.
\end{aligned}
$$
(3.12)

I now plug into the equation above the solutions expressed in terms of the new state variable y in equations (3.6)–(3.11), as well as the candidate value function and its derivatives in equations (1.12)–(3.5). Note that the term $e^{-\rho t}$ cancels out

$$
\begin{aligned}
0 =& \beta \ln \psi + \beta \ln \beta - \beta \ln \left[A_i + g_i'(y) (1 - y) \right] + (1 - \beta) \ln (1 - y) \\
&+ \ln (K + Z) - (\rho + \lambda_j) \left[A_i \ln (K + Z) + g_i(y) \right] + \lambda_j \left[A_j \ln (K + Z) + g_j(y) \right] \\
&+ \left[A_i + g_i'(y) (1 - y) \right] \frac{1}{(K + Z)} \left[\mu_i K - \beta \frac{(K + Z)}{A_i + g_i'(y) (1 - y)} - Z \left[\alpha \eta \frac{A_i - g_i'(y) y}{A_i + g_i'(y) (1 - y)} \right]^{\frac{1}{1-\eta}} \right] \\
&+ \left[A_i - g_i'(y) y \right] \frac{1}{(K + Z)} \left[\alpha Z \left[\alpha \eta \frac{A_i - g_i'(y) y}{A_i + g_i'(y) (1 - y)} \right]^{\frac{\eta}{1-\eta}} - \delta Z \right] \\
&+ \frac{1}{2} \sigma_{iK}^2 \left[g_i''(y) (1 - y)^2 - 2 g_i'(y) (1 - y) - A_i \right] \frac{1}{(K + Z)^2} K^2.
\end{aligned}
$$
(3.13)

Using $y = K/(K + Z)$ and $(1 - y) = Z/(K + Z)$, the equation above becomes

$$0 = \beta \ln (\psi \beta) - \beta \ln \left[A_i + g_i'(y)(1-y) \right] + (1-\beta) \ln (1-y)$$
$$+ \ln (K+Z) - (\rho + \lambda_j) \left[A_i \ln (K+Z) + g_i(y) \right] + \lambda_j \left[A_j \ln (K+Z) + g_j(y) \right]$$
$$+ \mu_i \left[A_i + g_i'(y)(1-y) \right] y - \beta - \left[A_i + g_i'(y)(1-y) \right] (1-y) \left[\alpha \eta \frac{A_i - g_i'(y)y}{A_i + g_i'(y)(1-y)} \right]^{\frac{1}{1-\eta}}$$
$$+ \alpha \left[A_i - g_i'(y)y \right] (1-y) \left[\alpha \eta \frac{A_i - g_i'(y)y}{A_i + g_i'(y)(1-y)} \right]^{\frac{\eta}{1-\eta}} - \delta \left[A_i - g_i'(y)y \right] (1-y)$$
$$+ \frac{1}{2} \sigma_{iK}^2 \left[g_i''(y)(1-y)^2 - 2g_i'(y)(1-y) - A_i \right] y^2. \tag{3.14}$$

The right-hand side of the equation above is equal to zero for all positive values of K, Z and y. Consequently, after grouping all terms in $\ln (K+Z)$, it must be the case that for each production regime $i, j \in \{h, \ell\}, j \neq i$ the following holds

$$0 = 1 - (\rho + \lambda_j) A_i + \lambda_j A_j$$
$$0 = 1 - (\rho + \lambda_i) A_j + \lambda_i A_i.$$

The system above is solved by setting $A_i = A_j = 1/\rho = \phi$. After plugging this result into equation (3.14), the system of ODE becomes

$$0 = -\beta (1 - \ln (\psi \beta)) - \beta \ln \left[\phi + g_i'(y)(1-y) \right] + (1-\beta) \ln (1-y) - (\rho + \lambda_j) g_i(y) + \lambda_j g_j(y)$$
$$+ \mu_i \left[\phi + g_i'(y)(1-y) \right] y - \left[\phi + g_i'(y)(1-y) \right] (1-y) \left[\alpha \eta \frac{\phi - g_i'(y)y}{\phi + g_i'(y)(1-y)} \right]^{\frac{1}{1-\eta}}$$
$$+ \alpha \left[\phi - g_i'(y)y \right] (1-y) \left[\alpha \eta \frac{\phi - g_i'(y)y}{\phi + g_i'(y)(1-y)} \right]^{\frac{\eta}{1-\eta}} - \delta \left[\phi - g_i'(y)y \right] (1-y)$$
$$+ \frac{1}{2} \sigma_{iK}^2 \left[g_i''(y)(1-y)^2 - 2g_i'(y)(1-y) - \phi \right] y^2. \tag{3.15}$$

I rewrite the equation above in a more compact form using the following notation for the marginal values of the capital stock and the durable service level for a unit size economy, or formally the marginal values of these state variables to the value function $J(K, Z, t)$ when $K + Z = 1$ and $t = 0$. Name these terms as

$$M_{iK}(y) \equiv \phi + g_i'(y)(1-y), \tag{3.16}$$
$$M_{iZ}(y) \equiv \phi - g_i'(y)y. \tag{3.17}$$

In the same spirit, I name the convexity term

$$M_{iKK}(y) \equiv g_i''(y) (1-y)^2 - 2g_i'(y) (1-y) - \phi. \tag{3.18}$$

As a result, the ODE system in its compact form becomes

$$
\begin{aligned}
0 = & -\beta \left(1 - \ln\left(\psi\beta\right)\right) - \beta \ln\left(M_{iK}(y)\right) + (1-\beta) \ln\left(1-y\right) - \left(\rho + \lambda_j\right) g_i(y) + \lambda_j g_j(y) \\
& + \mu_i y M_{iK}(y) - (\alpha\eta)^{\frac{1}{1-\eta}} M_{iZ}(y)^{\frac{1}{1-\eta}} (1-y) M_{iK}(y)^{\frac{-\eta}{1-\eta}} + \frac{1}{2}\sigma_{iK}^2 y^2 M_{iKK}(y) \\
& + \alpha \left(\alpha\eta\right)^{\frac{\eta}{1-\eta}} M_{iZ}(y)^{\frac{1}{1-\eta}} (1-y) M_{iK}(y)^{\frac{-\eta}{1-\eta}} - \delta\left(1-y\right) M_{iZ}(y).
\end{aligned}
\tag{3.19}
$$

3.1.3 Lower bounds and the limiting case of the value function

In this appendix, I investigate a non-optimal feasible strategy to both establish a lower bound for the value function and to characterize the behavior of the function $g_i(y)$ in the vicinity of the bound $y = 0$.

Consider a situation in which the social planner commits to an infinitesimal durable expenditure $K_{Dt} \to 0$, and allocate capital exclusively into the capital accumulation and nondurable production processes. As mentioned above, the representative agent would face a decreasing utility from the durable stock Z_t as it depreciates at a constant rate δ. More precisely,

$$
\begin{aligned}
\hat{J}_i(K, Z, t) = \sup_{\{K_{Ns}\}} E_t & \left[\int_t^\infty e^{-\rho s} \beta \ln\left(\psi K_{Ns}\right) ds \right] + (1-\beta) \int_t^\infty e^{-\rho s} \ln\left[Z e^{-\delta(s-t)}\right] ds \\
& \text{subject to} \\
& dK_t = \left(\mu_i K_t - K_{Nt}\right) dt + \sigma_{iK} K_t d\omega_t.
\end{aligned}
$$

The integral in the last term of the objective function can be evaluated directly

$$\int_t^\infty e^{-\rho s} \ln\left[Ze^{-\delta(s-t)}\right] ds = (\ln Z + \delta t) \int_t^\infty e^{-\rho s} ds - \delta \int_t^\infty e^{-\rho s} s\, ds$$

$$= (\ln Z + \delta t)\frac{e^{-\rho t}}{\rho} - \delta\frac{e^{-\rho t}}{\rho}t - \delta\frac{e^{-\rho t}}{\rho^2}$$

$$= \frac{e^{-\rho t}}{\rho}\left(\ln Z - \frac{\delta}{\rho}\right).$$

Define

$$V_i(K,t) = \sup_{\{K_{Ns}\}} E_t\left[\int_t^\infty e^{-\rho s}\beta \ln\left(\psi K_{Ns}\right) ds\right].$$

Then, the HJB equation for this problem becomes

$$0 = \sup_{\{K_N\}}\left[e^{-\rho t}\left(\beta \ln\psi + \beta \ln K_N\right) + V_{it} + V_{iK}\left(\mu_i K - K_N\right) + \frac{1}{2}V_{iKK}\sigma_{iK}^2 K^2 + \lambda_j\left(V_j - V_i\right)\right].$$

Assume a candidate solution of the form $V_i(K,t) = e^{-\rho t}\left(A_i \ln K + C_i\right)$. This yields the following partial derivatives of the value function

$$V_{it} = -\rho e^{-\rho t}\left(A_i \ln K + C_i\right)$$

$$V_{iK} = e^{-\rho t}\frac{A_i}{K}$$

$$V_{iKK} = -e^{-\rho t}\frac{A_i}{K^2}.$$

First-order condition on K_N implies

$$K_N^* = e^{-\rho t}\beta V_{iK}^{-1} = \frac{\beta K}{A_i}.$$

Plugging these results into the HJB equation gives

$$0 = \beta \left(\ln K + \ln \frac{\psi \beta}{A_i} \right) + \lambda_j \left(A_j \ln K + C_j \right) - \left(\lambda_j + \rho \right) \left(A_i \ln K + C_i \right) + \frac{A_i}{K} \left(\mu_i K - \frac{\beta K}{A_i} \right) - \frac{1}{2} \sigma_{iK}^2 A_i$$

$$= \ln K \left(\beta + \lambda_j A_j - \left(\lambda_j + \rho \right) A_i \right) + \beta \ln \frac{\psi \beta}{A_i} + \lambda_j C_j - \left(\lambda_j + \rho \right) C_i + A_i \mu_i - \beta - \frac{1}{2} \sigma_{iK}^2 A_i.$$

Similarly to the more general case, it must be that for each production regime $i, j \in \{h, \ell\}, j \neq i$ the following equations hold

$$0 = \beta + \lambda_j A_j - \left(\rho + \lambda_j \right) A_i$$
$$0 = \beta + \lambda_i A_i - \left(\rho + \lambda_i \right) A_j.$$

The system above is solved by setting $A_i = A_j = \beta / \rho$. Then, for $i, j \in \{h, \ell\}, j \neq i$

$$\rho C_i + \lambda_j \left(C_i - C_j \right) = -\beta \left(1 - \ln \left(\psi \rho \right) \right) + \frac{\beta}{\rho} \left(\mu_i - \frac{1}{2} \sigma_{iK}^2 \right) \tag{3.20}$$

$$\rho C_j + \lambda_i \left(C_j - C_i \right) = -\beta \left(1 - \ln \left(\psi \rho \right) \right) + \frac{\beta}{\rho} \left(\mu_j - \frac{1}{2} \sigma_{jK}^2 \right). \tag{3.21}$$

This is a system of two equations for the unknowns C_i and C_j. Subtracting the first equation from the second yields

$$C_i - C_j = \frac{\beta}{\rho} \left[\frac{\left(\mu_i - \frac{1}{2} \sigma_{iK}^2 \right) - \left(\mu_j - \frac{1}{2} \sigma_{jK}^2 \right)}{\rho + \lambda_i + \lambda_j} \right]. \tag{3.22}$$

Substitute this into the first equation and solve for C_i. This gives

$$C_i = \frac{\beta}{\rho^2} \left[\frac{\left(\lambda_i + \rho \right) \mu_i + \lambda_j \mu_j}{\rho + \lambda_i + \lambda_j} - \rho \left(1 - \ln \left(\psi \rho \right) \right) - \frac{1}{2} \frac{\left(\lambda_i + \rho \right) \sigma_{iK}^2 + \lambda_j \sigma_{jK}^2}{\rho + \lambda_i + \lambda_j} \right]. \tag{3.23}$$

Then, the value function for this strategy takes the form

$$\hat{J}_i(K, Z, t) = \frac{e^{-\rho t}}{\rho} \left[\beta \ln K + \rho C_i + (1 - \beta) \ln Z - (1 - \beta) \frac{\delta}{\rho} \right].$$

The equation above can be re-expressed in terms of the new state variable y. This gives

$$\hat{J}_i(K, Z, t) = \frac{e^{-\rho t}}{\rho} \left[\ln (K + Z) + \beta \left(\ln y + \frac{\rho}{\beta} C_i \right) + (1 - \beta) \left(\ln (1 - y) - \frac{\delta}{\rho} \right) \right]. \tag{3.24}$$

Since $J_i(K, Z, t) \geq \hat{J}_i(K, Z, t)$, a comparison of the expression above with the candidate value function for the general case in equation (1.12) establishes the following lower bound for the function $g_i(y)$

$$g_i(y) \geq \hat{g}_i(y) = \frac{\beta}{\rho} \left(\ln y + \frac{\rho}{\beta} C_i \right) + \frac{1 - \beta}{\rho} \left(\ln (1 - y) - \frac{\delta}{\rho} \right). \tag{3.25}$$

This result shows that at the extremes when $y \to 0$ and $y \to 1$, the function $g_i(y)$ cannot decrease at a faster rate than that of the logarithm functions in the first and second terms in the right-hand side of the equation above, respectively.

3.1.4 Recasting the system of ODEs

In this appendix, I derive the new system of ODEs for the scaled value function using the transformation

$$g_i(y) = \phi \left[\beta \ln y + (1 - \beta) \ln (1 - y) + f_i(y) \right]. \tag{3.26}$$

I start by assessing the derivatives

$$g_i'(y) = \phi \left[\frac{\beta - y}{y (1 - y)} + f_i'(y) \right],$$

$$g_i''(y) = \phi \left[-\frac{(\beta - 2\beta y + y^2)}{y^2 (1 - y)^2} + f_i''(y) \right],$$

and applying these results to the scaled marginal values

$$M_{iK}(y) = \frac{\phi}{y} \left[\beta + f_i(y)y\left(1-y\right) \right],$$

$$M_{iZ}(y) = \frac{\phi}{1-y} \left[(1-\beta) - f_i(y)y\left(1-y\right) \right],$$

$$M_{iK}(y) = \frac{\phi}{y^2} \left[-\beta - 2f_i'(y)y^2\left(1-y\right) - f_i''(y)y^2\left(1-y\right)^2 \right].$$

After plugging these partial results into the ODE system in equation (1.13), the terms $\beta \ln y$ and $(1-\beta)\ln(1-y)$ cancel out. The recasted ODE system becomes

$$0 = -\rho\beta\left(1 - \ln\left[\psi\rho\beta\right]\right) - \rho\beta\ln\left[\beta + f_i'(y)y\left(1-y\right)\right] - \left(\rho + \lambda_j\right)f_i(y) + \lambda_j f_j(y)$$

$$+ \mu_i\left[\beta + f_i'(y)y\left(1-y\right)\right] - (\alpha\eta)^{\frac{1}{1-\eta}}\left(\frac{y}{1-y}\right)^{\frac{\eta}{1-\eta}}\left[(1-\beta) - f_i'(y)y\left(1-y\right)\right]^{\frac{1}{1-\eta}}\left[\beta + f_i'(y)y\left(1-y\right)\right]^{\frac{-\eta}{1-\eta}}$$

$$+ \frac{1}{2}\sigma_{iK}^2 y^2\left[-\beta - 2f_i'(y)y^2\left(1-y\right) - f_i''(y)y^2\left(1-y\right)^2\right]$$

$$+ \alpha\left(\alpha\eta\right)^{\frac{\eta}{1-\eta}}\left(\frac{y}{1-y}\right)^{\frac{\eta}{1-\eta}}\left[(1-\beta) - f_i(y)y\left(1-y\right)\right]^{\frac{1}{1-\eta}}\left[\beta + f_i'(y)y\left(1-y\right)\right]^{\frac{-\eta}{1-\eta}}$$

$$- \delta\left[(1-\beta) - f_i(y)y\left(1-y\right)\right]. \tag{3.27}$$

It is easy to verify that the dependence of the efficient parameter α on the state variable y yields a numerically manageable system of ODEs. Applying the efficiency specification to equation (1.20) and the compact notation

$$M_{iK}^f(y) = \beta + f_i'(y)y\left(1-y\right), \tag{3.28}$$

$$M_{iZ}^f(y) = (1-\beta) - f_i'(y)y\left(1-y\right), \tag{3.29}$$

$$M_{iKK}^f(y) = -\beta - 2f_i'(y)y^2\left(1-y\right) - f_i''(y)y^2\left(1-y\right)^2, \tag{3.30}$$

results in

$$0 = -\rho\beta\left(1 - \ln\left[\psi\rho\beta\right]\right) - \rho\beta\ln\left(M_{iK}^f(y)\right) - \left(\rho + \lambda_j\right)f_i(y) + \lambda_j f_j(y) + \mu_i M_{iK}^f(y)$$

$$- (\theta\eta)^{\frac{1}{1-\eta}} y^{\frac{\eta}{1-\eta}}\left(1-y\right)M_{iZ}^f(y)^{\frac{1}{1-\eta}}M_{iK}^f(y)^{\frac{-\eta}{1-\eta}} + \frac{1}{2}\sigma_{iK}^2 y^2 M_{iKK}^f(y)$$

$$+ \theta\left(\theta\eta\right)^{\frac{\eta}{1-\eta}} y^{\frac{\eta}{1-\eta}}\left(1-y\right)M_{iZ}^f(y)^{\frac{1}{1-\eta}}M_{iK}^f(y)^{\frac{-\eta}{1-\eta}} - \delta M_{iZ}^f(y). \tag{3.31}$$

Note that the durable capital and durable expenditure terms (the first terms in the second and third lines in the equation above) converge to zero in the limits $y \to \{0, 1\}$. Note also that the newly transformed value function converges to bounded values in these limits since

$$\lim_{y \to \{0,1\}} M_{iK}^f(y) = \beta,$$

$$\lim_{y \to \{0,1\}} M_{iZ}^f(y) = 1 - \beta,$$

$$\lim_{y \to \{0,1\}} M_{iK}^f(y) = \beta,$$

given that first derivative of $f_i(y)$ does not grow at a faster rate than $1/y^{\frac{\eta}{1-\eta}}$ and $1/(1-y)$ when the state variable y approach the limits zero and one respectively.

3.1.5   Numerical techniques

I solve the ODE system by assuming a finitely lived representative agent. As a result, I use the explicit method for solving a system of partial differential equations (PDEs), which parallels the ODE system in equation (3.31). The solution for the infinitely lived agent is simply the steady state solution resulting from extending the life of the representative agent to infinity.

The system of PDEs for the finitely lived agent is analogous to equation (3.31) with an additional term $f_{it}(y, t)$ reflecting the fact that the scaled and transformed value functions are time dependent. Formally,

$$J_i(K, Z, t) = \phi \ln(K + Z) + g_i(y, t),$$
$$g_i(y, t) = \phi \left[\beta \ln y + (1 - \beta) \ln(1 - y) + f_i(y, t) \right].$$

The PDE system is written as

$$0 = - \rho\beta \left(1 - \ln[\psi\rho\beta] \right) - \rho\beta \ln\left(M_{iK}^f(y) \right) - (\rho + \lambda_j) f_i(y, t) + \lambda_j f_j(y, t) + f_{it}(y, t)$$

$$+ \mu_i M_{iK}^f(y, t) - (\theta\eta)^{\frac{1}{1-\eta}} y^{\frac{\eta}{1-\eta}} (1 - y) M_{iZ}^f(y, t)^{\frac{1}{1-\eta}} M_{iK}^f(y, t)^{\frac{-\eta}{1-\eta}} + \frac{1}{2}\sigma_{iK}^2 y^2 M_{iKK}^f(y, t)$$

$$+ \theta (\theta\eta)^{\frac{\eta}{1-\eta}} y^{\frac{\eta}{1-\eta}} (1 - y) M_{iZ}^f(y, t)^{\frac{1}{1-\eta}} M_{iK}^f(y, t)^{\frac{-\eta}{1-\eta}} - \delta M_{iZ}^f(y, t), \qquad (3.32)$$

where

$$M_{iK}^{f}(y,t) = \beta + f_{iy}(y,t)y\,(1-y)\,,$$
$$M_{iZ}^{f}(y,t) = (1-\beta) - f_{iy}(y,t)y\,(1-y)\,,$$
$$M_{iKK}^{f}(y,t) = -\beta - 2f_{iy}(y,t)y^{2}\,(1-y) - f_{iyy}(y,t)y^{2}\,(1-y)^{2}\,,$$

The space-time mesh used for the explicit method is carefully chosen to avoid numerical instability. Define an equally spaced mesh for the time dimension as $t \in \{t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \dots T\}$. As for the state space, define a more general discretization $y \in \{y_0, y_1, \dots, y_N\}$ were $\Delta y_n = y_n - y_{n-1}$. A successful implementation of the explicit method requires that the stability parameter

$$s_n \equiv \frac{\sigma_{y_n}^2 \Delta t}{\Delta y_n^2} = \sigma_K^2 \Delta t \frac{y_n^2\,(1-y_n)^2}{\Delta y_n^2} < \frac{1}{2},$$

where in the first equality I used equation (1.19), which defines the time dependency of σ_y on the state variable y. The expression above suggests that the state variable dependence on the stability parameter goes away by choosing an adaptive mesh for y. I choose the spacing

$$\Delta y_n = 4 \times y_n\,(1-y_n)\,\Delta y_{max},$$

which yields the following condition for numerical stability

$$s = \frac{\sigma_K^2 \Delta t}{(4\Delta y_{max})^2} < \frac{1}{2}.$$

The actual implementation uses values of Δt ranging from 30 to 90 days depending on the chosen model parameters. The value of Δy_{max} is set such that $s = 1/3$.

The backward recursion mechanism employed in the explicit method involves assessing the transformed value function at time t with the approximated time and space derivatives taken at time $t + \Delta t$. This gives the following recursive relationship for the transformed value function

$$f_i(y_n,t) = \frac{1}{1+\rho\Delta t}\Bigg[(1-\lambda_j\Delta t)\, f_i(y_n,t+\Delta t) + \lambda_j\Delta t f_j(y_n,t+\Delta t)$$

$$+ \Delta t\Bigg(-\rho\beta\left(1-\ln[\psi\rho\beta]\right) - \rho\beta\ln\left(M_{iK}^f(y_n,t+\Delta t)\right) + \mu_i M_{iK}^f(y_n,t+\Delta t)$$

$$- (\theta\eta)^{\frac{1}{1-\eta}} y_n^{\frac{\eta}{1-\eta}}(1-y_n)\, M_{iZ}^f(y_n,t+\Delta t)^{\frac{1}{1-\eta}} M_{iK}^f(y_n,t+\Delta t)^{\frac{-\eta}{1-\eta}} + \frac{1}{2}\sigma_{iK}^2 y_n^2 M_{iKK}^f(y_n,t+\Delta t)$$

$$+ \theta\,(\theta\eta)^{\frac{\eta}{1-\eta}} y_n^{\frac{\eta}{1-\eta}}(1-y_n)\, M_{iZ}^f(y_n,t+\Delta t)^{\frac{1}{1-\eta}} M_{iK}^f(y_n,t+\Delta t)^{\frac{-\eta}{1-\eta}} - \delta M_{iZ}^f(y_n,t+\Delta t)\Bigg)\Bigg].$$

The implemented algorithm starts by populating the ending nodes of the transformed value function with their corresponding lower bounds. The final solution is achieved when the absolute relative change on $\|f_i(y_n,t)-f_i(y_n,t+\Delta t)\|$ is negligible between backward iterations, which indicates numerical convergence to the value function for an infinitely lived agent.

3.1.6   Optimal consumption, expenditures and prices

In this section, I provide the derivation of the processes for the optimal consumption, optimal expenditure, the equilibrium durable price, and the state price deflator. Before proceeding, I extend the definitions in equations (1.14)-(1.16) in order to keep the notation compact

$$M_{iKZ}(y) \equiv -\phi - g_i'(y)\,(1-2y) - g_i''(y)y\,(1-y),$$
$$M_{iZZ}(y) \equiv -\phi + 2g_i'(y)y + g_i''(y)y^2,$$
$$M_{iKKK}(y) \equiv 2\phi + 6g_i'(y)\,(1-y) - 6g_i''(y)\,(1-y)^2 + g_i'''(y)\,(1-y)^3,$$
$$M_{iKKZ}(y) \equiv 2\phi + 2g_i'(y)\,(2-3y) - 2g_i''(y)\,(1-3y)\,(1-y) - g_i'''(y)y\,(1-y)^2.$$

All derivations involve a straightforward application of the generalized Ito's lemma and a long chain of algebraic manipulations. The following equations provide the stochastic process for nondurable consumption and durable expenditure respectively

$$dY_{iN_t}^* = Y_{iN_t}^*\left(\mu_{iN}dt + \sigma_{iN}d\omega_t\right) + \lambda_j\left(Y_{jN_t}^* - Y_{iN_t}^*\right)dn_t, \tag{3.33}$$

$$dY_{iD_t}^* = Y_{iD_t}^*\left(\mu_{iD}dt + \sigma_{iDN}d\omega_t\right) + \lambda_j\left(Y_{jD_t}^* - Y_{iD_t}^*\right)dn_t, \tag{3.34}$$

where the drift and volatility are expressed as

$$\mu_{iN} = -\frac{M_{iKK}(y)}{M_{iK}(y)}\left(\mu_i y - \frac{\beta}{M_{iK}(y)} - (\alpha(y)\eta)^{\frac{1}{1-\eta}}(1-y)\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{1}{1-\eta}}\right)$$

$$-(1-y)\frac{M_{iKZ}(y)}{M_{iK}(y)}\left(\alpha(y)(\alpha(y)\eta)^{\frac{\eta}{1-\eta}}\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{\eta}{1-\eta}} - \delta\right)$$

$$+\frac{1}{2}\sigma_K^2 y^2\left[2\left(\frac{M_{iKK}(y)}{M_{iK}(y)}\right)^2 - \frac{M_{iKKK}(y)}{M_{iK}(y)}\right],$$

$$\sigma_{iN} = -\sigma_K y\frac{M_{iKK}(y)}{M_{iK}(y)},$$

and

$$\mu_{iD} = \frac{\eta}{1-\eta}\left[\frac{M_{iKZ}(y)}{M_{iZ}(y)} - \frac{M_{iKK}(y)}{M_{iK}(y)}\right]\left(\mu_i y - \frac{\beta}{M_{iK}(y)} - (\alpha(y)\eta)^{\frac{1}{1-\eta}}(1-y)\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{1}{1-\eta}}\right)$$

$$+\left[\frac{\eta}{1-\eta}(1-y)\left(\frac{M_{iZZ}(y)}{M_{iZ}(y)} - \frac{M_{iKZ}(y)}{M_{iK}(y)}\right) - 1\right]\left(\alpha(y)(\alpha(y)\eta)^{\frac{\eta}{1-\eta}}\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{\eta}{1-\eta}} - \delta\right)$$

$$+\frac{1}{2}\sigma_K^2\frac{\eta}{1-\eta}y^2\left[\frac{\eta}{1-\eta}\left(\frac{M_{iKZ}(y)}{M_{iZ}(y)} - \frac{M_{iKK}(y)}{M_{iK}(y)}\right)^2 + \frac{M_{iKKZ}(y)}{M_{iZ}(y)} - \left(\frac{M_{iKZ}(y)}{M_{iZ}(y)}\right)^2\right.$$

$$\left.+\left(\frac{M_{iKK}(y)}{M_{iK}(y)}\right)^2 - \frac{M_{iKKK}(y)}{M_{iK}(y)}\right],$$

$$\sigma_{iD} = \sigma_K\frac{\eta}{1-\eta}y\frac{M_{iKK}(y)}{M_{iK}(y)},$$

The dynamic of the state price deflator becomes

$$d\zeta_{it} = \zeta_{it}\left(\mu_{i\zeta}dt + \sigma_{i\zeta}d\omega_t\right) + \lambda_j\left(\zeta_{jt} - \zeta_{it}\right)dn_t \tag{3.35}$$

where

$$\mu_{i\zeta} = \frac{M_{iKK}(y)}{M_{iK}(y)}\left(\mu_i y - \frac{\beta}{M_{iK}(y)} - (\alpha\eta)^{\frac{1}{1-\eta}}(1-y)\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{1}{1-\eta}}\right)$$

$$+(1-y)\frac{M_{iKZ}(y)}{M_{iK}(y)}\left(\alpha(\alpha\eta)^{\frac{\eta}{1-\eta}}\left(\frac{M_{iZ}(y)}{M_{iK}(y)}\right)^{\frac{\eta}{1-\eta}} - \delta\right) + \frac{1}{2}\sigma_K^2 y^2\frac{M_{iKKK}(y)}{M_{iK}(y)} - \rho, \tag{3.36}$$

$$\sigma_{i\zeta} = \sigma_K y\frac{M_{iKK}(y)}{M_{iK}(y)}. \tag{3.37}$$

Finally, the durable price dynamic is derived by a straightforward application of the generalized Ito's lemma on equation (1.26)

$$dp_{it} = p_{it} \left(\mu_{ip} dt + \sigma_{ip} d\omega_t \right) + \lambda_j \left(p_{jt} - p_{it} \right) dn_t. \tag{3.38}$$

where

$$
\mu_{ip} = \left[\frac{M_{iKZ}(y)}{M_{iZ}(y)} - \frac{M_{iKK}(y)}{M_{iK}(y)} \right] \left(\mu_i y - \frac{\beta}{M_{iK}(y)} - (\alpha \eta)^{\frac{1}{1-\eta}} (1-y) \left(\frac{M_{iZ}(y)}{M_{iK}(y)} \right)^{\frac{1}{1-\eta}} \right)
$$

$$
+ (1-y) \left[\frac{M_{iZZ}(y)}{M_{iZ}(y)} - \frac{M_{iKZ}(y)}{M_{iK}(y)} \right] \left(\alpha (\alpha \eta)^{\frac{\eta}{1-\eta}} \left(\frac{M_{iZ}(y)}{M_{iK}(y)} \right)^{\frac{\eta}{1-\eta}} - \delta \right)
$$

$$
+ \sigma_K^2 y^2 \left[-\frac{M_{iKKK}(y)}{M_{iZ}(y)} - 2 \frac{M_{iKK}(y)}{M_{iK}(y)} \left(\frac{M_{iKZ}(y)}{M_{iZ}(y)} - \frac{M_{iKK}(y)}{M_{iK}(y)} \right) + \frac{M_{iKKZ}(y)}{M_{iZ}(y)} \right], \tag{3.39}
$$

$$
\sigma_{ip} = \sigma_K y \left[\frac{M_{iKZ}(y)}{M_{iZ}(y)} - \frac{M_{iKK}(y)}{M_{iK}(y)} \right]. \tag{3.40}
$$

3.2 Second chapter

3.2.1 Time series plot of the percent long term debt share
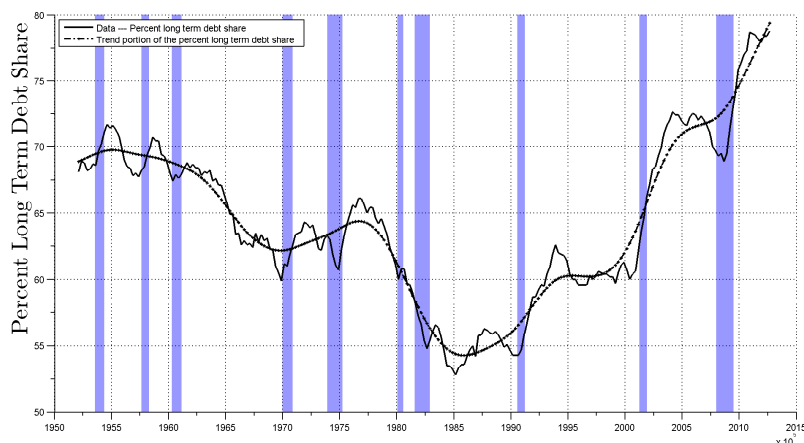
Figure 3.12: Long term debt share of non-financial corporate business. The solid line is the raw data and the dashed dotted line is the trend as calculated by the Hodrick-Prescott filter. Shaded bands in gray are NBER recession dates.

3.2.2 Time series plot of the term spread

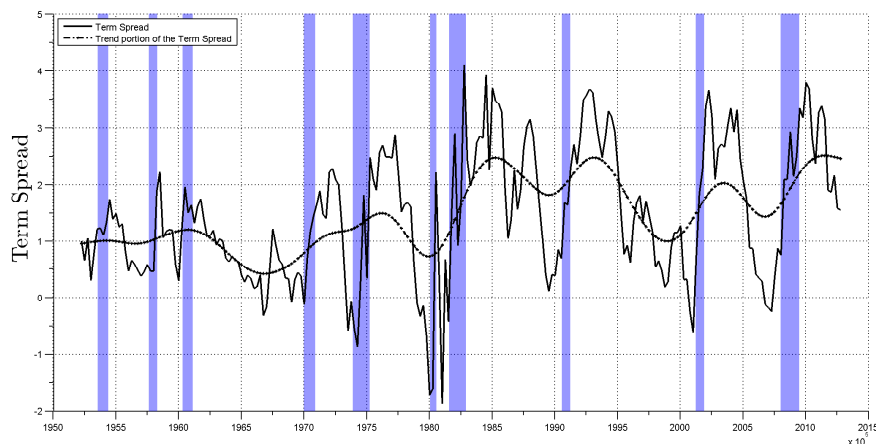Figure 3.13: Term Spread (difference between 10-year Treasure note yield and the 3-month Treasury bill). The solid line is the raw data and the dashed dotted line is the trend as calculated by the Hodrick-Prescott filter. Shaded bands in gray are NBER recession dates.

3.2.3 Derivation of the value of a risky zero coupon bond

$$
\begin{aligned}
D_{\text{zero}}(t, T, r_t; X_0) &= \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \{1 - \gamma \, \mathbb{I}(\tau \leq T)\} \right] \\
&= \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times 1 \right] - \gamma \, \mathbb{E}_t \left[e^{\int_t^T r_u \, du} \times \mathbb{I}(\tau \leq T) \right] \qquad (3.41)
\end{aligned}
$$

The first term represents the present value of one dollar upon no default. This expression is simply the value of a default free zero coupon bond $\Lambda(r_t, t, T)$. We use a change of measure to evaluate the second term.

Using Ito's Lemma, the dynamics of the zero coupon bond price are

$$
d\Lambda(r_t, t, T) = \Lambda(r_t, t, T) \left(r_t dt + \sigma_p(t, T) \, d\mathbb{W}_{rt} \right),
$$

where

$$
\sigma_p(t, T) = \frac{\sigma_r \, \Lambda_r(r_t, t, T)}{\Lambda(r_t, t, T)} = \sigma_r \, B(t, T).
$$

Let

$$
\eta_t = e^{\int_0^t \sigma_p(s,T) d\mathbb{W}_{rs} - \int_0^t \sigma_p^2(s,T) ds}.
$$

Note that $\eta_0 = 1$ and $d\eta_t = \eta_t \, \sigma_p(t, T) \, d\mathbb{W}_{rt}$. Furthermore, the since $\sigma_p(t, T)$ is deterministic, the Novikov condition

$$
\mathbb{E} \left[e^{\frac{1}{2} \int_0^T \sigma_p^2(s,T) \, ds} \right] < \infty
$$

is satisfied trivially. Therefore, it follows from Girsanov theorem that

$$
\begin{pmatrix} \mathbb{W}_{vt}^{\mathbb{T}} \\ \mathbb{W}_{rt}^{\mathbb{T}} \end{pmatrix} = \begin{pmatrix} \mathbb{W}_{vt} \\ \mathbb{W}_{rt} \end{pmatrix} - \int_0^t ds \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sigma_p(s, T) \end{pmatrix} \qquad (3.42)
$$

is a martingale under probability measure $\mathbb{Q}^{\mathbb{T}}$, which is given by

$$
\left. \frac{d\mathbb{Q}^{\mathbb{T}}}{d\mathbb{Q}} \right|_{\mathcal{F}_t} = \eta_t \quad \forall \quad t \leq T.
$$

Standard calculations yield that

$$
\Lambda(r_t, t, T) = \Lambda(r_0, 0, T) \times \eta_t \times e^{\int_0^t r_s ds}, \qquad (3.43)
$$

and

$$
e^{-\int_0^T r_s ds} = \Lambda(r_0, 0, T) \times \eta_T. \qquad (3.44)
$$

Then, we have that

$$
\begin{aligned}
\mathbb{E}_t\left[e^{-\int_t^T r_u\,du} \times \mathbb{I}(\tau \leq T)\right] &= \mathbb{E}_t\left[e^{-\int_0^T r_u\,du} \times e^{\int_0^t r_u\,du} \times \mathbb{I}(\tau \leq T)\right] \\
&= e^{\int_0^t r_u\,du}\,\mathbb{E}_t\left[e^{-\int_0^T r_u\,du} \times \mathbb{I}(\tau \leq T)\right] \\
&= e^{\int_0^t r_u\,du}\,\mathbb{E}_t\left[\Lambda(r_0,0,T) \times \eta_T \times \mathbb{I}(\tau \leq T)\right] \\
&= e^{\int_0^t r_u\,du}\,\Lambda(r_0,0,T)\,\mathbb{E}_t\left[\eta_T \times \mathbb{I}(\tau \leq T)\right] \\
&= e^{\int_0^t r_u\,du}\,\Lambda(r_0,0,T)\,\mathbb{E}_t^{\mathbb{T}}\left[\mathbb{I}(\tau \leq T) \times \mathbb{E}_t\left[\eta_T\right]\right] \\
&= e^{\int_0^t r_u\,du}\,\Lambda(r_0,0,T)\,\eta_t\,\mathbb{E}_t^{\mathbb{T}}\left[\mathbb{I}(\tau \leq T)\right] \\
&= \Lambda(r_t,t,T)\,\mathbb{E}_t^{\mathbb{T}}\left[\mathbb{I}(\tau \leq T)\right]
\end{aligned}
$$

Where we apply equation (3.44) in the third equality, abstract Bayes' rule in the fifth equality, and equation (3.43) in the last equality. Next we show that

$$
\mathbb{E}_t^{\mathbb{T}}\left[\mathbb{I}(\tau \leq T)\right] = G(T,T,X_0)
$$

where

$$
G(t;T,X_0) = \quad N\left(\frac{-X_0 - \mu_g(t;T)}{\sqrt{\Sigma(t;T)}}\right) + e^{-\frac{2X_0\mu_g(t;T)}{\Sigma(t;T)}}\,N\left(\frac{-X_0 + \mu_g(t;T)}{\sqrt{\Sigma(t;T)}}\right) \qquad (3.45)
$$

with

$$
\mu_g(t;T) = \frac{\int_0^t -\sigma^2(s;T)\,ds}{2} = \frac{-\Sigma(t;T)}{2};
$$

and

$$
\begin{aligned}
\Sigma(t;T) = \int_0^t \sigma^2(s;T)\,ds =& \sigma_V^2 t + \frac{\sigma_r^2}{\beta^2}\left(t + e^{-2\beta(T-t)}B_2(t) - 2e^{-\beta(T-t)}B_1(t)\right) \\
&+ \frac{2\rho\sigma_V\sigma_r}{\beta}\left(t - e^{-\beta(T-t)}B_1(t)\right).
\end{aligned}
$$

and

$$
\sigma(t;T) = \sqrt{\sigma_V^2 + \sigma_p^2(t;T) + 2\rho\sigma_V\sigma_p(t;T)};\ B_1(t) = \frac{(1 - e^{-\beta t})}{\beta};\ B_2(t) = \frac{(1 - e^{-2\beta t})}{2\beta}.
$$

Proof for the expression of $G(t,T,X_0)$

Given the dynamics of X_t, it is well known that the distribution of the first passage hitting times G() in equation (3.45) satisfies the Kolmogorov Backward Equation (KBE). Substituting the dynamics of X_t, KBE could be written as

$$\left(-\frac{1}{2}\sigma^2(t;T)\right)\frac{\partial G}{\partial X_0} + \frac{1}{2}\sigma^2(t;T)\frac{\partial^2 G}{\partial X_0^2} - \frac{\partial G}{\partial t} = 0$$

with the boundary conditions:

$$G(0;T,X_0) = 0 \text{ for } X_0 > 0 \text{ and } G(t;T,0) = 1.$$

It is sufficient to verify that equation (3.45) satisfies the Kolmogorov Backward Equation. A few tricks are useful. We define the pdf of a standard normal as

$$n(x) = n(-x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

. Straightforward calculations show that

$$n'(x) = -xn(x); \qquad \mu_g(t;T) = \frac{-\Sigma(t;T)}{2}.$$

Next, we evaluate the partial derivatives by brute force. Tedious algebra shows that

$$\frac{\partial G}{\partial X_0} = -n\left(\frac{X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\frac{1}{\sqrt{\Sigma(t;T)}} - n\left(\frac{X_0 + \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\frac{e^{X_0}}{\sqrt{\Sigma(t;T)}} + e^{X_0}N\left(\frac{-X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right),$$

$$\frac{\partial G}{\partial t} = n\left(\frac{X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\left(\frac{X_0\sigma^2(t;T)}{2\Sigma^{3/2}} + \frac{\sigma^2(t;T)}{4\Sigma(t;T)^{1/2}}\right) + e^{X_0}n\left(\frac{X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\left(\frac{X_0\sigma^2(t;T)}{2\Sigma^{3/2}} + \frac{\sigma^2(t;T)}{4\Sigma(t;T)^{1/2}}\right),$$

and

$$\begin{aligned}
\frac{\partial^2 G}{\partial X_0^2} &= n\left(\frac{X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\frac{X_0 - \frac{\Sigma(t;T)}{2}}{\Sigma(t;T)^{\frac{3}{2}}} + n\left(\frac{X_0 + \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\frac{(X_0 + \frac{\Sigma(t;T)}{2})e^{X_0}}{\Sigma(t;T)^{\frac{3}{2}}} \\
&- 2n\left(\frac{X_0 + \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right)\frac{e^{X_0}}{\sqrt{\Sigma(t;T)}} + e^{X_0}N\left(\frac{-X_0 - \frac{\Sigma(t;T)}{2}}{\sqrt{\Sigma(t;T)}}\right).
\end{aligned}$$

Substituting the partial derivatives in the Kolmogorov Backward Equation, we see that equation (3.45) is satisfied.

3.2.4   Proof of the expression of the bankruptcy cost in a one period debt issuance model

$$bc(r_0, \zeta, T, V_0) = \mathbb{E}_0 \left[\int_0^T ds\, \gamma\, V_D(r_s, s, T)\, \delta(s - \tau)\, e^{-\int_0^\tau r_u du} \right]$$

$$= \mathbb{E}_0 \left[\int_0^T ds\, \gamma\, \frac{P_0}{1 - \theta}\, \Lambda(s; T, X_0)\, e^{y(T-s)}\, \delta(s - \tau)\, e^{-\int_0^\tau r_u du} \right]$$

$$= \mathbb{E}_0 \left[\int_0^T ds\, \gamma\, \frac{P_0}{1 - \theta}\, \mathbb{E}_s \left[e^{\int_s^T r_u du} \right] e^{y(T-s)}\, \delta(s - \tau)\, e^{-\int_0^\tau r_u du} \right]$$

$$= \mathbb{E}_0 \left[\int_0^T ds\, \gamma\, \frac{P_0}{1 - \theta}\, \mathbb{E}_s \left[e^{\int_0^T r_u du} \right] e^{y(T-s)}\, \delta(s - \tau) \right].$$

We now move the outer expectation into the integral and apply the law of iterated expectations. The bankruptcy cost for a one period debt issuance becomes

$$bc(r_0, \zeta, T, V_0) = \gamma\, \frac{P_0}{1 - \theta} \times \int_0^T ds\, e^{y(T-s)}\, \mathbb{E}_0 \left[e^{\int_0^T r_u du}\, \delta(s - \tau) \right]$$

$$= \gamma\, \frac{P_0}{1 - \theta} \times \Lambda(r_0, 0, T) \int_0^T ds\, e^{y(T-s)}\, \mathbb{E}_0 \left[\frac{e^{\int_0^T r_u du}}{\Lambda(r_0, 0, T)}\, \delta(s - \tau) \right]$$

$$= \gamma\, \frac{P_0}{1 - \theta} \times \Lambda(r_0, 0, T) \int_0^T ds\, e^{y(T-s)}\, \mathbb{E}_0^{\mathbb{T}} \left[\delta(s - \tau) \right]$$

$$= \gamma\, \frac{P_0}{1 - \theta} \times \Lambda(r_0, 0, T) \int_0^T ds\, e^{y(T-s)}\, g(s, T, X_0)$$

$$= \gamma\, \frac{P_0}{1 - \theta} \times \Lambda(r_0, 0, T) \left[G(T; T, X_0) + \widehat{G}(T; T, X_0) \right].$$

We have used the following property of the dirac delta function. Suppose, we have a random variable $\tilde{x}$, then $\mathbb{E}[\delta(\tilde{x} - x)]$ yields the density at x. To see this, let the density of $\tilde{x}$ be $f(t)$, then $\mathbb{E}[\delta(\tilde{x} - x)] = \int_{-\infty}^{\infty} \delta(\tilde{x} - x) f(t) dt = f(x)$. In our case, $G(t; T, X_0)$ is equal to $Pr(\tau \leq t)$ in the $\mathbb{T}$ forward measure.

$$\mathbb{E}_0^{\mathbb{Q}} \left[\frac{e^{\int_0^T r_u du}}{\Lambda(0; T, X_0)} \delta(s - \tau) \right] = \mathbb{E}_0^{\mathbb{T}} \left[\delta(s - \tau) \right] = Pr(\tau = s) \equiv g(s, T, X_0).$$

3.2.5 Proof of the expression for tax benefits in a one period debt issuance model

The market value of debt is given by

$$
\begin{aligned}
L_0 &= \mathbb{E}_0\left[\int_0^T Ce^{-\int_0^s r(u)du}\mathbb{I}_{s<\tau}ds\right] \\
&+ \mathbb{E}_0\left[P_0\mathbb{I}_{\tau>T}e^{-\int_0^T r_u du}\right] \\
&+ \mathbb{E}_0\left[(1-\theta)(1-\gamma)\int_0^T V_D(r_s,s,T)\delta(s-\tau)e^{-\int_0^\tau r_u du}ds\right].
\end{aligned}
$$

The value of debt is composed of three parts: (i) present value of the flow of coupon payments prior to maturity while the firm remains solvent; (ii) present value of principal payment at time T conditional upon not defaulting prior to T and (iii) the present value of the recovery amount conditional upon defaulting at any time before T.

The expression for tax benefits is simply θ times the first expectation which represents the present value of the flow of coupon payments. We evaluate this expression as a difference equation using the fact that debt is issued at par, i.e. $L_0 = P_0$. So,

$$
\theta \times \mathbb{E}_0\left[\int_0^T Ce^{-\int_0^s r(u)du}\mathbb{I}_{s<\tau}ds\right] = \theta \times P_0 - \theta \times \underbrace{\mathbb{E}_0\left[P_0\mathbb{I}_{\tau>T}e^{-\int_0^T r_u du}\right]}_{\text{Term 1}}
$$

$$
- \theta \times \underbrace{\mathbb{E}_0\left[(1-\theta)(1-\gamma)\int_0^T V_D(r_s,s,T)\delta(s-\tau)e^{-\int_0^\tau r_u du}ds\right]}_{\text{Term2}}.
$$

Note that Term 2 is a $\frac{(1-\gamma)(1-\theta)}{\gamma}$ times $bc(r_0,\zeta,T,V_0)$ and hence we have an expression for it. Term 1 can be evaluated as follows:

$$
\begin{aligned}
\mathbb{E}_0\left[P_0\mathbb{I}_{\tau>T}e^{-\int_0^T r(u)du}\right] &= \mathbb{E}_0\left[P_0\,\mathbb{I}_{\tau>T}\frac{e^{-\int_0^T r_u du}}{\Lambda(r_0,0,T)}\right]\Lambda(r_0,0,T) \\
&= \mathbb{E}_0\left[\mathbb{I}_{\tau>T}\frac{e^{-\int_0^T r(u)du}}{\Lambda(r_0,0,T)}\right]P_0\,\Lambda(r_0,0,T) \\
&= \mathbb{E}_0^{\mathbb{T}}\left[\mathbb{I}_{\tau>T}\right]P_0\,\Lambda(r_0,0,T) \\
&= P_0\,\Lambda(r_0,0,T)\,(1-G(T,T,X_0)). \qquad (3.46)
\end{aligned}
$$

The third equality uses the change of measure formula. The expression for tax benefits follows.

3.2.6 Derivation of the expression of the coupon C

With the expression for tax benefits, the coupon rate C can be calculated immediately. Mathematically,

$$C = \frac{tb(r_0, \zeta, T, V_0)}{\theta \, \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u)du} \, \mathbb{I}_{s<\tau} \, ds \right]} = \frac{tb(r_0, \zeta, T, V_0)}{\theta \, \tilde{G}(T, T, X_0)}, \tag{3.47}$$

where

$$\tilde{G}(T, T, X_0) = \int_0^T ds \, \Lambda(r_0, 0, s) \, (1 - G(s, s, X_0)).$$

Proof of the expression of $\tilde{G}(T, T, X_0)$

$$\begin{aligned}
\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u)du} \, \mathbb{I}_{s<\tau} \, ds \right] &= \int_0^T ds \, \Lambda(r_0, 0, s) \, \mathbb{E}_0 \left[\frac{e^{-\int_0^s r(u)du}}{\Lambda(r_0, 0, s)} \, \mathbb{I}_{s<\tau} \, ds \right] \\
&= \int_0^T ds \, \Lambda(r_0, 0, s) \, (1 - G(s, s, X_0)) \\
&\equiv \tilde{G}(T, T, X_0)
\end{aligned}$$

The second inequality uses the definition of $G(t, T, X_0)$.

3.2.7 Expression for the present value of tax benefits at $(N-2)T$

First note that

$$
\mathbb{E}_{N-2}\left[\overline{TB_{j;N-1}} \times \mathbb{I}_{\tau_{N-2}>T}\, e^{\int_{(N-2)T}^{(N-1)T} r_u\, du}\Big|r_{(N-1)T} = r_k\right]
$$

$$
= \quad \mathbb{E}_{N-2}\left[V_{N-1} \times ntb_k \times \mathbb{I}_{\tau_{N-2}>T}\, e^{\int_{(N-2)T}^{(N-1)T} -r_u\, du}\Big|r_{(N-1)T} = r_k\right]
$$

$$
= \quad \mathbb{E}_{N-2}\left[V_{N-2}e^{\int_{(N-2)T}^{(N-1)T} (r_u - y - 0.5\sigma_v^2 du) + \int_{(N-2)T}^{(N-1)T} \sigma_v dW_{vu} du} \times ntb_k\, \mathbb{I}_{\tau_{N-2}>T}e^{\int_{(N-2)T}^{(N-1)T} -r_u\, du}\Big|r_{(N-1)T} = r_k\right]
$$

$$
= \quad V_{N-2}e^{-yT}\mathbb{E}_{N-2}\left[e^{\int_{(N-2)T}^{(N-1)T} -0.5\sigma_v^2 du + \int_{(N-2)T}^{(N-1)T} \sigma_v dW_{vu} du} \times ntb_k\, \mathbb{I}_{\tau_{N-2}>T}\Big|r_{(N-1)T} = r_k\right]
$$

$$
= \quad V_{N-2}\, e^{-yT}\, ntb_k\, H_{jk}
$$

where

$$
H_{jk} \doteq H(r_j, r_k, T) = \mathbb{E}_{N-2}\left[e^{\int_{(N-2)T}^{(N-1)T} -0.5\sigma_v^2 du + \int_{(N-2)T}^{(N-1)T} \sigma_v dW_{vu} du}\, \mathbb{I}_{\tau_{N-2}>T}\Big|r_{(N-1)T} = r_k\right].
$$

The present value of tax benefits at time $(N-2)T$ is

$$
\overline{TB_{j;N-2}} \quad \doteq \quad \overline{TB_{N-2}(r_j, \zeta, T, V_{N-2})}
$$

$$
= \quad V_{N-2} \times ntb_j + \sum_{k=1}^{M} \pi_{jk}\, \mathbb{E}_{N-2}\left[\overline{TB_{j;N-1}} \times \mathbb{I}_{\tau_{N-2}>T}\, e^{\int_{(N-2)T}^{(N-1)T} -r_u\, du}\Big|r_{(N-1)T} = r_k\right]
$$

$$
= \quad V_{(N-2)T} \times ntb_j + \sum_{k=1}^{M} V_{(N-2)T}\, e^{-yT}\, \pi_{jk}\, ntb_k\, H_{jk}. \tag{3.48}
$$

3.2.8 A Closed form approximation of H_{jk}

First, note the expression for H_{jk}:

$$
H_{jk} \doteq H(r_j, r_k, T) = \mathbb{E}_{N-2}\left[e^{\int_{(N-2)T}^{(N-1)T} -0.5\sigma_v^2 du + \int_{(N-2)T}^{(N-1)T} \sigma_v dW_{vu} du}\, \mathbb{I}_{\tau_{N-2}>T}\Big|r_{(N-1)T} = r_k\right].
$$

This expression involves the sample paths of both the interest rate process and the firm value. Additionally, the expectation is complicated by the fact that the expectation is only derived for paths that start at $r_{(N-2)T} = r_j$ and end up at $r_{(N-1)T} = r_k$. A appropriate way of evaluating this expression involves working with Brownian Bridges for the short rate paths.

We choose a different route. Empirically, the volatility of the firm value is on the order of 20% while the volatility of the interest rate process is on the order of 2%. That is, the firm value is significantly more volatile than the interest rate process. Therefore, chances are that the default would take place primarily because of the decline in firm value and not due to the changes in the short rate. Consequently, we ignore the conditional expectation involved in H_{jk}. We show that H_{jk} can be well approximated by $\tilde{H}(T, T, X_0)$ where

$$
\begin{aligned}
\tilde{H}(t, T, X_0) &= \mathbb{E}_0 \left[e^{-\sigma_V^2 t/2 + \sigma_V \mathbb{W}_{vt}} \mathbb{I}_{\tau > t} \right] \\
&= N \left(\frac{-X_0 - \mu_h(t, T)}{\sqrt{\Sigma(t, T)}} \right) + e^{-\frac{2 X_0 \mu_h(t; T)}{\Sigma(t, T)}} N \left(\frac{-X_0 + \mu_h(t, T)}{\sqrt{\Sigma(t, T)}} \right) \quad (3.49)
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma(t, T) &= \int_0^t \sigma^2(s; T) ds, \\
&= \sigma_v^2 t + \frac{\sigma_r^2}{\beta^2} \left(t + e^{-2\beta(T-t)} B_2(t) - 2 e^{-\beta(T-t)} B_1(t) \right) + \frac{2 \rho \sigma_v \sigma_r}{\beta} \left(t - e^{-\beta(T-t)} B_1(t) \right),
\end{aligned}
$$

$$
\mu_h(t, T) = \frac{\int_0^t \sigma^2(s, T) ds}{2} = \frac{\Sigma(t, T)}{2},
$$

$$
\sigma_h(t, T) = \sqrt{\sigma_v^2 + \sigma_p^2(t, T) + 2 \rho \sigma_V \sigma_p(t, T)},
$$

and

$$
B_1(t) = \frac{(1 - e^{-\beta t})}{\beta}; \quad B_2(t) = \frac{(1 - e^{-2\beta t})}{2\beta}
$$

The derivation is analogous to that of $G()$. We apply the following steps.

1. Upon inspection, $\tilde{H}(t, T, X_0) = \mathbb{E}\left[e^{-\sigma_v^2 t/2 + \sigma_V \mathbb{W}_{vt}} \mathbb{I}_{\tau > t} \right]$ is already in a change of measure form.

2. After applying the change of measure, we have that

$$
\tilde{H}(t, T, X_0) = \mathbb{E}\left[e^{-\sigma_V^2 t/2 + \sigma_V \mathbb{W}_{vt}} \mathbb{I}_{\tau > t} \right] = \mathbb{E}_0^{\mathbb{V}} \left[\mathbb{I}_{\tau > t} \right].
$$

The new measure $\mathbb{Q}^{\mathbb{V}}$ uses the unleveraged firm value as the numeraire.

3.

$$
\mathbb{E}_0^{\mathbb{V}} \left[\mathbb{I}_{\tau > t} \right] = \Pr^{\mathbb{V}}(\tau > t) = 1 - \Pr^{\mathbb{V}}(\tau < t) = 1 - \tilde{H}(t, T, X_0)
$$

As a measure of robustness, we checked the approximation of H_{jk} with $\tilde{H}(t, T, X_0)$ using Monte Carlo simulation. The approximation falls within 5% of the true value for a wide variety of parameter choices. The approximation worked well even when the interest rate process and the firm value are correlated.