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On Tensor Products of Demazure Modules for $sl_2[t]$

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To my nieces and nephews for making me smile everyday.

ABSTRACT OF THE DISSERTATION

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In this paper, we study tensor products of Demazure modules for the current algebra $\mathfrak{sl}_2[t]$. We establish a set of generators and relations for the tensor product of two local Weyl modules (which are also level 1 Demazure modules). We also establish a character formula for the tensor product of a level 2 Demazure module and a local Weyl module. To complete the proof of this character formula we also prove a short exact sequence of $V(\xi)$ modules. We further conjecture a character formula for the tensor product of any level ℓ Demazure module and a local Weyl module and provide a proof assuming the analogous short exact sequence holds.

Contents

1 Modules for the Current Algebra **3**
- 1.1 The Simple Lie Algebra $\mathfrak{sl}_2$ . . . 3
- 1.2 The Current Algebra $\mathfrak{sl}_2[t]$ . . . 4
- 1.3 Local Weyl Modules . . . 5
- 1.4 $V(\xi)$ Modules . . . 6
  - 1.4.1 Partitions . . . 6
  - 1.4.2 Presentations of $V(\xi)$ Modules . . . 7
  - 1.4.3 Fusion Products . . . 8
  - 1.4.4 Demazure Modules . . . 9

2 Demazure Flags **11**
- 2.1 Demazure Flags for $V(\xi)$ Modules . . . 11
- 2.2 Tensor Products of Demazure Modules . . . 12

3 Tensor Products of Local Weyl Modules **13**
- 3.1 Generator and Relations for $W_{loc}(n) \otimes W_{loc}(m)$ . . . 14
  - 3.1.1 Definition of $\widetilde{W}(n,m)$ . . . 14
  - 3.1.2 Surjective Map . . . 14
  - 3.1.3 Dimension of $\widetilde{W}(n,m)$ . . . 16
- 3.2 Filtration Examples . . . 19
  - 3.2.1 $m = 0$ . . . 19
  - 3.2.2 $m = 1$ . . . 20
  - 3.2.3 $m = 2$ . . . 22

4 A Short Exact Sequence **24**
- 4.1 Short Exact Sequence . . . 24
- 4.2 The map $\varphi^+(i)$ . . . 25
- 4.3 An Isomorphic Representation . . . 26
  - 4.3.1 The Representation $\rho \circ T$ . . . 27
- 4.4 Facts About $\varphi^-(i)$ . . . 29
  - 4.4.1 Local Weyl Module Relations . . . 29
  - 4.4.2 Number of Parts Relation . . . 30

4.4.3 Lemmas for Fusion Product Calculation 30
4.4.4 Extending the Map $\varphi^-(i)$ to any Partition 31
4.5 The map $\varphi^-(2)$ 36
4.6 The map $\varphi^-(3)$ 37
4.7 Dimension of $\ker \varphi^+(i)$ 37
4.8 Theorem 38

5 Character Formula 40
5.1 Conjecture 40
5.2 More Short Exact Sequences 40
5.3 Method for Proof of Conjecture 46
5.4 Theorem 56
5.4.1 $\mathrm{ch}_{gr}(W_{loc}(n) \otimes W_{loc}(m))$ 56
5.4.2 $\mathrm{ch}_{gr}(D(2, 2n + c) \otimes W_{loc}(m))$ 56

A Lemmas for $V(\xi^-(i))$ relations on the generator for $\ker \varphi^+$ 58
A.1 Number of Parts Relation 58
A.2 Lemmas for Fusion Product Calculation 61

Bibliography 80

Introduction

In recent years, representations of the current algebra, $\mathfrak{g}[t]$, have been a popular topic of study. Demazure modules are a family of representations that has been of particular interest. These modules were originally realized in studying irreducible integrable modules for affine Lie algebras, specifically as modules for the affine Borel subalgebra, but in certain conditions they are also modules for the current algebra. These Demazure modules are especially nice to study due their fairly simple presentations, especially in the $\mathfrak{sl}_2[t]$ case to which we restrict our attention in this paper. In [7], we are also introduced to $V(\xi)$ modules, another family of representations for the current algebra. The representations in this family are quotients of local Weyl modules and we can look at local Weyl modules and Demazure modules as special cases in this family.

Naoi shows in [12] that these Demazure modules, indexed by level, dominant integral weight, and grade, have a filtration by higher level Demazure modules in the simply laced cases. A constructive proof of this theorem was given in [6] for the $\mathfrak{sl}_2[t]$ case, and the result was extended to show that Demazure flags also exist for $V(\xi)$ modules. We turn our attention to representations for $\mathfrak{sl}_2[t]$ which we obtain by taking tensor products of Demazure modules. We would like to show that an analogous theorem is true for the existence of Demazure flags in these tensor products. In chapter 3, we will construct some examples for level 1 Demazure modules. However, the general case has proven difficult so we instead investigate the graded character of these tensor products to give evidence that Demazure flags exist.

In chapter 5, we prove a character formula for the tensor product of a level 2 Demazure module with a level 1 Demazure module. A generalized conjecture is also given but only proven in certain cases. The proof of this character formula relies on a short exact sequence of $V(\xi)$ modules which we address in chapter 4.

Chapter 1

Modules for the Current Algebra

In this chapter, we give the necessary notation and background from the representation theory of a simple Lie algebra $\mathfrak{g}$, specifically we will focus on $\mathfrak{g} = \mathfrak{sl}_2$. We will introduce the current algebra $\mathfrak{sl}_2[t]$ and discuss its representations that will be relevant to this paper, especially Demazure modules.

1.1 The Simple Lie Algebra $\mathfrak{sl}_2$

First we fix the notation $\mathbb{C}$ to represent the field complex numbers and $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ represent the integers, non-negative integers, and positive integers respectively.

The simple Lie algebra $\mathfrak{sl}_2$ is the algebra of 2x2 trace zero matrices over $\mathbb{C}$ and the standard basis is $\{x, y, h\}$ with $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$. The Cartan subalgebra of $\mathfrak{sl}_2$ is $\mathfrak{h} = \mathbb{C}h$.

The following family of modules defines all irreducible representations of sl_2. For $n \in \mathbb{Z}_+$, let V(n) be the sl_2-module generated by a nonzero vector v_n with the following defining relations:

$$x.v_n = 0, \qquad h.v_n = nv_n, \qquad y^{n+1}.v_n = 0.$$

We also note that the $\dim V(n) = n + 1$. Since $\mathfrak{sl}_2$ is simple, Weyl's theorem tells us that any finite dimensional representation of $\mathfrak{sl}_2$ can be decomposed into a direct sum of these irreducible representations.

The character of an $\mathfrak{sl}_2$-module is an invariant that indexes the dimensions of the weight spaces of a given representation V. It can be written as the Laurent polynomial

$$ch(V) = \sum_{m \in \mathbb{Z}} \dim V_m x^m$$

where $V_m = \{v \in V | h.v = mv\}$.

1.2 The Current Algebra $\mathfrak{sl}_2[t]$

The current algebra, $\mathfrak{sl}_2[t]$, is the Lie algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ with bracket defined as follows:

$$[g_1 \otimes t^r, g_2 \otimes t^s] = [g_1, g_2] \otimes t^{r+s},$$

for $g_i \in sl_2$ and $r, s \in \mathbb{Z}_+$. For ease of notation we may write an element $g \otimes t^r$ as g_r.

We will primarily be concerned with graded representations of $\mathfrak{sl}_2[t]$ so we recall that a representation V of $\mathfrak{sl}_2[t]$ is called $\mathbb{Z}$-graded if it is a vector space which has the following properties:

$$V = \bigoplus_{r \in \mathbb{Z}} V[r], \qquad (g \otimes t^s)V[r] \subset V[r+s], \; g \in sl_2, r \in \mathbb{Z}, s \in \mathbb{Z}_+.$$

For $r \in \mathbb{Z}$ and V, a $\mathbb{Z}$-graded $\mathfrak{sl}_2[t]$-module, $\tau_r V$ is the $\mathbb{Z}$-graded $\mathfrak{sl}_2[t]$-module with graded pieces shifted uniformly by r and the action of $\mathfrak{sl}_2[t]$ unchanged.

We notice that there is a natural inclusion of Lie algebras

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_2[t] \text{ where } g \mapsto g \otimes 1.$$

This means that every representation of $\mathfrak{sl}_2[t]$ is a representation of $\mathfrak{sl}_2$ as well. We also note a useful homomorphism that will be necessary in the representation theory of $\mathfrak{sl}_2[t]$. The evaluation homomorphism is the map of Lie algebras $ev_z : \mathfrak{sl}_2[t] \to \mathfrak{sl}_2$ defined by $ev_z(g \otimes f(t)) = f(z)g$ for $z \in \mathbb{C}$.

The irreducible representations of $\mathfrak{sl}_2[t]$ are the pullbacks $ev_z^* V(n)$ where V(n) is a finite dimensional irreducible $\mathfrak{sl}_2$-module, with action given by the evaluation homomorphism as:

$$(g \otimes t^s)v = z^s g.v, \text{ where } g \in \mathfrak{sl}_2, v \in V(n), s \in \mathbb{Z}_+, \text{ and } z \in \mathbb{C}.$$

We note that the only $\mathbb{Z}$-graded evaluation modules occur when $z = 0$, and so all finite dimensional graded irreducible representations of $\mathfrak{sl}_2[t]$ are of the form $\tau_r ev_0^* V(n)$ for all $r \in \mathbb{Z}_+$.

For graded representation, we also have the notion of graded characters. As before they are module invariants, however, in the graded case they index both the dimensions of the weight spaces and the graded pieces. For a graded $sl_2[t]$-module,

$$ch_{gr}(V) = \sum_{r \geq 0} chV[r]u^r.$$

1.3 Local Weyl Modules

We note the definition and some facts about a family of modules for $\mathfrak{sl}_2[t]$ called local Weyl modules that we will need in future sections.

The local Weyl module, $W_{loc}(n)$, is the $sl_2[t]$-module generated by a nonzero vector w_n with the following relations:

$$(x \otimes t^r)w_n = 0, \quad for \quad r \geq 0$$

$$(h \otimes t^r)w_n = n\delta_{0,r}w_n \quad for \quad r \geq 0$$

$$(y \otimes 1)^{n+1}w_n = 0.$$

We also note the following facts about local Weyl modules proven in [5]. First we have

$$W_{loc}(n) \cong \frac{\mathbb{C}[y_0, y_1, ...]}{(\mathbf{y}(r,s)|r+s \geq 1+rk+n-k)}w_n$$

as vector spaces.

Also the basis of $W_{loc}(n)$ is $\{y_{i_1}...y_{i_j}w_n | 0 \leq i_1 \leq ... \leq i_j \leq n-j, 0 \leq j \leq n\}$ where $j=0$ gives the basis element w_n, and dim $W_{loc}(n) = \sum_{j=0}^{n} \binom{n}{j} = 2^n$.

1.4 $V(\xi)$ Modules

1.4.1 Partitions

The modules $V(\xi)$ are indexed by partitions $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_s > 0)$ so we give some notation for these partitions.

First, the length a partition, denoted $|\xi|$, is the sum of all elements in the partition. So

$$|\xi| = \xi_1 + \xi_2 + \cdots + \xi_s.$$

We also let m_i represent the number of times i occurs in the partition, and ν_i represent the number of elements in the partition greater than or equal to i. When multiple partitions are involved we specify $\nu_i(\xi)$. Using this notation we have two alternate forms for writing our partitions:

$$\xi = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \dots 1^{m_1})$$

and

$$\xi = (\ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_\ell} \dots 1^{\nu_1-\nu_2}).$$

1.4.2 Presentations of $V(\xi)$ Modules

First we recall from [7] that, for $r, s \in \mathbb{Z}_+$,

$$x(r,s) = \sum (x \otimes 1)^{(b_0)}(x \otimes t)^{(b_1)} ... (x \otimes t^s)^{(b_s)}$$

where the sum is over $(b_i)_{i \geq 0}$, $b_i \in \mathbb{Z}_+$ such that

$$\sum_{i \geq 0} b_i = r \text{ and } \sum_{i \geq 0} i b_i = s,$$

where for $x \in sl_2[t]$, $x^{(p)} = \dfrac{x^p}{p!}$. The sum $y(r,s)$ is defined similarly. We also denote

$${}_j x(r,s) = \sum (x \otimes t^j)^{(b_j)}(x \otimes t^{j+1})^{(b_{j+1})} ... (x \otimes t^s)^{(b_s)},$$

where $j \leq s$ and the sum is over the same set as above.

Now, for a partition $\xi = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \dots 1^{m_1})$, $m_i \in \mathbb{Z}_+$, $V(\xi)$ is the $sl_2[t]$-module generated by a nonzero vector v_ξ with the following relations for $r, s, k \in \mathbb{Z}_+$:

$$(x \otimes t^r)v_\xi = 0$$

$$(h \otimes t^r)v_\xi = |\xi|\delta_{r,0}v_\xi,$$

$$(y \otimes 1)^{|\xi|+1}v_\xi = 0$$

$$(x \otimes 1)^{(s)}(y \otimes t)^{(r+s)}v_\xi = 0 \quad \text{for } r+s \geq 1 + rk + \sum_{j \geq k+1} \xi_j.$$

We note that since $(x\otimes 1)^{(s)}(y\otimes t)^{(r+s)} = \frac{1}{s!(r+s)!}(x\otimes 1)^s(y\otimes t)^{r+s}$ it is equivalent to use $(x\otimes 1)^s(y\otimes t)^{r+s}v_\xi = 0$.

A second presentation allows us to replace the last relation with

$$y(r,s)v_\xi = 0, \text{ for } r+s \geq 1 + rk + \sum_{j\geq k+1} \xi_j.$$

The following theorem in [11] gives a finite set of relations for $V(\xi)$.

Theorem 1.4.1. *The module $V(\xi)$ is isomorphic to the quotient of the local Weyl module $W_{\mathrm{loc}}(|\xi|)$ by the $\mathfrak{sl}_2[t]$–submodule generated by the elements*

$$y(r, -r+1+\sum_{j=1}^{r} \nu_j)w_{|\xi|}, \quad 1 \leq r < \ell.$$

Remark 1.4.2. In fact, the proof of the theorem implies that in $V(\xi)$ we have the relation

$$y(r,s)v_\xi = 0, \quad s \geq -r+1+\sum_{j=1}^{r} \nu_j, \quad r \geq 1.$$

Later we will use the notation

$$y(r, -r+1+\sum_{j=1}^{r} \nu_j)v_\xi = 0, \quad r \geq 1,$$

with the convention that $y(r,0)v_\xi = 0$ if $-r+1+\sum_{j=1}^{r} \nu_j < 0$.

1.4.3 Fusion Products

We recall the definition of fusion products of representations of the current algebra introduced in [8] and restated in [7].

Suppose that V is a finite-dimensional cyclic $\mathfrak{g}[t]$ module generated by an element v and for $r \in \mathbb{Z}_+$ define

$$F^r V = \sum_{0\leq s\leq r} \mathbf{U}(\mathfrak{g}[t])[s]v.$$

Each F^rV is a $\mathfrak{g}$-module and the associated graded space $\operatorname{gr} V$ is a cyclic graded $\mathfrak{g}[t]$-module with action

$$(x \otimes t^s)(\bar{w}) = \overline{(x \otimes t^s)w}, \quad \bar{w} \in F^rV/F^{r-1}V.$$

Now, given any $\mathfrak{g}[t]$-module V and $z \in \mathbb{C}$, let V^z be the $\mathfrak{g}[t]$-module with action defined by

$$(x \otimes t^r)v = (x \otimes (t+z)^r)v, \quad x \in \mathfrak{g}, \quad r \in \mathbb{Z}_+, \quad v \in V.$$

Then, the fusion product is $\operatorname{gr} \mathbf{V}$, denoted $V_1^{z_1} * \cdots * V_m^{z_m}$, where

$$\mathbf{V} = V_1^{z_1} \otimes \cdots V_m^{z_m},$$

for V_i cyclic finite-dimensional graded $\mathfrak{g}[t]$-modules generated by v_i, $1 \leq i \leq m$, and distinct parameters $z_i \in \mathbb{C}$.

In [7], it is shown that $V(\xi)$ modules can be recognized as the fusion product of evaluation modules $ev_0V(r)$ for $r \in \mathbb{Z}_+$. Also, by [9], the fusion product in this case is independent of choice of parameters.

1.4.4 Demazure Modules

Finally, we consider Demazure modules, which we can simplify for this paper since it is only necessary for us to understand the Demazure modules discussed in [7] and only for the algebra $\mathfrak{sl}_2[t]$.

Let $\ell, s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$. Then $D(\ell, s)$ is generated by an element v with defining relations:

$$(x \otimes t^r)v = 0, \quad (h \otimes t^r)v = s\delta_{0,r}v, \quad (y \otimes 1)^{s+1}v = 0, \quad r \geq 0,$$

$$(y \otimes t^{s_1+1})v = 0, \quad (y \otimes t^{s_1})^{s_0+1}v = 0, \quad \text{if} \;\; s_0 < \ell.$$

The module $\tau_m D(\ell, s)$ will be the graded $\mathfrak{sl}_2[t]$-module where the generating element v is defined to have grade m, for $m \in \mathbb{Z}$.

In [7], they also show that these Demazure modules can be recognized as $V(\xi)$ modules. So more specifically, we can write $s = n\ell + c$ for $\ell, n, c \in \mathbb{Z}_+$ and $c < \ell$ and then we have,

$$D(\ell, s) = D(\ell, n\ell + c) \cong V(\ell^n c).$$

Chapter 2

Demazure Flags

Since the current algebra, $\mathfrak{sl}_2[t]$, is not semisimple we attempt to understand its modules by finding flitrations or flags. In this section we recall that for certain families of $\mathfrak{sl}_2[t]$-modules we know that Demzaure flags exist.

2.1 Demazure Flags for $V(\xi)$ Modules

Let M be a finite–dimensional graded $\mathfrak{sl}_2[t]$–module. We say that an increasing sequence

$$\mathcal{F}(M) = \{0 = M_0 \subsetneq M_1 \subsetneq \cdots M_k \subsetneq M_{k+1} = M\}$$

of graded $\mathfrak{sl}_2[t]$-submodules of M is a Demazure flag of level ℓ, if

$$M_{i+1}/M_i \cong \tau_{p_i} D(\ell, n_i), \quad (n_i, p_i) \in \mathbb{Z}_+ \times \mathbb{Z}, \quad 0 \leq i \leq k.$$

The following theorem from [6] gives us criteria for the existence of Demazure flags for $V(\xi)$ modules.

Theorem 2.1.1. *For all $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_s > 0)$ and $m \in \mathbb{N}$ the module $V(\xi)$ has a Demazure flag of level m if and only if $m \geq \xi_1$. In particular, a level k-Demazure module has a Demazure flag of level m if and only if $m \geq k$.*

2.2 Tensor Products of Demazure Modules

We would like to try to understand a tensor products of Demazure modules in the same way that we understand $V(\xi)$ modules (and in particular Demazure modules) in the previous section. We believe the following conjecture should be true, although we are far from proving it, and the remainder of this paper is devoted to providing evidence in small cases to support the conjecture.

Conjecture 2.2.1. *A tensor product of Demazure modules for $\mathfrak{sl}_2[t]$,*

$$D(\ell_1, n_1\ell_1 + c_1) \otimes D(\ell_2, n_2\ell_2 + c_2) \otimes \cdots \otimes D(\ell_k, n_k\ell_k + c_k),$$

for $\ell_i, n_i, c_i \in \mathbb{Z}_+$, has a Demazure flag of level $\ell_1 + \ell_2 + \cdots + \ell_k$.

We note that by theorem 2.1.1, to prove that a tensor product of Demazure modules has a Demazure flag of level ℓ, it is enough to show it has a filtration by $V(\xi)$ modules with $\xi_1 \leq \ell$. Since each $V(\xi)$ in the filtration will have its own Demazure flag of level ℓ, we can construct a Demazure flag for the tensor product. This is the method we use throughout this paper.

Chapter 3

Tensor Products of Local Weyl Modules

In this chapter we investigate a tensor product of local Weyl modules. These modules are of interest since they are level 1 Demazure modules. We first prove a set of generators and relations for the tensor product and then we will construct examples for small values of m as evidence that a filtration for arbitrary m may exist.

3.1 Generator and Relations for $W_{loc}(n) \otimes W_{loc}(m)$

3.1.1 Definition of $\widetilde{W}(n,m)$

We denote by $\widetilde{W}(n,m)$ the $sl_2[t]$-module with generator $\tilde{w}$ and the following relations for $r,s>0, k\geq 0$:

$$(h\otimes 1)\tilde{w} = (n-m)\tilde{w}$$

$$(h\otimes t^r)\tilde{w} = 0, \quad for \quad r>0$$

$$y(r,s)\tilde{w} = 0, \quad for \quad r+s \geq 1+rk+n-k$$

$$x(r,s)\tilde{w} = 0, \quad for \quad r+s \geq 1+rk+m-k.$$

Now, we will show that this module is isomorphic to $W_{loc}(n)\otimes W_{loc}(m)$ as $sl_2[t]$-modules.

3.1.2 Surjective Map

Lemma 3.1.1. *There exists a surjective map of $sl_2[t]$-modules*

$$\phi : \widetilde{W}(n,m) \to W_{loc}(n)\otimes W_{loc}(m)$$

.

Proof. We know that $w_n \otimes y_0^m w_m$ generates $W_{loc}(n)\otimes W_{loc}(m)$, so we define $\phi(\tilde{w}) = w_n \otimes y_0^m w_m$. It suffices to check that the relations $\widetilde{W}(n,m)$ hold on $w_n\otimes y_0^m w_m$.

$$\begin{aligned}(h\otimes 1)(w_n\otimes y_0^m w_m) &= h_0 w_n \otimes y_0^m w_m + w_n \otimes h_0 y_0^m w_m \\ &= n w_n \otimes y_0^m w_m + w_n \otimes (-m) y_0^m w_m \\ &= (n-m) w_n \otimes y_0^m w_m\end{aligned}$$

For $r > 0$,

$$
\begin{aligned}
(h \otimes t^r)(w_n \otimes y_0^m w_m) &= h_r w_n \otimes y_0^m w_n + w_n \otimes h_r y_0^m w_m \\
&= 0 + w_n \otimes -2y_r y_0^{m-1} w_m + w_n \otimes y_0 h_r y_0^{m-1} w_m \\
&= w_n \otimes a y_r y_0^{m-1} w_m, \quad a \in \mathbb{Z} \\
&= 0, \qquad \text{since dim } W_{loc}(m)_{-m} = \text{ dim } W_{loc}(m)_{-m}[0] = 1.
\end{aligned}
$$

$$
\begin{aligned}
y(r,s)(w_n \otimes y_0^m w_m) &= x^-(r,s) w_n \otimes y_0^m w_m \\
&= 0, \qquad \text{by } W_{loc}(n) \ \textit{relations.}
\end{aligned}
$$

$$
\begin{aligned}
x(r,s)(w_n \otimes y_0^m w_m) &= w_n \otimes x^+(r,s) y_0^m w_m \\
&= 0, \qquad \text{by remark below.}
\end{aligned}
$$

□

Remark 3.1.2. Let $\bar{w}_m$ be a lowest weight element of $W_{loc}(m)$ then the following relations hold:

$$
\begin{aligned}
&(y \otimes t^r)\bar{w}_m = 0, \quad \forall r \geq 0 \\
&(h \otimes t^r)\bar{w}_m = \delta_{r,0}(-m)\bar{w}_m, \quad \forall r \geq 0 \\
&x(r,s)\bar{w}_m = 0, \quad for \quad r + s \geq 1 + rk + m - k
\end{aligned}
$$

This can be seen by applying the isomorphism of $\mathfrak{sl}_2[t]$ that sends $y_s \to x_s$, $x_s \to y_s$ and $h_s \to -h_s$, for $s \geq 0$.

3.1.3 Dimension of $\widetilde{W}(n,m)$

Now that we have a surjective map between our modules, it will suffice to show that $\dim \widetilde{W}(n,m) \leq \dim W_{loc}(n) \otimes W_{loc}(m) = 2^{n+m}$.

Lemma 3.1.3. *dim* $\widetilde{W}(n,m) \leq 2^{n+m}$.

Proof. First, we recall that $\widetilde{W}(n,m) = \mathbf{U}(sl_2[t])\tilde{w}$ with the given relations. So we investigate $\widetilde{W}(n,m)$ as a quotient of $\mathbf{U}(sl_2[t])\tilde{w}$. The PBW basis theorem allows us to write $\mathbf{U}(sl_2[t])\tilde{w}$ as $\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\mathbf{U}(\mathfrak{h}[t])\tilde{w}$. Now $\widetilde{W}(n,m)$ is also a quotient of $\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\mathbf{U}(\mathfrak{h}[t])\tilde{w}$ by the ideal generated by the elements $(h \otimes t^r) - (n-m)\delta_{r,0}$ for $r \geq 0$.

This means

$$\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\mathbf{U}(\mathfrak{h}[t])\tilde{w} = \mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\tilde{w}$$

and so we only need to consider $\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\tilde{w}$. We notice that since $\mathfrak{n}^+[t]$ and $\mathfrak{n}^-[t]$ are abelian Lie algebras their universal enveloping algebras coincide with the symmetric algebra which can be identified with the polynomial algebra. This means

$$\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])\tilde{w} \cong \mathbb{C}[x_0, x_1, ...]\mathbb{C}[y_0, y_1, ...]\tilde{w}$$

as vector spaces. So we also have the following isomorphism of vector spaces:

$$\frac{\mathbf{U}(\mathfrak{n}^+[t])\mathbf{U}(\mathfrak{n}^-[t])}{(y(r,s)|r+s \geq 1+rk+n-k)}\tilde{w} \cong \mathbb{C}[x_0, x_1, ...] \cdot \frac{\mathbb{C}[y_0, y_1, ...]}{(y(r,s)|r+s \geq 1+rk+n-k))}\tilde{w}.$$

The facts about local Weyl modules in chapter 1 give us that

$$\frac{\mathbb{C}[y_0, y_1, ...]}{(y(r,s)|r+s \geq 1+rk+n-k))}\tilde{w} \cong \mathbb{C}\text{-span}\{y_{i_1}...y_{i_j}|0 \leq i_1 \leq ... \leq i_j \leq n-j\}\tilde{w}.$$

We will denote $Y = \mathbb{C}\text{-span}\{y_{i_1}...y_{i_j}|0 \leq i_1 \leq ... \leq i_j \leq n-j\}$.

So we have shown that $\widetilde{W}(n,m)$ is a vector space quotient of $\mathbb{C}[x_0, x_1, ...] \cdot Y\tilde{w}$.

We now show that every element of $\widetilde{W}(n,m)$ can also be written as an element of $Y \cdot \mathbb{C}[x_0, x_1, ...]\tilde{w}$. More specifically, we want to show that each element in $\mathbb{C}[x_0, x_1, ...]y_{i_1}...y_{i_j}\tilde{w}$ can be written in the form suggested. We proceed with induction on j.

Consider the case where $j = 0$, then $\mathbb{C}[x_0, x_1, ...]\tilde{w}$ is already in the desired form.

Now, consider the case where $j = 1$. Without loss of generality we can consider the element $x_0^{b_0}...x_k^{b_k}y_i\tilde{w}$, where k is maximal such that $b_k > 0$. Then, we see that

$$\begin{aligned}
x_0^{b_0}...x_k^{b_k}y_i\tilde{w} &= x_0^{b_0}...x_k^{b_k-1}y_ix_k\tilde{w} + x_0^{b_0}...x_k^{b_k-1}h_{k+i}\tilde{w} \\
&= x_0^{b_0}...x_k^{b_k-2}y_ix_k^2\tilde{w} + x_0^{b_0}...x_k^{b_k-2}h_{k+i}x_k\tilde{w} + x_0^{b_0}...x_k^{b_k-1}h_{k+i}\tilde{w} \\
&\quad . \\
&\quad . \\
&\quad . \\
&= y_ix_0^{b_0}...x_k^{b_k}\tilde{w} + u, \qquad \text{where } u \in \mathbb{C}[x_0, x_1, ...]\tilde{w} \\
&\in Y \cdot \mathbb{C}[x_0, x_1, ...]\tilde{w}.
\end{aligned}$$

Assume for every $\ell < j$, $x_0^{b_0}...x_k^{b_k}y_{i_1}...y_{i_\ell}\tilde{w} \in Y \cdot \mathbb{C}[x_0, x_1, ...]\tilde{w}$.

Consider $x_0^{b_0}...x_k^{b_k}y_{i_1}...y_{i_j}\tilde{w}$. We see that

$$\begin{aligned}
x_0^{b_0}...x_k^{b_k}y_{i_1}...y_{i_j}\tilde{w} &= x_0^{b_0}...x_k^{b_k-1}y_{i_j}x_ky_{i_1}...y_{i_{j-1}}\tilde{w} + x_0^{b_0}...x_k^{b_k-1}h_{k+i_j}y_{i_1}...y_{i_{j-1}}\tilde{w} \\
&= x_0^{b_0}...x_k^{b_k-1}y_{i_j}x_ky_{i_1}...y_{i_{j-1}}\tilde{w} + u_{j-1}
\end{aligned}$$

where,

$u_{j-1} \in \mathbb{C}[x_0, x_1, ...] \cdot \mathbb{C}\text{-span}\{y_{i_1}...y_{i_{j-1}} | 0 \leq i_1 \leq ... \leq i_{j-1} \leq n - j + 1\}\tilde{w}$

since $h_{k+i_j} y_{i_1} ... y_{i_{j-1}} \tilde{w}$ must be a linear combination of basis terms that have j-1 y's to preserve the weight.

This process continues everytime we move y_{i_j} past some x_i. So once we get y_{i_j} to the front of the expression we have

$x_0^{b_0} ... x_k^{b_k} y_{i_1} ... y_{i_j} \tilde{w} = c y_{i_j} x_0^{b_0} ... x_k^{b_k - 1} x_k y_{i_1} ... y_{i_{j-1}} \tilde{w} + u'_{j-1}$, for some $c \in \mathbb{C}$ and

$u'_{j-1} \in \mathbb{C}[x_0, x_1, ...] \cdot \mathbb{C}\text{-span}\{y_{i_1} ... y_{i_{j-1}} | 0 \leq i_1 \leq ... \leq i_{j-1} \leq n - j + 1\} \tilde{w}$.

Next we move $y_{i_{j-1}}$ to the front in the same way and we get

$x_0^{b_0} ... x_k^{b_k} y_{i_1} ... y_{i_j} \tilde{w} = d y_{i_{j-1}} y_{i_j} x_0^{b_0} ... x_k^{b_k - 1} x_k y_{i_1} ... y_{i_{j-2}} \tilde{w} + u'_{j-2} + u'_{j-1}$, for some $d \in \mathbb{C}$ and

$u'_{j-2} \in \mathbb{C}[x_0, x_1, ...] \cdot \mathbb{C}\text{-span}\{y_{i_1} ... y_{i_{j-2}} | 0 \leq i_1 \leq ... \leq i_{j-2} \leq n - j + 2\} \tilde{w}$.

Continuing this process through all of the y_i's we finally get to

$x_0^{b_0} ... x_k^{b_k} y_{i_1} ... y_{i_j} \tilde{w} = a y_{i_1} ... y_{i_j} x_0^{b_0} ... x_k \tilde{w} + u'_0 + ... + u'_{j-1}$, for some $a \in \mathbb{C}$.

Now, by our induction hypothesis, each u'_i can be written in the correct form and our first term is already in the correct form, thus

$x_0^{b_0} ... x_k^{b_k} y_{i_1} ... y_{i_j} \tilde{w} \in Y \cdot \mathbb{C}[x_0, x_1, ...] \tilde{w}$ which completes our induction.

So, we have shown that every element in $\widetilde{W}(n,m)$ can be written as an element of $Y \cdot \mathbb{C}[x_0, x_1, ...] \tilde{w}$, and thus $\widetilde{W}(n,m)$ is a vector space quotient of $Y \cdot \mathbb{C}[x_0, x_1, ...] \tilde{w}$.

Finally, we consider the last relation for $\widetilde{W}(n,m)$ and we see that $\widetilde{W}(n,m)$ is also a vector space quotient of $Y \cdot \dfrac{\mathbb{C}[x_0, x_1, ...]}{(x(r,s) | r + s \geq 1 + rk + m - k))} \tilde{w}$. This means that

$$\dim \widetilde{W}(n,m) \leq \dim Y \cdot \frac{\mathbb{C}[x_0, x_1, ...]}{(x(r,s) | r + s \geq 1 + rk + m - k))} \tilde{w}$$

.

Using our facts from section 1 we already know that Y has 2^n elements and by the same argument $\frac{\mathbb{C}[x_0, x_1, ...]}{(x(r,s)|r+s \geq 1+rk+m-k))}\tilde{w}$ has a basis with 2^m elements. Thus

$$\dim \widetilde{W}(n,m) \leq \dim Y \cdot \frac{\mathbb{C}[x_0, x_1, ...]}{(x(r,s)|r+s \geq 1+rk+m-k))}\tilde{w} = 2^{n+m}$$

which proves our lemma.

□

So finally, we have a surjective map of $sl_2[t]$-modules $\widetilde{W}(n,m) \to W_{loc}(n) \otimes W_{loc}(m)$ and dim $\widetilde{W}(n,m) \leq$ dim $W_{loc}(n) \otimes W_{loc}(m)$. Therefore, $\widetilde{W}(n,m) \cong W_{loc}(n) \otimes W_{loc}(m)$ as $sl_2[t]$-modules giving our tensor product the generator and relations defined in this section.

3.2 Filtration Examples

In this section we construct some filtrations for the modules $W_{loc}(n) \otimes W_{loc}(m)$. We would like to prove that a Demazure flag always exists, however, as m gets larger the calculations get difficult so we have not been able to generalize this result. Again we note that it is enough to show that we can construct a filtration by $V(\xi)$ modules where $\xi_1 \leq 2$ due to theorem 2.1.1.

3.2.1 $m = 0$

First we consider the case where $m = 0$. In this case, $W_{loc}(m) \otimes W_{loc}(0) \cong W_{loc}(m)$. And by [7], we also know that $W_{loc}(n) \cong D(1,n) \cong V(1^n)$, and so the filtration is trivial.

3.2.2 $m = 1$

Now we consider the case where $m = 1$ and find a filtration of the tensor product by $V(\xi)$-modules.

We will show the $sl_2[t]$-module $W_{loc}(n) \otimes W_{loc}(1)$ has the filtration

$$0 \subset M_1 = \mathbf{U}(sl_2[t])(w_n \otimes w_1) \subset M_2 = \mathbf{U}(sl_2[t])(w_n \otimes y_0 w_1) = W_{loc}(n) \otimes W_{loc}(1),$$

where w_n generates $W_{loc}(n)$ and w_1 generates $W_{loc}(1)$, and

$$M_1 \cong V(21^{n-1}) \text{ and } M_2/M_1 \cong V(1^{n-1}),$$

as $sl_2[t]$-modules.

To prove this, we will show there exists surjective maps

$$V(21^{n-1}) \to M_1 \text{ and } V(1^{n-1}) \to M_2/M_1.$$

Let v_ξ be the generator of $V(21^{n-1})$ and let $m_1 = w_n \otimes w_1$ be the generator of M_1. Define the map $V(21^{n-1}) \to M_1$ by mapping v_ξ to m_1. Checking the relations we see

$$(x \otimes t^s)m_1 = x_s w_n \otimes w_1 + w_n \otimes x_s w_1 = 0, \quad for \quad s \geq 0$$

$$(h \otimes t^s)m_1 = h_s w_n \otimes w_1 + w_n \otimes h_s w_1 = 0, \quad for \quad s > 0$$

$$(h \otimes 1)m_1 = h w_n \otimes w_1 + w_n \otimes h w_1 = n w_n \otimes w_1 + w_n \otimes w_1 = (n+1)m_1$$

$$(y \otimes 1)^{n+2} m_1 = y_0^{n+2} w_n \otimes w_1 + b_1 y_0^{n+1} w_n \otimes y_0 w_1 + ... + w_n \otimes y_0^{n+2} w_1 = 0,$$

and finally checking the last relation for $V(\xi)$-modules we see that

$$(x\otimes t)^s(y\otimes 1)^{s+r}m_1 = x_1^s(y_0^{s+r}w_n\otimes w_1 + by_0^{s+r-1}w_n\otimes y_0w_1)$$

$$= x_1^s y_0^{s+r}w_n\otimes w_1 + bx_1^s y_0^{s+r-1}w_n\otimes y_0w_1$$

$$= 0 \qquad \text{for } r+s\geq 1+rk+\sum_{j\geq k+1}\xi_j.$$

So we have a surjective map.

Now similarly we show there exists a surjective map $V(1^{n-1})\to M_2/M_1$.

Let $v_{\xi'}$ be the generator of $V(1^{n-1})$ and let $m_2 = w_n\otimes y_0w_1 + M_1$ be the generator of M_2/M_1. Define the map $V(1^{n-1})\to M_2/M_1$ by mapping $v_{\xi'}$ to m_2. Checking the relations we see

$$(x\otimes t^s)m_2 = x_sw_n\otimes y_0w_1 + w_n\otimes x_sy_0w_1 = 0 + w_n\otimes h_sw_1 + w_n\otimes y_0x_sw_1 = 0, \quad for \quad s\geq 0$$

$$(h\otimes t^s)m_2 = h_sw_n\otimes y_0w_1 + w_n\otimes h_sy_0w_1$$

$$= 0 + w_n\otimes -2y_sw_1 + w_n\otimes y_0h_sw_1 = 0 \quad for \quad s>0$$

$$(h\otimes 1)m_2 = hw_n\otimes y_0w_1 + w_n\otimes hy_0w_1$$

$$= nw_n\otimes y_0w_1 + w_n\otimes -2y_0w_1 + w_n\otimes y_0hw_1$$

$$= n(w_n\otimes y_0w_1) - 2(w_n\otimes y_0w_1) + (w_n\otimes y_0w_1) = (n-1)m_2,$$

$$(y\otimes 1)^n m_2 = y_0^n w_n\otimes y_0w_1 + b_1y_0^{n-1}v\otimes y_0^2w_1 + ... + w_n\otimes y_0^{n+1}w_1$$

$$= y_0^{n+1}(v\otimes w_1) + 0 = 0 \quad \text{since } y_0^{n+1}(v\otimes w_1)\in M_1.$$

Since $V(1^{n-1})\cong W_{loc}(n-1)$ there are no further relations to check in this case, so the map exists.

So we have shown both surjective maps exist and now we show $\dim V(21^{n-1}) = \dim M_1$ and $\dim V(1^{n-1}) = \dim M_2/M_1$.

First we note that $\dim M_1 + \dim M_2/M_1 = \dim(W_{loc}(n) \otimes W_{loc}(1)) = 2^{n+1}$. Also $\dim V(21^{n-1}) = 3 \cdot 2^{n-1}$ and $\dim V(1^{n-1}) = 2^{n-1}$.

So,

$$\dim V(21^{n-1}) + \dim V(1^{n-1}) = 3 \cdot 2^{n-1} + 2^{n-1} = 2^{n+1}$$

Finally since we proved surjective maps above we already know that $\dim V(21^{n-1}) \geq \dim M_1$ and $\dim V(1^{n-1}) \geq \dim M_2/M_1$. So this fact together with the above calculation forces $\dim V(21^{n-1}) = \dim M_1$ and $\dim V(1^{n-1}) = \dim M_2/M_1$.

Thus since we have surjective maps and equal dimension, the modules are isomorphic and so we have the filtration we were looking for.

3.2.3 $m = 2$

In this case, we state the filtration without proof. The proof uses similar methods as the $m = 1$ case with more calculation but does not give us anymore insight into the general case.

The module $W_{loc}(n) \otimes W_{loc}(2)$ has the filtration

$$0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = W_{loc}(n) \otimes W_{loc}(2),$$

with

$$M_1 = \mathbf{U}(sl_2[t])(w_n \otimes w_2),$$

$$M_2 = \mathbf{U}(sl_2[t])(w_n \otimes y_1 w_2) + M_1,$$

$$M_3 = \mathbf{U}(sl_2[t])(w_n \otimes y_0 w_2) + M_2,$$

$$M_4 = \mathbf{U}(sl_2[t])(w_n \otimes y_2^2 w_2) + M_3,$$

where w_n generates $W_{loc}(n)$ and w_2 generates $W_{loc}(2)$, and

$$M_1 \cong V(2^2 1^{n-2}),$$

$$M_2/M_1 \cong \tau_1 V(21^{n-2}),$$

$$M_3/M_2 \cong V(21^{n-2}),$$

$$M_4/M_3 \cong V(1^{n-2}).$$

Chapter 4

A Short Exact Sequence

In this chapter we conjecture a useful short exact sequence that will be necessary for our character calculations in chapter 5. We complete the proof of the short exact sequence for two cases.

4.1 Short Exact Sequence

Let ξ be the partition $(\ell^{m_\ell}(\ell-1)^{m_{\ell-1}}\dots i^{m_i}\dots 1^{m_1})$, with $1 \le i \le \ell$ and $m_i \ge 2$. Then we define the following two partitions

$$\xi^+(i) = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}}\dots(i+2)^{m_{i+2}}(i+1)^{m_{i+1}+1}i^{m_i-2}(i-1)^{m_{i-1}+1}(i-2)^{m_{i-2}}\dots 1^{m_1})$$

and

$$\xi^-(i) = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}}\dots(i+2)^{m_{i+2}}(i+1)^{m_{i+1}}i^{m_i-2}(i-1)^{m_{i-1}}(i-2)^{m_{i-2}}\dots 1^{m_1}).$$

Recall that the notation $\nu_j(\xi)$ means the number of parts in the partition that are greater than or equal to j. For ease of notation, we may write $\nu_j(\xi) = \nu_j$ throughout this chapter.

Conjecture 4.1.1. *The following is a short exact sequence of graded $sl_2[t]$-modules:*

$$0 \to \tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i} V(\xi^-(i)) \to V(\xi) \to V(\xi^+(i)) \to 0.$$

The proof of this conjecture for $i = 2, 3$ will occupy the remainder of this chapter.

4.2 The map $\varphi^+(i)$

Lemma 4.2.1. *There exists a surjective map $\varphi^+(i) : V(\xi) \to V(\xi^+(i))$, and the kernel of $\varphi^+(i)$ is generated by $y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i)v_\xi$.*

Proof. Let $1 \le i \le \ell$. Assume $m_i \ge 2$. Define $\varphi^+(i) : V(\xi) \to V(\xi^+(i))$ by $v_\xi \mapsto v_{\xi^+(i)}$, where v_ξ generates $V(\xi)$ and $v_{\xi^+(i)}$ generates $V(\xi^+(i))$. Now, we need to show $\varphi^+(i)$ is well defined. Since $|\xi| = |\xi^+(i)|$, $V(\xi)$ and $V(\xi^+(i))$ are quotients of the same local Weyl module. Also, for all $k \ge 0$, $\sum_{j \ge k+1} \xi_j \ge \sum_{j \ge k+1} \xi^+(i)_j$, which implies that the last $V(\xi)$ relation holds on $v_{\xi^+(i)}$. Hence, $\varphi^+(i)$ is well defined. Also, since $\varphi^+(i)$ is surjective, $V(\xi^+(i))$ is a quotient of $V(\xi)$.

Since we know $V(\xi^+(i))$ and $V(\xi)$ are both quotients of the same local Weyl module, we note that, using the simplified relations presented earlier, the following are the remaining defining relations of $V(\xi^+(i))$ and $V(\xi)$ respectively:

$$y(r, \nu_1(\xi^+(i)) + \cdots + \nu_r(\xi^+(i)) - r + 1)v_{\xi^+(i)} = 0,$$

and

$$y(r, \nu_1(\xi) + \cdots + \nu_r(\xi) - r + 1)v_\xi = 0,$$

for all $1 \le r \le \ell$ (we also note that $r = \ell$ is actually only a relation and not a defining relation, but we use it here for the case when $i = \ell$). We also see that $\nu_j(\xi^+(i)) = \nu_j(\xi)$ for all $j \leq i-1$ and $j \geq i+2$, $\nu_i(\xi^+(i)) = \nu_i(\xi) - 1$, and $\nu_{i+1}(\xi^+(i)) = \nu_{i+1}(\xi) + 1$. Thus, we have

$$y(r, \nu_1(\xi^+(i)) + \cdots + \nu_r(\xi^+(i)) - r + 1) = y(r, \nu_1(\xi) + \cdots + \nu_r(\xi) - r + 1)$$

except at $r = i$.

Thus, since $V(\xi^+(i))$ is a quotient of $V(\xi)$, the kernel of $\varphi^+(i)$ is generated by

$$y(i, \nu_1(\xi^+(i)) + \cdots + \nu_i(\xi^+(i)) - i + 1)v_\xi = y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i)v_\xi.$$

□

4.3 An Isomorphic Representation

Lemma 4.3.1. *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Lie algebras with $\varphi : \mathfrak{a}_1 \to \mathfrak{a}_2$ surjective. Let V be an irreducible representation of $\mathfrak{a}_2$ then $\varphi^*(V)$ is an irreducible representation of $\mathfrak{a}_1$.*

Proof. Suppose, by way of contradiction, that $\varphi^*(V)$ is a reducible representation of $\mathfrak{a}_1$. Then there exists a nontrivial subrepresentation $W \subset \varphi^*(V)$. Now we choose any element $x \in \mathfrak{a}_2$. Since φ is surjective, we know that x has a preimage in $\mathfrak{a}_1$, call it y. We know that $yw \in W$, for all $w \in W$. But then, since the action of y is determined by the pullback of φ, $\varphi(y)w = xw \in W$ for all $w \in W$, and thus W is a subrepresentation of V as well, which is a contradiction. □

4.3.1 The Representation $\rho \circ T$

Define $T : \mathfrak{sl}_2[t] \to \mathfrak{sl}_2[t, t^{-1}]$ by

$$T(x_r) = x_{r+1}$$

$$T(y_r) = y_{r-1}$$

$$T(h_r) = h_r$$

for all $r \geq 0$. Also, define $\rho : \mathfrak{sl}_2[t, t^{-1}] \to \mathrm{end}(ev_{a_1}V(\xi_1) \otimes \cdots \otimes ev_{a_k}V(\xi_k))$ by

$$\rho(g \otimes f(t)) = \sum_{i=1}^{k} 1 \otimes \cdots \otimes f(a_i)g \otimes \cdots \otimes 1$$

for $a_i \neq a_j$, $a_i \in \mathbb{C}^\times$ and $g \in \mathfrak{sl}_2$.

Lemma 4.3.2. *The composite representation $\rho \circ T$ is an irreducible $\mathfrak{sl}_2[t]$ representation.*

Proof. We start by noting that $\rho : \mathfrak{sl}_2[t, t^{-1}] \to \mathrm{end}(ev_{a_1}V(\xi_1) \otimes \cdots \otimes ev_{a_k}V(\xi_k))$ by

$$\rho(g \otimes f(t)) = \sum_{i=1}^{k} 1 \otimes \cdots \otimes f(a_i)g \otimes \cdots \otimes 1$$

for $a_i \neq a_j$, $a_i \in \mathbb{C}^\times$ and $g \in \mathfrak{sl}_2$ defines a finite dimensional irreducible representation of the loop algebra (see [1], [3], [4], and [10]).

Now, since $\mathrm{end}(ev_{a_1}V(\xi_1) \otimes \cdots \otimes ev_{a_k}V(\xi_k))$ is finite dimensional and $\mathfrak{sl}_2[t, t^{-1}]$ is infinite dimensional, the kernel of ρ is nontrivial. This gives the irreducible representation

$$\frac{\mathfrak{sl}_2[t, t^{-1}]}{\ker \rho} \to \mathrm{end}(ev_{a_1}V(\xi_1) \otimes \cdots \otimes ev_{a_k}V(\xi_k)).$$

Next, we want to show that the map

$$\mathfrak{sl}_2[t] \xrightarrow{T} \mathfrak{sl}_2[t, t^{-1}] \to \frac{\mathfrak{sl}_2[t, t^{-1}]}{\ker \rho}$$

is surjective.

First we notice that $\ker\rho$ is an ideal of $\mathfrak{sl}_2[t,t^{-1}]$ so it must be of the form $\mathfrak{sl}_2\otimes(f)$ for some $f\in\mathbb{C}[t,t^{-1}]$. So

$$\frac{\mathfrak{sl}_2[t,t^{-1}]}{\ker\rho}\cong\mathfrak{sl}_2\otimes\frac{\mathbb{C}[t,t^{-1}]}{(f)}.$$

Now we want to find g_1, g_2, and $g_3\in\mathbb{C}[t]$ such that for any $\bar{g}\in\dfrac{\mathbb{C}[t,t^{-1}]}{(f)}$:

$$x\otimes g_1\mapsto x\otimes tg_1\mapsto x\otimes\bar{g}$$

$$h\otimes g_2\mapsto h\otimes g_2\mapsto h\otimes\bar{g}$$

$$y\otimes g_3\mapsto y\otimes t^{-1}g_3\mapsto y\otimes\bar{g}$$

Since t^s is a unit in $\mathbb{C}[t,t^{-1}]$ for all $s\in\mathbb{Z}$, it is easy to see that we can choose a coset representative for $\bar{g}$ so that $g_1, g_2,$ and $g_3\in\mathbb{C}[t]$ exist. So, we see that the map is surjective and applying lemma 4.3.1 we get our result. □

Lemma 4.3.3. *The composite representation $\rho\circ T$ is isomorphic to $\widetilde{\rho}:\mathfrak{sl}_2[t]\to end(ev_{a_1}V(\xi_1)\otimes\cdots\otimes ev_{a_k}V(\xi_k))$ defined by*

$$\widetilde{\rho}(g\otimes f(t))=\sum_{i=1}^{k}1\otimes\cdots\otimes f(a_i)g\otimes\cdots\otimes 1.$$

Proof. Now that we know the representation $\rho\circ T$ is irreducible, it is also cyclic. If we consider $v_{\xi_1}\otimes\cdots\otimes v_{\xi_k}$ where v_{ξ_i} is a highest weight generator of $ev_{a_i}V(\xi_i)$, we see that

$$x_r(v_{\xi_1}\otimes\cdots\otimes v_{\xi_k})=0$$

$$y_0^{\lambda+1}(v_{\xi_1}\otimes\cdots\otimes v_{\xi_k})=0$$

$$h_r(v_{\xi_1}\otimes\cdots\otimes v_{\xi_k})=\sum_{i=1}^{k}1\otimes\cdots\otimes(a_i)^r h\otimes\cdots\otimes 1$$

for all $r \geq 0$ and $\lambda = \xi_1 + ... + \xi_k$. Thus the representations $\rho \circ T$ and $\widetilde{\rho}$ are both quotients of the same local Weyl module. Since local Weyl modules have a unique irreducible quotient, the two representations must be isomorphic. □

4.4 Facts About $\varphi^-(i)$

4.4.1 Local Weyl Module Relations

Lemma 4.4.1. *The modules $V(\xi^-(i))$ and $\ker\varphi^+ = \mathbf{U}(\mathfrak{sl}_2[t])y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i)v_\xi$ are quotients of the same local Weyl module.*

Proof. We already know that the module $V(\xi^-(i))$ is the quotient of the local Weyl modules with highest weight $|\xi^-(i)|$. Now by section 4.2, we also know that we have the following short exact sequence of $\mathfrak{sl}_2$-modules:

$$0 \to \ker\varphi^+ \to V(\xi) \to V(\xi^+) \to 0.$$

However, this means that this short exact sequence must also be a short exact sequence of $\mathfrak{sl}_2$-modules and so splits when looked at as $\mathfrak{sl}_2$-modules. This means that $\ker\varphi^+$ is isomorphic as $\mathfrak{sl}_2$-modules to

$$V(\ell)^{\otimes m_\ell} \otimes \cdots \otimes V(i+1)^{\otimes m_{i+1}} \otimes V(i)^{\otimes m_i - 2} \otimes V(i-1)^{\otimes m_{i-1}} \otimes \cdots \otimes V(1)^{\otimes m_1}.$$

Since the kernel is generated by a highest weight element, the dimension forces the remainder of the local Weyl module relations to hold. Also since

$$\xi^- = (\ell^{m_\ell} \ldots (i+1)^{m_{i+1}} i^{m_i - 2} (i-1)^{m_{i-1}} \ldots 1^{m_1})$$

we see that $V(\xi^-(i))$ and $\ker\varphi^+$ are both quotients of $W_{loc}(|\xi^-(i)|)$. □

So, to prove that the map $\varphi^-(i) : V(\xi^-(i)) \to \mathbf{U}(\mathfrak{sl}_2[t])y(i, \nu_1(\xi)+\cdots+\nu_i(\xi)-1)v_\xi$ defined by $\varphi^-(i)(v_{\xi^-(i)}) = y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - 1)v_\xi$ exists, where $v_{\xi^-(i)}$ generates $V(\xi^-(i))$, it remains to show the remaining relations from $V(\xi^-(i))$ hold on $y(i, \nu_1(\xi)+\cdots+\nu_i(\xi)-1)v_\xi$. This however, has proven very difficult and we have only been able to show this for certain values of i. The remainder of this section is devoted to lemmas we will need to prove the relations hold and thus $\varphi^-(i)$ exists for $i = 2, 3$ in the next two sections.

Remark 4.4.2. The lemmas 4.4.3, 4.4.4, 4.4.5, and 4.4.6 are the unpublished work of Kayla Murray and may or may not appear in her future thesis. The proofs of these propositions and lemmas can be found in the appendix of this paper.

4.4.2 Number of Parts Relation

Lemma 4.4.3. *We have*

$$y(1, \nu_1 - 2)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

4.4.3 Lemmas for Fusion Product Calculation

Lemma 4.4.4. *We have*

$$y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = {}_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi.$$

Lemma 4.4.5. *For $1 \leq r \leq i$, if $\nu_r \geq 3$ and*

$${}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0,$$

for all $1 \le j \le r-1$, *then*

$$y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi$$

$$= \ {}_1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi.$$

Lemma 4.4.6. *For* $i + 1 \le r \le \ell - 1$, *if* $\nu_r \ge 1$,

$${}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0$$

for all $1 \le j \le i$, *and*

$${}_1y(\tilde{j}, \nu_1 + \nu_2 + \cdots + \nu_{\tilde{j}} - 2i - \tilde{j} + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0$$

for all $i + 1 \le \tilde{j} \le r - 1$, *then*

$$y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi$$

$$= \ {}_1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi.$$

4.4.4 Extending the Map $\varphi^-(i)$ to any Partition

Lemma 4.4.7. *If the map*

$$\varphi^-(i) : V(\xi'^-(i)) \to \ker \varphi^+(i)$$

defined by $\varphi^-(i)(v_{\xi'^-(i)}) = y(i, \nu_1(\xi') + \cdots + \nu_i(\xi') - i) v_{\xi'}$ *exists for a partition of the form* $\xi' = (i^2 (i-1)^{m_{i-1}} \ldots 1^{m_1})$, *then the* $V(\xi^-(i))$ *relations* $y(r, \nu_1(\xi^-(i)) + \cdots + \nu_r(\xi^-(i)) - r + 1)$ *will hold on the generator of the kernel of* $\varphi^+(i)$, $y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i) v_\xi$, *for any partition* $\xi = (\ell^{m_\ell} \ldots i^{m_i} (i-1)^{m_{i-1}} \ldots 1^{m_1})$.

Proof. We note that there are two cases. If $1 \le r \le i$ we have that,

$$y(r, \nu_1(\xi^-(i)) + \cdots + \nu_r(\xi^-(i)) - r + 1) = y(r, \nu_1(\xi) + \cdots + \nu_r(\xi) - 3r + 1)$$

and if $i \leq r \leq \ell - 1$,

$$y(r, \nu_1(\xi^-(i)) + \cdots + \nu_r(\xi^-(i)) - r + 1) = y(r, \nu_1(\xi) + \cdots + \nu_r(\xi) - 2i - r + 1).$$

So our proof will be in two steps. (Recall that we will let $\nu_j(\xi) = \nu_j$ for ease of notation.)

Suppose $1 \leq r \leq i$, we proceed by induction on r in order to meet the necessary conditions to apply the lemma 4.4.5. Let $r = 1$ for our base case. Then

$$y(1, \nu_1 - 2)y(i, \nu_1 + \cdots + \nu_i - i)v_\xi = 0$$

by lemma 4.4.3.

Now suppose for all $k < r$,

$$y(k, \nu_1 + \nu_2 + \cdots + \nu_k - 3k + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Finally we want to show that

$$y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

We will show this by inducting on, ν_i, the number of elements greater than or equal to i in the partition ξ, specifically we will induct by adding to the front of our partition at each step. For our base case, we will consider $\nu_i = 2$, more specifically the partition $(i^2(i-1)^{m_{i-1}} \cdots 1^{m_1})$. In this case, we know that the map $\varphi^-(i)$ exists by assumption. So

$$y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Now suppose the equality holds for $\nu_i \leq m_i + \cdots + m_{l-1} + (m_l - 1)$.

Lastly, we show the relation holds for $\nu_i = m_i + \cdots + m_{l-1} + m_l$, in other words for any partition, $\xi = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \ldots 1^{m_1})$.

We know by assumption that for $\xi' = (\ell^{m_\ell - 1}(\ell-1)^{m_{\ell-1}} \ldots 1^{m_1})$, we have the following, $y(r, \nu_1(\xi') + \cdots + \nu_r(\xi') - 3r + 1) y(i, \nu_1(\xi') + \cdots + \nu_i(\xi') - i) v_{\xi'} = 0$. We note that $\nu_j(\xi') = \nu_j - 1$ for all $j \leq \ell$, and so $y(r, \nu_1 + \cdots + \nu_r - 4r + 1) y(i, \nu_1 + \cdots + \nu_i - 2i) v_{\xi'} = 0$.

Let $Y = {}_1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1)_1 y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)$. Now since we have shown that the representation $\rho \circ T$ is isomorphic to our standard representation, we will use it to understand some facts about $Yv_{\xi'}$. Let $v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}$ be the generator of $ev_{a_1} V(\xi'_1) \otimes \cdots \otimes ev_{a_k} V(\xi'_k)$, then we see that

$$(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) = \rho(y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 4r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - 2i))(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})$$

by definition of T.

By assumption, $y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 4r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - 2i) v_{\xi'} = 0$ which implies that, in the fusion product, this element is in a grade strictly smaller than $2\nu_1 + \cdots + 2\nu_r + \nu_{r+1} + \cdots + \nu_i - 4r - 2i + 1$. Thus

$$\begin{aligned}(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) &= \rho\Big(\sum c y_{p_1} \cdots y_{p_{r+i}}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}), \\ &= (\rho \circ T)\Big(\sum c' y_{p_1+1} \cdots y_{p_{r+i}+1}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})\end{aligned}$$

where the sums are over the set $\{0 \leq p_1 \leq ... \leq p_{r+i}\}$ such that we have $p_1 + ... + p_{r+i} < 2\nu_1 + \cdots + 2\nu_r + \nu_{r+1} + \cdots + \nu_i - 4r - 2i + 1$. Then reindexing the grades we get

$$(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) = (\rho \circ T)\Big(\sum c' y_{p'_1} \cdots y_{p'_{r+i}}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})$$

where this time the sum is over the set $\{0 \leq p'_1 \leq ... \leq p'_{r+i}\}$ such that $p'_1 + ... + p'_{r+i} < 2\nu_1 + \cdots + 2\nu_r + \nu_{r+1} + \cdots + \nu_i - 3r - i + 1$.

So our induction hypothesis implies that $Yv_{\xi'}$ sits in two different grades and so is also zero in the fusion product.

Finally, consider Yv_ξ. Let $v_{\xi_1}\otimes\cdots\otimes v_{\xi_k}$ be the generator of $ev_{a_1}V(\xi_1)\otimes\cdots\otimes ev_{a_k}V(\xi_k)$. Since the fusion product is independent of choice of parameters, we can choose $a_1 = 0$. Working in the tensor product, we can write $Yv_\xi = Y(v_{\xi_1}\otimes \underline{v})$, where $\underline{v}$ meets the criteria of our induction hypothesis. Since y_0 is the only element of the form y_r that acts non trivally on v_{ξ_1} and since $Y = {}_1y(r, \nu_1+\nu_2+\cdots+\nu_r-3r+1)_1y(i,\nu_1+\nu_2+\cdots+\nu_i-i)$, we have $Yv_\xi = Y(v_{\xi_1}\otimes\underline{v}) = v_{\xi_1}\otimes Y\underline{v}$. Thus, $Yv_\xi = 0$ by the implication of the induction hypothesis proven above. Applying lemmas 4.4.4 and 4.4.5, we have our result for $1\le r\le i$.

So now we show that if $i\le r\le \ell-1$, $y(r,\nu_1+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\cdots+\nu_i-i)v_\xi = 0$. Again we proceed by induction on r in order to meet the necessary conditions to apply the lemma 4.4.6. Let $r = i$ for our base case. Then we have our result by the previous case.

Now suppose for all $k < r$,

$$y(k,\nu_1+\nu_2+\cdots+\nu_k-2i-k+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0.$$

Finally we want to show that

$$y(r,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0.$$

We will show this by inducting on, ν_i, in exactly the same way as in the first case. When $\nu_i = 2$ we know we have our result by assumption. Now suppose the equality holds for $\nu_i\le m_i+\cdots+m_{l-1}+(m_l-1)$. Lastly, we show the relation holds for $\nu_i = m_i+\cdots+m_{l-1}+m_l$, in other words for any partition, $\xi = (\ell^{m_\ell}(\ell-1)^{m_{\ell-1}}\dots 1^{m_1})$.

We know by assumption that for $\xi' = (\ell^{m_\ell - 1}(\ell-1)^{m_{\ell-1}} \dots 1^{m_1})$, we have the following, $y(r, \nu_1(\xi')+\cdots+\nu_r(\xi')-3r+1)y(i, \nu_1(\xi')+\cdots+\nu_i(\xi')-i)v_{\xi'} = 0$. We note that $\nu_j(\xi') = \nu_j - 1$ for all $j \leq \ell$, and so $y(r, \nu_1+\cdots+\nu_r-4r+1)y(i, \nu_1+\cdots+\nu_i-2i)v_{\xi'} = 0$.

Let $Y = {}_1y(r, \nu_1+\nu_2+\cdots+\nu_r-2i-r+1)_1y(i, \nu_1+\nu_2+\cdots+\nu_i-i)$. Now since we have shown that the representation $\rho \circ T$ is isomorphic to our standard representation, we will use it to understand some facts about $Yv_{\xi'}$. Let $v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}$ be the generator of $ev_{a_1}V(\xi'_1) \otimes \cdots \otimes ev_{a_k}V(\xi'_k)$, then we see that

$$(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) = \rho(y(r, \nu_1+\nu_2+\cdots+\nu_r-2i-2r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-2i))(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})$$

by definition of T.

By assumption, $y(r, \nu_1+\nu_2+\cdots+\nu_r-2i-2r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-2i)v_{\xi'} = 0$ which implies that, in the fusion product, this element is in a grade strictly smaller than $2\nu_1+\cdots+2\nu_i+\nu_{i+1}+\cdots+\nu_r-2r-4i+1$. Thus

$$\begin{aligned}(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) &= \rho\Big(\sum c y_{p_1} \cdots y_{p_{r+i}}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}), \\ &= (\rho \circ T)\Big(\sum c' y_{p_1+1} \cdots y_{p_{r+i}+1}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})\end{aligned}$$

where the sums are over the set $\{0 \leq p_1 \leq ... \leq p_{r+i}\}$ such that we have $p_1 + ... + p_{r+i} < 2\nu_1+\cdots+2\nu_i+\nu_{i+1}+\cdots+\nu_r-2r-4i+1$. Then reindexing the grades we get

$$(\rho \circ T)(Y)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k}) = (\rho \circ T)\Big(\sum c' y_{p'_1} \cdots y_{p'_{r+i}}\Big)(v_{\xi'_1} \otimes \cdots \otimes v_{\xi'_k})$$

where this time the sum is over the set $\{0 \leq p'_1 \leq ... \leq p'_{r+i}\}$ such that $p'_1 + ... + p'_{r+i} < 2\nu_1+\cdots+2\nu_r+\nu_{r+1}+\cdots+\nu_i-r-3i+1$.

So our induction hypothesis implies that $Yv_{\xi'}$ sits in two different grades and so is also zero in the fusion product.

Finally, consider Yv_ξ. Let $v_{\xi_1} \otimes \cdots \otimes v_{\xi_k}$ be the generator of $ev_{a_1}V(\xi_1) \otimes \cdots \otimes ev_{a_k}V(\xi_k)$. Since the fusion product is independent of choice of parameters, we can choose $a_1 = 0$. Working in the tensor product, we can write $Yv_\xi = Y(v_{\xi_1} \otimes \underline{v})$, where $\underline{v}$ meets the criteria of our induction hypothesis. Since y_0 is the only element of the form y_r that acts non trivally on v_{ξ_1} and since $Y = {}_1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1)_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)$, we have $Yv_\xi = Y(v_{\xi_1} \otimes \underline{v}) = v_{\xi_1} \otimes Y\underline{v}$. Thus, $Yv_\xi = 0$ by the implication of the induction hypothesis proven above. Applying lemmas 4.4.4 and 4.4.6, we have our result for $i \leq r \leq \ell - 1$.

□

4.5 The map $\varphi^-(2)$

Lemma 4.5.1. *For any partition ξ, there exists a surjective map*

$$\varphi^-(2) : V(\xi^-(2)) \to \ker \varphi^+(2) \subset V(\xi).$$

Proof. By lemma 4.4.1 we know that $V(\xi^-(2))$ and $\ker \varphi^+(2)$ are quotients of the same local weyl module. So if we let $\varphi^-(2)(v_{\xi^-(2)}) = y(2, \nu_1(\xi) + \nu_2(\xi) - 2)v_\xi$, where $v_{\xi^-(2)}$ generates $V(\xi^-(2))$ and $y(2, \nu_1(\xi) + \nu_2(\xi) - 2)v_\xi$ generates $\ker \varphi^+(2)$, it only remains to check the extra relations from $V(\xi^-(2))$ hold on $y(2, \nu_1(\xi) + \nu_2(\xi) - 2)v_\xi$. By lemma 4.4.7, it suffices to show this for a partition of the form $\xi = (2^2 1^{m_1})$. However, in this case we have $\xi^-(2) = (1^{m_1})$, which means it is only necessary to check the local Weyl module relations. So we have shown $\varphi^-(2)$ exists and is surjective. □

4.6 The map $\varphi^-(3)$

Lemma 4.6.1. *For any partition ξ, there exists a surjective map*

$$\varphi^-(3) : V(\xi^-(3)) \to \ker \varphi^+(3) \subset V(\xi).$$

Proof. By lemma 4.4.1 we know that $V(\xi^-(3))$ and $\ker \varphi^+(3)$ are quotients of the same local weyl module. So if we let $\varphi^-(3)(v_{\xi^-(3)}) = y(3, \nu_1(\xi) + \nu_2(\xi) + \nu_3(\xi) - 3)v_\xi$, where $v_{\xi^-(3)}$ generates $V(\xi^-(3))$ and $y(3, \nu_1(\xi)+\nu_2(\xi)+\nu_3(\xi)-3)v_\xi$ generates $\ker \varphi^+(3)$, it only remains to check the extra relations from $V(\xi^-(3))$ hold on $y(3, \nu_1(\xi) + \nu_2(\xi) + \nu_3(\xi) - 3)v_\xi$. By lemma 4.4.7, it suffices to show this for a partition of the form $\xi = (3^2 2^{m_2} 1^{m_1})$. However, in this case we have $\xi^-(3) = (2^{m_2} 1^{m_1})$, which means it is only necessary to check the relation when $r = 1$. So we see that

$$\begin{aligned}
&y(1, \nu_1(\xi^-(2)))y(3, \nu_1(\xi) + \nu_2(\xi) + \nu_3(\xi) - 3)v_\xi \\
&\qquad = y(1, \nu_1(\xi) - 2)y(3, \nu_1(\xi) + \nu_2(\xi) + \nu_3(\xi) - 3)v_\xi \\
&\qquad = 0,
\end{aligned}$$

by lemma 4.4.3. So we have shown $\varphi^-(3)$ exists and is surjective. □

4.7 Dimension of $\ker \varphi^+(i)$

Lemma 4.7.1. $\dim \ker \varphi^+(i) = \dim V(\xi^-(i))$.

Proof. From [7], we know how to calculate the dimension of $V(\xi)$ modules. So since $\xi^-(i) = (\ell^{m_\ell} \ldots (i+1)^{m_{i+1}} i^{m_i - 2} (i-1)^{m_{i-1}} \ldots 1^{m_1})$ we know

$$\dim V(\xi^-(i)) = (\ell+1)^{m_\ell} \cdots (i+2)^{m_{i+1}} \cdot (i+1)^{m_i - 2} \cdot (i)^{m_{i-1}} \cdots 2^{m_1}.$$

We also know that $\dim\ker\varphi^+(i) = \dim V(\xi) - \dim V(\xi^+(i))$. Now

$$\dim V(\xi) = (\ell+1)^{m_\ell}\cdots(i+2)^{m_{i+1}}\cdot(i+1)^{m_i}\cdot(i)^{m_{i-1}}\cdots 2^{m_1}$$

and

$$\dim V(\xi^+(i)) = (\ell+1)^{m_\ell}\cdots(i+2)^{m_{i+1}+1}\cdot(i+1)^{m_i-2}\cdot(i)^{m_{i-1}+1}\cdots 2^{m_1}.$$

So

$$\begin{aligned}\dim\ker\varphi^+(i) &= \dim V(\xi) - \dim V(\xi^+(i))\\ &= (\ell+1)^{m_\ell}\cdots(i+2)^{m_{i+1}}\cdot(i+1)^{m_i-2}\cdot(i)^{m_{i-1}}\cdots 2^{m_1}\\ &= \dim\ker\varphi^+(i).\end{aligned}$$

□

4.8 Theorem

The lemmas in the previous sections imply the proof of the following theorem for $i = 2, 3$. We note that the case where $i = 1$ was proven in [7]. It is also important to note that since we are considering graded modules, it is not enough that $V(\xi^-(i)) \cong \ker\varphi^+(i))$ for our short exact sequence. Since our kernel generator is $y(i, \nu_1(\xi)+\cdots+\nu_i(\xi)-i)v_\xi$, every element in $\ker\varphi^+(i)$ sits in grade $\nu_1(\xi)+\cdots+\nu_i(\xi)-i$ or higher since actions from $\mathfrak{sl}_2[t]$ can only raise the grade. So in the theorem below we need the grade shift $\tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i}$ to ensure we meet the requirements of graded modules.

Theorem 4.8.1. *The following is a short exact sequence of graded $sl_2[t]$-modules for $i = 1, 2, 3$:*

$$0 \to \tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i}V(\xi^-(i)) \to V(\xi) \to V(\xi^+(i)) \to 0.$$

There are also special cases of partitions where we can show this short exact sequence exists. In fact, the previous theorem generalizes to the following that will help us prove more cases of our character formula in chapter 5. Specifically this case occurs when the elements of the partition smaller than i form a rectangular partition or a consecutive fat hook (i.e. $\xi = k^{m_k}$ or $\xi = k^{m_k}(k-1)^{m_{k-1}}$).

Theorem 4.8.2. *The following is a short exact sequence of graded $sl_2[t]$-modules for $\xi = (\ell^{m_\ell} \dots (i+1)^{m_{i+1}} i^{m_i} k^{m_k}(k-1)^{m_{k-1}})$ where $k < i$, $m_j \in \mathbb{Z}_+$, and $m_i \geq 2$:*

$$0 \to \tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i} V(\xi^-(i)) \to V(\xi) \to V(\xi^+(i)) \to 0.$$

Proof. The lemmas in the previous sections imply that it suffices to show for $\xi = (i^2 k^{m_k}(k-1)^{m_{k-1}})$ the map

$$V(k^{m_k}(k-1)^{m_{k-1}}) \to \mathbf{U}(\mathfrak{sl}_2[t]) y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i) v_\xi$$

exists and is surjective. Let v_{ξ^-} be the generator of $V(k^{m_k}(k-1)^{m_{k-1}})$ and we consider the map $v_{\xi^-} \mapsto y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i) v_\xi$. By lemma 4.4.1, we already know the the local Weyl module relations hold, so it remains to show that the additional $V(\xi^-)$ relations hold. However, by [7] and [13], since the partition ξ^- is either rectangular or a consecutive fat hook we need only check that $y(1, \nu_1(\xi) - 2) y(i, \nu_1(\xi) + \cdots + \nu_i(\xi) - i) v_\xi = 0$. By lemma 4.4.3 we know that relation holds. So we have our result. □

Chapter 5

Character Formula

In this chapter, we conjecture a character formula for a tensor product of a Demazure module and a local Weyl module that gives us some evidence to believe that conjecture 2.2.1 is true. The proof of the conjecture will rely on the short exact sequence from the previous chapter and thus will only be considered a theorem for the cases where we have proven the short exact sequence holds.

5.1 Conjecture

Conjecture 5.1.1. *For $l, m, n, c \in \mathbb{Z}_+$, $n \geq m$ and $c < \ell$,*

$$\operatorname{ch}_{\mathrm{gr}}[D(\ell, n\ell + c) \otimes W_{loc}(m)] = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \operatorname{ch}_{\mathrm{gr}} V((\ell+1)^k \ell^{n-m} (\ell-1)^{m-k} c).$$

5.2 More Short Exact Sequences

Before we attempt to prove the conjecture, we prove some short exact sequences that will be needed.

Lemma 5.2.1. *For $n \geq m$, the following is a short exact sequence of $\mathfrak{sl}_2[t]$-modules:*

$$0 \to V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) \to V(\ell^{n-m+1}(\ell-1)^{m-1}c) \otimes W(1) \to V(\ell^{n-m}(\ell-1)^{m}c) \to 0.$$

Proof. First, we will show there exists a map

$$\phi : V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) \to V(\ell^{n-m+1}(\ell-1)^{m-1}c) \otimes W(1).$$

Let $v_{\tilde{\xi}}$ be the generator of $V((\ell+1)l^{n-m}(\ell-1)^{m-1}c)$ and v_ξ the generator of $V(\ell^{n-m+1}(\ell-1)^{m-1}c)$. Define ϕ by $\phi(v_{\tilde{\xi}}) = v_\xi \otimes w_1$. Checking the relations of $W_{loc}(n\ell - m + c + 2)$ is trivial, so we only need to show the additional relation from $V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c)$ as a quotient of the local Weyl module. Assume $s, r \in \mathbb{N}$ such that $s + r \geq 1 + rp + \sum_{j \geq p+1} \tilde{\xi}_j$ for some $p \in \mathbb{N} \cup \{0\}$. Then $x_1^s y_0^{s+r} v_{\tilde{\xi}} = 0$. We need to show $x_1^s y_0^{s+r}(v_\xi \otimes w_1) = 0$. We have

$$\begin{aligned} x_1^s y_0^{s+r}(v_\xi \otimes w_1) &= x_1^s(y_0^{s+r} v_\xi \otimes w_1 + y_0^{s+r-1} v_\xi \otimes y_0 w_1) \\ &= x_1^s y_0^{s+r} v_\xi \otimes w_1 + x_1^s y_0^{s+r-1} v_\xi \otimes y_0 w_1. \end{aligned}$$

Since $\sum_{j \geq p+1} \tilde{\xi}_j \geq \sum_{j \geq p+1} \xi_j$ for all p, then $x_1^s y_0^{s+r} v_\xi = 0$. For the other summand we consider the case when $p = 0$ and $p > 0$. If the inequality $s + r \geq 1 + rp + \sum_{j \geq p+1} \tilde{\xi}_j$ holds for $p = 0$, then we have

$$r + s \geq 1 + \sum_{j \geq 1} \tilde{\xi}_j = 1 + |\tilde{\xi}| = 1 + \sum_{j \geq 1} \xi_j + 1$$

which gives $r - 1 + s \geq 1 + \sum_{j \geq 1} \xi_j$. If the inequality holds for $p > 0$, we have

$$r + s \geq 1 + rp + \sum_{j \geq p+1} \tilde{\xi}_j \geq 1 + rp + \sum_{j \geq p+1} \tilde{\xi}_j + (1 - p) \geq 1 + rp + \sum_{j \geq p+1} \xi_j + (1 - p).$$

Rearranging this inequality gives us $(r-1)+s \geq 1+(r-1)p+\sum_{j\geq p+1}\xi_j$ so $r-1$ and s satisfy the $V(\xi)$-relation and thus $x_1^s y_0^{s+r-1} v_\xi = 0$. Hence, the map ϕ exists. Let $M = \text{Im } \phi$.

Next, we will show there exists a map

$$\psi : V(\ell^{n-m}(\ell-1)^m c) \to V(\ell^{n-m+1}(\ell-1)^{m-1}c) \otimes W(1)/M$$

where $\psi(v_{\xi'}) = v_\xi \otimes y_0 w_1 + M$ where $v_{\xi'}$ is the generator of $V(\ell^{n-m}(\ell-1)^m c)$. We have for $p \geq 0$,

$$x_p(v_\xi \otimes y_0 w_1) = v_\xi \otimes h_p w_1 = \delta_{p,0} v_\xi \otimes w_1 \in M$$

and

$$\begin{aligned} h_p(v_\xi \otimes y_0 w_1) &= h_p v_\xi \otimes y_0 w_1 + v_\xi \otimes h_p y_0 w_1 \\ &= \delta_{p,0}(\ell n - m + 1 + c) v_\xi \otimes y_0 w_1 + v_\xi \otimes y_0 h_p w_1 - 2 v_\xi \otimes y_p w_1 \\ &= \delta_{p,0}(\ell n - m + c) v_\xi \otimes y_0 w_1. \end{aligned}$$

Thus $v_\xi \otimes y_0 w_1 + M$ is a highest weight vector and $V(\ell^{n-m+1}(l-1)^{m-1}c) \otimes W(1)/M$ is finite dimensional thus $y_0^{\ell n-m+c+1}(v_\xi \otimes y_0 w_1) \in M$. Finally, we need to show the additional relation from $V(\ell^{n-m}(\ell-1)^m c)$ as a quotient of the local Weyl module.

We consider the partition $\xi' = (\ell^{n-m}(\ell-1)^m c)$, we know that the associated module $V(\xi')$ has the following relations for $0 \leq r < \ell$, $\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1) v_{\xi'} = 0$. We want to show $\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)(v_\xi \otimes y_0 w_1 + M) = 0 + M$ as well. We see that

$$\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)(v_\xi \otimes y_0 w_1 + M)$$

$$= \mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)v_\xi \otimes y_0 w_1 + M$$

$$= 0 \otimes y_0 w_1 + M$$

since for all $0 \leq r < \ell$, $\nu_r(\xi') = \nu_r(\xi)$. So the additional relations hold.

Finally considering dimension, we have

$$2(\ell+1)^{n-m+1}\ell^{m-1}(c+1)$$

$$= (\ell+2)(\ell+1)^{n-m}\ell^{m-1}(c+1) + (\ell+1)^{n-m}\ell^m(c+1)$$

$$= \dim V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \dim V(\ell^{n-m}(\ell-1)^m c)$$

$$\geq \dim V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + [\dim V(\ell^{n-m+1}(\ell-1)^{m-1}c) \otimes W_{loc}(1)]/M$$

$$= \dim V(\ell^{n-m+1}(\ell-1)^{m-1}c) \otimes W_{loc}(1)$$

$$= 2(\ell+1)^{n-m+1}\ell^{m-1}(c+1).$$

These are all equal thus we get the desired short exact sequence. □

Lemma 5.2.2. *For $n \geq m \geq k+1 \geq 2$, the following is a short exact sequence of $\mathfrak{sl}_2[t]$-modules:*

$$0 \to V(\tilde{\xi}) \to V(\xi) \otimes W(1) \to V(\xi') \to 0,$$

where $\tilde{\xi} = ((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c)$, $\xi = ((\ell+1)^k\ell^{n-m+1}(\ell-1)^{m-k-1}c)$ and $\xi' = ((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c)$.

Proof. First, we will show there exists a map

$$\phi : V((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c) \to V((\ell+1)^{k}\ell^{n-m+1}(\ell-1)^{m-k-1}c) \otimes W(1).$$

Let $v_{\tilde{\xi}}$ be the generator of $V((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c)$ and v_ξ the generator of $V((\ell+1)^{k}\ell^{n-m+1}(\ell-1)^{m-k-1}c)$. Define ϕ by $\phi(v_{\tilde{\xi}}) = v_\xi \otimes w_1$. Checking the relations of $W_{loc}(2k+n\ell-m+c+2)$ are trivial, so we only need to show the additional relation from $V((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c)$ as a quotient of the local Weyl module. Assume $s, r \in \mathbb{N}$ such that $s+r \geq 1+rp+\sum_{j\geq p+1}\tilde{\xi}_j$ for some $p \in \mathbb{N}\cup\{0\}$. Then $x_1^s y_0^{s+r} v_{\tilde{\xi}} = 0$. We need to show $x_1^s y_0^{s+r}(v_\xi \otimes w_1) = 0$. We have

$$\begin{aligned} x_1^s y_0^{s+r}(v_\xi \otimes w_1) &= x_1^s(y_0^{s+r} v_\xi \otimes w_1 + y_0^{s+r-1} v_\xi \otimes y_0 w_1) \\ &= x_1^s y_0^{s+r} v_\xi \otimes w_1 + x_1^s y_0^{s+r-1} v_\xi \otimes y_0 w_1. \end{aligned}$$

Since $\sum_{j\geq p+1}\tilde{\xi}_j \geq \sum_{j\geq p+1}\xi_j$ for all p, then $x_1^s y_0^{s+r} v_\xi = 0$. For the other summand we consider the case when $p=0$ and $p>0$. If the inequality $s+r \geq 1+rp+\sum_{j\geq p+1}\tilde{\xi}_j$ holds for $p=0$, then we have

$$r+s \geq 1+\sum_{j\geq 1}\tilde{\xi}_j = 1+|\tilde{\xi}| = 1+\sum_{j\geq 1}\xi_j + 1$$

which gives $r-1+s \geq 1+\sum_{j\geq 1}\xi_j$. If the inequality holds for $p>0$, we have

$$r+s \geq 1+rp+\sum_{j\geq p+1}\tilde{\xi}_j \geq 1+rp+\sum_{j\geq p+1}\tilde{\xi}_j + (1-p) \geq 1+rp+\sum_{j\geq p+1}\xi_j + (1-p).$$

Rearranging this gives us $(r-1)+s \geq 1+(r-1)p+\sum_{j\geq p+1}\xi_j$ so $r-1+s$ satisfies the $V(\xi)$-relation and thus $x_1^s y_0^{s+r-1} v_\xi = 0$. Hence, the map ϕ exists. Let $M = \text{Im } \phi$.

Next, we will show there exists a map

$$\psi : V((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c) \to V((\ell+1)^{k}\ell^{n-m+1}(\ell-1)^{m-k-1}c) \otimes W(1)/M$$

where $\psi(v_{\xi'}) = v_\xi \otimes y_0 w_1 + M$ where $v_{\xi'}$ is the generator of $V((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c)$. We have for $p \geq 0$,

$$x_p(v_\xi \otimes y_0 w_1) = v_\xi \otimes h_p w_1 = \delta_{p,0} v_\xi \otimes w_1 \in M$$

and

$$\begin{aligned} h_p(v_\xi \otimes y_0 w_1) &= h_p v_\xi \otimes y_0 w_1 + v_\xi \otimes h_p y_0 w_1 \\ &= \delta_{p,0}(2k + ln - m + c + 1) v_\xi \otimes y_0 w_1 + v_\xi \otimes y_0 h_p w_1 - 2 v_\xi \otimes y_p w_1 \\ &= \delta_{p,0}(2k + ln - m + c) v_\xi \otimes y_0 w_1. \end{aligned}$$

Thus $v_\xi \otimes y_0 w_1 + M$ is a highest weight vector and $V((\ell+1)^{k}\ell^{n-m+1}(\ell-1)^{m-k-1}c) \otimes W(1)/M$ is finite dimensional thus $y_0^{2k+n\ell-m+c+1}(v_\xi \otimes y_0 w_1) \in M$. Finally, we need to show the additional relation from $V((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c)$ as a quotient of the local Weyl module.

We consider the partition $\xi' = ((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c)$, we know that the associated module $V(\xi')$ has the following relations $\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1) v_{\xi'} = 0$, for $0 \leq r < \ell + 1$. We want to show $\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)(v_\xi \otimes y_0 w_1 + M) = 0 + M$ as well. We see that

$$\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)(v_\xi \otimes y_0 w_1 + M)$$

$$= \mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1) v_\xi \otimes y_0 w_1 + M$$

$$= 0 \otimes y_0 w_1 + M$$

since for all $0 \leq r < \ell + 1$, $\nu_r(\xi') = \nu_r(\xi)$. So the additional relations hold.

Finally considering dimension, we have

$$2(\ell+2)^k(\ell+1)^{n-m+1}\ell^{m-k-1}(c+1)$$

$$= (\ell+3)(\ell+2)^{k-1}(\ell+1)^{n-m+1}\ell^{m-k-1}(c+1) + (\ell+2)^{k-1}(\ell+1)^{n-m+2}\ell^{m-k-1}(c+1)$$

$$= \dim V((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c) + \dim V((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c)$$

$$\geq \dim V((\ell+2)(\ell+1)^{k-1}\ell^{n-m+1}(\ell-1)^{m-k-1}c)$$

$$+ [\dim V((\ell+1)^k\ell^{n-m+1}(\ell-1)^{m-k-1}c) \otimes W_{loc}(1)]/M$$

$$= \dim V((\ell+1)^k\ell^{n-m+1}(\ell-1)^{m-k-1}c) \otimes W_{loc}(1)$$

$$= 2(\ell+2)^k(\ell+1)^{n-m+1}\ell^{m-k-1}(c+1).$$

These are all equal thus we get the desired short exact sequence. □

5.3 Method for Proof of Conjecture

Now we prove the equality of graded characters in conjecture 5.1.1 assuming the short exact sequence in chapter 4 holds.

Conjecture. When $0 < m \leq n$ and $c < \ell$,

$$\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W(m)) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}} V((\ell+1)^k \ell^{n-m} (\ell-1)^{m-k} c).$$

Proof. We will proceed by induction on m to prove the proposition. We will need to consider the base cases $m = 0$ and $m = 1$.

First we consider the case where $m = 0$. In this case, $D(\ell, n\ell + c) \otimes W_{loc}(0) \cong D(\ell, n\ell + c)$. And by [7], we also know that $D(\ell, n\ell + c) \cong V(\ell^n c)$, which completes the case.

Now we consider the case where $m = 1$ and find a filtration of the tensor product by $V(\xi)$-modules.

We will show the $sl_2[t]$-module $D(\ell, n\ell + c) \otimes W_{loc}(1)$ has the filtration

$$0 \subset M_1 = \mathbf{U}(sl_2[t])(v \otimes w_1) \subset M_2 = \mathbf{U}(sl_2[t])(v \otimes y_0 w_1) = D(\ell, n\ell + c) \otimes W_{loc}(1),$$

where v generates $D(\ell, n\ell + c)$ and w_1 generates $W_{loc}(1)$, and

$$M_1 \cong V((\ell+1)\ell^{n-1} c) \text{ and } M_2/M_1 \cong V(\ell^{n-1}(\ell-1)c),$$

as $sl_2[t]$-modules.

To prove this, we will show there exists surjective maps

$$V((\ell+1)\ell^{n-1} c) \to M_1 \text{ and } V(\ell^{n-1}(\ell-1)c) \to M_2/M_1.$$

Let v_ξ be the generator of $V((\ell+1)\ell^{n-1}c)$ and let $m_1 = v \otimes w_1$ be the generator of M_1. Define the map $V((\ell+1)\ell^{n-1}c) \to M_1$ by mapping v_ξ to m_1. Checking the relations we see

$$(x \otimes t^s)m_1 = x_s v \otimes w_1 + v \otimes x_s w_1 = 0, \quad for \quad s \geq 0$$

$$(h \otimes t^s)m_1 = h_s v \otimes w_1 + v \otimes h_s w_1 = 0, \quad for \quad s > 0$$

$$(h \otimes 1)m_1 = hv \otimes w_1 + v \otimes hw_1 = (n\ell + c)v \otimes w_1 + v \otimes w_1 = (n\ell + c + 1)m_1$$

$$(y \otimes 1)^{n\ell+c+2} m_1 = y_0^{n\ell+c+2} v \otimes w_1 + b_1 y_0^{n\ell+c+1} v \otimes y_0 w_1 + ... + v \otimes y_0^{n\ell+c+2} w_1 = 0,$$

and finally checking the last relation for $V(\xi)$-modules we see that

$$\begin{aligned}(x \otimes t)^s (y \otimes 1)^{s+r} m_1 &= x_1^s (y_0^{s+r} v \otimes w_1 + b y_0^{s+r-1} v \otimes y_0 w_1) \\ &= x_1^s y_0^{s+r} v \otimes w_1 + b x_1^s y_0^{s+r-1} v \otimes y_0 w_1 \\ &= 0 \qquad \text{for } r + s \geq 1 + rk + \sum_{j \geq k+1} \xi_j.\end{aligned}$$

So we have a surjective map.

Now similarly we show there exists a surjective map $V(\ell^{n-1}(\ell-1)c) \to M_2/M_1$. Let $v_{\xi'}$ be the generator of $V(\ell^{n-1}(\ell-1)c)$ and let $m_2 = v \otimes y_0 w_1 + M_1$ be the generator of M_2/M_1. Define the map $V(\ell^{n-1}(\ell-1)c) \to M_2/M_1$ by mapping $v_{\xi'}$ to m_2. Checking the relations we see

$$(x \otimes t^s)m_2 = x_s v \otimes y_0 w_1 + v \otimes x_s y_0 w_1 = 0 + v \otimes h_s w_1 + v \otimes y_0 x_s w_1 = 0, \quad for \quad s \geq 0$$

$$(h \otimes t^s)m_2 = h_s v \otimes y_0 w_1 + v \otimes h_s y_0 w_1$$

$$= 0 + v \otimes -2y_s w_1 + v \otimes y_0 h_s w_1 = 0 \quad for \quad s > 0$$

$$(h \otimes 1)m_2 = hv \otimes y_0 w_1 + v \otimes h y_0 w_1$$

$$= (n\ell + c)v \otimes y_0 w_1 + v \otimes -2y_0 w_1 + v \otimes y_0 h w_1$$

$$= (n\ell + c)(v \otimes y_0 w_1) - 2(v \otimes y_0 w_1) + (v \otimes y_0 w_1) = (n\ell + c - 1)m_2,$$

$$(y \otimes 1)^{nl+c} m_2 = y_0^{n\ell+c} v \otimes y_0 w_1 + b_1 y_0^{n\ell+c-1} v \otimes y_0^2 w_1 + ... + v \otimes y_0^{n\ell+c+1} w_1$$

$$= y_0^{n\ell+c+1}(v \otimes w_1) + 0 = 0 \quad \text{since } y_0^{n\ell+c+1}(v \otimes w_1) \in M_1,$$

and finally for the partition $\xi' = (\ell^{n-1}(\ell-1)c)$, the associated module $V(\xi')$ has the following relations for $0 \leq r < \ell$, $\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)v_{\xi'} = 0$. We also see that

$$\mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)(v_\xi \otimes y_0 w_1 + M_1)$$

$$= \mathbf{y}(r, \nu_1(\xi') + ... + \nu_r(\xi') - r + 1)v_\xi \otimes y_0 w_1 + M_1$$

$$= 0 \otimes y_0 w_1 + M_1$$

since for all $0 \leq r < \ell$, $\nu_r(\xi') = \nu_r(\xi)$, where $\xi = (\ell^n c)$, the partition associated to the module $D(\ell, n\ell + c)$. So the additional relations hold.

So we have shown both surjective maps exist and now we show

$$\dim V((\ell+1)\ell^{n-1}c) = \dim M_1 \text{ and } \dim V(\ell^{n-1}(\ell-1)c) = \dim M_2/M_1.$$

First we note that $\dim M_1 + \dim M_2/M_1 = \dim(D(\ell, n\ell+c) \otimes W_{loc}(1)) = 2(\ell+1)^n(c+1)$. Also $\dim V((\ell+1)\ell^{n-1}c) = (\ell+2)(\ell+1)^{n-1}(c+1)$ and $\dim V(\ell^{n-1}(\ell-1)c) = (\ell+1)^{n-1}\ell(c+1)$.

So,

$$
\begin{aligned}
\dim V((\ell+1)\ell^{n-1}c) + \dim V(\ell^{n-1}(\ell-1)c) &= (\ell+2)(\ell+1)^{n-1}(c+1) + (\ell+1)^{n-1}\ell(c+1) \\
&= (\ell+1)^{n-1}(c+1)(\ell+2+\ell) \\
&= (\ell+1)^{n-1}(c+1)2(\ell+1) \\
&= 2(\ell+1)^{n}(c+1).
\end{aligned}
$$

Finally since we proved surjective maps above we already know that

$$\dim V((\ell+1)\ell^{n-1}c) \geq \dim M_1 \text{ and } \dim V(\ell^{n-1}(\ell-1)c) \geq \dim M_2/M_1.$$

So this fact together with the above calculation forces $\dim V((\ell+1)\ell^{n-1}c) = \dim M_1$ and $\dim V(\ell^{n-1}(\ell-1)c) = \dim M_2/M_1$.

Thus since we have surjective maps and equal dimension, the modules are isomorphic and so we have the character equivalence we were looking for.

Now, assume that the character equality holds for $\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell+c) \otimes W(k)), k \leq m-1$. We will use the following fact from [2], for $n \geq m$,

$$
\begin{aligned}
&\mathrm{ch}_{\mathrm{gr}}(W_{\mathrm{loc}}(n) \otimes W_{\mathrm{loc}}(m)) \\
&\qquad = \mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(n+m) + \sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (1-u)(1-u^2)\cdots(1-u^k)\, \mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(m+n-2k).
\end{aligned}
$$

Apply the fact to $\mathrm{ch}_{gr}(W_{\mathrm{loc}}(m-1) \otimes W_{\mathrm{loc}}(1))$. Multiplying by $\mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(n)$ for $n \geq m$, we have

$$
\begin{aligned}
&\mathrm{ch}_{\mathrm{gr}} D(\ell, n\ell+c)\, \mathrm{ch}_{\mathrm{gr}}(W(m-1) \otimes W(1)) \\
&\qquad\qquad = \mathrm{ch}_{\mathrm{gr}} D(\ell, n\ell+c)(\mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(m) + (1-u)[m-1]\, \mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(m-2)).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) = &\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m-1))\, \mathrm{ch}_{\mathrm{gr}}\, W_{\mathrm{loc}}(1) \\ &+ (u-1)[m-1]\, \mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m-2)).\end{aligned}$$

Using our inductive hypothesis, this gives us

$$\begin{aligned}&\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k \ell^{n-m+1} (\ell-1)^{m-1-k} c) \otimes W_{\mathrm{loc}}(1)) \\ &+ (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}\, V((\ell+1)^k \ell^{n-m+2} (\ell-1)^{m-2-k} c).\end{aligned}$$

Now using our the short exact sequences from lemmas 5.2.1 and 5.2.2 we get,

$$\begin{aligned}&\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\ &= \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1} c) + \mathrm{ch}_{\mathrm{gr}}(V(\ell^{n-m}(\ell-1)^m c) \\ &+ \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+2)(\ell+1)^{k-1} \ell^{n-m+1} (\ell-1)^{m-k-1} c) \\ &+ \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^{k-1} \ell^{n-m+2} (\ell-1)^{m-k-1} c) \\ &+ (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}\, V((\ell+1)^k \ell^{n-m+2} (\ell-1)^{m-2-k} c).\end{aligned}$$

Then using the short exact sequence from conjecture 4.1.1, *labelsescalc*1

$$\begin{aligned}
&\mathrm{ch_{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\
&\qquad = \mathrm{ch_{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \mathrm{ch_{gr}}(V(\ell^{n-m}(\ell-1)^{m}c) \\
&\qquad + \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch_{gr}}(V((\ell+1)^{k+1}\ell^{n-m}(\ell-1)^{m-k-1}c) \\
&\qquad - \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+1}\, \mathrm{ch_{gr}}(V((\ell+1)^{k-1}\ell^{n-m}(\ell-1)^{m-k-1}c) \\
&\qquad + \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch_{gr}}(V((\ell+1)^{k-1}\ell^{n-m+2}(\ell-1)^{m-k-1}c) \\
&\qquad + (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q \mathrm{ch_{gr}}\, V((\ell+1)^{k}\ell^{n-m+2}(\ell-1)^{m-2-k}c).
\end{aligned}$$

Reindexing gives us the following,

$$\begin{aligned}
&\mathrm{ch_{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\
&\qquad = \mathrm{ch_{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \mathrm{ch_{gr}}(V(\ell^{n-m}(\ell-1)^{m}c) \\
&\qquad + \sum_{k=2}^{m} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \mathrm{ch_{gr}}(V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k}c) \\
&\qquad - \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3}\, \mathrm{ch_{gr}}(V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k-2}c) \\
&\qquad + \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q \mathrm{ch_{gr}}(V((\ell+1)^{k}\ell^{n-m+2}(\ell-1)^{m-k-2}c) \\
&\qquad + (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q \mathrm{ch_{gr}}\, V((\ell+1)^{k}\ell^{n-m+2}(\ell-1)^{m-k-2}c).
\end{aligned}$$

Using the short exact sequence from conjecture 4.1.1 again we get,

$$\begin{aligned}
&\operatorname{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\
&\quad = \operatorname{ch}_{\mathrm{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \operatorname{ch}_{\mathrm{gr}}(V(\ell^{n-m}(\ell-1)^{m}c) \\
&\quad + \sum_{k=2}^{m} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \operatorname{ch}_{\mathrm{gr}}(V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k}c) \\
&\quad - \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} \operatorname{ch}_{\mathrm{gr}}(V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k-2}c) \\
&\quad + \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q \operatorname{ch}_{\mathrm{gr}}(V((\ell+1)^{k+1}\ell^{n-m}(\ell-1)^{m-k-1}c) \\
&\quad + \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{n\ell-\ell-m+k+c+2} \operatorname{ch}_{\mathrm{gr}}(V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k-2}c) \\
&\quad + (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q \operatorname{ch}_{\mathrm{gr}} V((\ell+1)^{k+1}\ell^{n-m}(\ell-1)^{m-k-1}c) \\
&\quad + (u-1)[m-1] \sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q u^{n\ell-\ell-m+k+c+2} \operatorname{ch}_{\mathrm{gr}} V((\ell+1)^{k}\ell^{n-m}(\ell-1)^{m-k-2}c).
\end{aligned}$$

Finally reindexing and rearranging the terms we see that two types of $V(\xi)$-modules occur.

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell+c)\otimes W_{\mathrm{loc}}(m)) \\
&\quad= \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \mathrm{ch}_{\mathrm{gr}}(V(\ell^{n-m}(\ell-1)^{m}c) \\
&\quad+ \sum_{k=2}^{m} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c) \\
&\quad+ \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c) \\
&\quad+ (u-1)[m-1]\sum_{k=1}^{m-1} \begin{bmatrix} m-2 \\ k-1 \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}} V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c) \\
&\quad- \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k-2}c) \\
&\quad+ \sum_{k=0}^{m-2} \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{k+n\ell-\ell-m+c+2} \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k-2}c) \\
&\quad+ (u-1)[m-1]\sum_{k=0}^{m-2} \begin{bmatrix} m-2 \\ k \end{bmatrix}_q u^{k+n\ell-\ell-m+c+2} \mathrm{ch}_{\mathrm{gr}} V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k-2}c).
\end{aligned}$$

If we compare the coefficients of the modules of the form $V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k-2}c)$ we see that

$$\begin{aligned}
&\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{k+n\ell-m-\ell+c+2} - \begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} + (u-1)[m-1]\begin{bmatrix} m-2 \\ k \end{bmatrix}_q u^{k+n\ell-m-\ell+c+2} \\
&\quad= -\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} + u^{k+n\ell-m-\ell+c+2}\left(\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q + (u^{m-1}-1)\begin{bmatrix} m-2 \\ k \end{bmatrix}_q\right) \\
&\quad= -\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} + u^{k+n\ell-m-\ell+c+2}\left(\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q + (u^{k+1}-1)\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q\right) \\
&\qquad= -\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q u^{2k+n\ell-m-\ell+c+3} + u^{k+n\ell-m-\ell+c+2}u^{k+1}\begin{bmatrix} m-1 \\ k+1 \end{bmatrix}_q = 0.
\end{aligned}$$

Thus, our formula can be reduced to

$$
\begin{aligned}
&\mathrm{ch}_{\mathrm{gr}}(D(\ell, n\ell + c) \otimes W_{\mathrm{loc}}(m)) \\
&\qquad = \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c) + \mathrm{ch}_{\mathrm{gr}}(V(\ell^{n-m}(\ell-1)^{m}c) \\
&\qquad + \sum_{k=2}^{m} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c) \\
&\qquad + \sum_{k=1}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}(V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c) \\
&\qquad + (u-1)[m-1] \sum_{k=1}^{m-1} \begin{bmatrix} m-2 \\ k-1 \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}\, V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c).
\end{aligned}
$$

Individually computing the coefficients of $\mathrm{ch}_{\mathrm{gr}}\, V(\ell^k(\ell-1)^{n-m}c)$ for $k = 0, 1, m$, we get, for $\mathrm{ch}_{\mathrm{gr}}\, V(\ell^{n-m}(\ell-1)^m c)$,

$$1 = \begin{bmatrix} m \\ 0 \end{bmatrix}_q,$$

for $\mathrm{ch}_{\mathrm{gr}}\, V((\ell+1)\ell^{n-m}(\ell-1)^{m-1}c)$

$$1 + \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q + (u-1)[m-1] \begin{bmatrix} m-2 \\ 0 \end{bmatrix}_q = \frac{1-u^m}{1-u} = \begin{bmatrix} m \\ 1 \end{bmatrix}_q$$

and for $\mathrm{ch}_{\mathrm{gr}}\, V((\ell+1)^m\ell^{n-m}c)$,

$$1 = \begin{bmatrix} m \\ m \end{bmatrix}_q,$$

Finally for $2 \le k \le m-1$, the coefficient of $\mathrm{ch}_{gr}\, V((\ell+1)^k\ell^{n-m}(\ell-1)^{m-k}c)$ is

$$
\begin{aligned}
&\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} m-1 \\ k \end{bmatrix}_q + (u-1)[m-1] \begin{bmatrix} m-2 \\ k-1 \end{bmatrix}_q \\
&\qquad = \frac{(1-u^{m-1})\cdots(1-u^{m-k+1})}{(1-u)\cdots(1-u^{k-1})} + \frac{(1-u^{m-1})\cdots(1-u^{m-k})}{(1-u)\cdots(1-u^{k})} \\
&\qquad - (1-u^{m-1}) \cdot \frac{(1-u^{m-2})\cdots(1-u^{m-k})}{(1-u)\cdots(1-u^{k-1})} \\
&\qquad = \frac{(1-u^{m-1})\cdots(1-u^{m-k+1})[(1-u^k) + (1-u^{m-k}) - (1-u^{m-k})(1-u^k)]}{(1-u)\cdots(1-u^k)} \\
&\qquad = \frac{(1-u^{m})\cdots(1-u^{m-k+1})}{(1-u)\cdots(1-u^{k})} = \begin{bmatrix} m \\ k \end{bmatrix}_q.
\end{aligned}
$$

And so we have our result.

□

5.4 Theorem

As stated at the beginning of the chapter, since we can only prove the short exact sequence in chapter 4 for certain cases, the previous sections imply the proofs of the following theorems.

5.4.1 $\mathrm{ch}_{gr}(W_{loc}(n) \otimes W_{loc}(m))$

Theorem 5.4.1. *For $m, n \in \mathbb{Z}_+$, $n \geq m$,*

$$\mathrm{ch}_{\mathrm{gr}}[W_{loc}(n) \otimes W_{loc}(m)] = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}} V(2^k 1^{n-m}).$$

Remark 5.4.2. Theorem 5.5.1 is an unpublished result of Matt O'Dell and Lisa Schneider, but it can proven by the method above.

5.4.2 $\mathrm{ch}_{gr}(D(2, 2n + c) \otimes W_{loc}(m))$

Theorem 5.4.3. *For $m, n \in \mathbb{Z}_+$, $n \geq m$ and $c \in \{0, 1\}$,*

$$\mathrm{ch}_{\mathrm{gr}}[D(2, 2n + c) \otimes W_{loc}(m)] = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}} V(3^k 2^{n-m} 1^{m-k} c).$$

We specify these two cases because it means we have the formula for any tensor product of two level 1 Demazure modules and for the tensor product of any level 2 Demazure module with any level 1 Demazure module. However, we can generalize this result using theorem 4.8.2 to give us other cases when the character formula holds.

Theorem 5.4.4. *For $m, n \in \mathbb{Z}_+$, $n \geq m$ and $c \in \{0, \ell - 1\}$,*

$$\operatorname{ch}_{\mathrm{gr}}[D(\ell, n\ell + c) \otimes W_{loc}(m)] = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \operatorname{ch}_{\mathrm{gr}} V((\ell + 1)^k \ell^{n-m} (\ell - 1)^{m-k} c).$$

Appendix A

Lemmas for $V(\xi^-(i))$ relations on the generator for $\ker \varphi^+$

Remark A.0.1. The results in this appendix are the unpublished work of Kayla Murray and may or may not appear in her future thesis.

A.1 Number of Parts Relation

Lemma A.1.1. *In the $\mathfrak{sl}_2$-module $V(r_1) \otimes V(r_2) \otimes \cdots \otimes V(r_k)$, the $r_1 + r_2 + \cdots + r_k - 2$ weight space has dimension k.*

Proof. For $1 \leq j \leq k$, let v_j be the generator of $V(r_j)$. We claim that

$$\{v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes yv_j \otimes v_{j+1} \otimes \cdots \otimes v_k \mid 1 \leq j \leq k\}$$

is a basis for the $r_1 + r_2 + \cdots + r_k - 2$ weight space.

These elements are in the $r_1 + r_2 + \cdots + r_k - 2$ weight space since

$$\begin{aligned} &h(v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes yv_j \otimes v_{j+1} \otimes \cdots \otimes v_k) \\ &\qquad = (r_1 + r_2 + \cdots + r_k - 2)v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes yv_j \otimes v_{j+1} \otimes \cdots \otimes v_k. \end{aligned}$$

Also, it is clear that these elements are linearly independent. Now, we need to show that these elements span the $r_1 + r_2 + \cdots + r_k - 2$ weight space. Suppose v is a vector in the $r_1 + r_2 + \cdots + r_k - 2$ weight space. It suffices to assume v is a simple tensor. So,

$$v = g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k$$

where $g_j \in \mathfrak{sl}_2$ for $1 \leq j \leq k$. By assumption, we have

$$hv = (r_1 + r_2 + \cdots + r_k - 2)v.$$

On the other hand,

$$\begin{aligned} hv &= h(g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k) \\ &= \sum_{j=1}^{k} g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_{j-1} v_{j-1} \otimes h(g_j v_j) \otimes g_{j+1} v_{j+1} \otimes \cdots \otimes g_k v_k. \end{aligned}$$

For all $1 \leq j \leq k$, we know that $h(g_j v_j) = r_j - 2k_j$ for some $k_j \in \mathbb{Z}_{\geq 0}$. Since $hv = r_1 + \cdots + r_k - 2$, there exists n such that

$$hg_n v_n = r_n - 2$$

$$hg_j v_j = r_j$$

for $j \neq n$. But, then in $V(r_n)$, the $r_n - 2$ weight space is one dimensional. Hence, $g_n v_n$ is a scalar multiple of yv_n. Also, for $j \neq n$, $g_j v_j$ is a highest weight vector in $V(r_j)$ and hence a scalar multiple of v_j. Thus, v is a scalar multiple of

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \otimes yv_n \otimes v_{n+1} \otimes \cdots \otimes v_k.$$

This proves that our set spans the $r_1 + r_2 + \cdots + r_k - 2$ weight space. Therefore, we have a basis for the $r_1 + r_2 + \cdots + r_k - 2$ weight space and the dimension of this weight space is k. □

Proposition A.1.2. *We have*

$$y(1, \nu_1 - 2)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Proof. As $\mathfrak{sl}_2$-modules, we know that the short exact sequence

$$0 \to V(\xi(i)^-) \to V(\xi(i)) \to V(\xi(i)^+) \to 0$$

where

$$\xi^-(i) = \ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_1} \cdots i^{\nu_i - \nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i} \cdots 1^{\nu_1-\nu_2}$$

$$\xi^+(i) = \ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_1} \cdots (i+1)^{\nu_{i+1}-\nu_{i+2}+1} i^{\nu_i - \nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i+1} \cdots 1^{\nu_1-\nu_2}$$

exists. As a $\mathfrak{sl}_2$-module,

$$V(\ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_\ell} \cdots (i+1)^{\nu_{i+1}-\nu_{i+2}} i^{\nu_i-\nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i} \cdots 1^{\nu_1-\nu_2})$$

$$= V(\ell)^{\otimes \nu_\ell} \otimes V(\ell-1)^{\otimes \nu_{\ell-1}-\nu_\ell} \otimes \cdots \otimes V(i+1)^{\otimes \nu_{i+1}-\nu_{i+2}} \otimes V(i)^{\otimes \nu_i - \nu_{i+1}-2}$$

$$\otimes V(i-1)^{\otimes \nu_{i-1}-\nu_i} \otimes \cdots \otimes V(1)^{\otimes \nu_1-\nu_2}$$

In this module, the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$ by the above lemma. Hence, in $V(\xi)$, the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$.

Now, consider the element

$$y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi,$$

which has weight $|\xi| - 2i$. Then, the element

$$y(1, v_1 - 2)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi$$

has weight $|\xi| - 2i - 2$.

Since $V(\xi)$ is finite dimensional, there exists m such that

$$y(1, m)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Let N be minimal with respect to this property. Now, consider the vectors

$$\{y(1, j)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi : 0 \leq j \leq N - 1\}$$

These vectors have different grades and hence are linearly independent. But, since the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$, $N \leq \nu_1 - 2$. Hence,

$$y(1, v_1 - 2)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

□

A.2 Lemmas for Fusion Product Calculation

Proposition A.2.1. *We have*

$$y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = {}_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi.$$

Proof. First, we write

$$\begin{aligned} y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) = {}& {}_1y(1, \nu_1 + \nu_2 + \cdots + \nu_i - i) \\ & + \sum_{j=1}^{i-1} y_0^{(j)}\ {}_1y(i - j, \nu_1 + \nu_2 + \cdots + \nu_i - i). \end{aligned}$$

Now, we claim

$$_1y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0$$

for all $1 \le j \le i-1$. We proceed by induction on j. For $j = 1$, we have

$$_1y(1, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = y_{\nu_1+\nu_2+\cdots+\nu_i-i}v_\xi.$$

Since $i \ge 2$ and $\nu_k \ge \nu_i \ge 2$ for all $2 \le k \le i$, we have

$$\begin{aligned}\nu_1 + \nu_2 + \cdots + \nu_i - i &\ge \nu_1 + 2(i-1) - i \\ &\ge \nu_1 + i - 2 \\ &\ge \nu_1.\end{aligned}$$

Hence,

$$_1y(1, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

For the inductive step, assume $2 \le j \le i-1$ and $_1y(k, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0$ for all $1 \le k < j$. First, we note

$$\nu_1 + \nu_2 + \cdots + \nu_i - i \ge \nu_1 + \nu_2 + \cdots + \nu_j - j + 1$$

since $i \ge j+1$ and $\nu_n \ge 2$ for all $1 \le n \le i$. Hence,

$$y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Now, we write

$$\begin{aligned}y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i) &= {}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i) \\ &\quad + \sum_{k=1}^{j-1} y_0^{(k)} \ {}_1y(j-k, \nu_1 + \nu_2 + \cdots + \nu_i - i).\end{aligned}$$

Hence, by the inductive hypothesis,

$$y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = \ {}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0,$$

which completes the claim. By the claim, we now have

$$y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = \ {}_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi.$$

□

Lemma A.2.2. *In* $\mathbf{U}(\mathfrak{sl}_2[t])$, *we have*

$${}_1y(r, s) = \frac{1}{r} \sum_{j=1}^{s-r+1} y_j \ {}_1y(r-1, s-j), \quad r \geq 2, \quad s \geq r.$$

Proof. First, we recall that for $r \geq 2$ and $s' \geq 0$, we have

$$y(r, s') = \frac{1}{r} \sum_{j=0}^{s'} y_j y(r-1, s'-j)$$

from a previously proved lemma. Now, we consider the algebra homomorphism

$$\gamma : \mathbb{C}[y_0, y_1, y_2, \ldots] \to \mathbb{C}[y_1, y_2, y_3, \ldots]$$

given by $\gamma(y_j) = y_{j+1}$ for all $j \geq 0$. Then,

$$\gamma(y(r, s')) = \ {}_1y(r, s' + r).$$

On the other hand, we have

$$
\begin{aligned}
{}_1y(r, s'+r) &= \gamma(y(r,s')) \\
&= \gamma\left(\frac{1}{r}\sum_{j=0}^{s'} y_j y(r-1, s'-j)\right) \\
&= \frac{1}{r}\sum_{j=0}^{s'} \gamma(y_j y(r-1, s'-j)) \\
&= \frac{1}{r}\sum_{j=0}^{s'} y_{j+1}\ {}_1y(r-1, s'-j+r-1) \\
&= \frac{1}{r}\sum_{j=1}^{s'+1} y_j\ {}_1y(r-1, s'-j+r)
\end{aligned}
$$

Now, we let $s = s'+r$. Then, $s \geq r$ and

$$
{}_1y(r,s) = \frac{1}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1, s-j),
$$

which completes the proof. □

Lemma A.2.3. *In* $\mathbf{U}(\mathfrak{sl}_2[t])$, *we have*

$$
[h_1,\ {}_1y(r,s)] = -2r\ {}_1y(r, s+1) + 2y_1\ {}_1y(r-1, s), \quad r > 0, \quad s \geq r.
$$

Proof. We proceed by induction on r. Assume $r = 1$ and $s \geq 1$. Then, we have

$$
\begin{aligned}
[h_1,\ {}_1y(1,s)] &= [h_1, y_s] \\
&= -2y_{s+1} \\
&= -2\ {}_1y(1, s+1).
\end{aligned}
$$

Now, assume $r = 2$ and $s \geq 2$. Then, we have

$$\begin{aligned}
[h_1,\ {}_1y(2,s)] &= \left[h_1, \frac{1}{2}\sum_{j=1}^{s-1} y_j\ {}_1y(1, s-j)\right] \\
&= \frac{1}{2}\sum_{j=1}^{s-1}[h_1, y_j y_{s-j}] \\
&= \frac{1}{2}\sum_{j=1}^{s-1}[h_1, y_j]y_{s-j} + y_j[h_1, y_{s-j}] \\
&= \frac{1}{2}\sum_{j=1}^{s-1} -2y_{j+1}y_{s-j} + \frac{1}{2}\sum_{j=1}^{s-1} -2y_j y_{s+1-j} \\
&= \frac{1}{2}\sum_{j=2}^{s} -2y_j y_{s+1-j} + \frac{1}{2}\sum_{j=1}^{s-1} -2y_j y_{s+1-j} \\
&= \frac{1}{2}\sum_{j=1}^{s}(-2y_j y_{s+1-j}) + y_1 y_s + \frac{1}{2}\sum_{j=1}^{s}(-2y_j y_{s+1-j}) + y_s y_1 \\
&= -2\ {}_1y(2, s+1) + -2\ {}_1y(2, s+1) + 2y_1 y_s \\
&= -4\ {}_1y(2, s+1) + 2y_1\ {}_1y(1, s).
\end{aligned}$$

Now assume $r \geq 3, s \geq r$. Also, assume that for all $s' \geq r-1$, we have

$$[h_1,\ {}_1y(r-1, s')] = -2(r-1)y(r-1, s'+1) + 2y_1\ {}_1y(r-2, s').$$

Then,

$$
\begin{aligned}
[h_1,\ {}_1y(r,s)] &= \left[h_1, \frac{1}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1,s-j)\right] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1} [h_1, y_j\ {}_1y(r-1,s-j)] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1} y_j[h_1,\ {}_1y(r-1,s-j)] + \frac{1}{r}\sum_{j=1}^{s-r+1} [h_1,y_j]\ {}_1y(r-1,s-j)] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1} y_j(-2(r-1)\ {}_1y(r-1,s+1-j) + 2y_1\ {}_1y(r-2,s-j)) \\
&\quad + \frac{1}{r}\sum_{j=1}^{s-r+1} -2y_{j+1}\ {}_1y(r-1,s-j) \\
&= \frac{-2(r-1)}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1,s+1-j) + \frac{2}{r}y_1\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-2,s-j) \\
&\quad + \frac{-2}{r}\sum_{j=1}^{s-r+1} y_{j+1}\ {}_1y(r-1,s-j).
\end{aligned}
$$

The first term is equal to

$$
\frac{2(r-1)}{r}y_{s-r+2}\ {}_1y(r-1,r-1) + \frac{-2(r-1)}{r}\sum_{j=1}^{s+1-r+1} y_j\ {}_1y(r-1,s+1-j).
$$

The second term is equal to

$$
\frac{-2}{r}y_{s-r+2}y_1\ {}_1y(r-2,r-2) + \frac{2}{r}y_1\sum_{j=1}^{s-r+2} y_j\ {}_1y(r-2,s-j).
$$

The third term is equal to

$$
\frac{2}{r}y_1\ {}_1y(r-1,s) + \frac{-2}{r}\sum_{j=1}^{s+1-r+1} y_j\ {}_1y(r-1,s+1-j).
$$

Now, if we add all these terms up, we get

$$\begin{aligned}[h_1,\ {}_1y(r,s)] &= \frac{2(r-1)}{r} y_{s-r+2} y_1^{(r-1)} - 2(r-1)\ {}_1y(r,s+1) \\ &+ \frac{-2}{r} y_{s-r+2} y_1^{(r-2)} y_1 + \frac{2(r-1)}{r} y_1\ {}_1y(r-1,s) \\ &+ \frac{2}{r} y_1\ {}_1y(r-1,s) - 2\ {}_1y(r,s+1) \\ &= -2r\ {}_1y(r,s+1) + 2y_1\ {}_1y(r-1,s).\end{aligned}$$

□

Lemma A.2.4. *For* $1 \le r \le i$, *if* $\nu_r \ge 3$ *and*

$${}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0,$$

for all $1 \le j \le r-1$, *then*

$$\begin{aligned} & y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\ = \ & {}_1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi. \end{aligned}$$

Proof. We proceed by induction on r.

Assume $r = 1$ and $\nu_1 \ge 3$. Then,

$$\begin{aligned} y(1, \nu_1 - 2) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi &= y_{\nu_1 - 2} y(i, \nu_1 + \nu + 2 + \cdots + \nu_i - i) v_\xi \\ &= {}_1y(1, \nu_1 - 2) y(i, \nu_1 + \nu + 2 + \cdots + \nu_i - i) v_x i. \end{aligned}$$

This completes the base case. Let $2 \le r \le i$. Assume $\nu_r \ge 3$ and

$${}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0$$

for all $1 \le j \le r-1$.

Now, we claim that

$$ {}_1y(m,s')y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi=0 $$

for all $1\le m\le r-1$ and $s'>\nu_1+\nu_2+\cdots+\nu_m-3m+1$.

To prove this claim, we begin by induction on m. Assume $m=1$ and $s'>\nu_1-2$. Then, $s'=\nu_1-2+n$, where $n\in\mathbb{Z}_{>0}$. Now, we continue by induction on n. Assume $n=1$. By assumption, we have

$$ {}_1y(1,\nu_1-2)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi=0. $$

Hence, by previous lemmas,

$$ \begin{aligned} 0=&h_1\ {}_1y(1,\nu_1-2)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi\\ =&[h_1,\ {}_1y(1,\nu_1-2)]y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi\\ &+\ {}_1y(1,\nu_1-2)h_1y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi\\ =&-2\ {}_1y(1,\nu_1-2+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi\\ &+\ {}_1y(1,\nu_1-2)[h_1,y(i,\nu_1+\nu_2+\cdots+\nu_i-i)]v_\xi\\ =&-2\ {}_1y(1,\nu_1-2+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi\\ &-2i\ {}_1y(1,\nu_1-2)y(i,\nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi\\ &+2y_0\ {}_1y(1,\nu_1-2)y(i-1,\nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi\\ =&-2\ {}_1y(1,\nu_1-2+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \end{aligned} $$

since $\nu_i\ge1$. Therefore,

$$ {}_1y(1,\nu_1-2+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi=0 $$

and this completes the base case $n = 1$.

Now, assume for $n \geq 1$, we have

$$_1y(1, \nu_1 - 2 + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Hence, by the inductive hypothesis and previous lemmas,

$$\begin{aligned}
0 =& h_1 \ _1y(1, \nu_1 - 2 + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
=& [h_1, \ _1y(1, \nu_1 - 2 + n)]y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+ \ _1y(1, \nu_1 - 2 + n)h_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
=& -2 \ _1y(1, \nu_1 - 2 + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+ \ _1y(1, \nu_1 - 2 + n)[h_1, y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)]v_\xi \\
=& -2 \ _1y(1, \nu_1 - 2 + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&- 2i \ _1y(1, \nu_1 - 2 + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\
&+ 2y_0 \ _1y(1, \nu_1 - 2 + n)y(i - 1, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\
=& -2 \ _1y(1, \nu_1 - 2 + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi
\end{aligned}$$

since $\nu_i \geq 1$. Therefore,

$$_1y(1, \nu_1 - 2 + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0,$$

which completes the induction on n and the base case $m = 1$.

Assume $2 \leq m \leq r - 1$. For the inductive hypothesis, assume

$$_1y(m - 1, \tilde{s})y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0$$

for all $\tilde{s} > \nu_1 + \nu_2 + \cdots + \nu_{m-1} - 3(m - 1) + 1$. Assume $s' > \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1$. Then, $s = \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n$ where $n \in \mathbb{Z}_{>0}$. We proceed by induction on n.

Since $m \leq r-1$, we have

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0. $$

Hence, by previous lemmas and the inductive hypothesis,

$$
\begin{aligned}
0 =& h_1 \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
=& [h_1, \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)]y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
&+ \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)h_1y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
=& -2m \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
&+ 2y_1 \, {}_1y(m-1, \nu_1+\cdots+\nu_m-3m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
&+ \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)[h_1, y(i, \nu_1+\nu_2+\cdots+\nu_i-i)]v_\xi \\
=& -2m \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\
&+ -2i \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi \\
&+ 2y_0 \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1)y(i-1, \nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi \\
=& -2m \, {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi
\end{aligned}
$$

since $\nu_m \geq \nu_r \geq 3$ and $\nu_i \geq 2$. Therefore,

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0 $$

and this completes the base case $n = 1$. Now, assume that for $n \geq 1$, we have

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-3m+1+n)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0. $$

Hence, by the inductive hypothesis and previous lemmas, we have

$$\begin{aligned}
0 =& h_1 \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
=& [h_1, \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n)] y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n) h_1 y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
=& -2m \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + 2y_1 \,{}_1y(m-1, \nu_1 + \cdots + \nu_m - 3m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n) [h_1, y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)] v_\xi \\
=& -2m \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + -2i \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1) v_\xi \\
& + 2y_0 \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n) y(i-1, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1) v_\xi \\
=& -2m \,{}_1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 3m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi
\end{aligned}$$

since $\nu_m \geq 3$ and $\nu_i \geq 2$. This completes the induction on n and the claim.

Now, we write

$$\begin{aligned}
& y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
&\quad = \sum_{j=1}^{r-1} y_0^{(j)} \,{}_1y(r-j, \nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
&\quad + \,{}_1y(r, \nu_1 + \cdots + \nu_r - 3r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi.
\end{aligned}$$

Also, for $1 \leq r - j \leq r - 1$, we have

$$\begin{aligned}
\nu_1 + \nu_2 + \cdots + \nu_r - 3r + 1 &\geq \nu_1 + \nu_2 + \cdots + \nu_{r-j} + 3(r - (r-j)) - 3r + 1 \\
&= \nu_1 + \nu_2 + \cdots + \nu_{r-j} - 3(r-j) + 1
\end{aligned}$$

since $\nu_r \geq 3$. Therefore, by the claim

$$_1y(r-j, \nu_1+\nu_2+\cdots+\nu_r-3r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0.$$

Hence, we have established

$$\begin{aligned} & y(r, \nu_1+\nu_2+\cdots+\nu_r-3r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ & \qquad = {}_1y(r, \nu_1+\nu_2+\cdots+\nu_r-3r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi. \end{aligned}$$

This completes the induction on r. □

Lemma A.2.5. *For $i+1 \leq r \leq \ell-1$, if $\nu_r \geq 1$,*

$$_1y(j, \nu_1+\nu_2+\cdots+\nu_j-3j+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0$$

for all $1 \leq j \leq i$, and

$$_1y(\tilde{j}, \nu_1+\nu_2+\cdots+\nu_{\tilde{j}}-2i-\tilde{j}+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0$$

for all $i+1 \leq \tilde{j} \leq r-1$, then

$$\begin{aligned} & y(r, \nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ & = {}_1y(r, \nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi. \end{aligned}$$

Proof. We proceed by induction on r.

Let $r = i+1$. Assume $\nu_{i+1} \geq 1, \nu_i - \nu_{i+1} \geq 2$, and

$$_1y(j, \nu_1+\nu_2+\cdots+\nu_j-3j+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0$$

for all $1 \leq j \leq i$. Then $\nu_i \geq 3$ and $\nu_j \geq 3$ for all $1 \leq j \leq i$. By the claim from that was proved in the previous lemma,

$$_1y(j, s)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0$$

for all $1 \leq j \leq i$ and $s \geq \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1$. Now, we write

$$
\begin{aligned}
y(i+1, &\nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
&= \sum_{k=1}^{i} y_0^{(k)} \ {}_1y(i+1-k, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
&+ \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi.
\end{aligned}
$$

Now, for all $1 \leq j \leq i$, we have

$$
\begin{aligned}
\nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i &\geq \nu_1 + \cdots + \nu_j + \nu_{j+1} + \cdots + \nu_{i+1} - 3i \\
&\geq \nu_1 + \nu_2 + \cdots + \nu_j + 3(i-j) + 1 - 3i \\
&= \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1
\end{aligned}
$$

since $\nu_i \geq 3$ and $\nu_{i+1} \geq 1$. Thus, by the previous lemma,

$$
{}_1y(i+1-k, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0
$$

for all $1 \leq k \leq i$. (Note: The claim in the previous lemma is only for $1 \leq k < i$, but a symmetric proof holds for $k = i$.) Hence,

$$
\begin{aligned}
y(i+1, &\nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
&= \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi.
\end{aligned}
$$

This completes the base case.

Let $i + 2 \leq r \leq \ell - 1$. Assume $\nu_r \geq 1$, $\nu_i - \nu_{i+1} \geq 2$,

$$
{}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_j - 3j + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0
$$

for all $1 \leq j \leq i$, and

$$
{}_1y(\tilde{j}, \nu_1 + \nu_2 + \cdots + \nu_{\tilde{j}} - 2i - \tilde{j} + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi = 0
$$

for all $i+1 \leq \tilde{j} \leq r-1$.

Now, write

$$\begin{aligned} y(r,&\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &=\sum_{k=1}^{r-1} y_0^{(k)}\ {}_1y(r-k,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+\ {}_1y(r,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi. \end{aligned}$$

Since $r \geq i+2, \nu_r \geq 1$, and $\nu_i - \nu_{i+1} \geq 2$, $\nu_i \geq 3$. Hence, $\nu_j \geq 3$ for all $1 \leq j \leq i$. Also, for all $1 \leq j \leq i$, we have

$$\begin{aligned} \nu_1+\nu_2+\cdots+\nu_r-2i-r+1 &= \nu_1+\nu_2+\cdots+\nu_j+\nu_{j+1}+\cdots+\nu_r-2i-r+1 \\ &\geq \nu_1+\nu_2+\cdots+\nu_j+3(i-j)+1(r-i)-2i-r+1 \\ &= \nu_1+\nu_2+\cdots+\nu_j-3j+1. \end{aligned}$$

Hence, for all $1 \leq j \leq i$,

$${}_1y(j,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_r-i)v_\xi = 0$$

by the claim from previous lemma. (Note: The claim in the previous lemma is only for $1 \leq j < i$, but a symmetric proof holds for $j = i$.)

Hence,

$$\begin{aligned} y(r,&\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &=\sum_{k=1}^{r-i-1} y_0^{(k)}\ {}_1y(r-k,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+\ {}_1y(r,\nu_1+\nu_2+\cdots+\nu_r-2i-r+1)y(i,\nu_1+\nu_2+\cdots+\nu_i-i)v_\xi. \end{aligned}$$

Now, we claim that

$$ {}_1y(m, s')y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0 $$

for all $i+1 \le m \le r-1$ and $s' > \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1$. To prove this claim, we begin by induction on m. Assume $m = i+1$ and $s' > \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i$. Then, $s' = \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n$, where $n \in \mathbb{Z}_{>0}$. Now, we continue by induction on n. Assume $n = 1$. By assumption, we have

$$ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0. $$

Hence, by previous lemmas,

$$ \begin{aligned} 0 =& h_1 \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ =& [h_1, \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)]y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ &+ \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)h_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ =& -2(i+1) \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ &+ 2y_1 \ {}_1y(i, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ &+ \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)[h_1, y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)]v_\xi \\ =& -2(i+1) \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\ &+ -2i \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\ &+ 2y_0 \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i)y(i-1, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\ =& -2(i+1) \ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \end{aligned} $$

since $\nu_i \ge 1$ and $\nu_{i+1} \ge 1$. Therefore,

$$ {}_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0 $$

and this completes the base case $n = 1$. Now, assume for $n \geq 1$, we have

$$_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0.$$

Hence, by the inductive hypothesis and previous lemmas,

$$\begin{aligned}
0 =& h_1\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
=& [h_1,\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)]y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)h_1y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
=& -2(i+1)\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+ 2y_1\ {_1y}(i, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)[h_1, y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)]v_\xi \\
=& -2(i+1)\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi \\
&+ -2i\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\
&+ 2y_0\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n)y(i-1, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1)v_\xi \\
=& -2(i+1)\ {_1y}(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi
\end{aligned}$$

since $\nu_i \geq 1$ and $\nu_{i+1} \geq 1$. Therefore,

$$_1y(i+1, \nu_1 + \nu_2 + \cdots + \nu_{i+1} - 3i + n + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0,$$

which completes the induction on n and the base case $m = 1$. Assume $i + 2 \leq m \leq r - 1$. For the inductive hypothesis, assume

$$_1y(m-1, \tilde{s})y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0$$

for all $\tilde{s} > \nu_1 + \nu_2 + \cdots + \nu_{m-1} - 2i - (m-1) + 1$. Assume $s' > \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1$. Then, $s = \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n$ where $n \in \mathbb{Z}_{>0}$. We proceed by induction on

n. Since $i+1 \leq m \leq r-1$, we have

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0. $$

Hence, by previous lemmas and the inductive hypothesis,

$$ \begin{aligned} 0 =& h_1\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ =& [h_1,\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)]y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)h_1y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ =& -2m\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+2y_1\ {}_1y(m-1, \nu_1+\cdots+\nu_m-2i-m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)[h_1, y(i, \nu_1+\nu_2+\cdots+\nu_i-i)]v_\xi \\ =& -2m\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \\ &+-2i\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi \\ &+2y_0\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1)y(i-1, \nu_1+\nu_2+\cdots+\nu_i-i+1)v_\xi \\ =& -2m\ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi \end{aligned} $$

since $\nu_m \geq \nu_r \geq 1$ and $\nu_i \geq 2$. Therefore,

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1+1)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0 $$

and this completes the base case $n=1$. Now, assume that for $n \geq 1$, we have

$$ {}_1y(m, \nu_1+\nu_2+\cdots+\nu_m-2i-m+1+n)y(i, \nu_1+\nu_2+\cdots+\nu_i-i)v_\xi = 0. $$

Hence, by the inductive hypothesis and previous lemmas, we have

$$
\begin{aligned}
0 =& h_1 \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
=& [h_1, \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n)] y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n) h_1 y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
=& - 2m \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + 2y_1 \ _1y(m - 1, \nu_1 + \cdots + \nu_m - 2i - m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n) [h_1, y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)] v_\xi \\
=& - 2m \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& + -2i \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1) v_\xi \\
& + 2y_0 \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n) y(i - 1, \nu_1 + \nu_2 + \cdots + \nu_i - i + 1) v_\xi \\
=& - 2m \ _1y(m, \nu_1 + \nu_2 + \cdots + \nu_m - 2i - m + 1 + n + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi
\end{aligned}
$$

since $\nu_m \geq 1$ and $\nu_i \geq 2$. This completes the induction on n and the claim.

Now, recall

$$
\begin{aligned}
y(r, & \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& = \sum_{k=1}^{r-i-1} y_0^{(k)} \ _1y(r - k, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi \\
& \quad + \ _1y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1) y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i) v_\xi .
\end{aligned}
$$

Also, for $i + 1 \leq j \leq r - 1$, we have

$$
\begin{aligned}
\nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1 &\geq \nu_1 + \nu_2 + \cdots + \nu_j + (r - j) - 2i - r + 1 \\
&= \nu_1 + \nu_2 + \cdots + \nu_j - 2i - j + 1
\end{aligned}
$$

since $\nu_r \geq 1$. Therefore, by the claim

$$ {}_1y(j, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi = 0 $$

for all $i + 1 \leq j \leq r - 1$.

Hence, we have established

$$ y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi $$
$$ =_1 y(r, \nu_1 + \nu_2 + \cdots + \nu_r - 2i - r + 1)y(i, \nu_1 + \nu_2 + \cdots + \nu_i - i)v_\xi. $$

This completes the induction on r. □

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