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**Author**
Stoffregen, Matthew Henry

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# Applications of Pin(2)-equivariant Seiberg-Witten Floer homology

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ABSTRACT OF THE DISSERTATION

Applications of Pin(2)-equivariant Seiberg-Witten Floer homology

by

Matthew Henry Stoffregen
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2017
Professor Ciprian Manolescu, Chair

We study Manolescu's Pin(2)-equivariant Seiberg-Witten Floer homology of rational homology three-spheres, with applications to the homology cobordism group $\theta_3^H$ in mind. We compute this homology theory for Seifert rational homology three-spheres in terms of their Heegaard Floer homology. We prove Manolescu's conjecture that $\beta = -\bar{\mu}$, the Neumann-Siebenmann invariant, for Seifert integral homology three-spheres. We establish the existence of integral homology spheres not homology cobordant to any Seifert space. We show that there is a naturally defined subgroup of the homology cobordism group, generated by certain Seifert spaces, which admits a $\mathbb{Z}^\infty$ summand, generalizing the theorem of Fintushel-Stern and Furuta on the infinite-generation of the homology cobordism group. In addition to the application of the Pin(2)-theory to Seifert spaces, we apply it to the full homology cobordism group. In this direction, we identify a $\mathbb{F}[U]$-submodule of Heegaard Floer homology, called connected Seiberg-Witten Floer homology, whose isomorphism class is a homology cobordism invariant.

The dissertation of Matthew Henry Stoffregen is approved.

Per Kraus

Ko Honda

Robert Brown

Ciprian Manolescu, Committee Chair

University of California, Los Angeles

2017

*To my parents, Patty and Roger,*

*and my brother Daniel,*

*and Laure*

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## VITA

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## PUBLICATIONS

*A remark on* Pin(2)*-equivariant Floer homology*, Michigan Mathematics Journal, *to appear*

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# CHAPTER 1

# Introduction

## 1.1 Introduction

Our story starts with *Seiberg-Witten Floer homology*, a functor that associates to a pointed, closed, oriented 3-manifold $Y$ with a spin$^c$ structure $\mathfrak{s}$ an abelian group, denoted $SWFH(Y, \mathfrak{s})$ and called the Seiberg-Witten Floer homology of $(Y, \mathfrak{s})$. Roughly speaking, $SWFH(Y, \mathfrak{s})$ is defined analgously to the construction of Morse homology for a finite-dimensional manifold. Recall that Morse homology associates to a (finite-dimensional, closed, oriented) Riemannian manifold $M$, equipped with a function $f : M \to \mathbb{R}$ satisfying certain transversality conditions (which are generically satisfied) a chain complex, with generators (over $\mathbb{Z}$) the critical points of the function $f$, and differentials given by counting the index 1 gradient trajectories between critical points. The homology of the chain complex is denoted by $H(M, f)$, and it turns out that the resulting homology theory is isomorphic to singular homology of the manifold $M$ (in particular, is independent of the function $f$).

The homology $SWFH(Y, \mathfrak{s})$ is thought of as the 'Morse Homology of the Chern-Simons-Dirac functional (csd functional) $\mathcal{L}$'. The csd function is defined on an infinite-dimensional space $\mathcal{B}$ of spin$^c$-connections and spinors over the 3-manifold $Y$, with its spin$^c$ structure $\mathfrak{s}$, in contrast to the Morse homology situation in finite dimensions. Roughly, $SWFH(Y, \mathfrak{s})$ can be thought of as the homology of a chain complex whose generators are the critical points of the csd functional, and whose differentials count formal gradient trajectories between critical points.

However, in the infinite-dimensional setting, it is not the case that just any functional $f$ determines a homology theory, but the csd functional has good properties that make it possible to define a homology theory following the general picture above. Typically, we call any homology theory constructed from an infinite-dimensional space by following this picture a *Floer theory.*

To be more precise, there are multiple definitions of a homology theory coming from the Chern-Simons-Dirac functional, and we will call any such theory a *monopole Floer homology.* Marcolli-Wang [32] provided a definition for a restricted class of 3-manifolds. The version we have been using above, $SWFH(Y, \mathfrak{s})$, was defined by Manolescu [28] and is only defined for 3-manifolds with first betti number $b_1(Y) = 0$. Kronheimer-Mrowka in [23] defined a monopole Floer homology for all closed oriented 3-manifolds with spin$^c$-structure, and their version is denoted $\widetilde{HM}(Y, \mathfrak{s})$. We will occasionally confound $SWFH(Y, \mathfrak{s})$ and $\widetilde{HM}(Y, \mathfrak{s})$, (Lidman-Manolescu [25] have shown that these abelian groups are canonically isomorphic), but during the course of the introduction we will also address the fact that their definitions are rather disparate.

For comparison, we note that the first Floer theory for 3-manifolds, instanton homology [10], is in some sense the dimensional reduction to 3-dimensions of Donaldson's polynomial invariant of closed 4-manifolds. Similarly, monopole Floer homology is the 3-dimensional cousin of the Seiberg-Witten (monopole) invariant of closed 4-manifolds, introduced in [54].

One of the key features of monopole Floer homology comes from the fact that the Chern-Simons-Dirac functional is invariant with respect to an $S^1$-action on $\mathcal{B}$. Pursuing the finite-dimensional analogy above, we would like to compare the Floer homology of $\mathcal{L}$ with the Morse homology of a function on a manifold with an $S^1$-action. For a manifold with $S^1$-action, we can take the *equivariant* (or *Borel*) *homology.* The Borel homology of a space $X$ with the action of a compact Lie group $G$, written $H^G_*(X)$, is a module over $H^*(BG)$, where $BG$ is the classifying space of $G$. In particular, for the case $G = S^1$, we have $BS^1 = \mathbb{C}P^\infty$, and $H^*(BS^1) = \mathbb{Z}[U]$, so $H^{S^1}_*(X)$ is equipped with a $\mathbb{Z}[U]$-module structure. To obtain the most general picture of Floer homology of $\mathcal{L}$, we then would like to have that the homology theory $SWFH(Y, \mathfrak{s})$ is a module over $\mathbb{Z}[U]$.

The chief difficulty in setting up such an *equivariant theory* is the presence of reducible points in the configuration space $\mathcal{B}$. We call a point $p \in \mathcal{B}$ reducible if the action by $S^1$ has nontrivial stabilizer (it turns out that having nontrivial stabilizer implies that $S^1$ acts trivially on $p$). In the monopole setting, reducible critical points in the configuration space correspond to $S^1$-flat connections. In particular, for integer homology spheres, there is a unique reducible point (and it is always a critical point for $\mathcal{L}$). In the setting of Kronheimer-Mrowka, the presence of reducibles is overcome by introducing the *blow-up construction*, where the configuration space is replaced with a new space (the blow-up) $\mathcal{B}^\sigma$ lying over $\mathcal{B}/S^1$, and one proceeds to construct Floer homology in the blow-up. However, in this process new difficulties are also created. The blow-up $\mathcal{B}^\sigma$ is analogous to a manifold-with-boundary in the finite-dimensional setting, and so one must develop a Floer theory in analogy with the case of finite-dimensional manifolds with boundary, generalizing the procedure we outlined above, for closed finite-dimensional manifolds.

Manolescu's construction of $SWFH(Y, \mathfrak{s})$ proceeds along different lines, and is limited to the setting where $b_1(Y) = 0$. To describe the construction, we first introduce an object called the *Conley index*, associated to a dynamical system on a finite-dimensional manifold $X$.

To describe this object, let $\phi_s$ for $s \in \mathbb{R}$ be the dynamical system on $X$. We call a compact subset $S$ of $X$ an *isolated invariant set* if

1. $S$ is invariant; namely $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$.
2. $S$ is the maximal invariant set in some compact neighborhood $N$ of $S$ for which $S \subset \text{int}(N)$.

Then the Conley index of $S$, denoted $I(S, \phi)$, is defined to be the pointed topological space $(N/L, [L])$, where $N$ is any isolating neighborhood of $S$ (that is, a neighborhood so that the above conditions are satisfied), and $L$ is an *exit set*. The Conley index is well-defined up to homotopy equivalence, independent of the choice of $N$ and $L$.

Manolescu constructs $SWFH(Y, \mathfrak{s})$ as the equivariant homology of a topological space $SWF(Y, \mathfrak{s})$ equipped with an $S^1$-action, which is built as the Conley index of *finite-dimensional* *approximations* of the Seiberg-Witten equations. Here, by a finite-dimensional approximation, we mean a finite-dimensional subspace of $\mathcal{B}$, along with a projection of the gradient of the Chern-Simons-Dirac functional to the finite-dimensional subspace. This gives us a vector field over a finite-dimensional manifold, and any such defines a dynamical system. From this dynamical system we can take a Conley index, which is roughly speaking $SWF(Y, \mathfrak{s})$. In particular, Manolescu shows that as one takes larger and larger approximations above, the homotopy-types are related by suspensions. The result is a well-defined stable-homotopy type $SWF(Y, \mathfrak{s})$ (with an $S^1$-action).

However, the issue of equivariance does not end with considering the $S^1$-action. In the case that a spin$^c$-structure actually comes from a spin structure, the Seiberg-Witten equations inherit a Pin(2)-symmetry, where Pin(2) is the subgroup of the unit quaternions generated by the unit circle in the complex plane, along with the quaternion $j$. In this case $SWF(Y, \mathfrak{s})$ is a Pin(2)-equivariant stable homotopy type. Then, its Pin(2)-equivariant homology, denoted $SWFH^{\mathrm{Pin}(2)}(Y, \mathfrak{s})$ is a module over $H^*(B\mathrm{Pin}(2)) = \mathbb{F}[U, q]/(q^3)$. Here and subsequently, $\mathbb{F}$ will be the field of two elements, and $SWFH^{\mathrm{Pin}(2)}(Y, \mathfrak{s})$ will be taken with $\mathbb{F}$-coefficients.

The Pin(2)-equivariance of the Seiberg-Witten equations in the presence of a spin structure was first used by Furuta [15] in order to prove the 10/8-Theorem. That is, the rank of $H_2(X)$ is at least 10/8 the signature of the intersection form on $H_2(X)$ for $X$ a spin simply-connected smooth closed 4-manifold. Furuta's technique required the Bauer-Furuta invariant of a 4-manifold, a homotopy refinement of the Seiberg-Witten invariant, and involved looking at its $K$-theory.

Manolescu introduced the Pin(2)-equivariant Seiberg-Witten Floer homotopy type, written $SWF(Y, \mathfrak{s})$, in [30] (upgrading the $S^1$-equivariance from [28]) and used it there to disprove the Triangulation conjecture. The study of this Pin(2)-equivariant Seiberg-Witten Floer homology is the topic of this thesis. In particular, we study the *Manolescu invariants*, $\alpha, \beta$, and $\gamma$, that arise as generalizations of the Frøyshov invariant (which we will introduce below).

In the remainder of this section, as motivation, we review Manolescu's disproof of the triangulation conjecture using these invariants. Let us first go over the statement of the triangulation conjecture, then we will address its connection to low-dimensional topology, and finally return to gauge theory to see how the disproof works.

**Question 1** (Kneser [22]). *Does every topological manifold admit a triangulation?*

Here, by a triangulation we mean a homeomorphism from a topological manifold $X$ to the realization of a simplicial complex.

### 1.1.1 The triangulation conjecture and low-dimensional topology

To explain this connection, we introduce the homology cobordism group $\theta_3^H$. We call two oriented, closed integer homology 3-spheres $Y_1$ and $Y_2$ *homology cobordant* if there exists a smooth oriented compact manifold $W$ so that $\partial W = Y_0 \amalg -Y_1$ and so that the maps on homology induced by inclusions $\iota_*\colon H_*(Y_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ are isomorphisms. Then $\theta_3^H$ is the set of equivalence classes of integer homology 3-spheres up to homology cobordism, and it inherits the structure of an abelian group as follows. We define addition using the connected sum operation $[Y_1] + [Y_2] = [Y_1 \# Y_2] \in \theta_3^H$. From this, it is clear that $[S^3]$ is the identity element of $\theta_3^H$, and we have inverse given by orientation reversal (one must of course check that $Y\# -Y$ is homology cobordant to $S^3$, as is readily verified).

The first invariant to distinguish elements of $\theta_3^H$ is the Rokhlin homomorphism $\mu\colon \theta_3^H \to \mathbb{Z}/2$. The construction of this invariant is made possible by the Theorem of Rokhlin:

**Theorem 1.1.1.** *(Rokhlin [43]) Any closed, smooth spin* 4*-manifold $X$ has signature $\sigma(X)$ divisible by* 16*.*

Then, we define $\mu(Y)$ to be $\sigma(X)/8 \bmod 2$, where $X$ is a smooth compact spin 4-manifold bounded by $Y$. Rokhlin's Thoerem guarantees that this quantity is independent of the choice of such $X$.

Galewski-Stern reduced the triangulation conjecture to a question about low-dimensional topology. They showed that there exist non-triangulable topological manifolds in all dimensions at least 5 if and only if there exists any non-triangulable topological manifold in dimension at least 5 if and only if there exists an element $[Y] \in \theta_3^H$ with $\mu(Y) = 1$ and $2[Y] = 0 \in \theta_3^H$.

### 1.1.2 The Frøyshov invariant

In order to disprove the Triangulation Conjecture, Manolescu introduces a version of the Frøyshov invariant, in analogy to the $h$-invariant from [13]. Previously, the $h$-invariant had been generalized from instanton homology to other versions of Floer homology for 3-manifolds, as in [38] for Heegaard Floer homology and [23] for monopole Floer homology. For convenience, we will review the $h$-invariant in the setting of the Seiberg-Witten Floer stable homotopy type of an oriented 3-manifold $Y$ with spin$^c$ structure $\mathfrak{s}$ and $b_1(Y) = 0$.

Recall that $SWFH^{S^1}(Y, \mathfrak{s})$ is the $S^1$-equivariant Borel homology of $SWF(Y, \mathfrak{s})$. As such, it comes with the action of $\mathbb{F}[U]$ (now working with $\mathbb{F}$ coefficients). However, the *equivariant localization theorem* (see [50] III) states that the localization of $H_*^{S^1}(SWF(Y, \mathfrak{s}))$ at the ideal $(U) \subset \mathbb{F}[U]$ is isomorphic to $\mathbb{F}[U]^{-1}H_*^{S^1}(SWF(Y, \mathfrak{s})^{S^1})$, where $SWF(Y, \mathfrak{s})^{S^1}$ is the subset of $SWF(Y, \mathfrak{s})$ fixed under the action of $S^1$. Roughly speaking, this says that the algebraic structure of the Borel homology module records information about the types of orbits in $SWF(Y, \mathfrak{s})$. Because we know that the fixed-point set of $SWF(Y, \mathfrak{s})$ is precisely a point, we have

$$\mathbb{F}[U]^{-1}H_*^{S^1}(SWF(Y, \mathfrak{s})) \cong \mathbb{F}[U, U^{-1}].$$

Since $H_*^{S^1}(SWF(Y, \mathfrak{s}))$ is bounded below, we obtain that there is some minimal degree $d$ for which the map

$$H_*^{S^1}(SWF(Y, \mathfrak{s})) \to \mathbb{F}[U]^{-1}H_*^{S^1}(SWF(Y, \mathfrak{s}))$$

is a surjection. We call $d/2$ the *Frøyshov invariant* of $Y$, and denote it $\delta(Y)$.

The utility of the Frøyshov invariant derives from knowing the reducible set of the Seiberg-Witten equations on a 4-manifold. In particular, Manolescu [28] showed that associated to a homology cobordism from $Y_1$ to $Y_2$, there is a map of stable homotopy types

$$SWF(Y_1, \mathfrak{s}_1) \to SWF(Y_2, \mathfrak{s}_2),$$

which induces a homotopy equivalence on fixed-point sets. In particular, such a cobordism map induces an isomorphism:

$$H_*^{S^1}(SWF(Y_1,\mathfrak{s}_1)^{S^1}) \to H_*^{S^1}(SWF(Y_2,\mathfrak{s}_2)^{S^1}),$$

We inherit a commutative diagram

$$\begin{array}{ccc} H_*^{S^1}(SWF(Y_1,\mathfrak{s}_1)) & \longrightarrow & H_*^{S^1}(SWF(Y_2,\mathfrak{s}_2)) \\ \uparrow & & \uparrow \\ H_*^{S^1}(SWF(Y_1,\mathfrak{s}_1)^{S^1}) & \longrightarrow & H_*^{S^1}(SWF(Y_2,\mathfrak{s}_2)^{S^1}), \end{array} \tag{1.1}$$

and reflecting on the commutative diagram we see that $\delta(Y)$ is an invariant of homology cobordism.

Manolescu performed a similar construction, using the Pin(2)-equivariant homology of $SWF(Y,\mathfrak{s})$ in place of the $S^1$-equivariant theory. His construction results in three separate invariants $\alpha, \beta$, and $\gamma$, corresponding to the fact that $H^*(B\mathrm{Pin}(2);\mathbb{F}) = \mathbb{F}[v,q]/(q^3)$, has three separate 'towers' corresponding to $1, q, q^2$. We will review the construction in the section on equivariant topology.

To finish a sketch of Manolescu's disproof of the triangulation conjecture, we only need a few more features of the invariant $\beta$. First,

$$\beta(Y,\mathfrak{s}) = \mu(Y,\mathfrak{s}) \bmod 2.$$

This follows essentially since the degree of the reducible in the Seiberg-Witten equations agrees with the Rokhlin invariant mod 2, and that $v$ is of degree 4. Moreover, Manolescu shows that

$$\beta(Y,\mathfrak{s}) = -\beta(-Y,\mathfrak{s})$$

where $-Y$ denotes orientation reversal. Then say, to obtain a contradiction, that $Y$ is 2-torsion in $\theta_3^H$, with $\mu(Y) = 1$, and hence $\beta(Y)$ is nonzero. We have $[Y] = [-Y] \in \theta_3^H$ and so $\beta(Y) = \beta(-Y) = -\beta(Y)$, contradicting $\beta(Y) \neq 0$. Thus, there is no such $Y$, finishing the proof.

## 1.2 Pin(2)-equivariant Floer homology of Seifert spaces

We next address the contents of this thesis. Let $Y$ be a closed, oriented three-manifold with $b_1 = 0$ and spin structure $\mathfrak{s}$, and let $G = \text{Pin}(2)$, the subgroup $S^1 \cup jS^1$ of the unit quaternions.

For now, also let $Y$ be a Seifert rational homology sphere, such that the base orbifold of the Seifert fibration of $Y$ has $S^2$ as underlying space[1]. We will use the description of the Seiberg-Witten moduli space given by Mrowka, Ozsváth, and Yu [33] to compute $SWFH^G(Y,\mathfrak{s})$, as a module over $\mathbb{F}[q,v]/(q^3)$ (Here, the action of $v$ decreases grading by 4, and that of $q$ decreases grading by 1). The description is in terms of the Heegaard Floer homology $HF^+(Y,\mathfrak{s})$, defined in [41],[40]. In particular, this description makes $SWFH^G(Y,\mathfrak{s})$ quickly computable, as Ozsváth-Szabó, Némethi, and Can-Karakurt [39],[34],[3] have developed algorithms to calculate $HF^+(Y,\mathfrak{s})$ for $Y$ a Seifert space. In order to obtain $SWFH^G(Y,\mathfrak{s})$ in terms of $HF^+(Y,\mathfrak{s})$, we use both the equivalence of $HF^+$ and $\widecheck{HM}$ due to Kutluhan-Lee-Taubes [24], and Colin-Ghiggini-Honda [4] and Taubes [49], and the equivalence of $\widecheck{HM}$ and $SWFH^{S^1}$ due to Lidman-Manolescu [25]. Here $SWFH^{S^1}(Y,\mathfrak{s})$ denotes the $S^1$-equivariant Borel homology of the stable homotopy type $SWF(Y,\mathfrak{s})$.

We will need to relate $SWFH^{S^1}(Y,\mathfrak{s})$ and $SWFH^G(Y,\mathfrak{s})$ when the underlying homotopy type $SWF(Y,\mathfrak{s})$ is simple enough. This should be compared with [27], in which Lin calculates the Pin(2)-monopole Floer homology in the setting of [26] for many classes of three-manifolds $Y$ obtained by surgery on a knot. The approach there is based, similarly, on extracting information from the $S^1$-equivariant theory $\widecheck{HM}(Y,\mathfrak{s})$ of [23], when $\widecheck{HM}(Y,\mathfrak{s})$ is simple enough.

To state the calculation of $SWFH^G(Y,\mathfrak{s})$, let $\mathcal{T}^+$ denote $\mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$, and $\mathcal{T}^+(i) = \mathbb{F}[U^{-i+1},U^{-i+2},...]/U\mathbb{F}[U]$. We also introduce the notation $\mathcal{V}^+$ to denote $\mathbb{F}[v,v^{-1}]/v\mathbb{F}[v]$, and $\mathcal{V}^+(i) = \mathbb{F}[v^{-i+1},v^{-i+2},...]/v\mathbb{F}[v]$. For any graded module $M$, let $M_n$ denote the submodule

[1]There are also Seifert fibered rational homology spheres with base orbifold $\mathbb{RP}^2$, and some of them do not have a Seifert structure over $S^2$. These are not considered here. None of them are integral homology spheres. Furthermore, in order for a Seifert fiber space $Y$ to be a rational homology sphere, it must fiber over an orbifold with underlying space either $\mathbb{RP}^2$ or $S^2$.

of homogeneous elements of degree $n$, and define $M[k]$ by $M[k]_n = M_{n+k}$. Let $\mathcal{T}_d^+(n) = \mathcal{T}^+(n)[-d]$ and $\mathcal{V}_d^+(n) = \mathcal{V}^+(n)[-d]$. The module $\mathcal{T}_d^+(n)$ is then supported in degrees from $d$ to $d+2(n-1)$, with the parity of $d$.

Fix $Y$ a Seifert rational homology three-sphere with negative fibration; that is, the orbifold line bundle of $Y$ is of negative degree (see Section 4.2). For example, the Brieskorn sphere $\Sigma(a_1,...,a_n)$, for coprime $a_i$, is of negative fibration. Using the graded roots algorithm of Némethi [34], we may write:

$$HF^+(Y,\mathfrak{s}) = \mathcal{T}^+_{s+d_1+2n_1-1} \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}\left(\frac{d_{i+1}+2n_{i+1}-d_i}{2}\right) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(n_i) \oplus J^{\oplus 2}[-s], \quad (1.2)$$

for some constants $s, d_i, n_i, N$ and some $\mathbb{F}[U]$-module $J$, all determined by $(Y,\mathfrak{s})$. Moreover, $d_{i+1} > d_i, n_{i+1} < n_i$ for all $i$. Roughly, in terms of Seiberg-Witten theory, the term $\mathcal{T}^+_{s+d_1+2n_1-1}$ accounts for the reducible critical point, and the modules $\mathcal{T}^+_{d_i}(n_i)$ and $\mathcal{T}^+_{d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2})$ account for the irreducibles which cancel against the bottom of the infinite $U$-tower. The term $J^{\oplus 2}$ accounts for the other irreducibles.

Let us denote by $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]}$ the restriction functor from the map of modules $\mathbb{F}[v] \to \mathbb{F}[U]$ given by $v \to U^2$. The restriction functor converts $\mathcal{T}^+_d(n)$ to $\mathcal{V}^+_d(\lfloor\frac{n+1}{2}\rfloor) \oplus \mathcal{V}^+_{d+2}(\lfloor\frac{n}{2}\rfloor)$.

**Theorem 1.2.1.** *Let $Y$ be a Seifert rational homology three-sphere of negative fibration, fibering over an orbifold with underlying space $S^2$, and let $\mathfrak{s}$ be a spin structure on $Y$. Let $HF^+(Y,\mathfrak{s})$ be as in (1.2). Then there exist constants $(a_i,b_i)$ and an $\mathbb{F}[q,v]/(q^3)$-module $J''$, specified in Corollary 4.2.4 and depending only on the sequence $(d_i,n_i)$, so that, as an $\mathbb{F}[v]$-module:*

$$\begin{aligned} SWFH^G(Y,\mathfrak{s}) &= \mathcal{V}^+_{s+4\lfloor\frac{d_1+2n_1+1}{4}\rfloor} \oplus \mathcal{V}^+_{s+1} \oplus \mathcal{V}^+_{s+2} \\ &\quad \oplus \bigoplus_{i=1}^{N'} \mathcal{V}^+_{s+a_i}\left(\frac{a_{i+1}+4b_{i+1}-a_i}{4}\right) \oplus J''[-s] \oplus \mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J[-s]. \end{aligned}$$

*The $q$-action is given by the isomorphism $\mathcal{V}^+_{s+2} \to \mathcal{V}^+_{s+1}$ and the map $\mathcal{V}^+_{s+1} \to \mathcal{V}^+_{s+4\lfloor\frac{d_1+2n_1+1}{4}\rfloor}$, which is an $\mathbb{F}$-vector space isomorphism in all degrees at least $s+4\lfloor\frac{d_1+2n_1+1}{4}\rfloor$ and vanishes otherwise. Further, $q$ annihilates $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J[-s]$ and $\bigoplus_{i=1}^{N'} \mathcal{V}^+_{s+a_i}(\frac{a_{i+1}+4b_{i+1}-a_i}{4})$. The action of $q$ on $J''$ is specified in Corollary 4.2.4.*

Theorem 1.2.1 specifies $\alpha, \beta$, and $\gamma$, which we state as Corollary 1.2.2. For $Y$ an integral homology three-sphere, let $d(Y)$ be the Heegaard Floer correction term [38]. Using Theorem 1.2.1 and Theorem 1.2.3 below we obtain:

**Corollary 1.2.2.** *(a) Let $Y$ be a Seifert integral homology sphere of negative fibration. Then $\beta(Y) = \gamma(Y) = -\bar{\mu}(Y)$, and*

$$\alpha(Y) = \begin{cases} d(Y)/2, & \text{if } d(Y)/2 \equiv -\bar{\mu}(Y) \text{ mod } 2 \\ d(Y)/2 + 1 & \text{otherwise.} \end{cases}$$

*(b) Let $Y$ be a Seifert integral homology sphere of positive fibration. Then $\alpha(Y) = \beta(Y) = -\bar{\mu}(Y)$, and*

$$\gamma(Y) = \begin{cases} d(Y)/2 & \text{if } d(Y)/2 \equiv -\bar{\mu}(Y) \text{ mod } 2 \\ d(Y)/2 - 1 & \text{otherwise.} \end{cases}$$

From Corollary 1.2.2, we see that for Seifert integral homology spheres the Manolescu invariants $\alpha, \beta$, and $\gamma$ are all determined by $d$ and $\bar{\mu}$. In particular, $\alpha$, $\beta$, and $\gamma$ provide no new obstructions to Seifert spaces bounding acyclic four-manifolds.

In [30], Manolescu also conjectured that for all spin Seifert rational homology spheres $\beta(Y, \mathfrak{s}) = -\bar{\mu}(Y, \mathfrak{s})$, where $\bar{\mu}$ is the Neumann-Siebenmann invariant defined in [36], [48]. We are able to prove part of this conjecture:

**Theorem 1.2.3.** *Let $Y$ be a Seifert integral homology three-sphere. Then $\beta(Y) = -\bar{\mu}(Y)$.*

We prove Theorem 1.2.3 by showing that $\beta$ is controlled by the degree of the reducible, and by using a result of Ruberman and Saveliev [44] that gives $\bar{\mu}$ as a sum of eta invariants.

Fukumoto-Furuta-Ue showed in [14] that $\bar{\mu}$ is a homology cobordism invariant for many classes of Seifert spaces, and Saveliev [47] extended this to show that Seifert integral homology spheres with $\bar{\mu} \neq 0$ have infinite order in $\theta_3^H$. Theorem 1.2.3 generalizes the result of Fukumoto-Furuta-Ue, showing that the Neumann-Siebenmann invariant $\bar{\mu}$, restricted to Seifert integral homology spheres, is a homology cobordism invariant.

For Seifert spaces with $HF^+(Y,\mathfrak{s})$ of a special form, $SWFH^G(Y,\mathfrak{s})$ may be expressed more compactly than is evident in the statement of Theorem 1.2.1. If $Y$ is of negative fibration and

$$HF^+(Y,\mathfrak{s}) = \mathcal{T}^+_d \oplus \mathcal{T}^+_{-2n+1}(m) \oplus \bigoplus_{i\in I} \mathcal{T}^+_{a_i}(m_i)^{\oplus 2}, \tag{1.3}$$

for some index set $I$, we say that $(Y,\mathfrak{s})$ is *of projective type*. We will say that $Y$ is of projective type if $Y$ is an integral homology sphere such that (1.3) holds. There are many examples of such Seifert spaces, among them $\Sigma(p,q,pqn\pm 1)$, by work of Némethi and Borodzik [35],[2] and Tweedy [51]. The condition (1.3) also admits a natural expression in terms of graded roots; see Section 4.2.2.

**Theorem 1.2.4.** *If $(Y,\mathfrak{s})$ is of projective type, as in (1.3), then:*
*If $d \equiv 2n+2$ mod 4,*

$$SWFH^G(Y,\mathfrak{s}) = \mathcal{V}^+_{d+2} \oplus \mathcal{V}^+_{-2n+1} \oplus \mathcal{V}^+_{-2n+2} \oplus \mathcal{V}^+_{-2n+3}(\lfloor \frac{m}{2} \rfloor) \oplus \bigoplus_{i\in I} \mathcal{V}^+_{a_i}(\lfloor \frac{m_i+1}{2} \rfloor) \oplus \bigoplus_{i\in I} \mathcal{V}^+_{a_i+2}(\lfloor \frac{m_i}{2} \rfloor). \tag{1.4}$$

*If $d \equiv 2n$ mod 4,*

$$SWFH^G(Y,\mathfrak{s}) = \mathcal{V}^+_{d} \oplus \mathcal{V}^+_{-2n+1} \oplus \mathcal{V}^+_{-2n+2} \oplus \mathcal{V}^+_{-2n+3}(\lfloor \frac{m}{2} \rfloor) \oplus \bigoplus_{i\in I} \mathcal{V}^+_{a_i}(\lfloor \frac{m_i+1}{2} \rfloor) \oplus \bigoplus_{i\in I} \mathcal{V}^+_{a_i+2}(\lfloor \frac{m_i}{2} \rfloor). \tag{1.5}$$

*The $q$-action is given by the isomorphism $\mathcal{V}^+_{-2n+2} \to \mathcal{V}^+_{-2n+1}$ and the map $\mathcal{V}^+_{-2n+1} \to \mathcal{V}^+_{d+2}$ (if $d \equiv 2n+2$ mod 4), or $\mathcal{V}^+_{-2n+1} \to \mathcal{V}^+_d$ (if $d \equiv 2n$ mod 4), which is an $\mathbb{F}$-vector space isomorphism in all degrees at least $d+2$ (respectively, $d$), and vanishes otherwise. In (1.4) and (1.5), $q$ acts on $\mathcal{V}^+_{-2n+3}(\lfloor \frac{m}{2} \rfloor)$ as the unique nonzero map $\mathcal{V}^+_{-2n+3}(\lfloor \frac{m}{2} \rfloor) \to \mathcal{V}^+_{-2n+2}$. The action of $q$ annihilates $\bigoplus_{i\in I} \mathcal{V}^+_{a_i}(\lfloor \frac{m_i+1}{2} \rfloor) \oplus \bigoplus_{i\in I} \mathcal{V}^+_{a_i+2}(\lfloor \frac{m_i}{2} \rfloor)$.*

For $X$ a topological space with $G$-action let $X^{S^1} \subset X$ denote the subset fixed by $S^1 \subset G$. We call $X$ a *j-split space* if

$$X/X^{S^1} = X_+ \vee jX_+. \tag{1.6}$$

That is, $X/X^{S^1}$ is a wedge sum of two components related by the action of $j$ (where $X_+$ and $jX_+$ are both $S^1$-spaces). We may think of $j$-split spaces as the simplest kind of (nontrivial) $G$-spaces which may occur as the Seiberg-Witten Floer spectrum $SWF(Y,\mathfrak{s})$ of some $(Y,\mathfrak{s})$.

To prove Theorem 1.2.1, we use [33] to show that a space representative of the stable homotopy type $SWF(Y,\mathfrak{s})$ is $j$-split. Then the chain complex of $EG \wedge_G SWF(Y,\mathfrak{s})$, used to compute the $G$-Borel homology, is closely related to the chain complex of $ES^1 \wedge_{S^1} SWF(Y,\mathfrak{s})$, whose homology is the $S^1$-Borel homology of $SWF(Y,\mathfrak{s})$. A careful, but entirely elementary, analysis of the differentials in these two complexes then yields Theorem 1.2.1.

### 1.2.1 Local Equivalence

Manolescu's construction of $SWF(Y,\mathfrak{s})$ contains more information about homology cobordism than the invariants $\alpha, \beta$, and $\gamma$. Namely, a spin cobordism $W$ from $Y_1$ to $Y_2$ with $b_2(W)=0$ induces a map $SWF(Y_1,\mathfrak{s}_1) \to SWF(Y_2,\mathfrak{s}_2)$ which is a homotopy equivalence on $S^1$-fixed point sets. We call two $G$-spaces $X_1, X_2$ *locally equivalent* if there exist $G$-equivariant stable maps $X_1 \to X_2$ and $X_2 \to X_1$ which induce homotopy equivalences on fixed point sets. The local equivalence class $[SWF(Y,\mathfrak{s})]_l$ is then a homology cobordism invariant of $(Y,\mathfrak{s})$. The local equivalence class $[SWF(Y,\mathfrak{s})]_l$ determines $\alpha(Y,\mathfrak{s}), \beta(Y,\mathfrak{s})$ and $\gamma(Y,\mathfrak{s})$. The construction of the local equivalence group is inspired by related constructions by Hom [20] in the context of knot Floer homology.

For a more computable version of local equivalence, we introduce *chain local equivalence*, using the $C_*(G)$-equivariant chain complex associated to a $G$-CW complex. The chain local equivalence class of a $G$-space $X$, denoted $[X]_{cl}$, takes values in the set $\mathfrak{CE}$ of homotopy-equivalence classes of chain complexes of a certain form. In particular, using the chain local equivalence class we have:

**Corollary 1.2.5.** *Let $Y$ be a rational homology three-sphere with spin structure $\mathfrak{s}$. Then there is a homology-cobordism invariant, $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$, the* connected Seiberg-Witten Floer homology *of $(Y,\mathfrak{s})$, taking values in isomorphism classes of $\mathbb{F}[U]$-modules. More specifically, $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$ is the isomorphism class of a summand of $HF_{\mathrm{red}}(Y,\mathfrak{s})$.*

The connected Seiberg-Witten Floer homology is constructed using the CW chain complex of a space representative $X$ of $SWF(Y,\mathfrak{s})$. The CW chain complex $C_*^{CW}(X)$ splits, as a module over $C_*^{CW}(G)$, into a direct sum of two subcomplexes, with one summand attached to the $S^1$-fixed-point set, and the other a free $C_*^{CW}(G)$-module. Roughly, the $S^1$-Borel homology of the former component is $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$.

In the calculation of $SWFH^G(Y,\mathfrak{s})$ for Seifert spaces, we provide enough information about the $G$-equivariant chain complex of $SWF(Y,\mathfrak{s})$ to calculate the chain local equivalence class $[SWF(Y,\mathfrak{s})]_{cl}$ of Seifert spaces. As a corollary, we obtain:

**Corollary 1.2.6.** *The sets $\{d_i\}_i, \{n_i\}_i$ in Theorem 1.2.1 are integral homology cobordism invariants of negative Seifert fiber spaces. That is, say $Y_1$ and $Y_2$ are negative Seifert integral homology spheres with $Y_1$ homology cobordant to $Y_2$. Let $S_i$ be the set of isomorphism classes of simple summands of $HF^+(Y_i)$ that occur an odd number of times in the decomposition (1.2). Then $S_1 = S_2$.*

We obtain Corollary 1.2.6 by showing that $\{d_i\}_i$ and $\{n_i\}_i$ determine $[SWF(Y,\mathfrak{s})]_{cl}$.

**Corollary 1.2.7.** *Let $(Y_1,\mathfrak{s}_1)$ be a negative Seifert rational homology three-sphere with spin structure, with $HF^+(Y_1,\mathfrak{s}_1)$ as in (1.2). Then*

$$SWFH_{\mathrm{conn}}(Y_1,\mathfrak{s}_1) = \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}\left(\frac{d_{i+1}+2n_{i+1}-d_i}{2}\right) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(n_i). \tag{1.7}$$

*In particular, if $Y_1$ is an integral homology sphere and $Y_2$ is any integral homology sphere homology cobordant to $Y_1$, then $\widetilde{HM}(Y_2) \simeq HF^+(Y_2)$ contains a summand isomorphic to (1.7), as $\mathbb{F}[U]$-modules.*

**Remark 1.2.8.** *In fact, $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$ is an invariant of spin rational homology cobordism, for $Y$ a rational homology three-sphere.*

From Corollary 1.2.7 and (1.2), we see that for Seifert integral homology spheres $Y$, $SWFH_{\mathrm{conn}}(Y,\mathfrak{s}) = 0$ if and only if $d(Y,\mathfrak{s})/2 = -\bar{\mu}(Y,\mathfrak{s})$. As an application of the Corollaries 1.2.5 and 1.2.7, we have:

**Corollary 1.2.9.** *The spaces $\Sigma(5,7,13)$ and $\Sigma(7,10,17)$ satisfy*

$$d(\Sigma(5,7,13)) = d(\Sigma(7,10,17)) = 2,$$

$$\bar{\mu}(\Sigma(5,7,13)) = \bar{\mu}(\Sigma(7,10,17)) = 0.$$

*However,* $SWFH_{\text{conn}}(\Sigma(5,7,13)) = \mathcal{T}_1^+(1)$*, while*

$$SWFH_{\text{conn}}(\Sigma(7,10,17)) = \mathcal{T}_{-1}^+(2) \oplus \mathcal{T}_{-1}^+(1). \tag{1.8}$$

*Thus* $\Sigma(5,7,13)$ *and* $\Sigma(7,10,17)$ *are not homology cobordant, despite having the same* $d$, $\bar{\mu}$, $\alpha$, $\beta$, *and* $\gamma$ *invariants.*

There are many other examples of homology cobordism classes that are distinguished by $d_i, n_i$, but not by $d$ and $\bar{\mu}$. As an example, we have the following Corollary.

**Corollary 1.2.10.** *The Seifert space* $\Sigma(7,10,17)$ *is not homology cobordant to* $\Sigma(p,q,pqn\pm1)$ *for any* $p, q, n$.

This result follows from Corollary 1.2.6. Indeed, since $\Sigma(p,q,pqn\pm 1)$ are of projective type, $SWFH_{\text{conn}}(\Sigma(p,q,pqn\pm 1))$ is a simple $\mathbb{F}[U]$-module, using the definition (1.3) and equation (1.7). Using (1.8), Corollary 1.2.10 follows.

Moreover, using a calculation from [29], we are able to show the existence of three-manifolds not homology cobordant to any Seifert fiber space. This result is also due to Frøyshov using instanton homology, and has been independently proved by Lin [27]. For example, we have:

**Corollary 1.2.11.** *The connected sum* $\Sigma(2,3,11)\#\Sigma(2,3,11)$ *is not homology cobordant to any Seifert fiber space.*

*Proof.* In [29], Manolescu shows $\alpha(\Sigma(2,3,11)\#\Sigma(2,3,11)) = \beta(\Sigma(2,3,11)\#\Sigma(2,3,11)) = 2$, while $\gamma(\Sigma(2,3,11)\#\Sigma(2,3,11)) = 0$. In addition, $d(\Sigma(2,3,11)) = 2$, so

$$d(\Sigma(2,3,11)\#\Sigma(2,3,11)) = 4.$$

To obtain a contradiction, say first that $\Sigma(2,3,11)\#\Sigma(2,3,11)$ is homology cobordant to a negative Seifert space $Y$. Corollary 1.2.2 implies

$$2 = \beta(\Sigma(2,3,11)\#\Sigma(2,3,11)) = \beta(Y) = \gamma(Y) = \gamma(\Sigma(2,3,11)\#\Sigma(2,3,11)) = 0.$$

a contradiction. Say instead that $\Sigma(2,3,11)\#\Sigma(2,3,11)$ is homology cobordant to a positive Seifert space $Y$. Then by Corollary 1.2.2, $\gamma(Y) = d(Y)/2 = d(\Sigma(2,3,11)\#\Sigma(2,3,11))/2 = 2$. However, $\gamma(Y) = 0$, again a contradiction, completing the proof. □

Note that Corollary 1.2.11 readily implies the following statement for knots.

**Corollary 1.2.12.** *There exist knots, such as the connected sum* $T(3,11)\#T(3,11)$ *of torus knots, which are not concordant to any Montesinos knot.*

We will also generalize this result to Theorem 1.3.5 in the next subsection, as part of a more general calculation of the Manolescu invariants of connected sums.

We also have that many Seifert integral homology spheres of negative fibration are not homology cobordant to any Seifert integral homology sphere of positive fibration. For instance:

**Corollary 1.2.13.** *The Seifert spaces* $\Sigma(2,3,12k+7)$*, for* $k \geqslant 0$*, are not homology cobordant to* $-\Sigma(a_1, a_2, ..., a_n)$ *for any choice of relatively prime* $a_i$*.*

This corollary is a direct consequence of Corollary 1.2.2, which shows that if $Y$ is a negative Seifert space with $d(Y)/2 \neq -\bar{\mu}(Y)$, then $Y$ is not homology cobordant to any positive Seifert space. We note $d(\Sigma(2,3,12k+7)) = 0$ and $\bar{\mu}(\Sigma(2,3,12k+7)) = 1$, and the corollary follows. This should be compared with a result of Fintushel-Stern [7] that gives a similar conclusion: If $R(a_1, ..., a_n) > 0$, then $\Sigma(a_1, ..., a_n)$ is not oriented cobordant to any connected sum of positive Seifert homology spheres by a positive definite cobordism $W$, where $H_1(W;\mathbb{Z})$ contains no 2-torsion. However, there are examples with $R < 0$, but $d/2 \neq -\bar{\mu}$, so we can apply Corollary 1.2.2. For instance, $\Sigma(2,3,7)$ has $R$-invariant $-1$, but $d \neq -\bar{\mu}$. Thus, Corollary 1.2.13 is not detected by the $R$-invariant.

## 1.3 Connected Sums

We investigate the behavior of the Manolescu invariants under the connected sum operation. In particular, we have the following theorems:

**Theorem 1.3.1.** *Let* $(Y_1, \mathfrak{s}_1), (Y_2, \mathfrak{s}_2)$ *be rational homology three-spheres with spin structure.*

*Then:*

$$\alpha(Y_1, \mathfrak{s}_1) + \gamma(Y_2, \mathfrak{s}_2) \leqslant \alpha(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \leqslant \alpha(Y_1, \mathfrak{s}_1) + \alpha(Y_2, \mathfrak{s}_2), \tag{1.9}$$

$$\gamma(Y_1, \mathfrak{s}_1) + \gamma(Y_2, \mathfrak{s}_2) \leqslant \gamma(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \leqslant \alpha(Y_1, \mathfrak{s}_1) + \gamma(Y_2, \mathfrak{s}_2), \tag{1.10}$$

$$\gamma(Y_1, \mathfrak{s}_1) + \beta(Y_2, \mathfrak{s}_2) \leqslant \beta(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \leqslant \alpha(Y_1, \mathfrak{s}_1) + \beta(Y_2, \mathfrak{s}_2), \tag{1.11}$$

$$\gamma(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \leqslant \beta(Y_1, \mathfrak{s}_1) + \beta(Y_2, \mathfrak{s}_2) \leqslant \alpha(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2). \tag{1.12}$$

**Theorem 1.3.2.** *Let $(Y, \mathfrak{s})$ be a rational homology three-sphere with spin structure. Then:*

$$\gamma(Y, \mathfrak{s}) \leqslant \delta(Y, \mathfrak{s}) \leqslant \alpha(Y, \mathfrak{s}). \tag{1.13}$$

We note, for comparison with Heegaard Floer theory, that the invariant $\delta(Y, \mathfrak{s})$ should correspond to the Heegaard Floer correction term $d(Y, \mathfrak{s})/2$.

If we regard Theorem 1.3.2 as a statement constraining the behavior of $\delta(Y, \mathfrak{s})$ in terms of the Manolescu invariants $\alpha, \beta$, and $\gamma$, then we may think of the following as a kind of converse statement, showing that $\delta(Y, \mathfrak{s})$ heavily constrains the behavior of the Manolescu invariants:

**Theorem 1.3.3.** *Let $(Y, \mathfrak{s})$ be a rational homology three-sphere with spin structure. Then:*

$$\alpha(\#_n(Y, \mathfrak{s})) - n\delta(Y, \mathfrak{s}), \ \beta(\#_n(Y, \mathfrak{s})) - n\delta(Y, \mathfrak{s}), \text{ and } \gamma(\#_n(Y, \mathfrak{s})) - n\delta(Y, \mathfrak{s}) \tag{1.14}$$

*are bounded functions of $n$, where $\#_n(Y, \mathfrak{s})$ denotes the connected sum of $n$ copies of $(Y, \mathfrak{s})$. In particular:*

$$\lim_{n \to \infty} \frac{\alpha(\#_n(Y, \mathfrak{s}))}{n} = \lim_{n \to \infty} \frac{\beta(\#_n(Y, \mathfrak{s}))}{n} = \lim_{n \to \infty} \frac{\gamma(\#_n(Y, \mathfrak{s}))}{n} = \delta(Y, \mathfrak{s}). \tag{1.15}$$

That is, one might think of the Manolescu invariants as perturbations of the $S^1$-Frøyshov invariant.

In order to obtain Theorem 1.3.3, we will make an explicit calculation of the Manolescu invariants of connected sums of negative Seifert spaces of projective type.

Recall that the $G$-equivariant Seiberg-Witten Floer stable homotopy type of a negative Seifert space, $SWF(Y, \mathfrak{s})$, is especially simple, namely, a $j$-split space.

The *projective type* condition further restricts what $X_+$ (as in Equation (1.6)) may be, and allows the following result.

**Theorem 1.3.4.** *Let $Y_1, \dots, Y_n$ be negative Seifert integral homology three-spheres of projective type. Define $\tilde{\delta}(Z) = d(Z)/2 + \bar{\mu}(Z)$, for $Z$ any Seifert fiber space, where $d$ is the Heegaard Floer correction term from [38], and where $\bar{\mu}$ is the Neumann-Siebenmann invariant defined in [36], [48]. Set $\tilde{\delta}_i := \tilde{\delta}(Y_i)$, and assume without loss of generality $\tilde{\delta}_1 \leqslant \cdots \leqslant \tilde{\delta}_n$. Then:*

$$\alpha(Y_1 \# \dots \# Y_n) = 2\left\lfloor \frac{(\sum_{i=1}^{n} \tilde{\delta}_i) + 1}{2} \right\rfloor - \sum_{i=1}^{n} \bar{\mu}(Y_i), \tag{1.16}$$

$$\beta(Y_1 \# \dots \# Y_n) = 2\left\lfloor \frac{(\sum_{i=1}^{n-1} \tilde{\delta}_i) + 1}{2} \right\rfloor - \sum_{i=1}^{n} \bar{\mu}(Y_i), \tag{1.17}$$

$$\gamma(Y_1 \# \dots \# Y_n) = 2\left\lfloor \frac{(\sum_{i=1}^{n-2} \tilde{\delta}_i) + 1}{2} \right\rfloor - \sum_{i=1}^{n} \bar{\mu}(Y_i), \tag{1.18}$$

*and*

$$\delta(Y_1 \# \dots \# Y_n) = (d(Y_1) + \cdots + d(Y_n))/2 = \sum_{i=1}^{n} \tilde{\delta}_i - \sum_{i=1}^{n} \bar{\mu}(Y_i). \tag{1.19}$$

To prove Theorem 1.3.4 we will investigate the Pin(2)-equivariant topology of joins of $j$-split spaces. To do so, we will make use of the Gysin sequence for Pin(2)-spaces, which provides a relationship between the Pin(2)-equivariant and $S^1$-equivariant homology of a Pin(2)-space. Lin has already used the Gysin sequence in [27] to study $\widetilde{HS}(Y, \mathfrak{s})$ for $Y$ a surgery on an alternating knot.

The proof of Theorem 1.3.4 also relies on the equivalence of several versions of Floer homologies: we employ the equivalence of $HF^+$ and $\widetilde{HM}$ from Colin-Ghiggini-Honda [4] and Taubes [49], and Kutluhan-Lee-Taubes [24], and the equivalence of $\widetilde{HM}$ and $SWFH^{S^1}$ due to Lidman-Manolescu [25].

To obtain Theorem 1.3.3 from Theorem 1.3.4 we will use the machinery of chain local equivalence, a refinement of the Manolescu invariants.

More specifically, to obtain Theorem 1.3.3, we will show that any CW chain complex associated to a Pin(2)-space admits some "large" $j$-split subcomplex (partly controlled by the $\delta$ invariant). Here, we call a Pin(2)-chain complex *$j$-split* if it is the CW chain complex of a $j$-split space. Using the "large" $j$-split subcomplex inside a given Pin(2)-complex, the calculation of Theorem 1.3.4 may be carried over, in part, to arbitrary rational homology three-spheres, yielding Theorem 1.3.3.

### 1.3.1 Applications

We apply Theorem 1.3.4 to study homology cobordisms among Seifert spaces.

A corollary of Theorem 1.3.4 is:

**Theorem 1.3.5.** *Let $Y_1, \dots, Y_n$ be negative Seifert integral homology spheres of projective type, with at least two of the $Y_i$ having $\frac{d(Y_i)}{2} \geqslant -\bar{\mu}(Y_i) + 2$. Then $Y := Y_1 \# \dots \# Y_n$ is not homology cobordant to any Seifert fiber space.*

We say that an integral homology three-sphere $Y$ is *$H$-split* if $\alpha(Y) = \beta(Y) = \gamma(Y)$. Theorem 1.3.1 implies that the set $\theta_{H\text{-split}} \subset \theta_3^H$ of $H$-split integral homology three-spheres is, in fact, a subgroup. We obtain from Theorem 1.3.4:

**Theorem 1.3.6.** *Let $\theta_{SFP}$ be the subgroup of $\theta_3^H$ generated by negative Seifert spaces of projective type, and let $\theta_{H\text{-split},SFP}$ be the subgroup consisting of $Y \in \theta_{SFP}$ such that $\alpha(Y) = \beta(Y) = \gamma(Y)$. Then:*

$$\theta_{SFP} = \theta_{H\text{-split},SFP} \oplus \mathbb{Z}^\infty. \tag{1.20}$$

*The $\mathbb{Z}^\infty$ summand is generated by $\{Y_p = \Sigma(p, 2p-1, 2p+1) \mid 3 \leqslant p, p \text{ odd}\}$. In particular, the elements $\{Y_p \mid 3 \leqslant p, p \text{ odd}\}$ are linearly independent in $\theta_3^H$.*

This implies the existence of a $\mathbb{Z}^\infty$ subgroup of $\theta_3^H$, a result originally due to Furuta [16] and Fintushel-Stern [8], both building on the $R$-invariant introduced by Fintushel and Stern [7] using instantons. Fintushel and Stern [8] show that the collection $\{\Sigma(p, q, pqn-1) \mid n \geqslant 1\}$ is linearly independent in $\theta_3^H$ for any relatively prime $p, q$, and Furuta's construction of $\mathbb{Z}^\infty \subseteq \theta_3^H$ is the special case $p = 2, q = 3$ of Fintushel and Stern's construction. However, we will see from Theorem 5.2.3 that the image of $\{\Sigma(p, q, pqn-1) \mid n \geqslant 1\}$ in $\theta_3^H$ is contained in $\theta_{H\text{-split}} \oplus \mathbb{Z}$, for any fixed $p, q$. In particular, the $\mathbb{Z}^\infty$ subgroups that Furuta and Fintushel-Stern originally identified are not detected by Pin(2)-techniques. We then obtain the following corollary:

**Corollary 1.3.7.** *The subgroup $\theta_{H\text{-split}} \subset \theta_3^H$ is infinitely-generated.*

To our knowledge, Theorem 1.3.6 is the first proof of the existence of a $\mathbb{Z}^\infty$ subgroup of $\theta_3^H$ using either monopoles or the technology of Heegaard Floer homology. The Fintushel-Stern $R$ invariant also shows that $Y_p$, for $p$ odd, are linearly independent in the homology cobordism group [6], but it does not show the splitting as in (1.20).

Theorem 1.3.6 follows from Theorem 1.3.4, once one finds a collection of Seifert integral homology spheres $Y$ of projective type with $d(Y)/2 + \bar{\mu}(Y)$ arbitrarily large:

**Theorem 1.3.8.** *Let $Y_p = \Sigma(p, 2p-1, 2p+1)$. For odd $p \geqslant 3$, $Y_p$ is of projective type, with $d(Y_p) = p-1$ and $\bar{\mu}(Y_p) = 0$.*

Theorem 1.3.8 is proved using the technology of graded roots, introduced by Némethi [34], and refinements of the method of graded roots for Seifert spaces in [3],[21]. The proof is essentially borrowed from the partial calculation of $HF^+(Y_p)$ for even $p$ by Hom, Karakurt, and Lidman [18].

Other convenient choices of the generating set for $\mathbb{Z}^\infty$ in Theorem 1.3.6 are possible, such as, for example, $\{\Sigma(2, q, 2q+1) \mid q \equiv 3 \bmod 4\}$. See Theorem 5.2.3 for a more precise statement.

Using Theorem 1.3.6, we may also obtain statements about knots. Endo showed in [6] that the smooth concordance group of topologically slice knots, denoted $\mathcal{C}_{TS}$, contains a $\mathbb{Z}^\infty$ subgroup, using the Fintushel-Stern $R$-invariant. Using Theorem 1.3.6, we are able to reproduce Endo's result:

**Corollary 1.3.9.** *The pretzel knots $K(-p, 2p-1, 2p+1)$, for odd $p \geqslant 3$, are linearly independent in $\mathcal{C}_{TS}$.*

*Proof.* We chose the Seifert spaces $Y_p$ in Theorem 1.3.6 instead of other possible generating sets because $Y_p$ are branched double covers of pretzel knots:

$$Y_p = \Sigma(K(-p, 2p-1, 2p+1))$$

where $K(-p, 2p-1, 2p+1)$ is the pretzel knot of type $(-p, 2p-1, 2p+1)$. We note that the Alexander polynomial

$$\Delta_K(K(-p, 2p-1, 2p+1)) = 1$$

for all odd $p$. Thus, by [11], $K(-p, 2p-1, 2p+1)$ are topologically slice. By Theorem 1.3.6, the present Corollary follows. □

The subgroup that Endo identifies in $\mathcal{C}_{TS}$ is identical to that of Corollary 1.3.9. Hom [20] much extended Endo's result, showing that $\mathcal{C}_{TS}$ has a $\mathbb{Z}^\infty$ summand, using the knot concordance invariant $\epsilon$ defined in [19]. Additionally, Ozsváth, Stipsicz, and Szabó [37] gave another proof that $\mathcal{C}_{TS}$ has a $\mathbb{Z}^\infty$ summand using the knot concordance invariant $\Upsilon$.

Furthermore, Friedl, Livingston, and Zentner [12] recently showed the following.

**Theorem 1.3.10** ([12]). *There is an infinitely-generated free subgroup $\mathcal{H} \subset \mathcal{C}_{TS}$ such that if $K$ represents a nontrivial class in $\mathcal{H}$, then $K$ is not concordant to any alternating knot.*

Theorem 1.3.6 provides an alternative proof of Theorem 1.3.10. Indeed, as for Heegaard-Floer homology, a quasi-alternating knot $K$ has $SWFH^G(\Sigma(K), \mathfrak{s}) = H_*(BG)$, perhaps with a grading shift, where $\Sigma(K)$ denotes the double-branched cover of $K$ and $\mathfrak{s}$ is the unique spin structure on $\Sigma(K)$. In particular, $\alpha(\Sigma(K), \mathfrak{s}) = \beta(\Sigma(K), \mathfrak{s}) = \gamma(\Sigma(K), \mathfrak{s})$. Then, in the decomposition of Theorem 1.3.6, no element of the $\mathbb{Z}^\infty$ subgroup is homology cobordant to a double-branched cover of a quasi-alternating knot. That is, the subgroup of $\mathcal{C}_{TS}$ generated by $K(-p, 2p-1, 2p+1)$ has no nontrivial element concordant to a quasi-alternating knot.

Another natural question is whether the Manolescu invariants of a pair of three-manifolds determine the Manolescu invariants of the connected sum. This is not the case, as may be seen using Theorem 1.3.4. We take $Y = \Sigma(2,3,7)$, noting

$$\alpha(Y) = 1,\ \beta(Y) = -1,\ \gamma(Y) = -1,\ \delta(Y) = 0, \text{ and } \bar{\mu}(Y) = 1. \tag{1.21}$$

Then we have $\tilde{\delta}(Y) = 1$, and, by Theorem 1.3.4, the Manolescu invariants of $2(n+1)Y$ and $(2n+1)Y$ are independent of $n \geqslant 0$. Specifically,

$$\alpha(2(n+1)Y) = 0,\ \beta(2(n+1)Y) = 0,\ \gamma(2(n+1)Y) = -2,$$

$$\alpha((2n+1)Y) = 1,\ \beta((2n+1)Y) = -1,\ \gamma((2n+1)Y) = -1.$$

Then the Manolescu invariants of $2nY$ and $2mY$ agree for $n > m \geqslant 1$. However,

$$\alpha(2nY\# -2nY) = \beta(2nY\# -2nY) = \gamma(2nY\# -2nY) = 0,$$

while

$$\alpha(2nY\# -2mY) = \beta(2nY\# -2mY) = 0,\ \gamma(2nY\# -2mY) = -2.$$

Thus, the Manolescu invariants of $Y_1$ and $Y_2$ do not determine those of the connected sum $Y_1\#Y_2$.

# CHAPTER 2

# Spaces of type SWF

## 2.1 Spaces of type SWF

### 2.1.1 $G$-CW Complexes

In this section we recall the definition of spaces of type SWF from [30], and introduce local equivalence. Spaces of type SWF are the output of the construction of the Seiberg-Witten Floer stable homotopy type of [30] and [31]; see Section 3.1.

First, we recall some basics of equivariant algebraic topology from [50]. The reader is encouraged to consult both [30] and [50] for a fuller discussion. For now, $G$ will denote a compact Lie group. We define a $G$-equivariant $k$-cell as a copy of $G/H \times D^k$, where $H$ is a closed subgroup of $G$. A (finite) equivariant $G$-CW decomposition of a relative $G$-space $(X, A)$, where the action of $G$ takes $A$ to itself, is a filtration $(X_n | n \in \mathbb{Z}_{\geqslant 0})$ such that

- $A \subset X_0$ and $X = X_n$ for $n$ sufficiently large.
- The space $X_n$ is obtained from $X_{n-1}$ by attaching $G$-equivariant $n$-cells.

When $A$ is a point, we call $(X, A)$ a pointed $G$-CW complex.

Let $EG$ be the total space of the universal bundle of $G$. For two pointed $G$-spaces $X_1$ and $X_2$, write:

$$X_1 \wedge_G X_2 = (X_1 \wedge X_2)/(gx_1 \times x_2 \sim x_1 \times gx_2).$$

The Borel homology of a pointed $G$-space $X$ is given by

$$\tilde{H}^G_*(X) = \tilde{H}_*(EG_+ \wedge_G X),$$

where $EG_+$ is $EG$ with a disjoint basepoint. Similarly, we define Borel cohomology:

$$\tilde{H}^*_G(X) = \tilde{H}^*(EG_+ \wedge_G X).$$

Additionally, we have a map given by projecting to the first factor:

$$f : EG_+ \wedge_G X \to BG_+.$$

From $f$ we have a map $p_G = f^* : H^*(BG) \to \tilde{H}^*_G(X)$. Then $H^*(BG)$ acts on $\tilde{H}^G_*(X)$, by composing $p_G$ with the cap product action of $\tilde{H}^*_G(X)$ on $\tilde{H}^G_*(X)$. We may also define the unpointed version of the above constructions in an apparent way.

As an example, consider the case $G = S^1$. Here $BS^1 = \mathbb{C}P^\infty$, so $H^*(BS^1) = \mathbb{F}[U]$, with $\deg U = 2$. Then $\mathbb{F}[U]$ acts on $H^{S^1}_*(X)$, for $X$ any $S^1$-space.

From now on we let $G = \text{Pin}(2)$. The group $G = \text{Pin}(2)$ is the set $S^1 \cup jS^1 \subset \mathbb{H}$, where $S^1$ is the unit circle in the $\langle 1, i \rangle$ plane. The group action of $G$ is induced from the group action of the unit quaternions. In order to agree with the conventions of [30] we deal with left $G$-spaces. Manolescu shows in [30] that $H^*(BG) = \mathbb{F}[q, v]/(q^3)$, where $\deg q = 1$ and $\deg v = 4$, so $\tilde{H}^G_*(X)$ is naturally an $\mathbb{F}[q, v]/(q^3)$-module for $X$ a pointed $G$-space. Moreover $S^\infty = S(\mathbb{H}^\infty)$ has a free action by the quaternions, making $S^\infty$ a free $G$-space. Since $S^\infty$ is contractible, we identify $EG = S^\infty$. We may view $EG = S^\infty$ also as $ES^1$ (as an $S^1$-space) by forgetting the action of $j$.

We will also need to relate $G$-Borel homology and $S^1$-Borel homology. Consider

$$f : \mathbb{C}P^\infty = BS^1 \to BG,$$

the map given by quotienting by the action of $j \in G$ on $BS^1 = ES^1/S^1$. Then we have the following fact (for a proof, see [30, Example 2.11]):

**Fact 2.1.1.** *The natural map*

$$\text{res}^G_{S^1} := f^* \colon \mathbb{F}[q, v]/(q^3) = H^*(BG) \to H^*(BS^1) = \mathbb{F}[U]$$

*is an isomorphism in degrees divisible by 4, and zero otherwise. In particular, $v \to U^2$. Similarly,*

$$f_* \colon H_*(BS^1) \to H_*(BG)$$

*has* $f_*(u^{-2n}) = v^{-n}$ *and* $f_*(u^{-2n+1}) = 0$*, where* $u^{-n}$ *is the unique nonzero element of* $H_*(BS^1)$ *in degree* $2n$*, and* $v^{-n}$ *is the unique nonzero element of* $H_*(BG)$ *in degree* $4n$*.*

Moreover, for $X$ a $G$-space, we have a natural map

$$g : EG_+ \wedge_{S^1} X \to EG_+ \wedge_G X.$$

The map $g$ induces a map

$$g_* = \mathrm{cor}_G^{S^1} : \tilde{H}_*^{S^1}(X) \to \tilde{H}_*^G(X),$$

called the corestriction map. As a Corollary of Fact 2.1.1, we have a relationship between the action of $U$ and $v$ (see [50, §III.1]):

**Fact 2.1.2.** *Let* $X$ *be a* $G$*-space. Then, for every* $x \in H_*^{S^1}(X)$*,*

$$v(\mathrm{cor}_G^{S^1}(x)) = \mathrm{cor}_G^{S^1}(U^2 x).$$

We shall use that Borel homology with $\mathbb{F}$ coefficients behaves well with respect to suspension. If $V$ is a finite-dimensional (real) representation of $G$, let $V^+$ be the one-point compactification, where $G$ acts trivially on $V^+ - V$. Then $\Sigma^V X = V^+ \wedge X$ will be called the suspension of $X$ by the representation $V$.

We mention the following representations of $G$:

- Let $\tilde{\mathbb{R}}^s$ be the vector space $\mathbb{R}^s$ on which $j$ acts by $-1$, and $e^{i\theta}$ acts by the identity, for all $\theta$.
- We let $\tilde{\mathbb{C}}$ be the representation of $G$ on $\mathbb{C}$ where $j$ acts by $-1$, and $e^{i\theta}$ acts by the identity for all $\theta$.
- The quaternions $\mathbb{H}$, on which $G$ acts by multiplication on the left.

**Definition 2.1.3.** Let $s \in \mathbb{Z}_{\geqslant 0}$. A *space of type SWF* at level $s$ is a pointed finite $G$-CW complex $X$ with

- The $S^1$-fixed-point set $X^{S^1}$ is $G$-homotopy equivalent to $(\tilde{\mathbb{R}}^s)^+$.

- The action of $G$ on $X - X^{S^1}$ is free.

As a source of examples of spaces of type SWF we have the following definition:

**Definition 2.1.4.** Let $G$ act freely on a finite $G$-CW complex $X$ (not a space of type SWF). We call

$$\tilde{\Sigma}X = ([0,1] \times X)/((0,x) \sim (0,x') \text{ and } (1,x) \sim (1,x') \text{ for all } x, x' \in X)$$

the unreduced suspension of $X$. The space $\tilde{\Sigma}X$ obtains a $G$-action by letting $G$ act trivially on the $[0,1]$ factor. We make $\tilde{\Sigma}X$ into a pointed space by setting $(0,x)$ as the basepoint. Then $\tilde{\Sigma}X$ is a space of type SWF, since $(\tilde{\Sigma}X)^{S^1} = S^0$ and $G$ acts freely away from $(\tilde{\Sigma}X)^{S^1}$.

We also find it convenient to recall the definition of *reduced Borel homology*, for spaces $X$ of type SWF:

$$\tilde{H}^{S^1}_{*,\mathrm{red}}(X) = \tilde{H}^{S^1}_{*}(X)/\mathrm{Im}\, U^N, \tag{2.1}$$

for $N \gg 0$. Indeed, for all $N$ sufficiently large $\mathrm{Im}\, U^N = \mathrm{Im}\, U^{N+1}$, so $\tilde{H}^{S^1}_{*,\mathrm{red}}(X)$ is well-defined.

Associated to a space $X$ of type SWF at level $s$, we take the Borel cohomology $\tilde{H}^*_G(X)$, from which we define $a(X), b(X)$, and $c(X)$ as in [30]:

$$a(X) = \min\{r \equiv s \bmod 4 \mid \exists\, x \in \tilde{H}^r_G(X), v^l x \neq 0 \text{ for all } l \geqslant 0\}, \tag{2.2}$$

$$b(X) = \min\{r \equiv s+1 \bmod 4 \mid \exists\, x \in \tilde{H}^r_G(X), v^l x \neq 0 \text{ for all } l \geqslant 0\} - 1,$$

$$c(X) = \min\{r \equiv s+2 \bmod 4 \mid \exists\, x \in \tilde{H}^r_G(X), v^l x \neq 0 \text{ for all } l \geqslant 0\} - 2.$$

The well-definedness of $a, b$, and $c$ follows from the Equivariant Localization Theorem (see [50] III). We list a version of this theorem for spaces of type SWF:

**Theorem 2.1.5** ([50] §III (3.8))**.** *Let $X$ be a space of type SWF. Then the inclusion $X^{S^1} \to X$, after inverting $v$, induces an isomorphism of $\mathbb{F}[q,v,v^{-1}]/(q^3)$-modules:*

$$v^{-1}\tilde{H}^*_G(X^{S^1}) \cong v^{-1}\tilde{H}^*_G(X).$$

For $X$ a space of type SWF, $X$ is a finite $G$-complex and so we have that $\tilde{H}^*_G(X)$ is finitely generated as an $\mathbb{F}[v]$-module. In particular, the $\mathbb{F}[v]$-torsion part of $\tilde{H}^*_G(X)$ is bounded above in grading. Theorem 2.1.5 then implies:

**Fact 2.1.6.** *Let $X$ be a space of type SWF. Then the inclusion $\iota\colon X^{S^1} \to X$ induces an isomorphism*

$$\iota^*\colon \tilde{H}^*_G(X) \to \tilde{H}^*_G(X^{S^1})$$

*in cohomology in sufficiently high degrees. Dualizing, $\iota_*$ induces an isomorphism in homology in sufficiently high degrees.*

We note that Fact 2.1.6 implies

$$\operatorname{Im} \iota_* = \{x \in \tilde{H}^G_*(X) \mid x \in \operatorname{Im} v^l \text{ for all } l \geqslant 0\}. \tag{2.3}$$

We also list an equivalent definition of $a, b$, and $c$ from [30], using homology:

$$\begin{aligned} a(X) &= \min \{r \equiv t \bmod 4 \mid \exists\, x \in \tilde{H}^G_r(X), x \in \operatorname{Im} v^l \text{ for all } l \geqslant 0\}, \\ b(X) &= \min \{r \equiv t+1 \bmod 4 \mid \exists\, x \in \tilde{H}^G_r(X), x \in \operatorname{Im} v^l \text{ for all } l \geqslant 0\} - 1, \\ c(X) &= \min \{r \equiv t+2 \bmod 4 \mid \exists\, x \in \tilde{H}^G_r(X), x \in \operatorname{Im} v^l \text{ for all } l \geqslant 0\} - 2. \end{aligned} \tag{2.4}$$

. We will see review the construction of $\alpha, \beta$ and $\gamma$ from $a, b, c$ shortly, from which the Manolescu invariants of a 3-manifold are defined.

**Definition 2.1.7** (see [31]). Let $X$ and $X'$ be spaces of type SWF, $m, m' \in \mathbb{Z}$, and $n, n' \in \mathbb{Q}$. We say that the triples $(X, m, n)$ and $(X', m', n')$ are *stably equivalent* if $n - n' \in \mathbb{Z}$ and there exists a $G$-equivariant homotopy equivalence, for some $r \gg 0$ and some nonnegative $M \in \mathbb{Z}$ and $N \in \mathbb{Q}$:

$$\Sigma^{r\mathbb{R}}\Sigma^{(M-m)\tilde{\mathbb{R}}}\Sigma^{(N-n)\mathbb{H}}X \to \Sigma^{r\mathbb{R}}\Sigma^{(M'-m')\tilde{\mathbb{R}}}\Sigma^{(N-n')\mathbb{H}}X'. \tag{2.5}$$

Let $\mathfrak{E}$ be the set of equivalence classes of triples $(X, m, n)$ for $X$ a space of type SWF, $m \in \mathbb{Z}$, $n \in \mathbb{Q}$, under the equivalence relation of stable $G$-equivalence[1]. The set $\mathfrak{E}$ may be considered as a subcategory of the $G$-equivariant Spanier-Whitehead category [30], by viewing $(X, m, n)$ as the formal desuspension of $X$ by $m$ copies of $\tilde{\mathbb{R}}^+$ and $n$ copies of $\mathbb{H}^+$.

[1] This convention is slightly different from that of [31]. The object $(X, m, n)$ in the set of stable equivalence classes $\mathfrak{E}$, as defined above, corresponds to $(X, \frac{m}{2}, n)$ in the conventions of [31].

For $(X, m, n), (X', m', n') \in \mathfrak{E}$, a map $(X, m, n) \to (X', m', n')$ is simply a map as in (2.5) that need not be a homotopy equivalence. We define Borel homology for $(X, m, n) \in \mathfrak{E}$ by

$$\tilde{H}_*^G((X, m, n)) = \tilde{H}_*^G(X)[m + 4n]. \tag{2.6}$$

The well-definedness of (2.6) follows from Proposition 2.1.8.

**Proposition 2.1.8** ([30] Proposition 2.2)**.** *Let $V$ be a finite-dimensional representation of $G$. Then, as $\mathbb{F}[q, v]/(q^3)$-modules:*

$$\begin{aligned} \tilde{H}_G^*(\Sigma^V X) &\cong \tilde{H}_G^{*-\dim V}(X) \\ \tilde{H}_*^G(\Sigma^V X) &\cong \tilde{H}_{*-\dim V}^G(X). \end{aligned} \tag{2.7}$$

**Definition 2.1.9.** For $[(X, m, n)] \in \mathfrak{E}$, we set

$$\alpha((X, m, n)) = \frac{a(X)}{2} - \frac{m}{2} - 2n, \ \beta((X, m, n)) = \frac{b(X)}{2} - \frac{m}{2} - 2n, \tag{2.8}$$

$$\gamma((X, m, n)) = \frac{c(X)}{2} - \frac{m}{2} - 2n.$$

The invariants $\alpha, \beta$ and $\gamma$ do not depend on the choice of representative of the class $[(X, m, n)]$.

**Definition 2.1.10.** We call $X_1, X_2 \in \mathfrak{E}$ *locally equivalent* if there exist $G$-equivariant (stable) maps

$$\phi : X_1 \to X_2,$$

$$\psi : X_2 \to X_1,$$

which are $G$-homotopy equivalences on the $S^1$-fixed-point set. For such $X_1, X_2$, we write $X_1 \equiv_l X_2$, and let $\mathfrak{LE}$ denote the set of local equivalence classes.

Local equivalence is easily seen to be an equivalence relation. The set $\mathfrak{LE}$ comes with an abelian group structure, with multiplication given by smash product. One may check that inverses are given by Spanier-Whitehead duality.

### 2.1.2 $G$-CW decompositions of $G$-spaces

Throughout this section $X$ will denote a space of type SWF. Here we will give example $G$-CW decompositions and construct a $G$-CW structure on smash products of $G$-spaces.

For $W$ a CW complex, we write $C_*^{CW}(W)$ for the corresponding cellular (CW) chain complex. We fix a convenient CW decomposition of $G$. The 0-cells are the points $1, j, j^2, j^3$ in $G$, and the 1-cells are $s, js, j^2s, j^3s$, where $s = \{e^{i\theta} \in S^1 \mid \theta \in (0, \pi)\}$. We identify each of the cells of this CW decomposition with its image in $C_*^{CW}(G)$, the corresponding $CW$ chain complex of $G$. Then $\partial(s) = 1 + j^2$. To ease notation, we will refer to $C_*^{CW}(G)$ by $\mathcal{G}$.

We will use that this CW decomposition also induces a CW decomposition of $S^1$, for which $C_*^{CW}(S^1)$ is the subcomplex of $\mathcal{G}$ generated by $1, j^2, s, j^2s$.

A $G$-CW decomposition of $X$ also induces a CW decomposition of $X$, using the decomposition of $G$ into cells as above, which we will call a $G$-compatible CW decomposition of $X$.

**Example 2.1.11.** *Note that the representation $(\tilde{\mathbb{R}}^s)^+$ admits a $G$-CW decomposition with 0-skeleton a copy of $S^0$ on which $G$ acts trivially, and an $i$-cell $c_i$ of the form $D^i \times \mathbb{Z}/2$ for $i \leqslant s$. One of the two points of the 0-skeleton of $(\tilde{\mathbb{R}}^s)^+$ is fixed as the basepoint.*

In particular, any space of type SWF has a $G$-CW decomposition with a subcomplex as in Example 2.1.11.

**Example 2.1.12.** *We find a CW decomposition for $\mathbb{H}^+$ as a $G$-space. We write elements of $\mathbb{H}$ as pairs of complex numbers $(z, w) = (r_1e^{i\theta_1}, r_2e^{i\theta_2})$ in polar coordinates. The action of $j$ is then given by $j(z, w) = (-\bar{w}, \bar{z})$. Fix the point at infinity as the base point. We let $(0, 0)$ be the ($G$-invariant) 0-cell labelled $r_0$. We let $y_1$ be the $G$-1-cell given by the orbit of $\{(r_1, 0) \mid r_1 > 0\}$:*

$$\{(r_1e^{i\theta}, r_2e^{i\theta}) \mid \ r_1r_2 = 0\}.$$

*We take $y_2$ the $G$-2-cell given by the orbit of $\{(r_1, r_2) \mid r_1r_2 \neq 0.\}$:*

$$\{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \mid \ \theta_1 = \theta_2 \bmod \pi, r_1r_2 \neq 0\}.$$

*Finally, $y_3$ consists of the orbit of $\{(r_1e^{i\theta_1}, r_2) \mid \theta_1 \in (0,\pi),\ r_1r_2 \neq 0.\}$ :*

$$\{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \mid \theta_1 \neq \theta_2 \bmod \pi, r_1r_2 \neq 0\}.$$

We now give $X_1 \wedge X_2$ a $G$-CW structure for $X_1$ and $X_2$ spaces of type SWF. To do so, we proceed cell by cell on both factors, so we need only find a $G$-CW structure on $G \times G$, $\mathbb{Z}/2 \times G$, and $\mathbb{Z}/2 \times \mathbb{Z}/2$, each with the diagonal $G$-action. The space $\mathbb{Z}/2 \times G$ has a $G$-CW decomposition as $G \amalg G$, as may be seen directly, and $\mathbb{Z}/2 \times \mathbb{Z}/2$ may be written as a disjoint union of $G$-0-cells $\mathbb{Z}/2 \amalg \mathbb{Z}/2$.

**Example 2.1.13.** *The $G$-CW structure on $G \times G$ is more complicated. Note that the product CW decomposition on $G \times G$ is not equivariant. We choose a homotopy $\phi_t : G \times G \to G \times G$ as in Figure 2.1, with $t \in [0,1]$, $\phi_0 = \mathrm{Id}$, and $\phi_1(G \times G)$ shown. The arrows depict the action of $S^1$. On the left, the diagonal lines show the $G$-action before homotopy. For example, the homotopy $\phi$ takes the line $\ell = \{(e^{i\theta} \times e^{i\theta} \mid \theta \in (0,\pi)\}$, the first half of the diagonal in $S^1 \times S^1$, to the sum of CW cells:*

$$s \otimes 1 + j^2 \otimes s.$$

*The arrows on the right show the $G$-action on $G \times G$ given by*

$$g(g_1 \times g_2) = \phi_1(g\phi_1^{-1}(g_1 \times g_2)). \tag{2.9}$$

*The action (2.9) is clearly cellular with respect to the product CW structure of $G \times G$. Then*

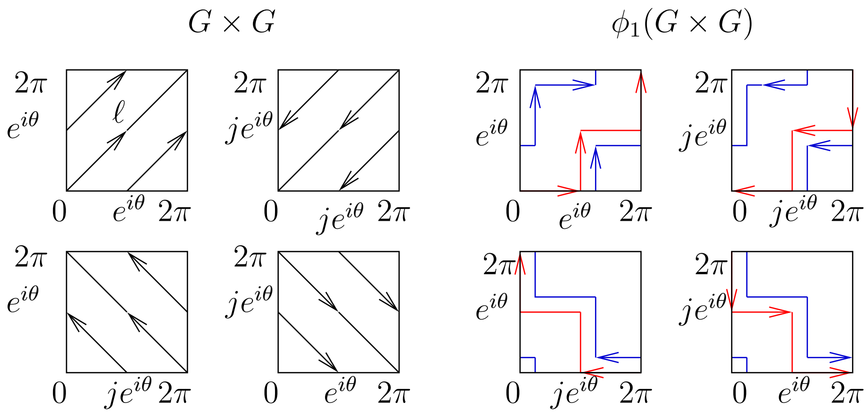


**Figure 2.1:** Homotopy of the action of $G$ on $G \times G$.

*$G \times G$ admits a $G$-CW-decomposition so that the induced CW decomposition is the product CW decomposition of $G \times G$.*

Now, let $X_1$ and $X_2$ be spaces of type SWF. We then give $X_1 \wedge X_2$ a $G$-CW decomposition proceeding cell-by-cell. That is, for $G$-cells $e_1 \subseteq X_1, e_2 \subseteq X_2$ we give $e_1 \wedge e_2$ the appropriate $G$-CW decomposition as constructed above. This is possible because the cells $e_i$ are necessarily of the form: $D^k, \mathbb{Z}/2 \times D^k$, or $G \times D^k$. In particular, the construction of a $G$-CW structure on $X_1 \wedge X_2$ gives us a $G$-CW structure for suspensions. For $V$ a finite-dimensional $G$-representation which is a direct sum of copies of $\mathbb{R}, \tilde{\mathbb{R}}$, and $\mathbb{H}$, we have $\Sigma^V X = V^+ \wedge X$, and so we give $\Sigma^V X$ the smash product $G$-CW decomposition.

Finally, we construct a CW structure for the $G$-smash product $X_1 \wedge_G X_2 = (X_1 \wedge X_2)/G$. More generally, we describe a CW structure for the quotient $W/G$ for $W$ a $G$-CW complex. Indeed, let $W = \bigcup e_i$ a $G$-CW complex, where $e_i = G/H_i \times D^{k(i)}$ are equivariant $G$-cells for some function $k$, and $H_i \subseteq G$ are subgroups. Then $W/G$ admits a CW decomposition given by $W = \bigcup e_i/G = \bigcup D^{k(i)}$.

### 2.1.3 Modules from $G$-CW decompositions.

Throughout this section $X$ will denote a space of type SWF. Here we will show that the CW chain complex of $X$ inherits a module structure from the action of $G$, and we will define chain local equivalence.

From the group structure of $G$, $C_*^{CW}(G) = \mathcal{G}$ acquires an algebra structure. Namely, the multiplication map $G \times G \to G$ gives a map $C_*^{CW}(G) \otimes_{\mathbb{F}} C_*^{CW}(G) \to C_*^{CW}(G)$. Here, we have used the product $G$-CW decomposition of $G \times G$, from Example 2.1.13, for which the multiplication map is cellular. A small calculation yields

$$C_*^{CW}(G) \cong \mathbb{F}[s,j]/(sj = j^3 s, s^2 = 0, j^4 = 1).$$

For any $G$-compatible decomposition of $X$, the relative CW chain complex $C_*^{CW}(X, \mathrm{pt})$ inherits the structure of a $\mathcal{G}$-chain complex, as the map $G \times X \to X$ gives a map $\mathcal{G} \times C_*^{CW}(X) \to C_*^{CW}(X)$. That is, $C_*^{CW}(X, \mathrm{pt})$ is a module over $\mathcal{G}$, such that, for $z \in C_*^{CW}(X, \mathrm{pt})$, and $a \in \mathcal{G}$, $\partial(az) = a\partial(z) + \partial(a)z$.

We find the module structure for the Examples 2.1.11-2.1.13 of Section 2.1.2.

**Example 2.1.14.** *Consider the $\mathcal{G}$-chain complex structure of $C_*^{CW}((\tilde{\mathbb{R}}^s)^+, \mathrm{pt})$ from Example 2.1.11. Identifying $c_i$ with its image in $C_*^{CW}((\tilde{\mathbb{R}}^s)^+, \mathrm{pt})$, we have $\partial(c_0) = 0, \partial(c_1) = c_0$, and $\partial(c_i) = (1+j)c_{i-1}$ for $i \geqslant 2$. One may check that the action of $\mathcal{G}$ is given by the relations $jc_0 = c_0$, $j^2c_i = c_i$ for $i \geqslant 1$, and $sc_i = 0$ for all $i$ (in particular, the CW cells of $((\tilde{\mathbb{R}}^s)^+, \mathrm{pt})$ are precisely $c_0, c_1, ...c_s$ and $jc_1, ..., jc_s$, and all of these are distinct).*

**Example 2.1.15.** *We also find the $\mathcal{G}$-chain complex structure of $C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$ from Example 2.1.12. One may check that the differentials are given by*

$$\partial(r_0) = 0,\ \partial y_1 = r_0,\ \partial y_2 = (1+j)y_1,\ \text{and}\ \partial y_3 = sy_1 + (1+j)y_2. \tag{2.10}$$

*The $\mathcal{G}$-action on the fixed-point set, $r_0$, is necessarily trivial. However, elsewhere the $G$-action on $(\mathbb{H}^+, \mathrm{pt})$ is free, and so the submodule (not a subcomplex, however) of $C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$ generated by $y_1, y_2, y_3$ is $\mathcal{G}$-free, specifying the $\mathcal{G}$-module structure of $C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$.*

**Example 2.1.16.** *The CW chain complex of the usual product CW structure on $G \times G$ becomes a $\mathcal{G}$-module via:*

$$C_*^{CW}(G \times G) = C_*^{CW}(G) \otimes_{\mathbb{F}} C_*^{CW}(G),$$

*where the action of $\mathcal{G}$ is given by*

$$s(a \otimes b) = sa \otimes b + j^2 a \otimes sb, \tag{2.11}$$

$$j(a \otimes b) = ja \otimes jb.$$

*The differentials are induced by those of the usual product CW structure on $G \times G$.*

For $X_1 \wedge X_2$ with the $G$-CW decomposition described in Section 2.1.2, we have:

$$C_*^{CW}(X_1 \wedge X_2, \mathrm{pt}) = C_*^{CW}(X_1, \mathrm{pt}) \otimes_{\mathbb{F}} C_*^{CW}(X_2, \mathrm{pt}), \tag{2.12}$$

as $\mathcal{G}$-chain complexes.

Furthermore the CW chain complex for the $G$-smash product $X_1 \wedge_G X_2$ is given by:

$$C_*^{CW}(X_1 \wedge_G X_2, \mathrm{pt}) \simeq C_*^{CW}(X_1 \wedge X_2, \mathrm{pt})/\mathcal{G}. \tag{2.13}$$

We will write elements of the latter as $x_1 \otimes_{\mathcal{G}} x_2$. Note that Borel homology $\tilde{H}^G_*(X)$ is calculated using a $G$-smash product, and so may be computed from the following chain complex:

$$\tilde{H}^G_*(X) = H(C^{CW}_*(EG) \otimes_{\mathcal{G}} C^{CW}_*(X, \mathrm{pt}), \partial). \tag{2.14}$$

In (2.14), we choose some fixed $G$-CW decomposition of $EG$ to define $C^{CW}_*(EG)$. Following (2.14), we make a definition.

**Definition 2.1.17.** Let $Z$ a $\mathcal{G}$-chain complex. We define the $G$-Borel homology of $Z$ by

$$H^G_*(Z) = H(C^{CW}_*(EG) \otimes_{\mathcal{G}} Z, \partial), \tag{2.15}$$

and similarly for $S^1$-Borel homology:

$$H^{S^1}_*(Z) = H(C^{CW}_*(EG) \otimes_{C^{CW}_*(S^1)} Z, \partial), \tag{2.16}$$

where $C^{CW}_*(S^1)$ is viewed as a subcomplex of $\mathcal{G}$.

By construction:

**Fact 2.1.18.** *If $Z = C^{CW}_*(X, \mathrm{pt})$ is the relative CW chain complex of a $G$-space $X$, then $H^G_*(Z) = \tilde{H}^G_*(X)$.*

Note then that $\mathcal{G}$-module $C^{CW}_*(X, \mathrm{pt})$ determines $\tilde{H}^G_*(X)$ for $X$ a space of type SWF.

For $R$ a ring and $M$ an $R$-module with a fixed basis $\{B_i\}$, we say that an element $m \in M$ *contains* $b \in \{B_i\}$ if when $m$ is written in the basis $\{B_i\}$ it has a nontrivial $b$ term.

**Definition 2.1.19.** We call a $\mathcal{G}$-chain complex $Z$ *a chain complex of type SWF* at level $s$ if $Z$ is isomorphic to a chain complex (perhaps with a grading shift) generated by

$$\{c_0, c_1, c_2, ..., c_s\} \cup \bigcup_{i \in I} \{x_i\}, \tag{2.17}$$

subject to the following conditions. The element $c_i$ is of degree $i$, and $I$ is some finite index set. The only relations are $j^2 c_i = c_i, s c_i = 0, j c_0 = c_0$. The differentials are given by $\partial c_1 = c_0$, and $\partial c_i = (1 + j) c_{i-1}$ for $2 \leqslant i \leqslant s - 1$. Further, $\partial(c_s)$ contains $(1 + j) c_{s-1}$. The submodule generated by $\{x_i\}_{i \in I}$ is free under the action of $\mathcal{G}$. We call the submodule generated by $\{c_i\}_i$ the *fixed-point set* of $Z$.

Chain complexes of type SWF are to be thought of as reduced $G$-CW chain complexes of spaces of type SWF. Indeed, all spaces $X$ of type SWF have a $G$-CW decomposition with reduced $G$-CW chain complex a complex of type SWF. To see this, we first decompose $X^{S^1} \simeq (\tilde{\mathbb{R}}^s)^+$ using the CW decomposition of Example 2.1.11 for $(\tilde{\mathbb{R}}^s)^+$. We note that $X^{S^1}$ is a $G$-CW subcomplex of $X$, and all cells of $(X, X^{S^1})$ are free $G$-cells, since $X$ is a space of type SWF. Label these cells $\{x_i\}$ for $i$ in some index set, and we obtain that the corresponding CW chain complex is as in Definition 2.1.19.

To introduce chain local equivalence, we will consider the CW chain complexes coming from suspensions. For a module $M$ and a submodule $S \subseteq M$, we let $\langle S \rangle \subseteq M$ denote the subset generated by $S$.

Note that, by Example 2.1.14 and the $G$-CW decomposition constructed in Section 2.1.2 for suspensions, for $X$ a complex of type SWF:

$$C_*^{CW}(\Sigma^{\tilde{\mathbb{R}}} X, \mathrm{pt}) = \langle c_0, c_1 \rangle \otimes_{\mathbb{F}} C_*^{CW}(X, \mathrm{pt}), \tag{2.18}$$

with relations $\partial c_1 = c_0$, $j^2 c_1 = c_1, jc_0 = c_0, sc_0 = sc_1 = 0$. The differential on the right is given by $\partial(a \otimes b) = \partial(a) \otimes b + a \otimes \partial(b)$. Similarly, using Example 2.1.15:

$$C_*^{CW}(\Sigma^{\mathbb{H}} X, \mathrm{pt}) = \langle r_0, y_1, y_2, y_3 \rangle \otimes_{\mathbb{F}} C_*^{CW}(X, \mathrm{pt}),$$

with the product differential on the right, and differentials for the $y_i$ given as in Example 2.1.15.

For $V = \mathbb{H}$, $\tilde{\mathbb{R}}$, or $\mathbb{R}$, we set:

$$\Sigma^V Z = C_*^{CW}(V^+, \mathrm{pt}) \otimes_{\mathbb{F}} Z, \tag{2.19}$$

with $\mathcal{G}$-action given by:

$$s(a \otimes b) = (sa \otimes b) + (j^2 a \otimes sb), \tag{2.20}$$

$$j(a \otimes b) = ja \otimes jb.$$

The chain complexes $C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$ and $C_*^{CW}(\tilde{\mathbb{R}}^+, \mathrm{pt})$ were given in Examples 2.1.14 and 2.1.15, respectively. Also, $C_*^{CW}(\mathbb{R}^+, \mathrm{pt}) = \langle c_1 \rangle$, where $jc_1 = c_1, sc_1 = 0$, and deg $c_1 = 1$. Hence, for example:

$$\Sigma^{\mathbb{R}} Z = Z[-1]. \tag{2.21}$$

**Lemma 2.1.20.** *Let $V = \mathbb{H}$, $\tilde{\mathbb{R}}$, or $\mathbb{R}$. If $Z = C_*^{CW}(X, \mathrm{pt})$ for $X$ a space of type SWF, then $\Sigma^V Z = C_*^{CW}(\Sigma^V X, \mathrm{pt})$.*

*Proof.* This follows from the CW chain complex structure given for suspensions in Section 2.1.2, and (2.12). □

For $V = \mathbb{H}^i \oplus \tilde{\mathbb{R}}^j \oplus \mathbb{R}^k$ for some constants $i, j, k$, we define $\Sigma^V Z$ by:

$$\Sigma^V Z = (\Sigma^{\mathbb{H}})^i (\Sigma^{\tilde{\mathbb{R}}})^j (\Sigma\mathbb{R})^k Z. \tag{2.22}$$

where $(\Sigma^{\mathbb{H}})^i$ denotes applying $\Sigma^{\mathbb{H}}$ $i$ times, and so for $\tilde{\mathbb{R}}$ and $\mathbb{R}$. It is then clear that:

$$\Sigma^V \Sigma^W Z \cong \Sigma^W \Sigma^V Z, \tag{2.23}$$

for two $G$-representations $V, W$, each a direct sum of copies of $\mathbb{H}, \mathbb{R}, \tilde{\mathbb{R}}$.

**Definition 2.1.21.** Let $Z_i$ be chain complexes of type SWF, $m_i \in \mathbb{Z}, n_i \in \mathbb{Q}$, for $i = 1, 2$. We call $(Z_1, m_1, n_1)$ and $(Z_2, m_2, n_2)$ *chain stably equivalent* if $n_1 - n_2 \in \mathbb{Z}$ and there exist $M \in \mathbb{Z}, N \in \mathbb{Q}$ and maps

$$\Sigma^{(N-n_1)\mathbb{H}} \Sigma^{(M-m_1)\tilde{\mathbb{R}}} Z_1 \to \Sigma^{(N-n_2)\mathbb{H}} \Sigma^{(M-m_2)\tilde{\mathbb{R}}} Z_2 \tag{2.24}$$

$$\Sigma^{(N-n_1)\mathbb{H}} \Sigma^{(M-m_1)\tilde{\mathbb{R}}} Z_1 \leftarrow \Sigma^{(N-n_2)\mathbb{H}} \Sigma^{(M-m_2)\tilde{\mathbb{R}}} Z_2, \tag{2.25}$$

which are chain homotopy equivalences.

**Remark 2.1.22.** *We do not consider suspensions by $\mathbb{R}$, unlike in the case of stable equivalence for spaces, since by (2.21), no new maps are obtained by suspending by $\mathbb{R}$.*

Chain stable equivalence is an equivalence relation, and we denote the set of chain stable equivalence classes by $\mathfrak{CE}$.

**Lemma 2.1.23.** *Associated to an element $(X, m, n) \in \mathfrak{C}$ there is a well-defined element $(C_*^{CW}(X, \mathrm{pt}), m, n) \in \mathfrak{CE}$.*

*Proof.* Say that $[(X_1, m_1, n_1)] = [(X_2, m_2, n_2)] \in \mathfrak{C}$ with $G$-CW decompositions $C_i$ of $X_i$. We will show that

$$[(C_*^{CW}(X_1, \mathrm{pt}), m_1, n_1)] = [(C_*^{CW}(X_2, \mathrm{pt}), m_2, n_2)] \in \mathfrak{C}\mathfrak{E}, \tag{2.26}$$

where $C_*^{CW}(X_i, \mathrm{pt})$ is the $\mathcal{G}$-chain complex associated to the $G$-CW decomposition $C_i$ of $X_i$. (In the case $X_1 \simeq X_2$, and $m_1 = m_2, n_1 = n_2$, we are showing that the corresponding element in $\mathfrak{C}\mathfrak{E}$ does not depend on the choice of $G$-CW decomposition). By hypothesis, there are homotopy equivalences $f$ and $g$:

$$f : \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}X_1 \to \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}X_2,$$

$$g : \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}X_2 \to \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}X_1.$$

By the Equivariant Cellular Approximation Theorem (see [52]), we may homotope $f$ and $g$ to cellular maps (where the cell structures of suspensions are given as in (2.19)):

$$f^{CW} : \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}C_1 \to \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}C_2,$$

$$g^{CW} : \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}C_2 \to \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}C_1.$$

Since $f$ and $g$ are homotopy equivalences, so are $f^{CW}$ and $g^{CW}$. The cellular maps $f^{CW}$ and $g^{CW}$ induce maps $f_*$ and $g_*$:

$$f_* : \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}C_*^{CW}(X_1, \mathrm{pt}) \to \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}C_*^{CW}(X_2, \mathrm{pt}),$$

$$g_* : \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}}C_*^{CW}(X_2, \mathrm{pt}) \to \Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}}C_*^{CW}(X_1, \mathrm{pt}).$$

These are chain homotopy equivalences, by construction, and so we obtain (2.26), as needed. □

In analogy with (2.6), we define Borel homology for elements of $\mathfrak{C}\mathfrak{E}$.

**Definition 2.1.24.** Let $(Z, m, n) \in \mathfrak{C}\mathfrak{E}$. We define $H_*^G((Z, m, n)) = H_*^G(Z)[m + 4n]$.

**Fact 2.1.25.** *For $Z \in \mathfrak{C}\mathfrak{E}$, $H_*^G(Z)$ is well-defined.*

*Proof.* It suffices to show, for $Z$ a chain complex of type SWF, that

$$H^G_*(\Sigma^V Z) = H^G_{*-\dim V}(Z). \tag{2.27}$$

By (2.15), we need to compute

$$H_*(C^{CW}_*(EG) \otimes_{\mathcal{G}} (C^{CW}_*(V^+, \mathrm{pt}) \otimes_{\mathbb{F}} Z)).$$

However, we have, by (2.12),

$$C^{CW}_*(EG) \otimes_{\mathbb{F}} (C^{CW}_*(V^+, \mathrm{pt}) \otimes_{\mathbb{F}} Z) = (C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt})) \otimes_{\mathbb{F}} Z,$$

as $\mathcal{G}$-modules. Recalling the definition of $\otimes_{\mathcal{G}}$ in (2.13) we have

$$C^{CW}_*(EG) \otimes_{\mathcal{G}} (C^{CW}_*(V^+, \mathrm{pt}) \otimes_{\mathbb{F}} Z) = (C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt})) \otimes_{\mathcal{G}} Z.$$

Then to show (2.27) we need only show

$$H_*((C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt})) \otimes_{\mathcal{G}} Z) = H_{*-\mathrm{dim} V}(C^{CW}_*(EG) \otimes_{\mathcal{G}} Z).$$

Indeed, $C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt})$ is the relative CW chain complex of $\Sigma^V EG_+$, a free $G$-space with nonzero homology only in degree $\dim V$. As any two $\mathcal{G}$-free resolutions are homotopy equivalent, we obtain $C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt}) \simeq C^{CW}_*(EG)[-\dim V]$. Then we have

$$H_*((C^{CW}_*(EG) \otimes_{\mathbb{F}} C^{CW}_*(V^+, \mathrm{pt})) \otimes_{\mathcal{G}} Z) = H_*((C^{CW}_*(EG) \otimes_{\mathcal{G}} Z)[-\dim V]) = H^G_{*-\dim V}(Z),$$

as needed. □

**Definition 2.1.26.** Let $Z_i$ be chain complexes of type SWF, $m_i \in \mathbb{Z}, n_i \in \mathbb{Q}$, for $i = 1, 2$. We call $(Z_1, m_1, n_1)$ and $(Z_2, m_2, n_2)$ *chain locally equivalent*, written $(Z_1, m_1, n_1) \equiv_{cl} (Z_2, m_2, n_2)$, if there exist $M \in \mathbb{Z}, N \in \mathbb{Q}$ and maps

$$\Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}} Z_1 \to \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}} Z_2 \tag{2.28}$$

$$\Sigma^{(N-n_1)\mathbb{H}}\Sigma^{(M-m_1)\tilde{\mathbb{R}}} Z_1 \leftarrow \Sigma^{(N-n_2)\mathbb{H}}\Sigma^{(M-m_2)\tilde{\mathbb{R}}} Z_2, \tag{2.29}$$

which are chain homotopy equivalences on the fixed-point sets.

We call a map as in (2.28) or (2.29) a *chain local equivalence.* Elements $Z_1, Z_2 \in \mathfrak{CE}$ are chain locally equivalent if and only if there are chain local equivalences $Z_1 \to Z_2$ and $Z_2 \to Z_1$. There are pairs of chain complexes with a chain local equivalence in one direction but not the other; these are not chain locally equivalent complexes. Chain local equivalence is an equivalence relation, and we write $[(Z, m, n)]_{cl}$ for the chain local equivalence class of $(Z, m, n) \in \mathfrak{CE}$. The set $\mathfrak{CLE}$ of chain local equivalence classes is naturally an abelian group, with multiplication given by the tensor product (over $\mathbb{F}$, with $\mathcal{G}$-action as above). (This abelian group structure on $\mathfrak{CLE}$ corresponds to connected sum in the homology cobordism group; see Fact 3.1.5). The inverse of an element $[(Z, 0, 0)]_{cl}$ of $\mathfrak{CLE}$ is $[(Z^*, 0, 0)]_{cl}$ where $Z^*$ denotes the chain complex dual to $Z$. The identity element 0 of $\mathfrak{CLE}$ is $[(\mathbb{F}, 0, 0)]_{cl}$, where $C^{CW}_*(S^0, \mathrm{pt}) = \mathbb{F} = \langle f_0 \rangle$ is the $\mathcal{G}$-module concentrated in degree 0 for which $jf_0 = f_0$ and $sf_0 = 0$.

**Definition 2.1.27.** For $[(Z, m, n)] \in \mathfrak{CLE}$, we call

$$\alpha((Z, m, n)) = \frac{a(Z)}{2} - \frac{m}{2} - 2n, \ \beta((Z, m, n)) = \frac{b(Z)}{2} - \frac{m}{2} - 2n, \tag{2.30}$$

$$\gamma((Z, m, n)) = \frac{c(Z)}{2} - \frac{m}{2} - 2n,$$

the *Manolescu invariants* of $(Z, m, n)$. The invariants $\alpha, \beta$ and $\gamma$ do not depend on the choice of representative of the class $[(Z, m, n)]$.

### 2.1.4 Calculating the chain local equivalence class

In this section we will obtain a description of $\mathfrak{CLE}$ more amenable to calculations than the definition. Throughout this section $Z$ will denote a chain complex of type SWF. The main result is Lemma 2.1.30, which allows us to determine if $(Z_1, m_1, n_1)$ and $(Z_2, m_2, n_2)$ are chain locally equivalent without checking all possible $M, N$.

To prove Lemma 2.1.30 we will first need Lemma 2.1.28, a result on chain homotopy classes of maps between fixed-point sets. For two $\mathcal{G}$-chain complexes $Z_1'$ and $Z_2'$, let $[Z_1', Z_2']$ denote the set of chain homotopy classes of maps from $Z_1'$ to $Z_2'$. We have an algebraic anologue of the Equivariant Freudenthal Suspension Theorem (Theorem 3.3 of [1]), as follows.

We recall that for $Z$ a chain complex of type SWF at level $s$, the fixed point set $R \subset Z$ is isomorphic, as a $\mathcal{G}$-chain complex, to

$$C_*^{CW}(\tilde{\mathbb{R}}^s, \mathrm{pt}) \cong \langle c_0, ..., c_s \rangle, \tag{2.31}$$

with relations $jc_0 = sc_0 = 0$ and, for $i > 0$, $j^2 c_i = c_i$, while $sc_i = 0$. The differentials in (2.31) are given by $\partial(c_i) = (1+j)c_{i-1}$ for $2 \leqslant i \leqslant s$, and $\partial(c_1) = c_0$, $\partial(c_0) = 0$.

**Lemma 2.1.28.** *Let $R_1 \cong R_2 \cong C_*^{CW}(\tilde{\mathbb{R}}^s, \mathrm{pt})$, where $\cong$ denotes isomorphism of $\mathcal{G}$-chain complexes. Then the map*

$$[R_1, R_2] \to [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2], \tag{2.32}$$

*obtained by suspension by $\mathbb{H}$ is an isomorphism.*

*Proof.* To show that the map in (2.32) is an isomorphism, we consider the commutative diagram:

$$\begin{array}{ccc} [\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}), \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})] & \xrightarrow{\Sigma^{\tilde{\mathbb{R}}^s}} & [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2] \\ \Sigma^{\mathbb{H}} \Big\uparrow & & \Sigma^{\mathbb{H}} \Big\uparrow \\ [C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] & \xrightarrow{\Sigma^{\tilde{\mathbb{R}}^s}} & [R_1, R_2] \end{array} \tag{2.33}$$

We have used the isomorphisms $R_1 \cong R_2 \cong \Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(S^0, \mathrm{pt})$ in writing the right column. In (2.33), the composition is precisely

$$\Sigma^{\mathbb{H} \oplus \tilde{\mathbb{R}}^s} : [C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] \to [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2].$$

We will show that the maps:

$$\Sigma^{\mathbb{H}} : [C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] \to [\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}), \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})], \tag{2.34}$$

$$\Sigma^{\tilde{\mathbb{R}}^s} : [C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] \to [R_1, R_2], \tag{2.35}$$

and

$$\Sigma^{\tilde{\mathbb{R}}^s} : [\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}), \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})] \to [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2] \tag{2.36}$$

are isomorphisms. Then, since three of the four maps in (2.33) are isomorphisms, so is the fourth, which is exactly the map from (2.32), proving the Lemma.

We show that (2.34) is an isomorphism. We use the notation of Example 2.1.15 for $\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})$, writing $c_0$ for the generator of $C_*^{CW}(S^0, \mathrm{pt})$. Let $f : \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}) \to \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})$. Then $f(r_0 \otimes c_0) = r_0 \otimes c_0$ or $f(r_0 \otimes c_0) = 0$, for degree reasons. In the former case, $f(y_1 \otimes c_0) = u_1 y_1 \otimes c_0$ where $u_1$ is a unit in $\mathcal{G}$. Indeed, this follows from the requirement:

$$\partial(f(y_1 \otimes c_0)) = f(\partial(y_1 \otimes c_0)) = f(r_0 \otimes c_0) = r_0 \otimes c_0.$$

Similarly, we obtain, perhaps after a homotopy,

$$f(y_i \otimes c_0) = u_i y_i \otimes c_0, \tag{2.37}$$

where $u_i$ is a unit in $\mathcal{G}$ for $i = 1, 2, 3$. Indeed, this follows from $H_*(\Sigma^{\mathbb{H}}\langle c_0\rangle)$ being concentrated in grading 4. For instance, $f(y_2 \otimes c_0) + u_1 y_2 \otimes c_0$ must be a cycle in $\Sigma^{\mathbb{H}}\langle c_0\rangle$, since $\partial(f(y_2 \otimes c_0)) = f(\partial(y_2 \otimes c_0)) = (1 + j)u_1 y_1 \otimes c_0$. Then, by $H_2(\Sigma^{\mathbb{H}}\langle c_0\rangle) = 0$, the element $f(y_2 \otimes c_0) + u_1 y_2 \otimes c_0$ is a boundary, and we may choose a homotopy $h$, vanishing in grading 1, so that $(\partial h + h\partial)(y_2 \otimes c_0) = \partial h(y_2 \otimes c_0) = f(y_2 \otimes c_0) + u_1 y_2 \otimes c_0$. This establishes (2.37) for $i = 2$, and $i = 3$ follows similarly.

We show that $f \simeq \mathrm{Id}_{\Sigma^{\mathbb{H}} C_*^{CW}(S^0,\mathrm{pt})}$. We define a homotopy $h : \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}) \to \Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})$ from $f$ to $\mathrm{Id}_{\Sigma^{\mathbb{H}} C_*^{CW}(S^0,\mathrm{pt})}$, proceeding by defining it in each grading. First, let $h(r_0 \otimes c_0) = 0$. Then choose $h$ in grading 1 so that $\partial h(y_1 \otimes c_0) = (1 + u_1) y_1 \otimes c_0$, and extend $\mathcal{G}$-linearly. This is possible, because $(1 + u_1) y_1 \otimes c_0$ is a boundary in $\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt})$ for any unit $u_1$. An elementary calculation shows that $h$ may be extended over degree 2 and degree 3. In the case that $f(r_0 \otimes c_0) = 0$, an explicit homotopy as above shows that $f$ is homotopic to the zero map. This shows that (2.34) is surjective.

To show that (2.34) is injective, we note that $[C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] = [\langle c_0\rangle, \langle c_0\rangle]$ is exactly $\mathbb{Z}/2$ as there is precisely one nontrivial map, $c_0 \to c_0$. Then we need only show the identity map has nontrivial suspension. But $\Sigma^{\mathbb{H}} \mathrm{Id}_{C_*^{CW}(S^0,\mathrm{pt})} = \mathrm{Id}_{\mathbb{H}^+}$, which induces an isomorphism in homology, and so is not null-homotopic. Then, indeed, we obtain that the map in (2.34) is an isomorphism.

The proof of the isomorphism (2.35) is parallel to the proof of (2.34), and is left to the reader.

We show that the map in (2.36) is an isomorphism. Note that $\Sigma^{\mathbb{H}} C_*^{CW}(S^0, \mathrm{pt}) \simeq C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$. We let $\tilde{\oplus}$ denote a direct sum of $\mathcal{G}$-modules that is not necessarily a direct sum of chain complexes (i.e. there may be differentials between the summands). Then $C_*^{CW}(\mathbb{H}^+, \mathrm{pt}) = \langle c_0 \rangle \tilde{\oplus} F$, for $F$ a $\mathcal{G}$-free submodule. We have:

$$\Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(\mathbb{H}^+, \mathrm{pt}) = \Sigma^{\tilde{\mathbb{R}}^s} \langle c_0 \rangle \tilde{\oplus} \Sigma^{\tilde{\mathbb{R}}^s} F. \tag{2.38}$$

However, $\Sigma^{\tilde{\mathbb{R}}^s} F \simeq F[-s]$. Indeed, we have a map $\gamma : F[-s] \to \Sigma^{\tilde{\mathbb{R}}^s} F$ defined by $\gamma(x[-s]) = C \otimes x$, where $C$ is the fundamental class of $(\tilde{\mathbb{R}}^s)^+$. If $Z$ is of type SWF at level 0, then $C = c_0$, while if $Z$ is of type SWF at level $s > 0$, we have $C = (1 + j)c_s$, where we use the notation from Example 2.1.14. Also, $\gamma$ is a chain map, as the reader may verify. Furthermore, it is clear that $\gamma$ induces a quasi-isomorphism. We show it is, in fact, a homotopy equivalence. We construct a homotopy inverse

$$\tau : \Sigma^{\tilde{\mathbb{R}}^s} F \to F[-s], \tag{2.39}$$

so that $\tau(C \otimes x) = x[-s]$ for $x \in F$. We treat the case $s = 1$; for $s > 1$ we apply:

$$\Sigma^{\tilde{\mathbb{R}}^s} F = (\Sigma^{\tilde{\mathbb{R}}})^s F \simeq F[-s]. \tag{2.40}$$

Fix a $\mathcal{G}$-basis $x_i$ of $F$. Assume we have defined $\tau(c_k \otimes x_i)$ for $k = 0, 1$, for all $x_i$ such that $\deg x_i \leqslant m - 1$, for some $m$. For generators $x_i$ of degree $m$ we define:

$$\tau(c_0 \otimes x_i) = \tau(c_1 \otimes \partial x_i), \tag{2.41}$$

$$\tau(c_1 \otimes x_i) = 0,$$

$$\tau(jc_1 \otimes x_i) = x_i[-1],$$

and extend by linearity. Further, $(1 + j)c_1 \otimes x \to x[-1]$ for all $x \in F$ by definition, so $\tau\gamma = 1_{F[-s]}$, where $1_{F[-s]}$ is the identity on $F[-s]$.

We find a homotopy $H$ from $\gamma\tau$ to $\mathrm{Id}_{\Sigma^{\tilde{\mathbb{R}}} F}$, to show that $\gamma$ is a homotopy equivalence. Fix generators $x_i$ as in the definition of $\tau$. Define $H$ by $H(c_0 \otimes x_i) = c_1 \otimes x_i$, for all $x_i$, and by $H(c_1 \otimes x) = 0 = H(jc_1 \otimes x)$ for all $x \in F$, and extend linearly. We must then show that $H$ is a chain homotopy between $\gamma\tau$ and $\mathrm{Id}_{\Sigma^{\tilde{\mathbb{R}}} F}$. That is, we need

$$(\partial H + H\partial)(c_0 \otimes x_i) = \gamma\tau(c_0 \otimes x_i) + c_0 \otimes x_i, \tag{2.42}$$

$$(\partial H + H\partial)(c_1 \otimes x_i) = \gamma\tau(c_1 \otimes x_i) + c_1 \otimes x_i, \tag{2.43}$$

and

$$(\partial H + H\partial)(jc_1 \otimes x_i) = \gamma\tau(jc_1 \otimes x_i) + jc_1 \otimes x_i. \tag{2.44}$$

We suppose inductively that (2.42)-(2.44) are true for all $x_i$ with deg $x_i \leqslant N$ for some $N$. The inductive hypothesis is true (vacuously) for $N$ sufficiently small, since $F$ is a bounded-below complex. Fix $x_i$ of degree $N+1$. We show that (2.42)-(2.44) hold for $x_i$.

First, consider:

$$(\partial H + H\partial)(c_0 \otimes x_i) = \partial(c_1 \otimes x_i) + H(c_0 \otimes \partial x_i), \tag{2.45}$$

where we have used the definition $H(c_0 \otimes x_i) = c_1 \otimes x_i$. Also:

$$(\partial H + H\partial)(c_1 \otimes \partial x_i) = \partial H(c_1 \otimes \partial x_i) + H(c_0 \otimes \partial x_i) = \gamma\tau(c_1 \otimes \partial x_i) + c_1 \otimes \partial x_i, \tag{2.46}$$

by the inductive hypothesis. Rearranging (2.46), we have:

$$H(c_0 \otimes \partial x_i) = \gamma\tau(c_1 \otimes \partial x_i) + c_1 \otimes \partial x_i + \partial H(c_1 \otimes \partial x_i).$$

By the definition of $\tau$, we have $\tau(c_1 \otimes \partial x_i) = \tau(c_0 \otimes x_i)$, so, using (2.45), we obtain:

$$(\partial H + H\partial)(c_0 \otimes x_i) = \gamma\tau(c_0 \otimes x_i) + c_0 \otimes x_i + \partial H(c_1 \otimes \partial x_i). \tag{2.47}$$

But $H(c_1 \otimes \partial x_i) = 0$ by definition, so:

$$(\partial H + H\partial)(c_0 \otimes x_i) = \gamma\tau(c_0 \otimes x_i) + c_0 \otimes x_i, \tag{2.48}$$

verifying (2.42).

Next, we investigate $(\partial H + H\partial)(c_1 \otimes x_i)$:

$$(\partial H + H\partial)(c_1 \otimes x_i) = \partial H(c_1 \otimes x_i) + H(c_0 \otimes x_i) + H(c_1 \otimes \partial x_i) = H(c_0 \otimes x_i) = c_1 \otimes x_i, \tag{2.49}$$

using $H(c_1 \otimes x_i) = 0$ and $H(c_1 \otimes \partial x_i) = 0$. Using $\tau(c_1 \otimes x_i) = 0$, we obtain (2.43) from (2.49).

We also check $(\partial H + H\partial)(jc_1 \otimes x_i)$ :

$$(\partial H + H\partial)(jc_1 \otimes x_i) = \partial H(jc_1 \otimes x_i) + H(c_0 \otimes x_i) + H(jc_1 \otimes \partial x_i) = H(c_0 \otimes x_i) = c_1 \otimes x_i, \tag{2.50}$$

since $H(jc_1 \otimes x_i) = 0$ and $H(jc_1 \otimes \partial x_i) = 0$. Additionally, $\tau(jc_1 \otimes x_i) = x_i[-1]$, and $\gamma(x_i[-1]) = (1+j)c_1 \otimes x_i$. Then $c_1 \otimes x_i = \gamma\tau(jc_1 \otimes x_i) + jc_1 \otimes x_i$, and (2.44) follows.

Then $H$ is a chain homotopy between $\gamma\tau$ and $\mathrm{Id}_{\Sigma^{\tilde{\mathbb{R}}}F}$, as needed, and so $\gamma$ and $\tau$ are homotopy equivalences.

We let $\mathbb{I}$ denote the identity map on $\Sigma^{\tilde{\mathbb{R}}^s}\langle c_0\rangle$. We have a homotopy equivalence:

$$\Sigma^{\tilde{\mathbb{R}}^s}\langle c_0\rangle \tilde{\oplus} \Sigma^{\tilde{\mathbb{R}}^s} F \xrightarrow{\mathbb{I}\tilde{\oplus}\tau} \Sigma^{\tilde{\mathbb{R}}^s}\langle c_0\rangle \tilde{\oplus} F[-s]. \tag{2.51}$$

Further, there is an isomorphism

$$[\Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(\mathbb{H}^+, \mathrm{pt}), \Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(\mathbb{H}^+, \mathrm{pt})] \to [\Sigma^{\tilde{\mathbb{R}}^s}\langle c_0\rangle \tilde{\oplus} F[-s], \Sigma^{\tilde{\mathbb{R}}^s}\langle c_0\rangle \tilde{\oplus} F[-s]],$$

given by

$$f \to (\mathbb{I}\tilde{\oplus}\tau) f (\mathbb{I}\tilde{\oplus}\gamma).$$

Here, the map $(\mathbb{I}\tilde{\oplus}\gamma)$ acts by the identity on the first summand, and by $\gamma$ on the second.

We first prove surjectivity of (2.36). Fix $f : \Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(\mathbb{H}^+, \mathrm{pt}) \to \Sigma^{\tilde{\mathbb{R}}^s} C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$. Let $f' = (\mathbb{I}\tilde{\oplus}\tau) f (\mathbb{I}\tilde{\oplus}\gamma)$. We find $g : C_*^{CW}(\mathbb{H}^+, \mathrm{pt}) \to C_*^{CW}(\mathbb{H}^+, \mathrm{pt})$ so that $\Sigma^{\tilde{\mathbb{R}}^s} g \simeq f$. We define $g$ separately on the two summands $C_*^{CW}(S^0, \mathrm{pt})$ and $F$.

Let $g_1 \in [C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})]$ so that $\Sigma^{\tilde{\mathbb{R}}^s} g_1 \simeq f|_{\langle c_0, \ldots, c_s\rangle}$. Such a $g_1$ exists by (2.35). Further, note that there is a natural isomorphism $[F, F] = [F[-s], F[-s]]$, and let $g_2 \in [F, F]$ be the element corresponding to $f'|_{F[-s]} \in [F[-s], F[-s]]$. Define a chain map by $g : \langle c_0\rangle \tilde{\oplus} F \to \langle c_0\rangle \tilde{\oplus} F$ by

$$g = g_1 \tilde{\oplus} g_2.$$

By construction, $\Sigma^{\tilde{\mathbb{R}}^s} g \simeq f$, as needed.

Finally, we check injectivity of (2.36). We have $[\langle c_0\rangle, \langle c_0\rangle] = [\Sigma^{\mathbb{H}}\langle c_0\rangle, \Sigma^{\mathbb{H}}\langle c_0\rangle]$ is $\mathbb{Z}/2$, with nontrivial map given by the identity $\mathrm{Id}_{\mathbb{H}^+}$. We need only show then that the map $\Sigma^{\tilde{\mathbb{R}}^s}\mathrm{Id}_{\mathbb{H}^+}$ is not null-homotopic. Indeed, it induces a nontrivial map on homology by construction, so is not null-homotopic. Then (2.36) is an isomorphism, as needed. □

**Remark 2.1.29.** *We have* $[C_*^{CW}(S^0, \mathrm{pt}), C_*^{CW}(S^0, \mathrm{pt})] = \mathbb{Z}/2$, *as remarked in the proof. Hence Lemma 2.1.28 implies* $[\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2] \cong \mathbb{Z}/2$.

**Lemma 2.1.30.** *Let $Z_1$ and $Z_2$ be locally equivalent chain complexes of type SWF. Let $R_i \subset Z_i$ be the corresponding fixed-point sets. Additionally, for all nonzero homogeneous $r \in R_i$, we require* $\deg r < \deg x$ *for all nonzero homogeneous*

$$x \in Z_i/R_i,$$

*for $i = 1, 2$. Then there exist chain maps*

$$Z_1 \to Z_2, \tag{2.52}$$

$$Z_1 \leftarrow Z_2,$$

*that are chain homotopy equivalences on the fixed-point sets.*

*Proof.* Let $Z_i(N, M)$ denote $\Sigma^{N\mathbb{H}}\Sigma^{M\tilde{\mathbb{R}}}Z_i$. By hypothesis there exist maps which are homotopy equivalences on the fixed-point sets:

$$Z_1(N, M) \to Z_2(N, M), \tag{2.53}$$

$$Z_1(N, M) \leftarrow Z_2(N, M),$$

for $M, N$ sufficiently large.

**Claim 1.** Let $V = \mathbb{H}$ or $\tilde{\mathbb{R}}$. Take $\phi$ a map which is a chain homotopy equivalence on fixed-point sets:

$$\phi : \Sigma^V Z_1 \to \Sigma^V Z_2.$$

Then $\phi$ is chain homotopic to the suspension of a map $\phi_0$, also a chain homotopy equivalence on fixed-point sets:

$$\phi_0 : Z_1 \to Z_2.$$

Since $\Sigma^V Z_i$ also satisfy the conditions of the Lemma, it follows from Claim 1 that any map which is a homotopy equivalence on fixed-point sets, for $M_0, N_0 \geqslant 0$:

$$\phi : Z_1(N_0, M_0) \to Z_2(N_0, M_0)$$

is homotopic to the suspension of a map:

$$\phi_0 : Z_1 \to Z_2,$$

which implies the existence of the maps as in (2.52).

We prove Claim 1 for $V = \mathbb{H}$; the case of $V = \tilde{\mathbb{R}}$ is similar, but easier.

We let $\tilde{\oplus}$ denote a direct sum of $\mathcal{G}$-modules that is not necessarily a direct sum of chain complexes.

Let $F_i$ be the $\mathcal{G}$-free submodule of $Z_i$ generated by elements $x$ of degree greater than deg $r$ for all $r$ in the fixed-point set $R_i$. We will also consider $F_i$ as a $\mathcal{G}$-chain complex so that the projection

$$Z_i \to Z_i/R_i \simeq F_i$$

is a map of complexes. Then we have $Z_i = R_i \tilde{\oplus} F_i$. For a given local equivalence $\phi : \Sigma^{\mathbb{H}} Z_1 \to \Sigma^{\mathbb{H}} Z_2$, we have the diagram:

$$\begin{array}{ccc} \Sigma^{\mathbb{H}}(R_1 \tilde{\oplus} F_1) & \xrightarrow{\phi} & \Sigma^{\mathbb{H}}(R_2 \tilde{\oplus} F_2) \\ \downarrow & & \downarrow \\ (\Sigma^{\mathbb{H}} R_1)\tilde{\oplus}(\Sigma^{\mathbb{H}} F_1) & \longrightarrow & (\Sigma^{\mathbb{H}} R_2)\tilde{\oplus}(\Sigma^{\mathbb{H}} F_2) \end{array}$$

However, $\Sigma^{\mathbb{H}} F_i$ is homotopy equivalent to $\Sigma^{\mathbb{R}^4} F_i = F_i[-4]$. To see this, we use the notation for suspension by $\mathbb{H}$ as in Example 2.1.15 and write $\gamma : F_i[-4] \to \Sigma^{\mathbb{H}} F_i$, where $\gamma(x[-4]) = s(1+j)^3 y_3 \otimes x$. The term $s(1+j)^3 y_3$ appears as it is the fundamental class of $S^4 \simeq \mathbb{H}^+$. Furthermore, $\gamma$ is a chain map, as the reader may verify. It is clear that $\gamma$ is a quasi-isomorphism, and it is, in fact, a homotopy equivalence. There is a homotopy inverse $\tau$, whose construction is analogous to that in (2.41), so that $\tau(s(1+j)^3 y_3 \otimes x) = x[-4]$. We obtain a map:

$$\phi' = (1_{\Sigma^{\mathbb{H}} R_2} \tilde{\oplus} \tau)\phi(1_{\Sigma^{\mathbb{H}} R_1} \tilde{\oplus} \gamma) : (\Sigma^{\mathbb{H}} R_1)\tilde{\oplus}(F_1[-4]) \to (\Sigma^{\mathbb{H}} R_2)\tilde{\oplus}(F_2[-4]).$$

For degree reasons, $\phi'$ sends $\Sigma^{\mathbb{H}} R_1 \to \Sigma^{\mathbb{H}} R_2$ and $F_1[-4] \to F_2[-4]$. By Lemma 2.1.28, we have:

$$[R_1, R_2] \xrightarrow{\Sigma^{\mathbb{H}}} [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2] \tag{2.54}$$

is an isomorphism. Also, $[F_1[-4], F_2[-4]] = [F_1, F_2]$, clearly. Define $\phi_0|_{R_1}$ by the element of $[R_1, R_2]$ corresponding to $\phi'|_{\Sigma^{\mathbb{H}} R_1} \in [\Sigma^{\mathbb{H}} R_1, \Sigma^{\mathbb{H}} R_2]$. Similarly, define $\phi_0|_{F_1}$ by the element of $[F_1, F_2]$ corresponding to $\phi'|_{F_1[-4]} \in [F_1[-4], F_2[-4]]$. Then we have a map, of $\mathcal{G}$-complexes:

$$\phi_0 : R_1 \tilde{\oplus} F_1 \to R_2 \tilde{\oplus} F_2.$$

By construction, $\Sigma^{\mathbb{H}} \phi_0 \sim \phi$, as needed.

□

For $Z$ a chain complex of type SWF, we will let $Z$ also denote the element $(Z, 0, 0) \in \mathfrak{CE}$.

**Definition 2.1.31.** Let $R$ be the fixed-point set of $Z$. If $\deg r < \deg x$ for all nonzero homogeneous $x \in (Z/R)$ and $r \in R$, we say that the chain complex $Z$ is a *suspensionlike complex.*

**Remark 2.1.32.** *Let $X$ be a free, finite G-CW complex. Then the reduced G-CW chain complex of $\tilde{\Sigma} X$, the unreduced suspension of $X$, is a suspensionlike chain complex. Conversely, any suspensionlike chain complex with fixed-point set $R = \langle c_0 \rangle$ may be realized as the G-CW chain complex of an unreduced suspension. Further, any suspensionlike chain complex of type SWF is chain stably equivalent to $C^{CW}_*(X, \mathrm{pt})$ for some space $X$ of type SWF.*

**Remark 2.1.33.** *For $X$ a space of type SWF, $C^{CW}_*(X, \mathrm{pt})$ need not be a suspensionlike chain complex of type SWF. However, any class in $\mathfrak{E}$ admits a representative $(X, m, n)$ with $C^{CW}_*(X, \mathrm{pt})$ a suspensionlike chain complex of type SWF.*

Lemma 2.1.30 states that if $\Sigma^{(N_0 - n_i)\mathbb{H}} \Sigma^{(M_0 - m_i)\tilde{\mathbb{R}}} Z_i$ are suspensionlike, then all local (stable) maps between $(Z_1, m_1, n_1)$ and $(Z_2, m_2, n_2)$ are realized as genuine chain maps by suspending the complexes $Z_i$ by $N_0 \mathbb{H} \oplus M_0 \tilde{\mathbb{R}}$.

Note that the tensor product $Z_1 \otimes_{\mathbb{F}} Z_2$ of suspensionlike chain complexes of type SWF, at levels $t_1, t_2$ respectively, is not suspensionlike unless $t_1 = 0$ or $t_2 = 0$. However, after quotienting $Z_1 \otimes_{\mathbb{F}} Z_2$ by a large acylic subcomplex, the resulting complex is suspensionlike.

To be more explicit, we note that any suspensionlike chain complex $Z$ of type SWF is quasi-isomorphic to a suspensionlike chain complex of type SWF at level 0, say $Z'$. We form $Z'$ by replacing the generators $c_0, \dots, c_t$ in $Z$ by $c_t'$ where $(1+j)c_t' = 0 = sc_t'$, and otherwise constructing $Z'$ just as $Z$. There is a quasi-isomorphism $Z' \to Z$ given by $c_t' \to (1+j)c_t$.

In particular, $Z_1' \otimes_{\mathbb{F}} Z_2'$ is quasi-isomorphic to $Z_1 \otimes Z_2$, and the quasi-isomorphism takes the fundamental class of $(Z_1' \otimes_{\mathbb{F}} Z_2')^{S^1}$ to $(1+j)c_{t_1} \otimes (1+j)c_{t_2}$. We may replace $(Z_1' \otimes_{\mathbb{F}} Z_2')^{S^1}$ with a copy of $C_*^{CW}(\tilde{\mathbb{R}}^{t_1+t_2}, \mathrm{pt})$, and the resulting complex $Z''$ is a summand of $Z_1 \otimes_{\mathbb{F}} Z_2$ for which inclusion is a chain homotopy equivalence. Thus, the tensor product of suspensionlike chain complexes is chain homotopy equivalent to a suspenionlike complex at the appropriate level, and the fundamental class of the fixed point set is $f_{t_1} \otimes f_{t_2}$, where $f_{t_i}$ are the fundamental classes of $Z_i^{S^1}$.

### 2.1.5 Inessential subcomplexes and connected quotient complexes

In this section, we show how Lemma 2.1.30 allows for a convenient characterization of chain locally equivalent complexes. We then define connected $S^1$-homology of spaces of type SWF, which we will use later to define $SWFH_{\mathrm{conn}}$ as in Corollary 1.2.5.

**Definition 2.1.34.** Take $Z$ a chain complex of type SWF, and let $R \subset Z$ be the fixed-point set. For any subcomplex $M \subset Z$ such that $M \cap R = \{0\}$, the projection $Z \to Z/M$ is a chain homotopy equivalence on $R$. If there exists a map of chain complexes $Z/M \to Z$ that is a chain homotopy equivalence on $R$, we say that $M$ is an *inessential subcomplex.*

If $M$ is inessential, then $Z/M \equiv_{cl} Z$. We order inessential subcomplexes by inclusion, $N \leqslant M$ if $N \subseteq M$. We show that there is a unique "minimal" model $Z/N$ locally equivalent to $Z$.

**Lemma 2.1.35.** *Let $M \subset Z$ be an inessential subcomplex, maximal with respect to inclusion. Then a map $f : Z/M \to Z$ which is a homotopy equivalence on fixed-point sets is injective.*

*Proof.* Indeed, say $f : Z/M \to Z$ is a local equivalence with nonzero kernel. Let $R_1$ denote the fixed-point set of $Z/M$ and $R_2$ denote the fixed-point set of $Z$. Since $f$ restricts to a homotopy equivalence on the fixed-point sets, $(\ker f) \cap R_1 = \{0\}$. Let $\pi : Z \to Z/M$ be the projection map. Then $f$ induces a map $Z/(\pi^{-1}(\ker f)) \to Z$, and by $(\ker f) \cap R_1 = \{0\}$, this map is a homotopy-equivalence on fixed-point sets. Additionally, we have $\pi^{-1}(\ker f) \cap R_2 = \{0\}$. Then $\pi^{-1}(\ker f)$ is an inessential submodule, and it (strictly) contains $M$, contradicting that $M$ was maximal. Then $f$ was injective, as needed. □

**Lemma 2.1.36.** *Let $Z$ be a chain complex of type SWF and let $M, N \subset Z$ be inessential subcomplexes, with $M$ and $N$ maximal with respect to inclusion. Then $Z/M \cong Z/N$, where $\cong$ denotes isomorphism of $\mathcal{G}$-chain complexes.*

*Proof.* Indeed, if there exist maps $\alpha : Z/M \to Z$, and $\beta : Z/N \to Z$ as above, consider the composition:

$$\phi : Z/N \to Z \to Z/M.$$

In particular, we have a map $\alpha\phi : Z/N \to Z$, which is injective by Lemma 2.1.35. It then follows that $\phi$ is injective. We also have:

$$\psi : Z/M \to Z \to Z/N.$$

As before, $\psi$ is injective. Then, since we have injective chain maps between $Z/N$ and $Z/M$, finite-dimensional $\mathbb{F}$-complexes, the two chain complexes must have the same dimension. An injective map between complexes of the same dimension is bijective, and, finally, a bijective $\mathcal{G}$-chain complex map is a $\mathcal{G}$-chain complex isomorphism. □

**Lemma 2.1.37.** *Let $Z$ be a chain complex of type SWF and $M$ a maximal inessential subcomplex of $Z$. We have a (noncanonical) decomposition of $Z$:*

$$Z = (Z/M) \oplus M, \tag{2.55}$$

*where the isomorphism class of $Z/M$ is an invariant of $Z$, independent of the choice of maximal inessential subcomplex $M \subseteq Z$.*

*Proof.* Let $\beta : Z/M \to Z$ be a homotopy equivalence on fixed-point sets. The map $\beta$ is injective by Lemma 2.1.35. Let $\pi$ be the projection $Z \to Z/M$. We note that $\beta\pi\beta$ is a map

$Z/M \to Z$ which is a homotopy equivalence on the fixed point set, and so by Lemma 2.1.35, $\beta\pi\beta$ is injective. Then $\pi\beta$ is also injective.

We have a map $\beta \oplus i : (Z/M) \oplus M \to Z$, where $i$ is the inclusion $i : M \to Z$. We check that $\beta \oplus i$ is injective. Indeed, if $(\beta \oplus i)(z \oplus m) = 0$, we have $\beta(z) = m$. By definition, $\pi(m) = 0$, while $\pi\beta$ is injective. It follows that $m = z = 0$, and $\beta \oplus i$ is injective. Then $Z/M \oplus M \to Z$ is an injective map of $\mathbb{F}$-vector spaces of the same dimension, and so is an isomorphism (of $\mathcal{G}$-chain complexes). Since, by Lemma 2.1.36, the isomorphism class of $Z/M$ is independent of $M$, we obtain that the isomorphism class of $Z/M$ is a well-defined invariant of $Z$. □

**Definition 2.1.38.** For $Z$ a chain complex of type SWF, let $Z_{\text{conn}}$ denote $Z/Z_{\text{iness}}$, for $Z_{\text{iness}} \subseteq Z$ a maximal inessential subcomplex. We call $Z_{\text{conn}}$ the *connected complex* of $Z$.

**Theorem 2.1.39.** *Let $Z$ be a suspensionlike chain complex of type SWF. Then for $W$ another suspensionlike complex of type SWF, $Z \equiv_{cl} W$ if and only if $Z_{\text{conn}} \cong W_{\text{conn}}$.*

*Proof.* By Lemma 2.1.37, we may write $Z = Z_{\text{conn}} \oplus Z_{\text{iness}}, W = W_{\text{conn}} \oplus W_{\text{iness}}$, with $Z_{\text{iness}}, W_{\text{iness}}$ maximal inessential subcomplexes. Say we have local equivalences (we need not consider suspensions, by Lemma 2.1.30)

$$\phi : Z_{\text{conn}} \oplus Z_{\text{iness}} \to W_{\text{conn}} \oplus W_{\text{iness}},$$

$$\psi : W_{\text{conn}} \oplus W_{\text{iness}} \to Z_{\text{conn}} \oplus Z_{\text{iness}}.$$

We restrict $\phi$ and $\psi$ to $Z_{\text{conn}}$ and $W_{\text{conn}}$, since it is clear that $Z_{\text{conn}} \oplus Z_{\text{iness}}$ is chain locally equivalent to $Z_{\text{conn}}$, and likewise for $W_{\text{conn}}$. Further, we project the image of $\phi$ and $\psi$ to $W_{\text{conn}}$ and $Z_{\text{conn}}$, respectively. Call the resulting maps $\phi_0$ and $\psi_0$. If $\phi_0$ had a nontrivial kernel, then we would obtain by composition a local equivalence:

$$\psi_0\phi_0 : Z_{\text{conn}}/\ker \phi_0 \to Z_{\text{conn}}.$$

Composing with the inclusion $\iota : Z_{\text{conn}} \to Z$ gives a chain local map $\iota\psi_0\phi_0 : Z_{\text{conn}}/\ker\,\phi_0 \to Z$, so by Lemma 2.1.35, $\iota\psi_0\phi_0$ is injective. Thus, $\phi_0$ is injective. Similarly $\psi_0$ is injective, so we obtain an isomorphism of chain complexes $Z_{\text{conn}} \cong W_{\text{conn}}$. Conversely, a homotopy equivalence $Z_{\text{conn}} \to W_{\text{conn}}$ yields a local equivalence $Z \to W$ by the composition

$$Z \xrightarrow{\pi} Z_{\text{conn}} \to W_{\text{conn}} \to W,$$

where $\pi : Z \to Z_{\text{conn}}$ is projection to the first summand. □

The next Corollary allows us to view the chain local equivalence type of a space of type SWF in $\mathfrak{CE}$ instead of $\mathfrak{CLE}$.

**Corollary 2.1.40.** *In the language of Theorem 2.1.39, there is an injection $B : \mathfrak{CLE} \to \mathfrak{CE}$ given by $[(Z, m, n)] \to [(Z_{\text{conn}}, m, n)]$, for $(Z, m, n)$ a representative of the class $[(Z, m, n)]$ with $Z$ suspensionlike.*

*Proof.* Fix $[(Z, m, n)] = [(Z', m', n')] \in \mathfrak{CLE}$ with $Z$ and $Z'$ suspensionlike; we will show that $[(Z_{\text{conn}}, m, n)] = [(Z'_{\text{conn}}, m', n')]$ in $\mathfrak{CE}$. First, we observe that, for $V = \mathbb{H}, \tilde{\mathbb{R}}$ :

$$\Sigma^V Z_{\text{conn}} \simeq (\Sigma^V Z)_{\text{conn}}. \tag{2.56}$$

We have, for $M, N$ sufficiently large:

$$\Sigma^{(M-m)\tilde{\mathbb{R}}}\Sigma^{(N-n)\mathbb{H}} Z \leftrightarrows \Sigma^{(M-m')\tilde{\mathbb{R}}}\Sigma^{(N-n')\mathbb{H}} Z'.$$

Here the maps in both directions are local equivalences. Choosing $M \geqslant \max\{m, m'\}$ and $N \geqslant \max\{n, n'\}$ guarantees that both

$$\Sigma^{(M-m)\tilde{\mathbb{R}}}\Sigma^{(N-n)\mathbb{H}} Z \text{ and } \Sigma^{(M-m')\tilde{\mathbb{R}}}\Sigma^{(N-n')\mathbb{H}} Z'$$

are suspensionlike. Then, by Theorem 2.1.39, we have a homotopy equivalence:

$$(\Sigma^{(M-m)\tilde{\mathbb{R}}}\Sigma^{(N-n)\mathbb{H}} Z)_{\text{conn}} \to (\Sigma^{(M-m')\tilde{\mathbb{R}}}\Sigma^{(N-n')\mathbb{H}} Z')_{\text{conn}}.$$

However, by (2.56), we obtain a homotopy equivalence:

$$\Sigma^{(M-m)\tilde{\mathbb{R}}}\Sigma^{(N-n)\mathbb{H}}(Z_{\text{conn}}) \to \Sigma^{(M-m')\tilde{\mathbb{R}}}\Sigma^{(N-n')\mathbb{H}}(Z'_{\text{conn}}).$$

Then $[(Z_{\text{conn}}, m, n)] = [(Z'_{\text{conn}}, m', n')] \in \mathfrak{CE}$, as needed. Finally, we show $B$ is injective. If $(Z_{\text{conn}}, m, n)$ is stably equivalent to $(Z'_{\text{conn}}, m', n')$, then $(Z, m, n)$ and $(Z', m', n')$ are locally equivalent, by Theorem 2.1.39 and (2.56). $\square$

By Corollary 2.1.40, instead of considering the relation given by chain local equivalence, we need only consider chain homotopy equivalences.

**Definition 2.1.41.** The *connected $S^1$-homology* of $(Z, m, n) \in \mathfrak{CE}$, denoted by $H^{S^1}_{\text{conn}}((Z, m, n))$, for $Z$ a suspensionlike chain complex of type SWF, is the quotient $(H^{S^1}_*(Z)/(H^{S^1}_*(Z^{S^1}) + H^{S^1}_*(Z_{\text{iness}})))[m + 4n]$, where $Z_{\text{iness}} \subseteq Z$ is a maximal inessential subcomplex. By Theorem 2.1.39, the graded $\mathbb{F}[U]$-module isomorphism class of $H^{S^1}_{\text{conn}}((Z, m, n))$ is an invariant of the chain local equivalence class of $(Z, m, n)$.

**Remark 2.1.42.** *We could have instead considered the quotient $(H^{S^1}_*(Z)/H^{S^1}_*(Z_{\text{iness}}))[m + 4n]$, which is isomorphic to $H^{S^1}_{\text{conn}}((Z, m, n)) \oplus \mathcal{T}^+_d$, for some $d$. As defined above, the group $H^{S^1}_{\text{conn}}((Z, m, n))$ has no infinite $\mathbb{F}[U]$-tower.*

### 2.1.6 Ordering $\mathfrak{CLE}$

In the following section we define a partial order on $\mathfrak{CLE}$.

**Definition 2.1.43.** The groups $\mathfrak{LE}$ and $\mathfrak{CLE}$ also come with a natural partial ordering. That is, we say $X_1 \preceq X_2$ if there exists a local equivalence $X_1 \to X_2$ or a local equivalence $\Sigma^{\frac{1}{2}\mathbb{H}} X_1 \to X_2$, for $X_1, X_2 \in \mathfrak{LE}$. For $(Z, m, n) \in \mathfrak{CLE}$, we write $\Sigma^{\frac{1}{2}\mathbb{H}}(Z, m, n) = (Z, m, n - \frac{1}{2})$. For $Z_1, Z_2 \in \mathfrak{CLE}$, we say $Z_1 \preceq Z_2$ if there exists a chain local equivalence $Z_1 \to Z_2$ or if there exists a chain local equivalence $\Sigma^{\frac{1}{2}\mathbb{H}} Z_1 \to Z_2$.

We have:

**Lemma 2.1.44.** *If $Z_1 \preceq Z_2 \in \mathfrak{CLE}$, then $\alpha(Z_1) \leqslant \alpha(Z_2), \beta(Z_1) \leqslant \beta(Z_2), \gamma(Z_1) \leqslant \gamma(Z_2)$.*

*Proof.* We assume without loss of generality $Z_1 = (Z_1, 0, 0)$, $Z_2 = (Z_2, 0, 0)$, for suspensionlike chain complexes of type SWF $Z_1$ and $Z_2$. A chain local equivalence $\phi\colon Z_1 \to Z_2$ induces a map $\phi_G\colon C_*^{CW}(EG)\otimes_{\mathcal{G}} Z_1 \to C_*^{CW}(EG)\otimes_{\mathcal{G}} Z_2$. We then have a commuting triangle, where $\iota_1$ and $\iota_2$ come from the inclusions $Z_1^{S^1} \to Z_1$ and $Z_2^{S^1} \to Z_2$.

$$\begin{array}{ccc} C_*^{CW}(EG)\otimes_{\mathcal{G}} Z_1 & \xrightarrow{\phi} & C_*^{CW}(EG)\otimes_{\mathcal{G}} Z_2 \\ & {}^{\iota_1}\nwarrow \quad \nearrow^{\iota_2} & \\ & C_*^{CW}(EG)\otimes_{\mathcal{G}} C_*^{CW}((\tilde{\mathbb{R}}^t)^+, \mathrm{pt}) & \end{array} \tag{2.57}$$

Diagram (2.57) also induces a commuting triangle in homology:

$$\begin{array}{ccc} H_*^G(Z_1) & \xrightarrow{\phi_*} & H_*^G(Z_2) \\ & {}^{\iota_{1,*}}\nwarrow \quad \nearrow^{\iota_{2,*}} & \\ & H_*^G(Z_1^{S^1}) & \end{array} \tag{2.58}$$

By Remark 2.1.32, a suspensionlike chain complex of type SWF is chain stably equivalent to some $C_*^{CW}(X, \mathrm{pt})$ for $X$ a space of type SWF. Then we may apply Fact 2.1.6 to see that $\iota_{1,*}$ and $\iota_{2,*}$ are isomorphisms in sufficiently high degree. Thus $\phi_*$ must be an isomorphism in sufficiently high degree. Furthermore,

$$\mathrm{Im}\,\iota_i = \{x \in H_*^G(Z_i) \mid x \in \mathrm{Im}\, v^l \text{ for all } l \geqslant 0\},$$

from (2.3). Thus, if $x \in H_*^G(Z_2)$ is in $\mathrm{Im}\, v^l$ for all $l \geqslant 0$, there exists some $y$ so that $x = \iota_{2,*}(y)$. By the commutativity of (2.58), $\iota_{1,*}(y) \neq 0$. Applying the definitions (2.4), we see $m(Z_2) \geqslant m(Z_1)$ where $m$ is any of $a, b, c$. Applying Definition 2.1.27, the Lemma follows.

A similar argument applies for a chain local equivalence $\phi\colon \Sigma^{\frac{1}{2}\mathbb{H}} Z_1 \to Z_2$, in which case one has:

$$\alpha(Z_1) \leqslant \alpha(Z_2) - 1,\ \beta(Z_1) \leqslant \beta(Z_2) - 1,\ \gamma(Z_1) \leqslant \gamma(Z_2) - 1.$$

□

**Lemma 2.1.45.** *Let $Z_1, Z_2, Z_3$ complexes of type SWF with $Z_1 \preceq Z_2$. Then $Z_1 \otimes Z_3 \preceq Z_2 \otimes Z_3$.*

*Proof.* If there exists a (stable) map:

$$\phi\colon Z_1 \to Z_2,$$

then $\phi \otimes \mathrm{Id}\colon Z_1 \otimes Z_3 \to Z_2 \otimes Z_3$ satisfies the conditions of Definition 2.1.43, establishing the Lemma (and similarly for suspensions by $\frac{1}{2}\mathbb{H}$). $\square$

## 2.2 Inequalities for the Manolescu Invariants

In this section we will obtain bounds on the Manolescu invariants of tensor products of suspensionlike chain complexes. In Section 3.1 we will apply these results to obtain bounds on the Manolescu invariants of three-manifolds.

### 2.2.1 Calculating Manolescu Invariants from a chain complex

We start by fixing a convenient $G$-CW decomposition of $EG = S(\mathbb{H}^\infty)$. Recalling Example 2.1.15, we have a $G$-CW decomposition for $\mathbb{H}^+ \cong S^4 = \langle r_0, y_1, y_2, y_3 \rangle$ with differentials as in (2.10). We then attach free $G$-cells $y_5, y_6, y_7$, with $\deg y_i = i$, where the attaching map of $y_i$ is the suspension of the attaching map of $y_{i-4}$. The result is a $G$-CW decomposition by cells $\{r_0, y_i\}$, for $i \leqslant 7, i \neq 4$, of $S^8 \cong (\mathbb{H}^2)^+$. We can repeat this procedure to obtain a $G$-CW decomposition of $((\mathbb{H}^n)^+, \mathrm{pt})$ for any $n$, by cells $\{r_0, y_i\}_{i \not\equiv 0 \bmod 4}$.

The unit sphere $S(\mathbb{H}^n)$ admits a $G$-CW decomposition with $G$-$(i-1)$-cells $e_{i-1} = y_i \cap S(\mathbb{H}^n)$ for $i \leqslant 4n-1$.

In the limit, the $e_i$ provide a $G$-CW decomposition of $S(\mathbb{H}^\infty) = EG$. That is, there is a $G$-CW decomposition of $EG$ with cells $e_{4i}$, $e_{4i+1}$, $e_{4i+2}$ for $i \geqslant 0$. The chain complex $C_*^{CW}(EG)$ is then the free $\mathcal{G}$-module on $e_i$ with

$$\begin{aligned} \partial(e_0) &= 0, \\ \partial(e_{4i}) &= s(1+j+j^2+j^3)e_{4i-2} \text{ for } i \geqslant 1, \\ \partial(e_{4i+1}) &= (1+j)e_{4i}, \\ \partial(e_{4i+2}) &= (1+j)e_{4i+1} + se_{4i}. \end{aligned} \tag{2.59}$$

The reader may check that $H(C_*^{CW}(EG))$, for $C_*^{CW}(EG)$ as above, is a copy of $\mathbb{F}$ concentrated in degree 0. As all contractible free $\mathcal{G}$-chain complexes are chain homotopy equivalent, all

$G$-CW complexes for $EG$ have CW chain complex chain homotopic to that given above.

Fix a space $X$ of type SWF so that $Z = C_*^{CW}(X, \mathrm{pt})$ is a suspensionlike chain complex of type SWF. (By Remark 2.1.33, for any class in $\mathfrak{E}$ there will be such a representative $X$). One may compute the reduced Borel homology of $X$ in terms of $Z$, using (2.15) and (2.16).

In particular, we show how to determine $a(Z), b(Z), c(Z)$ from $Z$.

**Lemma 2.2.1.** *Let $Z$ be a suspensionlike chain complex of type SWF at level $t$, with fundamental class $f_t \in Z^{S^1}$, of degree $t$, and $A, B, C \in \mathbb{Z}_{\geqslant 0}$. Then $a(Z) \geqslant 4A+t$ if and only if there exist elements $x_i \in Z$, $\deg x_i = i$, for all $i$ with $t+1 \leqslant i \leqslant t+4A-3$ and $i \not\equiv t+2 \bmod 4$, so that*

$$\partial(x_i) = \begin{cases} f_t & \text{if } i = t+1 \\ s(1+j+j^2+j^3)x_{i-2} & \text{if } i \equiv t+3 \bmod 4,\ i \leqslant t+4A-3 \\ (1+j)x_{i-1} & \text{if } i \equiv t \bmod 4, i \leqslant t+4A-3 \\ (1+j)x_{i-1} + sx_{i-2} & \text{if } i \equiv t+1 \bmod 4,\ t+1 < i \leqslant t+4A-3. \end{cases} \tag{2.60}$$

*Also, $b(Z) \geqslant 4B+t$ if and only if there exist elements $x_i \in Z$, $\deg x_i = i$, for all $i$ with $t+1 \leqslant i \leqslant t+4B-2$ and $i \not\equiv t+3 \bmod 4$ so that*

$$\partial(x_i) = \begin{cases} f_t & \text{if } i = t+1 \\ (1+j)x_{t+1} & \text{if } i = t+2 \\ s(1+j+j^2+j^3)x_{i-2} & \text{if } i \equiv t \bmod 4,\ i \leqslant t+4B-2 \\ (1+j)x_{i-1} & \text{if } i \equiv t+1 \bmod 4, t+1 < i \leqslant t+4B-2 \\ (1+j)x_{i-1} + sx_{i-2} & \text{if } i \equiv t+2 \bmod 4,\ t+2 < i \leqslant t+4B-2. \end{cases} \tag{2.61}$$

*Also, $c(Z) \geqslant 4C+t$ if and only if there exist elements $x_i \in Z$, $\deg x_i = i$, for all $i$ with $t+1 \leqslant i \leqslant t+4C-1$ and $i \not\equiv t \bmod 4$ so that*

$$\partial(x_i) = \begin{cases} f_t & \text{if } i = t+1 \\ (1+j)x_{i-1} & \text{if } i \equiv t+2 \bmod 4,\ i \leqslant t+4C-1 \\ (1+j)x_{i-1} + sx_{i-2} & \text{if } i \equiv t+3 \bmod 4, i \leqslant t+4C-1 \\ s(1+j+j^2+j^3)x_{i-2} & \text{if } i \equiv t+1 \bmod 4,\ t+1 < i \leqslant t+4C-1. \end{cases} \tag{2.62}$$

*Proof.* By (2.3), we have, where $\iota_*\colon H^G_*(Z^{S^1}) \to H^G_*(Z)$ is the map induced by inclusion,

$$\mathrm{Im}\ \iota_* = \{x \in H^G_*(Z) \mid x \in \mathrm{Im}\ v^l \text{ for all } l \geqslant 0\}. \tag{2.63}$$

Further, $H^G_*(Z^{S^1})$ is given by:

$$H^G_*(Z^{S^1}) = C^{CW}_*(EG) \otimes_{\mathbb{F}} f_t,$$

which is an $\mathbb{F}$-vector space with generators $e_i \otimes f_t$ in degree $i + t$ for $i$ such that $i \geqslant 0$ and $i \not\equiv 3 \bmod 4$. Then $a(Z) \geqslant 4A + t$ is equivalent to $e_{4A-4} \otimes f_t$ being a boundary in

$$C^G_*(Z) = C^{CW}_*(EG) \otimes_{\mathcal{G}} Z.$$

That is, $a(Z) \geqslant 4A + t$ is equivalent to the existence of some

$$x = \sum_{i=t+1}^{i=t+4A-3} e_{t+4A-3-i} \otimes x_i \in C^{CW}_*(EG) \otimes_{\mathcal{G}} Z,$$

so that $\partial(x) = e_{4A-4} \otimes f_t$, where $x_i \in Z$ is of degree $i$. Writing out the differential of $x$, one obtains the conditions (2.60) of the Lemma. Similarly, $b(Z) \geqslant 4B + t$ if and only if $e_{4B-3} \otimes f_t$ is a boundary, and $c(Z) \geqslant 4C + t$ if and only if $e_{4C-2} \otimes f_t$ is a boundary, from which one obtains (2.61) and (2.62). □

**Lemma 2.2.2.** *Let $Z$ be a suspensionlike chain complex of type SWF at level $t$, so that $c(Z) \geqslant 4C + t$. Then*

$$C^{CW}_*(\Sigma^{C\mathbb{H}}(\tilde{\mathbb{R}}^t)^+, \mathrm{pt}) \leq Z.$$

*Proof.* The chain complex $C^{CW}_*(\Sigma^{C\mathbb{H}}(\tilde{\mathbb{R}}^t)^+, \mathrm{pt})$ consists of cells $c_0, \ldots, c_t$ constituting the $S^1$-fixed point set, and has free cells $x_i$, of degree $i$, for $i = t+1, \ldots, t+4C-1$, for $i \not\equiv t \bmod 4$. The fundamental class of the subcomplex $C^{CW}_*((\tilde{\mathbb{R}}^t)^+, \mathrm{pt})$ is $f_t = (1+j)c_t$ (if $t > 0$, or $f_t = c_0$ if $t = 0$). The differentials of the $x_i$ in $C^{CW}_*(\Sigma^{C\mathbb{H}}(\tilde{\mathbb{R}}^t)^+, \mathrm{pt})$ are given exactly by the relations in (2.62). Then, since $Z$ has elements satisfying (2.62), there exists a chain local equivalence

$$C^{CW}_*(\Sigma^{C\mathbb{H}}(\tilde{\mathbb{R}}^t)^+, \mathrm{pt}) \to Z,$$

as needed. □

The problem of computing the Manolescu invariants of tensor products (and, thus, connected sums, using Fact 3.1.5) then amounts to asking how to find towers of elements of the form (2.60)-(2.62) in $Z_1 \otimes_{\mathbb{F}} Z_2$ from towers in $Z_1$ and $Z_2$.

**Remark 2.2.3.** *Say $\alpha(Z) = \gamma(Z) = 0$ for $Z$ a chain complex of type SWF. Then Lemma 2.2.2 implies $Z \geq 0 \in \mathfrak{CLE}$. By duality, $-\alpha(Z) = \gamma(Z^*) = 0$, where $Z^*$ is the dual of $Z$, so $Z^* \geq 0$. Combined, we see $Z = 0 \in \mathfrak{CLE}$. That is, if $Z \in \mathfrak{CLE}$ has $\alpha(Z) = \gamma(Z) = 0$, then $[Z]_{cl} = [C_*^{CW}(S^0, \mathrm{pt})]_{cl}$.*

**Theorem 2.2.4.** *For $Z_1, Z_2$ suspensionlike $\mathcal{G}$-chain complexes of type SWF, we have:*

$$\alpha(Z_1) + \alpha(Z_2) \geqslant \alpha(Z_1 \otimes_{\mathbb{F}} Z_2) \geqslant \alpha(Z_1) + \gamma(Z_2), \tag{2.64}$$

$$\alpha(Z_1) + \beta(Z_2) \geqslant \beta(Z_1 \otimes_{\mathbb{F}} Z_2) \geqslant \beta(Z_1) + \gamma(Z_2),$$

$$\alpha(Z_1) + \gamma(Z_2) \geqslant \gamma(Z_1 \otimes_{\mathbb{F}} Z_2) \geqslant \gamma(Z_1) + \gamma(Z_2).$$

*Proof.* Let $Z_i$ be at level $t_i$ for $i = 1, 2$. Then, by Lemma 2.2.2, $C_*^{CW}(\Sigma^{\frac{(c(Z_2)-t_2)}{4}\mathbb{H}}(\tilde{\mathbb{R}}^{t_2})^+, \mathrm{pt}) \preceq Z_2$. By Lemma 2.1.45,

$$Z_1 \otimes_{\mathbb{F}} C_*^{CW}(\Sigma^{\frac{(c(Z_2)-t_2)}{4}\mathbb{H}}(\tilde{\mathbb{R}}^{t_2})^+, \mathrm{pt}) \preceq Z_1 \otimes_{\mathbb{F}} Z_2.$$

However, $Z_1 \otimes_{\mathbb{F}} C_*^{CW}(\Sigma^{\frac{(c(Z_2)-t_2)}{4}\mathbb{H}}(\tilde{\mathbb{R}}^{t_2})^+, \mathrm{pt})$ is, by definition, $(Z_1, -t_2, \frac{-c(Z_2)+t_2}{4})$.

Then

$$(Z_1, -t_2, \frac{-c(Z_2) + t_2}{4}) \preceq Z_1 \otimes_{\mathbb{F}} Z_2.$$

By Lemma 2.1.44, $M((Z_1, -t_2, \frac{-c(Z_2)+t_2}{4})) \preceq M(Z_1 \otimes_{\mathbb{F}} Z_2)$ where $M$ is any of $\alpha, \beta$, or $\gamma$. By Definition 2.1.27, we have $\gamma(Z_2) = c(Z_2)/2$. Then, again using Definition 2.1.27, we see

$$\alpha(Z_1, -t_2, \frac{-c(Z_2) + t_2}{4}) = \alpha(Z_1) + \gamma(Z_2) \leqslant \alpha(Z_1 \otimes_{\mathbb{F}} Z_2),$$
$$\beta(Z_1, -t_2, \frac{-c(Z_2) + t_2}{4}) = \beta(Z_1) + \gamma(Z_2) \leqslant \beta(Z_1 \otimes_{\mathbb{F}} Z_2),$$
$$\gamma(Z_1, -t_2, \frac{-c(Z_2) + t_2}{4}) = \gamma(Z_1) + \gamma(Z_2) \leqslant \gamma(Z_1 \otimes_{\mathbb{F}} Z_2).$$

Thus, we have obtained the right-hand inequalities of (2.64).

To obtain the left-hand inequalities, we recall from [30][Proposition 2.13] that $\alpha(X) = -\gamma(X^*)$ and $\beta(X) = -\beta(X^*)$ where $X$ is a space of type SWF and $X^*$ is Spanier-Whitehead dual to $X$. The same argument as in [30][Proposition 2.13] implies that, for $Z$ a chain complex of type SWF, $\alpha(Z) = -\gamma(Z^*)$ and $\beta(Z) = -\beta(Z^*)$ where $Z^*$ is the dual chain complex. The left-hand inequalities of (2.64) then follow by applying the right-hand inequalities to $Z_1^*$ and $Z_2^*$, and using the above rules for duality. □

**Theorem 2.2.5.** *For $Z_1, Z_2$ suspensionlike $\mathcal{G}$-chain complexes of type SWF, we have:*

$$\gamma(Z_1 \otimes_{\mathbb{F}} Z_2) \leqslant \beta(Z_1) + \beta(Z_2) \leqslant \alpha(Z_1 \otimes_{\mathbb{F}} Z_2). \tag{2.65}$$

*Proof.* We construct a tower of elements in $Z_1 \otimes_{\mathbb{F}} Z_2$ satisfying (2.60) from towers in $Z_1$ and $Z_2$ satisfying (2.61). Say that $Z_1$ is at level $t_1$ and $Z_2$ is at level $t_2$, and denote the fundamental class of $Z_1^{S^1}$ by $f_{t_1}$ and that of $Z_2^{S^1}$ by $f_{t_2}$. We would like to apply Lemma 2.2.1, but, as explained after the introduction of the chain local equivalence group, the tensor product of suspensionlike chain complexes of type SWF is usually not suspensionlike. However, it becomes suspenionlike after removing a large acyclic subcomplex, and we can indeed apply Lemma 2.2.1, as follows.

Let $\{x_i\}_{i=t_1+1,\dots,b(Z_1)-2}$ and $\{y_i\}_{i=t_2+1,\dots,b(Z_2)-2}$ be sequences satisfying (2.61) for $Z_1, Z_2$, respectively. Then consider the sequence of elements:

$$\begin{gathered} x_{t_1+1} \otimes f_{t_2},\ s(1+j^2)x_{t_1+2} \otimes f_{t_2}, \\ x_{t_1+4} \otimes f_{t_2},\ x_{t_1+5} \otimes f_{t_2},\ s(1+j^2)x_{t_1+6} \otimes f_{t_2}, \\ x_{t_1+8} \otimes f_{t_2},\ x_{t_1+9} \otimes f_{t_2}, s(1+j^2)x_{t_1+10} \otimes f_{t_2}, \\ \dots, \\ x_{b(Z_1)-4} \otimes f_{t_2},\ x_{b(Z_1)-3} \otimes f_{t_2},\ s(1+j^2)x_{b(Z_1)-2} \otimes f_{t_2}. \end{gathered} \tag{2.66}$$

One may verify that the sequence in (2.66) satisfies (2.60). In fact, the sequence in (2.66) generates a subcomplex that is just a subcomplex of $Z_1$ satisfying (2.60) smashed against $Z_2^{S^1}$. To lengthen the sequence, we then incorporate chains coming from $Z_2$:

$$s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+1}, s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+2}, \tag{2.67}$$

$$s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+4},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+5},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+6},$$

$$s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+8},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+9},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{t_2+10}$$

$$\ldots,$$

$$s(1+j)^3 x_{b(Z_1)-2} \otimes y_{b(Z_2)-4},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{b(Z_2)-3},\ s(1+j)^3 x_{b(Z_1)-2} \otimes y_{b(Z_2)-2}.$$

One confirms that the sequence specified by (2.66)-(2.67) satisfies (2.60), and this establishes

$$a(Z_1 \otimes_{\mathbb{F}} Z_2) \geqslant b(Z_1) + b(Z_2).$$

Using Definition 2.1.27, we obtain the right-hand inequality of (2.65). The left-hand side follows from duality, as in the proof of Theorem 2.2.4. □

### 2.2.2 Relationship with $S^1$-invariants

We also recall the definition of the invariant $d$ from [30], analogous to the Frøyshov invariant of $S^1$-equivariant Seiberg-Witten Floer theory.

**Definition 2.2.6.** Let $Z$ be a suspensionlike chain complex of type SWF at level $t$.

$$d(Z) = \min \{ r \equiv t \bmod 2 \mid \exists\, x \in H_r^{S^1}(Z),\ x \in \operatorname{Im} u^l \text{ for all } l \geqslant 0 \}. \tag{2.68}$$

**Remark 2.2.7.** *In [30], $d_p$ is defined for coefficients in any field, rather than only $\mathbb{F} = \mathbb{Z}/2$. The invariant $d$ in our notation is $d_2$ of [30].*

Analogous to the the calculation for $a, b$, and $c$ in Lemma 2.2.1, we find a formula for $d(Z)$. We obtain:

**Lemma 2.2.8.** *Let $Z$ be a suspensionlike chain complex of type SWF at level $t$, and let $f_t$ denote the fundamental class of $Z^{S^1}$. Then $d(Z) \geqslant 2D + t$ if and only if there exist elements $x_i$ in $Z$, for $i = t+1, \ldots, t+2D-1$ with $i \equiv t+1 \bmod 2$, where $\deg x_i = i$, such that*

$$\partial(x_i) = \begin{cases} f_t & \text{if } i = t+1, \\ s(1+j^2)x_{i-2} & \text{if } t+3 \leqslant i \leqslant t+2D-1. \end{cases} \tag{2.69}$$

*Proof.* The proof is analogous to that of Lemma 2.2.1. □

**Definition 2.2.9.** We let $T_D(t)$ denote the chain complex given by

$$C_*^{CW}((\tilde{\mathbb{R}}^t)^+, \mathrm{pt}) \oplus \langle\{x_{t+2i-1}\}\rangle_{\{i=1,\ldots,D\}},$$

where $\langle\{x_i\}\rangle$ is the free $\mathcal{G}$-module with generators $x_i$, with the following requirements. We require that $C_*^{CW}((\tilde{\mathbb{R}}^t)^+, \mathrm{pt}) \subseteq T_D(t)$ is a subcomplex, where $C_*^{CW}((\tilde{\mathbb{R}}^t)^+, \mathrm{pt})$ is as in Example 2.1.14. Also, we set $\deg x_i = i$. The differentials of $T_D(t)$ are as in (2.69); namely, $\partial(x_{t+1})$ is the fundamental class of $(\tilde{\mathbb{R}}^t)^+$:

$$\partial(x_{t+1}) = \begin{cases} (1+j)c_t & \text{if } t > 0 \\ c_0 & \text{if } t = 0. \end{cases}$$

The differential of $x_i$ for $i > t+1$ is given by $\partial(x_i) = s(1+j)^2 x_{i-2}$.

**Fact 2.2.10.** *If $t = 0$, the chain complex $T_D(t)$ is the reduced CW complex of the unreduced suspension $\tilde{\Sigma}(S^{2D-1} \amalg S^{2D-1})$, where $S^1$ acts on $S^{2D-1}$ by complex multiplication, and $j$ interchanges the two copies of $S^{2D-1}$ (see Definition 2.1.4).*

**Lemma 2.2.11.** *We have $\beta(T_D(t)) = t/2$ and $\gamma(T_D(t)) = t/2$.*

*Proof.* Let $Q$ be the quotient complex $T_D(t)/T_D(t)^{S^1}$. By inspection $\partial Q \subseteq (1+j^2)Q$. Then there is no pair of elements $x_1, x_2 \in T_D(t)$ so that $\partial x_1 = f_t$ and $\partial x_2 = (1+j)x_1$. By (2.61) and (2.62), we obtain $b(T_D(t)) = c(T_D(t)) = t$. By Definition 2.1.27, the statement follows. □

The motivation for considering the complex $T_D(t)$ is that it is the "minimal" $\mathcal{G}$-chain complex for a fixed $d$-invariant, as made precise in the following lemma.

**Lemma 2.2.12.** *Let $Z$ be a suspensionlike chain complex at level $t$. Then $d(Z) \geqslant 2D + t$ if and only if $Z \succeq T_D(t)$.*

*Proof.* The lemma follows immediately from Lemma 2.2.8. □

We also recall the definition of the invariant $\delta$ from [30], analogous to Definition 2.1.9.

**Definition 2.2.13.** For $[(X, m, n)] \in \mathfrak{E}$, we set

$$\delta((X, m, n)) = d(C_*^{CW}(X, \mathrm{pt}))/2 - m/2 - 2n \tag{2.70}$$

The invariant $\delta$ does not depend on the choice of representative of the class $[(X, m, n)]$.

**Proposition 2.2.14.** *For $X_1, X_2 \in \mathfrak{E}$, $\delta(X_1 \otimes X_2) = \delta(X_1) + \delta(X_2)$.*

*Proof.* Entirely analogous to the proof of Theorem 2.2.4, we obtain

$$\delta(X_1 \otimes X_2) \geqslant \delta(X_1) + \delta(X_2).$$

Additionally, $\delta(X) = -\delta(X^*)$ using the properties of $\delta$ under duality, where $X^*$ denotes the dual of $X$. We then obtain:

$$\delta(X_1 \otimes X_2) \leqslant \delta(X_1) + \delta(X_2),$$

completing the proof. □

We next relate the Pin(2)-invariants to $d$.

**Proposition 2.2.15.** *Let $Z$ be a suspensionlike $\mathcal{G}$-chain complex of type SWF. Then $\alpha(Z) \geqslant \delta(Z)$.*

*Proof.* We will use the description of $\alpha$ from Lemma 2.2.1. Recall that $EG$ is the total space of the universal $S^1$-bundle, by forgetting the action of $j \in G$. Viewed thus, the chains

$$e_0, j((1+j)e_2 + se_1), e_4, j((1+j)e_6 + se_5), e_8, j((1+j)e_{10} + se_9), e_{12}, \dots \tag{2.71}$$

descend to generators of homology in $BS^1 = EG \times_{S^1} \{\mathrm{pt}\}$.

Say $Z$ is at level $t$ and let $f_t$ be the fundamental class of $Z^{S^1}$. Using (2.71) and repeating the proof of Lemma 2.2.1, $d(Z)$ is the degree of the minimal element of the form

$$e_{4i} \otimes f_t \text{ or } j((1+j)e_{4i+2} + se_{4i+1}) \otimes f_t$$

that is not a boundary in $C_*^{S^1}(Z) = C_*^{CW}(EG) \otimes_{C_*^{CW}(S^1)} Z$.

That is, $d(Z) \geqslant 4D + 2 + t$ if and only if $e_{4D} \otimes f_t$ is a boundary. Further, $d(Z) \geqslant 4D + t$ if and only if $j((1 + j)e_{4D-2} + se_{4D-3}) \otimes f_t$ is a boundary. In particular, if, for some $A \geqslant 0$, $d(Z) \geqslant 4A + t - 2$, we have $e_{4A-4} \otimes_{C_*^{CW}(S^1)} f_t$ is a boundary.

However, if

$$e_{4A-4} \otimes f_t \in C_*^{CW}(EG) \otimes_{C_*^{CW}(S^1)} Z$$

is a boundary, then $e_{4A-4} \otimes f_t \in C_*^{CW}(EG) \otimes_{\mathcal{G}} Z$ is also a boundary. Thus $a(Z) \geqslant 4A + t$, and so $a(Z) \geqslant d(Z)$. Thus, using Definition 2.1.27, the Proposition follows. $\square$

## 2.3 Manolescu Invariants of unreduced suspensions

In this section, we calculate the Manolescu invariants of certain smash products of unreduced suspensions.

### 2.3.1 Unreduced Suspensions

We draw from [30] the following calculation, which we will use in our application to Seifert fiber spaces. Recall the definition of unreduced suspensions from Definition 2.1.4.

For $X$ a free $G$-space, the cone of the inclusion map $(\tilde{\Sigma}X)^{S^1} \to \tilde{\Sigma}X$ is $\Sigma^{\mathbb{R}}X_+$, where $X_+$ is $X$ with a disjoint basepoint added. This gives the exact sequence, by taking Borel homology,

$$\ldots \longrightarrow \tilde{H}^G_{*+1}(\Sigma^{\mathbb{R}}X_+) \longrightarrow \tilde{H}^G_*(S^0) \longrightarrow \tilde{H}^G_*(\tilde{\Sigma}X) \longrightarrow \ldots \tag{2.72}$$

The term $\tilde{H}^G_{*+1}(\Sigma^{\mathbb{R}}X_+)$ is isomorphic to $\tilde{H}^G_*(X_+)$ because of suspension-invariance of Borel homology with $\mathbb{F}$-coefficients, from (2.7). Furthermore, $\tilde{H}^G_*(X_+) \simeq H_*(X/G)$ since $G$ acts freely on $X$. The exact sequence (2.72) becomes (as an exact sequence of $\mathbb{F}[q, v]/(q^3)$-modules):

$$\ldots \longrightarrow H_*(X/G) \xrightarrow{\kappa_*} H_*(BG) \longrightarrow \tilde{H}^G_*(\tilde{\Sigma}X) \longrightarrow \ldots \tag{2.73}$$

Here $\kappa_*$ is induced from $\kappa\colon X/G \to BG$, the classifying space map. Let $\kappa^d_*$ denote the restriction of $\kappa_*$ to degree $d$. From the exactness of (2.73), we have:

$$a(\tilde{\Sigma}X) = \min\{d \equiv 0 \bmod 4 \mid \kappa_*^d = 0\}, \tag{2.74}$$

$$b(\tilde{\Sigma}X) = \min\{d \equiv 1 \bmod 4 \mid \kappa_*^d = 0\} - 1, \tag{2.75}$$

$$c(\tilde{\Sigma}X) = \min\{d \equiv 2 \bmod 4 \mid \kappa_*^d = 0\} - 2. \tag{2.76}$$

### 2.3.2 Smash Products

In this section we compute the Manolescu invariants for smash products of the form

$$\bigwedge_{i=1}^{n} \tilde{\Sigma}(S^{2\tilde{\delta}_i - 1} \amalg S^{2\tilde{\delta}_i - 1}). \tag{2.77}$$

This calculation will enable us to find the Manolescu invariants for connected sums of certain Seifert spaces in Section 3.1.

We will find it convenient to write:

$$E(x) = 2\left\lfloor \frac{x+1}{2} \right\rfloor.$$

**Theorem 2.3.1.** *Fix $\tilde{\delta}_i \in \mathbb{Z}_{\geqslant 1}$, and $\tilde{\delta}_1 \leqslant \cdots \leqslant \tilde{\delta}_n$. Let $X_i = S^{2\tilde{\delta}_i - 1} \amalg S^{2\tilde{\delta}_i - 1}$ for $i = 1, \ldots, n$, where $X_i$ has a $G$-action given by $S^1$ acting by complex multiplication on each factor, and $j$ acting by interchanging the sphere factors. Then:*

$$\delta(\bigwedge_{i=1}^{n} \tilde{\Sigma}X_i) = \sum_{i=1}^{n} \tilde{\delta}_i, \tag{2.78}$$

$$\alpha(\bigwedge_{i=1}^{n} \tilde{\Sigma}X_i) = E(\sum_{i=1}^{n} \tilde{\delta}_i), \tag{2.79}$$

$$\beta(\bigwedge_{i=1}^{n} \tilde{\Sigma}X_i) = E(\sum_{i=1}^{n-1} \tilde{\delta}_i), \tag{2.80}$$

$$\gamma(\bigwedge_{i=1}^{n} \tilde{\Sigma}X_i) = E(\sum_{i=1}^{n-2} \tilde{\delta}_i), \tag{2.81}$$

We will use Gysin sequences in the proof of Theorem 2.3.1; for convenience we record the necessary fact here. As in [50][§III.2] there exists a Gysin sequence in homology for a $G$-space $X$:

$$H_*^G(X) \xrightarrow{(1+j)\cdot -} H_*^{S^1}(X) \xrightarrow{\pi_*} H_*^G(X) \xrightarrow{q\cap -} H_{*-1}^G(X) \longrightarrow \ldots \tag{2.82}$$

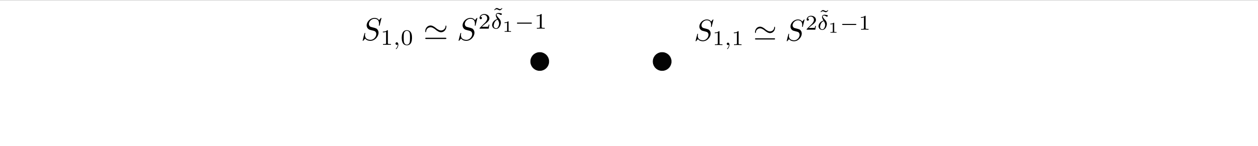


**Figure 2.2:** An image of $X$ for $n = 1$.

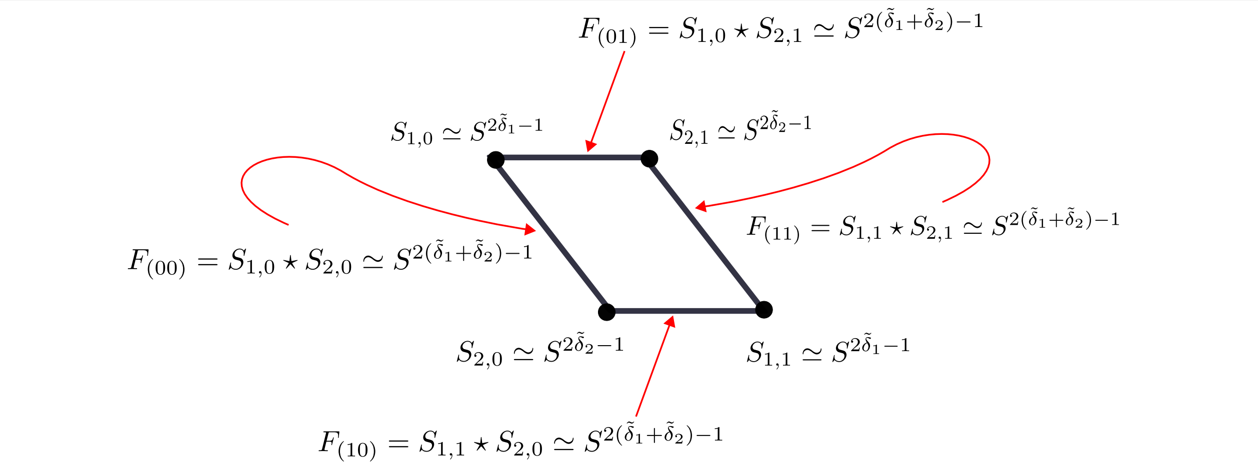


**Figure 2.3:** An image of $X$ for $n = 2$.

Here, the map $(1 + j) \cdot -$ is the map sending a cycle $[x] \in H_*^G(X)$, with chain representative (not necessarily a cycle) $x \in H_*^{S^1}(X)$, to $[(1 + j)x] \in H_*^{S^1}(X)$. The map $\pi_*$ comes from the quotient $\pi\colon EG \times_{S^1} X \to EG \times_G X$. From (4.17), we obtain immediately:

**Fact 2.3.2.** *Let* $[x] \in H_*^G(X)$ *so that* $(1 + j) \cdot [x] = 0$*. Then* $[x] \in \operatorname{Im} q$*.*

*Proof of Theorem 2.3.1.* We will use the description in Section 2.3.1 to perform the required calculation. Let $X = \star_{i=1}^n X_i$, where $\star_{i=1}^n$ denotes the join. We note

$$\bigwedge_{i=1}^{n} \tilde{\Sigma} X_i = \tilde{\Sigma}(\star_{i=1}^n X_i). \tag{2.83}$$

Further, for each $i$, label one of the disjoint spheres of $X_i$ by $S_{i,0}$ and the other by $S_{i,1}$. See Figures 2.2, 2.3, and 2.4 for visualization of $X$. As in the figures, we consider $X$ as if it were a polyhedron, with "points" the $X_i$ and "faces" (edges, etc.) the joins of subsets of $\{X_i\}$. We write

$$F_{(k_1,\ldots,k_n)} = \star_{i=1}^n S_{i,k_i},$$

where $k_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$, for the "face" spanned by $S_{i,k_i}$ (see Figure 2.4).

By Fact 2.2.10 and Lemma 2.2.12, $\delta(\tilde{\Sigma} X_i) = \tilde{\delta}_i$. Proposition 2.2.14 then implies (2.78).

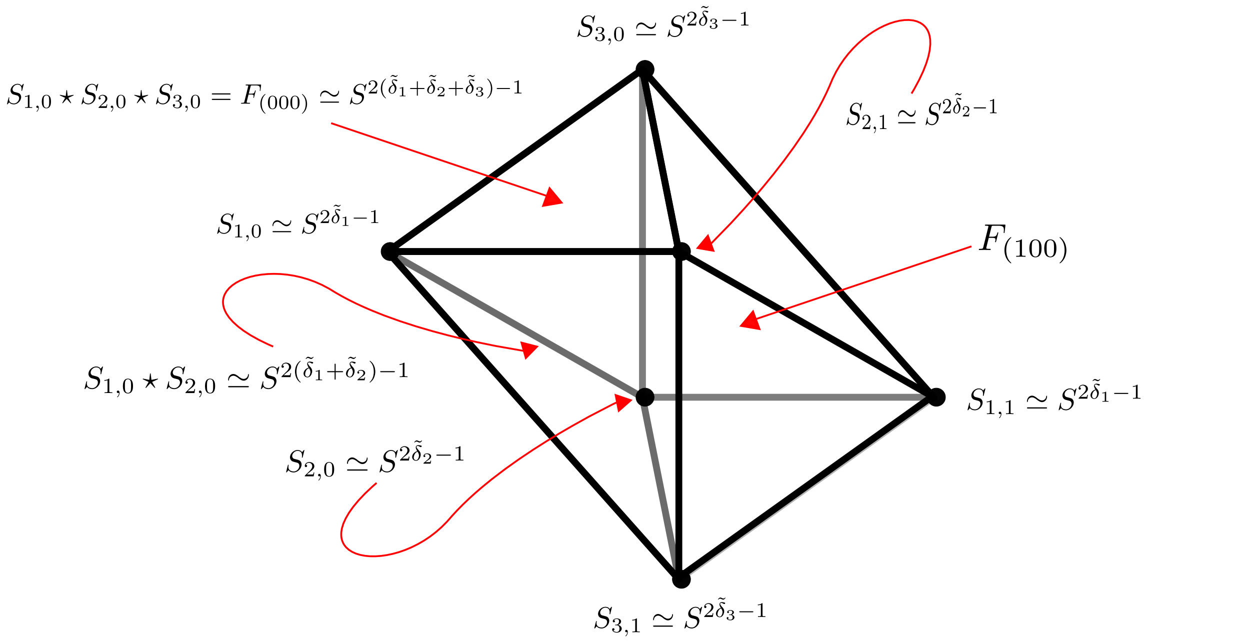


**Figure 2.4:** An image of $X$ for $n = 3$. Here, we only label a few of the faces.

**Proof of (2.79).** We observe that $S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \simeq \star_{i=1}^n S_{i,0} \subseteq X$, as $S^1$-spaces, where the action on both sides is given by complex multiplication. We then have a map:

$$S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \amalg S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \to \star_{i=1}^n S_{i,0} \amalg \star_{i=1}^n S_{i,1} \subseteq X$$

of $G$-spaces, where the action of $j$ interchanges the factors of $S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \amalg S^{2\sum_{i=1}^n \tilde{\delta}_i - 1}$. Taking the quotient by the action of $G$ we have a diagram:

$$\begin{array}{ccccc} S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \amalg S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} & \longrightarrow & X & & \\ \downarrow & & \downarrow & & \\ \mathbb{CP}^{\sum_{i=1}^n \tilde{\delta}_i - 1} & \longrightarrow & X/G & \longrightarrow & BG \end{array} \tag{2.84}$$

with vertical arrows given by $G$-quotient. The composition $H_*(\mathbb{CP}^{\sum_{i=1}^n \tilde{\delta}_i - 1}) \to H_*(BG)$ coming from the second line of (2.84) is the characteristic class map $\kappa_{*,\mathbb{CP}^{\sum_{i=1}^n \tilde{\delta}_i - 1}}$ of $S^{2\sum_{i=1}^n \tilde{\delta}_i - 1} \amalg S^{2\sum_{i=1}^n \tilde{\delta}_i - 1}$ as a $G$-bundle, so using Fact 2.1.1, we have:

$$\kappa_{*,\mathbb{CP}^{\sum_{i=1}^n \tilde{\delta}_i - 1}}(U^{-2\lfloor \frac{\sum_{i=1}^n \tilde{\delta}_i - 1}{2} \rfloor}) = v^{-\lfloor \frac{\sum_{i=1}^n \tilde{\delta}_i - 1}{2} \rfloor}.$$

Here $U^{-i}$, for $i, N \geqslant 0$, is the unique element of $H_*(\mathbb{CP}^N)$ so that $U^i(U^{-i}) = 1$, where $1$ is the unique nonzero element of $H_0(\mathbb{CP}^N)$, and similarly $q^{-i}, v^{-i}$ are, respectively, the unique elements of $H_*(BG)$ so that $q^i(q^{-i}) = 1 = v^i(v^{-i})$, where $1 \in H_0(BG)$ is nonzero. Then $\operatorname{Im}\kappa_*$ must be nonzero in degree $4\lfloor\frac{\sum_{i=1}^n \tilde{\delta}_i - 1}{2}\rfloor$, so

$$a(\tilde{\Sigma}X) \geqslant 4\lfloor\frac{\sum_{i=1}^n \tilde{\delta}_i - 1}{2}\rfloor + 4.$$

However, $\kappa_*^d$ must be zero in all degrees $d \geqslant 4\lfloor\frac{\sum_{i=1}^n \tilde{\delta}_i - 1}{2}\rfloor + 4$, since $\dim X = 2\sum_{i=1}^n \tilde{\delta}_i - 1$. Thus, using Definition 2.1.9:

$$\alpha(\tilde{\Sigma}X) = E(\sum_{i=1}^n \tilde{\delta}_i),$$

giving (2.79).

**Proof of (2.80).** We have a ($G$-equivariant) map $\phi_\beta \colon S^{2\sum_{i=1}^{n-1} \tilde{\delta}_i - 1} \times S^0 \to X$ (where $j$ acts by interchanging the factors $S^{2\sum_{i=1}^{n-1} \tilde{\delta}_i - 1}$) given by the inclusion

$$\star_{i=1}^{n-1} S_{i,0} \amalg \star_{i=1}^{n-1} S_{i,1} \subseteq \star_{i=1}^{n}(S_{i,0} \amalg S_{i,1}).$$

We will use the map $\phi_\beta$ to find classes in $H_*(BG)$ in the image of $\kappa_*$ in degree congruent to 1 mod 4.

Let

$$F^{n-1} = \coprod_{(l_1,\ldots,l_{n-1})\in\mathcal{L}} \star_{i=1}^{n-1}(S_{l_i,0} \amalg S_{l_i,1}),$$

where $\mathcal{L}$ is the set of all $(n-1)$-tuples of distinct elements of $\{1, \ldots, n\}$. In the analogy from the start of the proof, $F^{n-1}$ is the "$(n-1)$-skeleton" of $X$.

Note that associated to a linear subspace $\mathbb{C}^K \subseteq \mathbb{C}^N$, there is an $S^1$-equivariant submanifold $S^{2K-1} \subseteq S^{2N-1}$. That is, there is a map from $\mathrm{Gr}(K, N)$, the space of all $K$-planes in $\mathbb{C}^N$, to the space of all submanifolds $S^{2K-1} \subseteq S^{2N-1}$. We will call an embedded sphere obtained from a linear subspace this way a *linear* sphere. We also see that the inclusion

$$\star_{i=1}^{n-1} S_{i,0} \subseteq F_{(0,\ldots,0)} \tag{2.85}$$

corresponds to the inclusion of a linear subspace $\mathbb{C}^{\sum_{i=1}^{n-1} \tilde{\delta}_i} \subset \mathbb{C}^{\sum_{i=1}^{n} \tilde{\delta}_i}$ (i.e. (2.85) is linear).

Since $\mathrm{Gr}(K, N)$ is connected, we see that any two linear spheres $S^{2K-1} \to F_{(k_1,\ldots,k_n)}$, with $K \leqslant \sum_{i=1}^n \tilde{\delta}_i$, are homotopic in $F_{(k_1,\ldots,k_n)}$, through linear spheres.

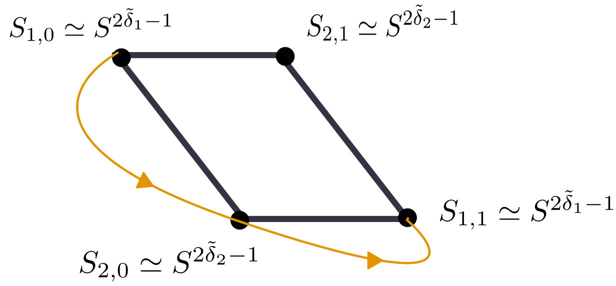


**Figure 2.5:** The homotopy from $S_{1,0}$ to $jS_{1,0}$ in the case $n = 2$. The sphere $S_{1,0}$ is homotopic to a copy of $S^{2\tilde{\delta}_1-1} \subseteq S_{2,0}$ in $F_{(00)} \simeq S_{1,0} \star S_{2,0} \simeq S^{2(\tilde{\delta}_1+\tilde{\delta}_2)-1}$. Furthermore, $S^{2\tilde{\delta}_1-1} \subseteq S_{2,0}$ is homotopic to $S_{1,1}$ in $F_{(10)}$. Thus, we have found a homotopy $\star_{i=1}^{n-1} S_{i,0} \to j(\star_{i=1}^{n-1} S_{i,0})$ for $n = 2$.

Further, we note that for any $\star_{i=1}^{n-1} S_{l_i,k_{l_i}} \subset F_{(k_1,\ldots,k_n)}$, there exists some linear sphere

$$S \simeq S^{2K-1} \subseteq \star_{i=1}^{n-1} S_{l_i,k_{l_i}}, \tag{2.86}$$

for all $K \leqslant \sum_{i=1}^{n-1} \tilde{\delta}_i$ (here we have used $\tilde{\delta}_1 \leqslant \cdots \leqslant \tilde{\delta}_n$).

In particular, fixing $K \leqslant \sum_{i=1}^{n-1} \tilde{\delta}_i$, we have a linear sphere $S$ as in (2.86). Then $S$ is $S^1$-equivariantly homotopic (through linear spheres, in $F_{(k_1,\ldots,k_n)}$) to a copy of $S^{2K-1}$ in $\star_{i=1}^{n-1} S_{l'_i,k_{l'_i}}$, for any other sequence of integers $1 \leqslant l'_1 < \cdots < l'_{n-1} \leqslant n$. Inductively then, $S$ is homotopic to a subset of

$$\star_{i=1}^{n-1} S_{l''_i,k'_{l''_i}},$$

for any sequences $l'' \in \mathcal{L}$, and $k'_i \in \{0,1\}$, in $X$. It follows that there exists a homotopy from $\star_{i=1}^{n-1} S_{i,0}^{2\tilde{\delta}_i-1}$ to $\star_{i=1}^{n-1} S_{i,1}^{2\tilde{\delta}_i-1} = j(\star_{i=1}^{n-1} S_{i,0}^{2\tilde{\delta}_i-1})$ in $X$. See Figures 2.5 and 2.6 for illustrations in the $n = 2, 3$ cases.

Now we take advantage of the Gysin sequence from (4.17). Let $\Phi_\alpha$ denote the fundamental class of the projective space

$$\mathbb{CP}^{\sum_{i=1}^{n-1} \tilde{\delta}_i - 1} \simeq (S^{2\sum_{i=1}^{n-1} \tilde{\delta}_i - 1} \times S^0)/G \simeq (\star_{i=1}^{n-1} S_{i,0})/S^1,$$

and let $\iota_*$ denote the map on homology induced by the inclusion:

$$\iota\colon (\star_{i=1}^{n-1} S_{i,0} \amalg \star_{i=1}^{n-1} S_{i,1})/G \to X/G.$$

Then, as in the argument proving (2.79), we have $\kappa_*(\iota_*(\Phi_\alpha)) = v^{-\lfloor \frac{\sum_{i=1}^{n-1} \tilde{\delta}_i - 1}{2} \rfloor}$.

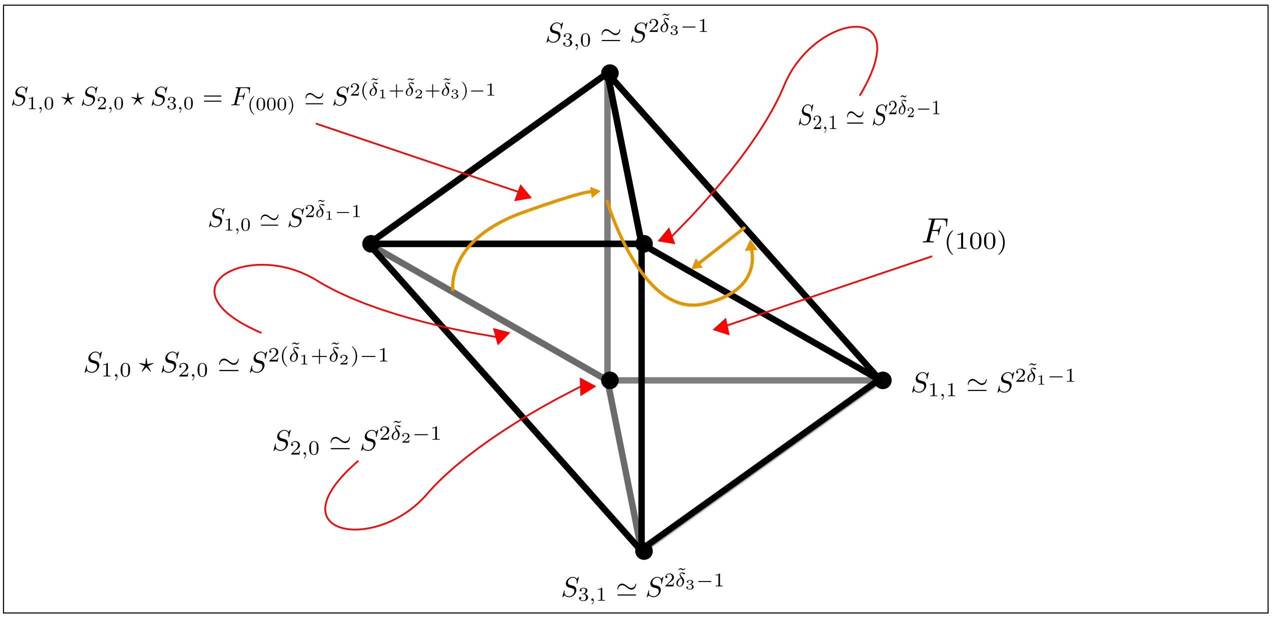


**Figure 2.6:** The homotopy $\star_{i=1}^{n-1} S_{i,0} \to j(\star_{i=1}^{n-1} S_{i,0})$, for $n = 3$. In $F_{(000)}$ we have $S_{1,0} \star S_{2,0} \simeq S^{2(\tilde{\delta}_1+\tilde{\delta}_2)-1}$ homotopic to a copy $S'$ of $S^{2(\tilde{\delta}_1+\tilde{\delta}_2)-1}$ contained in $S_{2,0} \star S_{3,0}$. The sphere $S'$ is then homotopic in $F_{(100)}$ to $S'' \subseteq S_{1,1} \star S_{3,0}$. In $F_{(110)}$, $S''$ is homotopic to $S''' \subseteq S_{1,1} \star S_{2,1}$, so we have constructed a homotopy $\star_{i=1}^{2} S_{i,0} \to j(\star_{i=1}^{2} S_{i,0})$, as needed. A similar procedure applies for $n \geqslant 4$.

We check that $\iota_*(\Phi_\alpha)$ is in the image of $q$ (for the action of $q$ on $H_*(X/G)$). Indeed, we have that $(1+j)\cdot\iota_*(\Phi_\alpha)$, viewed as a class of $X/S^1$, is zero by the above homotopy from $\star_{i=1}^{n-1}S_{i,0}$ to $\star_{i=1}^{n-1}S_{i,1} = j(\star_{i=1}^{n-1}S_{i,0})$. Then, by Fact 2.3.2, $\iota_*(\Phi_\alpha)$ is in the image of $q\cap -$.

Thus, there exists some class $\Phi_\beta^X \in H_*^G(X)$ so that $q\Phi_\beta^X = \iota_*(\Phi_\alpha)$. It follows that $\kappa_*(\Phi_\beta^X)$ must be nonzero, and we obtain $q^{-1}v^{-\lfloor\frac{\sum_{i=1}^{n-1}\tilde{\delta}_i-1}{2}\rfloor} \in \operatorname{Im}\kappa_*$. Using (2.75), we see:

$$b(\tilde{\Sigma}X) \geqslant 2E(\sum_{i=1}^{n-1}\tilde{\delta}_i). \tag{2.87}$$

Using the Definition 2.1.9 of $\beta$, that is:

$$\beta(\tilde{\Sigma}X) \geqslant E(\sum_{i=1}^{n-1}\tilde{\delta}_i). \tag{2.88}$$

By Theorem 2.2.4,

$$\begin{aligned}\beta(\tilde{\Sigma}X) &\leqslant \alpha(\tilde{\Sigma}(\star_{i=1}^{n-1}X_i)) + \beta(\tilde{\Sigma}X_n)\\ &\leqslant E(\textstyle\sum_{i=1}^{n-1}\tilde{\delta}_i) + 0.\end{aligned} \tag{2.89}$$

Here we have used Lemma 2.2.11 to see $\beta(\tilde{\Sigma}X_n) = 0$. Finally, (2.88) and (2.89) together imply (2.80).

**Proof of (2.81).** We again apply the Gysin sequence after constructing a homotopy. Repeating the argument from (2.80), we construct a homotopy, where $I$ is the unit interval:

$$\psi\colon I \times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1} \to X$$

so that $\psi(0,-)$ is a linear sphere:

$$S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1} \to \star_{i=1}^{n-2}S_{i,0},$$

and so that $\psi(1,-)$ is a linear sphere:

$$S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1} \to \star_{i=1}^{n-2}S_{i,1} = j(\star_{i=1}^{n-2}S_{i,0}).$$

Following the argument of (2.80), we see that we may choose $\psi$ to lie entirely within $F^{n-1}$, the "$(n-1)$-skeleton" of $X$. The construction of $\psi$ gives that it is a composition of homotopies in the faces:

$$F^{n-1}_{(k_1,\ldots,k_n)} = F^{n-1} \cap F_{(k_1,\ldots,k_n)},$$

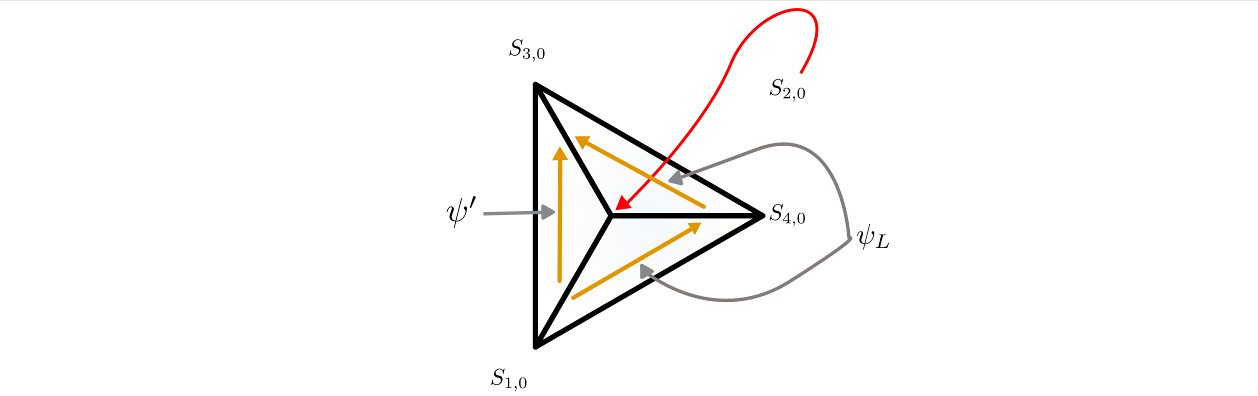


**Figure 2.7:** The tetrahedron corresponding to the face $F_{(0000)}$, where $n = 4$. In this example, the image of $\psi'$ is contained in $S_{1,0} \star S_{2,0} \star S_{3,0}$, and $\psi'$ takes a sphere in $S_{1,0} \star S_{2,0}$ to a sphere in $S_{2,0} \star S_{3,0}$. Further, for this example, $L_1 = (1, 2, 4)$, and $L_2 = (2, 3, 4)$. The path followed by $\psi_L$ is pictured.

so that in each $F_{(k_1,\ldots,k_n)}$, $\psi$ is a homotopy through linear spheres.

We will construct a homotopy from $\psi$ to $j\psi$ (perhaps up to reparameterization in the domain). Knowing that the homotopy $\psi$ was constructed by combining homotopies in the "faces" $F_{(k_1,\ldots,k_n)}$, we constuct a homotopy from $\psi$ to $j\psi$ by considering homotopies between homotopies in the "faces".

Let $S \subseteq \star_{i=1}^{n-2} S_{l_i,k_{l_i}} \subset F_{(k_1,\ldots,k_n)}$ where $1 \leqslant l_1 < \cdots < l_{n-2} \leqslant n$, and $S \simeq S^{2K-1}$ for some $K \leqslant \sum_{i=1}^{n-2} \tilde{\delta}_i$. Let $\psi'$ be a homotopy, through linear spheres, in $F_{(k_1,\ldots,k_n)}$, from $S$ to some $S' \simeq S^{2K-1} \subseteq \star_{i=1}^{n-2} S_{l'_i,k_{l'_i}}$, where $1 \leqslant l'_1 < \cdots < l'_{n-2} \leqslant n$.

Let $L_1, \ldots, L_m \in \mathcal{L}$ so that

$$(l_1, \ldots, l_{n-2}) \subset L_1 \text{ and } (l'_1, \ldots, l'_{n-2}) \subset L_m,$$

and so that $L_i$ and $L_{i+1}$ differ in only one place; see Figure 2.7. Then there exists a homotopy:

$$\psi_L \colon I \times S \to F_{(k_1,\ldots,k_n)},$$

so that $\psi_L(0, -)$ is the inclusion of $S$ and $\psi_L(1, -)$ is $\psi'(1, -)$, and so that

$$\psi_L([\frac{p-1}{m}, \frac{p}{m}], -) \subset \star_{l \in L_p} S_{l,k_l},$$

for $1 \leqslant p \leqslant m$. The homotopy $\psi_L|_{[\frac{p-1}{m}, \frac{p}{m}] \times S}$ is constructed exactly as in the proof of (2.80).

Next, let $\psi_L^-\colon I \times S \to F_{(k_1,\dots,k_n)}$ be given by $\psi_L^-(x,y) = \psi_L(1-x,y)$.

Consider the concatenation

$$H = \psi_L^- * \psi'\colon I \times S \to F_{(i_1,\dots,i_n)} \tag{2.90}$$

obtained by applying $\psi'$ and then running $\psi_L$ backwards. Since

$$\operatorname{Im} H(0,-) = \operatorname{Im} H(1,-),$$

we see that $H$ corresponds to a loop in $\mathrm{Gr}(K-1, \sum_{i=1}^n \tilde{\delta}_i - 1)$. However, $\pi_1(\mathrm{Gr}(K-1, \sum_{i=1}^n \tilde{\delta}_i - 1)) = 1$, from which we see that $H$ is null-homotopic. That is, $\psi'$ is homotopic to $\psi_L$ (again, perhaps up to reparameterization in the domain), as needed.

As in the proof of (2.80), we compose a sequence of the $\psi'$ to $\psi_L$ homotopies to see that $\psi$ is homotopic to $j\psi$, as in Figure 2.8. Concatenating the reverse $(j\psi)^-$ and $\psi$, we obtain a map:

$$(j\psi)^- * \psi\colon I \times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1} \to X.$$

Since Im $j\psi(1) =$ Im $\psi(0)$, by reparameterizing the domain $S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1}$ we obtain a map:

$$\iota := (j\psi)^- * \psi\colon S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1} \to X.$$

Here $S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1}$ is a space obtained by gluing the ends of $I \times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1}$.

The map $\iota$ descends to quotients by $S^1$ and $G$ to give maps $\iota_{S^1}$ and $\iota_G$, respectively.

Now that we have constructed the homotopy between $\psi$ and $j\psi$, we repeat the Gysin sequence argument we have already used in proving (2.80).

Let $\Phi_\alpha$ denote the fundamental class of

$$(1 \times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1})/S^1 \simeq \mathbb{CP}^{\sum_{i=1}^{n-2}\tilde{\delta}_i - 1} \subseteq (S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1})/G.$$

We have that $(1+j)\cdot\Phi_\alpha = 0$ as a homology class in $H_*^{S^1}(S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1})$, since the homotopy $\psi$ takes $\Phi_\alpha$ to $j\Phi_\alpha$. Then $\Phi_\alpha$, viewed as a class in $H_*^G(S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1})$, is in Im $q$. Let $\Phi_\beta$ denote the fundamental class of

$$(S^1 \tilde{\times} S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1})/G.$$

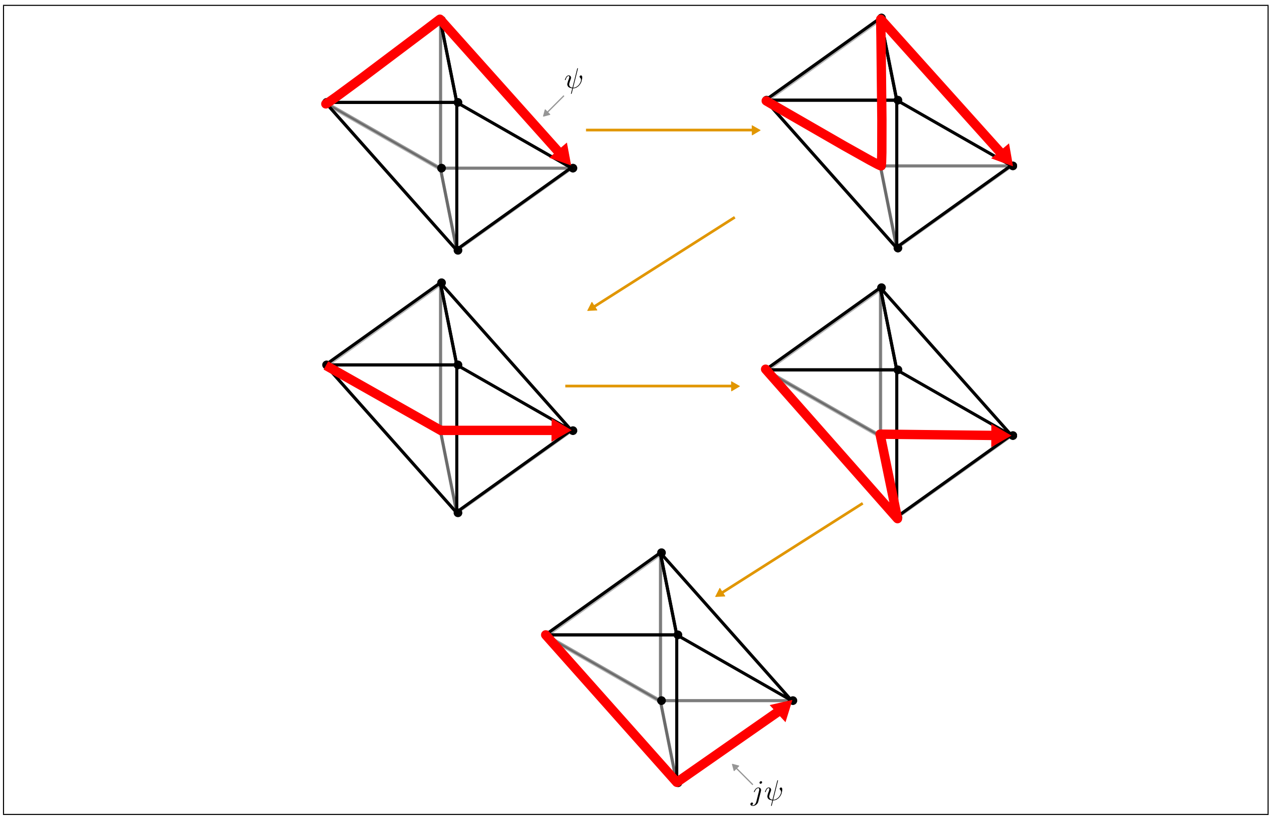


**Figure 2.8:** A homotopy from $\psi$ to $j\psi$ in the case $n = 3$. Here $\psi$ is a homotopy from $S_{1,0}$ to $jS_{1,0}$, and each stage pictured is one instance of the above construction of $\psi_L$. Composing these intermediate homotopies in the faces, we have the homotopy between $\psi$ and $j\psi$.

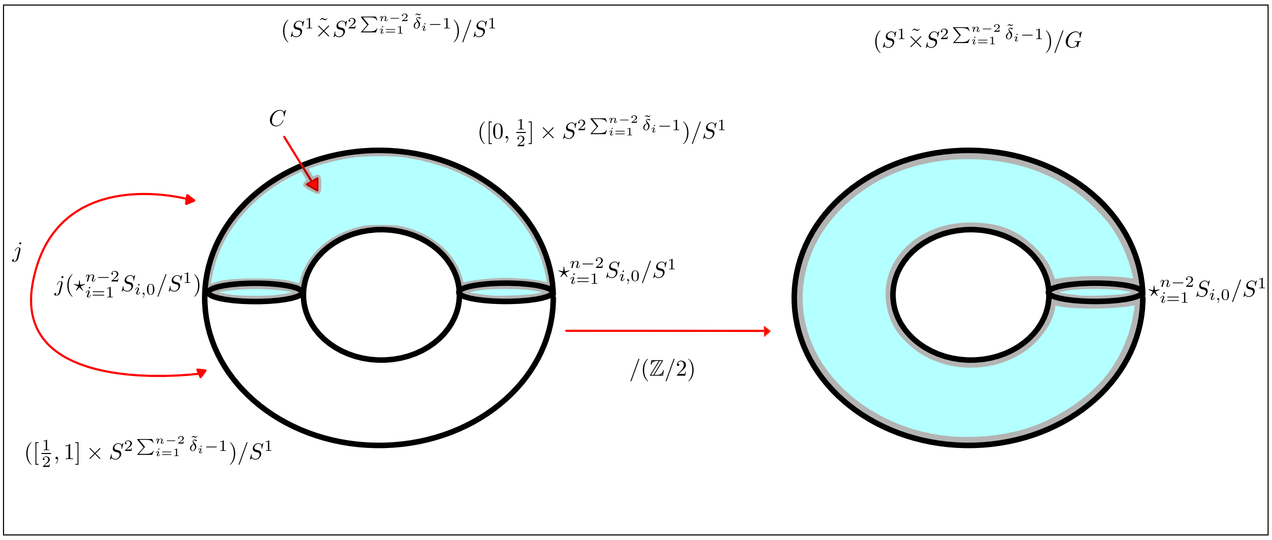


**Figure 2.9:** The shaded region $C$ denotes the relative fundamental class of $([0,\frac{1}{2}]\times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1$, the domain of $\psi$. We see from the figure that the quotient by the action of $\mathbb{Z}/2 = G/S^1$ takes $([0,\frac{1}{2}]\times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1)$ surjectively onto $(S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/G$. Thus $C$ is indeed a chain representative for $\Phi_\beta$, as a class in $(S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1$.

Then $\Phi_\beta \in H_*^G(S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})$ is the only class in degree $2\sum_{i=1}^{n-2}\tilde{\delta}_i - 1$, so $q\Phi_\beta = \Phi_\alpha$. Our next goal will be to show that $(1+j)\cdot\iota_{G,*}(\Phi_\beta) = 0$, as a class in $H_*^{S^1}(X)$.

Note that a chain representative $C$ of $\Phi_\beta$ in $(S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1$ is the relative fundamental class of $([0,\frac{1}{2}]\times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1$, as in Figure 2.9. Then we see that $(1+j)\cdot\Phi_\beta$ is the fundamental class of $(S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})/S^1$. It follows that

$$0 = \iota_{S^1,*}((1+j)\cdot\Phi_\beta) = (1+j)\cdot\iota_{G,*}(\Phi_\beta),$$

since

$$\psi([0,1]\times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1}) \text{ and } j\psi([0,1]\times S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1})$$

are homotopic in $X$. By Fact 2.3.2, we have $\iota_{G,*}(\Phi_\beta) = q\Phi_\gamma^X$ for some $\Phi_\gamma^X \in H_*^G(X)$.

As in the argument for (2.80) we note that $\kappa_*\iota_{G,*}(\Phi_\alpha) \neq 0$, since $\kappa_*\iota_{G,*}$ is the characteristic class map for $S^1\tilde{\times}S^{2\sum_{i=1}^{n-2}\tilde{\delta}_i-1}$ as a $G$-bundle. Then $\kappa_*\iota_{G,*}(\Phi_\beta)$ is nonzero, because $\kappa_*\iota_{G,*}$ must be $\mathbb{F}[q,v]/(q^3)$-equivariant. Similarly, we see $\kappa_*(\Phi_\gamma^X) \in H_*(BG)$ must be nonzero, from which we obtain

$$q^{-2}v^{-\lfloor\frac{\sum_{i=1}^{n-2}\tilde{\delta}_i-1}{2}\rfloor} \in \operatorname{Im}\kappa_*.$$

Thus:

$$c(\tilde{\Sigma}X) \geqslant 2E(\sum_{i=1}^{n-2} \tilde{\delta}_i),$$

so

$$\gamma(\tilde{\Sigma}X) \geqslant E(\sum_{i=1}^{n-2} \tilde{\delta}_i). \tag{2.91}$$

From Theorem 2.2.4, we have the inequalities (using $0 \leqslant \gamma(\tilde{\Sigma}(X_{n-1} \star X_n)) \leqslant \beta(\tilde{\Sigma}X_{n-1}) + \beta(\tilde{\Sigma}X_n) = 0$):

$$\begin{aligned} \gamma(\tilde{\Sigma}X) &\leqslant \alpha(\tilde{\Sigma}(\star_{i=1}^{n-2} X_i)) + \gamma(\tilde{\Sigma}(X_{n-1} \star X_n)) \\ &\leqslant E(\textstyle\sum_{i=1}^{n-2} \tilde{\delta}_i) + 0. \end{aligned} \tag{2.92}$$

Finally, (2.91) and (2.92) imply (2.81). □

# CHAPTER 3

# The Seiberg-Witten Floer Stable homotopy type

## 3.1 Seiberg-Witten Floer spectra and Floer homologies

### 3.1.1 Finite-dimensional approximation

In this section we review the finite-dimensional approximation to the Seiberg-Witten equations from Manolescu [28],[30].

Let $\mathbb{S}$ be the spinor bundle of the three-manifold with spin structure $(Y, \mathfrak{s})$, and $\Gamma(\mathbb{S})$ its space of sections. Let $D$ denote the Dirac operator. Let $W = \ker d^* \oplus \Gamma(\mathbb{S})$ be the global Coulomb slice, a Hilbert subspace of an appropriate Sobolev completion of $\Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathbb{S})$. For $\lambda \in (0, \infty)$, the Seiberg-Witten equations of $(Y, \mathfrak{s}, g)$ determine a sequence of vector fields $\mathcal{X}^{\mathrm{gC}}_\lambda$ on finite-dimensional vector spaces $W^\lambda$. Here $W^\lambda$ is the span of eigenvectors of the elliptic operator $*d + D$ acting on $W$, with eigenvalue in $(-\lambda, \lambda)$. The vector field $\mathcal{X}^{\mathrm{gC}}_\lambda$ on $W^\lambda$ is an approximation of the Seiberg-Witten equations restricted to $W^\lambda$. The action of $G = \mathrm{Pin}(2)$ on $\Gamma(\mathbb{S})$ restricts to a smooth action on $W^\lambda$ that commutes with the flow defined by $\mathcal{X}^{\mathrm{gC}}_\lambda$, and we define an action of $G$ on $\Omega^1$ by letting $j$ act by $-1$ and $S^1$ act trivially. There is a distinguished subspace $W(-\lambda, 0) \subset W^\lambda$ consisting of the span of the eigenvectors with eigenvalue in $(-\lambda, 0)$. Following [28], we will use the sequence of flows on the spaces $W^\lambda$ to define an invariant of $(Y, \mathfrak{s})$.

We next recall a few properties of the Conley Index. For a one-parameter family $\phi_t$ of diffeomorphisms of a manifold $M$ and a compact subset $A \subset M$, we define:

$$\mathrm{Inv}(A, \phi) = \{x \in A \mid \phi_t(x) \in A \text{ for all } t \in \mathbb{R}\}.$$

Then we say that a set $S \subset M$ is an isolated invariant set if there is some $A$ as above such that $S = \mathrm{Inv}(A, \phi) \subset \mathrm{int}(A)$. Conley proved in [5] that one may associate to any isolated invariant set $S$ a pointed homotopy type $I(S)$, an invariant of the triple $(M, \phi_t, S)$. Floer [9] and Pruszko [42] defined an equivariant version, so that if a compact Lie group $K$ acts smoothly on $M$ preserving the flow $\phi_t$, then we may associate a pointed $K$-equivariant homotopy type $I_K(S)$. The Conley Index, as well as its equivariant refinement, are invariant under continuous changes of the flow, if $S$ is isolated in an appropriate sense.

Manolescu showed in [30] that $S^\lambda$, the set of all critical points of $\mathcal{X}_\lambda^{\mathrm{gC}}$, along with all trajectories *of finite type* between them contained in a certain sufficiently large ball in $W^\lambda$, is an isolated invariant set, and that the flow $\mathcal{X}_\lambda^{\mathrm{gC}}$ is $G$-equivariant. We then write $I^\lambda(Y, \mathfrak{s}, g) = I_G(S^\lambda)$. To make this construction independent of $\lambda$, we desuspend by $W(-\lambda, 0)$. Then we can define a pointed stable homotopy type associated to a tuple $(Y, \mathfrak{s}, g)$:

$$SWF(Y, \mathfrak{s}, g) = \Sigma^{-W(-\lambda,0)} I^\lambda(Y, \mathfrak{s}, g). \tag{3.1}$$

The desuspension in (3.1) is interpreted in $\mathfrak{C}$. That is,

$$SWF(Y, \mathfrak{s}, g) = (I^\lambda(Y, \mathfrak{s}, g), \dim_{\mathbb{R}} W(-\lambda, 0)(\tilde{\mathbb{R}}), \dim_{\mathbb{H}} W(-\lambda, 0)(\mathbb{H})),$$

where $W(-\lambda, 0) \cong W(-\lambda, 0)(\tilde{\mathbb{R}}) \oplus W(-\lambda, 0)(\mathbb{H})$, and $W(-\lambda, 0)(\tilde{\mathbb{R}})$ is a direct sum of copies of $\tilde{\mathbb{R}}$. Similarly, $W(-\lambda, 0)(\mathbb{H})$ is a direct sum of copies of $\mathbb{H}$.

Manolescu showed in [30] that $SWF(Y, \mathfrak{s}, g)$ is well-defined, for $\lambda$ sufficiently large. Further, we must remove the dependence on the choice of metric $g$. We use $n(Y, \mathfrak{s}, g)$, a rational number which controls the spectral flow of the Dirac operator and may be expressed as a sum of eta invariants; for its definition, see [28]. We have:

$$SWF(Y, \mathfrak{s}) = \Sigma^{-\frac{1}{2}n(Y,\mathfrak{s},g)\mathbb{H}} SWF(Y, \mathfrak{s}, g). \tag{3.2}$$

Interpreted in $\mathfrak{C}$, if $SWF(Y, \mathfrak{s}, g) = (X, m, n)$, then $SWF(Y, \mathfrak{s}) = (X, m, n + \frac{1}{2}n(Y, \mathfrak{s}, g))$.

In addition to the approximate flow above, we may also consider perturbations of the flow as in [23].

For fixed $k \geqslant 1$, we call

$$\mathcal{C}(Y, \mathfrak{s}) = L_k^2\Omega^1(Y, i\mathbb{R}) \oplus L_k^2(Y; \mathbb{S})$$

the configuration space for the Seiberg-Witten equations, where $L^2_k\Omega^1(Y, i\mathbb{R})$ is the space of $L^2_k$ 1-forms. We write $\mathcal{L}$ for the Chern-Simons-Dirac functional and $\mathcal{G}$ for the $L^2_{k+1}$-gauge transformations. Let $\mathcal{X}$ be the $L^2$-gradient of $\mathcal{L}$ on $\mathcal{C}(Y, \mathfrak{s})$. We call a map

$$\mathfrak{q} : \mathcal{C}(Y, \mathfrak{s}) \to \mathcal{T}_0, \tag{3.3}$$

a perturbation, where $\mathcal{T}_j$ denotes the $L^2_j$ completion of the tangent bundle to $\mathcal{C}(Y, \mathfrak{s})$. Then we write

$$\mathcal{X}_{\mathfrak{q}} = \mathcal{X} + \mathfrak{q} : \mathcal{C}(Y, \mathfrak{s}) \to \mathcal{T}_0.$$

Let $W$ denote the global Coulomb slice in $\mathcal{C}(Y, \mathfrak{s})$ and $\mathcal{T}^{\mathrm{gC}}_k$ the $L^2_k$ completion of the tangent bundle to $W$. Lidman and Manolescu also consider a version of $\mathcal{X}_{\mathfrak{q}}$, obtained by projecting trajectories of $\mathcal{X}_{\mathfrak{q}}$ to $W$:

$$\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q}} : W \to \mathcal{T}^{\mathrm{gC}}_0.$$

Lidman and Manolescu prove that there is a bijective correspondence between finite-energy trajectories of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q}}$ and those of $\mathcal{X}_{\mathfrak{q}}$, modulo the appropriate gauges.

We write $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ for the finite-dimensional approximation of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q}}$ in $W^\lambda$ (recalling that $W^\lambda$ are finite-dimensional subspaces of $W$). For very tame perturbations in the sense of [25], we may define $I^\lambda(Y, \mathfrak{s}, g, \mathfrak{q})$ as above using $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ in place of $\mathcal{X}^{\mathrm{gC}}_\lambda$. Furthermore, from $I^\lambda(Y, \mathfrak{s}, g, \mathfrak{q})$ we may also define $SWF(Y, \mathfrak{s}, g, \mathfrak{q})$ analogously to the unperturbed case. Proposition 6.6 of [25] shows that the spectrum is independent of $\mathfrak{q}$. That is:

$$SWF(Y, \mathfrak{s}, g, \mathfrak{q}) = SWF(Y, \mathfrak{s}, g).$$

We also have the attractor-repeller sequence of [30]. For a generic perturbation $\mathfrak{q}$ we may arrange that the reducible critical point of $\mathcal{X}_{\mathfrak{q}}$ is nondegenerate and that there are no irreducible critical points $x$ with $\mathcal{L}(x) \in (0, \epsilon)$ for some $\epsilon > 0$. Denote the reducible critical point by $\Theta$. Let $T = T^\lambda$ be the set of all critical points of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ and points on flows of finite type between them. Then, for all $\omega > 0$, we have the following isolated invariant sets:

- $T^{\mathrm{irr}}_{>\omega}$: the set of irreducible critical points $x$ with $\mathcal{L}_{\mathfrak{q}}(x) > \omega$, together with all points on the flows between critical points of this type.

- $T_{\leqslant\omega}$ : Same, but with $\mathcal{L}_{\mathfrak{q}}(x) \leqslant \omega$, and allowing $x$ to be reducible.

Then we have the exact sequence:

$$I(T_{\leqslant\omega}) \to I(T) \to I(T^{\text{irr}}_{>\omega}) \to \Sigma I(T_{\leqslant\omega}) \to \dots \tag{3.4}$$

We record a theorem of [30].

**Theorem 3.1.1** (Manolescu [30],[31])**.** *Associated to a three-manifold with $b_1 = 0$ and a choice of spin structure $(Y, \mathfrak{s})$ there is an invariant $SWF(Y, \mathfrak{s})$, the Seiberg-Witten Floer spectrum class, in $\mathfrak{E}$. A spin cobordism $(W, \mathfrak{t})$ from $Y_1$ to $Y_2$, with $b_2(W) = 0$, induces a map $SWF(Y_1, \mathfrak{t}|_{Y_1}) \to SWF(Y_2, \mathfrak{t}|_{Y_2})$. The induced map is a homotopy-equivalence on the $S^1$-fixed-point set.*

**Remark 3.1.2.** *The three-manifold $Y$ in Theorem 3.1.1 may be disconnected.*

**Definition 3.1.3.** For $(Y, \mathfrak{s})$ a spin rational homology three-sphere, the Manolescu invariants $\alpha(Y, \mathfrak{s})$, $\beta(Y, \mathfrak{s})$, and $\gamma(Y, \mathfrak{s})$ are defined by $\alpha(SWF(Y, \mathfrak{s}))$, $\beta(SWF(Y, \mathfrak{s}))$, and $\gamma(SWF(Y, \mathfrak{s}))$, respectively.

**Theorem 3.1.4** ([30])**.** *Let $(Y, \mathfrak{s})$ be a spin rational homology three-sphere, and let $-Y$ denote $Y$ with orientation reversed. Then*

$$\alpha(Y, \mathfrak{s}) = -\gamma(-Y, \mathfrak{s}),\ \beta(Y, \mathfrak{s}) = -\beta(-Y, \mathfrak{s}),\ \gamma(Y, \mathfrak{s}) = -\alpha(-Y, \mathfrak{s}).$$

*Furthermore $\delta(Y, \mathfrak{s}) = -\delta(-Y, \mathfrak{s})$.*

From Theorem 3.1.1, the local and chain local equivalence classes of $SWF(Y, \mathfrak{s})$, namely $[SWF(Y, \mathfrak{s})]_l$ and $[SWF(Y, \mathfrak{s})]_{cl}$, respectively, are homology cobordism invariants of the pair $(Y, \mathfrak{s})$. Since the $G$-Borel homology of $SWF(Y, \mathfrak{s})$ depends only on $[SWF(Y, \mathfrak{s})]_{cl}$, we have that $\alpha(Y, \mathfrak{s}), \beta(Y, \mathfrak{s})$, and $\gamma(Y, \mathfrak{s})$ depend only on the chain local equivalence class $[SWF(Y, \mathfrak{s})]_{cl}$.

**Fact 3.1.5.** *Let $Y_1, Y_2$ be rational homology three-spheres with spin structures $\mathfrak{t}_1, \mathfrak{t}_2$ and $(X_i, m_i, n_i) = SWF(Y_i, \mathfrak{t}_i)$ for $i = 1, 2$. Then*

$$SWF(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) \equiv_l (X_1 \wedge X_2, m_1 + m_2, n_1 + n_2).$$

*Proof.* According to [30], the Seiberg-Witten Floer spectrum class of the disjoint union $Y_1 \amalg Y_2$ is given by:

$$SWF(Y_1 \amalg Y_2) \equiv_l (X_1 \wedge X_2, m_1 + m_2, n_1 + n_2).$$

On the other hand $Y_1 \amalg Y_2$ is homology cobordant to the connected sum $Y_1 \# Y_2$. Since the local equivalence class is a homology cobordism invariant, we obtain the claim. □

By Theorem 3.1.1 and Fact 3.1.5, we have a sequence of homomorphisms:

$$\theta_3^H \xrightarrow{SWF} \mathfrak{LE} \xrightarrow{C_*} \mathfrak{CLE}. \tag{3.5}$$

### 3.1.2 Approximate Trajectories

Fix $\mathfrak{q}$ a very tame admissible perturbation, as in Definitions 4.9 and 4.19 of [25]. Here we will record several results of Lidman-Manolescu [25] for use in Section 4.2. The first result is a corollary of Proposition 7.7 of [25]:

**Proposition 3.1.6.** *[25] For $\lambda$ sufficiently large, there is a grading-preserving isomorphism between the set of irreducible critical points of the finite-dimensional approximation $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ and the set of irreducible critical points of $\mathcal{X}_{\mathfrak{q}}$ on $\mathcal{C}(Y,\mathfrak{s})/\mathcal{G}$.*

For $x, y$ critical points of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$, let $M_\lambda([x],[y])$ denote the set of unparameterized trajectories of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ from $[x]$ to $[y]$ contained in the ball used to define $S^\lambda$. Similarly, we let $M([x],[y])$ be the set of unparameterized trajectories between critical points of $\mathcal{X}_{\mathfrak{q}}$ on $\mathcal{C}(Y,\mathfrak{s})/\mathcal{G}$.

**Proposition 3.1.7** ([25] Proposition 13.1)**.** *There is a correspondence of degree one trajectories compatible with Proposition 3.1.6. That is, if $[x_\lambda], [y_\lambda]$ are irreducible critical points, with $\mathrm{gr}(x_\lambda) = \mathrm{gr}(y_\lambda) + 1$, of $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$ corresponding to irreducible critical points $[x], [y]$ of $\mathcal{X}_{\mathfrak{q}}$, respectively, then there is an identification*

$$M([x],[y]) = M_\lambda([x_\lambda],[y_\lambda]).$$

The condition $\mathrm{gr}(x) = \mathrm{gr}(y) + 1$ allows the application of an inverse function theorem. However, without the grading assumption, a compactness result still holds. That is, Proposition 12.17 of [25] implies:

**Proposition 3.1.8.** *[25] Let* $[x]$ *and* $[y]$ *be critical points of* $\mathcal{X}_{\mathfrak{q}}$ *corresponding to critical points* $[x_\lambda], [y_\lambda]$ *of* $\mathcal{X}^{\mathrm{gC}}_{\mathfrak{q},\lambda}$*. If* $M([x],[y]) = \varnothing$*, then* $M_\lambda([x_\lambda],[y_\lambda]) = \varnothing$*.*

We will also need the following Theorem from [25].

**Theorem 3.1.9.** *[25] Let* $(Y,\mathfrak{s})$ *be a rational homology three-sphere with* spin$^c$ *structure. Then*

$$\widetilde{HM}(Y,\mathfrak{s}) = SWFH^{S^1}(Y,\mathfrak{s}),$$

*as absolutely graded* $\mathbb{F}[U]$*-modules, where* $\widetilde{HM}(Y,\mathfrak{s})$ *denotes the "to" version of monopole Floer homology defined in [23].*

### 3.1.3 Connected Seiberg-Witten Floer homology

**Definition 3.1.10.** Let $(Y,\mathfrak{s})$ be a rational homology three-sphere with spin structure, and

$$[SWF(Y,\mathfrak{s})] = (Z,m,n) \in \mathfrak{CE},$$

with $Z$ suspensionlike. The *connected Seiberg-Witten Floer homology* of $(Y,\mathfrak{s})$, written $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$, is the quotient $(H^{S^1}_*(Z)/(H^{S^1}_*(Z^{S^1})+H^{S^1}_*(Z_{\mathrm{iness}})))[m+4n]$, where $Z_{\mathrm{iness}} \subset Z$ is a maximal inessential subcomplex. By Theorems 2.1.39 and 3.1.1, the isomorphism class of $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$ is a homology cobordism invariant.

**Remark 3.1.11.** *We could have instead considered the quotient* $(H^{S^1}_*(Z)/H^{S^1}_*(Z_{\mathrm{iness}}))[m+4n]$*, which is isomorphic to* $SWFH_{\mathrm{conn}}(Y,\mathfrak{s}) \oplus \mathcal{T}^+_d$ *where* $d$ *is the Heegaard Floer correction term of* $(Y,\mathfrak{s})$*. As defined above,* $SWFH_{\mathrm{conn}}(Y,\mathfrak{s})$ *has no infinite* $\mathbb{F}[U]$*-tower, because of the quotient by* $H^{S^1}_*(Z^{S^1})$*. Further, let* $Z_{\mathrm{conn}}$ *denote the connected complex (Definition 2.1.38) of* $Z$*. It is clear from the construction that*

$$SWFH_{\mathrm{conn}}(Y,\mathfrak{s}) = (H^{S^1}_*(Z_{\mathrm{conn}})/H^{S^1}_*(Z^{S^1}))[m+4n].$$

**Remark 3.1.12.** *Let* $\phi$ *be the canonical isomorphism:*

$$\phi : H^{S^1}(SWF(Y,\mathfrak{s})) \to \widetilde{HM}(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s}),$$

*provided by, for the first map, [25], and for the second, [4] and [24]. Let* $\pi$ *be the projection* $\pi : HF^+(Y, \mathfrak{s}) \to HF_{\text{red}}(Y, \mathfrak{s})$. *We note that* $SWFH_{\text{conn}}(Y, \mathfrak{s})$ *is naturally isomorphic to the quotient*

$$\pi(\phi(H_*^{S^1}(SWF(Y, \mathfrak{s})))/\phi(H_*^{S^1}(Z_{\text{iness}})).$$

*Then* $SWFH_{\text{conn}}(Y, \mathfrak{s})$ *can be viewed as an* $\mathbb{F}[U]$*-summand of* $HF_{\text{red}}(Y, \mathfrak{s})$.

# CHAPTER 4

# Seiberg-Witten Floer homotopy of Seifert spaces

## 4.1 $j$-split spaces

In this section we introduce $j$-split spaces of type SWF, and compute their $G$-Borel homology. We will see in Lemma 4.2.3 that the Seiberg-Witten Floer spectra of Seifert spaces are $j$-split. The computation of this section will then provide the $G$-equivariant Seiberg-Witten Floer homology of Seifert spaces.

**Definition 4.1.1.** We call a space $X$ of type SWF *$j$-split* if $X/X^{S^1}$ may be written:

$$X/X^{S^1} \simeq X_+ \vee X_-,$$

for some $S^1$-space $X_+$, where $\simeq$ denotes $G$-equivariant homotopy equivalence, and $j$ acts on the right-hand side by interchanging the factors (that is, $jX_+ = X_-$). Similarly, we call a $\mathcal{G}$-chain complex $(Z, \partial)$ of type SWF *$j$-split* if (1) – (3) below are satisfied.

1. There exists $f_{\text{red}} \in Z$ such that $\langle f_{\text{red}} \rangle$ is the fixed-point set, $Z^{S^1}$, of $Z$. Furthermore $sf_{\text{red}} = 0, jf_{\text{red}} = f_{\text{red}}$. In particular, $Z$ is of type SWF at level 0.
2. The fixed-point set $Z^{S^1}$ is a subcomplex of $Z$ (that is, $\partial(f_{\text{red}}) = 0$).
3. We have:

   $$Z/Z^{S^1} = (Z_+ \oplus jZ_+),$$

   where $Z_+$ is a $C^{CW}_*(S^1)$ chain complex, and $j$ acts on the right-hand side by interchanging the factors.

Recall that $\tilde{\oplus}$ denotes a direct sum of $\mathcal{G}$-modules that is not necessarily a direct sum of chain complexes. For a $j$-split chain complex $Z$ we may write, referring to $jZ_+$ by $Z_-$:

$$Z = (Z_+ \oplus Z_-)\tilde{\oplus}\langle f_{\text{red}}\rangle.$$

In the above, $Z$ is to be thought of as the reduced CW chain complex of a $G$-space $X$, and $f_{\text{red}}$ is to be thought of as the chain corresponding to the $S^1$-fixed subset of $X$. The requirement that $Z$ be a chain complex of type SWF at level 0 will be used in Section 4.1.2 to calculate the chain local equivalence class of $j$-split chain complexes.

A $j$-split space $X$ with $X^{S^1} \simeq S^0$ admits a CW chain complex which is a $j$-split chain complex. For $X$ a $j$-split space of type SWF at level $s$, we use the following Lemma to relate the CW chain complex of $X$ to $j$-split complexes.

**Lemma 4.1.2.** *Let $X$ be a $j$-split space of type SWF at level $s$. Then*

$$[C^{CW}_*(X, \text{pt})] = [(Z, -s, 0)] \in \mathfrak{CE},$$

*for some $j$-split chain complex $Z$.*

*Proof.* The chain complex $C^{CW}_*(X, \text{pt})$ may be written

$$C^{CW}_*(X, \text{pt}) = R\tilde{\oplus}F, \tag{4.1}$$

where $R = C^{CW}_*(X^{S^1}, \text{pt}) \cong C^{CW}_*((\tilde{\mathbb{R}}^s)^+, \text{pt})$ is a subcomplex and $F$ is a free $\mathcal{G}$-chain complex. Since $X$ is $j$-split, the decomposition (4.1) may be chosen so that

$$F = F_+ \oplus jF_+, \tag{4.2}$$

where $F_+$ is a $C^{CW}_*(S^1)$-chain complex, and $j$ acts on $F$ by interchanging $F_+$ and $jF_+$.

We first show that we may choose $F$ satisfying (4.1) and (4.2) and so that, for $x \in F$ homogeneous,

$$(\partial x)|_R = 0, \tag{4.3}$$

if $\deg x \neq s + 1$.

Indeed, fix some $F$ satisfying (4.1) and (4.2), and let $\{x_i\}$ be a homogeneous basis for $F$. Let $F(n)$ denote the $\mathcal{G}$-chain complex generated by $x_i$ of degree less than or equal to $n$. We define new chain complexes $F'(n)$ so that $R\tilde{\oplus}F'(n) = R\tilde{\oplus}F(n)$, and so that $F' = \bigcup_n F'(n)$ satisfies (4.1)-(4.3). Let $\pi_n$ denote projection $\pi_n : R\tilde{\oplus}F'(n) \to R$ onto the first factor. Set $F'(0) = 0$. Assume we have defined $F'(n)$ for $n \leqslant N < s$, so that (4.3) holds for all $x \in F'(n)$.

We define $F'(N+1)$ by defining generators $x_i'$ of $F'(N+1)/F'(N)$ corresponding to the generators $x_i$ of $F(N+1)/F(N)$. For each $x_i$ of degree $N+1$ so that $\pi_N(\partial x_i) = 0$, let $x_i' = x_i$. If instead $x_i$ is of degree $N+1$ and $\pi_N(\partial x_i) \neq 0$, then

$$\partial(\pi_N(\partial x_i)) = \pi_N(\partial^2(x_i)) = 0.$$

So, $\pi_N(\partial x_i) = (1+j)c_N$, since $(1+j)c_N$ is the only nonzero cycle of $R$ in grading $N$ (or, when $N=0$, $\pi_N(\partial x_i) = c_0$). However, by assumption, $N < s$, so $\pi_N(\partial x_i) = \partial c_{N+1}$. Then, we let $x_i' = x_i + c_{N+1}$.

Let

$$F'(N+1) = \langle F'(N), \bigcup_{\{i|\deg x_i = N+1\}} x_i' \rangle.$$

By construction $R\tilde{\oplus}F'(N+1) = R\tilde{\oplus}F'(N)$, and (4.3) holds for all $x \in F'(N+1)$.

For $N \geqslant s$, define $F'(N+1)$ by $F'(N+1) = \langle F'(N), \bigcup_{\{i|\deg x_i = N+1\}} x_i \rangle$.

From the construction, it is clear that $F'$ satisfies (4.1)-(4.3), as needed.

Take $F$ satisfying (4.1)-(4.3). Consider the $\mathcal{G}$-chain complex $Z = C_*^{CW}(S^0, \mathrm{pt})\tilde{\oplus}F[s]$, where $C_*^{CW}(S^0, \mathrm{pt}) = \langle c_0 \rangle$ is a subcomplex. To define the differentials $F[s] \to C_*^{CW}(S^0, \mathrm{pt})$ in $Z$, we set, for $x[s] \in F[s]$:

$$(\partial x[s])|_{C_*^{CW}(S^0, \mathrm{pt})} = c_0, \tag{4.4}$$

if $(\partial x)|_R = (1+j)c_s$, and

$$(\partial x[s])|_{C_*^{CW}(S^0, \mathrm{pt})} = 0 \tag{4.5}$$

if $(\partial x)|_R = 0$.

By the construction of $F$, (4.4) and (4.5) determine the differential on $Z$.

Finally, consider the suspension:

$$\Sigma^{\tilde{\mathbb{R}}^s} Z = \Sigma^{\tilde{\mathbb{R}}^s}(C_*^{CW}(S^0, \mathrm{pt}))\tilde{\oplus}\Sigma^{\tilde{\mathbb{R}}^s}(F[s]) \simeq R\tilde{\oplus}\Sigma^{\tilde{\mathbb{R}}^s}F[s].$$

We note, as in the proof of Lemma 2.1.28, that $\Sigma^{\tilde{\mathbb{R}}^s}F[s] \simeq F[0] = F$. Then, there is a homotopy equivalence, constructed exactly as in the proofs of Lemmas 2.1.28 and 2.1.30:

$$\Sigma^{\tilde{\mathbb{R}}^s} Z \simeq R\tilde{\oplus}F. \tag{4.6}$$

It follows that $[(Z, -s, 0)] = [C_*^{CW}(X, \mathrm{pt})] \in \mathfrak{CE}$, as needed. □

Note also that any $j$-split chain complex occurs as the CW chain complex of some $j$-split space.

**Remark 4.1.3.** *$j$-splitness is not the same as Floer $K_G$-splitness of [31].*

### 4.1.1 Calculation of $\tilde{H}_*^G(X)$

In this section we will compute the $G$-equivariant homology of a $j$-split space in terms of its $S^1$-homology.

Let $X$ be a $j$-split space of type SWF at level $m$ with $X/X^{S^1} = X_+ \vee X_-$. The Puppe sequence

$$X^{S^1} \to X \to X/X^{S^1} \to \Sigma X^{S^1}$$

leads to a commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc} EG_+ \wedge_{S^1} X^{S^1} & \longrightarrow & EG_+ \wedge_{S^1} X & \longrightarrow & EG_+ \wedge_{S^1} (X_+ \vee X_-) & \longrightarrow & EG_+ \wedge_{S^1} \Sigma X^{S^1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge_G X^{S^1} & \longrightarrow & EG_+ \wedge_G X & \longrightarrow & EG_+ \wedge_G X/X^{S^1} & \longrightarrow & EG_+ \wedge_G \Sigma X^{S^1}. \end{array} \tag{4.7}$$

In (4.7) the vertical maps are obtained by taking the quotient by the action of $j \in G$. The diagram (4.7) itself yields a commutative diagram for Borel homology, where the rows are exact:

$$\begin{array}{ccccccc} \tilde{H}_*^{S^1}(X^{S^1}) & \longrightarrow & \tilde{H}_*^{S^1}(X) & \longrightarrow & \tilde{H}_*^{S^1}(X_+) \oplus \tilde{H}_*^{S^1}(X_-) & \xrightarrow{d_{S^1}[-1]} & \tilde{H}_*^{S^1}(\Sigma X^{S^1}) \\ {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_2}\downarrow & & {\scriptstyle \phi_3}\downarrow & & {\scriptstyle \Sigma\phi_1}\downarrow \\ \tilde{H}_*^G(X^{S^1}) & \xrightarrow{\iota_G} & \tilde{H}_*^G(X) & \xrightarrow{\pi_G} & \tilde{H}_*^G(X/X^{S^1}) & \xrightarrow{d_G[-1]} & \tilde{H}_*^G(\Sigma X^{S^1}). \end{array} \tag{4.8}$$

Specifically, we view (4.8) as a diagram of $\mathbb{F}[q,v]/(q^3)$ modules, where $v$ acts on the top row by $U^2$ and $q$ annihilates the top row. An $\mathbb{F}[U]$-module $M$ viewed as an $\mathbb{F}[q,v]/(q^3)$-module this way is denoted $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)}M$. More precisely, let $\phi : \mathbb{F}[q,v]/(q^3) \to \mathbb{F}[U]$ be $v \to U^2$, $q \to 0$, and let $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)}$ be the corresponding restriction functor. The restriction takes the simple $\mathbb{F}[U]$-module $\mathcal{T}^+_d(n)$ to

$$\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)}\mathcal{T}^+_d(n) = \mathcal{V}^+_d(\lfloor\frac{n+1}{2}\rfloor) \oplus \mathcal{V}^+_{d+2}(\lfloor\frac{n}{2}\rfloor). \tag{4.9}$$

We define the maps $d_{S^1} : \tilde{H}^{S^1}_*(X_+) \to \tilde{H}^{S^1}_*(X^{S^1})$ and $d_G : \tilde{H}^G_*(X/X^{S^1}) \to \tilde{H}^G_*(X^{S^1})$ by shifting by 1 the degree of the horizontal maps on the right of diagram (4.8). (So that $d_{S^1}$ and $d_G$ are maps of degree $-1$.)

**Fact 4.1.4.** *The map $\phi_1$ in (4.8) is precisely the corestriction map $\mathrm{cor}^{S^1}_G$, and is an isomorphism in degrees congruent to $m \bmod 4$, and vanishes otherwise.*

*Proof.* This follows from the construction of the $\phi_i$ and the dual of Fact 2.1.1. □

**Fact 4.1.5.**

$$\phi_3|_{\tilde{H}^{S^1}_*(X_+)} : \tilde{H}^{S^1}_*(X_+) \to \tilde{H}^G_*(X/X^{S^1}) \tag{4.10}$$

*is an isomorphism (of $\mathbb{F}[q,v]/(q^3)$-modules).*

*Proof.* Since $X$ is $j$-split, both domain and target are isomorphic, as vector spaces, to $H_*(X_+/S^1)$. The map $\phi_3$ is a bijection and an $\mathbb{F}[q,v]/(q^3)$-module map, and so is an isomorphism. □

In particular, Fact 4.1.5 shows that the $q$-action on $\tilde{H}^G_*(X/X^{S^1})$ is trivial. Since $\phi_3|_{\tilde{H}^{S^1}(X_+)}$ is an isomorphism, we have:

$$\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)}\tilde{H}^{S^1}_*(X_+) = \tilde{H}^G_*(X/X^{S^1}). \tag{4.11}$$

**Fact 4.1.6.** *The maps $d_{S^1}$ and $d_G$ are $\mathbb{F}[U]$ and $\mathbb{F}[q,v]/(q^3)$-equivariant, respectively.*

*Proof.* The fact follows since the maps $d_{S^1}$ and $d_G$ are induced on Borel homology, respectively, from $S^1$ and $G$-equivariant maps. □

By (4.8),

$$d_G\phi_3 = \phi_1 d_{S^1}. \tag{4.12}$$

**Lemma 4.1.7.** *We have:*

$$\tilde{H}^{S^1}_*(X) = \text{coker } d_{S^1} \oplus \text{ker } d_{S^1}. \tag{4.13}$$

*Proof.* Using the top row of (4.8), we have an exact sequence:

$$0 \to \text{coker } d_{S^1} \to \tilde{H}^{S^1}_*(X) \to \text{ker } d_{S^1} \to 0,$$

so $\tilde{H}^{S^1}_*(X)$ is an extension of ker $d_{S^1}$ by coker $d_{S^1}$. Note that coker $d_{S^1}$ is isomorphic to $\mathcal{T}^+_d$ for some integer $d$. A calculation shows $\text{Ext}^1_{\mathbb{F}[U]}(\mathcal{T}^+_{d_i}(n_i), \mathcal{T}^+_d) = 0$ for all $d, d_i, n_i$. Thus, any extension of ker $d_{S^1}$ by coker $d_{S^1}$ is trivial, and we obtain the Lemma. □

We also write (4.13) as the homology of the complex $\tilde{H}^{S^1}_*(X^{S^1}) \oplus \tilde{H}^{S^1}_*(X/X^{S^1})$ with differential $d_{S^1}$.

**Lemma 4.1.8.** *We have:*

$$\tilde{H}^G_*(X) \cong \text{coker}\, d_G \oplus \text{ker}\, d_G. \tag{4.14}$$

*as* $\mathbb{F}[v]$*-vector spaces. The subspace* coker $d_G$ *is a* $\mathbb{F}[q,v]/(q^3)$*-submodule, and* $q$ *acts on* $x \in \ker d_G$ *by* $qx = 0$ *if* $x \in \text{Im}\,\phi_2|_{\ker d_{S^1}}$ *(using the decomposition of* $\tilde{H}^{S^1}_*(X)$ *in Lemma 4.1.7). Also,* $qx \neq 0 \in \text{coker}\, d_G$ *if* $x \in \ker d_G$ *but* $x \notin \text{Im}\,\phi_2|_{\ker d_{S^1}}$*. As there is at most one homogeneous element of each degree in* coker $d_G$*,* $qx$ *is uniquely specified for all* $x \in \ker d_G$ *in the decomposition (4.14).*

*Proof.* As in the proof of Lemma 4.1.7, we see that $\tilde{H}^G_*(X)$ is an extension of

$$\ker d_G \subseteq \text{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)} \tilde{H}^{S^1}_*(X_+)$$

by coker $d_G = \tilde{H}^G_*(X^{S^1})/(\text{Im}\, d_G)$. We will first show that the extension is trivial as an $\mathbb{F}[v]$-extension.

We construct $M \subset \tilde{H}^G_*(X)$ a vector space lift of ker $d_G \subset \tilde{H}^G_*(X/X^{S^1})$, so that $\phi_2(\text{ker } d_{S^1}) \subseteq M$, using the decomposition of $\tilde{H}^{S^1}_*(X)$ in (4.13).

Specifically, we define $M$ in each degree $i$ by:

$$M_i = \begin{cases} (\phi_2(\ker d_{S^1}))_i \text{ for } i \not\equiv 3+m \bmod 4, \\ \tilde{H}_i^G(X) \text{ for } i \equiv 3+m \bmod 4. \end{cases}$$

We next show that $\pi_G|_M : M \to \ker d_G$ is an isomorphism.

We have $(\operatorname{coker} d_G)_i = 0$ for $i \equiv 3+m \bmod 4$, since $\tilde{H}_*^G(X^{S^1}) \cong H_*(BG)[-m]$, so

$$\pi_G : \tilde{H}_i^G(X) \to (\ker d_G)_i \tag{4.15}$$

is an isomorphism for all $i \equiv 3+m \bmod 4$.

We now show that $\pi_G : (\operatorname{Im} \phi_2|_{\ker d_{S^1}})_i \to (\ker d_G)_i$ is an isomorphism for $i \not\equiv 3+m \bmod 4$. It suffices to show $\ker d_G \subseteq \operatorname{Im} \phi_3|_{\ker d_{S^1}}$ in degrees not congruent to $3+m \bmod 4$. Indeed, $\phi_3$ is surjective by (4.10). Furthermore, by Fact 4.1.4, $\phi_1$ is injective in degrees not congruent to $2+m \bmod 4$. By (4.12), if $y \in \ker d_G$ with $\deg(y) \not\equiv 3+m \bmod 4$, and $y = \phi_3(x)$, for $x \in \tilde{H}_*^{S^1}(X/X^{S^1})$, then $\phi_1(d_{S^1}x) = 0$. By the injectivity of $\phi_1$, we have $d_{S^1}x = 0$, and we obtain:

$$y \in \operatorname{Im}(\phi_3|_{\ker d_{S^1}}).$$

That is, $(\operatorname{Im} \phi_3|_{\ker d_{S^1}})_i = (\ker d_G)_i$ for $i \not\equiv 3+m \bmod 4$. Then, $\pi_G(\operatorname{Im} \phi_2|_{\ker d_{S^1}})_i = (\ker d_G)_i$, as needed.

We have then established that $\tilde{H}_*^G(X) = \operatorname{coker} d_G \oplus M$ as $\mathbb{F}$-vector spaces.

We next determine the $\mathbb{F}[q,v]/(q^3)$-action on $M \subset \tilde{H}_*^G(X)$. Since $\ker d_{S^1} \subset \tilde{H}_*^{S^1}(X)$ is an $\mathbb{F}[q,v]/(q^3)$-submodule, so is its image in $\tilde{H}_*^G(X)$. Then, for $x \in M$ homogeneous of degree not congruent to $3+m \bmod 4$, we have $qx, vx \in M$. In fact, $qx = 0$, since $q$ acts trivially on $\tilde{H}_*^{S^1}(X)$. Moreover, for $x \in M$ of degree congruent to $3+m \bmod 4$, $vx \in \tilde{H}_*^G(X)$ is also of degree congruent to $3+m$, and, in particular, we see $vx \in M$. So we need only determine $qx$ for $x \in M$ with $\deg x \equiv 3+m \bmod 4$.

As in [50][III.2] there exists a Gysin sequence:

$$\tilde{H}_G^*(X) \longrightarrow \tilde{H}_{S^1}^*(X) \longrightarrow \tilde{H}_G^*(X) \xrightarrow{q\cup -} \tilde{H}_G^{*+1}(X) \longrightarrow \dots, \tag{4.16}$$

where $q \cup -$ denotes cup product with $q$. Dualizing, we obtain an exact sequence:

$$\tilde{H}^G_*(X) \xrightarrow{(1+j)\cdot -} \tilde{H}^{S^1}_*(X) \xrightarrow{\phi_2} \tilde{H}^G_*(X) \xrightarrow{q\cap -} \tilde{H}^G_{*-1}(X) \longrightarrow \dots, \tag{4.17}$$

where $(1 + j) \cdot -$ denotes the map obtained from multiplication (on the chain level) by $1 + j \in \mathcal{G}$, and $q \cap -$ denotes cap product with $q$.

From (4.17), we have that if $x \in M \subset \tilde{H}^G_*(X)$ is not in $\operatorname{Im} \phi_2|_{\ker d_{S^1}}$, then $qx \neq 0$. We will show that $qx \in \operatorname{coker} d_G$.

First, we see

$$(1 + j) \cdot \operatorname{coker} d_G \subset \operatorname{coker} d_{S^1}. \tag{4.18}$$

Indeed, (4.18) follows from the commutativity of the diagram

$$\begin{array}{ccc} \tilde{H}^G_*(X) & \xrightarrow{(1+j)\cdot} & \tilde{H}^{S^1}_*(X) \\ \uparrow & & \uparrow \\ \tilde{H}^G_*(X^{S^1}) & \xrightarrow{(1+j)\cdot} & \tilde{H}^{S^1}_*(X^{S^1}). \end{array}$$

Additionally, we see that

$$\ker d_G \xrightarrow{(1+j)\cdot -} \ker d_{S^1}$$

is injective by the $j$-splitness condition (Definition 4.1.1). Then $\ker\,(1 + j) \subset \tilde{H}^G_*(X)$ is, in fact, a subset of $\operatorname{coker} d_G$. Thus, if $x \notin \operatorname{Im} \phi_2|_{\ker d_{S^1}}$, $qx$ must be the unique nonzero element in grading $\deg x - 1$ in $\operatorname{coker} d_G$, completing the proof. □

Our goal will be to relate (4.13) and (4.14), relying on (4.11) and (4.12). From this relationship we will be able to show that the $S^1$-homology (4.13) determines the $G$-homology (4.14). In Lemmas 4.1.10 and 4.1.11 we compute $\tilde{H}^{S^1}_*(X)$ from $\tilde{H}^{S^1}_*(X/X^{S^1})$ and $d_{S^1}$. In Lemmas 4.1.12-4.1.15, we show how to compute $\tilde{H}^G_*(X)$ from the same information. Then in Theorem 4.1.16 we compute $\tilde{H}^G_*(X)$ directly from $\tilde{H}^{S^1}_*(X)$.

We begin by noting that any finite graded $\mathbb{F}[U]$-module may be written as a direct sum of copies of $\mathcal{T}^+_{d_i}(n_i)$, as $\mathbb{F}[U]$ is a principal ideal domain. In particular, $\tilde{H}^{S^1}_*(X/X^{S^1})$, since it has finite rank as an $\mathbb{F}$-module, is a direct sum of copies of the $\mathcal{T}^+_{d_i}(n_i)$.

**Lemma 4.1.9.** *On $\mathcal{T}_d^+(n) \subset \tilde{H}_*^{S^1}(X/X^{S^1})$, the differential $d_{S^1}$ vanishes unless $2n+d \geqslant 3+m$ and $d \leqslant m+1$.*

*Proof.* Let $U^{-k}$ denote the unique nonzero element of $\mathcal{T}_m^+$ in degree $m+2k$. Let $x_{d+2n-2}$ be an $\mathbb{F}[U]$-module generator of $\mathcal{T}_d^+(n)$, with deg $(x_{d+2n-2}) = d+2n-2$. Then either $d_{S^1}$ vanishes on $\mathcal{T}_d^+(n)$ or $d_{S^1}(x_{d+2n-2})$ is nonzero. In this latter case, because of the grading, $d_{S^1}(x_{d+2n-2}) = U^{-\frac{d+2n-m-3}{2}}$. If $2n+d < 3+m$, then $\mathcal{T}_d^+(n)$ has no elements in degree greater than $m$, and so has no nontrivial maps to $\mathcal{T}_m^+$. Similarly, for $d > m+1$, $d_{S^1}(\mathcal{T}_d^+(n)) = 0$. Indeed, if $d_{S^1}(\mathcal{T}_d^+(n)) \neq 0$, then

$$d_{S^1} x_{d+2n-2} = U^{-\frac{d+2n-m-3}{2}}.$$

Then, by Fact 4.1.6, $d_{S^1}(U^{\frac{d+2n-m-3}{2}} x_{d+2n-2}) = U^0 \neq 0 \in \mathcal{T}_m^+$. However, if $d > m+1$, then $U^{\frac{d+2n-m-3}{2}} x_{d+2n-2} = 0$, a contradiction. ☐

**Lemma 4.1.10.** *There exists a decomposition*

$$\tilde{H}_*^{S^1}(X_+) = J_1 \oplus J_2, \tag{4.19}$$

*as a direct sum of $\mathbb{F}[U]$-modules $J_1$ and $J_2$, where $d_{S^1}$ vanishes on $J_2$ and*

$$J_1 = \bigoplus_{i=1}^{N} \mathcal{T}_{d_i}^+(n_i),$$

*with $2n_i + d_i > 2n_{i+1} + d_{i+1}$, and $d_{i+1} > d_i$, for some $N$. Moreover, $d_N \leqslant 1+m, 2n_N + d_N \geqslant 3+m$, and $d_{S^1}$ is nonvanishing on each summand $\mathcal{T}_{d_i}^+(n_i)$.*

*Proof.* To begin, set $\tilde{H}_*^{S^1}(X_+) = J_1 \oplus J_2$ for some choices of $J_1$ and $J_2$ so that $d_{S^1}|_{J_2} = 0$, possibly by setting $J_2 = 0$. We introduce a partial ordering $\succeq$ of (graded) $\mathbb{F}[U]$-modules. We say

$$T_{d_1}(n_1) \succeq T_{d_2}(n_2)$$

if $2n_1 + d_1 \geqslant 2n_2 + d_2$ and $d_1 \geqslant d_2$. Our goal is to arrange that the summands of $J_1$ are not comparable under this relation. Suppose we have $\mathcal{T}_{d_1}^+(n_1) \oplus \mathcal{T}_{d_2}^+(n_2) \subset J_1$, and $\mathcal{T}_{d_1}^+(n_1) \succeq \mathcal{T}_{d_2}^+(n_2)$. If one of the $\mathcal{T}_{d_i}^+(n_i)$ has $d_{S^1}|_{\mathcal{T}_{d_i}^+(n_i)} = 0$, we move it to $J_2$. Otherwise, we have that $d_{S^1}$ is nontrivial on both $\mathcal{T}^+_{d_i}(n_i)$. Let $\mathcal{T}^+_{d_i}(n_i)$ be generated by $x_i$ for $i = 1, 2$. Then $\langle x_1, U^{n_1-n_2+(d_1-d_2)/2}x_1+x_2\rangle$ are new $\mathbb{F}[U]$-generators for $\mathcal{T}^+_{d_1}(n_1)\oplus\mathcal{T}^+_{d_2}(n_2) \subset J_1$, such that $d_{S^1}$ vanishes on $U^{n_1-n_2+(d_1-d_2)/2}x_1+x_2$, i.e. so that $d_{S^1}$ vanishes on the $\mathcal{T}^+_{d_2}(n_2)$ submodule. So we may choose a new decomposition $\tilde{H}^{S^1}_*(X_+) = J_1'\oplus J_2'$, where $J_2' \simeq J_2\oplus\mathcal{T}^+_{d_2}(n_2)$. Thus, we may choose $J_1$ such that there is no submodule $X \oplus Y$ of $J_1$ with $X \succeq Y$. Say $J_1 = \bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(n_i)$ has been chosen so that all its summands are incomparable under $\succeq$ (and so that $d_{S^1}$ is nonvanishing on each $\mathcal{T}^+_{d_i}(n_i)$). Perhaps by reordering, let $d_{i+1} \geqslant d_i$. If $d_{i+1} = d_i$, $\mathcal{T}^+_{d_i}(n_i)$ and $\mathcal{T}^+_{d_{i+1}}(n_{i+1})$ would be comparable, contradicting our choice of $J_1$. Thus $d_{i+1} > d_i$. Again using that the $\mathcal{T}^+_{d_i}(n_i)$ are incomparable, we obtain $2n_i + d_i > 2n_{i+1} + d_{i+1}$. Finally, we saw in Lemma 4.1.9 that $d_{S^1}$ vanishes on any summand $\mathcal{T}^+_d(n)$ with $d > 1+m$ or $2n+d < 3+m$, so by the condition that $d_{S^1}$ is nonvanishing, we have $d_N \leqslant 1 + m, 2n_N + d_N \geqslant 3 + m$. □

**Lemma 4.1.11.** *Let $\tilde{H}^{S^1}_*(X_+) = J_1 \oplus J_2$, with $J_1$ as in Lemma 4.1.10. Then*

$$\tilde{H}^{S^1}_*(X) = \mathcal{T}^+_{d_1+2n_1-1} \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(\frac{d_{i+1} + 2n_{i+1} - d_i}{2}) \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(n_i) \oplus J_2^{\oplus 2}. \tag{4.20}$$

*We interpret $d_{N+1} = m + 1, n_{N+1} = 0$. The expression $\frac{d_{N+1}+2n_{N+1}-d_N}{2}$ may vanish, in which case $\mathcal{T}^+_{d_N}(\frac{d_{N+1}+2n_{N+1}-d_N}{2})$ is the zero module.*

*Proof.* In the decomposition of Lemma 4.1.10, we write $x_i$ for the generator of $\mathcal{T}^+_{d_i}(n_i)$. We choose a basis for $\ker d_{S^1}$, given by $\{y_i\}_i$ for $y_i = x_{i+1}+U^{n_i-n_{i+1}+(d_i-d_{i+1})/2}x_i$ for $i = 1, ..., n-1$, and $y_N = U^{(d_N+2n_N-1)/2}x_N$. Note that $y_N$ may be zero.

We have seen that $J_2 \subset \ker d_{S^1}$, and also $jJ_2 \subset \ker d_{S^1}$, giving the two copies of the $J_2$ summand in (4.20). We see that $\mathbb{F}[U]U^{-\frac{d_1+2n_1-m-3}{2}} = \operatorname{Im} d_{S^1} \subset \mathcal{T}^+_m$, by Lemma 4.1.10. Then $\mathcal{T}^+_{d_1+2n_1-1} = \operatorname{coker} d_{S^1}$. Further, $(1 + j)J_1$ contributes the summand $\bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(n_i)$, since $d_{S^1}$ is $j$-invariant, and so vanishes on multiples of $(1 + j)$. Finally, the set $\{y_i\}$ generates the $\bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2})$ summand.

For an example of how the new basis gives the Lemma, see Figures 4.1 and 4.2. □

We now compute $\tilde{H}^G_*(X/X^{S^1})$. To find $\ker d_G$, we write $\tilde{H}^G_*(X/X^{S^1}) = J_1' \oplus J_2'$, where $d_G$ vanishes on $J_2'$ ($J_2'$ need not be maximal, currently). To find $J_1'$ and $J_2'$ in terms of $J_1$ and $J_2$, we use:

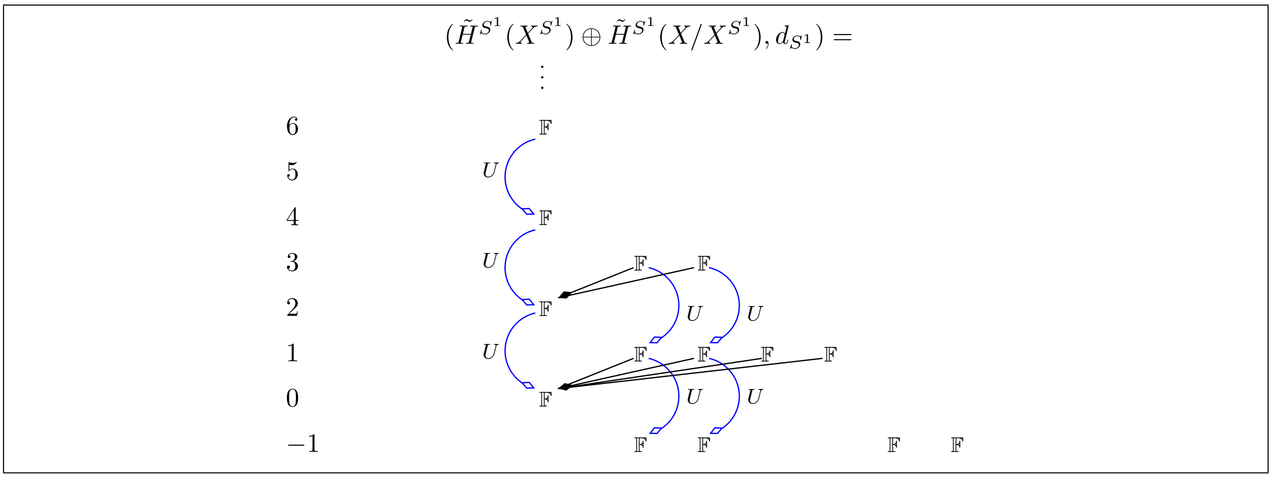


**Figure 4.1:** An example of $\tilde{H}^{S^1}_*(X)$ as in Lemma 4.1.11. The first four (finite) towers are $\mathcal{T}^+_{-1}(3)^{\oplus 2} \oplus \mathcal{T}^+_1(1)^{\oplus 2}$. Then $J_1 = \mathcal{T}^+_{-1}(3) \oplus \mathcal{T}^+_1(1)$ and $J_2 = \mathcal{T}^+_{-1}(1)$ in (4.19) (keeping in mind that the action of $j$ interchanges the pairs of copies $\mathcal{T}^+_{d_i}(n_i)$, so $\tilde{H}^{S^1}_*(X/X^{S^1}) \simeq J_1 \oplus J_2 \oplus J_1 \oplus J_2$ as an $\mathbb{F}[U]$-module). In particular, $d_1 = -1, n_1 = 3, d_2 = 1, n_2 = 1$. Here $m = 0$. The shaded-head arrows denote differentials while the open-head arrows denote $U$-actions.

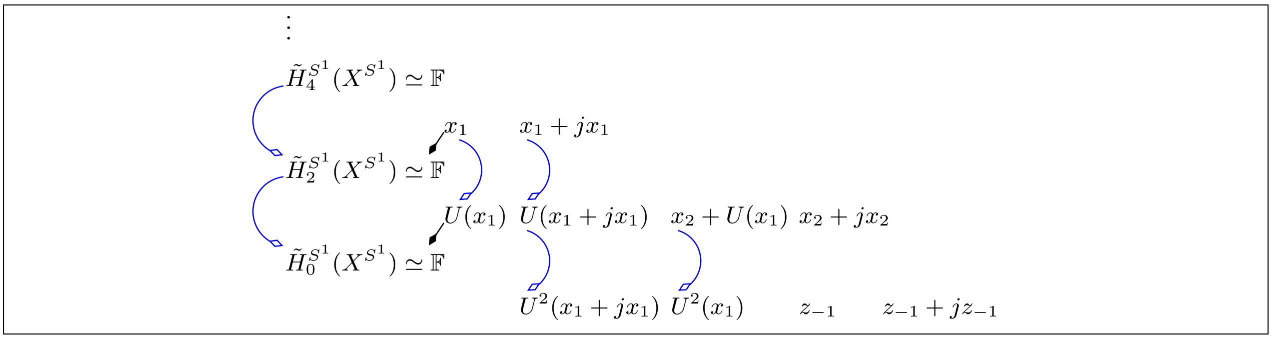


**Figure 4.2:** Using the basis in the proof of Lemma 4.1.11 for the complex of Figure 4.1. Here the generator of $J_2$ is written $z_{-1}$. The $x_i$ are generators of $\mathcal{T}^+_{d_i}(n_i)$ for $i = 1, 2$.

**Lemma 4.1.12.** *Let $J_1, J_2$ and $d_i, n_i$ be as in Lemma 4.1.10. Then we may set $\tilde{H}^G_*(X/X^{S^1}) = J'_1 \oplus J'_2$, where*

$$J'_1 = \bigoplus_{\{i \mid d_i \equiv m+1 \bmod 4\}} \mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor) \oplus \bigoplus_{\{i \mid d_i \equiv m+3 \bmod 4\}} \mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor),$$

$$J'_2 = \mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J_2 \oplus \bigoplus_{\{i \mid d_i \equiv m+1 \bmod 4\}} \mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor) \oplus \bigoplus_{\{i \mid d_i \equiv m+3 \bmod 4\}} \mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor).$$

*Moreover, $d_G$ is nonvanishing on each nontrivial summand of $J'_1$, and $d_G(J'_2) = 0$.*

*Proof.* We use (4.9) and (4.11) to conclude that

$$\phi_3 J_1 = \bigoplus_{i=1}^{N} \mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor) \oplus \bigoplus_{i=1}^{N} \mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor).$$

We also use

$$\mathrm{cor}_G^{S^1} d_{S^1} = d_G \phi_3,$$

as in (4.12) to obtain that $d_G$ is nonvanishing on each of $\mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor)$, with $d_i \equiv m+1 \bmod 4$ and $\mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor)$ with $d_i \equiv m+3 \bmod 4$. To find $J_2'$ we apply (4.11) again, to $J_2$, and we observe that $d_G$ is vanishing on each of $\mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor)$, with $d_i \equiv m+3 \bmod 4$ and $\mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor)$ with $d_i \equiv m+1 \bmod 4$. □

**Fact 4.1.13.** *The* $\mathbb{F}[v]$*-submodule*

$$\bigoplus_{\{i|d_i \equiv m+1 \bmod 4\}} \mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor) \oplus \bigoplus_{\{i|d_i \equiv m+3 \bmod 4\}} \mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor)$$

*in Lemma 4.1.12 is the component of* $\tilde{H}^G_*(X/X^{S^1})$ *not in the image of* $\phi_2|_{\ker d_{S^1}}$.

For an example of Lemma 4.1.12, see Figure 4.3. We define an order $\succeq$ on modules $\mathcal{V}^+_d(n)$

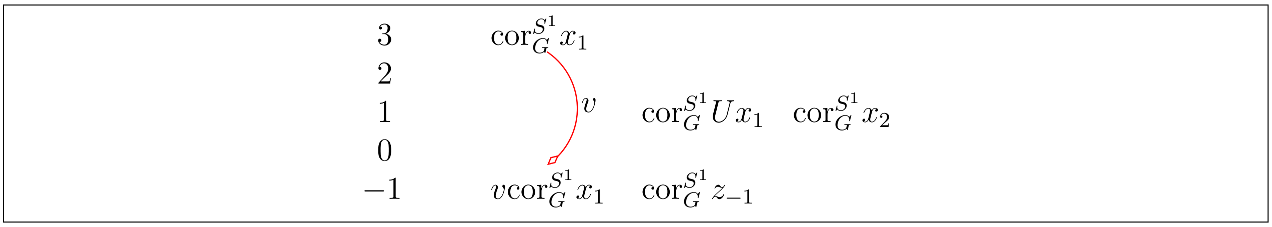

**Figure 4.3:** Computing $\tilde{H}^G_*(X/X^{S^1})$ for the complex of Figures 4.1 and 4.2. Here $J_1' = \mathcal{V}^+_1(1)^{\oplus 2}$, and $J_2' = \mathcal{V}^+_{-1}(2) \oplus \mathcal{V}^+_{-1}(1)$.

with $d \equiv m+1 \bmod 4$. Note that all simple submodules $\mathcal{V}^+_d(n)$ of $J_1'$ in Lemma 4.1.12 have $d \equiv m+1 \bmod 4$. Let $\mathcal{V}^+_{d_1}(n_1) \succeq \mathcal{V}^+_{d_2}(n_2)$ if $d_1 \geqslant d_2$ and $d_1 + 4n_1 \geqslant d_2 + 4n_2$. Let $\mathcal{J}$ denote the set of distinct pairs $(a,b)$ for which $\mathcal{V}^+_a(b)$ is a maximal summand of $J_1'$ as in Lemma 4.1.12. If $(a,b) \in \mathcal{J}$, set $m(a,b)+1$ to be the multiplicity with which $\mathcal{V}^+_a(b)$ occurs as a summand of $J_1'$. If $(a,b) \notin \mathcal{J}$, set $m(a,b)$ to be the multiplicity with which $\mathcal{V}^+_a(b)$ occurs in $J_1'$. Then we define:

$$J_{\mathrm{rep}} = \bigoplus_{(a,b)} \mathcal{V}^+_a(b)^{\oplus m(a,b)}, \tag{4.21}$$

where summands of multiplicity $0, -1$ do not contribute to the sum. That is, $J_{\text{rep}}$ counts the repeated summands (whence the "rep") in $J_1'$, as well as those which are not contributing "new" differentials targeting the reducible. In the example of Figure 4.3, $J_{\text{rep}} = \mathcal{V}_1^+(1)$.

Arguing as in Lemma 4.1.10, we obtain the following.

**Lemma 4.1.14.** *Let $\tilde{H}_*^{S^1}(X_+)$ be decomposed as in Lemma 4.1.10, and let $\mathcal{J}$ be as in the preceding paragraphs. Then we may set $\tilde{H}_*^G(X/X^{S^1}) = J_1'' \oplus J_2''$ with*

$$J_1'' \simeq \bigoplus_{(a_i,b_i)\in\mathcal{J}} \mathcal{V}_{a_i}^+(b_i),$$

$$J_2'' \simeq \operatorname{res}_{\mathbb{F}[v]}^{\mathbb{F}[U]} J_2 \oplus \bigoplus_{\{i|d_i\equiv m+1 \bmod 4\}} \mathcal{V}_{d_i+2}^+(\lfloor \frac{n_i}{2} \rfloor) \oplus \bigoplus_{\{i|d_i\equiv m+3 \bmod 4\}} \mathcal{V}_{d_i}^+(\lfloor \frac{n_i+1}{2} \rfloor) \oplus J_{\text{rep}}.$$

*Moreover, $d_G$ is nonvanishing on each nontrivial summand of $J_1''$, and $d_G(J_2'') = 0$. Further, $a_i < a_{i+1}$ and $a_i + 4b_i > a_{i+1} + 4b_{i+1}$ for $i = 1, ..., N_0 - 1$, where $N_0 = |\mathcal{J}|$.*

*Proof.* We argue as in Lemma 4.1.10, starting with the decomposition

$$\tilde{H}_*^G(X/X^{S^1}) = J_1' \oplus J_2'$$

given in Lemma 4.1.12. We will show that we may choose $J_1'' = \bigoplus_{(a_i,b_i)\in\mathcal{J}} \mathcal{V}_{a_i}^+(b_i)$, so that $\tilde{H}_*^G(X/X^{S^1}) = J_1'' \oplus J_2''$ with $d_G J_2'' = 0$. Fix a direct sum decomposition $J_1' = \bigoplus_i \mathcal{V}_{a_i}^+(b_i)$, for some $a_i, b_i$. Say that $\mathcal{V}_{e_1}^+(f_1) \subseteq J_1'$, where $(e_1, f_1) \notin \mathcal{J}$ and choose $(e_2, f_2) \in \mathcal{J}$, with $\mathcal{V}_{e_2}^+(f_2) \geq \mathcal{V}_{e_1}^+(f_1)$ and $\mathcal{V}_{e_1}^+(f_1) \oplus \mathcal{V}_{e_2}^+(f_2) \subseteq J_2'$. Further, assume that $d_G$ is nontrivial on $\mathcal{V}_{e_1}^+(f_1)$; if it were trivial, then we enlarge $J_2'$ by setting $J_2'' = J_2' \oplus \mathcal{V}_{e_1}^+(f_1)$. Let $x_i$ be the generator of $\mathcal{V}_{e_i}^+(f_i)$. We choose new $\mathbb{F}[v]$-generators, $x_2$ of $\mathcal{V}_{e_2}^+(f_2)$ and $v^{f_2-f_1+(e_2-e_1)/4}x_2 + x_1$ of $\mathcal{V}_{e_1}^+(f_1)$ so that $d_G$ vanishes on $\mathcal{V}_{e_1}^+(f_1)$. Again, then we may enlarge $J_2'$ by adding the $\mathcal{V}_{e_1}^+(f_1)$ factor. This shows that we can remove all summands $\mathcal{T}_a^+(b)$ with $(a, b) \notin \mathcal{J}$ from $J_1'$. Similarly, if $\mathcal{V}_a^+(b) \oplus \mathcal{V}_a^+(b) \subseteq J_1'$, with $(a, b) \in \mathcal{J}$ and with generators $x_1$ and $x_2$ such that $d_G(x_1) = d_G(x_2) \neq 0$, we choose the new basis $\langle x_1, x_2 + x_1 \rangle$. The differential $d_G$ is nonzero on the copy of $\mathcal{V}_a^+(b)$ generated by $x_1$, while $d_G$ vanishes on the copy of $\mathcal{V}_a^+(b)$ generated by $x_1 + x_2$, and $J_2'$ may be enlarged. Then we may choose $J_1'' \simeq \bigoplus_{(a,b)\in\mathcal{J}} \mathcal{V}_a^+(b)$. The formula for $J_2''$ also follows once $J_1''$ is specified. □

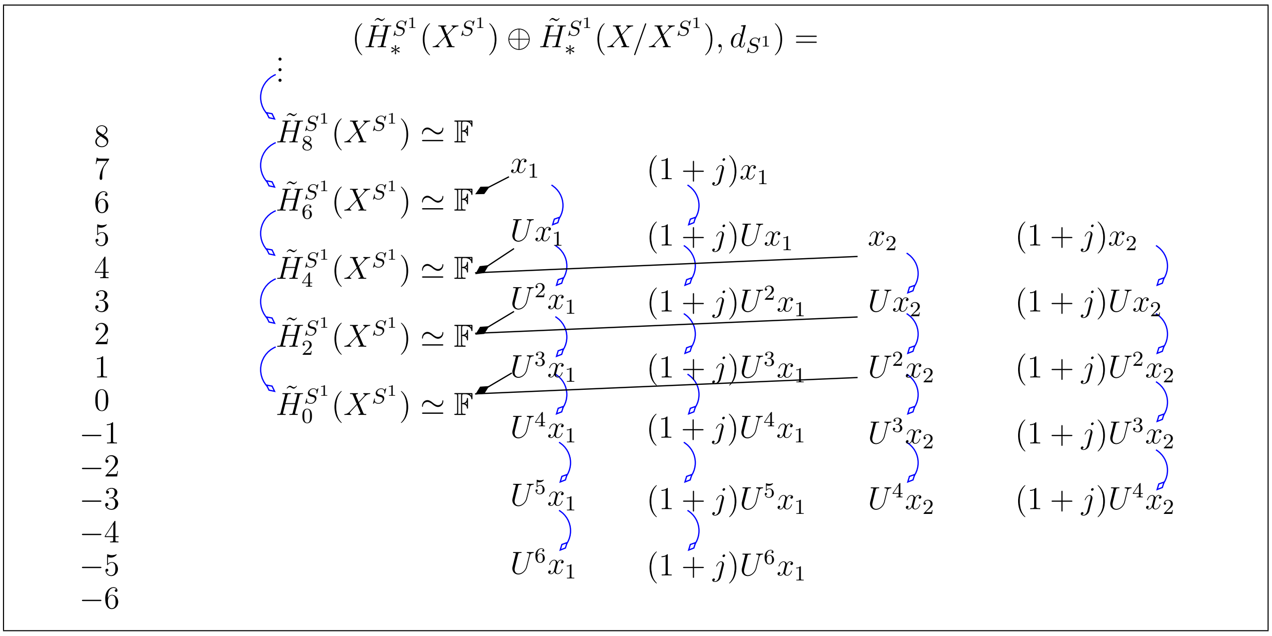


**Figure 4.4:** An example $\mathbb{F}[U]$-module $\tilde{H}^{S^1}_*(X^{S^1}) \oplus \tilde{H}^{S^1}_*(X/X^{S^1})$ for $X$ with $m = 0$. Here $d_1 = -5, n_1 = 7$ and $d_2 = -3, n_2 = 5$, and $J_2 = 0$.

In Figures 4.4 and 4.5, we provide an example illustrating the proof of Lemma 4.1.14. We may now compute $\tilde{H}^G_*(X)$ in terms of $\tilde{H}^{S^1}_*(X/X^{S^1})$ and the map $d_{S^1}$.

**Lemma 4.1.15.** *Let $\tilde{H}^{S^1}_*(X_+)$ be decomposed as in Lemma 4.1.10 and let $J''_1, J''_2$ be as in Lemma 4.1.14. Then:*

$$\begin{aligned} \tilde{H}^G_*(X) &= \mathcal{V}^+_{a_1+4b_1-1} \oplus \mathcal{V}^+_{1+m} \oplus \mathcal{V}^+_{2+m} \\ &\oplus \bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}\left(\frac{a_{i+1}+4b_{i+1}-a_i}{4}\right) \oplus J''_2, \end{aligned} \tag{4.22}$$

*as an $\mathbb{F}[v]$-module. The $q$-action is given by the isomorphism $q : \mathcal{V}^+_{2+m} \to \mathcal{V}^+_{1+m}$ and the map $\mathcal{V}^+_{1+m} \to \mathcal{V}^+_{a_1+4b_1-1}$, which is an $\mathbb{F}$-vector space isomorphism in all degrees at least $a_1 + 4b_1 - 1$. The action of $q$ annihilates $\bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}(\frac{a_{i+1}+4b_{i+1}-a_i}{4})$ and $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J_2 \oplus J_{\mathrm{rep}} \subseteq J''_2$.*

*To finish specifying the $q$-action, let $x_i$ be a generator of $\mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor)$ for $i$ such that $d_i \equiv m + 1 \bmod 4$ (respectively, let $x_i$ be a generator of $\mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor)$ if $d_i \equiv m + 3 \bmod 4$). Then $qx_i$ is the unique nonzero element of $H^G_*(X/X^{S^1})$ in grading $\deg x_i - 1$, for all $i$. In particular, $\tilde{H}^{S^1}_*(X/X^{S^1})$ and $d_{S^1}$ determine $\tilde{H}^G_*(X)$. Here $a_{N_0+1} = m + 1, b_{N_0+1} = 0$.*

*Proof.* The proof is analogous to that of Lemma 4.1.11. We choose a basis for $\ker d_G$ as

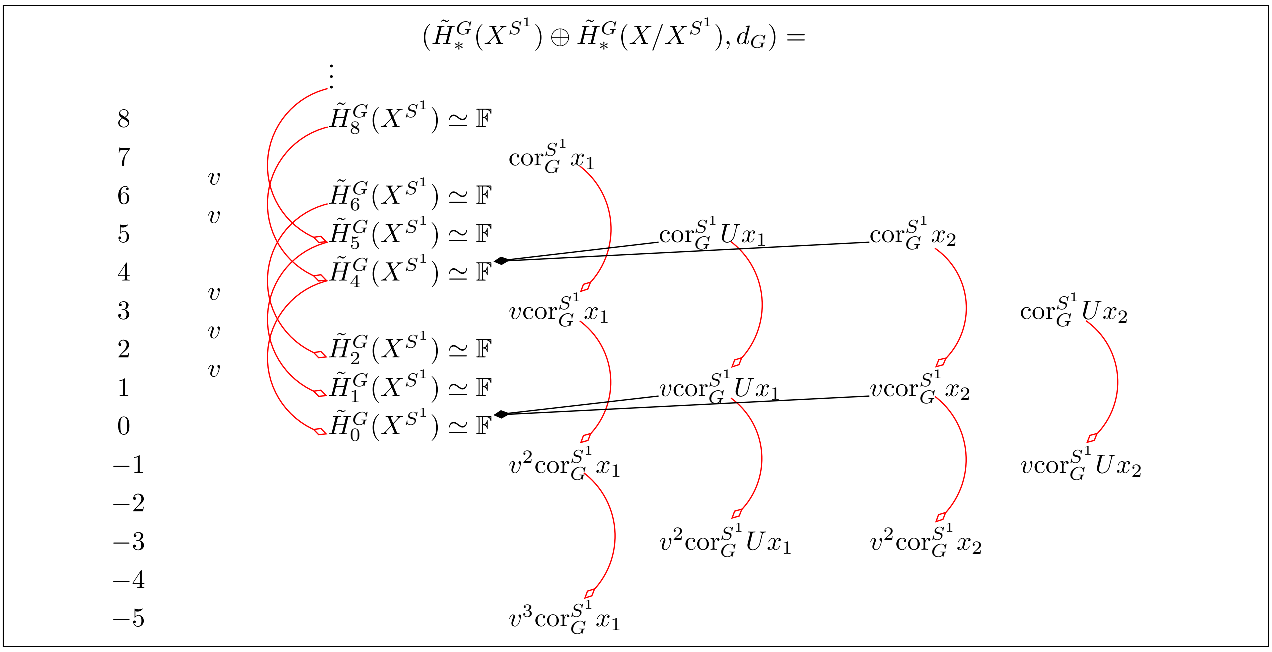


**Figure 4.5:** Here we show how to compute $(\tilde{H}^G_*(X^{S^1}) \oplus \tilde{H}^G_*(X/X^{S^1}), d_G)$, given $(\tilde{H}^{S^1}_*(X^{S^1}) \oplus \tilde{H}^{S^1}_*(X/X^{S^1}), d_{S^1})$, for the example complex given in Figure 4.4. The curved arrows denote the $v$-action. Here, $J_{\mathrm{rep}}$ is $\mathcal{V}^+_{-3}(3)$, and $J''_1 = \mathcal{V}^+_{-3}(3)$. Then we have also $J''_2 = \mathcal{V}^+_{-3}(3) \oplus \mathcal{V}^+_{-1}(2) \oplus \mathcal{V}^+_{-5}(4)$. If we have a basis of $\mathrm{cor}^{S^1}_G Ux_1$, $\mathrm{cor}^{S^1}_G x_2$ for $J'_1$, then $\mathrm{cor}^{S^1}_G Ux_1 + \mathrm{cor}^{S^1}_G x_2$ would be a basis for $J_{\mathrm{rep}}$ produced by Lemma 4.1.14.

follows. Write the generator of $\mathcal{V}^+_{a_i}(b_i)$ as $x_i$. Then set $y_i = x_{i+1} + v^{b_i - b_{i+1} + (a_i - a_{i+1})/4} x_i$ for $i = 1, ..., N_0 - 1$, and $y_{N_0} = v^{(a_{N_0} + 4b_{N_0} - 1)/4} x_{N_0}$. It is clear that $y_i \in \ker d_G$ for all $i$, and it is straightforward to check that $\{y_i\}$ generates $\ker d_G \cap J''_1$. The $y_i$ generate the term $\bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}(\frac{a_{i+1} + 4b_{i+1} - a_i}{4})$ in (4.22). Since $d_G$ is $q$-equivariant and $q$ annihilates $\tilde{H}^G_*(X/X^{S^1})$, the modules $\mathcal{V}^+_1$ and $\mathcal{V}^+_2 \subset H_*(BG)$ are disjoint from the image of $d_G$. Moreover, $v^{-\frac{a_1 + 4b_1 - 5 - m}{4}} = d_G(x_1)$, where $v^{-k}$ is the unique element $x$ of $H_*(BG)[-m]$ with $v^k x$ an $\mathbb{F}$-generator of $H_0(BG)[-m]$. Since there are no elements $x \in J''_1$ with grading greater than $a_1 + 4b_1 - 4$, the maximal $k$ for which $v^{-k} \in \mathrm{Im}\, d_G$ is $\frac{a_1 + 4b_1 - 5 - m}{4}$. It follows that

$$\mathrm{coker}\, d_G = \mathcal{V}^+_{a_1 + 4b_1 - 1} \oplus \mathcal{V}^+_{1+m} \oplus \mathcal{V}^+_{2+m}.$$

Furthermore, $J''_2 \subseteq \ker\, d_G$ by definition, contributing the $J''_2$ term of (4.22). To determine the $q$-action on $\ker d_G$, we use Lemma 4.1.8. Indeed, $q$ takes elements not in the image of $\phi_2|_{\ker d_{S^1}}$ to nontrivial elements of $\mathrm{coker}\, d_G$, and $q$ vanishes on $\mathrm{Im}\, \phi_2|_{\ker d_{S^1}}$. Using Fact 4.1.13, we obtain the $q$-action on $J_2''$ as in the Lemma. The $q$-action on $\operatorname{coker} d_G$ is given by that on $H_*(BG)$. □

We combine Lemmas 4.1.10-4.1.15 to determine $\tilde{H}^G_*(X)$ from $\tilde{H}^{S^1}_*(X)$. We record this as the following Theorem.

**Theorem 4.1.16.** *Let $X = (X', p, h/4) \in \mathfrak{E}$ and $X'$ be a $j$-split space of type SWF. Then:*

$$\tilde{H}^{S^1}_*(X) = \mathcal{T}^+_{s+d_1'+2n_1-1} \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i'}\left(\frac{d_{i+1}'+2n_{i+1}-d_i'}{2}\right) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i'}(n_i) \oplus J^{\oplus 2}[-s], \tag{4.23}$$

*for some constants $s, d_i', n_i, N$ and some $\mathbb{F}[U]$-module $J$, where $2n_i + d_i' > 2n_{i+1} + d_{i+1}'$ and $d_i' < d_{i+1}'$ for all $i$, $2n_N + d_N' \geqslant 3$, $d_N' \leqslant 1$, and $d_{N+1}' = 1, n_{N+1} = 0$. Let $\mathcal{J}_0 = \{(a_k, b_k)\}_k$ be the collection of pairs consisting of all $(d_i', \lfloor \frac{n_i+1}{2} \rfloor)$ for $d_i' \equiv 1 \bmod 4$ and all $(d_i'+2, \lfloor \frac{n_i}{2} \rfloor)$ for $d_i' \equiv 3 \bmod 4$, counting multiplicity. Let $(a,b) \succeq (c,d)$ if $a + 4b \geqslant c + 4d$ and $a \geqslant c$, and let $\mathcal{J}$ be the subset of $\mathcal{J}_0$ consisting of pairs maximal under $\succeq$ (not counted with multiplicity). If $(a,b) \in \mathcal{J}$, set $m(a,b)+1$ to be the multiplicity of $(a,b)$ in $\mathcal{J}_0$. If $(a,b) \notin \mathcal{J}$, set $m(a,b)$ to be the multiplicity of $(a,b)$ in $\mathcal{J}_0$. Let $|\mathcal{J}| = N_0$ and order the elements of $\mathcal{J}$ so that $\mathcal{J} = \{(a_i, b_i)\}_i$, with $a_i + 4b_i > a_{i+1} + 4b_{i+1}$. We interpret $a_{N_0+1} = 1, b_{N_0+1} = 0$. Then:*

$$\begin{aligned}\tilde{H}^G_*(X) \;=\; & (\mathcal{V}^+_{4\lfloor \frac{d_1'+2n_1+1}{4} \rfloor} \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \\ & \oplus \bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}\left(\frac{a_{i+1}+4b_{i+1}-a_i}{4}\right) \oplus \bigoplus_{(a,b)\in\mathcal{J}_0} \mathcal{V}^+_a(b)^{\oplus m(a,b)} \oplus \operatorname{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J \\ & \oplus \bigoplus_{\{i \mid d_i' \equiv 1 \bmod 4\}} \mathcal{V}^+_{d_i'+2}(\lfloor \frac{n_i}{2} \rfloor) \oplus \bigoplus_{\{i \mid d_i' \equiv 3 \bmod 4\}} \mathcal{V}^+_{d_i'}(\lfloor \frac{n_i+1}{2} \rfloor))[-s].\end{aligned} \tag{4.24}$$

*The $q$-action is given by the isomorphism $q : \mathcal{V}^+_2[-s] \to \mathcal{V}^+_1[-s]$ and the map $q : \mathcal{V}^+_1[-s] \to \mathcal{V}^+_{4\lfloor \frac{d_1'+2n_1+1}{4} \rfloor}[-s]$ which is an $\mathbb{F}$-vector space isomorphism in all degrees (in $\mathcal{V}^+_1[-s]$) greater than or equal to $4\lfloor \frac{d_1'+2n_1+1}{4} \rfloor + s + 1$, and vanishes on elements of $\mathcal{V}^+_1[-s]$ of degree less than $4\lfloor \frac{d_1'+2n_1+1}{4} \rfloor + s + 1$.*

*The action of $q$ annihilates $\bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}(\frac{a_{i+1}+4b_{i+1}-a_i}{4})[-s]$, as well as $(\bigoplus_{(a,b)\in\mathcal{J}_0} \mathcal{V}^+_a(b)^{\oplus m(a,b)} \oplus \operatorname{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J)[-s]$.*

*To finish specifying the $q$-action, let $x_i$ be a generator of $\mathcal{V}^+_{d_i'+2}(\lfloor \frac{n_i}{2} \rfloor)[-s]$ for $i$ such that $d_i' \equiv 1 \bmod 4$ (respectively, let $x_i$ be a generator of $\mathcal{V}^+_{d_i'}(\lfloor \frac{n_i+1}{2} \rfloor)[-s]$ if $d_i' \equiv 3 \bmod 4$). Then*

$qx_i$ *is the unique nonzero element of* $(\mathcal{V}^+_{4\lfloor\frac{d_1'+2n_1+1}{4}\rfloor}\oplus\mathcal{V}^+_1\oplus\mathcal{V}^+_2)[-s]$ *in grading* $\deg x_i - 1$*, for all* $i$*.*

*Proof.* We show that for $M$ an $\mathbb{F}[U]$-module of the form (4.20), the sets $\{n_i\}$,$\{d_i'\}$, and the module $J_2$, are determined by the (graded) isomorphism type of $M$, to establish that all the constants in (4.23) are well-defined (independent of the choice of direct sum decomposition of $\tilde{H}^{S^1}_*(X)$). For a fixed $d$, there are at most two distinct isomorphism classes $\mathcal{T}^+_d(x)$, each appearing as summands of $M$ that occur an odd number of times in the decomposition of $M$ into simple submodules (not including the infinite tower). Such a submodule $\mathcal{T}^+_d(x)$ will be called a submodule *occurring with odd multiplicity.* For any $d$ such that there is at least one isomorphism class $\mathcal{T}^+_d(x)$ with odd multiplicity, then $d=s+d_i'$ for some $i$, using (4.20). Consider the case that there are exactly two such isomorphism classes $\mathcal{T}^+_d(x_1)$ and $\mathcal{T}^+_d(x_2)$ with, say, $x_1<x_2$. Setting $d=s+d_i'$ for a fixed $i$, and using (4.20), we see that $x_2=n_i$, since $n_i>n_{i+1}+\frac{d_{i+1}'-d_i'}{2}$ for all $i$. If instead there is one (graded) isomorphism class $T_d(x)$ with odd multiplicity, Lemma 4.1.11 shows $x=n_N$. If, for a fixed $d$, there are no isomorphism classes $\mathcal{T}^+_d(x)$ occurring with odd multiplicity, then $d\notin\{s+d_i'\}$. Thus, we see that $\{d_i\}$ and $\{n_i\}$ are determined by the isomorphism type of $M$ as a graded $\mathbb{F}[U]$-module. It is then easy to see that $J_2$ is also determined by the isomorphism type of $M$.

In addition, we find that $s$ in (4.23) exists and is uniquely determined. First, we check that there is an $s$ so that (4.23) holds. Observe that $\tilde{H}^{S^1}_*(X)=\tilde{H}^{S^1}_*(X')[p+h]$. Say that $X'$ is a space of type SWF at level $m$, and set $d_i'=d_i-m$. Then Lemma 4.1.11 shows that (4.23) holds for this choice of $d_i'$, and $s=m-p-h$. We next show that there is a unique $s$ so that (4.23) holds. To see this, observe that $\tilde{H}^{S^1}_{*,\mathrm{red}}(X)$, as in (4.23), is an $\mathbb{F}$-module of *odd* rank in degrees $d$ such that $d\equiv s+1 \bmod 2$, with $s<d<s+d_1'+2n_1$, and of even rank (possibly zero) in all other degrees (Recall from (2.1) the definition of $\tilde{H}^{S^1}_{*,\mathrm{red}}$). Then, for $M$ an $\mathbb{F}[U]$-module that is the homology of $(X',p,h/4)$ with $X'$ $j$-split, we have that $s=m-p-h$ is determined by $M$.

As in (4.13),

$$\tilde{H}^{S^1}_*(X)=\mathrm{coker} d_{S^1}\oplus\ker d_{S^1}.$$

Additionally, given $M$, we have determined the sets $\{d'_i\}, \{n_i\}$ appearing in Lemma 4.1.10. Then Lemmas 4.1.12 and 4.1.14 show that $J''_1 = \oplus_{(a_i,b_i)\in\mathcal{J}}\mathcal{V}^+_{a_i}(b_i)$, for $a_i, b_i$ as in the statement of the Theorem, and that

$$J''_2 = \mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J \oplus \bigoplus_{\{i|d_i\equiv 1 \bmod 4\}} \mathcal{V}^+_{d_i+2}(\lfloor\frac{n_i}{2}\rfloor) \oplus \bigoplus_{\{i|d_i\equiv 3 \bmod 4\}} \mathcal{V}^+_{d_i}(\lfloor\frac{n_i+1}{2}\rfloor) \oplus \bigoplus_{(a,b)\in\mathcal{J}_0} \mathcal{V}^+_a(b)^{\oplus m(a,b)}. \tag{4.25}$$

Here we have replaced the notation $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[q,v]/(q^3)}$ by $\mathrm{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]}$ since $q$ acts by 0. Finally, Lemma 4.1.15 determines $\tilde{H}^G_*(X)$ given $J''_1$ and $J''_2$. This completes the proof of the Theorem.

□

**Remark 4.1.17.** *Since every $j$-split chain complex of type SWF is the cellular chain complex of some space of type SWF, Theorem 4.1.16 also applies to $j$-split chain complexes.*

We give an example illustrating the steps of the proof of Theorem 4.1.16. Let $X$ be a $j$-split space, and say that $\tilde{H}^{S^1}_*((X,p,h/4))$ is given as in Figure 4.6; that is:

$$\tilde{H}^{S^1}_*((X,p,h/4)) \simeq \mathcal{T}^+_6 \oplus \mathcal{T}^+_{-5}(6) \oplus \mathcal{T}^+_{-5}(5) \oplus \mathcal{T}^+_{-3}(4) \oplus \mathcal{T}^+_{-3}(3) \oplus \mathcal{T}^+_{-1}(2) \oplus \mathcal{T}^+_{-1}(1).$$

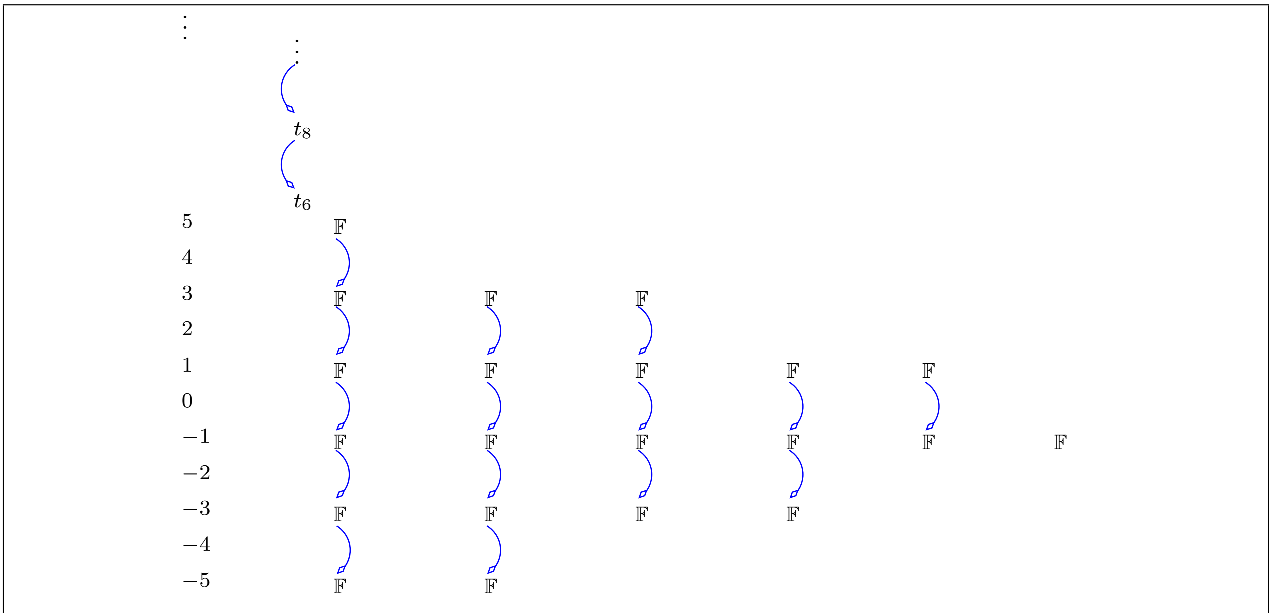


**Figure 4.6:** The $S^1$-Borel Homology of $(X,p,h/4) \in \mathfrak{C}$. The variables $t_i$ stand for entries of the infinite tower in grading $i$.

We calculate $d'_i, n_i$. As specified in the proof of Theorem 4.1.16, we see $\{d'_i + m - p - h\} = \{-5,-3,-1\}$, and $\{n_i\} = \{6,4,2\}$. We see that $m-p-h = 0$ because $\tilde{H}^{S^1}_{-1,\mathrm{red}}((X,p,h/4))$

(i.e. the contribution in degree $-1$ not coming from the tower) is of even rank, while $\tilde{H}^{S^1}_{1,\mathrm{red}}((X,p,h/4))$ has odd rank. So $s=0$ in Theorem 4.1.16. Then $\{d_i'\}=\{-5,-3,-1\}$. Furthermore, we see $J_2=0$. Then we recover $(\tilde{H}^{S^1}_*((X/X^{S^1},p,h/4))\oplus\tilde{H}^{S^1}_*((X^{S^1},p,h/4)),d_{S^1})$, as in Figure 4.7.

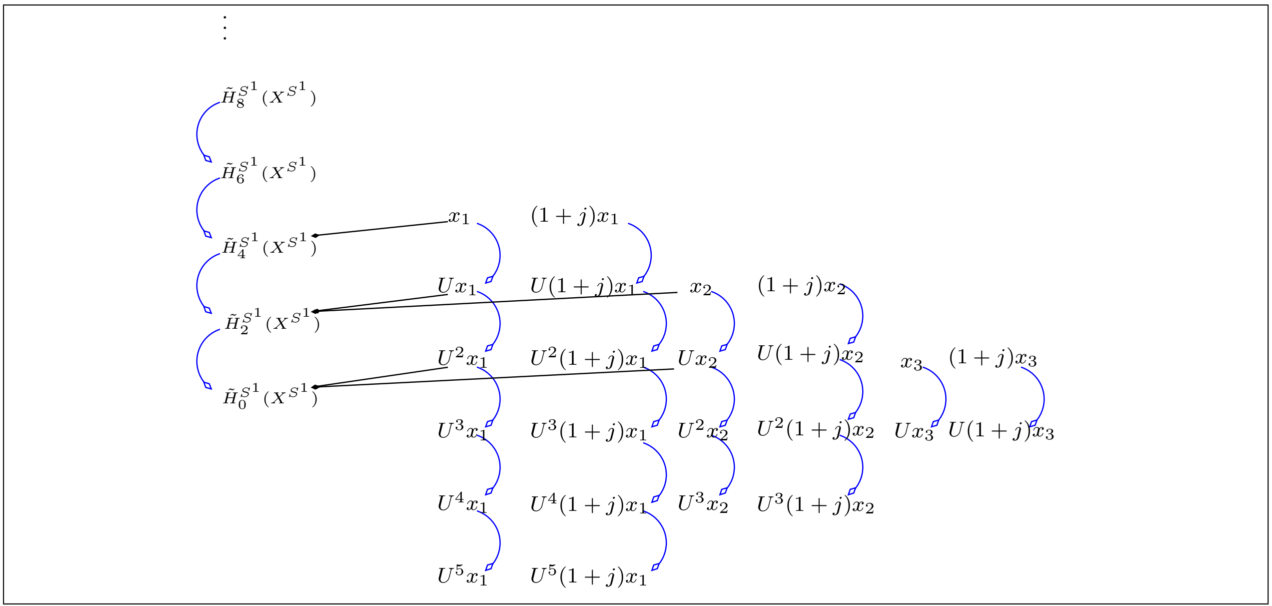


**Figure 4.7:** The complex $(\tilde{H}^{S^1}_*(X/X^{S^1})[p+h]\oplus\tilde{H}^{S^1}_*((X^{S^1},p,h/4)),d_{S^1})$ corresponding to Figure 4.6.

Using Lemma 4.1.12, we have $J_1'=\mathcal{V}^+_{-3}(3)\oplus\mathcal{V}^+_{-3}(2)\oplus\mathcal{V}^+_1(1)$ and $J_2'=\mathcal{V}^+_{-5}(3)\oplus\mathcal{V}^+_{-1}(2)\oplus\mathcal{V}^+_{-1}(1)$, as in Figure 4.8. We see that $\mathcal{V}^+_{-3}(2)$ is not maximal in $J_1'$, so $m(-3,2)=1$, while $m(-3,3)=0$, since $\mathcal{V}^+_{-3}(3)$ is maximal under $\succeq$. Similarly, $\mathcal{V}^+_1(1)$ is maximal, so $m(1,1)=0$. Then $J_{\mathrm{rep}}=\mathcal{V}^+_{-3}(2)$, using (4.21).

In Figure 4.8, $J_1''=\mathcal{V}^+_{-3}(3)\oplus\mathcal{V}^+_1(1)$. Then Lemma 4.1.15 allows us to compute $\tilde{H}^G_*(X)$, as in Figure 4.9.

We find $\tilde{H}^G_*(X)=\mathcal{V}^+_8\oplus\mathcal{V}^+_1\oplus\mathcal{V}^+_2\oplus\mathcal{V}^+_{-5}(3)\oplus\mathcal{V}^+_{-3}(2)^{\oplus 2}\oplus\mathcal{V}^+_{-1}(2)\oplus\mathcal{V}^+_{-1}(1)$, in accordance with Theorem 4.1.16.

### 4.1.2 Chain local equivalence and $j$-split spaces

Using Theorem 4.1.16, we can determine the chain local equivalence class of $j$-split spaces. We start with some results on $j$-split chain complexes. First, write $\mathcal{S}_d(n)$ for the free $\mathcal{G}$-

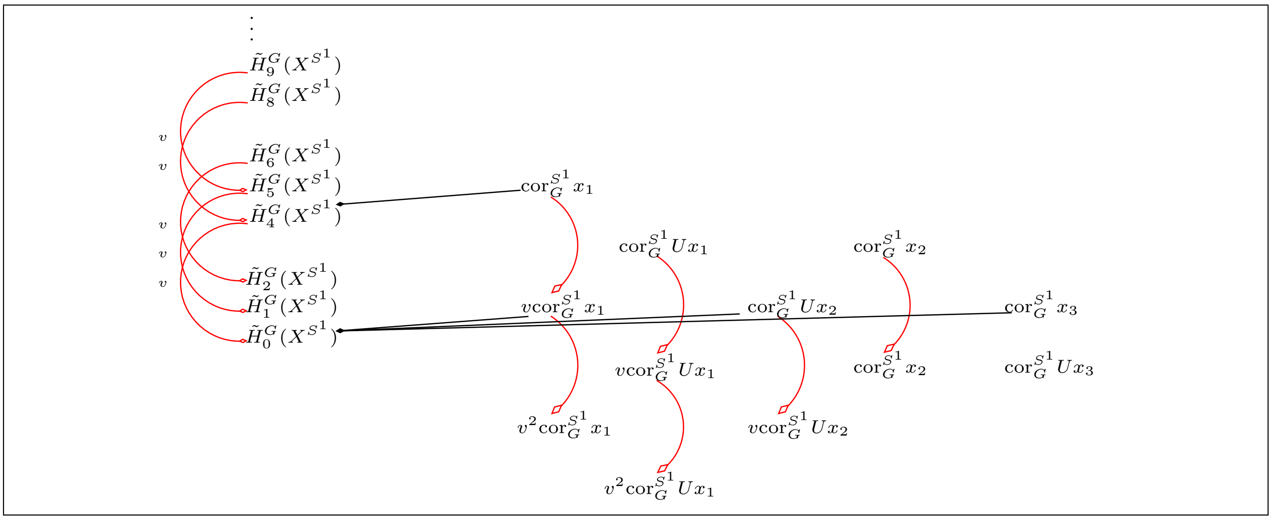


**Figure 4.8:** The complex $(\tilde{H}_*^G(X/X^{S^1})[p+h] \oplus \tilde{H}_*^G((X^{S^1}, p, h/4)), d_G)$ corresponding to Figure 4.6.

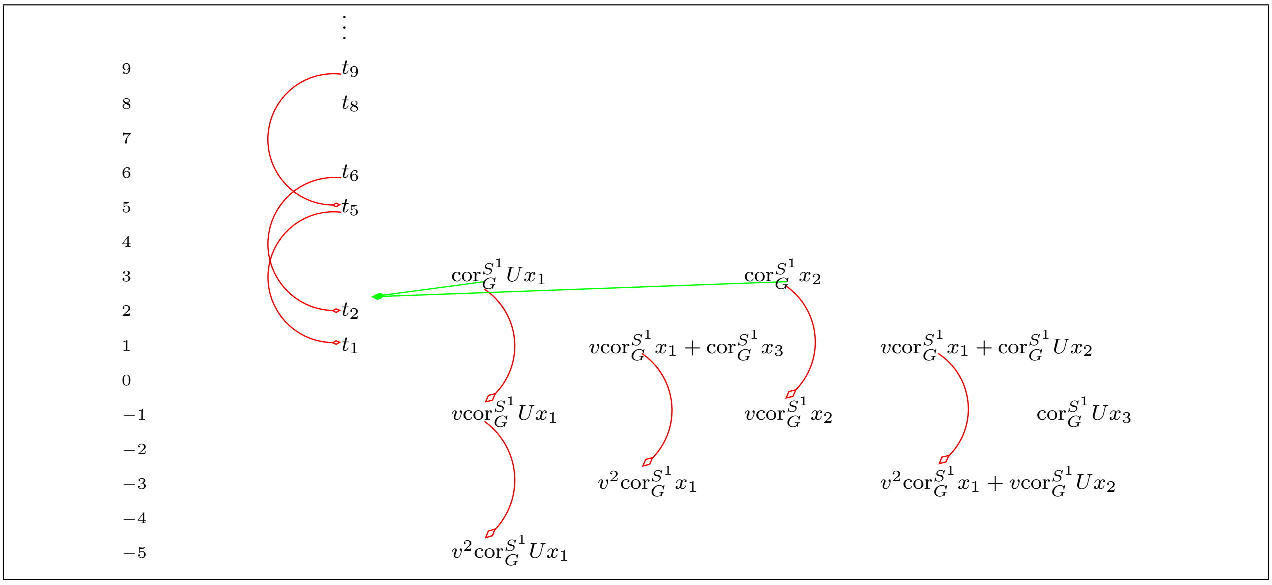


**Figure 4.9:** Finishing the calculation of $\tilde{H}_*^G(X)$ for the example of Figure 4.6. The curved arrows again represent the $v$-action. The straight arrows indicate a nontrivial $q$-action.

module generated by

$$\langle x_d, x_{d+2}, \ldots, x_{d+2n-2} \rangle,$$

with $x_i$ of degree $i$ and $\partial(x_i) = s(1+j^2)x_{i-2}$. A quick computation gives $H_*^{S^1}(\mathcal{S}_d(n)) = \mathcal{T}_d^+(n)^{\oplus 2}$ as $\mathbb{F}[U]$-modules, where $H_*^{S^1}(Z)$ is defined as in (2.16). Moreover, for an $\mathbb{F}[U]$-module $J = \bigoplus_i \mathcal{T}_{e_i}^+(m_i)$, let $S(J) = \bigoplus_i \mathcal{S}_{e_i}(n_i)$.

**Proposition 4.1.18.** *Let $C = \langle f_{\mathrm{red}}\rangle \tilde{\oplus}(C_+ \oplus C_-)$ be a $j$-split chain complex and*

$$H_*^{S^1}(C) = \mathcal{T}^+_{d_1+2n_1-1} \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2}) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{d_i}(n_i) \oplus J^{\oplus 2}, \tag{4.26}$$

*where $d_{i+1} > d_i$ and $2n_i + d_i > 2n_{i+1} + d_{i+1}$, $2n_N + d_N \geqslant 3$, and $d_N \leqslant 1$. We interpret $d_{N+1} = 1$, $n_{N+1} = 0$. Then $C$ is homotopy equivalent to the chain complex*

$$(\langle f_{\mathrm{red}}\rangle \tilde{\oplus}(\bigoplus_i \mathcal{S}_{d_i}(n_i))) \oplus S(J), \tag{4.27}$$

*where $\partial(f_{\mathrm{red}}) = 0$, $jf_{\mathrm{red}} = f_{\mathrm{red}}$, $sf_{\mathrm{red}} = 0$, and $\deg(f_{\mathrm{red}}) = 0$. Furthermore, let each factor $\mathcal{S}_{d_i}(n_i)$ have generators $x^i_j$, with $\deg x^i_j = j$. Then $\partial x^i_1 = f_{\mathrm{red}} + s(1+j^2)x^i_{-1}$ for all $i$.*

**Remark 4.1.19.** *By Lemma 4.1.10, for $C$ any $j$-split chain complex, a decomposition as in (4.26) is possible.*

Before giving the proof we establish a Lemma.

**Lemma 4.1.20.** *Let $F_1, F_2$ be two free, finite $C_*^{CW}(S^1)$-complexes such that $H_*^{S^1}(F_1) \cong H_*^{S^1}(F_2)$ as $\mathbb{F}[U]$-modules. Then $F_1 \simeq F_2$, where $\simeq$ denotes homotopy equivalence.*

*Proof.* First, we note that $C_*^{CW}(S^1)$ is chain homotopy equivalent to the algebra $\mathbb{F}[\bar{s}]/(\bar{s}^2)$ where $\deg(\bar{s}) = 1$ and $\partial(\bar{s}) = 0$. Koszul Duality [17] states that $F_1$ and $F_2$ are quasi-isomorphic as $\mathbb{F}[\bar{s}]/(\bar{s}^2)$ modules if and only if $H_*^{S^1}(F_1)$ and $H_*^{S^1}(F_2)$ are isomorphic as $\mathbb{F}[U]$-modules. Indeed, our original hypothesis was $H_*^{S^1}(F_1) \simeq H_*^{S^1}(F_2)$, so we see that $F_1$ and $F_2$ are quasi-isomorphic. Finally, by Theorem 10.4.8 of [53], quasi-isomorphic free chain complexes are chain homotopy equivalent, and so $F_1$ and $F_2$ are chain homotopy equivalent. This establishes the Lemma. □

*Proof of Proposition 4.1.18.* The proof is in two steps: first, we show that $C_+$ is chain homotopy equivalent to a chain complex of a certain form, and then we investigate differentials from $C_+$ to $\langle f_{\mathrm{red}}\rangle$.

Note that the complex $C_+$ is a $C_*^{CW}(S^1)$-complex. Let $\mathcal{S}_d^{S^1}(n)$ be the $C_*^{CW}(S^1)$-submodule of $\mathcal{S}_d(n)$ generated (as a $C_*^{CW}(S^1)$-module) by $\langle x_d, x_{d+2}, ..., x_{d+2n-2}\rangle$. As for $\mathcal{S}_d(n)$, a quick calculation shows $H_*^{S^1}(\mathcal{S}_d^{S^1}(n)) = \mathcal{T}_d^+(n)$. Similarly, for an $\mathbb{F}[U]$-module $J = \bigoplus_i \mathcal{T}_{e_i}^+(m_i)$, let $S^{S^1}(J) = \bigoplus_i \mathcal{S}_{e_i}^{S^1}(n_i)$. We see:

$$S(J) \cong S^{S^1}(J) \oplus S^{S^1}(J), \tag{4.28}$$

as $\mathcal{G}$-complexes, for all $\mathbb{F}[U]$-modules $J$, where the action of $j$ on the right is given by interchanging the factors.

Recall, by the proof of Theorem 4.1.16, that $H_*^{S^1}(C_+ \oplus C_-)$ is determined by $H_*^{S^1}(C)$ for $C$ a $j$-split chain complex (see Remark 4.1.17). That is, from (4.26):

$$H_*^{S^1}(C_+) = \bigoplus_{i=1}^{N} \mathcal{T}_{d_i}^+(n_i) \oplus J.$$

Lemma 4.1.20 then implies $C_+ = \mathcal{S}_d^{S^1}(n) \oplus S^{S^1}(J)$ as a $C_*^{CW}(S^1)$-complex. Since $j : C_+ \to C_-$ is an isomorphism, we have from (4.28):

$$C_+ \oplus C_- \cong \bigoplus_i \mathcal{S}_{d_i}(n_i) \oplus S(J). \tag{4.29}$$

Moreover, $H_*^{S^1}(C)$ determines the map $d_{S^1} : H_*^{S^1}(C_+) \to H_*^{S^1}(\langle f_{\mathrm{red}} \rangle)$. We compute $d_{S^1}$ a different way, by using the differential from $C_+$ to $\langle f_{\mathrm{red}} \rangle$, and the form of $C_+$ determined by (4.29). Fix a pair of integers $(d, n)$. If $x_i$ is the generator of a copy of $\mathcal{S}_d(n)$ in degree $i$ and $x_i \in C_+$, then $d_{S^1} : H_*^{S^1}(\mathcal{S}_d(n)) \cong \mathcal{T}_d^+(n) \to \mathcal{T}^+$ is nontrivial if and only if $\partial(x_1) = f_{\mathrm{red}} + s(1 + j^2)x_{-1}$. Thus, since $d_{S^1}$ is nonvanishing on the factors $\mathcal{T}_{d_i}^+(n_i) \subset H_*^{S^1}(C_+)$ and vanishing elsewhere, each generator $x_1^i$, with $\deg x_1^i = 1$ of $\mathcal{S}_{d_i}(n_i)$ in (4.29) must have $\partial(x_1^i) = f_{\mathrm{red}} + s(1+j^2)x_{-1}^i$, and all other differentials $C_+ \to \langle f_{\mathrm{red}} \rangle$ vanish. Thus, in particular, $\partial(S(J)) \subset S(J)$. The decomposition (4.27) follows. $\square$

**Proposition 4.1.21.** *Let $(X, p, h/4) \in \mathfrak{E}$ with $X$ a $j$-split space of type SWF at level $m$, and*

$$\tilde{H}_*^{S^1}((X, p, h/4)) = \mathcal{T}_{s+d_1+2n_1+1}^+ \oplus \bigoplus_{i=1}^{N} \mathcal{T}_{s+d_i}^+\left(\frac{d_{i+1} + 2n_{i+1} - d_i}{2}\right) \oplus \bigoplus_{i=1}^{N} \mathcal{T}_{s+d_i}^+(n_i) \oplus J^{\oplus 2}[-s], \tag{4.30}$$

*where $d_{i+1} > d_i$ and $2n_i + d_i > 2n_{i+1} + d_{i+1}$, as well as $2n_N + d_N \geqslant 3$, and $d_N \leqslant 1$. Then the chain local equivalence type $[(C_*^{CW}(X, \mathrm{pt}), p, h/4)]_{cl} \in \mathfrak{CLE}$ is the equivalence class of*

$$C(p - m, h/4, \{d_i\}_i, \{n_i\}_i) := ((\langle f_{\mathrm{red}} \rangle \tilde{\oplus} (\bigoplus_i \mathcal{S}_{d_i}(n_i))), p - m, h/4) \in \mathfrak{CLE}. \tag{4.31}$$

*The connected $S^1$-homology of $(X, p, h/4)$ is given by:*

$$H^{S^1}_{\text{conn}}((X, p, h/4)) = \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(\frac{d_{i+1} + 2n_{i+1} - d_i}{2}) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(n_i). \tag{4.32}$$

*Further, $s$ in (4.30) is $m - p - h$. Moreover, $C(p, h/4, \{d_i\}, \{n_i\})$ is chain locally equivalent to $C(p', h'/4, \{d'_i\}, \{n'_i\})$ if and only if $p = p'$, $h = h'$, $\{d_i\} = \{d'_i\}$, and $\{n_i\} = \{n'_i\}$.*

*Proof.* Write $[(A, b, c)]_{cl}$ for the chain local equivalence class of $(A, b, c) \in \mathfrak{CE}$. Let

$$[(Z, -m, 0)] = [C^{CW}_*(X, \text{pt})] \in \mathfrak{CE}$$

where $Z$ is a $j$-split chain complex, as allowed by Lemma 4.1.2. Using Proposition 4.1.18, we see:

$$[(Z, p, h/4)]_{cl} = ((\langle f_{\text{red}}\rangle \tilde{\oplus} \bigoplus \mathcal{S}_{d_i}(n_i)), p, h/4).$$

We have then:

$$C(p - m, h/4, \{d_i\}, \{n_i\}) = [(Z, p - m, h/4)]_{cl} = [(C^{CW}_*(X, \text{pt}), p, h/4)]_{cl},$$

as in (4.31).

To prove (4.32) we consider the complex $\Sigma^{\mathbb{H}^{\lfloor \frac{-d_1+3}{4} \rfloor}} C(0, 0, \{d_i\}, \{n_i\})[4\lfloor \frac{-d_1+3}{4} \rfloor]$ (we include the grading shift for convenience). We will see that it is a suspensionlike complex, so we may apply the results of Section 2.1.5. There is a homotopy equivalence:

$$\Sigma^{\mathbb{H}^{\lfloor \frac{-d_1+3}{4} \rfloor}} C(0, 0, \{d_i\}, \{n_i\})[4\lfloor \frac{-d_1+3}{4} \rfloor] \simeq \langle f_{\text{red}}\rangle \tilde{\oplus} \bigoplus_k \langle y_k \rangle \tilde{\oplus} \bigoplus_{i=1}^{N} \bigoplus_{\{k \equiv 1 \bmod 2,\ d_i \leqslant k \leqslant d_i + 2n_i - 2\}} \langle z^i_k \rangle, \tag{4.33}$$

where

$$\langle f_{\text{red}}\rangle \tilde{\oplus} \bigoplus_k \langle y_k \rangle \simeq \Sigma^{\mathbb{H}^{\lfloor \frac{-d_1+3}{4} \rfloor}} \langle f_{\text{red}} \rangle, \tag{4.34}$$

and $\deg z^i_k = \deg y_k = k$. Additionally, $\partial(z^i_k) = s(1 + j^2) z^i_{k-2}$ if $k \neq 1$, and $\partial(z^i_1) = s(1+j^2) z^i_{-1} + s(1+j)^3 y_{-1}$. The $y_k$ are defined for $k$ such that $k \not\equiv 3 \bmod 4$ and $-4\lfloor \frac{-d_1+3}{4} \rfloor + 1 \leqslant$

$k \leqslant -1$. Also,

$$\begin{aligned}
\partial(y_{4k}) &= s(1+j)^3 y_{4k-2}, && (4.35)\\
\partial(y_{4k+1}) &= (1+j)y_{4k}, \quad k \neq -\lfloor\frac{-d_1+3}{4}\rfloor, && (4.36)\\
\partial(y_{4k+2}) &= (1+j)y_{4k+1} + sy_{4k}, && (4.37)\\
\partial(y_{-4\lfloor\frac{-d_1+3}{4}\rfloor+1}) &= f_{\mathrm{red}}. && (4.38)
\end{aligned}$$

According to (4.34), the first two terms on the right of (4.33) account for the suspension of the reducible tower, and the $z_k^i$ correspond to the suspension of the free part. The $z_k^i$ are suspensions of $x_k^i \in \mathcal{S}_{d_i}(n_i) \subset C(0,0,\{d_i\},\{n_i\})$. From this presentation, it is clear that the chain complex $\Sigma^{\mathbb{H}\lfloor\frac{-d_1+3}{4}\rfloor} C(0,0,\{d_i\},\{n_i\})[4\lfloor\frac{-d_1+3}{4}\rfloor]$ is irreducible (that is, it may not be written as a non-trivial direct sum of $\mathcal{G}$-chain complexes). Then by Lemma 2.1.37 and Definition 2.1.38,

$$(\Sigma^{\mathbb{H}\lfloor\frac{-d_1+3}{4}\rfloor} C(0,0,\{d_i\},\{n_i\})[4\lfloor\frac{-d_1+3}{4}\rfloor])_{\mathrm{conn}} = \Sigma^{\mathbb{H}\lfloor\frac{-d_1+3}{4}\rfloor} C(0,0,\{d_i\},\{n_i\})[4\lfloor\frac{-d_1+3}{4}\rfloor]. \tag{4.39}$$

Then (4.32) follows from the definition of $H^{S^1}_{\mathrm{conn}}$, applied to $C(0,0,\{d_i\},\{n_i\})$. The calculation of $H^{S^1}_{\mathrm{conn}}((X,p,h/4))$ for nonzero $m,p,h$ follows, since

$$C(p-m,h/4,\{d_i\},\{n_i\}) = \Sigma^{(m-p)\tilde{\mathbb{R}}}\Sigma^{-\frac{h}{4}\mathbb{H}} C(0,0,\{d_i\},\{n_i\}).$$

The assertion that $s = m-p-h$ follows from the homology calculation of Theorem 4.1.16.

Recall that $H^{S^1}_{\mathrm{conn}}$ is a chain local equivalence invariant. Hence, if $[C(p,h/4,\{d_i\},\{n_i\})]_{cl} = [C(p',h'/4,\{d_i'\},\{n_i'\})]_{cl}$, we see from (4.32) that $\{d_i\} = \{d_i'\}$, $\{n_i\} = \{n_i'\}$, and $p+h = p'+h'$. Furthermore, if $C(p,h/4,\{d_i\},\{n_i\})$ and $C(p',h'/4,\{d_i'\},\{n_i'\})$ are chain locally equivalent, they must have chain homotopy equivalent fixed-point sets. That is, $p = p'$ and so also $h = h'$, completing the proof. □

## 4.2 Floer spectra of Seifert fiber spaces

### 4.2.1 The Seiberg-Witten equations on Seifert spaces

In this section we record some results of [33] to describe explicitly the monopole moduli space on Seifert fiber spaces. First we recall some notation associated with Seifert fiber spaces.

The standard fibered torus corresponding to a pair of integers $(a, b)$, for $a > 0$, is the mapping torus of the automorphism of the disk $D^2$ given by rotation by $2\pi b/a$. Let $D^2_a$ be the standard disk, given an orbifold structure by letting $\mathbb{Z}/a$ act by rotation by $2\pi/a$; the origin is then an orbifold point, with multiplicity $a$. The standard fibered torus is naturally a circle bundle over the orbifold $D^2_a$.

Let $f : Y \to P$ be a circle bundle over an orbifold $P$, and $x \in P$ an orbifold point with multiplicity $a$. If a neighborhood of the fiber over $x$ is equivalent, as an orbifold circle bundle, to the standard fibered torus corresponding to $(a, b)$, we say that $Y$ has *local invariant* $b$ at $x$.

For $a_i \in \mathbb{Z}_{\geqslant 1}$, let $S(a_1, \ldots, a_k)$ denote the orbifold with underlying space $S^2$ and $k$ orbifold points, with corresponding multiplicities $a_1, \ldots, a_k$. Fix $b_i \in \mathbb{Z}$ with $\gcd(a_i, b_i) = 1$ for all $i$. We let $\Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ denote the circle bundle over $S(a_1, \ldots, a_k)$ with first Chern class $b$ and local invariants $b_i$. We define the *degree* of the Seifert space $\Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ by $b + \sum \frac{b_i}{a_i}$. Finally, we call a space $\Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ negative (positive) if $b + \sum \frac{b_i}{a_i}$ is negative (positive). The spaces $\Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ of nonzero degree are rational homology spheres. As orbifold circle bundles, the orientation reversal $-\Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ is isomorphic to $\Sigma(-b, (-b_1, a_1), \ldots, (-b_k, a_k))$. We write $\Sigma(a_1, \ldots, a_k)$ for the unique negative Seifert integral homology sphere fibering over $S^2(a_1, \ldots, a_k)$.

Let $Y$ be a negative Seifert rational homology three-sphere fibering over a base orbifold $P$ with underlying space $S^2$. Equipping $Y$ with the metric for which $Y$ has the Seifert geometry, Mrowka, Ozsváth, and Yu [33] show that the Seiberg-Witten moduli space $\mathcal{M}(Y)$ is composed of the following:

- A finite set of points forming the reducible critical set, in bijection with $Hom(H_1(Y), S^1)$, and
- for each $(k+1)$-tuple of non-negative integers $\mathbf{e} = (e, \epsilon_1, ..., \epsilon_k)$, such that $0 \leqslant \epsilon_i < a_i$ and

$$e + \sum_{i=1}^{k} \frac{\epsilon_i}{a_i} \leqslant (\frac{k}{2} - 1) - \sum_{i=1}^{k} \frac{1}{2a_i},$$

there are two components, labelled $C^+(\mathbf{e})$ and $C^-(\mathbf{e})$, in $\mathcal{M}(Y)$.

Each component $C^+(\mathbf{e}), C^-(\mathbf{e})$ is a copy of $Sym^e(|\Sigma|)$, where $\Sigma$ is the base orbifold and $|\Sigma|$ its underlying manifold. Furthermore, $C^+(\mathbf{e})$ and $C^-(\mathbf{e})$ are related by the action of $j \in \mathrm{Pin}(2)$. That is, the restriction of $j$ to $C^+(\mathbf{e})$ acts as a diffeomorphism $C^+(\mathbf{e}) \to C^-(\mathbf{e})$, and vice versa. Then, in the quotient of the configuration space by the based gauge group, each $C^{\pm}(\mathbf{e})$ is diffeomorphic to $G \times Sym^e(|\Sigma|)$.

**Fact 4.2.1.** *All reducible critical points $x$ have $\mathcal{L}(x) = 0$, where $\mathcal{L}$ is the Chern-Simons-Dirac functional. All irreducible critical points have $\mathcal{L} > 0$.*

Mrowka, Ozsváth, and Yu do not use the Seiberg-Witten equations as in [23]. Instead, they replace the Dirac operator $\hat{D}$ associated to the Seifert metric in the equations with $D = \hat{D} - \frac{1}{2}\xi$ for $\xi$ some constant depending on the Seifert fibration. It is then clear that the Seiberg-Witten equations they consider differ from the usual equations by a tame perturbation $\mathfrak{q}_0$ in the sense of [23]. Abusing notation somewhat, we call the Seiberg-Witten equations as in [33] simply the Seiberg-Witten equations, or the *unperturbed* Seiberg-Witten equations in the sequel.

In the case of a negative Seifert space $Y$ with four or fewer singular fibers, the Seiberg-Witten equations are transverse in the sense of [23], so we may take $\mathfrak{q} = \mathfrak{q}_0$, as in [33].

We will further need:

**Fact 4.2.2.** *There are no trajectories between $C^+(\boldsymbol{e})$ and $C^-(\boldsymbol{f})$ for any $\boldsymbol{e}, \boldsymbol{f}$. The Seiberg-Witten equations on $Y$ is Morse-Bott, and if $Y$ has four or fewer singular fibers, the perturbation $\mathfrak{q} = \mathfrak{q}_0$ is* admissible *in the sense of Definition 22.1.1 of [23].*

Combining Propositions 3.1.6, 3.1.7, and Fact 4.2.2, we have:

**Lemma 4.2.3.** *Let $Y = \Sigma(b,(b_1,a_1),\ldots,(b_k,a_k))$ be a negative Seifert rational homology three-sphere. Then $SWF(Y,\mathfrak{s})$ has a representative $(X,m,n) \in \mathfrak{C}$ with $X$ a $j$-split space.*

*Proof.* We first treat the case where $Y$ has at most four singular fibers. Then the irreducibles are isolated, by Fact 4.2.2.

We recall the attractor-repeller sequence (3.4), which shows that $SWF(Y,\mathfrak{s})$ is obtained by successively attaching stable cells $G \times D^{\operatorname{ind} C^+(\mathbf{e})}$, corresponding to the irreducible critical point $C^+(\mathbf{e})$, to the reducible cell. Let $I_{\leqslant \omega}$ be the complex obtained by attaching all critical points with $\mathcal{L} \leqslant \omega$. We show by induction that $I_{\leqslant\omega}$ is $j$-split for all $\omega$. For $\omega = 0$, the only critical point is the reducible by Fact 4.2.1, so the statement is vacuous. Let

$$I_{\leqslant\omega_0}/I^{S^1}_{\leqslant\omega_0} = I^+_{\leqslant\omega_0} \vee jI^+_{\leqslant\omega_0}, \tag{4.40}$$

for some fixed $\omega_0$, where $I^+_{\leqslant\omega_0}$ contains all irreducible critical points $C^+(\mathbf{e})$ with $\mathcal{L} \leqslant \omega_0$. Fix $\mathbf{e}_1$ so that $\mathcal{L}(C^+(\mathbf{e}_1)) > \omega_0$ and $\mathcal{L}(C^+(\mathbf{e}_1))$ is minimal among $\mathcal{L}(x)$ for critical points $x$ with $\mathcal{L}(x) > \omega_0$. By Fact 4.2.2, and Proposition 3.1.8, $M_\lambda(x_\lambda, y_\lambda) = 0$, where $x_\lambda$ corresponds to $C^+(\mathbf{e}_1)$, and $y_\lambda$ corresponds to any critical point of $C^-(\mathbf{f})$. Additionally, the Conley Index satisfies:

$$I_{\leqslant\mathcal{L}(C^+(\mathbf{e}_1)\cup jC^+(\mathbf{e}_1))}/I_{\leqslant\omega_0} = G \times D^{\operatorname{ind} C^+(\mathbf{e}_1)} = S^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)} \vee jS^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)},$$

as $S^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)}$ and $jS^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)}$ are disjoint isolated invariant sets. Since $M_\lambda(x_\lambda,y_\lambda) = 0$ for all $y_\lambda \in jI^+_{\leqslant\omega_0}$ we have that the attaching map of the cell $S^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)}$ has target only in $I^+_{\leqslant\omega_0} \cup I^{S^1}_{\leqslant\omega_0}$; then we set

$$I^+_{\leqslant\mathcal{L}(C^+(\mathbf{e}_1))} = I^+_{\leqslant\omega_0} \cup (S^1 \times D^{\operatorname{ind} C^+(\mathbf{e}_1)}),$$

so that the analogue of the splitting (4.40) holds:

$$I_{\leqslant\mathcal{L}(C^+(\mathbf{e}_1)\cup C^-(\mathbf{e}_1))}/I^{S^1} = I^+_{\leqslant\mathcal{L}(C^+(\mathbf{e}_1)\cup C^-(\mathbf{e}_1))} \vee jI^+_{\leqslant\mathcal{L}(C^+(\mathbf{e}_1)\cup C^-(\mathbf{e}_1))}, \tag{4.41}$$

completing the induction.

In the case of five or more singular fibers, we perturb the Seiberg-Witten equations to be nondegenerate. We can arrange that for a small perturbation $\mathfrak{q}$ the analogue of Fact 4.2.2 continues to hold. That is, there exists some tame admissible perturbation $\mathfrak{q}$ such that the set of irreducible critical points of $\mathcal{X}_{\mathfrak{q}}$ may be partitioned into two sets $C^+$ and $C^-$, interchanged by the action of $j$, so that for all $x \in C^+, y \in C^-$, we have $M(x,y) = \varnothing$.

We show the existence of such a $j$-equivariant perturbation $\mathfrak{q}$. Choose a sequence of small $j$-equivariant tame admissible perturbations $\mathfrak{q}_i$, converging to 0 in $C^\infty$, so that for each $i$ the perturbed Seiberg-Witten equations have non-degenerate irreducible critical points. Lin establishes the existence of such perturbations in [26]. Choose disjoint neighbourhoods $\mathcal{U}^{\pm}(\mathbf{e})$ of $C^{\pm}(\mathbf{e})$ such that for $i$ sufficiently large all irreducible critical points of $\mathcal{L}_{\mathfrak{q}_i}$ lie in

$$\bigcup_{\mathbf{e}}(\mathcal{U}^+(\mathbf{e}) \cup \mathcal{U}^-(\mathbf{e})).$$

Let $C_i^+$ denote the set of irreducible critical points of $\mathcal{L}_{\mathfrak{q}_i}$ in $\cup_{\mathbf{e}}\mathcal{U}^+(\mathbf{e})$ and let $C_i^-$ denote the set of irreducible critical points of $\mathcal{L}_{\mathfrak{q}_i}$ in $\cup_{\mathbf{e}}\mathcal{U}^-(\mathbf{e})$. Let $C^{\pm}$ denote the union $\cup_{\mathbf{e}}C^{\pm}(\mathbf{e})$.

Say, to obtain a contradiction, that for all $i$ there exists some pair of critical points $x_i \in C_i^+$, $y_i \in C_i^-$, such that $M(x_i, y_i)$ is nonempty. The sequences $x_i, y_i$ have limit points $x \in C^+(\mathbf{e})$ and $y \in C^-(\mathbf{f})$, by Proposition 11.6.4 of [23]. Theorem 16.1.3 of [23] shows that the moduli space of unparameterized broken trajectories (for a fixed perturbation) is compact. The proof of Theorem 16.1.3 can be applied to a sequence of trajectories $\breve{\gamma}_i$ for perturbations $\mathfrak{q}_i$ with $\mathfrak{q}_i \to \mathfrak{q}$. That is, the sequence $\breve{\gamma}_i$ has a limit point a broken trajectory $(\breve{\tau}_1, ..., \breve{\tau}_n)$ from $x$ to $y$, for the perturbation $\mathfrak{q}$. Since $x \in C^+, y \in C^-$, there exists a trajectory $\breve{\tau}_k$ from $C^+$ to $C^-$, or there exists a trajectory $\breve{\tau}_k$ from $C^+$ to the reducible and a trajectory $\breve{\tau}_l$ from the reducible to $C^-$. The first case contradicts Fact 4.2.2. The second case contradicts the minimality of $\mathcal{L}$ on the reducible (Fact 4.2.1). Thus, for some perturbation $\mathfrak{q}$ as above we have the desired partition.

The Lemma then follows as in the case of three or four singular fibers. □

By Lemma 4.2.3, Theorem 4.1.16 applies to $SWF(Y, \mathfrak{s})$ for $Y$ a Seifert rational homology sphere, and we obtain the following corollary, from which Theorems 1.2.1 and 1.2.4 of the Introduction follow.

**Corollary 4.2.4.** *Let $Y = \Sigma(b, (\beta_1, \alpha_1), \ldots, (\beta_k, \alpha_k))$ be a negative Seifert rational homology sphere with a choice of spin structure $\mathfrak{s}$. Then*

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+_{s+d_1+2n_1-1} \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(\frac{d_{i+1} + 2n_{i+1} - d_i}{2}) \oplus \bigoplus_{i=1}^{N} \mathcal{T}^+_{s+d_i}(n_i) \oplus J^{\oplus 2}[-s], \quad (4.42)$$

*for some constants $s, d_i, n_i, N$ and some $\mathbb{F}[U]$-module $J$, all determined by $(Y, \mathfrak{s})$. Furthermore, $2n_i + d_i > 2n_{i+1} + d_{i+1}$ for all $i$, $2n_N + d_N \geqslant 3$, $d_N \leqslant 1$, and $d_{N+1} = 1$, $n_{N+1} = 0$. Let $\mathcal{J}_0 = \{(a_k, b_k)\}_k$ be the collection of pairs consisting of all $(d_i, \lfloor \frac{n_i+1}{2} \rfloor)$ for $d_i \equiv 1 \bmod 4$ and all $(d_i + 2, \lfloor \frac{n_i}{2} \rfloor)$ for $d_i \equiv 3 \bmod 4$, counting multiplicity. Let $(a, b) \succeq (c, d)$ if $a + 4b \geqslant c + 4d$ and $a \geqslant c$, and let $\mathcal{J}$ be the subset of $\mathcal{J}_0$ consisting of pairs maximal under $\succeq$ (not counted with multiplicity). If $(a, b) \in \mathcal{J}$, set $m(a, b) + 1$ to be the multiplicity of $(a, b)$ in $\mathcal{J}_0$. If $(a, b) \notin \mathcal{J}$, set $m(a, b)$ to be the multiplicity of $(a, b)$ in $\mathcal{J}_0$. Let $|\mathcal{J}| = N_0$ and order the elements of $\mathcal{J}$ so that $\mathcal{J} = \{(a_i, b_i)\}_i$, with $a_i + 4b_i > a_{i+1} + 4b_{i+1}$. Then:*

$$\begin{aligned} SWFH^G_*(Y, \mathfrak{s}) \quad = \quad & (\mathcal{V}^+_{4\lfloor \frac{d_1+2n_1+1}{4} \rfloor} \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \\ & \oplus \bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}(\frac{a_{i+1} + 4b_{i+1} - a_i}{4}) \oplus \bigoplus_{(a,b) \in \mathcal{J}_0} \mathcal{V}^+_a(b)^{\oplus m(a,b)} \oplus \operatorname{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J \\ & \oplus \bigoplus_{\{i | d_i \equiv 1 \bmod 4\}} \mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor) \oplus \bigoplus_{\{i | d_i \equiv 3 \bmod 4\}} \mathcal{V}^+_{d_i}(\lfloor \frac{n_i + 1}{2} \rfloor))[-s]. \end{aligned} \quad (4.43)$$

*The $q$-action is given by the isomorphism $\mathcal{V}^+_2[-s] \to \mathcal{V}^+_1[-s]$ and the map $\mathcal{V}^+_1[-s] \to \mathcal{V}^+_{4\lfloor \frac{d_1+2n_1+1}{4} \rfloor}[-s]$ which is an $\mathbb{F}$-vector space isomorphism in all degrees (in $\mathcal{V}^+_1[-s]$) greater than or equal to $4\lfloor \frac{d_1+2n_1+1}{4} \rfloor + s + 1$, and vanishes on elements of $\mathcal{V}^+_1[-s]$ of degree less than $4\lfloor \frac{d_1+2n_1+1}{4} \rfloor + s + 1$. We interpret $a_{N_0+1} = 1, b_{N_0+1} = 0$.*

*The action of $q$ annihilates*

$$\bigoplus_{i=1}^{N_0} \mathcal{V}^+_{a_i}(\frac{a_{i+1} + 4b_{i+1} - a_i}{4})[-s] \text{ and } (\bigoplus_{(a,b) \in \mathcal{J}_0} \mathcal{V}^+_a(b)^{\oplus m(a,b)} \oplus \operatorname{res}^{\mathbb{F}[U]}_{\mathbb{F}[v]} J)[-s].$$

*To finish specifying the $q$-action, let $x_i$ be a generator of $\mathcal{V}^+_{d_i+2}(\lfloor \frac{n_i}{2} \rfloor)[-s]$ for $i$ such that $d_i \equiv 1 \bmod 4$ (respectively, let $x_i$ be a generator of $\mathcal{V}^+_{d_i}(\lfloor \frac{n_i+1}{2} \rfloor)[-s]$ if $d_i \equiv 3 \bmod 4$). Then $qx_i$ is the unique nonzero element of $(\mathcal{V}^+_{4\lfloor \frac{d_1+2n_1+1}{4} \rfloor} \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2)[-s]$ in grading $\deg x_i - 1$, for all $i$.*

Theorem 1.2.4 follows by setting $N = 1$ and $d_1 = 1$; these conditions imply that $d_2 + 2n_2 - d_1 = 0$, and so the term $\bigoplus_{i=1}^N \mathcal{T}^+_{s+d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2})$ in (4.42) is the zero module in this case.

The constant $s$ is the grading of the reducible critical point, where the metric on $Y$ is that associated to the Seifert geometry on $Y$.

*Proof.* Let $(X', p, h/4)$ be a $j$-split representative for $SWF(Y, \mathfrak{s})$ at level $m$, and let $s = m - p - h$. We may choose such a representative for $SWF(Y, \mathfrak{s})$ by Lemma 4.2.3. Then, using Lemma 4.1.11, we have:

$$
\begin{aligned}
SWFH^{S^1}_*(Y, \mathfrak{s}) &= \tilde{H}^{S^1}_*(X')[-p-h] && (4.44)\\
&= (\bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2}) \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{d_i}(n_i) \oplus J_2^{\oplus 2} \oplus \mathcal{T}^+_{d_1+2n_1-1})[-s]. && (4.45)
\end{aligned}
$$

Applying the equivalence of $\widetilde{HM}$ and $SWFH^{S^1}$ of [25], and the equivalence of $\widetilde{HM}$ and $HF^+$ of [4] and [24], we obtain the expression (4.42). Then we apply Theorem 4.1.16 to obtain the calculation of $SWFH^G_*$ of the corollary. □

Further, using the results of Section 4.1.2, we prove the results of the Introduction on homology cobordisms of Seifert spaces. Corollaries 1.2.6 and 1.2.7 of the Introduction follow from Proposition 4.2.5 below.

**Proposition 4.2.5.** *Let $Y = \Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ be a negative Seifert rational homology three-sphere with a choice of spin structure $\mathfrak{s}$, and*

$$
HF^+(Y, \mathfrak{s}) = \mathcal{T}^+_{s+d_1+1} \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{s+d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2}) \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{s+d_i}(n_i) \oplus J^{\oplus 2}[-s], \qquad (4.46)
$$

*where $d_{i+1} > d_i$ and $2n_i + d_i > 2n_{i+1} + d_{i+1}$, as well as $2n_N + d_N \geqslant 3$ and $d_N \leqslant 1$. Then the chain local equivalence type $[SWF(Y, \mathfrak{s})]_{cl} \in \mathfrak{CLE}$ is the equivalence class of*

$$
C(s, \{d_i\}_i, \{n_i\}_i) = ((\langle f_{\text{red}} \rangle \tilde{\oplus} (\bigoplus_i \mathcal{S}_{d_i}(n_i))), 0, -s/4) \in \mathfrak{CLE}. \qquad (4.47)
$$

*Further, the connected Seiberg-Witten Floer homology of $(Y, \mathfrak{s})$ is:*

$$
SWFH_{\text{conn}}(Y, \mathfrak{s}) = \bigoplus_{i=1}^N \mathcal{T}^+_{s+d_i}(\frac{d_{i+1}+2n_{i+1}-d_i}{2}) \oplus \bigoplus_{i=1}^N \mathcal{T}^+_{s+d_i}(n_i). \qquad (4.48)
$$

*Moreover, if $s \neq t$, or $\{d_i\}_i \neq \{e_i\}_i$, or $\{n_i\}_i \neq \{m_i\}_i$, the complexes $C(s, \{d_i\}_i, \{n_i\}_i)$ and $C(t, \{e_i\}_i, \{m_i\}_i)$ are not locally equivalent.*

*Proof.* Let $SWF(Y, \mathfrak{s}) = (X, p, h/4) \in \mathfrak{E}$ with $X$ a $j$-split space of type SWF. By the construction of $SWF(Y, \mathfrak{s})$, $X^{S^1} \simeq (\tilde{\mathbb{R}}^p)^+$. By Lemma 4.1.2, $[(X, p, h/4)] \in \mathfrak{CE}$ admits a representative $(Z, p', h'/4)$ with $Z$ a $j$-split chain complex, for some $p', h'$. Since $[(X, p, h/4)] \in \mathfrak{CE}$ and $(Z, p', h'/4)$ must have chain homotopy equivalent fixed-point sets, we have:

$$\Sigma^{-\tilde{\mathbb{R}}^p}((\tilde{\mathbb{R}}^p)^+) = [(X^{S^1}, p, 0)] = (Z^{S^1}, p', 0) \in \mathfrak{CE}.$$

However, by the requirement that $Z$ is $j$-split, $Z^{S^1} \simeq \langle f_{\mathrm{red}} \rangle$, where $jf_{\mathrm{red}} = sf_{\mathrm{red}} = \partial(f_{\mathrm{red}}) = 0$. Thus, $p' = 0$. Furthermore, by the proof of Corollary 4.2.4, $-p' - h' = -h' = s$. Proposition 4.1.21 applied to $(Z, 0, -s/4)$ yields (4.47) from (4.31) and (4.48) from (4.32). □

### 4.2.2 Spaces of projective type

Let $Y = \Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ be a negative Seifert rational homology three-sphere. Consider the case that $HF^+(Y, \mathfrak{s})$ is given by:

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+_{2\delta} \oplus \mathcal{T}^+_d(n) \oplus J^{\oplus 2}, \tag{4.49}$$

for some $\mathbb{F}[U]$-module $J$, where possibly $n = 0$. In particular, by Corollary 4.2.4, this implies $d + 2n - 1 = 2\delta$. Let $(Z, 0, -s/4) = SWF(Y, \mathfrak{s}) \in \mathfrak{CE}$. Then by Proposition 4.1.18, we may write:

$$Z = (\langle f_{\mathrm{red}} \rangle \tilde{\oplus} \mathcal{S}_1(n)) \oplus S(J) \tag{4.50}$$

as a direct sum of $C^{CW}_*(S^1)$-chain complexes, with $\partial(x_1) = f_{\mathrm{red}}$, $\partial(x_{2i+1}) = s(1 + j^2)x_{2i-1}$ for $i = 1, ..., n-1$. Here $d = s + 1$, by Corollary 4.2.4. The complex $Z$ is evidently chain locally equivalent to $\langle f_{\mathrm{red}} \rangle \tilde{\oplus} \mathcal{S}_1(n)$. For $X$ a $G$-space, let $\tilde{\Sigma} X$ denote the unreduced suspension of $X$. The complex (4.50), for $\delta > 0$, may be realized as the $G$-CW complex associated to

$$(\tilde{\Sigma}(S^{2n-1} \amalg S^{2n-1}), 0, -s/4),$$

where $S^1$ acts by complex multiplication on each of the two factors, and $j$ interchanges the factors. Then

$$[SWF(Y, \mathfrak{s})]_{cl} \equiv [(\tilde{\Sigma}(S^{2n-1} \amalg S^{2n-1}), 0, -s/4)]_{cl}. \tag{4.51}$$

We call a negative Seifert rational homology sphere with spin structure $(Y, \mathfrak{s})$ *of projective type* if (4.51) holds or if the chain local equivalence class of $SWF(Y, \mathfrak{s})$ is $[\langle f_{\text{red}} \rangle]_{cl}$. Indeed, we have established that $(Y, \mathfrak{s})$ is of projective type if and only if $HF^+(Y, \mathfrak{s})$ takes the form (4.49) (where perhaps $n = 0$). The term *of projective type* refers to the fact:

$$(S^{2n-1} \amalg S^{2n-1})/G \simeq \mathbb{C}P^{n-1}.$$

We can rephrase the projective type condition (4.49) in terms of the *graded roots* of [34]. A graded root $(\Gamma, \chi)$ is an infinite tree $\Gamma$ with an action of $\mathbb{F}[U]$, together with a grading function $\chi : \Gamma \to \mathbb{Z}$. Associated to any positive Seifert rational homology sphere with spin structure there is a graded root, which, additionally, has an involution $\iota : \Gamma \to \Gamma$ that preserves the grading. We will provide a more detailed review of graded roots in Section 5.3.

We have the following characterization of spaces of projective type in terms of graded roots as a consequence of Corollary 4.2.4.

**Fact 4.2.6.** *Let $Y = \Sigma(b, (b_1, a_1), \ldots, (b_k, a_k))$ be a negative Seifert rational homology sphere with spin structure $\mathfrak{s}$. Let $(\Gamma_Y, \chi)$ be the graded root associated to $(-Y, \mathfrak{s})$, and let $\iota$ be the associated involution of $\Gamma_Y$. Let $v \in \Gamma_Y$ be the vertex of minimal grading which is invariant under $\iota$. The space $(Y, \mathfrak{s})$ is of projective type if and only if there exists a vertex $w$, and a path from $v$ to $w$ in $\Gamma_Y$ which is grading-decreasing at each step, with $\chi(w) = \min_{x \in \Gamma_Y} \chi(x)$. Moreover, $\delta(Y, \mathfrak{s}) - \beta(Y, \mathfrak{s}) = \chi(v) - \chi(w)$.*

For instance, we refer to Figure 4.10. We call a graded root *of projective type* if its homology is of the form (4.49), so that a Seifert integral homology sphere is of projective type if and only if its graded root is.

More generally, the sets $\{d_i\}$ and $\{n_i\}$ may be read from the graded root, in terms of the minimal grading elements $w$ that are leaves of vertices $v$ that are invariant under $\iota$.

For spaces $Y$ of projective type, the homology cobordism invariants $(d_i, n_i)$ are determined by $d(Y), \bar{\mu}(Y)$. The nice topological description of the Seiberg-Witten Floer spectrum of spaces of projective type simplifies calculations.

The spaces $\Sigma(p, q, pqn + 1)$ and $\Sigma(p, q, pqn - 1)$ are of projective type for all $p, q, n$, as

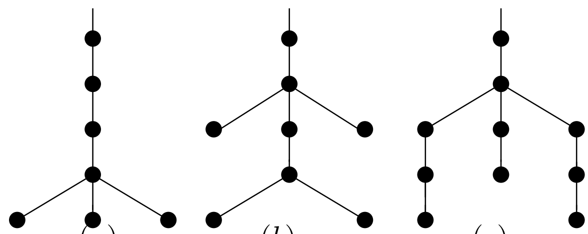


**Figure 4.10:** Three Graded Roots. The roots $(a)$ and $(b)$ are of projective type, while $(c)$ is not.

shown by Némethi [35] and Tweedy [51], respectively, building on work of Borodzik and Némethi [2].

However, not all Seifert fiber spaces are of projective type. The Brieskorn sphere $\Sigma(5,8,13)$ is a Seifert space not of projective type, for instance, as one may confirm using graded roots. Indeed, $SWFH_{\text{conn}}(\Sigma(5,8,13)) = \mathcal{T}_1^+(2) \oplus \mathcal{T}_1^+(1)$. By Corollary 1.2.6, any space not of projective type is not homology cobordant to a space of projective type. In particular, $\Sigma(5,8,13)$ is not homology cobordant to any $\Sigma(p,q,pqn \pm 1)$.

### 4.2.3 Calculation of Beta

By the construction of $SWF(Y,\mathfrak{s})$, the grading of the reducible element is $-2n(Y,\mathfrak{s},g)$. We also saw that the constant $s$ (depending on $(Y,\mathfrak{s})$) in Corollary 4.2.4 is the grading of the reducible (with respect to the Seifert metric). Also in Corollary 4.2.4, we saw $s/2 = \beta(Y,\mathfrak{s})$ for Seifert rational homology spheres. We then obtain:

**Corollary 4.2.7.** *Let $Y = \Sigma(b,(b_1,a_1),\dots,(b_k,a_k))$ be a negative Seifert rational homology sphere and $\mathfrak{s}$ a spin structure on $Y$. Then $\beta(Y,\mathfrak{s}) = -n(Y,\mathfrak{s},g)$, where $g$ is a metric for which $Y$ has the Seifert geometry.*

Ruberman and Saveliev [44] show $n(Y,g) = \bar{\mu}(Y)$ for Seifert integral homology spheres for the Seifert metric, from which we establish Theorem 1.2.3.

We have established that $\bar{\mu}$ restricted to Seifert integral homology three-spheres extends to a homology cobordism invariant, but not necessarily that $\bar{\mu}$ extends to a homology cobordism invariant. In [29] it is shown that $\beta$ is not additive; on the other hand, $\bar{\mu}$ is additive.

Similarly, $\beta$ does not agree with the Saveliev $\nu$ invariant of [45],[46], although the two agree on Seifert fiber spaces.

## 4.3 Manolescu Invariants for Connected Sums of Seifert Spaces

We will take advantage of Theorem 3.1.1 again

We can now prove Theorems 1.3.1 and 1.3.2 of the Introduction.

*Proof of Theorem 1.3.1.* By Definition, $M(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = M(SWF(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2))$, where $M$ is any of $\alpha, \beta$ and $\gamma$. By Fact 3.1.5, $M(SWF(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)) = M(SWF(Y_1, \mathfrak{s}_1) \wedge SWF(Y_2, \mathfrak{s}_2))$. Theorems 2.2.4 and 2.2.5 applied to $SWF(Y_1, \mathfrak{s}_1)$ and $SWF(Y_2, \mathfrak{s}_2)$ yield Theorem 1.3.1. □

*Proof of Theorem 1.3.2.* It follows from Definition 3.1.3 and Proposition 2.2.15 that $\delta(Y, \mathfrak{s}) \leqslant \alpha(Y, \mathfrak{s})$. The inequality $\gamma(Y, \mathfrak{s}) \leqslant \delta(Y, \mathfrak{s})$ then follows from Theorem 3.1.4. □

Next, we specialize to Seifert spaces to acquire Theorem 1.3.4 of the Introduction.

We focus on Seifert spaces of projective type because their chain local equivalence class is simplest. Recall that a Seifert rational homology three-sphere $(Y, \mathfrak{s})$ is of projective type if (4.49) holds, which is equivalent to

$$[SWF(Y, \mathfrak{s})]_{cl} = [(\tilde{\Sigma}(S^{d(Y,\mathfrak{s})+2s-1} \amalg S^{d(Y,\mathfrak{s})+2s-1}), 0, s/2)]_{cl}. \tag{4.52}$$

where $d(Y, \mathfrak{s})$ the Heegaard Floer correction term, for some $s \in \mathbb{Q}$. If $Y$ is an integral homology three-sphere, the quantity $s$ is $n = \bar{\mu}(Y)$.

Applying Theorem 2.3.1, we obtain Theorem 1.3.4 of the Introduction:

*Proof of Theorem 1.3.4.* By (4.52) and Fact 3.1.5, we have:

$$[SWF(Y_1\# \dots \#Y_n)]_{cl} = [(\wedge_{i=1}^n(\tilde{\Sigma}(S^{2(d(Y_i)/2+\bar{\mu}(Y_i))-1} \amalg S^{2(d(Y_i)/2+\bar{\mu}(Y_i))-1}), 0, \bar{\mu}(Y_1\# \dots \#Y_n)/2)]_{cl}. \tag{4.53}$$

In Theorem 2.3.1, we computed $\alpha, \beta$, and $\gamma$ for the right-hand side of (4.53), completing the proof. □

# CHAPTER 5

# Applications to the Homology Cobordism Group

## 5.1 Seifert Spaces

First, we see that Corollary 1.2.2 follows from Corollary 4.2.4 and Theorem 1.2.3. Indeed, the negative fibration case follows immediately, and the positive fibration statement follows by using the properties of $\alpha, \beta, \gamma, \bar{\mu}$, and $d$ under orientation reversal.

We also obtain:

**Theorem 5.1.1.** *Let $Y$ be a Seifert integral homology sphere. If $-\bar{\mu}(Y)/2 \neq d(Y)$, then $Y$ is not homology cobordant to any Seifert integral homology sphere with fibration of sign opposite that of $Y$.*

*Proof.* If $Y$ is a negative Seifert fibration, and $-\bar{\mu}(Y)/2 \neq d(Y)$, then $\alpha(Y) \neq \beta(Y)$, but for all positive fibrations $\alpha = \beta$. One performs a similar check for positive fibrations. □

This statement is expressed only in terms of $\bar{\mu}$ and $d$, but the proof comes from the properties of $\alpha, \beta, \gamma$. As a particular example, we have $\Sigma(2, 3, 12k-5)$ and $\Sigma(2, 3, 12k-1)$, for all $k \geqslant 1$, have $\alpha \neq \beta$ and so are not homology cobordant to any positive Seifert fibration.

We remark that Némethi's algorithm [34] for Heegaard Floer homology of Seifert fiber spaces makes $SWFH_*^G$ of Seifert spaces computable. Using Tweedy's computations in [51], we provide calculations of $SWFH_*^G$ for the following infinite families as an example. In the following tables, there are nontrivial $q$-actions between infinite towers. The only other nontrivial $q$-actions are for $\Sigma(2, 7, 28k-1)$ and $\Sigma(2, 7, 28k+15)$, where $q$ sends each summand of $\mathcal{V}_3^+(1)^{\oplus k}$ (respectively $\mathcal{V}_{-1}^+(1)^{\oplus k+1}$) to $\mathcal{V}_2^+$ (respectively $\mathcal{V}_{-2}^+$).

| $Y$ | $SWFH^G_*(Y)$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|
| $\Sigma(2,5,20k+11)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1} \oplus \mathcal{V}^+_0 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k} \oplus \bigoplus_{i=1}^{2k+1} \mathcal{V}^+_{-1-2i}(1)$ | 1 | -1 | -1 | 0 |
| $\Sigma(2,5,20k+1)$ | $\mathcal{V}^+_0 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k} \oplus \bigoplus_{i=1}^{2k} \mathcal{V}^+_{-1-2i}(1)$ | 0 | 0 | 0 | 0 |
| $\Sigma(2,5,20k-11)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_3 \oplus \mathcal{V}^+_4 \oplus \mathcal{V}^+_1(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-2} \mathcal{V}^+_{-1-2i}(1)$ | 1 | 1 | 1 | 1 |
| $\Sigma(2,5,20k-1)$ | $\mathcal{V}^+_4 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_1(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-1} \mathcal{V}^+_{-1-2i}(1)$ | 2 | 0 | 0 | 1 |
| $\Sigma(2,5,20k-13)$ | $\mathcal{V}^+_0 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-2} \mathcal{V}^+_{-1-2i}(1)$ | 0 | 0 | 0 | 0 |
| $\Sigma(2,5,20k-3)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1} \oplus \mathcal{V}^+_0 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-1} \mathcal{V}^+_{-1-2i}(1)$ | 1 | -1 | -1 | 0 |
| $\Sigma(2,5,20k+3)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_3 \oplus \mathcal{V}^+_4 \oplus \mathcal{V}^+_1(1)^{\oplus k} \oplus \bigoplus_{i=0}^{2k-1} \mathcal{V}^+_{-1-2i}(1)$ | 1 | 1 | 1 | 1 |
| $\Sigma(2,5,20k+13)$ | $\mathcal{V}^+_4 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_1(1)^{\oplus k} \oplus \bigoplus_{i=0}^{2k} \mathcal{V}^+_{-1-2i}(1)$ | 2 | 0 | 0 | 1 |

Table 5.1: The Pin(2)-equivariant Floer homology of $\Sigma(2,5,p)$.

## 5.2 Connected Sums

We use Theorem 1.3.4 to obtain Theorem 1.3.5 of the Introduction:

*Proof of Theorem 1.3.5.* Define $\tilde{\delta}(Y_i)$ by $d(Y_i)/2 + \bar{\mu}(Y_i)$. Assume without loss of generality that $\tilde{\delta}(Y_1) \leqslant \cdots \leqslant \tilde{\delta}(Y_n)$. We have, by Theorem 1.3.4:

$$\beta(Y) - \gamma(Y) = E(\sum_{i=1}^{n-1} \tilde{\delta}(Y_i)) - E(\sum_{i=1}^{n-2} \tilde{\delta}(Y_i)).$$

Since we assumed $\tilde{\delta}(Y_i) \geqslant 2$ for at least two distinct $i$, we have $\tilde{\delta}(Y_{n-1}) \geqslant 2$, so:

$$\beta(Y) - \gamma(Y) \geqslant 2.$$

Negative Seifert integral homology spheres $Z$ have $\beta(Z) - \gamma(Z) = 0$, so $Y$ is not homology cobordant to any negative Seifert integral homology sphere.

Using Theorem 1.3.4 again, we similarly obtain $\alpha(Y) - \beta(Y) \geqslant 2$. But positive Seifert spaces have $\alpha(Z) = \beta(Z)$, using Corollary 1.2.2. Thus $Y$ is not homology cobordant to any positive Seifert space, completing the proof. □

| $Y$ | $SWFH^G_*(Y)$ |
|---|---|
| $\Sigma(2,7,28k-1)$ | $\mathcal{V}^+_4 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_3(1)^{\oplus k} \oplus \mathcal{V}^+_1(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-1} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=0}^{2k-1} \mathcal{V}^+_{-1-4k-4i}(1)$ |
| $\Sigma(2,7,28k-15)$ | $\mathcal{V}^+_4 \oplus \mathcal{V}^+_5 \oplus \mathcal{V}^+_6 \oplus \mathcal{V}^+_3(1)^{\oplus k-1} \oplus \mathcal{V}^+_1(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{2k-2} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=0}^{2k-2} \mathcal{V}^+_{1-4k-4i}(1)$ |
| $\Sigma(2,7,28k+1)$ | $\mathcal{V}^+_0 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_{-3}(1)^{\oplus k} \oplus \mathcal{V}^+_{-1}(1)^{\oplus k} \oplus \bigoplus_{i=1}^{2k} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=1}^{2k} \mathcal{V}^+_{-1-4k-4i}(1)$ |
| $\Sigma(2,7,28k+15)$ | $\mathcal{V}^+_0 \oplus \mathcal{V}^+_{-3} \oplus \mathcal{V}^+_{-2} \oplus \mathcal{V}^+_{-3}(1)^{\oplus k} \oplus \mathcal{V}^+_{-1}(1)^{k+1} \oplus \bigoplus_{i=1}^{2k+1} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=1}^{2k+1} \mathcal{V}^+_{-3-4k-4i}(1)$ |
| $\Sigma(2,7,14k-3)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_3 \oplus \mathcal{V}^+_4 \oplus \mathcal{V}^+_1(1)^{\oplus k-1} \oplus \bigoplus_{i=0}^{k-1} \mathcal{V}^+_{1-2i}(1) \oplus \bigoplus_{i=0}^{k-1} \mathcal{V}^+_{1-2k-4i}(1)$ |
| $\Sigma(2,7,14k+3)$ | $\mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1} \oplus \mathcal{V}^+_0 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k} \oplus \bigoplus_{i=1}^{k} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=1}^{k} \mathcal{V}^+_{-1-2k-4i}(1)$ |
| $\Sigma(2,7,14k-5)$ | $\mathcal{V}^+_4 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_1(1)^{\oplus k-2} \oplus \bigoplus_{i=0}^{k-1} \mathcal{V}^+_{1-2i}(1) \oplus \bigoplus_{i=0}^{k-1} \mathcal{V}^+_{1-2k-4i}(1)$ |
| $\Sigma(2,7,14k+5)$ | $\mathcal{V}^+_0 \oplus \mathcal{V}^+_1 \oplus \mathcal{V}^+_2 \oplus \mathcal{V}^+_{-1}(1)^{\oplus k+1} \oplus \bigoplus_{i=1}^{k} \mathcal{V}^+_{-1-2i}(1) \oplus \bigoplus_{i=1}^{k} \mathcal{V}^+_{-1-2k-4i}(1)$ |

Table 5.2: The Pin(2)-equivariant Floer homology of $\Sigma(2,7,p)$.

**Definition 5.2.1.** We call a rational homology three-sphere with spin structure $(Y,\mathfrak{s})$ *$H$-split* if $\alpha(Y,\mathfrak{s}) = \beta(Y,\mathfrak{s}) = \gamma(Y,\mathfrak{s})$, in analogy to the concept of $K$-split from [31]. We note from Theorem 1.3.1 that the subset $\theta_{H\text{-split}}$ of $H$-split homology cobordism classes is a subgroup of $\theta^H_3$.

**Lemma 5.2.2.** *Let $Y = Y_1\#\ldots\#Y_n$ be a connected sum of negative Seifert integral homology spheres of projective type $Y_i$, with $\tilde{\delta}(Y_1) \leqslant \cdots \leqslant \tilde{\delta}(Y_n)$. Then $\tilde{\delta}(Y_n)$ is determined by $[Y] \in \theta^H_3$. That is, $\tilde{\delta}(Y_n)$ is a homology cobordism invariant of $Y_1\#\ldots\#Y_n$ among connected sums of negative Seifert integral homology spheres of projective type.*

*Proof.* We show how to determine $\tilde{\delta}(Y_n)$ from $Y$. First, we note that $Y$ is $H$-split if and only if $\tilde{\delta}(Y_n) = 0$ using (1.16)-(1.19), so we may assume from now on that $\tilde{\delta}(Y_n) \geqslant 1$. Consider $Y\#\Sigma(2,3,11)$ (recalling that $d(\Sigma(2,3,11)) = 2$, and $\bar{\mu}(\Sigma(2,3,11)) = 0$). We have:

$$\alpha(Y) - \beta(Y) = E(\sum_{i=1}^{n} \tilde{\delta}(Y_i)) - E(\sum_{i=1}^{n-1} \tilde{\delta}(Y_i)) \tag{5.1}$$

$$\alpha(Y\#\Sigma(2,3,11)) - \beta(Y\#\Sigma(2,3,11)) = E(\sum_{i=1}^{n} \tilde{\delta}(Y_i) + 1) - E(\sum_{i=1}^{n-1} \tilde{\delta}(Y_i) + 1) \tag{5.2}$$

If $\tilde{\delta}(Y_n)$ is even, then the difference in (5.1) is $\tilde{\delta}(Y_n)$, while if $\tilde{\delta}(Y_n)$ is odd, (5.1) is $\tilde{\delta}(Y_n)+1$ if $\sum_{i=1}^{n-1}\tilde{\delta}(Y_i)$ is even, or $\tilde{\delta}(Y_n)-1$ otherwise. If $\tilde{\delta}(Y_n)$ is even, the difference in (5.2) is $\tilde{\delta}(Y_n)$, while if $\tilde{\delta}(Y_n)$ is odd, (5.2) is $\tilde{\delta}(Y_n)-1$ if $\sum_{i=1}^{n-1}\tilde{\delta}(Y_i)$ is even, or $\tilde{\delta}(Y_n)+1$ otherwise.

In particular, we observe that $\alpha(Y), \beta(Y)$, $\alpha(Y\#\Sigma(2,3,11))$, and $\beta(Y\#\Sigma(2,3,11))$ determine $\tilde{\delta}(Y_n)$. $\square$

We show the existence of a summand of a certain subgroup of the homology cobordism group. Let $\theta_{SFP}$ denote the subgroup of $\theta_3^H$ generated by negative Seifert spaces of projective type.

**Theorem 5.2.3.** *Let $\theta_{H\text{-split},SFP} = \theta_{H\text{-split}} \cap \theta_{SFP}$. The group $\theta_{SFP}$ splits into a direct sum*

$$\theta_{SFP} = \theta_{H\text{-split},SFP} \oplus \bigoplus_{\{x>0 \mid \exists Y, \tilde{\delta}(Y)=x\}} \mathbb{Z}. \tag{5.3}$$

*Proof.* Here the rightmost direct sum runs over all positive $x$ for which there exists a negative Seifert integral homology sphere $Y$ of projective type with $\tilde{\delta}(Y) = x$. Let $H$ be the free abelian group with generators $e_i$, for each $i \in \mathbb{Z}_{>0}$. The group $H$ is isomorphic to $\mathbb{Z}^\infty$.

We define a homomorphism $\psi\colon \theta_{SFP} \to H$. For $Y$ a negative Seifert integral homology sphere of projective type with $\tilde{\delta}(Y) > 0$, we define $\psi(Y) = e_{\tilde{\delta}(Y)}$, while if $\tilde{\delta}(Y) = 0$, we set $\psi(Y) = 0$. To define $\psi$ on all of $\theta_{SFP}$ we extend linearly. To establish that $\psi$ is a homomorphism, we need only show that the set (with multiplicity) $\{\tilde{\delta}(Y_1), \dots, \tilde{\delta}(Y_n)\}$ associated to $Y \sim Y_1\# \dots \#Y_n$ is indeed a homology cobordism invariant of $Y$, i.e. that it does not depend on how we express $Y$ as a connected sum of Seifert integral homology spheres in $\theta_{SFP}$.

Say we have an identity in $\theta_{SFP}$ among (not necessarily negative) Seifert spaces of projective type:

$$Y_1\# \dots \#Y_n \sim Z_1\# \dots \#Z_m. \tag{5.4}$$

We need to show $\sum \psi(Y_i) = \sum \psi(Z_i)$. To do so, by rearranging (5.4) we may assume that all the $Y_i, Z_j$ are negative Seifert spaces. We assume without loss of generality that $\tilde{\delta}(Y_1) \leqslant \dots \leqslant \tilde{\delta}(Y_n)$ and $\tilde{\delta}(Z_1) \leqslant \dots \leqslant \tilde{\delta}(Z_m)$, and that $n \leqslant m$.

By Lemma 5.2.2, $\tilde{\delta}(Y_n) = \tilde{\delta}(Z_m)$, and so

$$[SWF(Z_m\# - Y_n)]_{cl} = [(S^0, 0, \frac{\bar{\mu}(Z_m) - \bar{\mu}(Y_n)}{2})]_{cl}.$$

Thus, subtracting $Y_n$ from both sides of (5.4), we obtain:

$$[SWF(Y_1\# \dots \# Y_{n-1})]_{cl} = [(SWF(Z_1\# \dots \# Z_{m-1})) \wedge (S^0, 0, \frac{\bar{\mu}(Z_m) - \bar{\mu}(Y_n)}{2})]_{cl} \tag{5.5}$$

The right-hand side of (5.5) is

$$[SWF((\#_{(\bar{\mu}(Y_n)-\bar{\mu}(Z_m))}\Sigma(2,3,5))\#Z_1\# \dots \# Z_{m-1})]_{cl},$$

using $d(\Sigma(2,3,5)) = 2$ and $\bar{\mu}(\Sigma(2,3,5)) = -1$.

We repeat the use of Lemma 5.2.2 to find $\tilde{\delta}(Y_{n-i}) = \tilde{\delta}(Z_{m-i})$ for all $i \leqslant n$. This gives finally that $Z_1\# \dots \# Z_{m-n}$ must be $H$-split, and so in particular $\tilde{\delta}(Z_i) = 0$ for all $i \leqslant m-n$. This shows that $\sum_{i=1}^m \psi(Z_i) = \sum_{i=1}^n \psi(Y_i)$, whence $\psi$ is well-defined on $\theta_{SFP}$. It is clear that $\psi$ is surjective onto the $\bigoplus_{\{x>0|\exists Y, \tilde{\delta}(Y)=x\}} \mathbb{Z}$ factor, with kernel $\theta_{H\text{-split},SFP}$, giving the splitting stated in the Theorem. □

*Proof of Theorem 1.3.6.* By Theorem 1.3.8, for all $N > 0$ there exists some negative Seifert space of projective type $Y$ for which $\tilde{\delta}(Y) = N$. Theorem 1.3.6 then follows from Theorem 5.2.3.

However, other generators for

$$\bigoplus_{\{x>0|\exists Y, \tilde{\delta}(Y)=x\}} \mathbb{Z}$$

are easier to find, using results of Némethi (we use $Y_p$ from Theorem 1.3.8 in order to obtain Corollary 1.3.9).

We record a different generating set, starting with some notation from [35]. Let, for relatively prime $p$ and $q$, $\mathcal{S}_{p,q} \subset \mathbb{Z}_{\geqslant 0}$ denote the semigroup

$$\mathcal{S}_{p,q} = \{ap + bg \mid (a,b) \in \mathbb{Z}^2_{\geqslant 0}\},$$

and

$$\alpha_i = \#\{s \notin \mathcal{S}_{p,q} \mid s > i\}.$$

Also, set

$$g = \frac{(p-1)(q-1)}{2}.$$

Then Némethi [35] shows

$$HF^+(-\Sigma(p,q,pqn+1)) = \mathcal{T}_0^+ \oplus \mathcal{T}_0^+(\alpha_{g-1})^{\oplus n} \oplus \bigoplus_{i=1}^{n(g-1)} \mathcal{T}^+_{(\lfloor \frac{i}{n} \rfloor+1)(\{\frac{i}{n}\}n+i)}(\alpha_{g-1+\lceil \frac{i}{n} \rceil})^{\oplus 2}.$$

Reversing orientation, we have:

$$HF^+(\Sigma(p,q,pqn+1)) = \mathcal{T}_0^+ \oplus \mathcal{T}^+_{1-2\alpha_{g-1}}(\alpha_{g-1})^{\oplus n} \oplus \bigoplus_{i=1}^{n(g-1)} \mathcal{T}^+_{1-(\lfloor \frac{i}{n} \rfloor+1)(\{\frac{i}{n}\}n+i)-2\alpha_{g-1+\lceil \frac{i}{n} \rceil}}(\alpha_{g-1+\lceil \frac{i}{n} \rceil})^{\oplus 2}.$$

This implies that $\Sigma(p,q,pqn+1)$ is of projective type, and the discussion following (4.49) gives, for $n$ odd, $\alpha_{g-1} = d(\Sigma(p,q,pqn+1))/2 + \bar{\mu}(\Sigma(p,q,pqn+1))$.

Fixing $p=2$, we note that the complement of $\mathcal{S}_{p,q}$ is precisely $\{s \mid s < q,\ s \text{ odd}\}$. We see from the definition of $\alpha_{g-1}$ that $\alpha_{g-1} = \lfloor \frac{q+1}{4} \rfloor$. We then have that $\{\Sigma(2,q,2q+1) \mid q>1, \text{ odd}\}$ attains all positive values of $\tilde{\delta} = d/2 + \bar{\mu}$. By Theorem 5.2.3, $\Sigma(2,4k+3,8k+7)$ then span a $\mathbb{Z}^\infty$ summand of $\theta_{SFP}$. □

*Proof of Corollary 1.3.7.* By the calculation in [30], for all $k \geqslant 1$,

$$d(\Sigma(2,3,12k-1)) = 2,\ \bar{\mu}(\Sigma(2,3,12k-1)) = 0,$$

$$d(\Sigma(2,3,12k-7)) = 2,\ \bar{\mu}(\Sigma(2,3,12k-7)) = -1.$$

In particular, $[\Sigma(2,3,12k-7)]_{cl}$ is independent of $k$. Furthermore,

$$[\Sigma(2,3,12k-7)] \in \theta_{H\text{-split}}$$

for all $k \geqslant 1$. However, Furuta [16] shows $\Sigma(2,3,6k-1)$ are linearly independent in $\theta_3^H$. Then $\{\Sigma(2,3,12k-7)\}_{k\geqslant 1}$ generates a $\mathbb{Z}^\infty$ subgroup of $\theta_{H\text{-split}}$, as needed. □

We establish Theorem 1.3.3 of the Introduction, using Theorem 2.3.1.

*Proof of Theorem 1.3.3.* By Lemma 2.2.12, for $X$ a space of type SWF at level $t$ the complex $C_*^{CW}(X)$ must contain a copy of $T = T_{(d(X)-t)/2}(t)$. We recall, by Fact 2.2.10, that $T$ is chain locally equivalent to

$$\Sigma^{t\tilde{\mathbb{R}}}\tilde{\Sigma}(S^{d(X)-t-1} \amalg S^{d(X)-t-1}).$$

Theorem 2.3.1 then shows:

$$a(T^{\otimes_n}) = 2E(n(d(X)-t)/2) + nt, \tag{5.6}$$

$$b(T^{\otimes_n}) = 2E((n-1)(d(X)-t)/2) + nt, \tag{5.7}$$

$$c(T^{\otimes_n}) = 2E((n-2)(d(X)-t)/2) + nt. \tag{5.8}$$

Let $(X, g, h) = SWF(Y, \mathfrak{s})$, and let $X$ be of type SWF at level $t$. Then $\delta(Y, \mathfrak{s}) = d(X)/2 - g/2 - 2h$. From

$$\bigwedge^n (T_{(d(X)-t)/2}(t), g, h) \leqslant \bigwedge^n (X, g, h)$$

and (5.6)-(5.8) we obtain:

$$\begin{aligned}
\alpha(\bigwedge^n (X,g,h)) &\geqslant E(n(d(X)-t)/2) + \frac{nt - ng - 4nh}{2}, \\
\beta(\bigwedge^n (X,g,h)) &\geqslant E((n-1)(d(X)-t)/2) + \frac{nt - ng - 4nh}{2}, \\
\gamma(\bigwedge^n (X,g,h)) &\geqslant E((n-2)(d(X)-t)/2) + \frac{nt - ng - 4nh}{2}, \\
\delta(\bigwedge^n (X,g,h)) &= nd(X)/2 - ng/2 - 2nh.
\end{aligned}$$

Using $E(x) \geqslant x$, we see:

$$\begin{aligned}
\alpha(\#_n(Y,\mathfrak{s})) &\geqslant n\delta(Y,\mathfrak{s}), \\
\beta(\#_n(Y,\mathfrak{s})) &\geqslant (n-1)\delta(Y,\mathfrak{s}) + \frac{(t-g-4h)}{2}, \\
\gamma(\#_n(Y,\mathfrak{s})) &\geqslant (n-2)\delta(Y,\mathfrak{s}) + 2\frac{(t-g-4h)}{2}, \\
\delta(\#_n(Y,\mathfrak{s})) &= n\delta(Y,\mathfrak{s}).
\end{aligned} \tag{5.9}$$

From (5.9), we obtain:

$$\gamma(\#_n(Y,\mathfrak{s})) \geqslant n\delta(Y,\mathfrak{s}) + C \tag{5.10}$$

where $C$ is some constant depending on $Y$ (but not $n$). However, by Theorem 1.3.2, $\gamma(\#_n(Y,\mathfrak{s})) \leqslant \delta(\#_n(Y,\mathfrak{s})) = n\delta(Y,\mathfrak{s})$, from which we obtain that $\gamma(\#_n(Y,\mathfrak{s})) - n\delta(Y,\mathfrak{s})$ is a bounded function of $n$. Using the properties of $\alpha, \beta$, and $\gamma$ under orientation reversal we find that $\alpha(\#_n(Y,\mathfrak{s})) - n\delta(Y,\mathfrak{s})$ is also a bounded function of $n$. Since $\gamma(\#_n(Y,\mathfrak{s})) \leqslant \beta(\#_n(Y,\mathfrak{s})) \leqslant \alpha(\#_n(Y,\mathfrak{s}))$, we also obtain that $\beta(\#_n(Y,\mathfrak{s})) - n\delta(Y,\mathfrak{s})$ is a bounded function of $n$. □

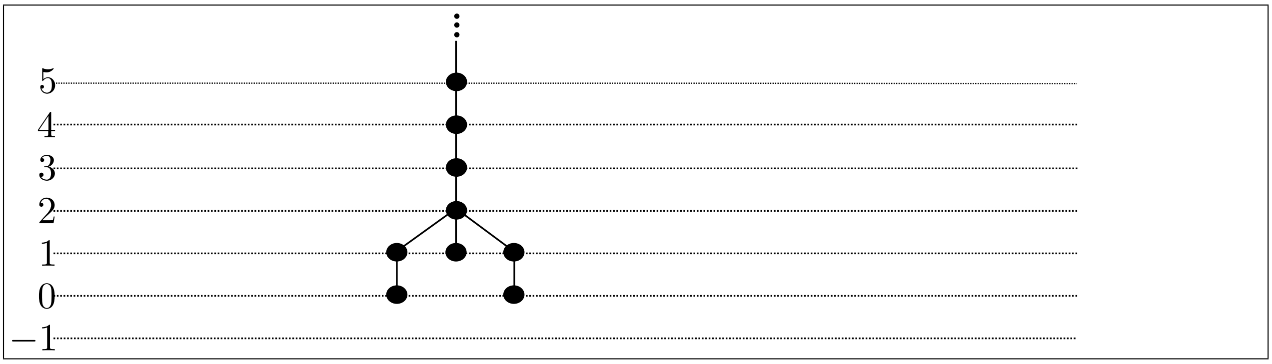


**Figure 5.1:** Example of a graded root, with $\Delta$ sequence $\{2, -1, 1, -2\}$.

## 5.3 Graded Roots

In this section we collect the preliminaries needed to show Theorem 1.3.8. We use graded roots, which were introduced by Némethi [34] in order to study the Heegaard Floer homology of plumbed manifolds. The graded roots of Seifert spaces were studied in [3],[21]. Our brief introduction to graded roots will follow [18, §4] extremely closely.

### 5.3.1 Definitions

**Definition 5.3.1** ([34])**.** A *graded root* consists of a pair $(\Gamma, \chi)$, where $\Gamma$ is an infinite tree, and $\chi\colon \mathrm{Vert}(\Gamma) \to \mathbb{Z}$ satisfies the following.

- $\chi(u) - \chi(v) = \pm 1$, if $u, v$ are adjacent.
- $\chi(u) > \min\{\chi(v), \chi(w)\}$ if $u$ and $v$ are adjacent and $u$ and $w$ are adjacent.
- $\chi$ is bounded below.
- For all $k \in \mathbb{Z}$, $\chi^{-1}(k)$ is finite.
- For $k$ sufficiently large, $|\chi^{-1}(k)| = -1$.

An example graded root is featured in Figure 5.1.

Graded roots are specified, up to degree shift, by a finite sequence, as follows. Let $\Delta\colon \{0, \ldots, N\} \to \mathbb{Z}$, and define $\tau_\Delta\colon \{0, \ldots, N\} \to \mathbb{Z}$ by the recurrence:

$$\tau_\Delta(n+1) - \tau_\Delta(n) = \Delta(n), \text{ with } \tau_\Delta(0) = 0. \tag{5.11}$$

For each $n \in \{0, \ldots, N+1\}$, let $R_n$ be the graph with vertex set $\{\tau_\Delta(n), \tau_\Delta(n)+1, \ldots\}$, with edges between $k$ and $k+1$ for all $k \geqslant \tau_\Delta(n)$. The graded root associated to $\tau_\Delta$ is the infinite tree obtained by identifying the common edges and vertices of $R_n$ and $R_{n+1}$ for each $n \in \{0, \ldots, N+1\}$; call this tree $\Gamma_\Delta$. We define the grading function $\chi_\Delta$ on $\Gamma_\Delta$ by setting $\chi_\Delta(v)$ to be the integer corresponding to $v$ (this integer is independent of which tree $R_n$ we consider $v$ as a vertex of, by the construction). Notice that lengthening $\Delta$ by assigning 0 to $\{N+1, \ldots, M\}$, for some $M > N$ does not change the graded root determined by $\Delta$.

To a graded root $(\Gamma, \chi)$ is associated a graded $\mathbb{F}[U]$-module $\mathbb{H}(\Gamma, \chi)$. We define $\mathbb{H}(\Gamma, \chi)$ by the $\mathbb{F}$-vector space with generators the vertices of $\Gamma$. The element of $\mathbb{H}(\Gamma, \chi)$ corresponding to a vertex $v \in \Gamma$ has grading $2\chi(v)$. The $\mathbb{F}[U]$-module structure is given by setting $Uv$ to be the sum of all vertices $w$ adjacent to $v$ with $\chi(w) = \chi(v) - 1$.

### 5.3.2 Delta Sequences

Karakurt and Lidman [21] define an *abstract delta sequence* as a pair $(X, \Delta)$ with $X$ a well-ordered finite set, and $\Delta\colon X \to \mathbb{Z} - \{0\}$, with $\Delta$ positive on the minimal element of $X$. As we saw in §5.3.1, an abstract delta sequence specifies a graded root up to a grading shift.

To connect graded roots back to topology: Némethi associates a graded root to any manifold belonging to a large family of plumbed manifolds (including Brieskorn spheres). The corresponding $\mathbb{F}[U]$-module $\mathbb{H}(\Gamma, \chi)$ is isomorphic to $HF^+(-Y)$ up to a grading shift. Can and Karakurt [3] simplify the method for Seifert homology spheres. In the proof of Theorem 1.3.8 we will use their reformulation.

In particular, we review the abstract delta sequence $(X_Y, \Delta_Y)$ of an arbitrary Brieskorn sphere $Y = \Sigma(p, q, r)$, following [3]. We follow the convention that the Seifert space $\Sigma(p, q, r)$ is the circle bundle over the orbifold $S^2(p, q, r)$ with orbifold degree $-1/pqr$. Here $S^2(p, q, r)$ is the orbifold with underlying space $S^2$ and cone singularities modelled on the actions of $\mathbb{Z}/p$, $\mathbb{Z}/q$, and $\mathbb{Z}/r$. This convention for $\Sigma(p, q, r)$ agrees with the notation of [3], but is opposite the notation of [18]. Set $N_Y = pqr - pq - pr - qr$. Let $S_Y$ be the intersection of the semigroup on the generators $pq, pr, qr$ with $[0, N_Y]$. Set

$$Q_Y = \{N_Y - s \mid s \in S_Y\},$$

and

$$X_Y = S_Y \cup Q_Y.$$

Can and Karakurt show $S_Y$ and $Q_Y$ are disjoint. Define $\Delta_Y : X_Y \to \{-1, 1\}$ by $\Delta_Y = 1$ on $S_Y$ and $-1$ on $Q_Y$. It is clear that $(X_Y, \Delta_Y)$ is an abstract delta sequence.

**Theorem 5.3.2** ([3] Theorem 1.3,[34] Section 11 ,[39] Theorem 1.2)**.** *Let $Y = \Sigma(p, q, r)$ for coprime $p, q, r$. Let $(\Gamma_Y, \chi_Y)$ be the graded root associated to the abstract delta sequence $(X_Y, \Delta_Y)$ described above. Then $\mathbb{H}(\Gamma_Y, \chi_Y) \cong HF^+(-Y)$ as relatively graded $\mathbb{F}[U]$-modules.*

Note furthermore that $\Delta_Y(x) = -\Delta_Y(N_Y - x)$ for $x \in X_Y$.

### 5.3.3 Operations on Delta Sequences

Different abstract delta sequences may correspond to the same graded root. For instance, let $(X, \Delta)$ be an abstract delta sequence. Fix $t \geqslant 2$ and $z \in X$ with $|\Delta(z)| \geqslant t$. Choose $n_1, \ldots, n_t \in \mathbb{Z}$, so that the sign of all $n_i$ is the same as that of $\Delta(z)$ and so that $n_1 + \cdots + n_t = \Delta(z)$. From this data we construct an abstract delta sequence with the same graded root as $(X, \Delta)$. Let $X' = X/z \cup \{z_1, \ldots, z_t\}$ for some new elements $z_1 \leqslant \cdots \leqslant z_t$ taking the place of $z$ in $X$. Define $\Delta' : X' \to \mathbb{Z}$ by $\Delta'(x) = \Delta(x)$ for $x \in X/\{z\}$ and by $\Delta'(z_i) = n_i$ for all $i$. We call $(X', \Delta')$ a *refinement* of $(X, \Delta)$, and $(X, \Delta)$ a *merge* of $(X', \Delta')$.

**Definition 5.3.3.** We call an abstract delta sequence $(X, \Delta)$ *reduced* if it has no consecutive positive or negative values of $\Delta$ (this is the same as $(X, \Delta)$ not admitting any merges). Every abstract delta sequence admits a unique reduced form. We call an abstract delta sequence *expanded* if it does not admit any refinement (this is equivalent to all values of $\Delta$ being $\pm 1$).

It is more convenient to work with reduced delta sequences, but we saw in Section 5.3.2 that the abstract delta sequence associated to Brieskorn spheres is expanded, so we will need a way to explicitly write the reduced form of $(X_Y, \Delta_Y)$. This will be handled in Section 5.4 using several lemmas from [18].

### 5.3.4 Successors and Predecessors

Let $(X, \Delta)$ be an abstract delta sequence. Let $S \subset X$ be the set on which $\Delta$ is positive, and $Q \subset X$ the set on which $\Delta$ is negative. For $x \in X$, we define the positive successor

$$\mathrm{suc}_+(x) = \min\{x' \in S \mid x < x'\}$$

and negative successor $\mathrm{suc}_-(x) = \min\{x' \in Q \mid x < x'\}$.

The sequence $(X, \Delta)$ is reduced if and only if for all $x \in S$:

$$x < \mathrm{suc}_-(x) \leqslant \mathrm{suc}_+(x),$$

and, for all $x \in Q$:

$$x < \mathrm{suc}_+(x) \leqslant \mathrm{suc}_-(x).$$

We also define $\mathrm{pre}_\pm(x)$, the positive and negative *predecessors*, analogously.

We will need a specific model for the reduced form of $(X, \Delta)$. First, we need a few further pieces of notation. For $x \in S$, let

$$\pi_+(x) = \max\{z \in S \mid z < \mathrm{suc}_-(x)\} \text{ and } \pi_-(x) = \min\{z \in S \mid z > \mathrm{pre}_-(x)\}.$$

For $y \in Q$, let

$$\eta_+(y) = \max\{z \in Q \mid z < \mathrm{suc}_+(y)\} \text{ and } \eta_-(y) = \min\{z \in Q \mid z > \mathrm{pre}_+(y)\}.$$

Now define $\tilde{S} = \{\pi_+(x) \mid x \in S\}$ (noting that $S$ contains one element for each maximal interval of elements of $X$ on which $\Delta$ is positive). Similarly, define $\tilde{Q} = \{\eta_-(y) \mid y \in Q\}$. Then set $\tilde{X} = \tilde{S} \cup \tilde{Q}$. We define $\tilde{\Delta}$ on $\tilde{S}$ by

$$\tilde{\Delta}(\pi_+(x)) = \sum_{z \mid \pi_-(x) \leqslant z \leqslant \pi_+(x)} \Delta(z),$$

and on $\tilde{Q}$ by

$$\tilde{\Delta}(\eta_-(y)) = \sum_{z \mid \eta_-(y) \leqslant z \leqslant \eta_+(y)} \Delta(z).$$

The pair $(\tilde{X}, \tilde{\Delta})$ is the reduced form of $(X, \Delta)$.

Note, in particular, that we may consider $\tilde{X}$ as a subset of $X$.

### 5.3.5 Tau Functions and Sinking Delta Sequences

Let $\mathrm{suc}(x)$ be $\min\{x' \in X \mid x < x'\}$, and let $x_{\min}$, $x_{\max}$ be the minimal and maximal elements of $X$. For an abstract delta sequence $(X, \Delta)$, we define $\tau_\Delta$ as in (5.11) by:

$$\tau_\Delta(\mathrm{suc}(x)) - \tau_\Delta(x) = \Delta(x), \text{ with } \tau_\Delta(x_{\min}) = 0.$$

Let $X^+ = X \cup \{x^+\}$ where $x^+ = \mathrm{suc}(x_{\max})$. The function $\tau_\Delta$ is then defined on $X^+$.

We call $\tau_\Delta$ the *tau function* associated to the abstract delta sequence $(X, \Delta)$.

**Definition 5.3.4** ([18])**.** Let $(X, \Delta)$ be an abstract delta sequence and $(\tilde{X}, \tilde{\Delta})$ its reduced form. We call $(X, \Delta)$ sinking if the following hold.

1. The maximal element $x_{\max}$ of $X$ belongs to $Q$ (i.e. $\Delta(x_{\max}) < 0$).
2. For all $x \in \tilde{S}$, $\tilde{\Delta}(x) \leqslant |\tilde{\Delta}(\mathrm{suc}_-(x))|$.
3. $\tilde{\Delta}(\mathrm{pre}_+(x_{\max})) < |\tilde{\Delta}(x_{\max})|$.

Sinking delta sequences will be significant to us because of the following Proposition, which follows immediately from Definition 5.3.4.

**Proposition 5.3.5** (Proposition 4.7 [18])**.** *A sinking delta sequence attains its minimum at and only at its last element.*

### 5.3.6 Symmetric Delta Sequences

There is a symmetry in Figure 5.1 obtained by reflecting the graded root across the vertical axis. This symmetry holds for graded roots of all Seifert integral homology spheres. For simplicity, write $\Delta = \langle k_1, k_2, \ldots, k_n \rangle$ for the function $\Delta\colon X \to \mathbb{Z}/\{0\}$, where $X$ is a finite well-ordered set, and $k_1$ is the value of $\Delta$ on the minimal element of $X$, $k_2$ is the value of $\Delta$ on the successor of the minimal element of $X$, and so on.

**Definition 5.3.6.** Let $(X, \Delta)$ be an abstract delta sequence with $\Delta = \langle k_1, \ldots, k_n \rangle$. Define the *symmetrization* of $(X, \Delta)$ by the abstract delta sequence $\Delta^{\mathrm{sym}} = \langle k_1, \ldots, k_n, -k_n, \ldots, -k_1 \rangle$. We call a delta sequence $\Delta$ symmetric if $\Delta = (\Delta')^{\mathrm{sym}}$ for some delta sequence $\Delta'$.

**Definition 5.3.7.** For delta sequences $\Delta_1 = \langle k_1, \ldots, k_n\rangle$ and $\Delta_2 = \langle \ell_1, \ldots, \ell_m\rangle$, we define the *join* delta sequence $\Delta_1 * \Delta_2$ by

$$\Delta_1 * \Delta_2 = \langle k_1, \ldots, k_n, \ell_1, \ldots, \ell_m\rangle.$$

For $\Delta$ a symmetric delta sequence, the $\mathbb{F}[U]$-module $\mathbb{H}(\Gamma_\Delta)$ admits an involution $\iota_\Delta$, given as follows. The delta sequence $\Delta$ gives a map:

$$\Delta\colon \{0, \ldots, 2n+1\} \to \mathbb{Z}.$$

Let $\iota\colon \{0, \ldots, 2n+2\} \to \{0, \ldots, 2n+2\}$ be $\iota(k) = 2n+2-k$. Then $\tau_\Delta$ is $\iota$-equivariant:

$$\begin{aligned} \Delta(\iota(k)) =& \tau_\Delta(\iota(k+1)) - \tau_\Delta(\iota(k)) \\ =& \tau_\Delta(2n+2-(k+1)) - \tau_\Delta(2n+2-k) \\ =& -(\tau_\Delta(2n+2-k) - \tau_\Delta(2n+1-k)) \\ =& -\Delta(2n+1-k) \\ =& \Delta(k). \end{aligned} \tag{5.12}$$

where in the last equality we have used that $\Delta$ is symmetric. We may then define $\iota_\Delta$ on each of the $R_{\tau_\Delta(k)}$ by acting as the identity map:

$$\iota_\Delta\colon R_{\tau_\Delta(k)} \to R_{\tau_\Delta(\iota(k))}.$$

Then $\iota_\Delta$ induces an involution of $\Gamma_\Delta$, and so also of $\mathbb{H}(\Gamma_\Delta)$, as an $\mathbb{F}[U]$-module.

We use the definition of symmetrization for delta sequences to further specify the form of the abstract delta sequence (and its reduction) associated to Brieskorn spheres.

Since $x \in S_Y$ if and only if $N_Y - x \in Q_Y$ (so, in particular, $\Delta_Y(x) = -\Delta_Y(N_Y - x)$), we have $N_Y/2 \notin X_Y$, and

$$\Delta_Y = (\Delta_Y|_{[0,N_Y/2]})^{\text{sym}}. \tag{5.13}$$

We also need a version of (5.13) for the reduction. By $\Delta_Y(x) = -\Delta_Y(N_y - x)$, if the maximal element of $X_Y \cap [0, N_Y/2]$ is in $S_Y$ (respectively $Q_Y$), then the minimal element of $X_Y \cap [N_Y/2, N_Y]$ is in $Q_Y$ ($S_Y$). Then

$$\tilde{\Delta}_Y = (\tilde{\Delta}_Y|_{[0,N_Y/2]})^{\text{sym}}. \tag{5.14}$$

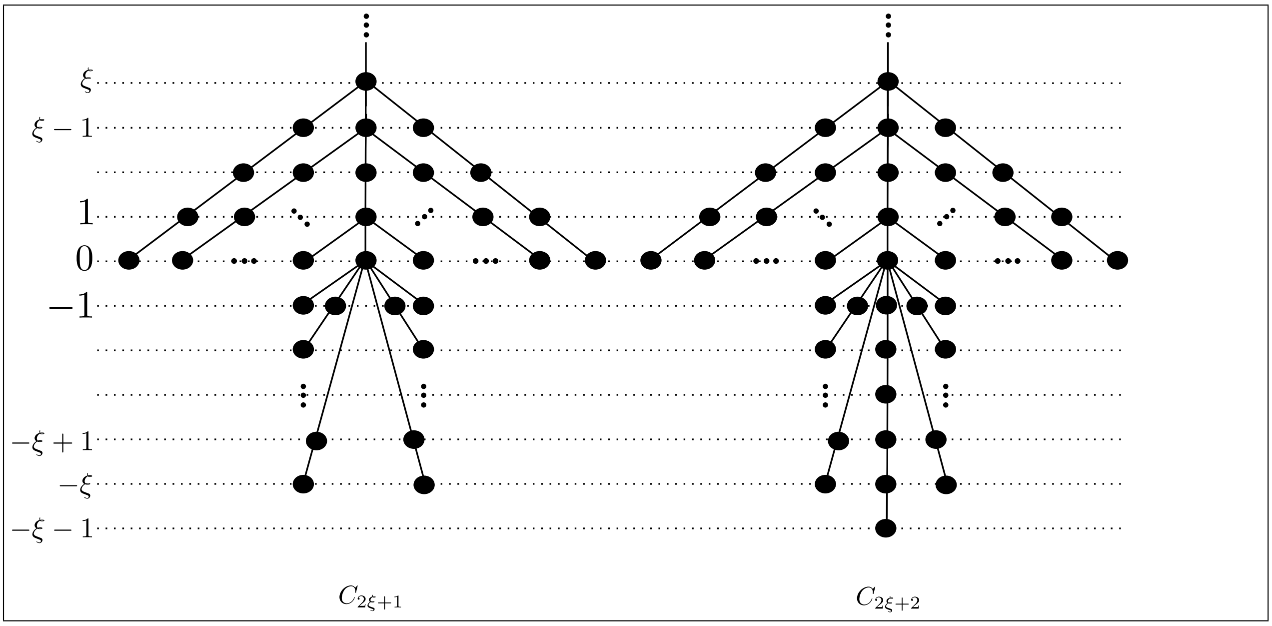


**Figure 5.2:** Creatures $\Gamma_{C_p}$. Based on Figure 5 of [18].

## 5.4 Semigroups and Creatures

In this section we will prove Theorem 1.3.8. First, we will introduce the *creatures* from [18] and write their delta sequences. Then we will prove a technical decomposition result (Lemma 5.4.2) for the graded roots of the Brieskorn spheres $\Sigma(p, 2p-1, 2p+1)$, for $p$ odd. Hom, Lidman and Karakurt were concerned with this family of Brieskorn spheres, but with $p$ even, and the proof of Lemma 5.4.2 is adapted from their proof of an analogous decomposition result, for $p$ even. We will quote, without proof, the lemmas from [18] that do not depend on parity, and suitably modify several other lemmas from that paper to account for the change in parity. We will then verify that $\Sigma(p, 2p-1, 2p+1)$ is of projective type, and calculate its $\beta$ and $d$. As in Section 5.3, we will be following [18] extremely closely.

### 5.4.1 Creatures

Hom, Karakurt, and Lidman [18] observe via examples that there are certain sub-graded roots occuring in $\Sigma(p, 2p-1, 2p+1)$, as shown in Figure 5.2. The two graded roots $\Gamma_{C_p}$ in Figure 5.2 are both called *creatures*.

The abstract delta sequence for the creature $\Gamma_{C_p}$ for $p = 2\xi + 2$, $\xi \in \mathbb{Z}_{\geqslant 1}$ is the symmetrization of

$$\Delta_{C_p} = \langle \xi, -\xi, (\xi-1), -(\xi-1), \ldots, 2, -2, 1, -2, 1, -2, 2, \ldots, -(\xi-1), \xi-1, -\xi, \xi, -(\xi+1) \rangle,$$

as observed in [18].

**Definition 5.4.1.** For every $p = 2\xi + 1$, with $\xi \in \mathbb{Z}_{\geqslant 1}$, the creature $\Gamma_{C_p}$ is the graded root defined by the symmetrization of the abstract delta sequence:

$$\Delta_{C_p} = \langle \xi, -\xi, (\xi-1), -(\xi-1), \ldots, 2, -2, 1, -2, 1, -2, 2, \ldots, -(\xi-1), \xi-1, -\xi, \xi \rangle. \quad (5.15)$$

Set $Y_p = \Sigma(p, 2p-1, 2p+1)$, and $\Delta_{Y_p}$ the abstract delta sequence corresponding to $Y_p$, with reduced form $\tilde{\Delta}_{Y_p}$. We have the following technical lemma, the analogue of [18][Lemma 5.3].

**Lemma 5.4.2.** *For every odd integer $p \geqslant 3$, we have the decomposition:*

$$\tilde{\Delta}_{Y_p} = (\Delta_{Z_p} * \Delta_{C_p})^{\mathrm{sym}}, \quad (5.16)$$

*where $\Delta_{Z_p}$ is a sinking delta sequence.*

Set $r_\pm = p(2p \pm 1)$ and $w = (2p+1)(2p-1)$. We work with the semigroup $S(r_-, r_+, w)$ on the generators $r_-, r_+$, and $w$ in studying the graded root associated to $Y_p$. The next three lemmas are verbatim from [18] and apply to both even and odd $p$.

**Lemma 5.4.3** ([18] Lemma 5.4)**.** *Let $S(r_-, r_+)$ be the semigroup generated by $r_-$ and $r_+$. The intersection $S(r_-, r_+) \cap [0, (p-1)r_+]$, as an ordered set, is given by:*

$$\begin{aligned}\{0,&\\ &r_-, r_+,\\ &2r_-, r_- + r_+, 2r_+,\\ &3r_-, 2r_- + r_+, r_- + 2r_+, 3r_+,\\ &\vdots\\ &(p-1)r_-, (p-2)r_- + r_+, \ldots, (p-1)r_+\}. \quad (5.17)\end{aligned}$$

**Lemma 5.4.4** ([18] Lemma 5.6)**.** *Say that $x \in S_{Y_p}$ is of the form $x = ar_- + br_+$, with $a, b \geqslant 0$, and $x \leqslant 2r_- + (p-3)r_+$. Then,*

1. $x < N_{Y_p} - (p-a-1)r_- - (p-b-3)r_+ < \mathrm{suc}_+(x)$.

2. $[\pi_-(x), \pi_+(x)] \cap S_{Y_p} = \{x - \min\{a,b\}, \ldots, x\}$, *unless* $x = (p-2)r_+$ *or* $(p-1)r_-$. *In either of these exceptional cases,* $[\pi_-(x), \pi_+(x)] \cap S_{Y_p} = \{(p-2)r_+, (p-1)r_-\}$.

**Lemma 5.4.5** ([18] Proposition 5.7 )**.** *The reduced form $\tilde{\Delta}_{Y_p}$ of $\Delta_{Y_p}$ satisfies:*

1. *As ordered subsets of* $\mathbb{N}$, $\tilde{S}_{Y_p} \cap [0, 2r_- + (p-3)r_+] = S(r_-, r_+) \cap [0, 2r_- + (p-3)r_+] \backslash \{(p-2)r_+\}$.

2. *Let* $x \in S(r_-, r_+) \cap [0, 2r_- + (p-3)r_+] \backslash \{(p-2)r_+, (p-1)r_-\}$ *be written* $x = ar_- + br_+$. *Then* $\tilde{\Delta}_{Y_p}(x) = \min\{a,b\} + 1$. *Further,* $\tilde{\Delta}_{Y_p}((p-1)r_-) = 2$.

3. *Let* $x \in \tilde{S}_{Y_p}$ *and say* $x < N_{Y_p} - cr_- - dr_+ < \mathrm{suc}_+(x)$, *where* $c, d \geqslant 0$. *Then* $\tilde{\Delta}_{Y_p}(\mathrm{suc}_-(x)) \leqslant -\min\{c,d\} - 1$.

Fix $p = 2\xi + 1$ for a positive integer $\xi$. Define

$$K = (\xi - 1)r_- + (\xi - 1)r_+. \tag{5.18}$$

We note two inequalities:

$$(p-1)r_- + (p-3)r_+ < N_{Y_p}, \tag{5.19}$$

$$(p-2)r_- + (p-2)r_+ > N_{Y_p}. \tag{5.20}$$

Note

$$K < (p-3)r_+ < N_{Y_p}/2, \tag{5.21}$$

by (5.19). By (5.14),

$$\tilde{\Delta}_{Y_p} = (\tilde{\Delta}_{Y_p}|_{\tilde{X}_{Y_p} \cap [0,K)} * \tilde{\Delta}_{Y_p}|_{\tilde{X}_{Y_p} \cap [K, N_{Y_p}/2]})^{\mathrm{sym}}. \tag{5.22}$$

Let $S(r_-, r_+)$ be the semigroup generated by $r_-, r_+$. Observe that $K \in S(r_-, r_+) \cap [0, 2r_- + (p-3)r_+]$ and $K \neq (p-2)r_+$, so $K \in \tilde{S}_{Y_p}$ by Lemma 5.4.5. Set:

$$\Delta_{Z_p} = \tilde{\Delta}_{Y_p}|_{\tilde{X}_{Y_p} \cap [0,K)} \tag{5.23}$$

$$\Delta_{W_p} = \tilde{\Delta}_{Y_p}|_{\tilde{X}_{Y_p} \cap [K, N_{Y_p}/2]}. \tag{5.24}$$

**Lemma 5.4.6** (cf. Lemma 5.8 of [18])**.** *For $p \geqslant 3$ odd, the abstract delta sequence $\Delta_{Z_p}$ is sinking.*

*Proof.* We check (1)-(3) of Definition 5.3.4. For (1), we recall that $\Delta_{Z_p}$ is in reduced form. We saw above that $K \in \tilde{S}_{Y_p}$, so if the last element of the delta sequence $\Delta_{Z_p}$ were positive, $\tilde{\Delta}_{Y_p}$ would have two consecutive positive values, contradicting that $\tilde{\Delta}_{Y_p}$ is reduced. This establishes (1) in Definition 5.3.4.

As in [18], we denote predecessors and successors taken with respect to $\tilde{X}_{Y_p}$ with a tilde, and those with respect to $X_{Y_p}$ without a tilde. By the construction of the reduced delta sequence as in Section 5.3.4,

$$\mathrm{suc}_+(x) \leqslant \widetilde{\mathrm{suc}}_+(x) \text{ for every } x \in \tilde{X}_{Y_p}. \tag{5.25}$$

We will next show:

$$\tilde{\Delta}_{Y_p}(x) \leqslant -\tilde{\Delta}_{Y_p}(\widetilde{\mathrm{suc}}_-(x)) \text{ for all } x \in \tilde{S}_{Y_p} \cap [0, K), \tag{5.26}$$

to establish (2) of Definition 5.3.4. Let $x \in \tilde{S}_{Y_p} \cap [0, K)$. Then $x \in S(r_-, r_+) \cap [0, (p-3)r_+]$ by (5.21) and Lemma 5.4.5(1). Writing $x = ar_- + br_+$, Lemma 5.4.5(2) gives $\tilde{\Delta}_{Y_p}(x) = \min\{a, b\} + 1$. Set

$$y = (p - a - 1)r_- + (p - b - 3)r_+.$$

Lemma 5.4.4 and (5.25) give:

$$x < N_{Y_p} - y < \mathrm{suc}_+(x) \leqslant \widetilde{\mathrm{suc}}_+(x).$$

By $x \in S(r_-, r_+) \cap [0, (p-3)r_+]$, we see that $a + b \leqslant p - 3$. Thus $p - a - 1 \geqslant 0$ and $p - b - 3 \geqslant 0$. Then, by the definition of $Q_{Y_p}$, $N_{Y_p} - y \in Q_{Y_p}$. Lemma 5.4.5(3) gives

$$\tilde{\Delta}_{Y_p}(\widetilde{\mathrm{suc}}_-(x)) \leqslant -\min\{p - a - 1, p - b - 3\} - 1.$$

Then, to prove (5.26) it is sufficient to show

$$\min\{a, b\} \leqslant \min\{p - a - 1, p - b - 3\}. \tag{5.27}$$

But $a + b \leqslant p - 3$, so $a \leqslant p - b - 3$ and $b \leqslant p - a - 3$, showing (5.27).

We must check that Definition 5.3.4(3) holds for $\Delta_{Z_p}$. The last positive value of $\Delta_{Z_p}$ occurs at $\widetilde{\text{pre}}_+(K) = \xi r_- + (\xi - 2)r_+$ by Lemma 5.4.3 and Lemma 5.4.5(1). Thus $\widetilde{\text{suc}}_-(\xi r_- + (\xi - 2)r_+)$ is the largest element of $Z_p$. Then to show Definition 5.3.4(3) holds for $\Delta_{Z_p}$, we need to show:

$$\tilde{\Delta}_{Y_p}(\xi r_- + (\xi - 2)r_+) < -\tilde{\Delta}_{Y_p}(\widetilde{\text{suc}}_-(\xi r_- + (\xi - 2)r_+)). \tag{5.28}$$

By Lemma 5.4.5(2), $\tilde{\Delta}_{Y_p}(\xi r_- + (\xi - 2)r_+) = \xi - 1$. However, Lemma 5.4.4(1) gives:

$$\xi r_- + (\xi - 2)r_+ < N_{Y_p} - (p - \xi - 1)r_- - (p - \xi - 1)r_+ < \text{suc}_+(\xi r_- + (\xi - 2)r_+) \leqslant K.$$

Then from Lemma 5.4.5(3):

$$-\tilde{\Delta}_{Y_p}(\widetilde{\text{suc}}_-(\xi r_- + (\xi - 2)r_+)) \geqslant \min\{p - \xi - 1, p - \xi - 1\} + 1 = p - \xi.$$

Then to show (5.28), we need only show $\xi - 1 < p - \xi$, which is clear since $p = 2\xi + 1$. □

**Lemma 5.4.7** (cf. Lemma 5.9 of [18])**.** *Let $p \geqslant 3$ odd. As abstract delta sequences $\Delta_{W_p} \cong \Delta_{C_p}$ where $\Delta_{C_p}$ is as in Definition 5.4.1.*

*Proof.* We must explicitly compute $\Delta_{W_p}$. We begin by describing $\tilde{S}_{Y_p} \cap [K, N_{Y_p}/2]$. By (5.21), $K < N_{Y_p}$, and by (5.20), $N_{Y_p}/2 < (p - 2)r_+$. By Lemma 5.4.5, we see $\tilde{S}_{Y_p} \cap [K, N_{Y_p}/2] = S(r_-, r_+) \cap [K, N_{Y_p}/2]$. Then Lemma 5.4.3 gives:

$$\begin{aligned}\tilde{S}_{Y_p} \cap [K, N_{Y_p}/2] = \{&(\xi - 1)r_- + (\xi - 1)r_+, (\xi - 2)r_- + \xi r_+, \dots, r_- + (2\xi - 3)r_+, \\ &(2\xi - 2)r_+, (2\xi - 1)r_-, (2\xi - 2)r_- + r_+, \dots, \xi r_- + (\xi - 1)r_+\}.\end{aligned} \tag{5.29}$$

To check that the last term of the sequence (5.29) is as written, we need to show

$$\xi r_- + (\xi - 1)r_+ < N_{Y_p}/2, \tag{5.30}$$

and

$$(\xi - 1)r_- + \xi r_+ > N_{Y_p}/2. \tag{5.31}$$

To see (5.30), note that (5.19) gives $2\xi r_- + (2\xi - 2)r_+ < N_{Y_p}$, so $\xi r_- + (\xi - 1)r_+ < N_{Y_p}/2$. To see (5.31), note that (5.20) gives $(2\xi - 1)r_- + (2\xi - 1)r_+ > N_{Y_p}$, so $(\xi - \frac{1}{2})r_- + (\xi - \frac{1}{2})r_+ > N_{Y_p}/2$, and observe $(\xi - 1)r_- + \xi r_+ > (\xi - \frac{1}{2})r_- + (\xi - \frac{1}{2})r_+$. Thus, (5.29) holds.

We also find $\tilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$, which is the same as finding $\tilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K]$. By (5.20) and (5.21),

$$N_{Y_p}/2 < N_{Y_p} - K < 2r_- + (p-3)r_+. \tag{5.32}$$

By Lemma 5.4.5(1), $\tilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K] = S(r_-, r_+) \cap [N_{Y_p}/2, N_{Y_p} - K] \backslash \{(2\xi - 1)r_+\}$. Then, by Lemma 5.4.3,

$$\begin{aligned}\tilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K] = \{&(\xi-1)r_- + \xi r_+, (\xi-2)r_- + (\xi+1)r_+, \dots, r_- + (2\xi-2)r_+,\\ &2\xi r_-, (2\xi-1)r_- + r_+, \dots, (\xi+1)r_- + (\xi-1)r_+\}.\end{aligned} \tag{5.33}$$

Note that $(2\xi - 1)r_+$ is not present in (5.33). To verify that $(\xi+1)r_- + (\xi-1)r_+$ is the last element in $\tilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K]$, we must show

$$(\xi+1)r_- + (\xi-1)r_+ < N_{Y_p} - K, \text{ and} \tag{5.34}$$

$$\xi r_- + \xi r_+ > N_{Y_p} - K. \tag{5.35}$$

Inequality (5.34) follows from (5.19) and the definition of $K$, while (5.35) follows from (5.20). Thus (5.33) holds.

We find the positions of elements of $\tilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$ relative to the elements of $\tilde{S}_{Y_p} \cap [K, N_{Y_p}/2]$. To do so, we use the following inequalities, all obtained from (5.19) and (5.20). For $0 \leqslant j \leqslant \xi - 1$, we have:

$$(\xi-1-j)r_- + (\xi-1+j)r_+ < N_{Y_p} - (\xi+1+j)r_- - (\xi-1-j)r_+. \tag{5.36}$$

For $0 \leqslant j \leqslant \xi - 2$, we have:

$$N_{Y_p} - (\xi+1+j)r_- - (\xi-1-j)r_+ < (\xi-2-j)r_- + (\xi+j)r_+, \tag{5.37}$$

$$jr_+ + (2\xi-1-j)r_- < N_{Y_p} - (j+1)r_- - (2\xi-2-j)r_+, \tag{5.38}$$

$$N_{Y_p} - (j+1)r_- - (2\xi-2-j)r_+ < (j+1)r_+ + (2\xi-2-j)r_-. \tag{5.39}$$

We observe

$$N_{Y_p} - 2\xi r_- < (2\xi-1)r_- \tag{5.40}$$

directly from the definitions, and

$$N_{Y_p} - (\xi-1)r_- - \xi r_+ < \xi r_- + (\xi-1)r_+ \tag{5.41}$$

from (5.20).

It follows from (5.29), (5.33), (5.36)-(5.39), (5.40), and (5.41) that $\tilde{X}_{Y_p} \cap [K, N_{Y_p}/2]$ is:

$$\begin{gathered}\tilde{X}_{Y_p} \cap [K, N_{Y_p}/2] = \{(\xi-1)r_- + (\xi-1)r_+, N_{Y_p} - (\xi+1)r_- - (\xi-1)r_+, (\xi-2)r_- + \xi r_+, \\ N_{Y_p} - (\xi+2)r_- - (\xi-2)r_+, \ldots, r_- + (2\xi-3)r_+, \\ N_{Y_p} - (2\xi-1)r_- - r_+, (2\xi-2)r_+, N_{Y_p} - 2\xi r_-, (2\xi-1)r_-, \\ N_{Y_p} - r_- - (2\xi-2)r_+, (2\xi-2)r_- + r_+, N_{Y_p} - 2r_- - (2\xi-3)r_+, \ldots, (\xi+2)r_- + (\xi-3)r_+, \\ N_{Y_p} - (\xi-2)r_- - (\xi+1)r_+, (\xi+1)r_- + (\xi-2)r_+, N_{Y_p} - (\xi-1)r_- - \xi r_+, \xi r_- + (\xi-1)r_+\}.\end{gathered} \tag{5.42}$$

Now we need to calculate $\tilde{\Delta}_{Y_p}$ on $\tilde{X}_{Y_p} \cap [K, N_{Y_p}/2]$, and verify that it agrees with $\Delta_{C_p}$. By Lemma 5.4.5(2) and $N_{Y_p}/2 < (p-2)r_+$,

$$\tilde{\Delta}_{Y_p}(cr_- + dr_+) = \min\{c,d\} + 1 \text{ for } cr_- + dr_+ \in \tilde{S}_{Y_p} \cap [K, N_{Y_p}/2]. \tag{5.43}$$

Similarly, for $N_{Y_p} - cr_- - dr_+ \in \tilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$ such that $cr_- + dr_+ \neq 2\xi r_-$:

$$\tilde{\Delta}_{Y_p}(N_{Y_p} - cr_- - dr_+) = -\tilde{\Delta}_{Y_p}(cr_- + dr_+) = -\min\{c,d\} - 1 \tag{5.44}$$

by Lemma 5.4.5(2), using (5.32) to obtain $cr_- + dr_+ < N_{Y_p} - K < 2r_- + (p-3)r_+$. Also, Lemma 5.4.5 gives

$$-2 = -\tilde{\Delta}_{Y_p}(2\xi r_-) = \tilde{\Delta}_{Y_p}(N_{Y_p} - 2\xi r_-). \tag{5.45}$$

Computing $\tilde{\Delta}_{Y_p}$ using (5.43),(5.44), and (5.45), we see that $\Delta_{W_p}$ agrees with $\Delta_{C_p}$ from Definition 5.4.1. This completes the proof of Lemma 5.4.2. □

*Proof of Theorem 1.3.8.* By Remark 3.3 of [18], $d(Y_p) = p-1$, so we need only show that $Y_p$ is of projective type, and that $\beta(Y_p) = 0$.

Let $\Gamma_{Y_p}$ have its grading shifted so that it agrees with the grading of $HF^+(-Y_p)$ (using Theorem 5.3.2). The decomposition in Lemma 5.4.2 implies that $\Gamma_{C_p}$ embeds into $\Gamma_{Y_p}$ as a subgraph. Since $d(-Y_p) = 1-p$, we see that the embedding of $\Gamma_{C_p}$ is degree-preserving. Since $\Delta_{Z_p}$ is sinking, by Proposition 5.3.5 the minimal value of $\tau_{Z_p}$ is 0. Thus

$$\mathbb{H}_{\leqslant 0}(\Gamma_{C_p}) = \mathbb{H}_{\leqslant 0}(\Gamma_{Y_p}).$$

By Fact 4.2.6 applied to the graded root $\Gamma_{\Delta_{Y_p}}$ (see Figure 5.2), we have that $Y_p$ is of projective type. It is clear from Figure 5.2 that the vertex of minimal grading which is invariant under $\iota$ is in degree 0, from which we obtain $\beta(Y_p) = 0$. $\square$

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