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On the representation and boundary behavior of certain classes of holomorphic functions in several variables

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On the representation and boundary behavior of certain classes of holomorphic functions in several variables

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by

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The dissertation of Ryan Keddie Tully-Doyle is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

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DEDICATION

To Lauren.

EPIGRAPH

He who would learn to fly must first learn

to stand and walk and run and climb and dance;

one cannot fly into flying.

—F. Nietzsche

TABLE OF CONTENTS

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R. Tully-Doyle, D. Cushing, J. E. Pascoe, "Free functions with symmetry", *in submission*, 2015

R. Tully-Doyle, J. W. Helton, J. E. Pascoe, V. Vinnikov "Convex entire noncommutative functions are polynomials of degree two or less", to appear in *Integral Equations Operator Theory*. 2015

ABSTRACT OF THE DISSERTATION

On the representation and boundary behavior of certain classes of holomorphic functions in several variables

by

Ryan Keddie Tully-Doyle

Doctor of Philosophy in Mathematics

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Professor Jim Agler, Chair

This dissertation concerns the investigation of function theoretic properties of certain classes of holomorphic functions in two or more variables by means of operator theoretic methods. Of primary concern will be the *Schur class*, the class of holomorphic functions from the complex polydisk into the complex unit disk, and the *Pick class*, the class of holomorphic functions from the complex poly-upperhalfplane into the complex upperhalfplane.

In more than two variables, our results will concern certain large subclasses of these functions that satisfy an operator-theoretic condition analogous to a classical

inequality of functions of one variable due to von Neumann [vN51]. These subclasses are typically referred to as the Schur-Agler subclass of the Schur functions (introduced in [Agl90]), and the Löwner subclass of the Pick functions (introduced in [AMY12b]). (In one or two variables, these subclasses coincide with the whole class.) These functions are amenable to investigation by means of an operator-theoretic construct called a Hilbert space model, introduced in [Agl90], which relates operator theoretic properties with function theoretic behavior. Hilbert space models are associated with and closely related to the notion of a transfer function realization from engineering and control theory [Hel87].

In Chapter 2, we describe a generalization of Hilbert space models for Schur functions on the bidisk that is well-suited to the investigation of boundary behavior of a function at a class of singular points for the function on the 2-torus. We prove that generalized models with certain regularity properties exist at these singularities. We then solve two function theoretic problems. First, we characterize the directional derivatives of a function in the Schur class at a singular point on the torus where a Carathéodory condition holds (following the generalization of the Julia-Carathédory theorem in [AMY12a]). Second, we develop a representation theorem for functions in the two-variable Pick class analogous to the Nevanlinna representation theorem characterizing the Cauchy transforms of positive measures on the real line.

In Chapter 3, we investigate more closely the structure of the generalized Hilbert space model. We characterize the directional derivatives in terms of a rational function depending on the structure of a positive contraction associated with a generalized model of a given Schur function. We describe classes of generalized models corresponding to different classes of singular points in the boundary for a Schur function in two variables.

In Chapter 4, we generalize to several variables the Nevanlinna representation first investigated in Chapter 2. We show that for the Löwner class, there are representation formulae in terms of densely-defined self-adjoint operators on a Hilbert space that classify completely the Löwner class. We identify four types of such representations, and we obtain function-theoretic conditions that are necessary and sufficient for a given function to possess a representation of each of the four types.

Chapter 1

Introduction

1.1 Background

In the early 20th century, mathematicians such as K. Kraus, R. Nevanlinna, C. Carathéodory, G. Pick, and G. Herglotz for example, investigated classes of holomorphic functions by means of a set of tools that exploited the connection between function theory and integral representations (see e.g. [Kra36, Nev22, Car29, Pic16, Her11]). The ability to cast function theoretic questions in the language of measure theory provided deep insights into the nature of a number of important classes of functions in one variable, as well as opening the door to investigating functions on matrices. The modern history of this approach, in terms of operators, goes back to von Neumann, Korányi, Sz.-Nagy, and Sarason (e.g. [vN51, Sar67, Kor61, Sar67]). Operator theoretic methods led to extensions of classical results to several variables, as, for example, in the work of Taylor, Putinar, Agler, Douglas, Paulsen, and Curto (e.g. [Tay70b, Tay70a, Put83, Agl90]).

In a preprint from the late 1980s and a paper of 1990, J. Agler introduced a methodology of operator theoretic representation of holomorphic functions that proved well-suited to the generalization of classical one variable theorems from complex analysis

[Agl, Agl90]. Exploiting analogues of classical integral representations, of which there are myriad examples for classical families of holomorphic functions, Agler's method involves the representation of functions in terms of operators on Hilbert spaces, essentially using the transfer function realization of control theory to build operator theoretic models that contain geometric encodings of function theoretic data.

This dissertation will use Hilbert space methods to refine an operator theoretic model for functions that take the several variable analogue of the unit disk into itself. With these tools, we will investigate function theoretic properties, in particular differential structure at the distinguished boundary. We will also generalize a classical one variable integral representation due to Nevanlinna to several variables and use it to establish function theoretically determined classes of functions that take the several variable analogue of the upper half-plane into itself.

1.2 The classical Julia-Carathéodory theorem

The *Schur class* is the set of analytic functions φ that map the complex unit disk $\mathbb{D}$ into itself. Say that a Schur function φ satisfies the *Carathéodory condition* at a point τ in the unit circle $\mathbb{T}$ if

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - |\lambda|} < \infty. \tag{1.2.1}$$

A classical theorem of Julia and Carathéodory [Jul20, Car29] relates this regularity condition at a boundary point τ in the circle to function theoretic properties of φ at τ.

For a set $S \subset \mathbb{D}$ and $\tau \in D^-$, say that S approaches τ nontangentially if $\tau \in S^-$ and there exists a constant $c > 0$ so that for all $\lambda \in S$,

$$|\tau - \lambda| \leq c(1 - |\lambda|).$$

A sequence $\{\lambda_n\}$ tends to τ nontangentially if the set $\{\lambda_n : n \geq 1\}$ approaches τ nontangentially.

Theorem 1.2.1 (Julia, Carathéodory)**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic and nonconstant. Let $\tau \in \mathbb{T}$. The following are equivalent.*

1. *There exists a sequence $\{\lambda_n\} \subset \mathbb{D}$ tending to τ such that*

$$\frac{1 - |\varphi(\lambda_n)|}{1 - |\lambda_n|} \tag{1.2.2}$$

is bounded.

2. *for every sequence $\{\lambda_n\}$ tending to τ nontangentially, the quotient* (1.2.2) *is bounded.*

3. *the nontangential limit*

$$\omega := \lim_{\lambda \overset{\text{nt}}{\to} \tau} \varphi(\lambda)$$

and the angular derivative

$$\varphi'(\tau) := \lim_{\lambda \overset{\text{nt}}{\to} \tau} \frac{\varphi(\lambda) - \omega}{\lambda - \tau}$$

exist.

4. *There exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}$ such that $\varphi(\lambda) \to \omega$ and $\varphi'(\lambda) \to \eta$ as $\lambda \overset{\text{nt}}{\to} \tau$.*

1.3 Carapoints

C. Carathéodory in [Car29] proved that if a function φ in the one-variable Schur class satisfies

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - |\lambda|} < \infty \tag{1.3.1}$$

for some $\tau \in \mathbb{T}$ then not only does φ have a nontangential limit at τ, but it also has an angular derivative $\varphi'(\tau)$ at τ, and $\varphi'(\lambda) \to \varphi'(\tau)$ as λ tends nontangentially to τ in $\mathbb{D}$. Here nontangential limits are defined as follows. For any domain U and for τ in the topological boundary ∂U of U we say that a set $S \subset U$ *approaches* τ *nontangentially* if $\tau \in S^-$, the closure of S, and

$$\left\{ \frac{\|\lambda - \tau\|}{\operatorname{dist}(\lambda, \partial U)} : \lambda \in S \right\} \text{ is bounded.}$$

We say that a function φ on U *has nontangential limit* ℓ *at* τ, in symbols

$$\lim_{\lambda \xrightarrow{\text{nt}} \tau} \varphi(\lambda) = \ell,$$

if

$$\lim_{\substack{\lambda \to \tau \\ \lambda \in S}} \varphi(\lambda) = \ell$$

for every set $S \subset U$ that approaches τ nontangentially.

Carathéodory's result has been generalized by several authors, notably by K. Włodarczyk [Wlo87], W. Rudin [Rud80], F. Jafari [Jaf93], M. Abate [Aba98] and two of us with J. E. McCarthy [AMY12a]. Carathéodory's condition (1.3.1) generalizes naturally to holomorphic maps $\varphi : U \to V$ for any pair of bounded domains U, V in complex Euclidean spaces of finite dimensions.

Definition 1.3.1. *For any* τ *in* ∂U *we say that* φ satisfies the Carathéodory condition at τ, *or that* τ *is a* carapoint *for* φ*, if*

$$\liminf_{\substack{\lambda \to \tau \\ \lambda \in U}} \frac{\operatorname{dist}(\varphi(\lambda), \partial V)}{\operatorname{dist}(\lambda, \partial U)} < \infty. \tag{1.3.2}$$

In particular, when $U=\mathbb{D}^d$, $V=\mathbb{D}$ *and* $\varphi\in\mathcal{S}_d$, τ *is a carapoint for* φ *if*

$$\liminf_{\lambda\to\tau}\frac{1-|\varphi(\lambda)|}{1-\|\lambda\|_\infty}<\infty.$$

Likewise, if φ *is a contractive operator-valued analytic function on* $\mathbb{D}^d$, $\tau\in\mathbb{T}^d$ *is a carapoint for* φ *if*

$$\liminf_{\lambda\to\tau}\frac{1-\|\varphi(\lambda)\|}{1-\|\lambda\|_\infty}<\infty.$$

Of course any point in $\mathbb{T}^d$ at which φ is analytic is a carapoint for φ, but we are concerned here with *singular* carapoints. We say that an analytic function φ on a domain U is *singular* at a point $\tau\in\partial U$ if there is no neighborhood W of τ such that φ extends to an analytic function on $U\cup W$.

Not all the conclusions of Carathéodory's Theorem hold even for $\mathcal{S}_2$. However it is true for all the cases considered in this dissertation that if $\tau\in\partial U$ is a carapoint for $\varphi:U\to V$ then φ has a nontangential limit at τ [Wlo87]. This limit will be denoted by $\varphi(\tau)$; it is obvious that $\varphi(\tau)\in\partial V$.

1.4 Models and realizations

A primary tool that can be used to study the boundary behavior of functions in $\mathcal{S}_2$ is the notion of a *Hilbert space model*.

Definition 1.4.1. *Let* $\varphi\in\mathcal{S}_2$. *A pair* $(\mathcal{M},u)$ *is a* model *for* φ *if* $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ *is an orthogonally decomposed separable Hilbert space and* $u:\mathbb{D}^2\to\mathcal{M}$ *is an analytic map such that*

$$1-\overline{\varphi(\mu)}\varphi(\lambda)=\left\langle(1-\mu^*\lambda)u_\lambda,u_\mu\right\rangle \tag{1.4.1}$$

holds for every $\lambda, \mu \in \mathbb{D}^2$*, where* $u_\lambda = u(\lambda)$ *and* λ *is the operator*

$$\lambda = \lambda^1 P_{\mathcal{M}_1} + \lambda^2 P_{\mathcal{M}_2}.$$

Every function in $\mathcal{S}_2$ has a model [Agl90, AM02]. A lurking isometry argument on the model equation (1.4.1) gives rise to the following related operator theoretic construct.

Theorem 1.4.2. *If* $(\mathcal{M}, u)$ *is a model of* $\varphi \in \mathcal{S}_2$*, then there exist* $a \in \mathbb{C}$*, vectors* $\beta, \gamma \in \mathcal{M}$*, and a linear operator* $D : \mathcal{M} \to \mathcal{M}$ *such that the operator*

$$\begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix}$$

is a contraction on $\mathbb{C} \oplus \mathcal{M}$*, and, for all* $\lambda \in \mathbb{D}^2$*,*

$$(1 - D\lambda)u_\lambda = \gamma,$$
$$\varphi(\lambda) = a + \langle \lambda u_\lambda, \beta \rangle .$$

In this case, we can write

$$\varphi(\lambda) = a + \left\langle \lambda (1 - D\lambda)^{-1} \gamma, \beta \right\rangle .$$

The 4-tuple (a, β, γ, D) is called a realization of a model of φ.

1.5 Connections between operator- and function-theoretic properties

Hilbert space models and realizations have great utility in the analysis of boundary behavior of functions in the Schur class because they encode function theoretic data in the geometrical structure of the model. Two particularly important characteristics of models are boundedness and continuity of the model function u_λ, captured in the followed definition.

Definition 1.5.1. *For a given function $\varphi \in \mathcal{S}_2$, a point $\tau \in \mathbb{D}^d$ is a B-point of the model if u is bounded on every subset of $\mathbb{D}^d$ that approaches τ nontangentially. The point τ is a C-point of the model if, for every subset S of D^d that approaches τ nontangentially, u extends continuously so $S \cup \{\tau\}$ (with respect to the norm topology on $\mathcal{M}$).*

In practice, one merely needs to check a single set of nontangential approach [AMY12a].

For a function $\varphi \in \mathcal{S}_2$, a B-point for a model of φ corresponds to a carapoint for the function φ.

Theorem 1.5.2 (Agler, M^c^Carthy, Young)**.** *Let $\varphi \in \mathcal{S}_2$, $\tau \in \mathbb{T}^2$, and $\{\lambda_n\}$ a sequence in $\mathbb{D}^2$ that approaches τ nontangentially. The following are equivalent:*

1. *τ is a carapoint for φ;*
2. *there exists a model $(\mathcal{M}, u)$ of φ such that τ is a B-point;*
3. *for every model $(\mathcal{M}, u)$ of φ, τ is a B-point.*

While the classical Theorem 1.2.1 does not hold in two variables, (noted in Lemma 3.3.1 below), in [AMY12a], Agler, McCarthy, and Young used Hilbert space model techniques to prove a two part generalization of Theorem 1.2.1 in terms of the properties of a model at a boundary point. The following theorems are a simplified, qualitative version of those results.

Theorem 1.5.3 (Agler, McCarthy, Young)**.** *If* τ *is a B-point for a model* $(\mathcal{M}, u)$ *of* φ*, then the nontangential limit of* φ *at* τ *given by*

$$\varphi(\tau) := \lim_{\lambda \stackrel{\mathrm{nt}}{\to} \tau} \varphi(\lambda)$$

exists.

Theorem 1.5.4 (Agler, McCarthy, Young)**.** τ *is a C-point for* φ *if and only if* φ *is nontangentially differentiable at* τ*.*

In short, a function $\varphi \in \mathcal{S}_2$ has a nontangential limit at τ when the model function is bounded there, and φ is differentiable at τ when the model function is continuous there. More can be said about the differential structure of functions at carapoints (the subject of [AMY12a]), which will be discussed in the following sections.

Chapter 2

Boundary behavior of Schur functions in two variables

2.1 Introduction

In this chapter, we solve two problems about analytic functions of two variables using a variant of the notion of a Hilbert space model of a function. One problem concerns the generalization to two variables of a classical representation theorem of Nevanlinna, while the other is quite unlike any question that arises for functions of a single variable. Both relate to behavior of functions at boundary points of their domains.

The first problem is: what directional derivatives are possible for a function in the two-variable Schur class $\mathcal{S}_2$ at a singular point on the 2-torus $\mathbb{T}^2$? To clarify this question let us consider the rational function

$$\varphi(\lambda) = \frac{\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 - \lambda_1\lambda_2}{1 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2}, \quad \lambda \in \mathbb{D}^2, \tag{2.1.1}$$

where $\mathbb{D}$ denotes the open unit disc. This function belongs to $\mathcal{S}_2$ (that is, it is analytic and bounded by 1 in modulus on $\mathbb{D}^2$). It has a singularity at the point $\chi = (1,1) \in \mathbb{T}^2$, in that φ does not extend analytically (or even continuously) to χ. Nevertheless φ has nontangential limit 1 at χ, and so we may define $\varphi(\chi)$ to be 1. Despite the fact that φ is discontinuous at χ, the directional derivative $D_{-\delta}\varphi(\chi)$ exists for every direction $-\delta$ pointing into the bidisc at χ, and

$$\begin{aligned} D_{-\delta}\varphi(\chi) &= -\frac{2\delta_1\delta_2}{\delta_1+\delta_2} \\ &= \varphi(\chi)\delta_2 h(\delta_2/\delta_1) \end{aligned} \tag{2.1.2}$$

where $h(z) = -2/(1+z)$.

Remarkably enough, a similar statement holds in great generality [AMY12a, Theorem 2.10]. If $\varphi \in \mathcal{S}_2$ has a singularity at χ and φ satisfies a weak regularity condition at χ (the Carathéodory condition, explained in Section 1.3) then $D_{-\delta}\varphi(\chi)$ exists for all relevant directions δ, and furthermore there exists an analytic function h on the upper halfplane

$$\Pi = \{z : \text{Im } z > 0\}$$

such that both $h(z)$ and $-zh(z)$ have non-negative imaginary part and the directional derivative $D_{-\delta}\varphi(\chi)$ is given by equation (2.1.2). We call h the *slope function* for φ at χ. The problem, then, is to find necessary and sufficient conditions for a function h on Π to be the slope function of some member of $\mathcal{S}_2$. It transpires that the stated necessary conditions on h are also sufficient for h to be a slope function (Theorem 2.5.2 below).

The second problem is to generalize to two variables a theorem of Nevanlinna which plays an important role in one proof of the spectral theorem for self-adjoint operators. Nevanlinna's theorem gives an integral representation formula for the functions in the Pick class $\mathcal{P}$ that satisfy a growth condition on the imaginary axis; it states that such functions are the Cauchy transforms of the finite positive measures on the real line $\mathbb{R}$. Nevanlinna's growth condition can be regarded as a regularity condition at the point

∞ on the boundary of Π. We obtain an analogous representation for functions in the two-variable Pick class that satisfy a suitable regularity condition at ∞, but rather than an integral formula we get an expression involving the two-variable resolvent of a densely defined self-adjoint operator on a Hilbert space (Theorem 2.6.3).

To solve these two problems we modify the notion of model so as to focus on the behavior of a function $\varphi \in \mathcal{S}_2$ near a boundary point at which φ satisfies Carathéodory's condition (see Definition 1.3.2 below).

2.2 Generalized models of Schur-class functions

In the definition of a model of a function $\varphi : \mathbb{D}^d \to \mathbb{C}$ (see Definition 1.4.1 above), the function $\lambda = \lambda^1 P_{\mathcal{M}_1} + \lambda^2 P_{\mathcal{M}_2}$ is linear in the coordinates of the point $\lambda = (\lambda^1, \lambda^2)$. One consequence is that any singular behavior of φ at a boundary point must be reflected in singular behavior of u near that point. A simple relaxation of the definition of model to allow an inner function $I(\cdot)$ in place of λ enables us to concentrate information about singular behavior in $I(.)$ instead of u, and this proves helpful for the two problems we study here.

Definition 2.2.1. *Let* $\varphi : \mathbb{D}^d \to \mathbb{C}$ *be analytic. The triple* $(\mathcal{M}, u, I)$ *is a* generalized model of φ *if*

1. *$\mathcal{M}$ is a separable Hilbert space,*

2. *$u : \mathbb{D}^d \to \mathcal{M}$ is analytic, and*

3. *I is a contractive analytic $\mathcal{L}(\mathcal{M})$-valued function on $\mathbb{D}^d$*

such that the equation

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = \left\langle (1 - I(\mu)^* I(\lambda)) u_\lambda, u_\mu \right\rangle \tag{2.2.1}$$

holds for all $\lambda, \mu \in \mathbb{D}^d$.

The generalized model $(\mathcal{M}, u, I)$ *is* inner *if* $I(.)$ *is inner.*

Clearly, in the case that $I(\lambda) = \lambda_1 P_1 + \cdots + \lambda_d P_d$, we recapture the original notion of model as in Definition 1.4.1.

A well-known lurking isometry argument proceeds from a model $(\mathcal{M}, u)$ of a function $\varphi \in \mathcal{S}_d$ to a realization of φ [Agl90]. The identical argument applied to a generalized model $(\mathcal{M}, u, I)$ produces a generalized notion of realization (compare with Theorem 1.4.2).

Theorem 2.2.2. *If* $(\mathcal{L}, u, I)$ *is a generalized model of* $\varphi \in \mathcal{S}_d$ *then there exist a Hilbert space* $\mathcal{M}$ *containing* $\mathcal{L}$*, a scalar* $a \in \mathbb{C}$*, vectors* $\beta, \gamma \in \mathcal{M}$ *and a linear operator* $D : \mathcal{M} \to \mathcal{M}$ *such that the operator*

$$L = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix} \tag{2.2.2}$$

is unitary on $\mathbb{C} \oplus \mathcal{M}$ *and, for all* $\lambda \in \mathbb{D}^d$*,*

$$L \begin{pmatrix} 1 \\ I(\lambda) u_\lambda \end{pmatrix} = \begin{pmatrix} \varphi(\lambda) \\ u_\lambda \end{pmatrix}, \tag{2.2.3}$$

and consequently, for all $\lambda \in \mathbb{D}^d$*,*

$$\varphi(\lambda) = a + \left\langle I(\lambda)(1 - D I(\lambda))^{-1} \gamma, \beta \right\rangle. \tag{2.2.4}$$

Proof. By equation (2.2.1), for all $\lambda, \mu \in \mathbb{D}^d$,

$$1 + \left\langle I(\lambda) u_\lambda, I(\mu) u_\mu \right\rangle = \overline{\varphi(\mu)} \varphi(\lambda) + \left\langle u_\lambda, u_\mu \right\rangle.$$

We may interpret this equation as an equality between the gramians of two families of vectors in $\mathbb{C} \oplus \mathcal{L}$. Accordingly we may define an isometric operator

$$L_0 : \text{span}\left\{ \begin{pmatrix} 1 \\ I(\lambda)u_\lambda \end{pmatrix} : \lambda \in \mathbb{D}^d \right\} \to \text{span}\left\{ \begin{pmatrix} \varphi(\lambda) \\ u_\lambda \end{pmatrix} : \lambda \in \mathbb{D}^d \right\}$$

by equation (2.2.3). If necessary we may enlarge $\mathbb{C} \oplus \mathcal{L}$ to a space $\mathbb{C} \oplus \mathcal{M}$ in which the domain and range of L_0 have equal codimension, and then we may extend L_0 to a unitary operator L on $\mathbb{C} \oplus \mathcal{M}$. □

The ordered 4-tuple (a, β, γ, D), as in equation (2.2.2), will be called a *realization* of the (generalized) model $(\mathcal{L}, u, I)$ of φ if L is a contraction and equation (2.2.3) holds. It will be called a *unitary realization* if in addition L is unitary on $\mathbb{C} \oplus \mathcal{L}$.

Realizations provide an effective tool for the study of boundary behavior. Here is a preliminary observation.

Lemma 2.2.3. *Suppose that* $\varphi \in \mathcal{S}_d$ *has a model* $(\mathcal{M}, u)$ *with realization* (a, β, γ, D)*. Let* $\tau \in \mathbb{T}^d$ *be a carapoint for* φ *and let* $\mathcal{N} = \ker(1 - D\tau)$*. Then*

$$\gamma \in \text{ran}(1 - D\tau) \subset \mathcal{N}^\perp \textit{ and } \tau^* \beta \in \mathcal{N}^\perp. \tag{2.2.5}$$

Proof. First we show that $\tau^*\beta \in \mathcal{N}^\perp$. Let L be given by equation (2.2.2). Choose any $x \in \mathcal{N}$. Then $x = D\tau x$ and so

$$L \begin{pmatrix} 0 \\ \tau x \end{pmatrix} = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix} \begin{pmatrix} 0 \\ \tau x \end{pmatrix} = \begin{pmatrix} \langle \tau x, \beta \rangle \\ D\tau x \end{pmatrix} = \begin{pmatrix} \langle x, \tau^* \beta \rangle \\ x \end{pmatrix}.$$

Since L is a contraction and τ is an isometry,

$$\left\| \begin{pmatrix} \langle x, \tau^* \beta \rangle \\ x \end{pmatrix} \right\| = \left\| L \begin{pmatrix} 0 \\ \tau x \end{pmatrix} \right\| \leq \|\tau x\| = \|x\|,$$

and so $\langle x, \tau^* \beta \rangle = 0$. Since $x \in \mathcal{N}$ is arbitrary, $\tau^* \beta \in \mathcal{N}^\perp$.

Proposition 5.17 of [AMY12a] asserts that τ is a carapoint for φ if and only if $\gamma \in \operatorname{ran}(1 - D\tau)$. Now since $D\tau$ is a contraction, every eigenvector of $D\tau$ corresponding to an eigenvalue λ of unit modulus is also an eigenvector of $(D\tau)^*$ with eigenvalue $\bar{\lambda}$. Hence $\ker(1 - D\tau) = \ker(1 - \tau^* D^*)$, and we have

$$\gamma \in \operatorname{ran}(1 - D\tau) \subset \ker(1 - \tau^* D^*)^\perp = \ker(1 - D\tau)^\perp = \mathcal{N}^\perp.$$

□

We are interested in the behavior of models at carapoints of $\varphi \in \mathcal{S}_d$. Here are two relevant notions.

Definition 2.2.4. *Let $(\mathcal{M}, u, I)$ be a generalized model of a function $\varphi \in \mathcal{S}_d$. A point $\tau \in \partial \mathbb{D}^d$ is a B-*point *of the model if u is bounded on every subset of $\mathbb{D}^d$ that approaches τ nontangentially. The point τ is a C-*point *of the model if, for every subset S of $\mathbb{D}^d$ that approaches τ nontangentially, u extends continuously to $S \cup \{\tau\}$ (with respect to the norm topology of $\mathcal{M}$).*

As is well known, not all functions in $\mathcal{S}_d$ have models when $d \geq 3$. For the rest of the paper we restrict attention to the case $d = 2$; in this case it *is* true that every function in the Schur class has a model [Agl90].

Our next task is to show that if a function $\varphi \in \mathcal{S}_2$ has a singularity at a B-point τ, then we can construct a generalized model of φ in which the singularity of φ is encoded in an $I(\lambda)$ that is singular at τ, in such a way that the model has a C-point at τ. The device that leads to this conclusion is to write vectors in and operators on $\mathcal{M}$ in terms of the orthogonal decomposition $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ where $\mathcal{N} = \ker(1 - D\tau)$ and D comes from a realization of $(\mathcal{M}, u)$. The following observation is straightforward.

Lemma 2.2.5. *Let $\mathcal{N}$ be a subspace of $\mathcal{M}$ and let P_1 be a Hermitian projection on $\mathcal{M}$. With respect to the decomposition $\mathcal{N} \oplus \mathcal{N}^{\perp}$ the operator P_1 has operator matrix*

$$P_1 = \begin{bmatrix} X & B \\ B^* & Y \end{bmatrix} \tag{2.2.6}$$

for some operators X, Y, B, where

1. $0 \le X, Y \le 1$
2. $BB^* = X(1-X), \quad B^*B = Y(1-Y)$
3. $BY = (1-X)B, \quad B(1-Y) = XB$
4. $B^*X = (1-Y)B^*, \quad B^*(1-X) = YB^*.$

We now construct a generalized model corresponding to a carapoint of $\varphi \in \mathcal{S}_2$.

Theorem 2.2.6. *Let $\tau \in \mathbb{T}^2$ be a carapoint for $\varphi \in \mathcal{S}_2$. There exists an inner generalized model $(\mathcal{M}, u, I)$ of φ such that*

1. *τ is a C-point for $(\mathcal{M}, u, I)$,*
2. *I is analytic at every point $\lambda \in \mathbb{T}^2$ such that $\lambda_1 \neq \tau_1$ and $\lambda_2 \neq \tau_2$, and*
3. *τ is a carapoint for I and $I(\tau) = 1_{\mathcal{M}}$.*

Furthermore, we may express I in the form

$$I(\lambda) = \frac{\bar{\tau}_1\lambda_1 Y + \bar{\tau}_2\lambda_2(1-Y) - \bar{\tau}_1\bar{\tau}_2\lambda_1\lambda_2}{1-\bar{\tau}_1\lambda_1(1-Y) - \bar{\tau}_2\lambda_2 Y} \tag{2.2.7}$$

for some positive contraction Y on $\mathcal{M}$.

Proof. Choose any model $(\mathcal{L}, v)$ of φ and any realization (a, β_0, γ, D) of $(\mathcal{L}, v)$. By definition, $\mathcal{L}$ comes with an orthogonal decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$: let P_1 be the orthogonal projection on $\mathcal{L}_1$. Since τ is a carapoint for φ we may apply Lemma 2.2.3 to deduce that $\gamma \in \operatorname{ran}(1-D\tau)$ and $\tau^*\beta_0, \gamma \in \ker(1-D\tau)^\perp$.

Consider first the case that $\ker(1-D\tau) = \{0\}$. This relation implies that there is a *unique* vector $v_\tau \in \mathcal{L}$ such that $(1-D\tau)v_\tau = \gamma$. Let (λ_n) be any sequence in $\mathbb{D}^2$ that converges nontangentially to τ. We claim that $v_{\lambda_n} \to v_\tau$. Suppose not: then since (v_{λ_n}) is bounded, by [AMY12a, Corollary 5.7], we can assume on passing to a subsequence that (v_{λ_n}) tends weakly to a limit $x \in \mathcal{L}$ different from v_τ. By [AMY12a, Proposition 5.8] it follows that $v_{\lambda_n} \to x$ in norm. Take limits in the equation

$$(1-D\lambda_n)v_{\lambda_n} = \gamma$$

to deduce that $(1-D\tau)x = \gamma$. Since $x \neq v_\tau$, this contradicts the fact that $(1-D\tau)^{-1}\gamma = \{v_\tau\}$. Hence $v_{\lambda_n} \to v_\tau$. In other words v extends continuously to $S \cup \{\tau\}$ for any set S in $\mathbb{D}^2$ that tends nontangentially to τ, which is to say that τ is a C-point for the model $(\mathcal{L}, v)$. The conclusion of the theorem therefore holds if we simpy take $\mathcal{M} = \mathcal{L}$, $u = v$ and $I(\lambda) = \lambda_1 P_1 + \lambda_2 P_2$.

Now consider the case that $\ker(1-D\tau) \neq \{0\}$. Let $\mathcal{N} = \ker(1-D\tau)$. With respect to the decomposition $\mathcal{L} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ we may write

$$D\tau = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \tag{2.2.8}$$

and $v_\lambda = \begin{pmatrix} w_\lambda \\ u_\lambda \end{pmatrix}$. Note that $\ker(1-Q) = \{0\}$.

Let us express λ, acting as an operator on $\mathcal{L}$ by

$$\lambda = \lambda_1 P_1 \oplus \lambda_2(1 - P_1),$$

as an operator matrix with respect to the decomposition $\mathcal{L} = \mathcal{N} \oplus \mathcal{N}^{\perp}$, as in Lemma 2.2.5:

$$\lambda = \lambda_1 P_1 + \lambda_2(1 - P_1) = \begin{bmatrix} \lambda_1 X + \lambda_2(1-X) & (\lambda_1 - \lambda_2)B \\ (\lambda_1 - \lambda_2)B^* & \lambda_1 Y + \lambda_2(1-Y) \end{bmatrix} \tag{2.2.9}$$

where X, Y are the compressions of P_1 to $\mathcal{N}, \mathcal{N}^{\perp}$ respectively, so that $0 \leq X, Y \leq 1$. Thus

$$1 - D\lambda = 1 - D\tau\tau^*\lambda = \begin{bmatrix} 1 - (\lambda_1' X + \lambda_2'(1-X)) & -(\lambda_1' - \lambda_2')B \\ -(\lambda_1' - \lambda_2')QB^* & 1 - Q(\lambda_1' Y + \lambda_2'(1-Y)) \end{bmatrix}$$

where $\lambda_1' = \bar{\tau}_1\lambda_1$, $\lambda_2' = \bar{\tau}_2\lambda_2$. Since $(1 - D\lambda)v_\lambda = \gamma$,

$$\begin{pmatrix} 0 \\ \gamma \end{pmatrix} = (1 - D\lambda)v_\lambda = \begin{bmatrix} 1 - (\lambda_1' X + \lambda_2'(1-X)) & -(\lambda_1' - \lambda_2')B \\ -(\lambda_1' - \lambda_2')QB^* & 1 - Q(\lambda_1' Y + \lambda_2'(1-Y)) \end{bmatrix} \begin{pmatrix} w_\lambda \\ u_\lambda \end{pmatrix},$$

from which we have the equations

$$(1-\lambda_1' X - \lambda_2'(1-X))w_\lambda - (\lambda_1' - \lambda_2')Bu_\lambda = 0 \tag{2.2.10}$$

$$-(\lambda_1' - \lambda_2')QB^* w_\lambda + (1 - Q(\lambda_1' Y + \lambda_2'(1-Y)))u_\lambda = \gamma. \tag{2.2.11}$$

By Lemma 2.2.5 and equation (2.2.10) we have

$$\begin{aligned} 0 &= B^*\left((1-\lambda_1' X - \lambda_2'(1-X))w_\lambda - (\lambda_1' - \lambda_2')Bu_\lambda\right) \\ &= (B^* - \lambda_1' B^* X - \lambda_2' B^*(1-X))w_\lambda - (\lambda_1' - \lambda_2')B^*Bu_\lambda \\ &= (B^* - \lambda_1'(1-Y)B^* - \lambda_2' Y B^*)w_\lambda - (\lambda_1' - \lambda_2')Y(1-Y)u_\lambda \\ &= (1 - \lambda_1'(1-Y) - \lambda_2' Y)B^* w_\lambda - (\lambda_1' - \lambda_2')Y(1-Y)u_\lambda. \end{aligned} \tag{2.2.12}$$

Since $0 \le Y \le 1$ it is clear from the spectral mapping theorem that

$$1 \notin \sigma(\lambda_1'(1-Y) + \lambda_2' Y)$$

for all $\lambda \in \mathbb{D}^2$, and thus equation (2.2.12) tells us that

$$B^* w_\lambda = \frac{(\lambda_1' - \lambda_2')Y(1-Y)}{1 - \lambda_1'(1-Y) - \lambda_2' Y} u_\lambda. \tag{2.2.13}$$

Substituting the relation (2.2.13) into (2.2.11) we obtain

$$\begin{aligned} \gamma &= -(\lambda_1' - \lambda_2')QB^* w_\lambda + (1 - Q(\lambda_1' Y + \lambda_2'(1-Y))u_\lambda \\ &= -(\lambda_1' - \lambda_2')Q\frac{(\lambda_1' - \lambda_2')Y(1-Y)}{1 - \lambda_1'(1-Y) - \lambda_2' Y}u_\lambda + (1 - Q(\lambda_1' Y + \lambda_2'(1-Y))u_\lambda \\ &= \left[1 - Q\left(\frac{(\lambda_1' - \lambda_2')^2 Y(1-Y)}{1 - \lambda_1'(1-Y) - \lambda_2' Y} + \lambda_1' Y + \lambda_2'(1-Y)\right)\right]u_\lambda \\ &= (1 - QI(\lambda))u_\lambda \end{aligned} \tag{2.2.14}$$

where

$$I(\lambda) = \frac{\lambda_1' Y + \lambda_2'(1-Y) - \lambda_1'\lambda_2'}{1 - \lambda_1'(1-Y) - \lambda_2' Y} \tag{2.2.15}$$
$$= \frac{\bar\tau_1\lambda_1 Y + \bar\tau_2\lambda_2(1-Y) - \bar\tau_1\bar\tau_2\lambda_1\lambda_2}{1 - \bar\tau_1\lambda_1(1-Y) - \bar\tau_2\lambda_2 Y} \in \mathcal{L}(\mathcal{M}),$$

which agrees with equation (2.2.7).

Let $\mathcal{M} = \mathcal{N}^\perp$: we claim that $(\mathcal{M}, u, I)$ is an inner generalized model of φ having the properties described in Theorem 2.2.6.

Firstly, it is clear from the formula (2.2.15) that I is analytic on $\mathbb{D}^2$ and at every point $\lambda \in \mathbb{T}^2$ such that $1 \notin \sigma(\lambda_1'(1-Y) + \lambda_2' Y)$. By the spectral mapping theorem and the fact that $0 \le Y \le 1$, the spectrum of $\lambda_1'(1-Y) + \lambda_2' Y$ is contained in the convex hull of the points λ_1', λ_2'. Hence $\sigma(\lambda_1'(1-Y) + \lambda_2' Y)$ contains the point 1 if and only if either $\lambda_1' = 1$ and $0 \in \sigma(Y)$ or $\lambda_2' = 1$ and $1 \in \sigma(Y)$. Thus I is analytic at points $\lambda \in \mathbb{T}^2$ for which $\lambda_1' \neq 1, \lambda_2' \neq 1$, that is, such that $\lambda_1 \neq \tau_1, \lambda_2 \neq \tau_2$. The function I therefore satisfies condition (2) of the theorem.

We must show that I is an inner function. Indeed, if $d(\lambda)$ denotes the denominator of $I(\lambda)$ in equation (2.2.15), we find that, for all $\lambda \in \Delta^2$ such that $1 \notin \sigma(\lambda_1'(1-Y) + \lambda_2' Y)$,

$$d(\lambda)^*(1 - I(\lambda)^* I(\lambda))d(\lambda) =$$
$$|1-\lambda_1'|^2(1-|\lambda_2'|^2) + 2\{\text{Re }(\lambda_1' - \lambda_2') - |\lambda_1'|^2 + |\lambda_2'|^2 + \text{Re }(\overline{\lambda_1'\lambda_2'}(\lambda_1' - \lambda_2'))\}Y.$$

Hence $I(\lambda)^* I(\lambda) = 1_{\mathcal{M}}$ for all $\lambda \in \mathbb{T}^2$ such that $\lambda_1 \neq \tau_1$, $\lambda_2 \neq \tau_2$, and therefore for almost all $\lambda \in \mathbb{T}^2$ with respect to 2-dimensional Lebesgue measure on $\mathbb{T}^2$. Since $I(\lambda)$ is clearly a normal operator for all such λ, it follows that I is an inner $\mathcal{L}(\mathcal{M})$-valued function.

Next we prove the model relation (2.2.1) for $(\mathcal{M}, u, I)$. Let us calculate $\tau^*\lambda v_\lambda$

using equation (2.2.9):

$$\tau^*\lambda v_\lambda = \tau^*\lambda \begin{pmatrix} w_\lambda \\ u_\lambda \end{pmatrix} = \begin{pmatrix} (\lambda_1' X + \lambda_2'(1-X))w_\lambda + (\lambda_1' - \lambda_2')Bu_\lambda \\ (\lambda_1' - \lambda_2')B^* w_\lambda + (\lambda_1' Y + \lambda_2'(1-Y))u_\lambda \end{pmatrix}_{\mathcal{N}\oplus\mathcal{N}^\perp} . \qquad (2.2.16)$$

By equations (2.2.13) and (2.2.16),

$$\begin{aligned} P_{\mathcal{N}^\perp}\tau^*\lambda v_\lambda &= \left(\frac{(\lambda_1' - \lambda_2')^2 Y(1-Y)}{1-(\lambda_1'(1-Y)+\lambda_2' Y)} + \lambda_1' Y + \lambda_2'(1-Y) \right) u_\lambda \\ &= I(\lambda)u_\lambda . \end{aligned}$$

By equation (2.2.10),

$$(\lambda_1' X + \lambda_2'(1-X))w_\lambda = w_\lambda - (\lambda_1' - \lambda_2')Bu_\lambda ,$$

which, in combination with equation (2.2.16), yields the relation

$$P_{\mathcal{N}}\tau^*\lambda v_\lambda = (\lambda_1' X + \lambda_2'(1-X))w_\lambda + (\lambda_1' - \lambda_2')Bu_\lambda = w_\lambda$$

and therefore

$$\tau^*\lambda v_\lambda = \begin{pmatrix} w_\lambda \\ I(\lambda)u_\lambda \end{pmatrix}_{\mathcal{N}\oplus\mathcal{N}^\perp} .$$

Hence

$$
\begin{aligned}
1-\overline{\varphi(\mu)}\varphi(\lambda) &= \left\langle (1-\mu^*\lambda)v_\lambda, v_\mu \right\rangle_{\mathcal{L}} \\
&= \left\langle v_\lambda, v_\mu \right\rangle_{\mathcal{L}} - \left\langle \lambda v_\lambda, \mu v_\mu \right\rangle_{\mathcal{L}} \\
&= \left\langle w_\lambda, w_\mu \right\rangle_{\mathcal{N}} + \left\langle u_\lambda, u_\mu \right\rangle_{\mathcal{N}^\perp} - \left\langle \tau^*\lambda v_\lambda, \tau^*\mu v_\mu \right\rangle_{\mathcal{L}} \\
&= \left\langle w_\lambda, w_\mu \right\rangle_{\mathcal{N}} + \left\langle u_\lambda, u_\mu \right\rangle_{\mathcal{N}^\perp} - \left(\left\langle w_\lambda, w_\mu \right\rangle_{\mathcal{N}} + \left\langle I(\lambda)u_\lambda, I(\mu)u_\mu \right\rangle_{\mathcal{N}^\perp} \right) \\
&= \left\langle (1-I(\mu)^*I(\lambda))u_\lambda, u_\mu \right\rangle_{\mathcal{M}}.
\end{aligned}
$$

Thus $(\mathcal{M}, u, I)$ is an inner generalized model of φ.

We show next that τ is a C-point for $(\mathcal{M}, u, I)$. To establish this we must produce a vector $u_\tau \in \mathcal{M}$ such that $u_{\lambda_n} \to u_\tau$ as $n \to \infty$ for every sequence (λ_n) in $\mathbb{D}^2$ that converges nontangentially to τ.

As we observed above, τ is a B-point for the model $(\mathcal{L}, v)$ and $\gamma \in \operatorname{ran}(1-D\tau)$. Let u_τ be the unique element of smallest norm in the nonempty closed convex set $(1-D\tau)^{-1}\gamma$. Then $u_\tau \in \ker(1-D\tau)^\perp = \mathcal{N}^\perp$, and every element of $(1-D\tau)^{-1}\gamma$ has the form $e \oplus u_\tau$ for some $e \in \mathcal{N}$.

Let X_τ be the nontangential cluster set of v at τ in the model $(\mathcal{L}, v)$; that is, X_τ comprises the limits in $\mathcal{L}$ of all convergent sequences (v_{λ_n}) for all sequences (λ_n) in $\mathbb{D}^2$ that converge nontangentially to τ. Recall that, by [AMY12a, Proposition 5.8], a sequence (v_{λ_n}) converges in norm if and only if it converges weakly in $\mathcal{L}$. If $x \in X_\tau$ is the limit of u_{λ_n} for some sequence (λ_n) that converges nontangentially to τ then, since $(1-D\lambda_n)v_{\lambda_n} = \gamma$, on letting $n \to \infty$ we find that $(1-D\tau)x = \gamma$. Thus

$$
X_\tau \subset (1-D\tau)^{-1}\gamma \subset \left\{ \begin{pmatrix} e \\ u_\tau \end{pmatrix} : e \in \mathcal{N} \right\}.
$$

We claim that $u_{\lambda_n} \to u_\tau$ as $n \to \infty$ for every sequence (λ_n) in $\mathbb{D}^2$ that converges nontangentially to τ. For suppose that u_{λ_n} does not converge to u_τ. Since v_{λ_n}, and hence also u_{λ_n}, is bounded, on passing to a subsequence we may suppose that $u_{\lambda_n} \to \xi$ for some vector $\xi \neq u_\tau$, and by passing to a further subsequence, we may suppose that v_{λ_n} converges to some vector $x \in X_\tau$. But then

$$v_{\lambda_n} = \begin{pmatrix} w_{\lambda_n} \\ u_{\lambda_n} \end{pmatrix} \to x \in \left\{ \begin{pmatrix} e \\ u_\tau \end{pmatrix} : e \in \mathcal{N} \right\},$$

and hence $u_{\lambda_n} \to u_\tau$, which is a contradiction. We have shown that $u_{\lambda_n} \to u_\tau$ for every sequence (λ_n) in $\mathbb{D}^2$ that converges to τ nontengentially; hence τ is a C-point for the generalized model $(\mathcal{M}, u, I)$.

To see that τ is a carapoint for I, observe that if $\lambda = r\tau$, where $0 < r < 1$, then $\lambda' = (r, r)$, and so by equation (2.2.15),

$$I(r\tau) = 1 - \frac{(1-r)^2}{1-r} = r.$$

Hence

$$\liminf_{\lambda \to \tau} \frac{1 - \|I(\lambda)\|}{1 - \|\lambda\|_\infty} \leq \liminf_{r \to 1} \frac{1 - \|I(r\tau)\|}{1-r} = 1.$$

Thus τ is a carapoint for I.

To complete the proof of condition (3) of Theorem 2.2.6 we must show that $I(\tau) = 1_{\mathcal{M}}$, which by definition means that $I(\lambda) \to 1_{\mathcal{N}^\perp}$ as $\lambda \xrightarrow{\text{nt}} \tau$. Observe that

$$I(\lambda) - 1 = -\frac{(\lambda_1' - 1)(\lambda_2' - 1)}{1 - \lambda_1'(1-Y) - \lambda_2' Y} = -\bar{\tau}_1 \bar{\tau}_2 \frac{(\lambda_1 - \tau_1)(\lambda_2 - \tau_2)}{1 - \bar{\tau}_1 \lambda_1 (1-Y) - \bar{\tau}_2 \lambda_2 Y}. \qquad (2.2.17)$$

Since the spectrum of the normal operator $Z = \lambda_1'(1-Y) + \lambda_2' Y$ is contained in the convex hull of the points λ_1', λ_2',

$$\begin{aligned}\operatorname{dist}(1, \sigma(Z)) &\geq \operatorname{dist}(\mathbb{T}, \sigma(Z)) \geq \operatorname{dist}(\mathbb{T}, \operatorname{conv}\{\lambda_1', \lambda_2'\}) = \operatorname{dist}((\lambda_1', \lambda_2'), \partial\mathbb{D}^2) \\ &= \operatorname{dist}(\lambda, \partial\mathbb{D}^2).\end{aligned}$$

It follows that

$$\left\|(1 - \lambda_1'(1-Y) - \lambda_2' Y)^{-1}\right\| \leq \frac{1}{\operatorname{dist}(\lambda, \partial\mathbb{D}^2)}$$

and therefore

$$\|I(\lambda) - 1\| \leq \frac{|\lambda_1 - \tau_1|\,|\lambda_2 - \tau_2|}{\operatorname{dist}(\lambda, \partial\mathbb{D}^2)}.$$

If λ approaches τ in a set S on which

$$\frac{\|\lambda - \tau\|}{\operatorname{dist}(\lambda, \partial\mathbb{D}^2)} \leq c < \infty,$$

then, by the inequality of the means,

$$\|I(\lambda) - 1\| \leq \tfrac{1}{2} c \, \|\lambda - \tau\|$$

for $\lambda \in S$. Thus $I(\lambda) \to 1$ as $\lambda \xrightarrow{\text{nt}} \tau$. □

A consequence of Theorem 2.2.6 is that φ has a generalized realization, as in Theorem 2.2.2. The preceding proof yields slightly more.

Corollary 2.2.7. *If $\tau \in \mathbb{T}^2$ is a carapoint for $\varphi \in \mathcal{S}_2$ then φ has a generalized realization*

$$\varphi(\lambda) = a + \left\langle I(\lambda)(1 - QI(\lambda))^{-1}\gamma, \beta \right\rangle_{\mathcal{M}}$$

for some $\beta, \gamma \in \mathcal{M}$ and some contraction Q on $\mathcal{M}$ satisfying $\ker(1-Q) = \{0\}$*, where I is the inner function given by equation* (2.2.7)*, having the properties described in Theorem*

2.2.6.

Proof. In the proof of Theorem 2.2.6 it is clear from the definition (2.2.8) of Q that Q is a contraction and that $\ker(1-Q)=\{0\}$. From equation (2.2.14) we have

$$\gamma + QI(\lambda)u_\lambda = u_\lambda,$$

and from the realization (a,β_0,γ,D) of the model $(\mathcal{L},v)$,

$$\varphi(\lambda) = a + \langle \lambda v_\lambda, \beta_0\rangle .$$

Note that, since $\tau^*\beta_0 \in \mathcal{N}^\perp$,

$$\begin{aligned}\varphi(\lambda) &= a + \langle \lambda v_\lambda, \beta_0\rangle_{\mathcal{L}} = a + \langle \tau^*\lambda v_\lambda, \tau^*\beta_0\rangle_{\mathcal{L}} = a + \left\langle P_{\mathcal{N}^\perp}\tau^*\lambda v_\lambda, \tau^*\beta_0\right\rangle_{\mathcal{N}^\perp}\\ &= a + \langle I(\lambda)u_\lambda, \tau^*\beta_0\rangle_{\mathcal{M}} .\end{aligned}$$

Let $\beta = \tau^*\beta_0 \in \mathcal{M}$. We then have

$$\begin{bmatrix} a & 1\otimes\beta \\ \gamma\otimes 1 & Q \end{bmatrix} \begin{pmatrix} 1 \\ I(\lambda)u_\lambda \end{pmatrix} = \begin{pmatrix} a + \langle I(\lambda)u_\lambda, \beta\rangle \\ \gamma + QI(\lambda)u_\lambda \end{pmatrix} = \begin{pmatrix} \varphi(\lambda) \\ u_\lambda \end{pmatrix},$$

and so (a,β,γ,Q) is a generalized realization of the generalized model $(\mathcal{M},u,I)$ of φ. □

We shall call the model $(\mathcal{M},u,I)$ constructed in the foregoing proof of Theorem 2.2.6 the *desingularization* of the model $(\mathcal{L},v)$ at τ. The construction depends on the choice of a realization of the model $(\mathcal{L},v)$, and so where appropriate we should more precisely speak of the desingularization relative to a particular realization. Of course the singularity of φ at τ, if there is one, does not disappear; it is shifted into the inner function I, where it becomes accessible to analysis by virtue of the formula (2.2.7) for I.

Example 2.2.8. *The inner function $I(.)$ given by equation* (2.2.7) *is not in general analytic on* $\mathbb{T}^2 \setminus \{\tau\}$.

Let Y be the operation of multiplication by the independent variable t on $L^2(0,1)$ with Lebesgue measure: then $0 \leq Y \leq 1$. Let $\tau = (1,1)$. Suppose that I is analytic at the point $(1,-1)$: then the scalar function

$$f(\lambda) = \langle I(\lambda)\mathbf{1}, \mathbf{1} \rangle$$

is analytic at $(1,-1)$, where $\mathbf{1}$ denotes the constant function equal to 1. We have, for $\lambda \in \mathbb{D}^2$,

$$\begin{aligned} f(\lambda) &= \int_0^1 \frac{t\lambda_1 + (1-t)\lambda_2 - \lambda_1\lambda_2}{1-(1-t)\lambda_1 - t\lambda_2} \, \mathrm{d}t \\ &= \int_0^1 1 - \frac{(1-\lambda_1)(1-\lambda_2)}{(\lambda_1-\lambda_2)t + 1 - \lambda_1} \, \mathrm{d}t \\ &= 1 - \frac{(1-\lambda_1)(1-\lambda_2)}{\lambda_1 - \lambda_2} [\log((\lambda_1-\lambda_2)t + 1 - \lambda_1)]_0^1 \\ &= 1 - \frac{(1-\lambda_1)(1-\lambda_2)}{\lambda_1 - \lambda_2} [\log(1-\lambda_2) - \log(1-\lambda_1)]. \end{aligned}$$

Here we may take any branch of log that is analytic in $\{z : \mathrm{Re}\, z > 0\}$. Since f is analytic in a neighborhood of $(1,-1)$, we may let $\lambda_2 \to -1$ and deduce that, for some neighborhood U of 1 and for $\lambda_1 \in U \cap \mathbb{D}$,

$$f(\lambda_1, -1) = 1 + 2\frac{1-\lambda_1}{1+\lambda_1}[\log 2 - \log(1-\lambda_1)].$$

It is then clear that $f(., -1)$ is not analytic at 1, contrary to assumption. Thus $I(.)$ is not analytic at $(1,-1)$, even though $(1,-1) \neq \tau$.

2.3 Directional derivatives and slope functions

In this section we study the directional derivatives of a function $\varphi \in \mathcal{S}_2$ at a carapoint on the boundary. One of the main results of [AMY12a], namely Theorem 7.14, asserts the following[1].

Theorem 2.3.1. *Let $\tau \in \mathbb{T}^2$ be a carapoint for $\varphi \in \mathcal{S}$. There exists a function h in the Pick class, analytic and real-valued on $(0,\infty)$, such that the function $z \mapsto -zh(z)$ also belongs to the Pick class,*

$$h(1) = -\liminf_{\lambda \to \tau} \frac{1-|\varphi(\lambda)|}{1-\|\lambda\|_\infty} \tag{2.3.1}$$

and, for all $\delta \in \mathbb{H}$,

$$D_{-\delta}\varphi(\tau) = \varphi(\tau)\overline{\tau_2}\delta_2 h\left(\frac{\overline{\tau_2}\delta_2}{\overline{\tau_1}\delta_1}\right). \tag{2.3.2}$$

With the aid of generalized models we shall present an alternative, more algebraic, proof of this result. At the same time we obtain further information about directional derivatives at carapoints. We need a simple preliminary observation.

Lemma 2.3.2. *If $\mathcal{H}$ is a Hilbert space and Y is a positive contraction on $\mathcal{H}$, then*

$$H(z) = -\frac{1}{1-Y+zY}$$

is a well-defined $\mathcal{L}(\mathcal{H})$-valued analytic function on $\mathbb{C}\setminus(-\infty,0]$. Furthermore, $\operatorname{Im} H(z)$ and $-\operatorname{Im} zH(z)$ are both positive operators for all $z \in \Pi$, and $H(z)$ is Hermitian for $z \in (0,\infty)$.

Proof. For any $z \in \mathbb{C}$, the spectrum $\sigma(1-Y+zY)$ is contained in the convex hull of the points $1,\ z$, by the spectral mapping theorem, and therefore $(1-Y+zY)^{-1}$ is an analytic

[1] Actually the theorem is slightly more general in that it treats carapoints of φ in the *topological* boundary of $\mathbb{D}^2$.

function of z on the set $\mathbb{C} \setminus (-\infty, 0]$; it clearly takes Hermitian values on the interval $(0, \infty)$.

For any $z \in \Pi$ we have

$$\text{Im } (1 - Y + zY) = (\text{Im } z)Y \geq 0,$$

and since $-\text{Im } T^{-1}$ is congruent to $\text{Im } T$ for any invertible operator T, it follows that

$$\text{Im } H(z) = -\text{Im } (1 - Y + zY)^{-1} \geq 0.$$

Similarly

$$-\text{Im } (zH(z)) = -\text{Im } \frac{1 - Y + zY}{z} = -\text{Im } \frac{1 - Y}{z} = (1 - Y)\,\text{Im } \left(-\frac{1}{z}\right) \geq 0.$$

□

Proof of Theorem 2.3.1. Let (a, β, γ, D) be a realization of φ, associated with a model $(\mathcal{L}, v)$, and let (M, u, I) be the desingularization of this realization at τ. By Theorem 2.2.6, τ is a C-point of (M, u, I), and so there exists $u_\tau \in \mathcal{M}$ such that

$$\lim_{\lambda \overset{\text{nt}}{\to} \tau} u_\lambda = u_\tau$$

and, for all $\lambda, \mu \in \mathbb{D}^2$,

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = \left\langle (1 - I^*(\mu)I(\lambda))u_\lambda, u_\mu \right\rangle. \tag{2.3.3}$$

Take limits in the last equation as $\mu \overset{\text{nt}}{\to} \tau$ to obtain

$$1 - \overline{\varphi(\tau)}\varphi(\lambda) = \langle (1 - I(\lambda))u_\lambda, u_\tau \rangle.$$

On multiplying through by $-\varphi(\tau)$ we deduce that

$$\begin{aligned}\varphi(\lambda)-\varphi(\tau) &= \varphi(\tau)\left\langle (I(\lambda)-1)u_\lambda, u_\tau\right\rangle \\ &= \varphi(\tau)\left\langle (I(\lambda)-1)u_\tau, u_\tau\right\rangle + \varphi(\tau)\left\langle (I(\lambda)-1)(u_\lambda - u_\tau), u_\tau\right\rangle. \end{aligned} \tag{2.3.4}$$

Let $\delta \in \mathbb{H}(\tau)$, so that $\lambda_t \stackrel{\text{def}}{=} \tau - t\delta \in \mathbb{D}^2$ for small enough $t>0$. Then, from equation (2.2.17),

$$\begin{aligned} I(\lambda_t)-1 = I(\tau-t\delta)-1 &= -\bar\tau_1\bar\tau_2 \frac{t^2\delta_1\delta_2}{1-\bar\tau_1(\lambda_t)_1(1-Y)-\bar\tau_2(\lambda_t)_2 Y} \\ &= -\bar\tau_1\bar\tau_2\frac{t\delta_1\delta_2}{\bar\tau_1\delta_1(1-Y)+\bar\tau_2\delta_2 Y}. \end{aligned} \tag{2.3.5}$$

In combination with equation (2.3.4) this relation yields

$$\begin{aligned} \frac{\varphi(\lambda_t)-\varphi(\tau)}{t} &= -\varphi(\tau)\left\langle \frac{\delta_1\delta_2}{\tau_2\delta_1(1-Y)+\tau_1\delta_2 Y}u_\tau, u_\tau\right\rangle \\ &\qquad -\varphi(\tau)\left\langle \frac{\delta_1\delta_2}{\tau_2\delta_1(1-Y)+\tau_1\delta_2 Y}(u_{\lambda_t}-u_\tau), u_\tau\right\rangle, \end{aligned}$$

and on letting $t \to 0+$ we conclude that

$$\begin{aligned} D_{-\delta}\varphi(\tau) &= -\varphi(\tau)\left\langle \frac{\delta_1\delta_2}{\tau_2\delta_1(1-Y)+\tau_1\delta_2 Y}u_\tau, u_\tau\right\rangle \\ &= \varphi(\tau)\bar\tau_2\delta_2 h\left(\frac{\bar\tau_2\delta_2}{\bar\tau_1\delta_1}\right) \end{aligned}$$

where, for any $z \in \Pi$,

$$h(z) = -\left\langle \frac{1}{1-Y+zY}u_\tau, u_\tau\right\rangle = \left\langle H(z)u_\tau, u_\tau\right\rangle; \tag{2.3.6}$$

here $H(z)$ is as defined in Lemma 2.3.2. It is then immediate from Lemma 2.3.2 that h and $-zh(z)$ belong to the Pick class and that h is analytic on $\mathbb{C}\setminus(-\infty,0]$ and is real-valued on $(0,\infty)$.

It remains to prove equation (2.3.1). From the definition (2.3.6) we have

$$h(1) = -\|u_\tau\|^2, \tag{2.3.7}$$

while from the model equation (2.3.3), for any $\lambda \in \mathbb{D}^2$,

$$1 - |\varphi(\lambda)|^2 = \|u_\lambda\|^2 - \|I(\lambda)u_\lambda\|^2 .$$

Let $\lambda_t = \tau - t\tau$ for $t > 0$. By equation (2.3.5) we have

$$I(\lambda_t) - 1 = -t,$$

and so, for small enough $t > 0$,

$$1 - |\varphi(\lambda_t)|^2 = \left\|u_{\lambda_t}\right\|^2 - \left\|(1-t)u_{\lambda_t}\right\|^2 = (2t - t^2)\left\|u_{\lambda_t}\right\|^2 .$$

We also have

$$\|\lambda_t\|_\infty = \|\tau - t\tau\|_\infty = (1-t)\|\tau\|_\infty = 1 - t,$$

and so $1 - \|\lambda_t\|_\infty^2 = 2t - t^2 > 0$ for small t. Hence

$$\frac{1 - |\varphi(\lambda_t)|^2}{1 - \|\lambda_t\|_\infty^2} = \left\|u_{\lambda_t}\right\|^2$$

and therefore

$$\lim_{t\to 0+} \frac{1 - |\varphi(\lambda_t)|}{1 - \|\lambda_t\|_\infty} = \lim_{t\to 0+} \frac{1 - |\varphi(\lambda_t)|^2}{1 - \|\lambda_t\|_\infty^2} = \lim_{t\to 0+} \left\|u_{\lambda_t}\right\|^2 = \|u_\tau\|^2 .$$

Hence, by equation (2.3.7),

$$h(1) = -\lim_{t\to 0+} \frac{1-|\varphi(\lambda_t)|}{1-\|\lambda_t\|_\infty}.$$

However, it is known that, for any carapoint τ of φ,

$$\lim_{t\to 0+} \frac{1-|\varphi(\lambda_t)|}{1-\|\lambda_t\|_\infty} = \liminf_{\lambda\to\tau} \frac{1-|\varphi(\lambda)|}{1-\|\lambda\|_\infty},$$

(see for example [Jaf93] or [AMY12a, Corollary 4.14]). Equation (2.3.1) follows. □

We shall call the function h described in Theorem 2.3.1 the *slope function* of φ at the point τ. Thus h is the slope function of φ at a carapoint $\tau \in \mathbb{T}^2$ if, for all $\delta \in \mathbb{H}(\tau)$,

$$D_{-\delta}\varphi(\tau) = \varphi(\tau)\overline{\tau_2}\delta_2 h\left(\frac{\overline{\tau_2}\delta_2}{\overline{\tau_1}\delta_1}\right). \tag{2.3.8}$$

The foregoing proof shows that slope functions have the following representation.

Proposition 2.3.3. *Let $\tau \in \mathbb{T}^2$ be a carapoint for a function $\varphi \in \mathcal{S}_2$. There exists a Hilbert space $\mathcal{M}$, a vector $u_\tau \in \mathcal{M}$ and a positive contractive operator Y on $\mathcal{M}$ such that, for all $z \in \Pi$,*

$$h(z) = -\left\langle \frac{1}{1-Y+zY} u_\tau, u_\tau \right\rangle. \tag{2.3.9}$$

2.4 Integral representations of slope functions

Theorem 2.3.1 tells us that the directional derivative of a function $\varphi \in \mathcal{S}_2$ at a carapoint is encoded in a slope function h belonging to the Pick class $\mathcal{P}$ such that $-zh$ is also in $\mathcal{P}$. In this section we derive a representation of functions h with this property. To obtain such a description we shall need the following well-known theorem of Nevanlinna [Nev22], or see [Don74, Section II.2, Theorem I].

Theorem 2.4.1. *For every holomorphic function F on Π such that $\operatorname{Im} F(z) \geq 0$ there exist $c \in \mathbb{R}$, $d \geq 0$ and a finite non-negative Borel measure μ on $\mathbb{R}$ such that*

$$F(z) = c + dz + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \mathrm{d}\mu(t), \tag{2.4.1}$$

for all $z \in \Pi$. Moreover, the c, d and μ in the representation (2.4.1) *are uniquely determined, subject to $c \in \mathbb{R}$, $d \geq 0$, $\mu \geq 0$ and $\mu(\mathbb{R}) < \infty$.*

Conversely, any function F of the form (2.4.1) *is in the Pick class.*

We shall also need another classical theorem – the Stieltjes Inversion Formula [Don74, Section II.2, Lemma I].

Theorem 2.4.2. *Let V be a nonnegative harmonic function on Π, and suppose that V is the Poisson integral of a positive measure μ on $\mathbb{R}$:*

$$V(x + \mathrm{i}y) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{(t-x)^2 + y^2} \tag{2.4.2}$$

for some $c \geq 0$ and all $y > 0$, where

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty. \tag{2.4.3}$$

Then

$$\lim_{y \to 0+} \int_a^b V(x + \mathrm{i}y)\, \mathrm{d}x = \mu((a,b)) + \tfrac{1}{2}\mu(\{a\}) + \tfrac{1}{2}\mu(\{b\}) \tag{2.4.4}$$

whenever $-\infty < a < b < \infty$.

We can now identify the class of $h \in \mathcal{P}$ such that $-zh \in \mathcal{P}$.

Theorem 2.4.3. *The following are equivalent for any analytic function h on Π.*

(i) $h, -zh \in \mathcal{P}$;

(ii) $h \in \mathcal{P}$ *and the Nevanlinna representation of* h *has the form*

$$h(z) = c + dz + \frac{1}{\pi}\int \frac{1+tz}{t-z}\mathrm{d}\mu(t)$$

where

(a) $d = 0$*,*

(b) $\mu((0,\infty)) = 0$ *and*

(c) $c \le \frac{1}{\pi}\int t\,\mathrm{d}\mu(t)$*;*

(iii) *there exists a positive Borel measure* ν *on* $[0,1]$ *such that*

$$h(z) = -\int \frac{1}{1-s+sz}\mathrm{d}\nu(s).$$

Proof. (i)⇒(ii) Let h and $-zh$ be in the Pick class. Then there exist unique $c, c' \in \mathbb{R}, d, d' \ge 0$, and finite positive Borel measures μ, ν on $\mathbb{R}$ such that

$$h(z) = c + dz + \frac{1}{\pi}\int \frac{1+tz}{t-z}\mathrm{d}\mu(t) \tag{2.4.5}$$

and

$$-zh(z) = c' + d'z + \frac{1}{\pi}\int \frac{1+tz}{t-z}\mathrm{d}\nu(t). \tag{2.4.6}$$

If $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, then

$$\operatorname{Im}\frac{1+tz}{t-z} = \operatorname{Im}\left(\frac{1+t^2}{t-z} - t\right) = (1+t^2)\operatorname{Im}\frac{1}{t-z} = (1+t^2)\frac{y}{(t-x)^2+y^2}.$$

Hence

$$\operatorname{Im} h(z) = dy + \frac{y}{\pi} \int \frac{1}{(t-x)^2 + y^2} (1+t^2) \mathrm{d}\mu(t),$$
$$\operatorname{Im}(-zh(z)) = d'y + \frac{y}{\pi} \int \frac{1}{(t-x)^2 + y^2} (1+t^2) \mathrm{d}\nu(t).$$

Since $\operatorname{Im} h$ is nonnegative and harmonic, Theorem 2.4.2 implies that

$$\lim_{y \to 0^+} \int_a^b \operatorname{Im} h(x+iy) \mathrm{d}x = \mu((a,b)) + \frac{\mu(\{a\}) + \mu(\{b\})}{2} \tag{2.4.7}$$

and

$$\lim_{y \to 0^+} \int_a^b \operatorname{Im}(-zh(z)) \mathrm{d}x = \nu((a,b)) + \frac{\nu(\{a\}) + \nu(\{b\})}{2}. \tag{2.4.8}$$

Note that

$$\operatorname{Im}(-zh) = -\operatorname{Im}((x+iy)h) = -x \operatorname{Im} h - y \operatorname{Re} h, \tag{2.4.9}$$

and so

$$\lim_{y \to 0^+} \int_a^b \operatorname{Im}(-zh(z)) \, \mathrm{d}x = -\lim_{y \to 0^+} \int_a^b x \operatorname{Im} h(x+iy) \, \mathrm{d}x - \lim_{y \to 0^+} y \int_a^b \operatorname{Re} h(x+iy) \, \mathrm{d}x. \tag{2.4.10}$$

Now let

$$A_y = \int_a^b x \operatorname{Im} h(x+iy) \, \mathrm{d}x$$

and

$$B_y = y \int_a^b \operatorname{Re} h(x+iy) \, \mathrm{d}x,$$

so that

$$\lim_{y \to 0^+} \int_a^b \operatorname{Im}(-zh(z)) \, \mathrm{d}x = -\lim_{y \to 0^+} A_y - \lim_{y \to 0^+} B_y. \tag{2.4.11}$$

Lemma 2.4.4. *For any $a,b \in \mathbb{R}$ such that $a < b$,*

$$\lim_{y\to 0^+} B_y = \lim_{y\to 0^+} y \int_a^b \operatorname{Re} h(z)\, \mathrm{d}x = 0.$$

Proof. In view of the representation (2.4.5) of h we have

$$\begin{aligned}
B_y &= y \int_a^b \operatorname{Re} \int \frac{1+t(x+\mathrm{i}y)}{t-x-\mathrm{i}y}\, \mathrm{d}\mu(t)\, \mathrm{d}x \\
&= y \int_a^b \operatorname{Re} \int \frac{(1+t(x+\mathrm{i}y))(t-x+\mathrm{i}y)}{(t-x)^2+y^2}\, \mathrm{d}\mu(t)\, \mathrm{d}x \\
&= y \int_a^b \int \frac{(1+tx)(t-x)-ty^2}{(t-x)^2+y^2}\, \mathrm{d}\mu(t)\, \mathrm{d}x \\
&= y \int_a^b \int \frac{(t-x)(1+(t-x)x+x^2)-(t-x)y^2-xy^2}{(t-x)^2+y^2}\, \mathrm{d}\mu(t)\, \mathrm{d}x \\
&= y \int_a^b \int xC_2 + (1+x^2-y^2)C_1 - xy^2C_0\, \mathrm{d}\mu(t)\, \mathrm{d}x
\end{aligned} \tag{2.4.12}$$

where

$$C_2 = \frac{(t-x)^2}{(t-x)^2+y^2}, \quad C_1 = \frac{t-x}{(t-x)^2+y^2} \quad \text{and} \quad C_0 = \frac{1}{(t-x)^2+y^2}.$$

For all t,x in $\mathbb{R}$ and $y > 0$ we have $C_2 \le 1$ and $C_0 \le 1/y^2$, and so

$$\left| xC_2 - xy^2C_0 \right| \le 2|x|. \tag{2.4.13}$$

Choose $N \ge 1+\max\{|a|,|b|\}$ such that $\mu(\{N,-N\}) = 0$. Then

$$\begin{aligned}
\int_a^b \int |xC_2 - xy^2C_0|\, \mathrm{d}\mu(t)\, \mathrm{d}x &\le \int_a^b \int 2|x|\, \mathrm{d}\mu(t)\, \mathrm{d}x \\
&\le \mu(\mathbb{R}) \int_{-N}^{N} 2|x| \mathrm{d}x \\
&= 2N^2\mu(\mathbb{R}).
\end{aligned} \tag{2.4.14}$$

It is then immediate that

$$\lim_{y\to 0+} y\int_a^b \int |xC_2 - xy^2C_0|\ \mathrm{d}\mu(t)\ \mathrm{d}x = 0. \tag{2.4.15}$$

For $|t| \geq N,\ a \leq x \leq b$ we have $|t-x| \geq 1$, hence $|C_1| \leq 1$ and so

$$\begin{aligned}\int_a^b \int_{|t|\geq N} |(1+x^2+y^2)C_1|\ \mathrm{d}\mu(t)\ \mathrm{d}x &\leq \int_a^b \int 1+x^2+y^2\ \mathrm{d}\mu(t)\ \mathrm{d}x \\ &\leq \mu(\mathbb{R})(1+N^2+y^2)(b-a). \end{aligned}\tag{2.4.16}$$

On the other hand, when $|t| \leq N$ and $a \leq x \leq b$,

$$|(1+x^2+y^2)C_1| \leq (1+N^2+y^2)\frac{|t-x|}{(t-x)^2+y^2}.$$

On making the change of variable $s = |t-x|$ and observing that $0 \leq s \leq 2N$ when $|t| \leq N$ and $a \leq x \leq b$, we find that

$$\begin{aligned}\int_a^b |(1+x^2+y^2)C_1|\ \mathrm{d}x &\leq 2(1+N^2+y^2)\int_0^{2N} \frac{s\ \mathrm{d}s}{s^2+y^2} \\ &= (1+N^2+y^2)\left(\log(4N^2+y^2) - 2\log y\right),\end{aligned}$$

and therefore

$$\int_{|t|\leq N} \mathrm{d}\mu(t) \int_a^b |(1+x^2+y^2)C_1|\ \mathrm{d}x \leq \mu(\mathbb{R})(1+N^2+y^2)\left(\log(4N^2+y^2) - 2\log y\right) < \infty. \tag{2.4.17}$$

It follows from the Fubini-Tonelli theorem that the order of integration can be reversed, and on combining the estimates (2.4.16) and (2.4.17) we find that

$$\int_a^b \int |(1+x^2+y^2)C_1|\ \mathrm{d}\mu(t)\ \mathrm{d}x \leq \mu(\mathbb{R})(1+N^2+y^2)\left[b-a+\log(4N^2+y^2) - 2\log y\right],$$

from which it is clear that

$$\lim_{y\to 0+} y \int_a^b \int |(1+x^2+y^2)C_1| \, \mathrm{d}\mu(t) \, \mathrm{d}x = 0.$$

On combining this statement with (2.4.15) we conclude that

$$\lim_{y\to 0+} y \int_a^b \int |xC_2 + (1+x^2+y^2)C_1 - xy^2 C_0| \, \mathrm{d}\mu(t) \, \mathrm{d}x = 0$$

and hence, by equation (2.4.12), that $B_y \to 0$ as $y \to 0+$. $\square$

Since Im $h \geq 0$ we have

$$a \int_a^b \operatorname{Im} h(x+iy) \, \mathrm{d}x \leq A_y \leq b \int_a^b \operatorname{Im} h(x+iy) \mathrm{d}x.$$

Combining this inequality with (2.4.7) we find that

$$a\left(\mu(a,b) + \frac{\mu(\{a\})+\mu(\{b\})}{2}\right) \leq \lim_{y\to 0^+} A_y \leq b\left(\mu(a,b) + \frac{\mu(\{a\})+\mu(\{b\})}{2}\right)$$

and so, in view of equation (2.4.8), for all $a < b$,

$$-b\left(\mu(a,b) + \frac{\mu(\{a\})+\mu(\{b\})}{2}\right) \leq \nu((a,b)) + \frac{\nu(\{a\})+\nu(\{b\})}{2} \tag{2.4.18}$$

$$\leq -a\left(\mu(a,b) + \frac{\mu(\{a\})+\mu(\{b\})}{2}\right). \tag{2.4.19}$$

As this inequality holds for all $a < b \in \mathbb{R}$, we can let $a = 0$ and $b > 0$. Then

$$\nu((0,b)) + \frac{\nu(\{0\})+\nu(\{b\})}{2} \leq 0.$$

But as ν is a positive measure, this implies that $\nu((0,\infty)) = 0$ and $\nu(\{0\}) = 0$, i.e.

$\nu([0,\infty)) = 0$.

Now let $0 < a < b$. Then

$$a\left(\mu(a,b) + \frac{\mu(\{a\}) + \mu(\{b\})}{2}\right) \leq \left(\nu(a,b) + \frac{\nu(\{a\}) + \nu(\{b\})}{2}\right) = 0.$$

But $\mu \geq 0$, and so $\mu((a,b)) = 0$. It follows that $\mu((0,\infty)) = 0$, which is to say that condition (b) holds.

Fact 1. *For $t < 0, \nu(\{t\}) = -t\mu(\{t\})$.*

Since μ, ν are finite and positive, they can only have at most countably many point masses, and so we may choose a sequence of intervals $(a_n, b_n) \subset (-\infty, 0)$ such that $\mu(\{a_n\}) = \mu(\{b_n\}) = \nu(\{a_n\}) = \nu(\{b_n\}) = 0$, $t \in (a_n, b_n)$ for all n and $\bigcap (a_n, b_n) = \{t\}$. Inequality (2.4.18) implies that

$$-b_n\mu((a_n,b_n)) \leq \nu((a_n,b_n)) \leq -a_n\mu((a_n,b_n)),$$

and in the limit we obtain

$$-t\mu(\{t\}) = \nu(\{t\}) \leq -t\mu(\{t\}).$$

If σ is a finite positive measure on $(-\infty,0)$, we shall call a finite partition $P = \{x_1,\ldots,x_n\}$, where $x_1 < x_2 < \ldots < x_n < 0$, *special for* σ if $\sigma(P) = 0$.

Fact 2. *If f is continuous on $(-\infty,0)$ with compact support and $\varepsilon > 0$, there exists a partition P that is special for σ such that*

$$\left|\int f \, \mathrm{d}\sigma - S(f,P)\right| < \varepsilon,$$

where $S(f,P)$ denotes the Riemann sum of f over P.

Lemma 2.4.5. *If f is a continuous function of compact support on $(-\infty,0)$ then*

$$\int f \, \mathrm{d}\mu = \int f(t)\, t\mathrm{d}\nu(t).$$

Equations (2.4.5) and (2.4.6) give us two different expressions for $-zh(z)$:

$$-z\left(c+dz+\frac{1}{\pi}\int\frac{1+tz}{t-z}\,\mathrm{d}\mu(t)\right)=c'+d'z+\frac{1}{\pi}\int\frac{1+tz}{t-z}\,\mathrm{d}\nu(t).$$

Hence, by Lemma 2.4.5,

$$-z\left(c+dz+\frac{1}{\pi}\int\frac{1+tz}{t-z}\,\mathrm{d}\mu(t)\right)=c'+d'z+\frac{1}{\pi}\int\frac{1+tz}{t-z}\,(-t\mathrm{d}\mu(t)).$$

We may rearrange this equation, in order to compare polynomials, obtaining

$$\begin{aligned} c'+(d'+c)z+dz^2 &= \frac{1}{\pi}\int\frac{1+tz}{t-z}(t-z)\,\mathrm{d}\mu(t)\\ &= \frac{1}{\pi}\int(1+tz)\,\mathrm{d}\mu(t)\\ &= \frac{1}{\pi}\int\,\mathrm{d}\mu(t)+\left(\frac{1}{\pi}\int t\mathrm{d}\mu(t)\right)z. \end{aligned}$$

We immediately see that

$$c'=\frac{1}{\pi}\int\,\mathrm{d}\mu,\quad d'+c=\frac{1}{\pi}\int t\,\mathrm{d}\mu(t)\quad\text{and}\quad d=0.$$

The last of these statements is condition (a) in (ii). Since $d'>0$ the second statement tells us that

$$c\le d'+c=\int t\,\mathrm{d}\mu(t),$$

which is condition (c). This concludes the proof that (i)⇒(ii).

(ii)⇒(iii) Suppose that $h\in\mathcal{P}$ has a Nevanlinna representation that satisfies conditions (a)-(c) of (ii). Then, for $z \in \Pi$,

$$\begin{aligned} h(z) &= c + \frac{1}{\pi}\int \frac{1+tz}{t-z}\,\mathrm{d}\mu(t) \\ &= c + \frac{1}{\pi}\int \left(\frac{1+t^2}{t-z} - t\right)\,\mathrm{d}\mu(t) \\ &= c - \frac{1}{\pi}\int t\,\mathrm{d}\mu(t) + \frac{1}{\pi}\int \frac{1-t}{t-z}\frac{1+t^2}{1-t}\,\mathrm{d}\mu(t). \end{aligned} \tag{2.4.20}$$

Since the indefinite integral of $\frac{1+t^2}{1-t}\,\mathrm{d}\mu(t)$ is a finite positive measure on $(-\infty,0]$, we may define a finite positive Borel measure ν on $[0,1]$ by

$$\nu(\{0\}) = \frac{1}{\pi}\int t\,\mathrm{d}\mu(t) - c, \tag{2.4.21}$$

$$\nu(E) = \frac{1}{\pi}\int_{\tilde{E}} \frac{1+t^2}{1-t}\,\mathrm{d}\mu(t) \tag{2.4.22}$$

for any Borel set $E \subset (0,1]$, where $\tilde{E} \stackrel{\text{def}}{=} \{1-1/s : s \in E\}$.

With this definition, if ψ is a continuous bounded function on $(-\infty,0]$,

$$\frac{1}{\pi}\int \psi(t)\frac{1+t^2}{1-t}\,\mathrm{d}\mu(t) = \int_{(0,1]} \psi\left(1-\frac{1}{s}\right)\,\mathrm{d}\nu(s).$$

From equations (2.4.20) and (2.4.22),

$$\begin{aligned} h(z) &= c - \frac{1}{\pi}\int t\,\mathrm{d}\mu(t) + \frac{1}{\pi}\int \left(\frac{1-t}{t-z}\right)\frac{1+t^2}{1-t}\,\mathrm{d}\mu(t) \\ &= -\nu(\{0\}) + \int_{(0,1]} \frac{1-(1-\frac{1}{s})}{1-\frac{1}{s}-z}\,\mathrm{d}\nu(s) \\ &= -\nu(\{0\}) + \int_{(0,1]} \frac{1}{s-1-sz}\,\mathrm{d}\nu(s) \\ &= -\left[\nu(\{0\}) + \int_{(0,1]} \frac{1}{1-s+sz}\,\mathrm{d}\nu(s)\right] \\ &= -\int_{[0,1]} \frac{1}{1-s+sz}\,\mathrm{d}\nu(s), \end{aligned}$$

which completes the proof that (ii)⇒(iii).

(iii)⇒(i) Suppose that ν is a positive finite Borel measure on $[0,1]$ and

$$h(z) = -\int \frac{1}{1-s+sz}\, d\nu(s)$$

for all $z \in \Pi$. Let Y be the operator of multiplication by the independent variable s on $L^2(\nu)$. Evidently Y is a positive contraction, and hence, by Lemma 2.3.2, for any $z \in \Pi$, the operators

$$-\operatorname{Im}\,(1-Y+Yz)^{-1} \quad \text{and} \quad \operatorname{Im}\,\left(z(1-Y+Yz)^{-1}\right)$$

on $L^2(\nu)$ are positive definite. Since

$$\operatorname{Im}\, h(z) = -\operatorname{Im}\int \frac{1}{1-s+sz}\, d\nu(s) = \left\langle -\operatorname{Im}\, \frac{1}{1-Y+Yz} 1, 1 \right\rangle_{L^2(\nu)} \geq 0$$

and likewise

$$\operatorname{Im}\,(-zh(z)) = \operatorname{Im}\int \frac{z}{1-s+sz}\, d\nu(s) = \left\langle \operatorname{Im}\, \frac{z}{1-Y+Yz} 1, 1 \right\rangle_{L^2(\nu)} \geq 0,$$

it follows that (i) holds. □

The proof shows that if h and $-zh$ belong to $\mathcal{P}$ then h is analytic on $(0,\infty)$.

2.5 Functions with prescribed slope function

In this section we prove a converse to Theorem 2.3.1: we construct, for any function $h \in \mathcal{P}$ such that $-zh \in \mathcal{P}$, a function $\varphi \in \mathcal{S}_2$ with slope function h at a carapoint.

We shall need the following simple observation about the Cayley transform (an aplication of the quotient rule). The *two-variable Herglotz class* is defined to be the set of analytic functions on $\mathbb{D}^2$ with non-negative real part.

Lemma 2.5.1. *If f is a function in the two-variable Herglotz class then the function φ on $\mathbb{D}^2$ given by*

$$\varphi = \frac{1-f}{1+f}$$

belongs to $\mathcal{S}_2$. Furthermore, if $\tau \in \mathbb{T}^2$ is such that the radial limit

$$f(\tau) \stackrel{\text{def}}{=} \lim_{r \to 1-} f(r\tau)$$

exists and is not -1 and if the directional derivative $D_{-\delta} f(\tau)$ exists for some direction δ, then so does $D_{-\delta}\varphi(\tau)$, and

$$D_{-\delta}\varphi(\tau) = \frac{-2D_{-\delta} f(\tau)}{(1+f(\tau))^2}. \tag{2.5.1}$$

Recall that χ denotes the point $(1,1)$.

Theorem 2.5.2. *If $h \in \mathcal{P}$ and $-zh \in \mathcal{P}$ then there exists $\varphi \in \mathcal{S}_2$ such that χ is a carapoint for φ, $\varphi(\chi) = 1$ and h is the slope function for φ at χ.*

Proof. By Theorem 2.3.1 there exists a positive Borel measure ν on $[0,1]$ such that

$$h(z) = -\int \frac{1}{1-s+sz}\, \mathrm{d}\nu(s).$$

Define a family of functions f_s on $\mathbb{D}^2$ for $s \in [0,1]$ by

$$f_s(\lambda) = \left(s\frac{1+\lambda_1}{1-\lambda_1} + (1-s)\frac{1+\lambda_2}{1-\lambda_2} \right)^{-1}.$$

For any $\lambda \in \mathbb{D}^2$ the denominator on the right hand side is a convex combination of two points in $\mathbb{H}$, hence belongs to $\mathbb{H}$. Thus f_s lies in the two-variable Herglotz class for $0 \le s \le 1$. Moreover, for $0 < r < 1$ and every $s \in [0,1]$,

$$f_s(r\chi) = \frac{1-r}{1+r} \tag{2.5.2}$$

and hence the radial limit

$$f_s(\chi) \stackrel{\text{def}}{=} \lim_{r \to 1-} f_s(r\chi) = \lim_{r \to 1-} \frac{1-r}{1+r}$$

exists and is zero. We compute $D_{-\delta} f_s(\chi)$.

$$\begin{aligned} \frac{f_s(\chi - t\delta) - f(\chi)}{t} &= \frac{1}{t}\left(s\frac{1+1-t\delta_1}{1-(1-t\delta_1)} + (1-s)\frac{1+1-t\delta_2}{1-(1-t\delta_2)} \right)^{-1} \\ &= \left(s\frac{2}{\delta_1} + (1-s)\frac{2}{\delta_2} - t \right)^{-1} \\ &= \frac{1}{2} \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2 - \frac{1}{2}t\delta_1\delta_2}. \end{aligned} \tag{2.5.3}$$

On letting $t \to 0$ we obtain

$$D_{-\delta} f_s(\chi) = \frac{1}{2} \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}. \tag{2.5.4}$$

Define a function f on $\mathbb{D}^2$ by

$$f(\lambda) = \int f_s(\lambda)\, \mathrm{d}\nu(s).$$

Since $\operatorname{Re} f_s(\lambda) > 0$ for every $s \in [0,1]$, it is clear that f lies in the two-variable Herglotz class. Furthermore, in view of equation (2.5.2), for $0 < r < 1$,

$$f(r\chi) = \frac{1-r}{1+r} \nu[0,1] \tag{2.5.5}$$

and f has radial limit 0 at χ:

$$f(\chi) \stackrel{\text{def}}{=} \lim_{r\to 1-} \int f_s(r\chi)\, \mathrm{d}\nu(s) = \lim_{r\to 1-} \int \frac{1-r}{1+r}\, \mathrm{d}\nu(s) = \lim_{r\to 1-} \nu[0,1]\frac{1-r}{1+r} = 0. \quad (2.5.6)$$

Let us calculate the directional derivative of f at χ in the direction $-\delta$ where $\delta \in \mathbb{H}\times\mathbb{H}$. Equation (2.5.4) suggests that

$$D_{-\delta}f(\chi) = \frac{1}{2}\int \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}\, \mathrm{d}\nu(s). \quad (2.5.7)$$

We must verify that this is correct. By equation (2.5.3), we have, for small $t>0$,

$$\begin{aligned}\frac{f(\chi - t\delta) - f(\chi)}{t} &- \frac{1}{2}\int \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}\, \mathrm{d}\nu(s) \\ &= \frac{1}{2}\int \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2 - \frac{1}{2}t\delta_1\delta_2} - \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}\, \mathrm{d}\nu(s) \\ &= \frac{\delta_1\delta_2}{2}\int \frac{\frac{1}{2}t\delta_1\delta_2\, \mathrm{d}\nu(s)}{\left((1-s)\delta_1 + s\delta_2 - \frac{1}{2}t\delta_1\delta_2\right)\left((1-s)\delta_1 + s\delta_2\right)}. \quad (2.5.8)\end{aligned}$$

Since $\delta_1, \delta_2 \in \mathbb{H}$, the distance K from 0 to the convex hull of $\{\delta_1, \delta_2\}$ is positive. For sufficiently small $t>0$ we have, for all $s\in[0,1]$,

$$|(1-s)\delta_1 + s\delta_2 - \tfrac{1}{2}t\delta_1\delta_2| \geq \tfrac{1}{2}K,$$

and for such t the denominator of the integrand in equation (2.5.8) is at least $\frac{1}{2}K^2$. It follows that, for small enough t,

$$\left|\frac{f(\chi - t\delta) - f(\chi)}{t} - \frac{1}{2}\int \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}\, \mathrm{d}\nu(s)\right| \leq \frac{|\delta_1\delta_2|^2\nu[0,1]}{2K^2}t,$$

and hence equation (2.5.7) is correct.

Let φ be defined by

$$\varphi = \frac{1-f}{1+f}.$$

We claim that χ is a carapoint for φ. For any $\lambda \in \mathbb{D}^2$,

$$\frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|_\infty^2} = \frac{4\,\mathrm{Re}\; f(\lambda)}{(1-\|\lambda\|_\infty^2)|1+f(\lambda)|^2}$$

and so, by equation (2.5.5),

$$\begin{aligned}\frac{1-|\varphi(r\chi)|^2}{1-\|r\chi\|_\infty^2} &= \frac{4\nu[0,1]}{(1+r+(1-r)\nu[0,1])^2} \\ &\to \nu[0,1] \quad \text{as } r \to 1-.\end{aligned}$$

Hence

$$\liminf_{\lambda\to\chi} \frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|_\infty^2} \le \nu[0,1] < \infty$$

and χ is a carapoint for φ.

In view of equation (2.5.6) it is clear that φ has radial limit 1 at χ, that is to say, $\varphi(\chi) = 1$. By Lemma 2.5.1, φ lies in $\mathcal{S}_2$ and has directional derivative at χ given by

$$\begin{aligned}D_{-\delta}\varphi(\chi) &= \frac{-2D_{-\delta}f(\chi)}{(1+f(\chi))^2} \\ &= (-2)\frac{1}{2}\int \frac{\delta_1\delta_2}{(1-s)\delta_1 + s\delta_2}\, \mathrm{d}\nu(s) \\ &= -\delta_2 h\left(\frac{\delta_2}{\delta_1}\right) \\ &= -\varphi(\chi)\delta_2 h\left(\frac{\delta_2}{\delta_1}\right).\end{aligned}$$

Thus h is the slope function for $\varphi \in \mathcal{S}_2$ at the point χ. □

By a simple change of variable we obtain the following.

Corollary 2.5.3. *Let* $\omega \in \mathbb{T}$*, let* $\tau \in \mathbb{T}^2$ *and let* $h,\ -zh(z) \in \mathcal{P}$*. There exists a function* $\varphi \in \mathcal{S}_2$ *having a carapoint at* τ *such that* $\varphi(\tau) = \omega$ *and* h *is the slope function of* φ *at* τ*.*

2.6 Nevanlinna representations in two variables

The following refinement of Theorem 2.4.1, also due to Nevanlinna, is the main tool in one of the standard proofs of the Spectral Theorem for unbounded self-adjoint operators [Lax02].

Proposition 2.6.1. *Let* $h \in \mathcal{P}$*. If*

$$\lim_{y \to \infty} y \operatorname{Im} h(iy) < \infty \tag{2.6.1}$$

then there exists a finite positive measure μ *on* $\mathbb{R}$ *such that, for all* $z \in \Pi$*,*

$$h(z) = \int \frac{\mathrm{d}\mu(t)}{t - z} \ . \tag{2.6.2}$$

For a proof see [Lax02].

In this section we shall generalize Proposition 2.6.1 to two variables. We need an analog for the Cauchy transform formula (2.6.2). The closest one we can find involves the two-variable resolvent of a self-adjoint operator B on a Hilbert space $\mathcal{M}$, to wit

$$h(z_1, z_2) = b - \left\langle (B + z_1 Y + z_2(1 - Y))^{-1} \alpha, \alpha \right\rangle$$

for some $b \in \mathbb{R},\ \alpha \in \mathcal{M}$ and some positive contraction Y on $\mathcal{M}$. In an earlier paper [AMY12b, Theorem 6.9] we obtained a somewhat similar result, but with the unsatisfactory feature that the representation obtained was not of h itself but rather of a "twist" of h. The use of generalized models enables us to remedy this defect.

The growth condition (2.6.1) is expressible in terms of carapoints of the Schur-class function φ associated with h by the definition

$$\varphi(\lambda) = \frac{h(z)-i}{h(z)+i} \quad \text{where} \quad z = i\frac{1+\lambda}{1-\lambda}. \tag{2.6.3}$$

Let us establish the corresponding assertion for functions of two variables. We denote by $\mathcal{P}_2$ the two-variable Pick class, that is the set of analytic functions on Π^2 with non-negative imaginary part and we recall that χ denotes $(1,1)$.

Proposition 2.6.2. *Let $h \in \mathcal{P}_2$ and let $\varphi \in \mathcal{S}_2$ be defined by*

$$\varphi(\lambda) = \frac{h(z)-i}{h(z)+i} \quad \textit{where } z_j = i\frac{1+\lambda_j}{1-\lambda_j}, \quad j = 1,2, \tag{2.6.4}$$

for $\lambda \in \Pi^2$. The following conditions are equivalent.

1. $\liminf_{y\to\infty} y \operatorname{Im} h(iy\chi) < \infty$;
2. $\lim_{y\to\infty} y \operatorname{Im} h(iy\chi)$ *exists and is finite;*
3. χ *is a carapoint for* φ *and* $\varphi(\chi) \neq 1$;
4. $(0,0)$ *is a carapoint for the function* $H \in \mathcal{P}_2$ *given by* $H(z) = h(-1/z_1, -1/z_2)$.

Proof. (2)⇒(1) is trivial.

(1)⇒(3) Suppose (1) holds and let β be the limit inferior in (1). There is a sequence (y_n) in $\mathbb{R}^+$ such that $y_n \to \infty$ and

$$\lim_{n\to\infty} y_n \operatorname{Im} h(iy_n\chi) = \beta.$$

Let

$$r_n = \frac{iy_n - i}{iy_n + i} = \frac{y_n - 1}{y_n + 1}.$$

Then $y_n = \frac{1+r_n}{1-r_n}$ and $r_n \to 1-$ as $n \to \infty$. From the relation

$$1 - |\varphi(\lambda)|^2 = \frac{4\,\mathrm{Im}\ h(z)}{|h(z)+i|^2}$$

we have

$$1 - |\varphi(r_n\chi)|^2 = \frac{4\,\mathrm{Im}\ h(iy_n\chi)}{|h(iy_n\chi)+i|^2}.$$

Since $|h(z)+i| \geq 1$ for all $z \in \Pi^2$,

$$1 - |\varphi(r_n\chi)|^2 \leq 4\,\mathrm{Im}\ h(iy_n\chi).$$

Similarly

$$\begin{aligned} 1 - \|r_n\chi\|_\infty^2 = 1 - r_n^2 &= \frac{4\,\mathrm{Im}\ iy_n}{|iy_n+i|^2} \\ &= \frac{4y_n}{(1+y_n)^2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1 - |\varphi(r_n\chi)|^2}{1 - \|r_n\chi\|_\infty^2} &\leq 4\,\mathrm{Im}\ h(iy_n\chi)\frac{(1+y_n)^2}{4y_n} \\ &\to \beta \quad \text{as } n \to \infty. \end{aligned}$$

Consequently

$$\liminf_{\lambda\to\chi} \frac{1 - |\varphi(\lambda)|^2}{1 - \|\lambda\|_\infty^2} \leq \beta < \infty,$$

and so χ is a carapoint for φ.

By the Carathéodory-Julia theorem for the bidisc [Jaf93, AMY12a],

$$\alpha \stackrel{\text{def}}{=} \liminf_{\lambda\to\chi} \frac{1 - |\varphi(\lambda)|^2}{1 - \|\lambda\|_\infty^2} = \lim_{r\to 1-} \frac{1 - |\varphi(r\chi)|^2}{1 - r^2} > 0 \tag{2.6.5}$$

and $\beta \neq 0$ since $\alpha \leq \beta$. Now for any $r \in (0,1)$, $y = \frac{1+r}{1-r}$, a simple calculation shows that

$$\begin{aligned} y\,\mathrm{Im}\; h(iy\chi) &= \frac{1+r}{1-r}\frac{1-|\varphi(r\chi)|^2}{|1-\varphi(r\chi)|^2} \\ &= \frac{(1+r)^2}{|1-\varphi(r\chi)|^2}\frac{1-|\varphi(r\chi)|^2}{1-r^2}. \end{aligned} \tag{2.6.6}$$

On putting $r = r_n$ and letting $n \to \infty$ we find that

$$\lim_{n\to\infty} |1-\varphi(r_n\chi)|^2 = \frac{4\alpha}{\beta} \neq 0.$$

Thus $\varphi(\chi) \neq 1$. Hence (1)⇒(3).

(3)⇒(2) Suppose (3). Then the quantity α defined by equation (2.6.5) satisfies $0 < \alpha < \infty$. On letting $y \to \infty$ (and hence $r \to 1-$) in equation (2.6.6) we obtain

$$\lim_{y\to\infty} y\,\mathrm{Im}\; h(iy\chi) = \frac{4\alpha}{|1-\varphi(\chi)|^2},$$

and so (2) holds.

(2)⇔(4) According to Defintion 1.3.1, the point $(0,0)$ is a carapoint for $H \in \mathcal{P}_2$ if

$$\liminf_{z\to(0,0)} \frac{\mathrm{Im}\; H(z)}{\min\{\mathrm{Im}\; z_1, \mathrm{Im}\; z_2\}} < \infty,$$

and by the two-variable Carathéodory-Julia theorem [Jaf93, AMY12a], this is so if and only if

$$\liminf_{\eta\to 0} \frac{\mathrm{Im}\; H(i\eta\chi)}{\eta} = \liminf_{\eta\to 0} \frac{\mathrm{Im}\; h(i\chi/\eta)}{\eta} < \infty.$$

On setting $y = 1/\eta$ we deduce that (2)⇔(4). □

We shall say that ∞ is a *carapoint for $h \in \mathcal{P}_2$ with finite value* if the equivalent conditions of Proposition 2.6.2 hold. We define the value $h(\infty)$ in this case by

$$h(\infty) = \lim_{y\to\infty} h(iy\chi) = H(0,0) = i\frac{1+\varphi(\chi)}{1-\varphi(\chi)}$$

where H, φ are as in Proposition 2.6.2. There is also a notion of carapoint of h with infinite value: see [ATDY, Section 7].

Here is our generalization of the Nevanlinna representation (2.6.2) to the two-variable Pick class.

Theorem 2.6.3. *The following statements are equivalent for a function* $h : \Pi^2 \to \mathbb{C}$.

1. *h is in the Pick class $\mathcal{P}_2$, ∞ is a carapoint for h with finite value;*

2. *there exist a scalar $b \in \mathbb{R}$, a Hilbert space $\mathcal{M}$, a vector $\alpha \in \mathcal{M}$, a positive contraction Y on $\mathcal{M}$ and a densely defined self-adjoint operator B on $\mathcal{M}$ such that, for all $z \in \Pi^2$,*

$$h(z) = b - \left\langle (B + z_1 Y + z_2(1-Y))^{-1}\alpha, \alpha \right\rangle. \tag{2.6.7}$$

Proof. We begin by observing that the inverse in equation (2.6.7) exists for any $z \in \Pi^2$. Write $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$, with $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 > 0$ and let $T = B + z_1 Y + z_2(1-Y)$. We have, for any $u \in \mathcal{M}$,

$$\begin{aligned} \text{Im } \langle Tu, u\rangle &= y_1 \langle Yu, u\rangle + y_2 \langle (1-Y)u, u\rangle \\ &\geq \min\{y_1, y_2\} \|u\|^2, \end{aligned}$$

and therefore

$$\|Tu\|\,\|u\| \geq |\langle Tu, u\rangle| \geq \text{Im } \langle Tu, u\rangle \geq \min\{y_1, y_2\}\ \|u\|^2.$$

Thus the operator T has the positive lower bound $\min\{y_1, y_2\}$, and so has a left inverse.

A similar argument with z_j replaced by its complex conjugate shows that T^* also has a left inverse. Hence $B+z_1Y+z_2(1-Y)$ is invertible for all $z\in\Pi^2$, and clearly the two-variable resolvent $(B+z_1Y+z_2(1-Y))^{-1}$ is analytic on Π^2.

(2)⇒(1) Suppose that a representation of the form (2.6.7) holds for h. Then h is analytic on Π^2. For any invertible operator T, Im (T^{-1}) is congruent to $-$Im T, and so

$$\text{Im }(B+z_1Y+z_2(1-Y))^{-1} \text{ is congruent to } -(\text{Im } z_1)Y-(\text{Im } z_2)(1-Y).$$

Since the last operator is negative, it follows that Im $h(z)\geq 0$ for all $z\in\Pi^2$, and so $h\in\mathcal{P}_2$.

To see that ∞ is a carapoint for h note that

$$y\,\text{Im }h(iy\chi) = -y\,\text{Im }\left\langle (B+iy)^{-1}\alpha,\alpha\right\rangle.$$

Now

$$\text{Im }(B+iy)^{-1} = -y(B+iy)^{-1}(B-iy)^{-1}.$$

Let the spectral representation of B be

$$B=\int t\,\mathrm{d}E(t).$$

Then

$$\begin{aligned}
y\,\text{Im }h(iy\chi) &= y^2\left\langle (B+iy)^{-1}(B-iy)^{-1}\alpha,\alpha\right\rangle\\
&= y^2\int \frac{1}{(t+iy)(t-iy)}\,\langle \mathrm{d}E(t)\alpha,\alpha\rangle\\
&= \int \frac{y^2}{t^2+y^2}\,\langle \mathrm{d}E(t)\alpha,\alpha\rangle\\
&\to \int \langle \mathrm{d}E(t)\alpha,\alpha\rangle = \|\alpha\|^2 \quad \text{as } y\to\infty
\end{aligned}$$

by the Dominated Convergence Theorem. Hence ∞ is a carapoint for h with finite value.

(1)⇒(2) Suppose that (1) holds and let $\varphi \in \mathcal{S}_2$ be the Schur-class function associated with h by equations (2.6.4). By Proposition 2.6.2, $\chi = (1,1)$ is a carapoint for φ and $\varphi(\chi) \neq 1$.

By Theorem 2.2.6 there exists a generalized model $(\mathcal{M}, u, I)$ of φ having χ as a C-point and an accompanying unitary realization (a, β, γ, Q) of $(\mathcal{M}, u, I)$ with $\ker(1-Q) = \{0\}$. Moreover I is expressible by the formula (2.2.7) (with $\tau_1 = \tau_2 = 1$) for some positive contraction Y on $\mathcal{M}$. Thus

$$L = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & Q \end{bmatrix}$$

is unitary on $\mathbb{C} \oplus \mathcal{M}$ and

$$L \begin{pmatrix} 1 \\ I(\lambda) u_\lambda \end{pmatrix} = \begin{pmatrix} \varphi(\lambda) \\ u_\lambda \end{pmatrix}. \tag{2.6.8}$$

We wish to define the Cayley transform J of L:

$$J = i \frac{1+L}{1-L}.$$

Of course $1-L$ may not be invertible, and so we define J as an operator from $\operatorname{ran}(1-L)$ to $\operatorname{ran}(1+L)$ by

$$J(1-L)x = i(1+L)x. \tag{2.6.9}$$

This equation does define J as an operator, in view of the following observation.

Proposition 2.6.4. *If χ is a B-point for φ such that $\varphi(\chi) \neq 1$ and (a, β, γ, Q) is a realization of a generalized model of φ such that* $\ker(1-Q) = \{0\}$, *then* $\ker(1-L) = \{0\}$.

Proof. Let $x \in \ker(1-L) \subset \mathbb{C} \oplus \mathcal{M}$ and suppose $x \neq 0$. Since $\ker(1-Q) = \{0\}$, it cannot be that $x \in \mathcal{M}$, and so we can suppose that $x = \begin{pmatrix} 1 \\ x_0 \end{pmatrix}$ for some $x_0 \in \mathcal{M}$. Then

$$\begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & Q \end{bmatrix} \begin{pmatrix} 1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 1 \\ x_0 \end{pmatrix},$$

which implies that

$$\begin{aligned} a + \langle x_0, b \rangle &= 1 \qquad (2.6.10) \\ \gamma + Q x_0 &= x_0, \end{aligned}$$

and hence

$$(1 - Q) x_0 = \gamma. \qquad (2.6.11)$$

By equation (2.6.8),

$$u_\lambda = \gamma + Q I(\lambda) u_\lambda. \qquad (2.6.12)$$

Since χ is a C-point of the generalized model $(\mathcal{M}, u, I)$, there is a vector $u_\chi \in \mathcal{M}$ such that $u_\lambda \to u_\chi$ as $\lambda \stackrel{\text{nt}}{\to} \chi$. On taking nontangential limits in equation (2.6.12) we obtain

$$u_\chi = \gamma + Q u_\chi,$$

and so

$$(1 - Q) u_\chi = \gamma. \qquad (2.6.13)$$

On comparing this relation with equation (2.6.11) and using the fact that $\ker(1 - Q) = \{0\}$ we deduce that $x_0 = u_\chi$. Again by equation (2.6.8),

$$\varphi(\lambda) = a + \langle I(\lambda) u_\lambda, \beta \rangle .$$

Let $\lambda \overset{\text{nt}}{\to} \chi$: then $I(\lambda) \to 1$ and so

$$\varphi(\chi) = a + \langle u_\chi, \beta \rangle = a + \langle x_0, \beta \rangle \,.$$

In view of equation (2.6.10) we have $\varphi(\chi) = 1$, contrary to hypothesis. Thus $\ker(1-L) = \{0\}$. □

We have shown that $J : \operatorname{ran}(1-L) \to \mathbb{C} \oplus \mathcal{M}$ is well defined by equation (2.6.9). Moreover $\operatorname{ran}(1-L)$ is dense in $\mathbb{C} \oplus \mathcal{M}$, since

$$\operatorname{ran}(1-L)^\perp = \ker(1-L^*) = \ker(1-L) = \{0\}.$$

Thus J is a densely defined operator on $\mathbb{C} \oplus \mathcal{M}$, and since L is unitary, J is self-adjoint.

The next step is to derive a matricial representation of J on $\mathbb{C} \oplus \mathcal{M}$. By the definition (2.6.9) of J and equation (2.6.8),

$$J(1-L) \begin{pmatrix} 1 \\ I(\lambda) u_\lambda \end{pmatrix} = i(1+L) \begin{pmatrix} 1 \\ I(\lambda) u_\lambda \end{pmatrix}$$

and therefore

$$J \begin{pmatrix} 1 - \varphi(\lambda) \\ (I(\lambda) - 1) u_\lambda \end{pmatrix} = i \begin{pmatrix} 1 + \varphi(\lambda) \\ (I(\lambda) + 1) u_\lambda \end{pmatrix}.$$

Divide through by $1 - \varphi(\lambda)$ to get

$$J \begin{pmatrix} 1 \\ \dfrac{I(\lambda) - 1}{1 - \varphi(\lambda)} u_\lambda \end{pmatrix} = \begin{pmatrix} i \dfrac{1 + \varphi(\lambda)}{1 - \varphi(\lambda)} \\ i \dfrac{I(\lambda) + 1}{1 - \varphi(\lambda)} u_\lambda \end{pmatrix}. \tag{2.6.14}$$

Define $v : \Pi^2 \to \mathcal{M}$ by

$$v_z = -\frac{I(\lambda)-1}{1-\varphi(\lambda)} u_\lambda. \tag{2.6.15}$$

Recall that (compare equation (2.2.17))

$$I(\lambda)-1 = -\frac{(\lambda_1-1)(\lambda_2-1)}{1-\lambda_1(1-Y)-\lambda_2 Y},$$

and hence $I(\lambda)-1$ is invertible for $\lambda \in \mathbb{D}^2$. We have

$$\begin{aligned} i\frac{I(\lambda)+1}{1-\varphi(\lambda)} u_\lambda &= i\frac{I(\lambda)+1}{I(\lambda)-1}\left[\frac{I(\lambda)-1}{1-\varphi(\lambda)} u_\lambda\right] \\ &= i\frac{1+I(\lambda)}{1-I(\lambda)} v_z. \end{aligned}$$

A straightforward calculation now yields the appealing formula

$$i\frac{1+I(\lambda)}{1-I(\lambda)} = z_1 Y + z_2(1-Y).$$

Thus equation (2.6.14) becomes

$$J\begin{pmatrix} 1 \\ -v_z \end{pmatrix} = \begin{pmatrix} h(z) \\ (z_1 Y + z_2(1-Y))\, v_z \end{pmatrix}. \tag{2.6.16}$$

We wish to write J as an operator matrix

$$J = \begin{bmatrix} b & 1\otimes\alpha \\ \alpha\otimes 1 & B \end{bmatrix} \tag{2.6.17}$$

on $\mathbb{C}\oplus\mathcal{M}$, but in order for this to make sense we require that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be in the domain of

J, which is $\operatorname{ran}(1-L)$. We must show that there exists a vector $\begin{pmatrix} c \\ x \end{pmatrix}$ such that

$$\begin{bmatrix} 1-a & -1\otimes\beta \\ -\gamma\otimes 1 & 1-Q \end{bmatrix} \begin{pmatrix} c \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which is to say that there exist $c \in \mathbb{C}$ and $x \in \mathcal{M}$ such that

$$\begin{aligned} c(1-a) - \langle x, \beta \rangle &= 1, \\ -c\gamma + (1-Q)x &= 0. \end{aligned} \tag{2.6.18}$$

Since $\varphi(\chi) \neq 1$ we may choose

$$c = \frac{1}{1-\varphi(\chi)}, \qquad x = cu_\chi,$$

and by virtue of equation (2.6.13), c, x then satisfy equations (2.6.18). Accordingly equation (2.6.17) is a *bona fide* matricial representation of J on $\mathbb{C} \oplus \mathcal{M}$ for some $b \in \mathbb{R}$, some $\alpha \in \mathcal{M}$ and some operator B on $\mathcal{M}$. One can show that in fact B is a densely defined self-adjoint operator on $\mathcal{M}$; the details can be found in, for example, [AMY12b, Lemma 6.24].

Equation (2.6.16) now becomes

$$\begin{bmatrix} b & 1\otimes\alpha \\ \alpha\otimes 1 & B \end{bmatrix} \begin{pmatrix} 1 \\ -v_z \end{pmatrix} = J \begin{pmatrix} 1 \\ -v_z \end{pmatrix} = \begin{pmatrix} h(z) \\ (z_1 Y + z_2(1-Y))\, v_z \end{pmatrix}$$

and so

$$h(z) = b - \langle v_z, \alpha \rangle \,,$$
$$(z_1 Y + z_2(1-Y))\, v_z = \alpha - B v_z.$$

Thus

$$v_z = (B + z_1 Y + z_2(1-Y))^{-1}\, \alpha,$$

and finally

$$h(z) = b - \left\langle (B + z_1 Y + z_2(1-Y))^{-1}\, \alpha, \alpha \right\rangle.$$

Therefore (1)⇒(2). □

Some generalizations of Nevanlinna's representation theorems to several variables can be found in [ATDY].

Chapter 2 contains material as it appears in Indagationes Mathematicae, 2012. The dissertation author was a co-author with J. Agler and N.J. Young on this paper.

Chapter 3

Geometry of generalized models

3.1 Introduction

Generalized models provide a method for investigating local behavior of Schur functions at the boundary. Though we lose the global data of the standard model, we gain access to refined information about the differential structure of the function at a large class of singular boundary points. In this chapter, we seek to further develop our investigation of the way in which the differential structure of a function is encoded into the objects of a generalized model. In particular, we will show that the structure of the positive contraction Y associated with a generalized model at a carapoint determines the behavior of the directional derivative of φ. We classify the types of behavior possible in terms of the geometry of certain invariant spaces for Y, and prove a local version of Agler, M^{c}Carthy, and Young's generalization of the Julia-Carathéodory Theorem in terms of the spectral structure of Y.

While the operator λ in (1.4.1) has the virtue of simplicity, the function u_λ can exhibit quite complicated behavior. This behavior can become an issue when trying to model a function with even a mild boundary singularity. In this case, the model $(\mathcal{M}, u)$

encodes the singularity into the structure of the modeling Hilbert space itself. This gives rise to ambiguity in the nature of the regularity of the function φ at the point.

Note 3.1.1. *Let φ be a function in $\mathcal{S}_2$ with a carapoint at τ, and let (α,β,γ,D) be a realization for φ. If $\ker(1-D\tau)=\{0\}$, then the model has a C-point at τ and therefore by Theorem 1.5.4, φ is nontangentially differentiable there. On the other hand, if $\ker(1-D\tau)\neq\{0\}$, the situation is ambiguous.*

Recall that the generalized model function I_Y has the formula

$$I_Y(\lambda)=\frac{\bar{\tau}^1\lambda^1Y+\bar{\tau}^2\lambda^2(1-Y)-\bar{\tau}^1\bar{\tau}^2\lambda^1\lambda^2}{1-\bar{\tau}^1\lambda^1(1-Y)-\bar{\tau}^2\lambda^2Y} \tag{3.1.1}$$

This function has a well-behaved singularity on the boundary that can be used to model singular behavior in Schur functions. Moving the singular behavior of the model from the Hilbert space into I reduces the size of the modeling space and regularizes the behavior of the modeling function u_λ. We will use this regularity in u to refine our understanding of the differential structure of φ at a singular boundary carapoint.

3.2 Generalized models and directional derivatives

3.2.1 Directional derivatives

In Chapter 2, generalized models were used to investigate the differentiable structure of Schur functions at boundary points. The connection between directional derivatives at carapoints and generalized models will be the main subject of this chapter.

Note 3.2.1. *Unless otherwise indicated, the calculations and theorems proved in this chapter are proved at the boundary point $\chi=(1,1)\in\mathbb{T}^2$. There is nothing special about the point χ, other than ease of exposition. Under the change of variables $\lambda\to\bar{\tau}\lambda$, the arguments hold for any carapoint τ in $\mathbb{T}^2$.*

Recall that at a carapoint, the generalized model function u_λ extends continuously to the boundary on nontangential sets, and thus has a nontangential limit u_χ as $\lambda \to \chi$. We begin by characterizing the directional derivative of a function φ in terms of the positive contraction Y.

Lemma 3.2.2. *If φ has a carapoint at χ and a generalized model $(\mathcal{M}, v, I)$ as in Theorem 2.2.6, then the directional derivative of φ for δ pointing into the bidisc at χ is given by the formula*

$$D_\delta \varphi(\chi) = \left\langle \frac{\delta_1 \delta_2}{\delta_1(1-Y)+\delta_2 Y} v_\chi, v_\chi \right\rangle .$$

Proof. Let $\lambda_t = \chi + t\delta$ where $\delta = (\delta_1, \delta_2) \in \mathbb{C}^2$ and $\operatorname{Re}\delta_1, \operatorname{Re}\delta_2 < 0$ (so that $\lambda_t \in \mathbb{D}^2$ for small enough $t > 0$.)

Let φ have a generalized model such that

$$1 - \varphi(\lambda)\overline{\varphi}\mu = \left\langle (1 - I_Y(\mu)^* I_(\lambda)) v_\lambda, v_\mu \right\rangle . \tag{3.2.1}$$

As χ is a carapoint, v_λ is nontangentially continuous at χ. Then applying limits to (3.2.1) as $\mu \overset{\text{nt}}{\to} \chi$ gives

$$1 - \overline{\varphi(\chi)}\varphi(\lambda) = \left\langle (1 - I(\chi)^* I(\lambda)) v_\lambda, v_\chi \right\rangle .$$

Multiplying through by $-\varphi(\chi)$ gives

$$\begin{aligned} \varphi(\lambda) - \varphi(\chi) &= \phi(\chi)\left\langle (I(\lambda)-1)v_\lambda, v_\chi \right\rangle \\ &= \varphi(\chi)\left\langle (I(\lambda)-1)v_\chi, v_\chi \right\rangle + \varphi(\chi)\left\langle (I(\lambda)-1)(v_\lambda - v_\chi), v_\chi \right\rangle . \end{aligned} \tag{3.2.2}$$

The difference $I(\lambda_t) - I(\chi)$ is given by

$$
\begin{aligned}
I(\lambda_t) - I(\chi) &= \left[\frac{(\lambda_t^1)Y + \lambda_t^2(1-Y) - \lambda_t^1\lambda_t^2}{1 - \lambda_t^1(1-Y) + \lambda_t^2 Y} - 1\right] \\
&= \left[\frac{(1+t\delta_1)Y + (1+t\delta_2)(1-Y) - (1+t\delta_1)(1+t\delta_2)}{1 - (1+t\delta_1)(1-Y) - (1+t\delta_2)Y} - 1\right] \\
&= \left[\frac{t\delta_1 Y + t\delta_2(1-Y) - t\delta_1 - t\delta_2 - t^2\delta_1\delta_2 + t\delta_1(1-Y) + t\delta_2 Y}{-t\delta_1(1-Y) - t\delta_2 Y}\right] \\
&= \frac{t\delta_1\delta_2}{\delta_1(1-Y) + \delta_2 Y},
\end{aligned} \tag{3.2.3}
$$

Upon dividing by t and applying the limit as $t \to 0^+$, we get

$$
D_\delta I(\chi) = \frac{\delta_1\delta_2}{\delta_1(1-Y) + \delta_2 Y}. \tag{3.2.4}
$$

Combining with (3.2.2), we calculate a difference quotient.

$$
\begin{aligned}
\frac{\varphi(\lambda_t) - \varphi(\chi)}{t} &= \varphi(\chi)\frac{1}{t}\left\langle (I(\lambda_t) - 1)v_{\lambda_t}, v_\chi \right\rangle \\
&= \varphi(\chi)\left\langle \frac{I(\lambda_t) - I(\chi)}{t} v_\chi, v_\chi \right\rangle \\
&\quad + \varphi(\chi)\left\langle \frac{I(\lambda_t) - I(\chi)}{t}(v_{\lambda_t} - v_\chi), v_\chi \right\rangle.
\end{aligned}
$$

Finally, letting $t \to 0^+$, we conclude

$$
D_\delta \varphi(\chi) = \left\langle \frac{\delta_1\delta_2}{\delta_1(1-Y) + \delta_2 Y} v_\chi, v_\chi \right\rangle. \tag{3.2.5}
$$

□

3.3 Structure of rational model functions

By Theorem 2.2.6, any Schur function φ with a carapoint at χ has a continuous generalized model at χ. We are interested in exploring the inverse problem: when given a continuous generalized model for φ, what can we say about the differential structure of φ at χ?

We begin with an example of a family of simple rational functions that possess a single nondifferentiable carapoint.

Lemma 3.3.1. *Let*

$$\varphi_y(\lambda) = \frac{\lambda^1 y + \lambda^2(1-y) - \lambda^1\lambda^2}{1 - \lambda^1(1-y) - \lambda^2 y}.$$

For all $y \in (0,1)$*, the function* φ_y *has a nondifferentiable carapoint at* $\chi = (1,1)$.

Proof. By calculation in the proof of Lemma 3.2.2, the directional derivative of φ_y in the direction δ is not linear, and so φ_y is not nontangentially differentiable at χ. (In fact, the singularity that $\varphi_y(\lambda)$ has at χ is complicated enough that it is impossible to extend $\varphi_y(\lambda)$ even continuously across the boundary at χ.) To see that φ_y has a carapoint at χ, it is enough to check the Carathédory condition along the ray (r,r) has $r \to 1$. On this ray,

$$\varphi_y(r,r) = r.$$

Hence, if $\lambda = (r,r)$ tends to χ,

$$\begin{aligned} \liminf_{\lambda \to \chi} \frac{1 - |\varphi_y(\lambda)|}{1 - \|\lambda\|_\infty} &= \liminf_{r \to 1} \frac{1-|r|}{1-r} \\ &= 1 \end{aligned}$$

Thus the Julia quotient for φ_y at χ is bounded, and so φ_y has a carapoint at χ. □

Note that the function φ_y is the scalar case of the generalized model function I_Y

in Theorem 2.2.6. Consider the following special case, where Y is a contractive diagonal matrix.

Note 3.3.2. *If Y is a diagonal matrix*

$$Y = \begin{bmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{bmatrix}$$

then $I_Y(\lambda)$ has the form

$$I_Y(\lambda) = \bigoplus_{i=1}^{n} \varphi_{y_i}(\lambda) E_i,$$

where

$$\varphi_y(\lambda) = \frac{\lambda^1 y + \lambda^2(1-y) - \lambda^1\lambda^2}{1 - \lambda^1(1-y) - \lambda^2 y},$$

and E_i is the projection onto the eigenspace corresponding to y_i.

Thus, in the case where Y is a finite dimensional diagonal matrix, we get that the function I_Y is a direct sum of the scalar-valued functions φ_y. We will show that this statement holds for a general positive contraction Y in terms of a spectral integral of scalar functions φ_y. Accordingly, we first develop some properties of the one parameter family of scalar functions φ_y. By Lemma 3.3.1, such functions are the simplest degree $(1,1)$ rational functions that can be constructed with a singular point at χ. Every function φ_y has an explicit model (a statement that appears without proof as Proposition 6.3 in [AMY12a]).

Proposition 3.3.3. *For a real number y, $0 < y < 1$, let φ_y be the inner function on $\mathbb{C}^2$ given by*

$$\varphi_y(\lambda) = \frac{y\lambda^1 + (1-y)\lambda^2 - \lambda^1\lambda^2}{1 - (1-y)\lambda^1 - y\lambda^2}. \tag{3.3.1}$$

Then any model $(\mathcal{M}, u)$ *of* φ_y *has a B-point at* $\chi = (1,1)$. *Furthermore,* $(\mathbb{C}^2, u_y)$ *is a model for* φ_y*, where* $u_{y,\lambda}$ *has the form*

$$u_{y,\lambda} = \frac{1}{1-(1-y)\lambda^1 - y\lambda^2} \begin{pmatrix} \sqrt{y}(1-\lambda^2) \\ \sqrt{1-y}(1-\lambda^1) \end{pmatrix}. \tag{3.3.2}$$

With respect to the orthonormal basis of $\mathbb{C}^2$ *given by*

$$e_+ = \begin{pmatrix} \sqrt{1-y} \\ \sqrt{y} \end{pmatrix}, \quad e_- = \begin{pmatrix} \sqrt{y} \\ -\sqrt{1-y} \end{pmatrix},$$

we can write the model as

$$u_{y,\lambda} = \frac{\sqrt{(1-y)y}(\lambda^1-\lambda^2)}{1-(1-y)\lambda^1-y\lambda^2} e_+ + e_-. \tag{3.3.3}$$

Then $(0, e_-, e_-, e_+ \otimes e_+)$ *is a self-adjoint unitary realization for* $u_{y,\lambda}$.

Proof. A straightforward calculation shows that

$$1 - \varphi_y(\lambda)^* \varphi_y(\lambda) = \left\langle (1-\mu^*\lambda) u_{y,\lambda}, u_{y,\mu} \right\rangle.$$

To show that χ is a B-point for φ_t, we need to show that $u_{y,\lambda}$ is bounded as $\lambda \to \chi$ nontangentially. Let S be a set in $\mathbb{D}^2$ that approaches χ nontangentially. Then there exists a $c > 0$ so that for $\lambda \in S$,

$$\|\chi - \lambda\| \leq c(1 - \|\lambda\|).$$

We will show that the coefficient of e_+ in (3.3.3) is bounded on S. To do so, notice that

$$\begin{aligned}
\left|\lambda^1-\lambda^2\right| &= \left|(1-\lambda^2)+(\lambda^1-1)\right| \\
&\leq \left|1-\lambda^1\right| + \left|1-\lambda^2\right| \\
&\leq 2\max\{\left|1-\lambda^1\right|, \left|1-\lambda^2\right|\} \\
&\leq 2c\min\{(1-\left|\lambda^1\right|), (1-\left|\lambda^2\right|)\} \\
&\leq 2c[(1-y)(1-\left|\lambda^1\right|)+y(1-\left|\lambda^2\right|)] \\
&= 2c[(1-y)-(1-y)\left|\lambda^1\right|+y-y\left|\lambda^2\right|] \\
&= 2c[1-(1-y)\left|\lambda^1\right|-y\left|\lambda^2\right|] \\
&\leq 2c\left|1-(1-y)\lambda^1-y\lambda^2\right|.
\end{aligned}$$

Then $u_{y,\lambda}$ is bounded on the set S, as

$$\begin{aligned}
\|u_{y,\lambda}\| &= \left\|\frac{\sqrt{y(1-y)}(\lambda^1-\lambda^2)}{1-(1-y)\lambda^1-y\lambda^2}e_+ + e_-\right\| \\
&\leq 2c\sqrt{y(1-y)}\,\|e_+\| + \|e_-\| \\
&= 2c\sqrt{y(1-y)}+1 \qquad (3.3.4)
\end{aligned}$$

which depends only on y. Then $u_{y,\lambda}$ is bounded as $\lambda \to \chi$ nontangentially, and so by Theorem 1.5.3, χ is a B-point for φ_y. □

In particular, we note that this implies that the model $u_{t,\lambda}$ is not continuous at χ as $\lambda \stackrel{\text{nt}}{\to} \chi$, which gives the following simple observation.

Lemma 3.3.4. *Let $y \in (0,1)$ and let φ_y be a function in the family* (3.3.1) *with a model $(\mathcal{M}, u_{y,\lambda})$. Let S be a set approaching χ nontangentially. There exist sequences*

$\{\lambda_n\}, \{\mu_n\} \subset S$ so that $\lambda_n \overset{\text{nt}}{\to} \chi, \mu_n \overset{\text{nt}}{\to} \chi$ and for any $t \in (0,1)$,

$$\left| u^i_{t,\lambda_n} - u^i_{t,\mu_n} \right| > 0 \tag{3.3.5}$$

for at least one of $i = 1$ or $i = 2$.

Proof. The point χ is a B-point for the model that is not a C-point, and so $u_{y,\lambda}$ does not extend continuously at χ. Thus there must exist distinct sequences that approach χ on which $u_{y,\lambda}$ approaches different vectors. □

Note 3.3.5. *The scalar functions φ_0 and φ_1 are well behaved, as*

$$\varphi_1(\lambda) = \lambda^1, \qquad \varphi_0(\lambda) = \lambda^2, \tag{3.3.6}$$

respectively, and in these bounding cases, the singularity at χ disappears.

We now formalize the observation we made about the structure of I_Y for diagonal Y and extend the observation to the case of general positive contractions.

Lemma 3.3.6. *Let $\varphi \in \mathcal{S}_2$ have a carapoint at χ, and let $(\mathcal{M}, v, I)$ be a generalized model so that χ is a C-point. There exists a projection-valued measure E such that*

$$I_Y(\lambda) = \int_0^1 \varphi_y(\lambda)\, \mathrm{d}E(t),$$

Proof. For the positive contraction Y, let E_0 designate the projection onto $\ker Y$ and let E_1 denote the projection onto $\ker(1-Y)$, so that Y has the decomposition

$$Y = \begin{bmatrix} 1 & & \\ & 0 & \\ & & Y_0 \end{bmatrix}.$$

The operator Y_0 is a strictly positive strict contraction, so by the spectral theorem, there exists a spectral measure $E(t)$ so that for $y \in (0,1)$,

$$Y_0 = \int_{(0,1)} y \, \mathrm{d}E(y),$$

so that

$$Y = 0E_0 + 1E_1 + \int_{(0,1)} y \, \mathrm{d}E(y). \tag{3.3.7}$$

Then substituting (3.3.7) and (3.3.6) into formula 3.1.1,

$$\begin{aligned} I_Y(\lambda) &= \frac{\lambda^1 Y + \lambda^2(1-Y) - \lambda^1\lambda^2}{1 - \lambda^1(1-Y) - \lambda^2 Y} \\ &= \lambda^1 E_1 + \lambda^2 E_0 + \int_{(0,1)} \varphi_y(\lambda) \, \mathrm{d}E(y) \qquad (3.3.8) \\ &= \int_0^1 \varphi_y(\lambda) \mathrm{d}E(y). \qquad (3.3.9) \end{aligned}$$

□

With this observation, we are now prepared to examine the relationship between the geometry of the model $(\mathcal{M}, u, I)$ and the differentiability of φ. We begin by addressing the trivial case in which $\varphi \in \mathcal{S}_2$ has a generalized model where the contraction Y in the formula for I is in fact a projection.

Lemma 3.3.7. *Suppose that $\varphi \in \mathcal{S}_2$ has a generalized model $(\mathcal{M}, v, I)$ such that*

1. *v_λ is continuous at χ, and*

2. *$I(\lambda)$ is given by*

$$\frac{\lambda^1 P + \lambda^2(1-P) - \lambda^1\lambda^2}{1 - \lambda^1(1-P) - \lambda^2 P}$$

   where P is a projection acting on $\mathcal{M}$.

Then φ has a differentiable carapoint at χ.

Proof. If P is a projection, then $Y_0 = 0$, and the formula for $I(\lambda)$ simplifies to

$$I(\lambda) = \lambda^1 P + \lambda^2(1-P).$$

In this case, the generalized model is a standard model for φ with a C-point at χ, and so φ has a differentiable carapoint at χ by Theorem 1.5.4. □

We need the following geometrical Lemma about the behavior of the model function at χ. Recall that if a model has a C-point at χ then the model function is continuous on sets approaching χ nontangentially (Definition 1.5.1). In this case, a sequence v_λ as $\lambda \to \chi$ will have a nontangential limit at χ, which we denote $\lim_{\lambda \overset{\text{nt}}{\to} \chi} v_\lambda = v_\chi$.

Proposition 3.3.8. *Let $\varphi \in \mathcal{S}_2$ have a carapoint at χ. Then for a generalized model $(\mathcal{M}, v, I)$ with a C-point at χ,*

$$\|v_\chi\| > 0.$$

Proof. On taking limits as $\mu \to \lambda$, the model equation

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = \left\langle (1 - I(\mu)^* I(\lambda)) v_\lambda, v_\mu \right\rangle$$

becomes

$$1 - \|\varphi(\lambda)\|^2 = \|v_\lambda\|^2 - \|I(\lambda)v_\lambda\|^2. \tag{3.3.10}$$

From (3.2.3),

$$I(\lambda_t) - I(\chi) = \frac{t\delta_1\delta_2}{\delta_1(1-Y) + \delta_2 Y}.$$

When $\lambda_t = \chi + t(-\chi)$, this becomes

$$I(\lambda_t) - 1 = -t$$

and so $I(\lambda_t) = 1 - t$. Plugging into (3.3.10),

$$1 - |\varphi(\lambda_t)|^2 = \left\|v_{\lambda_t}\right\|^2 - \left\|(1-t)v_{\lambda_t}\right\|^2 = (2t - t^2)\left\|v_{\lambda_t}\right\|^2. \tag{3.3.11}$$

Additionally,

$$1 - \|\lambda_t\|_\infty^2 = 1 - \|(\chi + t(-\chi)\|_\infty^2 = (2t - t^2)\|\chi\|_\infty^2 = (2t - t^2). \tag{3.3.12}$$

Combining 3.3.11 with 3.3.12 yields

$$\left\|v_{\lambda_t}\right\|^2 = \frac{1 - |\varphi(\lambda_t)|^2}{1 - \|\lambda_t\|_\infty^2}.$$

On application of limits, we get

$$\left\|v_\chi\right\|^2 = \lim_{t\to 0^+} \frac{1 - |\varphi(\lambda_t)|^2}{1 - \|\lambda_t\|_\infty^2} = \lim_{t\to 0^+} \frac{1 - |\varphi(\lambda_t)|}{1 - \|\lambda_t\|_\infty}.$$

However, as χ is a carapoint of φ,

$$\lim_{t\to 0^+} \frac{1 - |\varphi(\lambda_t)|}{1 - \|\lambda_t\|_\infty} = \liminf_{\lambda \overset{\text{nt}}{\to} \chi} \frac{1 - |\varphi(\lambda)|}{1 - \|\lambda\|_\infty}.$$

(see, for example, [AMY12a] or [Jaf93]). Finally, so long as φ is not constant, as χ is a carapoint for φ, by [AMY12a, Theorem 4.9],

$$\liminf_{\lambda \overset{\text{nt}}{\to} \chi} \frac{1 - |\varphi(\lambda)|}{1 - \|\lambda\|_\infty} = \alpha > 0,$$

which gives $\left\|v_\chi\right\| > 0$. □

We are now prepared to state and prove the a converse of Theorem 2.2.6.

Theorem 3.3.9. *Let φ be a function in $\mathcal{S}_2$ and $(\mathcal{M}, v, I)$ a model for φ with a C-point at χ. Then χ is a carapoint for φ.*

Proof. If Y is a projection, then by Lemma 3.3.7, the model $(\mathcal{M}, v, I)$ is a standard model with a C-point at χ and we are done.

Assume that Y is not a projection. By Lemma 3.3.6, there exists a spectral measure E such that

$$I_Y(\lambda) = \lambda^1 E_1 + \lambda^2 E_0 + \int \varphi_y(\lambda)\, \mathrm{d}E(y),$$

where the integral is present.

As $(\mathcal{M}, v, I)$ is a model,

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = \left\langle (1 - I_Y(\mu)^* I_Y(\lambda)) v_\lambda, v_\mu \right\rangle. \tag{3.3.13}$$

We will show that χ is a carapoint for ϕ by deriving a standard model for φ and then proving that the model is bounded at χ, that is that χ is a B-point. First, we derive an expression for $1 - I_Y(\mu)^* I_Y(\lambda)$:

$$1 - I_Y(\mu)^* I(\lambda) \tag{3.3.14}$$

$$= 1 - \left(\mu^1 E_1 + \mu^2 E_2 + \int \varphi_y(\mu)\, \mathrm{d}E(y)\right)^* \times \tag{3.3.15}$$

$$\left(\lambda^1 E_1 + \lambda^2 E_0 + \int \varphi_y(\lambda)\, \mathrm{d}E(y)\right)$$

$$= 1 - \left(\bar{\mu}^1 \lambda^1 E_1 + \bar{\mu}^2 \lambda^2 E_0 + \int \overline{\varphi_y(\mu)}\varphi_y(\lambda)\, \mathrm{d}E(y)\right)$$

$$= (1 - \bar{\mu}^1 \lambda^1) E_1 + (1 - \bar{\mu}^2 \lambda^2) E_0 \tag{3.3.16}$$

$$+ \int (1 - \overline{\varphi_y(\mu)}\varphi_y(\lambda))\, \mathrm{d}E(y). \tag{3.3.17}$$

Each function φ_y can be modeled with $(\mathbb{C}^2, u_{y,\lambda})$ as given in Proposition 3.3.3, so continuing from (3.3.17), we get

$$\begin{aligned}
&(1-\overline{\mu}^1\lambda^1)E_1+(1-\overline{\mu}^2\lambda^2)E_0+\int(1-\overline{\varphi_y(\mu)}\varphi_y(\lambda))\,\mathrm{d}E(y)\\
&=(1-\overline{\mu}^1\lambda^1)E_1+(1-\overline{\mu}^2\lambda^2)E_0+\int\left\langle(1-\mu^*\lambda)u_{y,\lambda},u_{y,\mu}\right\rangle\,\mathrm{d}E(y)\\
&=(1-\overline{\mu}^1\lambda^1)E_1+(1-\overline{\mu}^2\lambda^2)E_0\\
&\qquad+\int\left\langle(1-\overline{\mu}_1\lambda^1)u^1_{y,\lambda},u^1_{y,\mu}\right\rangle\,\mathrm{d}E(y)\\
&\qquad\qquad+\int\left\langle(1-\overline{\mu}_2\lambda^2)u^2_{y,\lambda},u^2_{y,\mu}\right\rangle\,\mathrm{d}E(y)\\
&=(1-\overline{\mu}_1\lambda^1)\left(E_1+\int\left\langle u^1_{y,\lambda},u^1_{y,\mu}\right\rangle\,\mathrm{d}E(t)\right)\\
&\qquad+(1-\overline{\mu}^2\lambda^2)\left(E_0+\int\left\langle u^2_{y,\lambda},u^2_{y,\mu}\right\rangle\,\mathrm{d}E(y)\right).
\end{aligned}\tag{3.3.18}$$

If we let

$$\begin{aligned}
U_1(\lambda)&=1E_1+0E_0+\int u^1_{y,\lambda}\,\mathrm{d}E(y),\\
U_2(\lambda)&=0E_1+1E_0+\int u^2_{y,\lambda}\,\mathrm{d}E(y)
\end{aligned}$$

then we can substitute into (3.3.18) to get

$$1-I(\mu)^*I(\lambda)=(1-\overline{\mu}^1\lambda^1)U^1(\mu)^*U^1(\lambda)+(1-\overline{\mu}^2\lambda^2)U^2(\mu)^*U^2(\lambda).\tag{3.3.19}$$

Upon substitution of this expression into the generalized model equation (3.3.13), we get

$$\begin{aligned}
1-\overline{\varphi(\mu)}\varphi(\lambda)&=\left\langle(1-I(\mu)^*I(\lambda))v_\lambda,v_\mu\right\rangle\\
&=\left\langle((1-\overline{\mu}^1\lambda^1)U_1(\mu)^*U_1(\lambda)+(1-\overline{\mu}^2\lambda^2)U_2(\mu)^*U_2(\lambda))v_\lambda,v_\mu\right\rangle\\
&=(1-\overline{\mu}^1\lambda^1)\left\langle U_1(\lambda)v_\lambda,U_1(\mu)v_\mu\right\rangle+(1-\overline{\mu}^2\lambda^2)\left\langle U_2(\lambda)v_\lambda,U_2(\mu)v_\mu\right\rangle.
\end{aligned}$$

Then we have shown that $(\mathcal{M} \oplus \mathcal{M}, U)$ is a model for φ, where U is the function

$$U(\lambda) = \begin{pmatrix} U_1(\lambda) v_\lambda \\ U_2(\lambda) v_\lambda \end{pmatrix}. \tag{3.3.20}$$

To show that χ is a B-point for φ, by Theorem 1.5.3 it is enough to show that $U(\lambda)$ is bounded as $\lambda \stackrel{\text{nt}}{\to} \chi$. We will show that the component $U_1(\lambda)v_\lambda$ is bounded on a set that approaches χ nontangentially (that $U_2(\lambda)v_\lambda$ is bounded follows similarly). First, $U_1(\lambda)$ is a bounded operator. To see this, let S be a set that approaches χ nontangentially such that for all $\lambda \in S$,

$$|\tau - \lambda| \leq c(1 - |\lambda|).$$

Trivially, the operator $1E_1$ is bounded. By Proposition 3.3.3, for any y with $0 < y < 1$, for all $\lambda \in S$,

$$\left\| u_{y,\lambda} \right\| \leq 2c\sqrt{y(1-y)} + 1.$$

As the maximum of the function $f(x) = \sqrt{y(1-y)}$ is $1/2$, for all $y \in (0,1)$,

$$\left\| u_{y,\lambda} \right\| \leq c + 1.$$

Thus, the family $\{u_{y,\lambda}\}$ is uniformly bounded on S. Let u, v be arbitrary vectors in $\mathcal{M}$.

Since E is a spectral measure,

$$\begin{aligned}\left|\left\langle\left(\int u^i_{y,\lambda}\,\mathrm{d}E(y)\right)u,v\right\rangle\right| &= \left|\int u^i_{y,\lambda}\,\mathrm{d}E_{u,v}(y)\right| \\ &\leq \int\left|u^i_{y,\lambda}\right|\mathrm{d}\left|E_{u,v}(y)\right| \\ &\leq \int(c+1)\,\mathrm{d}\left|E_{u,v}(y)\right| \\ &\leq (c+1)\left\|E_{u,v}(y)\right\| \\ &\leq (c+1)\left\|u\right\|\left\|v\right\|. \end{aligned} \tag{3.3.21}$$

Then $U_1(\lambda)$ is a bounded operator, and as the bound does not depend on the choice of $\lambda\in S$, the family of operators $\{U_1(\lambda)\}_{\lambda\in S}$ is uniformly bounded on S. To show that $U_1(\lambda)v_\lambda$ is bounded on S, recall that by hypothesis, the generalized model function v_λ has a C-point at χ. By (3.3.21), for all $\lambda\in S$,

$$\left\|U^i(\lambda)v_\lambda\right\| \leq \left\|U^i(\lambda)\right\|\left\|v_\lambda\right\| \leq \sqrt{c+1}\left\|v_\lambda\right\|. \tag{3.3.22}$$

By Proposition 3.3.8, the estimate (3.3.22) is non-zero, and on any sequence $\lambda_n \stackrel{\text{nt}}{\to} \chi$ in S,

$$\left\|U^i(\lambda)v_\lambda\right\| \to \sqrt{c+1}\left\|v_\chi\right\| = (\sqrt{c+1})\alpha.$$

As each component of the model function U is bounded on S as $\lambda\to\chi$, so too is U, and so the model $(\mathcal{M}\oplus\mathcal{M},U)$ has a B-point at χ, and thus φ has a carapoint there by Theorem 1.5.3. □

We are interested in refining the Theorem above into a statement about the nature of the carapoint χ. In particular, we are interested in discovering a condition that will detect the differentiable structure of φ at χ in terms of the model. It turns out that the positive contraction Y encodes this information in its spectral structure.

Theorem 3.3.10. *Let φ be a function in $\mathcal{S}_2$ and $(\mathcal{M}, v, I)$ a model for φ with a C-point at χ. If Y is not a projection and*

$$\lim_{\lambda \overset{\mathrm{nt}}{\to} \chi} P_{\ker Y(1-Y)} v_\lambda = 0, \tag{3.3.23}$$

then φ has a nondifferentiable carapoint at χ.

Proof. Denote by $\mathcal{N}$ the space

$$\mathcal{N} = \ker Y(1-Y).$$

Let $(\mathcal{M} \oplus \mathcal{M}, U)$ be the standard model given in (3.3.20). We will show that U_λ is not continuous at χ.

We proceed by contradiction. Suppose that $U(\lambda)$ is continuous on any set S that approaches χ nontangentially. Then there exists a vector $U(\chi)$ so that

$$\lim_{\lambda \overset{\mathrm{nt}}{\to} \chi} U(\lambda) = U(\chi) = \begin{pmatrix} U_1(\chi) \\ U_2(\chi) \end{pmatrix}.$$

In particular, this will imply that

$$\lim_{\lambda \to \chi} U_1(\lambda) v_\lambda = \left(1E_1 + 0E_0 + \int u^1_{y,\lambda} \, \mathrm{d}E(y) \right) v_\lambda = U_1(\chi),$$

and

$$\lim_{\lambda \to \chi} U_2(\lambda) v_\lambda = \left(0E_1 + 1E_0 + \int u^2_{y,\lambda} \, \mathrm{d}E(y) \right) v_\lambda = U_2(\chi).$$

Since by assumption $\lim_{\lambda \overset{\mathrm{nt}}{\to} \chi} P_{\mathcal{N}} v_\lambda = 0$, these equations reduce to

$$\int u^1_{y,\lambda} \, \mathrm{d}E(y) v_\lambda = U_1(\chi),$$

and

$$\int u_{y,\lambda}^2 \, \mathrm{d}E(y) v_\lambda = U_2(\chi).$$

For the sake of notational simplicity, let $A_\lambda^i = \int u_{y,\lambda}^i \mathrm{d}E(y)$ for $i = 1, 2$. Then

$$A_\lambda^i v_\lambda = A_\lambda^i v_\chi + A_\lambda^i (v_\lambda - v_\chi).$$

Since χ is a C-point for (M, v, I), the map v_λ is continuous at χ and by assumption $U(\lambda)$ is continuous at χ, and so for any sequence $\lambda_n \overset{\mathrm{nt}}{\to} \chi$,

$$A_{\lambda_n}^i v_\chi \to U_i(\chi).$$

By Lemma 3.3.4, for at least one of $i = 1, 2$, there exist sequences λ_n and μ_n satisfying (3.3.5). For these sequences, as $n \to \infty$,

$$A_{\lambda_n}^i v_\chi - A_{\mu_n}^i v_\chi = (A_{\lambda_n}^i - A_{\mu_n}^i) v_\chi \to 0.$$

Then

$$\begin{aligned}
\left\| (A_{\lambda_n}^i - A_{\mu_n}^i) v_\chi \right\|^2 &= \left\langle (A_{\lambda_n}^i - A_{\mu_n}^i) v_\chi, (A_{\lambda_n}^i - A_{\mu_n}^i) v_\chi \right\rangle \\
&= \left\langle \overline{(A_{\lambda_n}^i - A_{\mu_n}^i)} (A_{\lambda_n}^i - A_{\mu_n}^i) v_\chi, v_\chi \right\rangle \\
&= \left\langle \int \left| u_{y,\lambda_n}^i - u_{y,\mu_n}^i \right| \, \mathrm{d}E(y) v_\chi, v_\chi \right\rangle \\
&= \int \left| u_{y,\lambda_n}^i - u_{y,\mu_n}^i \right| \, \mathrm{d}E_{v_\chi, v_\chi}(y).
\end{aligned}$$

Since the expression in the norm in the above calculation is composed of objects that are nontangentially continuous at χ, an application of limits implies that

$$\lim_{n\to\infty}\int\left|u^i_{y,\lambda_n}-u^i_{y,\mu_n}\right|\mathrm{d}E_{v_\chi,v_\chi}(y)=\lim_{n\to\infty}\left\|(A^i_{\lambda_n}-A^i_{\mu_n})v_\chi\right\|^2=0.$$

As $\|v_\chi\|>0, E_{v_\chi,v_\chi}$ is a finite, positive measure supported on $\sigma(Y)$, for $y\in\sigma(Y)$,

$$\lim_{n\to\infty}\left|u^i_{y,\lambda_n}-u^i_{y,\mu_n}\right|=0.$$

But this contradicts the conclusion of Lemma 3.3.4. Therefore U cannot be continuous at χ, and so χ is not a C-point for the model $(\mathcal{M}\oplus\mathcal{M},U)$. Then by Theorem 1.5.4, χ is a nondifferentiable carapoint for φ. □

Applying the notion of examining the differentiability of the function in terms of the spectral structure of Y gives a refinement of Theorem 2.2.6 that is a partial converse to Theorem 3.3.10.

Theorem 3.3.11. *Let $\chi=(1,1)$ be a nondifferentiable carapoint for $\varphi\in\mathcal{S}_2$. Then there exists a generalized model $(\mathcal{M},v,I_Y)$ of φ with a C-point at χ such that*

$$\lim_{\lambda\overset{\text{nt}}{\to}\chi}P_{\mathcal{N}^\perp}v_\lambda\neq 0.$$

where $\mathcal{N}=\ker Y(1-Y)$.

Proof. Suppose that $\varphi\in\mathcal{S}_2$ has a nondifferentiable carapoint at χ. By Theorem 2.2.6, there exists a generalized model $(\mathcal{M},v,I)$ with a C-point at χ.

To show that

$$P_{\mathcal{N}^\perp}v_\chi\neq 0,$$

we will use facts about the directional derivative of φ at χ. From Lemma 3.2.2, for δ

pointing into the bidisc,

$$D_\delta\varphi(\chi) = \left\langle \frac{\delta_1\delta_2}{\delta_1(1-Y)+\delta_2 Y} v_\chi, v_\chi \right\rangle. \tag{3.3.24}$$

Decompose Y as $1E_1 + 0E_0 + Y_0$, where E_1 and E_0 are projections onto $\ker Y$ and $\ker 1-Y$ respectively. Let $E = 1 - E_0 - E_1$. Then Y can be written in block matrix form as

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y_0 \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_0 \\ \mathcal{M}_s \end{matrix}$$

where $\mathcal{M}_1 = E_1\mathcal{M}, \mathcal{M}_0 = E_0\mathcal{M}$, and $\mathcal{M}_s = E\mathcal{M}$. Then

$$\begin{aligned}(\delta_1(1-Y)+\delta_2(Y))^{-1} &= \begin{bmatrix} \delta_2 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_1(1-Y_0)+\delta_2 Y_0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{\delta_2} & 0 & 0 \\ 0 & \frac{1}{\delta_1} & 0 \\ 0 & 0 & (\delta_1(1-Y_0)+\delta_2 Y_0)^{-1} \end{bmatrix},\end{aligned}$$

and so

$$\frac{\delta_1\delta_2}{\delta_1(1-Y)+\delta_2 Y} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \frac{\delta_1\delta_2}{\delta_1(1-Y_0)+\delta_2 Y_0} \end{bmatrix} = \delta_1 E_1 + \delta_2 E_0 + \frac{\delta_1\delta_2}{\delta_1(1-Y_0)+\delta_2 Y_0} E.$$

Then the formula given in (3.3.24) decomposes as

$$D_\delta\varphi(\chi) = \langle \delta_1 E_1 v_\chi, E_1 v_\chi \rangle + \langle \delta_2 E_0 v_\chi, E_0 v_\chi \rangle \\ + \left\langle \frac{\delta_1\delta_2}{\delta_1(1-Y_0)+\delta_2 Y_0} E v_\chi, E v_\chi \right\rangle. \tag{3.3.25}$$

As φ has a nondifferentiable carapoint at χ, the directional derivative cannot be linear in δ. This implies that Ev_χ must be non-zero, but this is precisely the condition

$$\lim_{\lambda \overset{\text{nt}}{\to} \chi} P_{\mathcal{N}}^{\perp} v_\lambda \neq 0.$$

Note that this also precludes the case that Y is a projection. □

3.4 Regularity of generalized models at carapoints

The results of the previous section indicate that the structure of the generalized model of a function at a point in the distinguished boundary has bearing on the behavior of the function at that point. With this in mind, we make the following definitions.

Definition 3.4.1. *For a Schur function φ with a carapoint at χ, let the generalized model $(\mathcal{M}, v, I_Y)$ be as in Theorem 2.2.6. Let $\mathcal{N} = \ker Y(1-Y)$ and denote the orthogonal complement by $\mathcal{N}^{\perp}$. We call a generalized model* regular *if $P_{\mathcal{N}^{\perp}} v_\chi = 0$. Otherwise, the model is* singular. *If instead $P_{\mathcal{N}} v_\chi = 0$, then the generalized model is* purely singular.

Note 3.4.2. *We should point out that by the above definitions, if Y is a projection then $(\mathcal{M}, v, I_Y)$ as above is automatically a regular generalized model.*

Every φ with a carapoint at χ has a generalized model that is either regular or singular. Then these definitions allow us to make an explicit classification of carapoints by looking at the connections between geometrical conditions on a generalized model and the analyticity of the function at a boundary point in $\mathbb{T}^2$. The following two theorems are restatements of Theorems 3.3.9 and 3.3.11 with respect to the above definition.

Theorem 3.4.3. *Let $\varphi \in \mathcal{S}_2$ have a carapoint at χ. If φ has a purely singular generalized model $(\mathcal{M}, v, I_Y)$ at χ then φ has a non-differentiable carapoint at χ.*

Theorem 3.4.4. *Let $\varphi \in \mathcal{S}_2$ have a nondifferentiable carapoint at χ. Then φ has a singular generalized model $(\mathcal{M}, v, I_Y)$ at χ.*

We have an even stronger statement for differentiable carapoints.

Theorem 3.4.5. *Let $\varphi \in \mathcal{S}_2$ have a carapoint at χ. φ has a regular generalized model if and only if χ is a differentiable carapoint for φ.*

Proof. $(\Rightarrow)$: Suppose that φ has a regular generalized model $(\mathcal{M}, v, I_Y)$ at χ. From (3.3.25),

$$\begin{aligned} D_\delta\varphi(\chi) = &\left\langle \delta_1 E_1 v_\chi, E_1 v_\chi \right\rangle + \left\langle \delta_2 E_0 v_\chi, E_0 v_\chi \right\rangle \\ &+ \left\langle \frac{\delta_1\delta_2}{\delta_1(1-Y_0)+\delta_2 Y_0} E v_\chi, E v_\chi \right\rangle, \end{aligned}$$

but as the model is regular, this reduces to

$$D_\delta\varphi(\chi) = \left\langle \delta_1 E_1 v_\chi, E_1 v_\chi \right\rangle + \left\langle \delta_2 E_0 v_\chi, E_0 v_\chi \right\rangle.$$

Clearly the directional derivative is linear in δ, and thus χ is a differentiable carapoint for φ.

$(\Leftarrow)$: Assume that φ has a differentiable carapoint. By Theorem 2.2.6, there is a generalized model $(\mathcal{M}, v, I_Y)$ of φ. Any expression for the directional derivative will have to be linear in δ, but this means that $P_{\mathcal{N}^\perp} v_\chi = 0$, and so the model must be regular. $\square$

In the case where a boundary point is analytic, the differentiabilty of φ is immediate. However, there are relatively simple functions that possess singular C-points. In [ATDY12], the author with Agler and Young developed a method for constructing generalized models that uses the idea of cutting a realization (a,β,γ,D) down by removing the kernel of the operator $(1-D)$ from the model space. If $\ker(1-D)=\{0\}$, then the differentiabilty of φ at χ follows directly. However, this is not a necessary condition for checking differentiability. Indeed, consider the example stated in the following proposition [You12].

Proposition 3.4.6. *Let $\varphi(\lambda)$ be the rational inner function given by the formula*

$$\psi(\lambda)=\frac{-4\lambda^1(\lambda^2)^2+(\lambda^2)^2+3\lambda^1\lambda^2-\lambda^1+\lambda^2}{(\lambda^2)^2-\lambda^1\lambda^2-\lambda^1-3\lambda^2+4}. \tag{3.4.1}$$

ψ is in the Schur class, has a C-point at $\chi=(1,1)$, satisfies $\psi(1,1)=1$ and $D_{-\delta}\varphi(1,1)=-2\delta^2$. ψ is not analytic at χ. Further, there exists a realization (a,β,γ,D) of ψ so that $\ker(1-D)$ *is non-trivial.*

The nature of the singularity at $\chi=(1,1)$ is not immediately obvious. One way to conclude that χ is a C-point for ψ is by a laborious calculation of the directional derivative. However, a generalized model reveals the nature of the boundary point χ.

Proposition 3.4.7. *For ψ, a diagonal generalized model $(\mathcal{M},v,I_Y)$ is given by*

$$Y=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix}, \qquad I_Y(\lambda)=\begin{bmatrix}\phi_{\frac{1}{2}}(\lambda) & 0\\ 0 & \lambda^2\end{bmatrix},$$

and

$$v_\lambda=\frac{\sqrt{2}}{(\lambda^2)^2-\lambda^1\lambda^2-\lambda^1-3\lambda^2+4}\begin{bmatrix}2-\lambda^1-3\lambda^2+\lambda^1\lambda^2+(\lambda^2)^2\\ 1-\lambda^1\lambda^2\end{bmatrix}.$$

This generalized model is regular at χ.

Proof. That $(\mathcal{M}, v, I_Y)$ is a generalized model for ψ follows from computation. To show that the model is regular, note that $P_{\mathcal{N}^\perp} v_\lambda$ is given by

$$P_{\mathcal{N}^\perp} v_\lambda = \frac{\sqrt{2}(2 - \lambda^1 - 3\lambda^2 + \lambda^1\lambda^2 + (\lambda^2)^2)}{(\lambda^2)^2 - \lambda^1\lambda^2 - \lambda^1 - 3\lambda^2 + 4}.$$

On the ray (r, r) with $r < 1$, this becomes

$$\frac{\sqrt{2}(2 - r - 3r + r^2 + r^2)}{r^2 - r^2 - r - 3r + 4} = \frac{2\sqrt{2}(1 - 2r + r^2)}{-4(r-1)} = \frac{2\sqrt{2}(r-1)}{-4},$$

and taking the limit as $r \to 1^-$, we get

$$P_{N^\perp} v_\chi = 0.$$

The non-zero component of v_χ is contained entirely inside $\ker Y(1-Y)$ and so the model $(\mathcal{M}, v, I_Y)$ is regular. Thus ψ has a differentiable carapoint at χ. $\square$

Chapter 4

Nevanlinna representations in several variables

4.1 Introduction

This chapter will be concerned with the generalization of the following classical integral representation of Pick functions, proved by R. Nevanlinna in [Nev22].

Theorem 4.1.1 (Nevanlinna's Representation)**.** *Let h be a function defined on Π. There exists a finite positive measure μ on $\mathbb{R}$ such that*

$$h(z) = \int \frac{d\mu}{t - z} \tag{4.1.1}$$

if and only if $h \in \mathcal{P}$ and

$$\liminf_{y \to \infty} y\,|h(iy)| < \infty. \tag{4.1.2}$$

A closely related theorem, also referred to in the literature as Nevanlinna's Representation, provides an integral representation for a general element of $\mathcal{P}$.

Theorem 4.1.2. *A function $h : \Pi \to \mathbb{C}$ belongs to the Pick class $\mathcal{P}$ if and only if there exist $a \in \mathbb{R},\ b \geq 0$ and a finite positive Borel measure μ on $\mathbb{R}$ such that*

$$h(z) = a + bz + \int \frac{1+tz}{t-z}\, \mathrm{d}\mu(t) \tag{4.1.3}$$

for all $z \in \Pi$. Moreover, for any $h \in \mathcal{P}$, the numbers $a \in \mathbb{R},\ b \geq 0$ and the measure $\mu \geq 0$ in the representation (4.1.3) *are uniquely determined.*

What are the several-variable analogs of Nevanlinna's theorems? In this chapter, we shall propose four types of Nevanlinna representation for various subclasses of the n-variable Pick class $\mathcal{P}_n$. In addition, we shall present necessary and sufficient conditions for a function defined on Π^n to possess a representation of a given type in terms of asymptotic growth conditions at ∞.

The integral representation (4.1.1) of those functions in the Pick class that satisfy condition (4.1.2) can be written in the form

$$h(z) = \left\langle (A-z)^{-1}\mathbf{1}, \mathbf{1} \right\rangle_{L^2(\mu)},$$

where A is the operation of multiplication by the independent variable on $L^2(\mu)$ and $\mathbf{1}$ is the constant function 1. We propose that an appropriate n-variable analog of the Cauchy transform is the formula

$$h(z_1, \ldots, z_n) = \left\langle (A - z_1 Y_1 - \cdots - z_n Y_n)^{-1} v, v \right\rangle_{\mathcal{H}} \qquad \text{for } z_1, \ldots, z_n \in \Pi, \tag{4.1.4}$$

where $\mathcal{H}$ is a Hilbert space, A is a densely defined self-adjoint operator on $\mathcal{H}$, $Y_1, \ldots, Y_n$ are positive contractions on $\mathcal{H}$ summing to 1 and v is a vector in $\mathcal{H}$.

Theorem 4.1.6 below characterizes those functions on Π^n that have a representation of the form (4.1.4). To state this theorem we require a notion based on the following classical result of Pick [Pic16].

Theorem 4.1.3. *A function h defined on Π belongs to $\mathcal{P}$ if and only if the function A defined on $\Pi \times \Pi$ by*

$$A(z,w) = \frac{h(z) - \overline{h(w)}}{z - \overline{w}}$$

is positive semidefinite, that is, for all $n \geq 1, z_1, \ldots, z_n \in \Pi, c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum A(z_j, z_i)\overline{c_i}c_j \geq 0.$$

The following theorem, proved in [Agl90], leads to a generalization of Theorem 4.1.3 to two variables. The *Schur class of the polydisc*, denoted by $\mathcal{S}_2$, is the set of analytic functions on the polydisc $\mathbb{D}^n$ that are bounded by 1 in modulus.

Theorem 4.1.4. *A function φ defined on $\mathbb{D}^2$ belongs to $\mathcal{S}_2$ if and only if there exist positive semidefinite functions A_1 and A_2 on $\mathbb{D}^2$ such that*

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = (1 - \overline{\mu}_1\lambda_1)A_1(\lambda,\mu) + (1 - \overline{\mu}_2\lambda_2)A_2(\lambda,\mu). \tag{4.1.5}$$

By way of the transformations

$$z = i\frac{1+\lambda}{1-\lambda}, \quad \lambda = \frac{z-i}{z+i}, \tag{4.1.6}$$

and

$$h(z) = i\frac{1+\varphi(\lambda)}{1-\varphi(\lambda)}, \quad \varphi(\lambda) = \frac{h(z)-i}{h(z)+i}, \tag{4.1.7}$$

there is a one-to-one correspondence between functions in the Schur and Pick classes. Under these transformations, Theorem 4.1.4 becomes the following generalization of Pick's theorem to two variables.

Theorem 4.1.5. *A function h defined on* Π^2 *belongs to* $\mathcal{L}_2$ *if and only if there exist positive semidefinite functions* A_1 *and* A_2 *on* Π^2 *such that*

$$h(z) - \overline{h(w)} = (z_1 - \overline{w_1})A_1(z,w) + (z_2 - \overline{w_2})A_2(z,w).$$

In the light of Theorems 4.1.3 and 4.1.5 we define the *Loewner class* $\mathcal{L}_2$ to be the set of analytic functions h on Π^n with the property that there exist n positive semidefinite functions $A_1, \ldots, A_n$ on Π^n such that

$$h(z) - \overline{h(w)} = \sum_{j=1}^{n} (z_j - \overline{w_j})A_j(z,w) \tag{4.1.8}$$

for all $z, w \in \Pi^n$. The Loewner class $\mathcal{L}_2$ played a key role in [AMY12b], which gave a generalization to several variables of Loewner's characterization of the one-variable operator-monotone functions [Löw34]. As the following theorem makes clear, $\mathcal{L}_2$ also has a fundamental role to play in the understanding of Nevanlinna representations in several variables.

Theorem 4.1.6. *A function h defined on* Π^n *has a representation of the form* (4.1.4) *if and only if* $h \in \mathcal{L}_2$ *and*

$$\liminf_{y\to\infty} y|h(iy, \ldots, iy)| < \infty. \tag{4.1.9}$$

In the cases when $n = 1$ and $n = 2$, Theorems 4.1.3 and 4.1.5 assert that that $\mathcal{L}_2 = \mathcal{P}_n$, and so for $n = 1$, Theorem 4.1.6 is Nevanlinna's classical Theorem 4.1.1, and when $n = 2$, Theorem 4.1.6 is a straightforward generalization of that result to two variables. When there are more than two variables, it is known that the Loewner class is a proper subset of the Pick class, $\mathcal{L}_2 \neq \mathcal{P}_n$ [Par70, Var74]. Nevertheless, Nevanlinna's result survives as a theorem about the representation of elements of $\mathcal{L}_2$. Other than the work in [GKVVW08] very little is known about the representation of functions in $\mathcal{P}_n$ for three or more variables.

For a function h on Π^n, we call the formula (4.1.4) a *Nevanlinna representation of type* 1. Thus, Theorem 4.1.6 can be rephrased as the assertion that h has a Nevanlinna representation of type 1 if and only if $h \in \mathcal{P}_2$ and h satisfies condition (4.1.9). Somewhat more complicated representation formulae are needed to generalize Theorem 4.1.2. We identify three further representation formulae, of increasing generality, and show that every function in $\mathcal{P}_2$ has a representation of one or more of the four types.

For a function h defined on Π^n, we refer to a formula

$$h(z_1,\dots,z_n) = a + \left\langle (A - z_1Y_1 - \cdots - z_nY_n)^{-1}v, v\right\rangle_{\mathcal{H}} \qquad \text{for } z_1,\dots,z_n \in \Pi, \quad (4.1.10)$$

where a is a constant, $\mathcal{H}$ is a Hilbert space, A is a densely defined self-adjoint operator on $\mathcal{H}$, $Y_1,\dots,Y_n$ are positive contractions on $\mathcal{H}$ summing to 1 and v is a vector in $\mathcal{H}$, as a *Nevanlinna representation of type* 2.

Theorem 4.1.7. *A function h defined on Π^n has a Nevanlinna representation of type* 2 *if and only if $h \in \mathcal{P}_2$ and*

$$\liminf_{y\to\infty} y \operatorname{Im}\, h(iy,\dots,iy) < \infty. \qquad (4.1.11)$$

A *Nevanlinna representation of type* 3 of a function h defined on Π^n is of the form

$$h(z) = a + \left\langle (1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1}v, v\right\rangle \qquad \text{for all } z \in \Pi^n$$

for some real a, some self-adjoint operator A and some vector v, where $Y_1,\dots,Y_n$ are operators as in equation (4.1.4) above and $z_Y = z_1Y_1 + \cdots + z_nY_n$.

Theorem 4.1.8. *A function h defined on Π^n has a Nevanlinna representation of type* 3 *if* *and only if $h \in \mathcal{P}_2$ and*

$$\liminf_{y\to\infty} \frac{1}{y} \mathrm{Im}\ h(iy,\dots,iy) = 0.$$

Finally, *Nevanlinna representations of type* 4 are given by the formula

$$h(z) = \langle M(z)v, v\rangle\,, \tag{4.1.12}$$

where $M(z)$ is an operator of the form

$$\begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1}, \tag{4.1.13}$$

acting on an orthogonal direct sum of Hilbert spaces $\mathcal{N} \oplus \mathcal{M}$. In (4.1.12), v is a vector in $\mathcal{N} \oplus \mathcal{M}$. In (4.1.13), A is a densely-defined self-adjoint operator acting on $\mathcal{M}$ and z_P is the operator acting on $\mathcal{N} \oplus \mathcal{M}$ via the formula

$$z_P = \sum z_i P_i$$

where $P_1, \dots, P_n$ are pairwise orthogonal projections acting on $\mathcal{N} \oplus \mathcal{M}$ that sum to 1.

Theorem 4.1.9. *Let h be a function defined on Π^n. Then h has a Nevanlinna representation of type* 4 *if and only if $h \in \mathcal{P}_2$.*

A weaker, "generic" version of Theorem 4.1.9 appeared in [AMY12b, Theorem 6.9], where it was used to show that elements in $\mathcal{P}_2$ are locally operator-monotone.

It turns out that for $1 \le k \le 4$, if h is a function on Π^n and h has a Nevanlinna representation of type k, then for $k \le j \le 4$, h also has a Nevanlinna representation of type j. Thus, it is natural to define the *type* of a function in $\mathcal{P}_2$ to be the smallest k such that h has a Nevalinna representation of type k.

As the point $\chi = (1,\dots,1)$ is transformed to the point $\infty = (\infty,\dots,\infty)$ by (4.1.6), it is natural to say that a function $h \in \mathcal{P}_2$ has a carapoint at ∞ if the associated Schur function φ, given by the transformation in (4.1.7), has a carapoint at χ, and in that case to define $h(\infty)$ by

$$h(\infty) = i\frac{1+\varphi(\chi)}{1-\varphi(\chi)}. \tag{4.1.14}$$

The connection between carapoints and function types is given in the following theorem.

Theorem 4.1.10. *For a function $h \in \mathcal{P}_2$,*

1. *h is of type 1 if and only if ∞ is a carapoint of h and $h(\infty) = 0$;*
2. *h is of type 2 if and only if ∞ is a carapoint of h and $h(\infty) \in \mathbb{R} \setminus \{0\}$;*
3. *h is of type 3 if and only if ∞ is not a carapoint of h;*
4. *h is of type 4 if and only if ∞ is a carapoint of h and $h(\infty) = \infty$.*

The paper is structured as follows. As is clear from the formulae used to define the various Nevanlinna representations, Nevanlinna representations are generalizations of the resolvent of a self-adjoint operator. These *structured resolvents*, studied in Sections 4.2 and 4.3, are analytic operator-valued functions on the polyhalfplane Π^n with non-negative imaginary part, fully analogous to the familiar resolvent operator.

In modern texts Nevanlinna's representation is derived from the Herglotz Representation with the aid of the Cayley transform [Lax02, Don74]. In Section 4.4 we introduce the n-variable *strong Herglotz class* and then prove Theorem 4.1.12 by applying the Cayley transform to Theorem 1.8 of [Agl90].

In Section 4.5 we derive the Nevanlinna representations of type $3, 2$, and 1, we show how they arise naturally from the underlying Hilbert space geometry and we prove slight strengthenings of Theorems 4.1.6, 4.1.7 and 4.1.8. In Section 4.6 we give function-theoretic conditions for a function $h \in \mathcal{P}_2$ to possess a representation of a given type.

In Section 4.7 we introduce the notion of carapoints for functions in the Pick class and in Section 4.8 we establish the criteria in Theorem 4.1.10 for the type of a function using the language of carapoints.

Results related to ours from a system-theoretic perspective have been obtained in ongoing work of J. A. Ball and D. Kalyuzhnyi-Verbovetzkyi [BKVa, BKVb]. See also [BS06], where Krein space methods are applied to similar problems.

4.2 Structured resolvents of operators

The resolvent operator $(A-z)^{-1}$ of a densely defined self-adjoint operator A on a Hilbert space plays a prominent role in spectral theory. It has the following properties.

1. It is an analytic bounded operator-valued function of z in the upper halfplane Π;
2. it satisfies the growth estimate $\|(A-z)^{-1}\| \leq 1/\text{Im } z$ for $z \in \Pi$;
3. $(A-z)^{-1}$ has non-negative imaginary part for all $z \in \Pi$;
4. it satisfies the "resolvent identity".

Here we are interested in several-variable analogs of the resolvent. These will again be operator-valued analytic functions with non-negative imaginary part, but now on the polyhalfplane Π^n. Because of the additional complexities in several variables we encounter three different types of resolvent; all of them have the four listed properties, with very slight modifications, and therefore deserve the name *structured resolvent*.

For any Hilbert space $\mathcal{H}$, a *positive decomposition* of $\mathcal{H}$ will mean an n-tuple $Y = (Y_1, \ldots, Y_n)$ of positive contractions on $\mathcal{H}$ that sum to the identity operator. For any

$z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and any n-tuple $T = (T_1, \ldots, T_n)$ of bounded operators we denote by z_T the operator $\sum_j z_j T_j$. Here each T_j is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, for some Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, so that z_T is also a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Definition 4.2.1. *Let A be a closed densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and let Y be a positive decomposition of $\mathcal{H}$. The* structured resolvent of A of type 2 *corresponding to Y is the operator-valued function*

$$z \mapsto (A - z_Y)^{-1} : \Pi^n \to \mathcal{L}(\mathcal{H}).$$

The following observation is essentially [AMY12b, Lemma 6.25].

Proposition 4.2.2. *For A and Y as in Definition* 4.2.1 *the structured resolvent $(A - z_Y)^{-1}$ is well defined on Π^n and satisfies, for all $z \in \Pi^n$,*

$$\|(A - z_Y)^{-1}\| \leq \frac{1}{\min_j \operatorname{Im} z_j}. \tag{4.2.1}$$

Moreover

$$\begin{aligned} \operatorname{Im} \left((A - z_Y)^{-1}\right) &= (A - z_Y^*)^{-1} (\operatorname{Im} z_Y) (A - z_Y)^{-1} \\ &= (A - z_Y)^{-1} (\operatorname{Im} z_Y) (A - z_Y^*)^{-1} \\ &\geq 0. \end{aligned} \tag{4.2.2}$$

The range of the bounded operator $(A - z_Y)^{-1}$ is of course $\mathcal{D}(A)$, the domain of A.

Proof. For any vector ξ in the domain of A,

$$\begin{aligned}\|(A-z_Y)\xi\|\,\|\xi\| &\geq |\langle (A-z_Y)\xi,\xi\rangle| \\ &\geq |\text{Im }\langle (A-z_Y)\xi,\xi\rangle| \\ &= \langle (\text{Im } z_Y)\xi,\xi\rangle \\ &= \sum_j (\text{Im } z_j)\,\langle Y_j\xi,\xi\rangle \\ &\geq (\min_j \text{Im } z_j)\left\langle \sum_j Y_j\xi,\xi\right\rangle \\ &= (\min_j \text{Im } z_j)\|\xi\|^2.\end{aligned}$$

Thus $A-z_Y$ has lower bound $\min_j \text{Im } z_j > 0$, and so has a bounded left inverse. A similar argument with z replaced by $\bar{z}$ shows that $(A-z_Y)^*$ also has a bounded left inverse, and so $A-z_Y$ has a bounded inverse and the inequality (4.2.1) holds.

The identities (4.2.2) are easy. □

Resolvents of type 2 are the simplest several-variable analogues of the familiar one-variable resolvent but they are not sufficient for the analysis of the several-variable Pick class. To this end we introduce two further generalizations. Let us first recall some basic facts about closed unbounded operators.

Lemma 4.2.3. *Let T be a closed densely defined operator on a Hilbert space $\mathcal{H}$, with domain $\mathcal{D}(T)$. The operator $1+T^*T$ is a bijection from $\mathcal{D}(T^*T)$ to $\mathcal{H}$, and the operators*

$$B \stackrel{\text{def}}{=} (1+T^*T)^{-1}, \qquad C \stackrel{\text{def}}{=} T(1+T^*T)^{-1}$$

are everywhere defined and contractive on $\mathcal{H}$. Moreover B is self-adjoint and positive, and $\operatorname{ran} C \subset \mathcal{D}(T^*)$.

Proof. All these statements are proved in [RSN90, Sections 118, 119], although the final statement about $\operatorname{ran} C$ is not explicitly stated. We must show that for all $v \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that, for all $h \in \mathcal{H}$,

$$\langle Th, Cv \rangle = \langle h, y \rangle .$$

It is straightforward to check that this relation holds for $y = v - Bv$, and so $\operatorname{ran} C \subset \mathcal{D}(T^*)$. □

Definition 4.2.4. *Let A be a closed densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and let Y be a positive decomposition of $\mathcal{H}$. The* structured resolvent of A of type 3 *corresponding to Y is the operator-valued function $M : \Pi^n \to \mathcal{L}(\mathcal{H})$ given by*

$$M(z) = (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1}. \tag{4.2.3}$$

We denote the ℓ_1 norm on $\mathbb{C}^n$ by $\|\cdot\|_1$. Note that $\|z_Y\| \leq \|z\|_1$ for all $z \in \mathbb{C}^n$ and all positive decompositions Y.

Proposition 4.2.5. *For A and Y as in Definition* 4.2.4 *the structured resolvent $M(z)$ of type* 3 *given by equation* (4.2.3) *is well defined as a bounded operator on $\mathcal{H}$ for all $z \in \Pi^n$ and satisfies*

$$\|M(z)\| \leq (1 + 2\|z\|_1)\left(1 + \frac{1 + \|z\|_1}{\min_j \operatorname{Im} z_j}\right). \tag{4.2.4}$$

Proof. Since

$$1 + z_Y A = 1 - iz_Y + iz_Y(1 - iA) : \mathcal{D}(A) \to \mathcal{H}$$

and $(1 - iA)^{-1}$ is a contraction on all of $\mathcal{H}$, with range $\mathcal{D}(A)$, the operator $(1 + z_Y A)(1 -$

$iA)^{-1}$ is well defined as an operator on $\mathcal{H}$ and

$$\begin{aligned}\|(1+z_Y A)(1-iA)^{-1}\| &= \|(1-iz_Y)(1-iA)^{-1}+iz_Y\| \\ &\le \|1-iz_Y\|+\|z_Y\| \\ &\le 1+2\|z_Y\| \\ &\le 1+2\|z\|_1. \end{aligned} \tag{4.2.5}$$

Similarly $(1-iA)(A-z_Y)^{-1}$ is well defined on $\mathcal{H}$, and since

$$i(A-z_Y) = -(1-iA)+(1-iz_Y) : \mathcal{D}(A) \to \mathcal{H}$$

we have

$$i = -(1-iA)(A-z_Y)^{-1}+(1-iz_Y)(A-z_Y)^{-1} : \mathcal{H} \to \mathcal{H}.$$

Thus, by virtue of the bound (4.2.1),

$$\begin{aligned}\|(1-iA)(A-z_Y)^{-1}\| &= \|i-(1-iz_Y)(A-z_Y)^{-1}\| \\ &\le 1+\|1-iz_Y\|\,\|(A-z_Y)^{-1}\| \\ &\le 1+\frac{1+\|z\|_1}{\min_j \mathrm{Im}\ z_j}. \end{aligned} \tag{4.2.6}$$

On combining the estimates (4.2.6) and (4.2.5) we obtain the bound (4.2.4). □

The following alternative formula for the structured resolvent of type 3, valid on the dense subspace $\mathcal{D}(A)$ of $\mathcal{H}$, allows us to show that $\mathrm{Im}\ M(z) \ge 0$.

Proposition 4.2.6. *For A and Y as in Definition* 4.2.4 *and* $z \in \Pi^n$

$$M(z)|\mathcal{D}(A) = (1-iA)\left\{(A-z_Y)^{-1} - A(1+A^2)^{-1}\right\}(1+iA) \tag{4.2.7}$$

$$= (1-iA)(A-z_Y)^{-1}(1+iA) - A : \mathcal{D}(A) \to \mathcal{H}. \tag{4.2.8}$$

Moreover, for every $v \in \mathcal{D}(A)$,

$$\text{Im } \langle M(z)v, v\rangle = \left\langle (1-iA)(A-z_Y^*)^{-1}(\text{Im } z_Y)(A-z_Y)^{-1}(1+iA)v, v\right\rangle \geq 0. \tag{4.2.9}$$

Proof. By Lemma 4.2.3 the operator $A(1+A^2)^{-1}$ is contractive on $\mathcal{H}$ and has range contained in $\mathcal{D}(A)$. On $\mathcal{D}(A^2)$ we have the identity

$$1 + z_Y A = 1 + A^2 - (A - z_Y)A.$$

Since $(1+A^2)^{-1}$ maps $\mathcal{H}$ into $\mathcal{D}(A^2)$ we have

$$(1+z_Y A)(1+A^2)^{-1} = 1 - (A-z_Y)A(1+A^2)^{-1} : \mathcal{H} \to \mathcal{H},$$

and therefore

$$(A-z_Y)^{-1}(1+z_Y A)(1+A^2)^{-1} = (A-z_Y)^{-1} - A(1+A^2)^{-1} : \mathcal{H} \to \mathcal{D}(A). \tag{4.2.10}$$

Clearly

$$(1+A^2)^{-1}(1+iA) = (1-iA)^{-1} \quad \text{ on } \mathcal{D}(A)$$

and so, on multiplying equation (4.2.10) fore and aft by $1 \pm iA$, we deduce that, as operators from $\mathcal{D}(A)$ to $\mathcal{H}$,

$$\begin{aligned} M(z)|\mathcal{D}(A) &= (1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1} \\ &= (1-iA)(A-z_Y)^{-1}(1+z_YA)(1+A^2)^{-1}(1+iA) \\ &= (1-iA)\left\{(A-z_Y)^{-1} - A(1+A^2)^{-1}\right\}(1+iA). \end{aligned}$$

This establishes equation (4.2.7).

The expression (4.2.8) follows from equation (4.2.7) since

$$(1-iA)A(1+A^2)^{-1}(1+iA) = A \quad \text{on } \mathcal{D}(A).$$

By equation (4.2.8) we have, for any $z \in \Pi^n$ and $v \in \mathcal{D}(A)$,

$$\begin{aligned} \text{Im } \langle M(z)v, v\rangle &= \text{Im } \left\langle (1-iA)(A-z_Y)^{-1}(1+iA)v, v\right\rangle - \text{Im } \langle Av, v\rangle \\ &= \text{Im } \left\langle (A-z_Y)^{-1}(1+iA)v, (1+iA)v\right\rangle \end{aligned}$$

and hence, by equation (4.2.2),

$$\text{Im } \langle M(z)v, v\rangle = \left\langle (A-z_Y^*)^{-1}(\text{Im } z_Y)(A-z_Y)^{-1}(1+iA)v, (1+iA)v\right\rangle,$$

and so equation (4.2.9) holds. □

Corollary 4.2.7. *For A and Y as in Definition* 4.2.4 *the structured resolvent* $M(z)$ *given by equation* (4.2.3) *satisfies* $\text{Im } M(z) \geq 0$ *for all* $z \in \Pi^n$.

For, by Propositions 4.2.5 and 4.2.6, $M(z)$ is a bounded operator on $\mathcal{H}$ and $\text{Im } \langle M(z)v, v\rangle \geq 0$ for $v \in \mathcal{D}(A)$. The conclusion follows by density of $\mathcal{D}(A)$ and continuity.

In the case of bounded A there is yet another expression for the structured resolvent of type 3.

Proposition 4.2.8. *If A is a* bounded *self-adjoint operator on $\mathcal{H}$ and Y is a positive decomposition of $\mathcal{H}$ then, for $z \in \Pi^n$,*

$$M(z) = (1+iA)^{-1}(1+Az_Y)(A-z_Y)^{-1}(1+iA) \tag{4.2.11}$$

Proof. Since A is bounded it is defined on all of $\mathcal{H}$. We have

$$1 + Az_Y = 1 + A^2 - A(A - z_Y)$$

and hence

$$(1+Az_Y)(A-z_Y)^{-1} = (1+A^2)(A-z_Y)^{-1} - A.$$

Thus

$$\begin{aligned}(1+iA)^{-1}(1+Az_Y)(A-z_Y)^{-1}(1+iA) &= (1-iA)(A-z_Y)^{-1}(1+iA) - A \\ &= M(z)\end{aligned}$$

by equation (4.2.8). □

Remark 4.2.9. In the case of unbounded A the expression (4.2.11) for $M(z)$ is valid wherever it is defined, but it is not to be expected that this will be a dense subspace of $\mathcal{H}$ in general.

Here are two examples of structured resolvents of type 3, one on $\mathbb{C}^2$ and one on an infinite-dimensional space.

Example 4.2.10. Let

$$\mathcal{H}=\mathbb{C}^2, \quad A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad Y_1=\tfrac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \quad Y_2=1-Y_1, \quad Y=(Y_1,Y_2).$$

Then

$$\begin{aligned} M(z) &= (1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1} \\ &= \frac{1}{1-z_1z_2}\begin{bmatrix}(1+z_1)(1+z_2) & -i(z_1-z_2)\\ i(z_1-z_2) & -(1-z_1)(1-z_2)\end{bmatrix}. \end{aligned}$$

Example 4.2.11. Let $\mathcal{H}=L^2(\mathbb{R})$, let A be the operation of multiplication by the independent variable t and let $Y=(P,Q)$ where P,Q are the orthogonal projection operators onto the subspaces of even and odd functions respectively in L^2. Thus

$$Pf(t)=\tfrac{1}{2}\{f(t)+f(-t)\}, \qquad Qf(t)=\tfrac{1}{2}\{f(t)-f(-t)\}.$$

Let $Y'=(Q,P)$. Note that

$$PA=AQ, \qquad QA=AP$$

and hence

$$z_YA=Az_{Y'}, \qquad z_{Y'}A=Az_Y, \qquad z_Yz_{Y'}=z_1z_2=z_{Y'}z_Y.$$

It follows that z_Y and $z_{Y'}$ commute with A^2, and it may be checked that

$$(A-z_Y)^{-1}=(A^2-z_1z_2)^{-1}(z_{Y'}+A)=(z_{Y'}+A)(A^2-z_1z_2)^{-1}$$

and hence

$$(A - z_Y)^{-1}(1 + z_Y A) = (A^2 - z_1 z_2)^{-1}\left((1 + A^2) z_{Y'} + (1 + z_1 z_2) A\right).$$

A straightforward calculation now shows that the structured resolvent $M(z)$ of A corresponding to Y is given by

$$(M(z)f)(t) = \frac{\left(\frac{1}{2}(z_1 + z_2)(1 + t^2) + (1 + z_1 z_2)t\right) f(t) + \frac{1}{2}(z_2 - z_1)(1 - it)^2 f(-t)}{t^2 - z_1 z_2}$$

for all $z \in \Pi^2$, $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. In particular, we note for future use that if f is an even function,

$$(M(z)f)(t) = \frac{t(1 + z_1 z_2) + (1 - it)(itz_1 + z_2)}{t^2 - z_1 z_2} f(t). \tag{4.2.12}$$

4.3 The matricial resolvent

The third and last form of structured resolvent that we consider has a 2×2 matricial form. As will become clear, this extra complication is needed for the description of the most general type of function in the several-variable Loewner class.

By an *orthogonal decomposition* of a Hilbert space $\mathcal{H}$ we shall mean an n-tuple $P = (P_1, \dots, P_n)$ of orthogonal projection operators with pairwise orthogonal ranges such that $\sum_{j=1}^n P_j$ is the identity operator.

Proposition 4.3.1. *Let $\mathcal{H}$ be the orthogonal direct sum of Hilbert spaces $\mathcal{N}, \mathcal{M}$, let A be a densely defined self-adjoint operator on $\mathcal{M}$ with domain $\mathcal{D}(A)$ and let P be an orthogonal decomposition of $\mathcal{H}$. For every $z \in \Pi^n$ the operator on $\mathcal{H}$ given with respect* *to the decomposition $\mathcal{N} \oplus \mathcal{M}$ by the matricial formula*

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \tag{4.3.1}$$

$$\times \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1} \tag{4.3.2}$$

is a bounded operator defined on all of $\mathcal{H}$, and

$$\|M(z)\| \leq (1+\sqrt{10}\|z\|_1) \left(1 + \frac{1+\sqrt{2}\|z\|_1}{\min_j \mathrm{Im}\ z_j} \right) \tag{4.3.3}$$

Proof. Let $z \in \Pi^n$. Let the projection P_j have operator matrix

$$P_j = \begin{bmatrix} X_j & B_j \\ B_j^* & Y_j \end{bmatrix} \tag{4.3.4}$$

with respect to the decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}$. Then

$$X = (X_1, \ldots, X_n), \quad Y = (Y_1, \ldots, Y_n)$$

are positive decompositions of $\mathcal{N}$, $\mathcal{M}$ respectively, and

$$B = (B_1, \ldots, B_n), \quad B^* = (B_1^*, \ldots, B_n^*)$$

are n-tuples of contractions summing to 0, from $\mathcal{M}$ to $\mathcal{N}$ and from $\mathcal{N}$ to $\mathcal{M}$ respectively. Since the B_j are contractions we have

$$\|z_B\| \leq \|z\|_1.$$

For any $z \in \mathbb{C}^n$,

$$z_P = \begin{bmatrix} z_X & z_B \\ z_{B^*} & z_Y \end{bmatrix}. \tag{4.3.5}$$

Consider the third and fourth factors in the product on the right hand side of equation (4.3.1); the product of these two factors is well defined as an operator on $\mathcal{H}$ since $(1-iA)^{-1}$ maps $\mathcal{M}$ to $\mathcal{D}(A)$. It is even a bounded operator, since, by virtue of equation (4.3.5),

$$\left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1} = \begin{bmatrix} iz_X & z_B A(1-iA)^{-1} \\ iz_{B^*} & (1+z_Y A)(1-iA)^{-1} \end{bmatrix}. \tag{4.3.6}$$

Since

$$\|A(1-iA)^{-1}\| = \|i\left(1-(1-iA)^{-1}\right)\| \leq 2$$

we can immediately see that the operator (4.3.6) is bounded. We can get an estimate of the norm of the operator matrix (4.3.6) if we replace each of the four operator entries by an upper bound for its norm. We find that

$$\begin{aligned} \left\| \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1} \right\| &\leq \left\| \begin{bmatrix} \|z\|_1 & 2\|z\|_1 \\ \|z\|_1 & 1+2\|z\|_1 \end{bmatrix} \right\| \\ &\leq 1 + \|z\|_1 \left\| \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right\| \\ &= 1 + \sqrt{10}\|z\|_1. \end{aligned} \tag{4.3.7}$$

Now consider the second factor in the definition (4.3.1) of $M(z)$. We find that

$$\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - z_P\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{bmatrix}1 & -z_B\\ 0 & A-z_Y\end{bmatrix}^{-1}$$
$$= \begin{bmatrix}1 & z_B(A-z_Y)^{-1}\\ 0 & (A-z_Y)^{-1}\end{bmatrix}, \tag{4.3.8}$$

which maps $\mathcal{H}$ into $\mathcal{N} \oplus \mathcal{D}(A)$. Hence the product of the first two factors in the product on the right hand side of equation (4.3.1) is

$$\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - z_P\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{bmatrix}-i & -iz_B(A-z_Y)^{-1}\\ 0 & (1-iA)(A-z_Y)^{-1}\end{bmatrix}. \tag{4.3.9}$$

Since

$$\begin{aligned}\|(1-iA)(A-z_Y)^{-1}\| &= \|(1-iz_Y)(A-z_Y)^{-1} - i\|\\ &\le 1 + \|1-iz_Y\|\,\|(A-z_Y)^{-1}\|\\ &\le 1 + \frac{1+\|z\|_1}{\min_j \operatorname{Im}\, z_j}\end{aligned}$$

we deduce from equation (4.3.9) that

$$\begin{aligned}\left\|\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - z_P\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}\right\| &\le \left\|\begin{bmatrix}1 & \|z\|_1\,\|(A-z_Y)^{-1}\|\\ 0 & 1+(1+\|z\|_1)\|(A-z_Y)\|^{-1}\end{bmatrix}\right\|\\ &\le 1 + \left\|\begin{bmatrix}0 & \|z\|_1\\ 0 & 1+\|z\|_1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \|(A-z_Y)^{-1}\|\end{bmatrix}\right\|\\ &\le 1 + \frac{1+\sqrt{2}\|z\|_1}{\min_j \operatorname{Im}\, z_j}.\end{aligned} \tag{4.3.10}$$

On combining the estimates (4.3.10) and (4.3.7) we obtain the bound (4.3.3) for $\|M(z)\|$.

□

Remark 4.3.2. On multiplying together the expressions (4.3.9) and (4.3.6) we obtain the formula

$$M(z) = \begin{bmatrix} z_X + z_B(A - z_Y)^{-1} z_{B^*} & -iz_B(A - z_Y)^{-1}(1 + iA) \\ i(1 - iA)(A - z_Y)^{-1} z_{B^*} & (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1} \end{bmatrix}.$$

Notice in particular that the $(2,2)$ entry (that is, the compression of $M(z)$ to $\mathcal{M}$) is the structured resolvent of A of type 3 corresponding to Y, the compression of P to $\mathcal{M}$, as in equation (4.2.3).

Definition 4.3.3. *Let $\mathcal{H}$ be the orthogonal direct sum of Hilbert spaces $\mathcal{N}, \mathcal{M}$, let A be a densely defined self-adjoint operator on $\mathcal{M}$ with domain $\mathcal{D}(A)$ and let P be an orthogonal decomposition of $\mathcal{H}$. The* structured resolvent of A of type 4 *corresponding to P is the operator-valued function $M : \Pi^n \to \mathcal{L}(\mathcal{H})$ given by equation* (4.3.1).

We shall also refer to $M(z)$ as the *matricial resolvent of A with respect to P*. The important property that Im $M(z) \geq 0$ is not at once apparent from the formula (4.3.1); as with structured resolvents of type 3, there are alternative formulae from which this property is more easily shown. Once again the alternatives suffer the minor drawback that they give $M(z)$ only on a dense subspace of $\mathcal{H}$.

Proposition 4.3.4. *With the notation of Definition* 4.3.3*, as operators on* $\mathcal{N} \oplus \mathcal{D}(A)$*,*

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A(1+A^2)^{-1} \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} \right) \times \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} \tag{4.3.11}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \tag{4.3.12}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \tag{4.3.13}$$

for all $z \in \Pi^n$*. Moreover, for all* $z,\ w \in \Pi^n$*,*

$$M(z) - M(w)^* = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - w_P^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \times (z_P - w_P^*) \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} \tag{4.3.14}$$

on $\mathcal{N} \oplus \mathcal{D}(A)$*.*

Proof. By Lemma 4.2.3 the operators $(1+A^2)^{-1}$ and

$$C \stackrel{\text{def}}{=} \text{Im}\,(1-iA)^{-1} = A(1+A^2)^{-1}$$

are self-adjoint contractions defined on all of $\mathcal{M}$. Furthermore,

$$\operatorname{ran}(1+A^2)^{-1} = \mathcal{D}(A^2), \qquad \operatorname{ran} C \subset \mathcal{D}(A).$$

We claim that, as operators on $\mathcal{N} \oplus \mathcal{D}(A)$,

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix}\right) \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} z_P\right)^{-1}. \tag{4.3.15}$$

We have

$$
\begin{aligned}
&\left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} z_P\right) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} + z_P \begin{bmatrix} 1 & 0 \\ 0 & AC \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} z_P - z_P \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} z_P \\
&= \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} + z_P \left(\begin{bmatrix} 1 & 0 \\ 0 & AC \end{bmatrix} - 1\right) + \left(1 - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix}\right) z_P - z_P \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} z_P \\
&= \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & AC \end{bmatrix} z_P - z_P \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} z_P \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix}\right).
\end{aligned}
$$

This is an identity between operators on $\mathcal{H}$, in both cases a composition $\mathcal{H} \to \mathcal{N} \oplus \mathcal{D}(A) \to \mathcal{H}$, and moreover the first factor on the left hand side and the second factor on the right hand side are invertible, from $\mathcal{N} \oplus \mathcal{D}(A)$ to $\mathcal{H}$ and from $\mathcal{H}$ to $\mathcal{N} \oplus \mathcal{D}(A)$ respectively. We may pre- and post-multiply appropriately to obtain equation (4.3.15), but note that the equation is then only valid as an identity between operators on $\mathcal{N} \oplus \mathcal{D}(A)$.

On combining equations (4.3.1) and (4.3.15) we deduce that

$$
\begin{aligned}
M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} &\left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix}\right) \times \\
&\left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} z_P\right)^{-1} \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1}.
\end{aligned}
$$

Since

$$\begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1+A^2 \end{bmatrix}^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1+A^2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} z_P \right) = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P,$$

we deduce further that

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} \right) \times \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix}, \tag{4.3.16}$$

which proves equation (4.3.11). It is straightforward to verify that

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \tag{4.3.17}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & A(1+A^2)^{-1} \end{bmatrix}. \tag{4.3.18}$$

Clearly

$$\begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A(1+A^2)^{-1} \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix},$$

and so on suitably pre- and post-multiplying equation (4.3.17), we obtain equation

(4.3.12).

To prove equation (4.3.13), check first that

$$\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - z_P \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} z_P + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) =$$

$$\left(z_P \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} z_P\right)$$

as operators on $\mathcal{N} \oplus \mathcal{D}(A)$. It follows that

$$\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} z_P + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} z_P\right)^{-1} =$$

$$\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix} - z_P \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}\left(z_P \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)$$

as operators from $\mathcal{H}$ to $\mathcal{N} \oplus \mathcal{D}(A)$. On combining this relation with equation (4.3.12) we derive the expression (4.3.13) for $M(z)|\mathcal{N} \oplus \mathcal{D}(A)$.

We now derive the identity (4.3.14). Let

$$D = \begin{bmatrix}i & 0\\ 0 & 1+iA\end{bmatrix}$$

and consider $z,\ w \in \Pi^n$. By equation (4.3.11)

$$M(z) = D^* W(z) D \tag{4.3.19}$$

on $\mathcal{N} \oplus \mathcal{D}(A)$, where

$$W(z) = R(z)S(z)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & A(1+A^2)^{-1} \end{bmatrix} \tag{4.3.20}$$

and

$$R(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S(z) = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P.$$

We have seen that $S(z)$ is invertible for any $z \in \Pi^n$, so that $W(z)$ is a bounded operator on $\mathcal{H}$. Clearly

$$\begin{aligned} M(z) - M(w)^* &= D^* \left(R(z)S(z)^{-1} - S(w)^{*-1}R(w)^* \right) D \\ &= D^* S(w)^{*-1} \left(S(w)^* R(z) - R(w)^* S(z) \right) S(z)^{-1} D. \end{aligned}$$

Here

$$\begin{aligned} S(w)^*R(z) - R(w)^*S(z) = &\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_P + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - w_P^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \\ &\left(w_P^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right) \\ = &\, z_P - w_P^*. \end{aligned}$$

Hence

$$M(z) - M(w)^* = D^* S(w)^{*-1} (z_P - w_P^*) S(z)^{-1} D,$$

which is equation (4.3.14). □

The next result shows that the matricial resolvent belongs not just to the operator

Pick class, but to the smaller *operator Loewner class*.

Proposition 4.3.5. *With the notation of Definition* 4.3.3*, there exists an analytic operator-valued function* $F : \Pi^n \to \mathcal{L}(\mathcal{H})$ *such that for all* $z,\ w \in \Pi^n$*,*

$$M(z) - M(w)^* = F(w)^*(z-\bar{w})_P F(z) \tag{4.3.21}$$

on $\mathcal{H}$*.*

Proof. The identity (4.3.14) shows that such a relation holds on $\mathcal{N} \oplus \mathcal{D}(A)$; we must extend it to all of $\mathcal{H}$. Write P_j as an operator matrix with respect to the decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}$, as in equation (4.3.4). Then z_P has the matricial expression (4.3.5). For $z \in \Pi^n$ let

$$F^\sharp(z) = \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_P \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix}.$$

Then $F^\sharp(z)$ is an operator from $\mathcal{N} \oplus \mathcal{D}(A)$ to $\mathcal{H}$, and we find that

$$\begin{aligned} F^\sharp(z) &= \begin{bmatrix} 1 & 0 \\ -z_{B^*} & A - z_Y \end{bmatrix}^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} \\ &= \begin{bmatrix} i & 0 \\ i(A-z_Y)^{-1} z_{B^*} & (A-z_Y)^{-1}(1+iA) \end{bmatrix} : \mathcal{N} \oplus \mathcal{D}(A) \to \mathcal{H}. \end{aligned}$$

Let

$$F(z) = \begin{bmatrix} i & 0 \\ i(A-z_Y)^{-1} z_{B^*} & i + (A-z_Y)^{-1}(1+iz_Y) \end{bmatrix} : \mathcal{N} \oplus \mathcal{M} \to \mathcal{H}. \tag{4.3.22}$$

Since

$$(A-z_Y)^{-1}(1+iA) = i + (A-z_Y)^{-1}(1+iz_Y)$$

on $\mathcal{N} \oplus \mathcal{D}(A)$ and the right hand side of the last equation is a bounded operator on all of $\mathcal{H}$, it is clear that, for every $z \in \Pi^n$, $F(z)$ is a continuous extension to $\mathcal{H}$ of $F^\sharp(z)$ and is a bounded operator. Furthermore F is analytic on Π^n.

By Proposition 4.3.4, equation (4.3.14), the relation (4.3.21) holds on the dense subspace $\mathcal{N} \oplus \mathcal{D}(A)$ of $\mathcal{H}$ for every $z,\ w \in \Pi^n$. Since the operators on both sides of equation (4.3.21) are continuous on $\mathcal{H}$, the equation holds throughout $\mathcal{H}$. □

Corollary 4.3.6. *A matricial resolvent has a non-negative imaginary part at every point of* Π^n.

Proof. In the notation of Proposition 4.3.5, on choosing $w = z$ in equation (4.3.21) and dividing by $2i$ we obtain the relation

$$\operatorname{Im} M(z) = F(z)^*(\operatorname{Im} z_P)F(z)$$

on $\mathcal{H}$. We have

$$\operatorname{Im} z_P = \sum_j (\operatorname{Im} z_j)P_j \geq 0,$$

and so $\operatorname{Im} M(z) \geq 0$ on $\mathcal{H}$ for all $z \in \Pi^n$. □

Here is a concrete example of a matricial resolvent.

Example 4.3.7. The function

$$M(z) = \frac{1}{z_1 + z_2} \begin{bmatrix} 2z_1z_2 & i(z_1 - z_2) \\ -i(z_1 - z_2) & -2 \end{bmatrix} \tag{4.3.23}$$

is the matricial resolvent corresponding to

$$\mathcal{H} = \mathbb{C}^2, \quad \mathcal{N} = \mathcal{M} = \mathbb{C}, \quad A = 0 \text{ on } \mathbb{C}, \quad P_1 = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P_2 = 1 - P_1.$$

4.4 Nevanlinna representations of type 4

In this section we derive a multivariable analog of the most general form of Nevanlinna representation for functions in the one-variable Pick class (Theorem 4.1.2). We start with a multivariable Herglotz theorem [Agl90, Theorem 1.8]. We shall say that an analytic operator-valued function F on $\mathbb{D}^n$ is a *Herglotz function* if $\operatorname{Re} F(\lambda) \geq 0$ for all $\lambda \in \mathbb{D}^n$. For present purposes we need the following modification of the notion.

Definition 4.4.1. *An analytic function $F : \mathbb{D}^n \to \mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space, is a* strong Herglotz function *if, for every commuting n-tuple $T = (T_1, \dots, T_n)$ of operators on a Hilbert space and for* $0 \leq r < 1$, $\operatorname{Re} F(rT) \geq 0$.

In [Agl90] these functions were called $\mathcal{F}_n$-Herglotz functions. The class of strong Herglotz functions has also been called the *Herglotz-Agler class* (for example [KV05, BKVb]). It is clear that every strong Herglotz function is a Herglotz function, and in the cases $n = 1$ and 2 the converse is also true [Agl90].

Theorem 4.4.2. *Let $\mathcal{K}$ be a Hilbert space and let $F : \mathbb{D}^2 \to \mathcal{L}(\mathcal{K})$ be a strong Herglotz function such that $F(0) = 1$. There exist a Hilbert space $\mathcal{H}$, an orthogonal decomposition P of $\mathcal{H}$, an isometric linear operator $V : \mathcal{K} \to \mathcal{H}$ and a unitary operator U on $\mathcal{H}$ such that, for all $\lambda \in \mathbb{D}^n$,*

$$F(\lambda) = V^* \frac{1 + U\lambda_P}{1 - U\lambda_P} V. \tag{4.4.1}$$

Conversely, every function $F : \mathbb{D}^n \to \mathcal{L}(\mathcal{K})$ expressible in the form (4.4.1) *for some $\mathcal{H}$, P, V and U with the stated properties is a strong Herglotz function and satisfies* $F(0) = 1$.

Note that $\lambda_P = \sum_j \lambda_j P_j$ has operator norm at most $\|\lambda\|_\infty < 1$ for $\lambda \in \mathbb{D}^n$, and hence equation (4.4.1) does define F as an analytic operator-valued function on $\mathbb{D}^n$.

On specialising to scalar-valued functions in the n-variable Herglotz class we obtain the following consequence.

Corollary 4.4.3. *Let f be a scalar-valued strong Herglotz function on $\mathbb{D}^n$. There exists a Hilbert space $\mathcal{H}$, a unitary operator L on $\mathcal{H}$, an orthogonal decomposition P of $\mathcal{H}$, a real number a and a vector $v \in \mathcal{H}$ such that, for all $\lambda \in \mathbb{D}^n$,*

$$f(\lambda) = -ia + \left\langle (L - \lambda_P)^{-1}(L + \lambda_P)v, v \right\rangle . \tag{4.4.2}$$

Conversely, for any $\mathcal{H}, L, P, a$ and v with the properties described, equation (4.4.2) *defines f as an n-variable strong Herglotz function.*

Again, the right hand side of equation (4.4.2) is an analytic function of $\lambda \in \mathbb{D}^n$ since

$$(L - \lambda_P)^{-1} = L^{-1}(1 - \lambda_P L^{-1})^{-1}$$

is a bounded operator and is analytic in λ.

Definition 4.4.4. *A* Nevanlinna representation of type 4 *of a function $h : \Pi^n \to \mathbb{C}$ consists of an orthogonally decomposed Hilbert space $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}$, a self-adjoint densely defined operator A on $\mathcal{M}$, an orthogonal decomposition P of $\mathcal{H}$, a real number a and a vector $v \in \mathcal{H}$ such that*

$$h(z) = a + \langle M(z)v, v \rangle \tag{4.4.3}$$

for all $z \in \Pi^n$, where $M(z)$ is the structured resolvent of A of type 4 *corresponding to P (given by the formula* (4.3.1)*).*

We wish to convert Corollary 4.4.3 to a representation theorem for suitable analytic functions on Π^n. The fact that the corollary only applies to *strong* Herglotz functions results in representation theorems for a subclass of the Pick class $\mathcal{P}_n$. Recall from the introduction:

Definition 4.4.5. *The* Loewner class $\mathcal{P}_2$ *is the set of analytic functions h on Π^n with the property that there exist n positive semi-definite functions $A_1,\ldots,A_n$ on Π^n, analytic in the first argument, such that*

$$h(z)-\overline{h(w)}=\sum_{j=1}^{n}(z_j-\overline{w_j})A_j(z,w)$$

for all $z,w\in\Pi^n$.

A function h on Π^n belongs to $\mathcal{P}_2$ if and only if it corresponds under conjugation by the Cayley transform to a function in the Schur-Agler class of the polydisc [AMY12b, Lemma 2.13]. Another characterization: $h\in\mathcal{P}_2$ if and only if, for every commuting n-tuple T of bounded operators with strictly positive imaginary parts, $h(T)$ has positive imaginary part.

We can now prove Theorem 4.1.9 from the introduction: *a function h defined on Π^n has a Nevanlinna representation of type* 4 *if and only if $h\in\mathcal{P}_2$.*

Proof. Let $h\in\mathcal{P}_2$. Define an n-variable Herglotz function $f:\mathbb{D}^n\to\mathbb{C}$ by

$$f(\lambda)=-ih(z) \tag{4.4.4}$$

where

$$z_j=i\frac{1+\lambda_j}{1-\lambda_j}\qquad \text{for } j=1,\ldots,n. \tag{4.4.5}$$

When $\lambda\in\mathbb{D}^n$ the point z belongs to Π^n, and so $f(\lambda)$ is well defined, and since $\operatorname{Im}\ h(z)\geq 0$ we have $\operatorname{Re}\ f(\lambda)\geq 0$, so that f is indeed a Herglotz function. In fact f is even a strong Herglotz function: since $h\in\mathcal{P}_2$, the function $\varphi\in\mathcal{S}_2$ corresponding to h lies in the

Schur-Agler class of the polydisc, and so $f = (1+\varphi)/(1-\varphi)$ is a strong Herglotz function.

By Corollary 4.4.3 there exist a real number a, a Hilbert space $\mathcal{H}$, a vector $v \in \mathcal{H}$, a unitary operator L on $\mathcal{H}$ and an orthogonal decomposition P on $\mathcal{H}$ such that, for all $z \in \Pi^n$,

$$\begin{aligned} h(z) = if(\lambda) &= a + \left\langle i(L-\lambda)^{-1}(L+\lambda)v, v\right\rangle \\ &= a + \left\langle i[L-(z-i)(z+i)^{-1}]^{-1}[L+(z-i)(z+i)^{-1}]v, v\right\rangle. \end{aligned} \tag{4.4.6}$$

Here and in the rest of this section z, λ are identified with the operators z_P, λ_P on $\mathcal{H}$, and in consequence the relation

$$\lambda = \frac{z-i}{z+i}$$

is meaningful and valid.

For $z \in \Pi^n$ let

$$M(z) = i\,(L-\lambda)^{-1}\,(L+\lambda) = i\left(L - \frac{z-i}{z+i}\right)^{-1}\left(L + \frac{z-i}{z+i}\right). \tag{4.4.7}$$

Since L is unitary on $\mathcal{H}$ and $\lambda \in \mathbb{D}^n$, the operator $M(z)$ is bounded on $\mathcal{H}$ for every $z \in \Pi^n$ and, by equation (4.4.6), we have

$$h(z) = a + \langle M(z)v, v\rangle \tag{4.4.8}$$

for all $z \in \Pi^2$. Theorem 4.1.9 will follow provided we can show that $M(z)$ is given by equation (4.3.1) for a suitable self-adjoint operator A.

Observe that

$$\begin{aligned} M(z) &= i((z+i)L-(z-i))^{-1}((z+i)L+(z-i)) \\ &= i\left(z(L-1)+i(L+1)\right)^{-1}\left(z(L+1)+i(L-1)\right). \end{aligned} \tag{4.4.9}$$

We wish to take out a factor $1-L$ from both factors in equation (4.4.9), but this may be impossible since $1-L$ can have a nonzero kernel. Accordingly we decompose $\mathcal{H}$ into $\mathcal{N}\oplus\mathcal{M}$ where $\mathcal{N}=\ker(1-L),\ \mathcal{M}=\mathcal{N}^{\perp}$. With respect to this decomposition we can write L as an operator matrix

$$L = \begin{bmatrix} 1 & 0 \\ 0 & L_0 \end{bmatrix},$$

where L_0 is unitary and $\ker(1-L_0)=\{0\}$. Substituting into equation (4.4.9) we have

$$\begin{aligned} M(z) &= i\left(z\begin{bmatrix} 0 & 0 \\ 0 & L_0-1 \end{bmatrix} + i\begin{bmatrix} 2 & 0 \\ 0 & L_0+1 \end{bmatrix}\right)^{-1}\left(z\begin{bmatrix} 2 & 0 \\ 0 & L_0+1 \end{bmatrix} + i\begin{bmatrix} 0 & 0 \\ 0 & L_0-1 \end{bmatrix} z\right) \\ &= \left(-z\begin{bmatrix} 0 & 0 \\ 0 & 1-L_0 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ 0 & i(1+L_0) \end{bmatrix}\right)^{-1}\left(z\begin{bmatrix} 2i & 0 \\ 0 & i(1+L_0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1-L_0 \end{bmatrix}\right) \end{aligned} \tag{4.4.10}$$

Formally we may now write

$$\begin{aligned} M(z) = \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & (1-L_0)^{-1} \end{bmatrix} \left(-z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & i\frac{1+L_0}{1-L_0} \end{bmatrix}\right)^{-1} \times \\ \left(z\begin{bmatrix} 1 & 0 \\ 0 & i\frac{1+L_0}{1-L_0} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 2i & 0 \\ 0 & 1-L_0 \end{bmatrix}, \end{aligned} \tag{4.4.11}$$

but whereas equation (4.4.10) is a relation between bounded operators defined on all of

$\mathcal{H}$, equation (4.4.11) involves unbounded, partially defined operators and we must verify that the product of operators on the right hand side is meaningful.

Let

$$A = i\frac{1+L_0}{1-L_0}.$$

Since L_0 is unitary on $\mathcal{M}$ and $\ker(1-L_0) = \{0\}$, the operator A is self-adjoint and densely defined on $\mathcal{M}$ [RSN90, Section 121]. The domain $\mathcal{D}(A)$ of A is the dense subspace $\operatorname{ran}(1-L_0)$ of $\mathcal{M}$. It follows from the definition of A that

$$(1-L_0)^{-1} = \tfrac{1}{2}(1-iA), \tag{4.4.12}$$

which is an equation between bijective operators from $\mathcal{D}(A)$ to $\mathcal{M}$. Likewise

$$1+L_0 = -2iA(1-iA)^{-1} : \mathcal{M} \to \mathcal{D}(A) \tag{4.4.13}$$

are bounded operators.

Let us continue the calculation from the first factor on the right hand side of equation (4.4.10). Since $\ker(1-L_0) = \{0\}$, the right hand side of the relation

$$-z\begin{bmatrix} 0 & 0 \\ 0 & 1-L_0 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ 0 & i(1+L_0) \end{bmatrix} = \left(-z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \right) \begin{bmatrix} 2i & 0 \\ 0 & 1-L_0 \end{bmatrix}$$

comprises a bijective map from $\mathcal{H}$ to $\mathcal{N} \oplus \mathcal{D}(A)$ followed by a bijection from $\mathcal{N} \oplus \mathcal{D}(A)$ to $\mathcal{H}$ (recall the equation (4.3.8)). We may therefore take inverses in the equation to obtain

$$\left(-z\begin{bmatrix}0 & 0\\ 0 & 1-L_0\end{bmatrix}+\begin{bmatrix}2i & 0\\ 0 & i(1+L_0)\end{bmatrix}\right)^{-1} \tag{4.4.14}$$

$$=\begin{bmatrix}-\frac{1}{2}i & 0\\ 0 & (1-L_0)^{-1}\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}-\frac{1}{2}i & 0\\ 0 & \frac{1}{2}(1-iA)\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1} \tag{4.4.15}$$

as operators on $\mathcal{N}\oplus\mathcal{D}(A)$.

Similar reasoning applies to the equation

$$z\begin{bmatrix}2i & 0\\ 0 & i(1+L_0)\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1-L_0\end{bmatrix}=\left(z\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}2i & 0\\ 0 & 1-L_0\end{bmatrix}$$

$$=\left(z\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}-\frac{1}{2}i & 0\\ 0 & \frac{1}{2}(1-iA)\end{bmatrix}^{-1}; \tag{4.4.16}$$

it is valid as an equation between operators on $\mathcal{H}$. The right hand side comprises an operator from $\mathcal{H}$ to $\mathcal{N}\oplus\mathcal{D}(A)$ followed by an operator from $\mathcal{N}\oplus\mathcal{D}(A)$ to $\mathcal{H}$, and so both sides of the equation denote an operator on $\mathcal{H}$.

On combining equations (4.4.10), (4.4.14) and (4.4.16) we obtain

$$M(z) = \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}(1-iA) \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \times \left(z \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}(1-iA) \end{bmatrix}^{-1}.$$

Premultiply this equation by 2 and postmultiply by $\frac{1}{2}$ to deduce that $M(z)$ is indeed the structured resolvent of A of type 4 corresponding to P, as defined in equation (4.3.1). Thus the formula (4.4.8) is a Nevanlinna representation of h of type 4.

Conversely, let $h \in \mathcal{P}_2$ have a type 4 representation (4.4.3). By Proposition 4.3.5 there exists an analytic operator-valued function $F : \Pi^n \to \mathcal{L}(\mathcal{H})$ such that, for all $z,\ w \in \Pi^n$,

$$M(z) - M(w)^* = F(w)^*(z - \bar{w})_P F(z) \tag{4.4.17}$$

on $\mathcal{H}$. Hence

$$\begin{aligned} h(z) - \overline{h(w)} &= \langle (M(z) - M(w)^*)v, v \rangle \\ &= \langle F(w)^*(z - \bar{w})_P F(z)v, v \rangle \\ &= \sum_{j=1}^{n} (z_j - \bar{w}_j) A_j(z,w) \end{aligned}$$

for all $z,\ w \in \Pi^n$, where

$$A_j(z,w) = \left\langle P_j F(z)v, F(w)v \right\rangle.$$

The A_j are clearly positive semidefinite on Π^n, and hence h belongs to the Loewner class $\mathcal{L}_n$. □

4.5 Nevanlinna representations of types 3, 2 and 1

Nevanlinna representations of type 4 have the virtue of being general for functions in $\mathcal{P}_2$, but they are undeniably cumbersome. In this section we shall show that there are three simpler representation formulae, corresponding to increasingly stringent growth conditions on $h \in \mathcal{P}_2$.

In Nevanlinna's one-variable representation formula of Theorem 4.1.2,

$$h(z) = a + bz + \int \frac{1+tz}{t-z} \, \mathrm{d}\mu(t), \tag{4.5.1}$$

it may be the case for a particular $h \in \mathcal{P}$ that the bz term is absent. The analogous situation in two variables is that the space $\mathcal{N}$ in a type 4 representation may be zero. Equivalently, in the corresponding Herglotz representation, the unitary operator L does not have 1 as an eigenvalue. This suggests the following notion.

Definition 4.5.1. *A* Nevanlinna representation of type 3 *of a function h on Π^n consists of a Hilbert space $\mathcal{H}$, a self-adjoint densely defined operator A on $\mathcal{H}$, a positive decomposition Y of $\mathcal{H}$, a real number a and a vector $v \in \mathcal{H}$ such that, for all $z \in \Pi^n$,*

$$h(z) = a + \left\langle (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1} v, v \right\rangle. \tag{4.5.2}$$

Thus h has a type 3 representation if $h(z) = a + \langle M(z)v, v\rangle$ where $M(z)$ is the structured resolvent of A of type 3 corresponding to Y, as given by equation (4.2.3).

In [ATDY12] the authors derived a somewhat simpler representation which can also be regarded as an analog of the case $b = 0$ of Nevanlinna's one-variable formula (4.5.1).

Definition 4.5.2. *A* Nevanlinna representation of type 2 *of a function h on Π^n consists of a Hilbert space $\mathcal{H}$, a self-adjoint densely defined operator A on $\mathcal{H}$, a positive* *decomposition Y of $\mathcal{H}$, a real number a and a vector $\alpha \in \mathcal{H}$ such that, for all $z \in \Pi^n$*

$$h(z) = a + \left\langle (A - z_Y)^{-1}\alpha, \alpha \right\rangle. \tag{4.5.3}$$

This means of course that, for all $z \in \Pi^n$,

$$h(z) = a + \langle M(z)\alpha, \alpha \rangle$$

where $M(z)$ is the structured resolvent of A of type 2 corresponding to Y (compare equation (4.2.1)).

We wish to understand the relationship between type 3 and type 2 representations.

Proposition 4.5.3. *If $h \in \mathcal{P}_n$ has a type 2 representation then h has a type 3 representation. Conversely, if $h \in \mathcal{P}_n$ has a type 3 representation as in equation (4.5.2) with the additional property that $v \in \mathcal{D}(A)$ then h has a type 2 representation.*

Proof. Suppose that $h \in \mathcal{P}_n$ has the type 2 representation

$$h(z) = a_0 + \left\langle (A - z_Y)^{-1}\alpha, \alpha \right\rangle$$

for some $a_0 \in \mathbb{R}$, positive decomposition Y and $\alpha \in \mathcal{H}$. We must show that h has a representation of the form (4.5.2) for some $a \in \mathbb{R}$ and $v \in \mathcal{H}$. By Proposition 4.2.6, it suffices to find $a \in \mathbb{R}$ and $v \in \mathcal{D}(A)$ such that

$$h(z) = a + \left\langle (1 - iA)\left\{ (A - z_Y)^{-1} - A(1 + A^2)^{-1} \right\} (1 + iA)v, v \right\rangle$$

for all $z \in \Pi^n$.

To this end, let $C = A(1+A^2)^{-1}$ and let

$$a = a_0 + \langle C\alpha, \alpha \rangle . \tag{4.5.4}$$

Since $1+iA$ is invertible on $\mathcal{H}$ and $\operatorname{ran}(1+iA)^{-1} \subset \mathcal{D}(A)$ we may define

$$v = (1+iA)^{-1}\alpha \in \mathcal{D}(A). \tag{4.5.5}$$

Then

$$\begin{aligned} h(z) &= a_0 + \left\langle (A - z_Y)^{-1}\alpha, \alpha \right\rangle \\ &= a - \langle C\alpha, \alpha \rangle + \left\langle (A - z_Y)^{-1}\alpha, \alpha \right\rangle \\ &= a + \left\langle \left\{ (A - z_Y)^{-1} - C \right\} (1+iA)v, (1+iA)v \right\rangle \\ &= a + \left\langle (1-iA) \left\{ (A - z_Y)^{-1} - C \right\} (1+iA)v, v \right\rangle \end{aligned}$$

as required. Thus h has a type 3 representation.

Conversely, let h have a type 3 representation (4.5.2) such that $v \in \mathcal{D}(A)$, that is

$$h(z) = a + \langle M(z)v, v \rangle$$

where $a \in \mathbb{R}$ and M is the structured resolvent of A of type 3 corresponding to Y, as in equation (4.2.3). Since $v \in \mathcal{D}(A)$ we may define the vector $\alpha \stackrel{\text{def}}{=} (1+iA)v \in \mathcal{H}$, and furthermore, by Proposition 4.2.6,

$$\begin{aligned} h(z) &= a + \left\langle (1-iA)\left\{(A-z_Y)^{-1} - C\right\}(1+iA)v, v\right\rangle \\ &= a + \left\langle \left\{(A-z_Y)^{-1} - C\right\}\alpha, \alpha\right\rangle \\ &= a - \langle C\alpha, \alpha\rangle + \left\langle (A-z_Y)^{-1}\alpha, \alpha\right\rangle \\ &= a_0 + \left\langle (A-z_Y)^{-1}\alpha, \alpha\right\rangle, \end{aligned}$$

where $a_0 \in \mathbb{R}$ is given by equation (4.5.4). Thus h has a representation of type 2. □

A special case of a type 2 representation occurs when the constant term a in equation (4.5.3) is 0. In one variable, this corresponds to Nevanlinna's characterization of the Cauchy transforms of positive finite measures on $\mathbb{R}$. Accordingly we define a *type* 1 *representation* of $h \in \mathcal{P}_2$ to be the special case of a type 2 representation of h in which $a = 0$ in (4.5.3).

Definition 4.5.4. *An analytic function h on Π^n has a* Nevanlinna representation of type 1 *if there exist a Hilbert space $\mathcal{H}$, a densely defined self-adjoint operator A on $\mathcal{H}$, a positive decomposition Y of $\mathcal{H}$ and a vector $\alpha \in \mathcal{H}$ such that, for all $z \in \Pi^n$,*

$$h(z) = \left\langle (A-z_Y)^{-1}\alpha, \alpha\right\rangle. \tag{4.5.6}$$

A representation of type 1 is obviously a representation of type 2. The following proposition is an immediate corollary of Proposition 4.5.3.

Proposition 4.5.5. *A function $h \in \mathcal{P}_2$ has a type* 1 *representation if and only if h has a type* 3 *representation as in equation* (4.5.2) *with the additional properties that $v \in \mathcal{D}(A)$ and*

$$a - \left\langle A(1+A^2)^{-1}\alpha, \alpha\right\rangle = 0.$$

For consistency with our earlier terminology for structured resolvents and representations we should have to define a structured resolvent of type 1 to be the same as a structured resolvent of type 2. We refrain from making such a confusing definition.

We conclude this section by giving examples of the four types of Nevanlinna representation in two variables.

Example 4.5.6. (1) The formula

$$h(z) = -\frac{1}{z_1+z_2} = \left\langle (0-z_Y)^{-1}v, v \right\rangle_{\mathbb{C}},$$

where $Y = (\frac{1}{2}, \frac{1}{2})$ and $v = 1/\sqrt{2}$, exhibits a representation of type 1, with $A = 0$.

(2) Likewise

$$h(z) = 1 - \frac{1}{z_1+z_2} = 1 + \left\langle (0-z_Y)^{-1}v, v \right\rangle_{\mathbb{C}}$$

is a representation of type 2.

(3) Let

$$h(z) = \begin{cases} \dfrac{1}{1+z_1z_2}\left(z_1 - z_2 + \dfrac{iz_2(1+z_1^2)}{\sqrt{z_1z_2}}\right) & \text{if } z_1z_2 \neq -1 \\ & \text{if} \\ \frac{1}{2}(z_1+z_2) & \text{if } z_1z_2 = -1 \end{cases} \tag{4.5.7}$$

where we take the branch of the square root that is analytic in $\mathbb{C} \setminus [0,\infty)$ with range Π. We claim that $h \in \mathcal{P}_2$ and that h has the type 3 representation

$$h(z) = \langle M(z)v, v \rangle_{L^2(\mathbb{R})}, \tag{4.5.8}$$

where $M(z)$ is the structured resolvent of type 3 given in Example 4.2.11 and $v(t) = 1/\sqrt{\pi(1+t^2)}$. To see this, let h be temporarily defined by equation (4.5.8). Since v is an even function in $L^2(\mathbb{R})$, equation (4.2.12) tells us that

$$h(z) = \int_{-\infty}^{\infty} \frac{t(1+z_1z_2)+(1-it)(itz_1+z_2)}{\pi(t^2-z_1z_2)(1+t^2)}\,\mathrm{d}t.$$

Since the denominator is an even function of t, the integrals of all the odd powers of t in the numerator vanish, and we have, provided $z_1z_2 \neq -1$,

$$\begin{aligned} h(z) &= \frac{2}{\pi}\int_0^{\infty} \frac{z_2+t^2z_1}{(t^2-z_1z_2)(1+t^2)}\,\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\infty} \frac{z_2(1+z_1^2)}{1+z_1z_2}\,\frac{1}{t^2-z_1z_2} + \frac{z_1-z_2}{1+z_1z_2}\,\frac{1}{1+t^2}\,\mathrm{d}t. \end{aligned}$$

Now, for $w \in \Pi$,

$$\int_0^{\infty} \frac{\mathrm{d}t}{t^2-w^2} = \frac{i\pi}{2w},$$

and so we find that h is indeed given by equation (4.5.7) in the case that $z_1z_2 \neq -1$. When $z_1z_2 = -1$ we have

$$\begin{aligned} h(z) &= \frac{2}{\pi}\int_0^{\infty} \frac{z_2+z_1t^2}{(1+t^2)^2}\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\infty} \frac{z_1}{1+t^2} + \frac{z_2-z_1}{(1+t^2)^2}\mathrm{d}t \\ &= \tfrac{1}{2}(z_1+z_2). \end{aligned}$$

Thus equation (4.5.8) is a type 3 representation of the function h given by equation (4.5.7). This function is *constant* and equal to i on the diagonal $z_1 = z_2$.

(4) The function

$$h(z) = \frac{z_1z_2}{z_1+z_2} = -\left(-\frac{1}{z_1} - \frac{1}{z_2}\right)^{-1}$$

clearly belongs to $\mathcal{P}_2$. It has the representation of type 4

$$h(z) = \langle M(z)v, v\rangle_{\mathbb{C}^2}$$

where $M(z)$ is the matricial resolvent given in Example 4.3.7 and

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We claim that each of the above representations is of the simplest available type for the function in question; for example, the function h in part (4) does not have a Nevanlinna representation of type 3. To prove this claim we need characterizations of the types of functions – the subject of the next two sections.

4.6 Asymptotic behavior and types of representations

In this section we shall give function-theoretic conditions for a function in $\mathcal{P}_2$ to have a representation of a given type. These conditions will be in terms of the asymptotic behavior of the function at ∞.

Every function in $\mathcal{P}_2$ has a type 4 representation, by Theorem 4.1.9. Let us characterize the functions that possess a type 3 representation. We denote by χ the vector $(1, \ldots, 1)$ of ones in $\mathbb{C}^n$. The following statement contains Theorem 4.1.8.

Theorem 4.6.1. *The following three conditions are equivalent for a function $h \in \mathcal{P}_2$.*

1. *The function h has a Nevanlinna representation of type 3;*

2.
$$\liminf_{s\to\infty} \frac{1}{s} \operatorname{Im}\ h(is\chi) = 0; \tag{4.6.1}$$

3.

$$\lim_{s\to\infty}\frac{1}{s}\operatorname{Im}\ h(is\chi)=0. \tag{4.6.2}$$

Proof. (1)⇒(3) Suppose that h has a Nevanlinna representation of type 3:

$$h(z)=a+\left\langle(1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1}v,v\right\rangle \tag{4.6.3}$$

for suitable $a\in\mathbb{R},\mathcal{H},A,Y$ and $v\in\mathcal{H}$. Since

$$(is\chi)_Y=\sum_j isY_j=is$$

we have

$$h(is\chi)=a+\left\langle(1-iA)(A-is)^{-1}(1+isA)(1-iA)^{-1}v,v\right\rangle.$$

Let ν be the scalar spectral measure for A corresponding to the vector $v\in\mathcal{H}$. By the Spectral Theorem

$$\begin{aligned}h(is\chi)&=a+\int(1-it)(t-is)^{-1}(1+ist)(1-it)^{-1}\ \mathrm{d}\nu(t)\\&=a+\int\frac{1+ist}{t-is}\ \mathrm{d}\nu(t).\end{aligned}$$

Since

$$\operatorname{Im}\ \frac{1+ist}{t-is}=\frac{s(1+t^2)}{s^2+t^2},$$

we have

$$\frac{1}{s}\operatorname{Im}\ h(is\chi)=\int\frac{1+t^2}{s^2+t^2}\ \mathrm{d}\nu(t).$$

The integrand decreases monotonically to 0 as $s\to\infty$ and so, by the Monotone Convergence Theorem, equation (4.6.2) holds.

(3)⇒(2) is trivial.

(2)⇒(1) Now suppose that $h \in \mathcal{P}_2$ and

$$\liminf_{s\to\infty} \frac{1}{s} \operatorname{Im} h(is\chi) = 0.$$

By Theorem 4.1.9, h has a Nevanlinna representation of type 4: that is, there exist $a, \mathcal{H}, \mathcal{N} \subset \mathcal{H}$, operators A, Y on $\mathcal{N}^\perp$ and a vector $v \in \mathcal{H}$ with the properties described in Definition 4.5.1 such that

$$h(z) = a + \langle M(z)v, v\rangle$$

for all $z \in \Pi^n$, where

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \times \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}^{-1}. \tag{4.6.4}$$

Thus, for $s > 0$, since once again $(is\chi)_P = is$,

$$\begin{aligned} M(is\chi) &= \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (A-is)^{-1} \end{bmatrix} \begin{bmatrix} is & 0 \\ 0 & 1+isA \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & (1-iA)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} is & 0 \\ 0 & (1-iA)(A-is)^{-1}(1+isA)(1-iA)^{-1} \end{bmatrix}. \end{aligned}$$

Let the projections of v onto $\mathcal{N}$, $\mathcal{N}^\perp$ be v_1, v_2 respectively. Then

$$\begin{aligned} h(is\chi) &= a + \langle M(is\chi)v, v\rangle \\ &= a + is\|v_1\|^2 + \left\langle (1-iA)(A-is)^{-1}(1+isA)(1-iA)^{-1}v_2, v_2 \right\rangle \end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{s}\operatorname{Im}\, h(is\chi) &= \|v_1\|^2 + \frac{1}{s}\operatorname{Im}\,\left\langle (1-iA)(A-is)^{-1}(1+isA)(1-iA)^{-1}v_2, v_2\right\rangle \\ &\geq \|v_1\|^2\end{aligned}$$

by Corollary 4.2.7. Hence

$$\begin{aligned}0 &= \liminf_{s\to\infty}\frac{1}{s}\operatorname{Im}\, h(is\chi) \\ &\geq \|v_1\|^2.\end{aligned}$$

It follows that $v_1 = 0$.

Let the compression of the projection P_j to $\mathcal{N}^\perp$ be Y_j: then $Y = (Y_1, \ldots, Y_n)$ is a positive decomposition of $\mathcal{N}^\perp$, and the compression of z_P to $\mathcal{N}^\perp$ is z_Y. By Remark 4.3.2 the (2,2) block $M_{22}(z)$ in $M(z)$ is

$$M_{22}(z) = (1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1}.$$

Since $v_1 = 0$ it follows that

$$\begin{aligned}h(z) &= a + \langle M(z)v, v\rangle \\ &= a + \langle M_{22}(z)v_2, v_2\rangle \\ &= a + \left\langle (1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1}v_2, v_2\right\rangle,\end{aligned}$$

which is the desired type 3 representation of h. Hence (2)⇒(1). □

In [BKVb] it is shown that condition (3) in the above theorem is also a necessary and sufficient condition that $-ih$ have a Π^n*-impedance-conservative realization.*

Type 2 representations were characterized by the following theorem in [ATDY12] in the case of two variables. The following result, which contains Theorem 4.1.7, shows that the result holds generally.

Theorem 4.6.2. *The following three conditions are equivalent for a function $h \in \mathcal{P}_2$.*

1. *The function h has a Nevanlinna representation of type* 2*;*

2. $$\liminf_{s\to\infty} s\,\mathrm{Im}\, h(is\chi) < \infty; \tag{4.6.5}$$

3. $$\lim_{s\to\infty} s\,\mathrm{Im}\, h(is\chi) < \infty. \tag{4.6.6}$$

Proof. (1)⇒(3) Suppose that h has the type 2 representation $h(z) = a + \left\langle (A - z_Y)^{-1} v, v \right\rangle$ for a suitable real a, self-adjoint A, positive decomposition Y and vector v. Let ν be the scalar spectral measure for A corresponding to the vector v. Then, for $s > 0$, $A - (is\chi)_Y = A - is$ and so

$$\begin{aligned} s\,\mathrm{Im}\; h(is\chi) &= s\,\mathrm{Im} \int \frac{\mathrm{d}\nu(t)}{t - is} \\ &= \int \frac{s^2\,\mathrm{d}\nu(t)}{t^2 + s^2}. \end{aligned}$$

The integrand is positive and increases monotonically to 1 as $s \to \infty$. Hence, by the Dominated Convergence Theorem

$$\lim_{s\to\infty} s\,\mathrm{Im}\; h(is\chi) = \nu(\mathbb{R}) = \|v\|^2 < \infty.$$

Hence (1)⇒(3).

(3)⇒(2) is trivial.

(2)⇒(1) Suppose (2) holds. *A fortiori*,

$$\liminf_{s\to\infty} \frac{1}{s} \operatorname{Im}\ h(is\chi) = 0.$$

By Theorem 4.6.1 h has a type 3 representation (4.6.3) for suitable $a \in \mathbb{R}, \mathcal{H}, A, Y$ and $v \in \mathcal{H}$. Let ν be the scalar spectral measure for A corresponding to the vector v. Then for $s > 0$

$$\begin{aligned} s\operatorname{Im}\ h(is\chi) &= s\operatorname{Im} \int \frac{1+ist}{t-is}\, \mathrm{d}\nu(t) \\ &= \int \frac{s^2(1+t^2)}{t^2+s^2}\, \mathrm{d}\nu(t). \end{aligned}$$

As $s \to \infty$ the integrand increases monotonically to $1+t^2$. Condition (2) now implies that

$$\int 1+t^2\, \mathrm{d}\nu(t) < \infty.$$

It follows that $v \in \mathcal{D}(A)$. Hence, by Proposition 4.5.3, h has a representation of type 2. □

In [ATDY12] we proved Theorem 4.6.2 for $n = 2$ using a different approach from the present one.

From this theorem the characterization of type 1 representations follows just as in the one-variable case. We obtain a strengthening of Theorem 4.1.6.

Theorem 4.6.3. *The following three conditions are equivalent for a function $h \in \mathcal{P}_2$.*

1. *The function h has a Nevanlinna representation of type* 1*;*

2. $$\liminf_{s\to\infty} s\,|h(is\chi)| < \infty;$$

3.

$$\lim_{s\to\infty} s\,|h(is\chi)| < \infty. \tag{4.6.7}$$

Proof. We follow Lax's treatment [Lax02] of the one-variable Nevanlinna theorem.

(1)⇒(3) Suppose that h has a type 1 representation as in equation (4.5.6) for some $\mathcal{H}$, A, Y and v. Then

$$\begin{aligned} h(is\chi) &= \left\langle (A-is)^{-1}\alpha, \alpha \right\rangle \\ &= \left\langle (A+is)(A^2+s^2)^{-1}\alpha, \alpha \right\rangle, \end{aligned}$$

and so

$$\text{Re } sh(is\chi) = \left\langle sA(A^2+s^2)^{-1}\alpha, \alpha \right\rangle, \quad \text{Im } sh(is\chi) = \left\langle s^2(A^2+s^2)^{-1}\alpha, \alpha \right\rangle.$$

Let ν be the scalar spectral measure for A corresponding to the vector $\alpha \in \mathcal{H}$. Then

$$\text{Re } sh(is\chi) = \int \frac{st}{t^2+s^2}\, \mathrm{d}\nu(t), \quad \text{Im } sh(is\chi) = \int \frac{s^2}{t^2+s^2}\, \mathrm{d}\nu(t).$$

The integrand in the first integral tends pointwise in t to 0 as $s \to \infty$, and by the inequality of the means it is no greater than $\frac{1}{2}$; thus the integral tends to 0 as $s \to \infty$ by the Dominated Convergence Theorem. The integrand in the second integral increases monotonically to 1 as $s \to \infty$. Thus

$$\text{Re } sh(is\chi) \to 0, \qquad \text{Im } sh(is\chi) \to \|\alpha\|^2 \quad \text{as } s \to \infty.$$

Hence the inequality (4.6.7) holds. Thus (1)⇒(3).

(3)⇒(2) is trivial.

(2)⇒(1) Suppose that

$$\liminf_{s\to\infty} s\,|h(is\chi)| < \infty. \tag{4.6.8}$$

As

$$\liminf_{s\to\infty} s\,\mathrm{Im}\,h(is\chi) \le \liminf_{s\to\infty} s\,|h(is\chi)| < \infty,$$

h satisfies condition (4.6.5) of Theorem 4.6.2. Therefore h has a representation of type 2, say

$$h(z) = a + \left\langle (A - z_Y)^{-1}\alpha, \alpha \right\rangle.$$

It remains to show that $a = 0$. The inequality (4.6.8) implies that there exists a sequence s_n tending to ∞ such that $h(is_n\chi) \to 0$. But

$$\mathrm{Re}\,h(is_n\chi) = a + \left\langle A(A^2 + s_n^2)^{-1}\alpha, \alpha \right\rangle \to a.$$

Hence $a = 0$ and h has a type 1 representation. This establishes (2)⇒(1). □

4.7 Carapoints at infinity

How can we recognise from function-theoretic properties whether a given function in the n-variable Loewner class admits a Nevanlinna representation of a given type? In the preceding section it was shown that it depends on growth along a single ray through the origin. In this section we describe the notion of carapoints at infinity for a function in the Pick class, and in the next section we shall give succinct criteria for the four types in the language of carapoints.

Carapoints (though not with this nomenclature) were first introduced by Carathéodory in 1929 [Car29] for a function φ on the unit disc, as a hypothesis in the "Julia-Carathéodory

Lemma". Recall that for any $\tau \in \mathbb{T}$, a function φ in the Schur class *satisfies the Carathéodory condition at* τ if

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - |\lambda|} < \infty. \tag{4.7.1}$$

The notion has been generalized to other domains by many authors. Consider domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ and an analytic function φ from U to the closure of V. The function φ is said to satisfy Carathéodory's condition at $\tau \in \partial U$ if

$$\liminf_{\lambda \to \tau} \frac{\operatorname{dist}(\varphi(\lambda), \partial V)}{\operatorname{dist}(\lambda, \partial U)} < \infty.$$

Thus, for example, when $U = \Pi^n, V = \Pi$, a function $h \in \mathcal{P}_n$ satisfies Carathéodory's condition at the point $x \in \mathbb{R}^n$ if

$$\liminf_{z \to x} \frac{\operatorname{Im}\ h(z)}{\min_j \operatorname{Im}\ z_j} < \infty. \tag{4.7.2}$$

This definition works well for finite points in ∂U, but for our present purpose we need to consider points at infinity in the boundaries of Π^n and Π. We shall introduce a variant of Carathéodory's condition for the class $\mathcal{P}_n$ with the aid of the Cayley transform

$$z = i\frac{1+\lambda}{1-\lambda}, \qquad \lambda = \frac{z-i}{z+i}, \tag{4.7.3}$$

which furnishes a conformal map between $\mathbb{D}$ and Π, and hence a biholomorphic map between $\mathbb{D}^n$ and Π^n by co-ordinatewise action. We obtain a one-to-one correspondence between $\mathcal{S}_2 \setminus \{\mathbf{1}\}$ and $\mathcal{P}_n$ via the formulae

$$h(z) = i\frac{1+\varphi(\lambda)}{1-\varphi(\lambda)}, \quad \varphi(\lambda) = \frac{h(z)-i}{h(z)+i} \tag{4.7.4}$$

where **1** is the constant function equal to 1 and λ, z are related by equations (4.7.3). For $\varphi \in \mathcal{S}_2$ we define $\tau \in \mathbb{T}^n$ to be a *carapoint* of φ if

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - \|\lambda\|_\infty} < \infty. \tag{4.7.5}$$

We can now extend the notion of carapoints to points at infinity. The point $(\infty, \ldots, \infty)$ in the boundary of Π^n corresponds to the point χ in the closed unit disc; as in the last section, χ denotes the point $(1, \ldots, 1) \in \mathbb{C}^n$.

Definition 4.7.1. *Let h be a function in the Pick class $\mathcal{P}_n$ with associated function φ in the Schur class $\mathcal{S}_2$ given by equation* (4.7.4). *Let $\tau \in \mathbb{T}^n$, $x \in (\mathbb{R} \cup \infty)^n$ be related by*

$$x_j = i\frac{1+\tau_j}{1-\tau_j} \quad \textit{for } j = 1, \ldots, n. \tag{4.7.6}$$

We say that x is a carapoint *for h if τ is a carapoint for φ. We say that h has a* carapoint at ∞ *if h has a carapoint at $(\infty, \ldots, \infty)$, that is, if φ has a carapoint at χ.*

Note that, for a point $x \in \mathbb{R}^n$, to say that x is a carapoint of h is *not* the same as saying that h satisfies the Carathéodory condition (4.7.2) at x. Consider the function $h(z) = -1/z_1$ in $\mathcal{P}_n$. Clearly h does not satisfy Carathéodory's condition at $0 \in \mathbb{R}^n$. However, the function φ in $\mathcal{S}_2$ corresponding to h is $\varphi(\lambda) = -\lambda_1$, which does have a carapoint at $-\chi$, the point in $\mathbb{T}^n$ corresponding to $0 \in \mathbb{R}^n$. Hence h has a carapoint at 0.

We shall be mainly concerned with carapoints at 0 and ∞. The following observation will help us identify them. For any $h \in \mathcal{P}_n$ we define $h^\flat \in \mathcal{P}_n$ by

$$h^\flat(z) = h\left(-\frac{1}{z_1}, \ldots, -\frac{1}{z_n}\right) \quad \text{for } z \in \Pi^n.$$

For $\varphi \in \mathcal{S}_2$ we define

$$\varphi^\flat(\lambda) = \varphi(-\lambda).$$

If h and φ are corresponding functions, as in equations (4.7.4), then so are $h^\flat$ and $\varphi^\flat$.

Proposition 4.7.2. *The following conditions are equivalent for a function $h \in \mathcal{P}_n$.*

1. ∞ *is a carapoint for h;*

2. 0 *is a carapoint for $h^\flat$;*

3. $$\liminf_{y \to 0+} \frac{\text{Im } h^\flat(iy\chi)}{y|h^\flat(iy\chi)+i|^2} < \infty;$$

4. $$\liminf_{y \to \infty} \frac{y \text{Im } h(iy\chi)}{|h(iy\chi)+i|^2} < \infty.$$

Proof. (1)$\Leftrightarrow$(2) Since $-\chi \in \mathbb{T}^n$ corresponds under the Cayley transform to $0 \in \mathbb{R}^n$, we have

$$\begin{aligned} \infty \text{ is a carapoint of } h \quad &\Leftrightarrow \quad \chi \text{ is a carapoint of } \varphi \\ &\Leftrightarrow \quad -\chi \text{ is a carapoint of } \varphi^\flat \\ &\Leftrightarrow \quad 0 \text{ is a carapoint of } h^\flat. \end{aligned}$$

(2)$\Leftrightarrow$(3) A consequence of the n-variable Julia-Carathéodory Theorem [Jaf93, Aba98], is that $\tau \in \mathbb{T}^n$ is a carapoint of $\varphi \in \mathcal{S}_2$ if and only if

$$\liminf_{r \to 1-} \frac{1-|\varphi(r\tau)|}{1-r} < \infty.$$

It follows that

$$
\begin{aligned}
0 \text{ is a carapoint for } h^\flat \quad &\Leftrightarrow \quad -\chi \text{ is a carapoint for } \varphi^\flat \\
&\Leftrightarrow \quad \liminf_{r\to 1-} \frac{1-|\varphi^\flat(-r\chi)|}{1-r} < \infty \\
&\Leftrightarrow \quad \liminf_{r\to 1-} \frac{1-|\varphi^\flat(-r,-r)|^2}{1-r^2} < \infty.
\end{aligned}
$$

Let $iy \in \Pi$ be the Cayley transform of $-r \in (-1,0)$, so that $y \to 0+$ as $r \to 1-$. In view of the identity

$$
\frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|_\infty^2} = \left(\max_j \frac{|z_j+i|^2}{\text{Im } z_j}\right) \frac{\text{Im } h(z)}{|h(z)+i|^2} \tag{4.7.7}
$$

we have

$$
\begin{aligned}
0 \text{ is a carapoint for } h^\flat \quad &\Leftrightarrow \quad \liminf_{y\to 0+} \frac{|iy+i|^2}{y} \frac{\text{Im } h^\flat(iy\chi)}{|h^\flat(iy\chi)+i|^2} < \infty \\
&\Leftrightarrow \quad \liminf_{y\to 0+} \frac{\text{Im } h^\flat(iy\chi)}{y|h^\flat(iy\chi)+i|^2} < \infty.
\end{aligned}
$$

(3)⇔(4) Replace y by $1/y$. □

Corollary 4.7.3. *If $f \in \mathcal{P}_n$ satisfies Carathéodory's condition*

$$
\liminf_{z\to x} \frac{\text{Im } f(z)}{\text{Im } z} < \infty \tag{4.7.8}
$$

at $x \in \mathbb{R}^n$ then x is a carapoint for f. If

$$
\liminf_{y\to\infty} y \, \text{Im } f(iy\chi) < \infty
$$

then ∞ is a carapoint for f.

Proof. Let $h = f^\flat \in \mathcal{P}_n$. Clearly $|h^\flat(z) + i| \geq 1$ for all $z \in \Pi^n$. If the condition (4.7.8) holds for $x = 0$ then

$$\liminf_{z \to 0} \frac{\text{Im } h^\flat(z)}{|h^\flat(z) + i|^2 \min_j \text{Im } z_j} \leq \liminf_{z \to 0} \frac{\text{Im } h^\flat(z)}{\min_j \text{Im } z_j} < \infty$$

and hence, by (2)⇔(3) of Proposition 4.7.2, 0 is a carapoint for $h^\flat = f$. The case of a general $x \in \mathbb{R}^n$ follows by translation. □

If $h \in \mathcal{P}_n$ has a carapoint at $x \in (\mathbb{R} \cup \infty)^n$ then it has a value at x in a natural sense. If $\varphi \in \mathcal{S}_2$ has a carapoint at $\tau \in \mathbb{T}^n$, then by [Jaf93] there exists a unimodular constant $\varphi(\tau)$ such that

$$\lim_{\lambda \stackrel{\text{nt}}{\to} \tau} \varphi(\lambda) = \varphi(\tau). \tag{4.7.9}$$

Here $\lambda \stackrel{\text{nt}}{\to} \tau$ means that λ tends nontangentially to τ in $\mathbb{D}^n$.

Definition 4.7.4. *If $h \in \mathcal{P}_n$ has a carapoint at $x \in (\mathbb{R} \cup \infty)^n$ then we define*

$$h(x) = \begin{cases} \infty & \textit{if } \varphi(\tau) = 1 \\ & \textit{if} \\ i\dfrac{1+\varphi(\tau)}{1-\varphi(\tau)} & \textit{if } \varphi(\tau) \neq 1 \end{cases}$$

where $\tau \in \mathbb{T}^n$ corresponds to x as in equation (4.7.6).

Thus $h(\infty) \in \mathbb{R} \cup \{\infty\}$ when ∞ is a carapoint of h.

In the example $h(z) = -1/z_1$, since the value of $\varphi(-\lambda)$ at $-\chi$ is 1, we have $h(0) = \infty$.

Although the value of $h(\infty)$ is defined in terms of the Schur class function φ, it can be expressed more directly in terms of h.

Proposition 4.7.5. *If ∞ is a carapoint of h then*

$$h(\infty) = h^\flat(0) = \lim_{z \stackrel{\text{nt}}{\to} \infty} h(z). \tag{4.7.10}$$

Here we say that $z \stackrel{\text{nt}}{\to} \infty$ if $z \to (\infty, ..., \infty)$ in the set $\{z \in \Pi^n : (-1/z_1, \dots, -1/z_n) \in S\}$ for some set $S \subset \Pi^n$ that approaches 0 nontangentially, or equivalently, if $z \to (\infty, \dots, \infty)$ in a set on which $\|z\|_\infty / \min_j \operatorname{Im} z_j$ is bounded.

Proof. Clearly

$$h(\infty) = \infty \quad \Leftrightarrow \quad \varphi(\chi) = 1 \quad \Leftrightarrow \quad \varphi^\flat(-\chi) = 1 \quad \Leftrightarrow \quad h^\flat(0) = \infty.$$

Similarly, for $\xi \in \mathbb{R}$,

$$h(\infty) = \xi \quad \Leftrightarrow \quad \varphi(\chi) = \frac{\xi - i}{\xi + i} \quad \Leftrightarrow \quad \varphi^\flat(-\chi) = \frac{\xi - i}{\xi + i} \quad \Leftrightarrow \quad h^\flat(0) = \xi.$$

Thus, whether $h(\infty)$ is finite or infinite, $h(\infty) = h^\flat(0)$. Equation (4.7.10) follows from the relation (4.7.9). □

4.8 Types of functions in the Loewner class

In this section we shall show that the type of a function $h \in \mathcal{P}_2$ is entirely determined by whether or not ∞ is a carapoint of h and by the value of $h(\infty)$. Let us make precise the notion of the *type* of a function in $\mathcal{P}_2$.

Definition 4.8.1. *A function $h \in \mathcal{P}_2$ is of* type 1 *if it has a Nevanlinna representation of type* 1*. For $n = 2, 3$ or* 4 *we say that h is of* type *n if h has a Nevanlinna representation of type n but has no representation of type $n-1$.*

Clearly every function in $\mathcal{P}_2$ is of exactly one of the types 1 to 4. We shall now prove Theorem 4.1.10. Recall that it states the following, for any function $h \in \mathcal{P}_2$.

1. h is of type 1 if and only if ∞ is a carapoint of h and $h(\infty) = 0$;

2. h is of type 2 if and only if ∞ is a carapoint of h and $h(\infty) \in \mathbb{R} \setminus \{0\}$;

3. h is of type 3 if and only if ∞ is not a carapoint of h;

4. h is of type 4 if and only if ∞ is a carapoint of h and $h(\infty) = \infty$.

Proof. (2) Let $h \in \mathcal{P}_2$ have a type 2 representation $h(z) = a + \left\langle (A - z_Y)^{-1} v, v \right\rangle$ with $a \neq 0$. By Theorem 4.6.2,

$$\liminf_{y \to \infty} y \operatorname{Im}\ h(iy\chi) < \infty.$$

By Corollary 4.7.3, ∞ is a carapoint for h. Furthermore, by Proposition 4.7.5

$$h(\infty) = \lim_{y \to \infty} h(iy\chi) = a \in \mathbb{R} \setminus \{0\}.$$

Conversely, suppose that ∞ is a carapoint for h and $h(\infty) \in \mathbb{R} \setminus \{0\}$. By Proposition 4.7.2

$$\liminf_{y \to \infty} \frac{y \operatorname{Im}\ h(iy\chi)}{|h(iy\chi) + i|^2} < \infty$$

while by Proposition 4.7.5

$$\lim_{y \to \infty} |h(iy\chi) + i|^2 = h(\infty)^2 + 1 \in (1, \infty).$$

On combining these two limits we find that

$$\liminf_{y \to \infty} y \operatorname{Im}\ h(iy\chi) < \infty,$$

and so, by Theorem 4.6.2, h has a representation of type 2. Since $h(\infty) \neq 0$ it is clear that h does not have a representation of type 1. Thus (2) holds.

A trivial modification of the above argument proves that (1) is also true.

(4) Let h be of type 4. Then h has no type 3 representation, and so, by Theorem 4.6.1, there exists $\delta > 0$ and a sequence (s_n) of positive numbers tending to ∞ such that

$$\frac{1}{s_n} \operatorname{Im} h(is_n\chi) \geq \delta > 0.$$

Let $y_n = 1/s_n$; then $-1/(is_n) = iy_n$, and we have

$$y_n \operatorname{Im}\ h^\flat(iy_n\chi) \geq \delta \quad \text{for all } n \geq 1. \tag{4.8.1}$$

Since $|h^\flat(z) + i| > \operatorname{Im}\ h^\flat(z)$ for all z, we have

$$\begin{aligned}\liminf_{z\to 0} \frac{\operatorname{Im}\ h^\flat(z)}{|h^\flat(z)+i|^2 \min_j \operatorname{Im}\ z_j} &\leq \liminf_{z\to 0} \frac{1}{\operatorname{Im}\ h^\flat(z) \min_j \operatorname{Im}\ z_j} \\ &\leq \liminf_{n\to\infty} \frac{1}{y_n \operatorname{Im}\ h^\flat(iy_n\chi)} \\ &\leq 1/\delta.\end{aligned}$$

Hence $(0,0)$ is a carapoint of $h^\flat$, and so ∞ is a carapoint of h.

Since $y_n \to 0$ it follows from the inequality (4.8.1) that $\operatorname{Im}\ h^\flat(iy_n\chi) \to \infty$, hence that $h^\flat(0) = \infty$, and therefore that $h(\infty) = \infty$.

Conversely, suppose that ∞ is a carapoint of h and that $h(\infty) = \infty$. We shall show that

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{Im} h(is\chi) \neq 0, \tag{4.8.2}$$

and it will follow from Theorem 4.6.1 that h does not have a representation of type 3, that is, h is of type 4.

Let $\varphi \in \mathcal{S}_2$ correspond to h and let $r \in (0,1)$ correspond to $is \in \Pi$. Then

$$\begin{aligned}\frac{1}{s}\operatorname{Im}\, h(is\chi) &= \frac{1-r}{1+r} \quad \frac{1-|\varphi(r\chi)|^2}{|1-\varphi(r\chi)|^2} \\ &= \frac{1-|\varphi(r\chi)|^2}{1-r^2}\,\frac{(1-r)^2}{|1-\varphi(r\chi)|^2}. \end{aligned} \tag{4.8.3}$$

By hypothesis, χ is a carapoint for φ and $\varphi(\chi) = 1$. By definition of carapoint,

$$\liminf_{z\to\chi} \frac{1-|\varphi(z)|^2}{1-\|z\|_\infty^2} = \alpha < \infty \quad \text{for all } s > 0.$$

The n-variable Julia-Carathéodory Lemma (see [Jaf93, Aba98]) now tells us that $\alpha > 0$ and

$$\frac{|1-\varphi(r\chi)|^2}{|1-r|^2} \le \alpha \frac{1-|\varphi(r\chi)|^2}{1-r^2} \quad \text{for all } r \in (0,1). \tag{4.8.4}$$

On combining equations (4.8.3) and (4.8.4) we obtain

$$\frac{1}{s}\operatorname{Im}\, h(is\chi) \ge \frac{1}{\alpha} > 0 \quad \text{for all } s > 0.$$

Thus the relation (4.8.2) is true, and so, by Theorem 4.6.1, h is of type 4.

Statement (3) now follows easily. The function $h \in \mathcal{P}_2$ is of type 3 if and only if it is not of types $1, 2$ or 4, hence if and only if it is not the case that ∞ is a carapoint for h and $h(\infty) \in \mathbb{R} \cup \{\infty\}$, hence if and only if ∞ is not a carapoint of h. □

Chapter 4 contains material as it may appear in the Proceedings of the London Mathematical Society, 2014. The dissertation author was a co-author with J. Agler and N.J. Young on this material.

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