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DIGITAL-COMPUTER ANALYSIS OF DATA FROM BUBBLE CHAMBER III. THE KINEMATICAL ANALYSIS OF INTERACTION VERTICES

J. Peter Berge and Frank T. Solmitz

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ABSTRACT

A high-speed computer program for the kinetmatical analysis of bubble chamber events is described. The program treats individual particle interaction or decay vertices, subjecting the measured variables to the equations of energy and momentum according to the least-squares criterion. This is done in four different cases in which the problem is overdetermined, and a fifth case is calculated in which the problem is just determined. The adjusted or computed variables corresponding to each particle are obtained in all cases as well as the first-order error matrices for each type of variable. A connected chain of vertices may be processed in sequence.

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I. INTRODUCTION

The analysis of large amounts of data obtained from the observation and measurement of particle interactions, especially in bubble chambers, may be accomplished in roughly three stages. The first stage may consist of an analysis of the measurements in order to determine the most probable values of the various quantities--such as coordinates, space angles, curvature or momentum, and energy or velocity--to be assigned to each particle both before and after the interaction. In addition, the errors to be associated with each of these quantities must be determined. Although a mass assignment for each particle must be made if the data are to be completely utilized, this first stage of analysis is usually kept as independent as possible of any hypothesis as to the nature of the interaction in order to preserve its generality.

In the second stage, the output of such a "one-track" analysis or spatial reconstruction program may be used in conjunction with kinematical requirements to interpret the interaction under consideration. Such a kinematical analysis may be used in the choice of one of several possible interpretations, as well as to reduce the uncertainties in the kinematical variables to be assigned to each particle. Finally, in the third stage, the

results from separate events may be combined and the usual statistical analysis performed upon a group of events.

A possible method for performing the second stage, namely the kinematical analysis, has been formulated and is described in this article. The treatment is sufficiently general that a large part of the computation may be applied to almost any kinematical situation, leaving a minimum amount of logical control to be constructed for any particular experiment. This generality has been achieved by reducing the problem to a consideration of one interaction or decay vertex at a time. When the problem has been solved at a vertex, the information acquired may be propagated to any connected vertex and used in fitting a second interaction or decay. The mathematical problem of applying the kinematical equations at a vertex is thus isolated, and considerable flexibility is achieved without great complication.

The single-vertex kinematical analysis program to be described here has been coded under the title GUTS for the IBM 704 computer as a closed subroutine approximately 5000 words long, and has been tested under a large variety of conditions. This description is limited to the general principles and equations involved. Details of the coding may be obtained from the authors.

II. NOTATION

In this article the following notation conventions are used:

P The total number of outgoing particles plus one (if there is an incoming particle); for the program GUTS, $2 \leq P \leq 7$.

L The number of analytic constraints to be applied at the interaction vertex. Here $0 \leq L \leq 4$.

I The number of measured variables. $I=3P + (L-4)$, and, for GUTS, $2 \leq I \leq 21$.

D The distance back from the interaction vertex at which the curvature k and the uncertainty in curvature δk are specified for the first (incident) particle.

ϕ_q The azimuthal angle of the qth particle at the vertex, measured from some arbitrary axis.

$\tan \lambda_q$ The tangent of the latitude of the qth particle at the vertex, measured from the plane in which ϕ_q is measured.

k_q The "projected curvature" of the qth particle, defined by $k_q = [P_q \cos \lambda_q]^{-1}$.

p_q The momentum of the qth particle, in Mev/c.

E_q The energy of the qth particle (including rest energy), in Mev.

m_q The mass of the qth particle, in Mev.

m_T The mass of the target, in Mev.

x_i Any measured variable (ϕ_q, $\tan \lambda_q$, k_q).

y_a Any unmeasured variable computed by the program.

Subscripts:

s Refers to a completely measured particle.

ℓ For the 3c case, refers to the particle with unmeasured momentum.

For the 2c case, refers to the first particle with unmeasured momentum.

For the 1c case, refers to the completely unmeasured particle.

For the 0c case, refers to the completely unmeasured particle.

m For the 2c case, refers to the second particle with unmeasured momentum.

u For the 0c case, refers to the particle with unmeasured momentum.

i, j, k, ... Refer to any variable x.

λ, μ, σ, ... Refer to the λth constraint.

III. FORMULATION OF THE FITTING PROBLEM

A. Variables

For the purposes of the present program, it is assumed that three quantities are associated with each track, namely the "curvature" or momentum, and two space angles. Not all these quantities need be measured. Since the further assumption is to be made that the measured variables are normally distributed, it is important to pick a set of variables that approximately fulfill this requirement. The variable tan λ is chosen here because of its convenience, although it is obviously not even approximately normally distributed in the limit as $\lambda \longrightarrow \pm 90^{\circ}$. However, the method outlined below is applicable to other choices of variable.

With these considerations in mind, the following variables are used:

$$x_1 = \phi_1; x_2 = \tan \lambda_1; x_3 = k_1; x_4 = \phi_2; x_5 = \tan \lambda_2; \ldots \qquad \text{(III-1)}$$

The measured values of these variables are denoted by x^m_j, where j runs from 1 to I and it is assumed that the errors on the measured values are known. With these errors denoted by δx^m_j, the usual error matrix is then given by

$$\overline{\delta x^m_i \, \delta x^m_j} = G^{-1}_{ij} \quad , \tag{III-2}$$

and must be supplied to the program. As presently set up, the program assumes that individual particles are not correlated with one another, i.e., that G_{ij}^{-1} consists of a set of matrices on the diagonal, none of which is larger than 3×3. However, no assumption is made as to whether or not these individual-particle error matrices are diagonal.

B. Method

A tentative interpretation of the vertex is then made consisting of a decision as to the number of tracks associated with the vertex and a mass assignment for each track. The assumption of normally distributed variables then leads to the usual principle of least squares, i.e., the optimum set of variables x_i is that set which minimizes the function

$$\chi^2 = \sum_{\substack{i=1 \\ j=1}}^{I} (x_i - x^m_i) \, G_{ij} \, (x_j - x^m_j), \tag{III-3}$$

subject to whatever constraints are required to satisfy energy and momentum conservation.[1] Let these constraints be denoted by

$$F_\lambda(x_i) = 0 \quad \text{for} \quad \lambda = 1, 2, \ldots \, L, \tag{III-4}$$

where λ runs from 1 to L. Two procedures are available. The equations of constraint may be used to eliminate L variables from the function $\chi^2(x_1, x_2, \ldots x_I)$ and the I-L minimizing equations may be solved, or L Lagrange multipliers a_λ may be introduced and the problem may be reduced

to finding the stationary value of the function

$$M = \sum_{\substack{i=1\\j=1}}^{I} (x_i - x^m_i)\, G_{ij}\, (x_j - x^m_j) + 2 \sum_{\lambda=1}^{L} a_\lambda F_\lambda(x) . \qquad \text{(III-5)}$$

Since it can be shown that if the constraint functions are linear the latter procedure involves the solution of only L rather than $I-L$ simultaneous equations, it seems reasonable to expect that the latter procedure is superior at least over regions of variation of x sufficiently small that the $F_\lambda(x)$ may be considered to be approximately linear.

Using the variables listed above, one can write the usual equations of constraint in the following form:

$$F_1 = \sum_{q=1}^{P} (\pm)_q \cos \phi_q / k_q = 0 , \qquad \text{(III-6a)}$$

$$F_2 = \sum_{q=1}^{P} (\pm)_q \sin \phi_q / k_q = 0 , \qquad \text{(III-6b)}$$

$$F_3 = \sum_{q=1}^{P} (\pm)_q \tan \lambda_q / k_q = 0 , \qquad \text{(III-6c)}$$

$$F_4 = \sum_{q=1}^{P} (\pm)_q E_q - m_T = 0 , \qquad \text{(III-6d)}$$

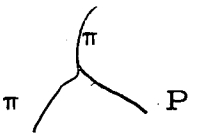
where $(\pm)_q$ is $+1$ if particle q is outgoing and -1 if particle q is incoming. These equations then represent the conservation of momentum along three mutually perpendicular axes in the chamber and the conservation of energy. A vertex at which all quantities are measured is subject to all four constraints and may be classified as a 4c vertex. If the momentum of one particle is not measured, Eq. (III-6d) may be used to solve for this momentum, and since only three constraints remain to be satisfied, this may be classified as a 3c vertex. Similar procedures are used for 2c and 1c vertices,

and a least-squares fit may be performed in each case. If four variables are unmeasured, the number of constraint equations is just sufficient to calculate the missing variables and no least-squares fit can be made. If more than four variables are unmeasured, the problem is undetermined.

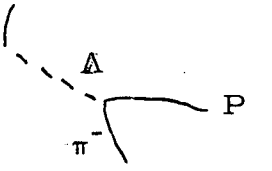
In the above cases in which fitting or calculation is possible, the program GUTS finds the optimum values for the measured variables x_i and computes the corresponding missing variables y_α. GUTS also computes the three error matrices $\overline{\delta x_i \delta x_j}$, $\overline{\delta x_i \delta y_\alpha}$, and $\overline{\delta y_\alpha \delta y_\beta}$. All these quantities are evaluated at the interaction vertex. In each of the five cases, however, only certain variables are allowed to be missing. These restrictions have been made on the grounds of physical plausibility. We list each case, describe what variables are allowed to be missing, and give one example of each, as follows.

Case

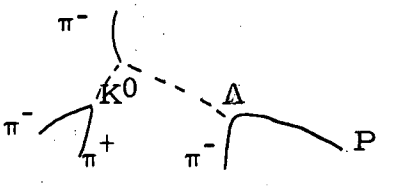
4c: All variables for all tracks are measured. Example is a π-p scattering in which all tracks are measurable.



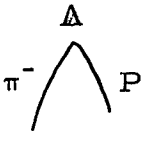
3c: One momentum is unmeasurable. Example a Λ decay when the direction of the Λ is known.



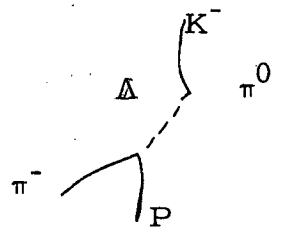
2c: Two momenta are unmeasurable. Example is an associated-production vertex for which both the Λ and the k^0 direction are known.



1c: All variables corresponding to one particle are unmeasurable. Example is a Λ decay for which the direction of the Λ is not known.



0c: All variables corresponding to one particle are unmeasurable and the momentum of a second particle is unmeasurable. Example is the reaction $K^- + p \longrightarrow \Lambda + \pi^0$ for which the Λ direction is known.



IV. SOLUTION OF THE FITTING PROBLEM

A. Iteration Equations

The problem of finding a stationary value for the function may be solved by a simple iteration. The equations to be solved are

$$\frac{\partial M}{\partial x_i} = \left\{ 2 \sum_{j=1}^{I} G_{ij} (x_j - x^m_j) + \sum_{\lambda=1}^{L} F_{i\lambda}(x)\, a_\lambda \right\} = 0 \text{ for } i = 1, 2, \ldots I, \tag{IV-1}$$

$$\frac{\partial M}{\partial a_\lambda} = 2F_\lambda(x) = 0 \text{ for } \lambda = 1, 2, \ldots L, \tag{IV-2}$$

where we use the notation

$$\partial F_\lambda(x)/\partial x_i \equiv F_{i\lambda}(x), \tag{IV-3}$$

and (IV-2) is of course just the constraint conditions. The problem is now reduced to solving the I+L equations (IV-1) and (IV-2) for the I+L variables x_i and a_λ. Equation (IV-1) may be rewritten

$$x_i = x^m_i - \sum_{\lambda=1}^{L} E_{i\lambda}\, a_\lambda \text{ for } i = 1, 2, \ldots I, \tag{IV-4}$$

where

$$E_{i\lambda}(x) = \sum_{j=1}^{I} G^{-1}_{ij}\, F_{j\lambda}(x). \tag{IV-5}$$

Equation (IV-4) may then be used to define an iterative procedure. We expand the constraint equations around some approximate solution x_i^{ν}, where the superscripts refer to the order of the iteration, getting from (IV-2) and (IV-4)

$$F_\lambda^\nu + \sum_{j=1}^{I} (x_j^{\nu+1} - x^\nu{}_j)\, F_{j\lambda}^\nu = 0 \quad \text{for} \quad \lambda = 1, 2, \ldots L \tag{IV-6}$$

and

$$x_i^{\nu+1} = x_i^m - \sum_{\lambda=1}^{L} E_{i\lambda}^\nu \; a_\lambda^{\nu+1} \quad \text{for} \quad i = 1, 2, \ldots I\,, \tag{IV-7}$$

where we use the notation

$$F_{j\lambda}^\nu \equiv F_{j\lambda}(x^\nu) \;;\; E_{j\lambda}^\nu = E_{j\lambda}(x^\nu)\,. \tag{IV-8}$$

We may now eliminate $x_i^{\nu+1}$ from Eqs. (IV-6) and (IV-7) and solve for $a_\lambda^{\nu+1}$. Defining for convenience

$$b_\lambda^\nu \equiv F_\lambda^\nu + \sum_{j=1}^{I} (x^m{}_j - x_j)\, F_{j\lambda}^\nu \quad \text{for} \quad \lambda = 1, 2, \ldots L \tag{IV-9}$$

and

$$H_{\lambda\mu}^\nu \equiv \sum_{j=1}^{I} (E^\nu)_{\lambda j}^T \, F_{j\mu}^\nu\,, \tag{IV-10}$$

we find

$$a_\lambda^{\nu+1} = \sum_{\mu=1}^{L} (H^\nu)_{\lambda\mu}^{-1} \; b_\mu^\nu \quad \text{for} \quad \lambda = 1, 2, \ldots L\,. \tag{IV-11}$$

This solution for the Lagrange multipliers may then be used in conjunction with Eq. (IV-7) to solve for the entire set of $x_i^{\nu+1}$.

To start the iteration, we use $x_i^0 = x_i^m$ and $a_\lambda^0 = 0$, and we continue the iteration until a "fit" is found. At every step we have

$$(\chi^2)^{\nu+1} = \sum_{\substack{i=1 \\ j=1}}^{I} (x_i^{\nu+1} - x^m{}_i)\, G_{ij}\, (x_j^{\nu+1} - x_j^m)\,, \tag{IV-12}$$

and, from (IV-7),

$$(x_i^{\nu+1} - x_i^m) = -\sum_{\lambda=1}^{L} E_{i\lambda}^{\nu} a_{\lambda}^{\nu+1} \quad \text{for } i = 1, 2, \ldots I . \tag{IV-7a}$$

Substituting into (IV-12) gives

$$(\chi^2)^{\nu+1} = \sum_{\substack{i=1 \\ j=1}}^{I} \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} \left\{ a_{\lambda}^{\nu+1} (E^{\nu})_{\lambda i}^{T} G_{ij} E_{j\mu}^{\nu} a_{\mu}^{\nu+1} \right\}$$

$$= \sum_{\lambda=1}^{L} \left\{ a_{\lambda}^{\nu+1} \left[\sum_{\mu=1}^{L} \sum_{i=1}^{I} (E^{\nu})_{\lambda i}^{T} F_{i\mu}^{\nu} a_{\mu}^{\nu+1} \right] \right\}$$

$$= \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} E_{\lambda}^{\nu+1} H_{\lambda\mu}^{\nu} a_{\mu}^{\nu+1} . \tag{IV-13}$$

Thus, using the definition of b_{λ}^{ν}, Eq. (IV-9), at every step we have the algebraic identity

$$(\chi^2)_a^{\nu+1} = \sum_{\substack{i=1 \\ j=1}}^{I} (x_i^{\nu+1} - x_i^m) G_{ij} (x_j^{\nu+1} - x_j^m) \equiv \sum_{\lambda=1}^{L} a_{\lambda}^{\nu+1} b_{\lambda}^{\nu} = (\chi^2)_b^{\nu+1} , \tag{IV-14}$$

which is independent of the linearity assumption. The program GUTS uses this identity to check the rounding errors inherent in so much numerical work. We insist that, on each step,

$$\left| \frac{(\chi^2)_a^{\nu+1} - (\chi^2)_b^{\nu+1}}{(\chi^2)_a^{\nu+1}} \right| < \epsilon_1 . \tag{IV-15}$$

This fails to be satisfied only if bad rounding errors creep in or if the matrix $H_{\lambda\mu}$ is very badly conditioned.

B. The Fit: Definition of Convergence

Each step of the iteration yields an approximate solution to the problem and it is necessary to set criteria that determine at what step the approximation is good enough, i. e., to define a "fit".

It is important to note here and in the following sections that in reducing expressions involving statistical averages we assume that the constraint functions are linear over the regions of interest. Thus all equations involving these averages are only approximate.

Since a minimum value of χ^2 subject to the kinematical constraints will have been reached when (a) all the constraint functions, which are the derivatives of M(x, a) with respect to the Lagrange multipliers, and (b) all the derivatives of M(x, a) with respect to the variables x_i are vanishingly small, it is only necessary to find reasonable criteria which these functions are to satisfy. For the first set of derivatives we introduce the function

$$\overline{F^{\nu+1}} = \left\{ \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} \overline{\left(\frac{\partial M}{\partial a_\lambda}^{\nu+1} \delta a_\lambda^{\nu+1}\right)\left(\frac{\partial M}{\partial a_\mu}^{\nu+1} \delta a_\mu^{\nu+1}\right)} \right\} . \qquad \text{(IV-16)}$$

Using the linearity assumption and (IV-9), we find

$$\delta b_\lambda^\nu = \sum_{j=1}^{I} \delta x_j^m F_{j\lambda}^\nu \qquad \text{(IV-17)}$$

and

$$\delta b_\lambda^\nu \delta b_\mu^\nu = H_{\lambda\mu}^\nu , \qquad \text{(IV-18)}$$

where we have used (IV-10) and (IV-5).

Similarly, using (IV-11) and (IV-18), we have

$$\delta a_\lambda^{\nu+1} = \sum_{\mu=1}^{L} (H^\nu)^{-1}_{\lambda\mu} \delta b_\mu^\nu \qquad \text{(IV-19)}$$

and

$$\overline{\delta a_\lambda^{\nu+1} \delta a_\mu^{\nu+1}} = (H^\nu)^{-1}_{\lambda\mu} . \qquad \text{(IV-20)}$$

One may now use this result to show

$$\overline{F^{\nu+1}} = 2 \left\{ \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} F_\lambda^{\nu+1} (H^\nu)^{-1}_{\lambda\mu} F_\mu^{\nu+1} \right\} . \qquad \text{(IV-21)}$$

Thus $\overline{F^{\nu+1}}$ is a positive definite function of the constraints. In the linear approximation to the problem, where only one step is necessary to arrive at the solution, it is easily shown that this function evaluated at the measured values x^m is equal to χ^2 and hence has the expectation value L as shown in Section IVC. If the constraints are to be satisfied much better than at $x=x^m$ it seems reasonable to demand that $\overline{F^{\nu+1}}$ be small compared with L. We therefore require

$$\overline{F^{\nu+1}} < \epsilon_2 , \qquad \text{(IV-22)}$$

where ϵ_2 is to be chosen $<<$ L.

For the second set of derivatives we define a positive definite function

$$\delta M^{\nu+1} = \frac{1}{4} \left\{ \sum_{\substack{i=1 \\ j=1}}^{I} \overline{\left(\frac{\partial M^{\nu+1}}{\partial x_i}\right)_{\nu+1} \delta x_i^m \left(\frac{\partial M^{\nu+1}}{\partial x_j}\right)_{\nu+1} \delta x_j^m} \right\} . \qquad \text{(IV-23)}$$

In the same linear approximation discussed above, one may show that at any point x for which

$$F_\lambda(x) = 0$$

the difference of the value of $M(x, a)$ from its minimum value M^* is given by

$$M(x, a) - M^* = \delta M^{\nu+1} . \qquad \text{(IV-24)}$$

If a point x is such that $\delta M^{\nu+1} << 1$, then its probability differs by a negligible amount from that of the "optimum" point x^*. We then require

$$\delta M^{\nu+1} < \epsilon_3 , \qquad \text{(IV-25)}$$

where ϵ_3 is to be chosen $<< 1$.

This last requirement may be put into explicit form by considering

$$\frac{\partial M^{\nu+1}}{\partial x_i^{\nu+1}} = 2\left\{\sum_{j=1}^{I} G_{ij}(x_i^{\nu+1} - x_j^{m}) + \sum_{\lambda=1}^{L} F_{i\lambda}^{\nu+1} a_\lambda^{\nu+1}\right\} . \qquad \text{(IV-26)}$$

Substituting Eq. (IV-7) into the above, we get

$$\frac{\partial M^{\nu+1}}{\partial x_i^{\nu+1}} = 2\sum_{\lambda=1}^{L}(F_{i\lambda}^{\nu+1} - F_{i\lambda}^{\nu})\, a_\lambda^{\nu+1} . \qquad \text{(IV-27)}$$

Substituting (IV-27) into (IV-23), we can show

$$\delta M^{\nu+1} = \left\{\sum_{\substack{i=1\\ j=1}}^{I} \psi_i^{\nu+1} G_{ij}^{-1} \psi_j^{\nu+1}\right\} ,$$

where

$$\psi_i = 2\sum_{\lambda=1}^{L}(F_{i\lambda}^{\nu+1} - F_{i\lambda}^{\nu})\, a_\lambda^{\nu+1} .$$

Our requirement then becomes

$$\partial M^{\nu+1} = \sum_{\substack{i=1\\ j=1}}^{I} \psi_i^{\nu+1} G_{ij}^{-1} \psi_j^{\nu+1} < \epsilon_3 \qquad \text{(IV-29)}$$

C. The Stationary Value of M

Equations (IV-13) and (IV-20) also lead to an expression for the average value of $M(x, a)$ at its stationary value, say M^*. Since the constraint functions have been brought to zero, this corresponds to the expected minimized value of χ^2. Since we have $\bar{a}_\lambda = 0$, we have from (IV-20)

$$\overline{a_\lambda^{\nu+1} a_\mu^{\nu+1}} = \overline{\delta a_\lambda^{\nu+1} \delta a_\mu^{\nu+1}} = (H^\nu)^{-1}_{\lambda\mu} , \qquad \text{(IV-30)}$$

and substituting into (IV-13) gives

$$\overline{M^*} = \sum_{\substack{\lambda=1\\ \mu=1}}^{L} \overline{a_\lambda^{\nu+1} a_\mu^{\nu+1}}\, H_{\lambda\mu}^{\nu} = L. \qquad \text{(IV-31)}$$

Thus the average value of M^* equals the number of constraints.

D. Summary

In the last three sections we have defined an iterative procedure by Eqs. (IV-7) and (IV-11). At the end of each step, we check rounding errors by computing $(\chi^2)_a$ and $(\chi^2)_b$ and requiring that Eq. (IV-15) be satisfied. After having taken a step and checked the calculation we ask if the new values of the variables x_i lead us to acceptable values of the constraints as imposed by Eq. (IV-22). If so, we ask if a stationary value has been found as determined by Eq. (IV-25). If both these conditions are satisfied, we accept the values of x_i. If not, we take another step.

We have been using the values

$$\epsilon_1 = 10^{-4}, \ \epsilon_2 = 10^{-2}, \ \epsilon_3 = 10^{-2} . \qquad \text{(IV-28)}$$

These values are of course adjustable. Experience thus far indicates that ϵ_1 is occasionally exceeded, especially if variables with highly disproportionate errors are used. The value of ϵ_3 seems to impose about as strict a condition as does ϵ_2 in most cases. For fitting the correct hypotheses to a reasonably well-measured event about four iterations seem to be required, on the average. To impose some sort of stability restriction on the solution we do not accept a fit until all restrictions have been satisfied twice.

It should be noted at this point that the method is not guaranteed to converge, although we have found that it does converge in the overwhelming majority of cases in which the correct interpretation is chosen and even in the majority of cases in which the wrong interpretation is used.

V. PROPAGATION OF ERRORS

Denoting the fitted values of the measured variables by $x_i^{\nu+1}$ and the values of the unmeasured variables computed from these by $y_\alpha^{\nu+1}$, we wish to compute the three error matrices $\overline{\delta x_i^{\nu+1}\,\delta x_j^{\nu+1}}$, $\overline{\delta x_i^{\nu+1}\,\delta y_\alpha^{\nu+1}}$ and $\overline{\delta y_\alpha^{\nu+1}\,\delta y_\beta^{\nu+1}}$. On the assumption that the constraint functions are linear over the region covered by the errors, we may compute the first of these from Eq. (IV-7). This gives us

$$\delta x_i^{\nu+1} = \delta x_i^{m} - \sum_{\lambda=1}^{L} E_{i\lambda}^{\nu}\,\delta a_\lambda^{\nu+1} \qquad \text{(V-1)}$$

Thus we have

$$\overline{\delta x_i^{\nu+1}\,\delta x_j^{\nu+1}} = \overline{\delta x_i^{m}\,\delta x_j^{m}} - \sum_{\lambda=1}^{L} E_{i\lambda}^{\nu}\,\overline{\delta x_i^{m}\,\delta a_\lambda^{\nu+1}} - \sum_{\mu=1}^{L} E_{j\mu}^{\nu}\,\overline{\delta x_j^{m}\,\delta a_\mu^{\nu+1}}$$

$$+ \sum_{\substack{\lambda=1\\ \mu=1}}^{L} E_{i\lambda}^{\nu}\,E_{j\mu}^{\nu}\,\overline{\delta a_\lambda^{\nu+1}\,\delta a_\mu^{\nu+1}} \qquad \text{(V-2)}$$

Equations (IV-5), (IV-17), and (IV-19) may be combined to give

$$\overline{\delta x_i^{m}\,\delta a_\lambda^{\nu+1}} = \sum_{\mu=1}^{L} E_{i\mu}^{\nu}\,(H^\nu)^{-1}_{\mu\lambda} \quad , \qquad \text{(V-3)}$$

and substituting (V-3) and (IV-20) into (V-2) gives finally

$$\overline{\delta x_i^{\nu+1}\,\delta x_j^{\nu+1}} = G_{ij}^{-1} - \sum_{\substack{\mu=1\\ \lambda=1}}^{L} E_{i\lambda}^{\nu}\,(H^\nu)^{-1}_{\lambda\mu}\,(E^\nu)^{T}_{\mu j} \quad , \qquad \text{(V-4)}$$

which is the expression for the error matrix of the fitted variables. We see that the adjusted variables will in general be correlated even if the measured variables are not. One can also prove in general

$$\sum_{\substack{\lambda=1\\ \mu=1}}^{L} E^{\nu}{}_{i}\,(H^\nu)^{-1}_{\lambda\mu}\,(E^\nu)^{T}_{\mu i} \geq 0 \quad , \qquad \text{(V-5)}$$

so that the errors in the adjusted variables are all smaller than those in the measured variables.

The other two error matrices $\overline{\delta x_i^{\nu+1}\, \delta y_\alpha^{\nu+1}}$ and $\overline{\delta y_\alpha^{\nu+1}\, \delta y_\beta^{\nu+1}}$ are constructed by using

$$\delta y_\alpha^{\nu+1} = \sum_{i=1}^{I} \partial y_\alpha / \partial x_i \Big|_{x_i = x_i^{\nu+1}} \delta x_i^{\nu+1} \quad . \tag{V-6}$$

We define for convenience

$$U_{i\alpha}^{\nu+1} = \partial y_\alpha / \partial x_i \Big|_{x_i = x_i^{\nu+1}} \quad , \tag{V-7}$$

and then

$$\overline{\delta x_i^{\nu+1}\, \delta y_\alpha^{\nu+1}} = \sum_{j=1}^{I} U_{j\alpha}^{\nu+1}\, \overline{(\delta x_i^{\nu+1}\, \delta x_j^{\nu+1})} \quad , \tag{V-8}$$

$$\overline{\delta y_\alpha^{\nu+1}\, \delta y_\beta^{\nu+1}} = \sum_{\substack{i=1 \\ j=1}}^{I} (U^{\nu+1})^T_{\alpha i}\, \overline{(\delta x_i^{\nu+1}\, \delta x_j^{\nu+1})}\, U_{j\beta}^{\nu+1} \quad . \tag{V-9}$$

We may further produce a set of very useful test functions with our output. Consider the rms value of the quantity $(x_i^{\nu+1} - x_i^m)$. Using Eqs. (IV-7) and (IV-30), we find

$$\begin{aligned} \overline{(x_i^{\nu+1} - x_i^m)\,(x_j^{\nu+1} - x_j^m)} &= \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} E^\nu{}_{i\lambda}\, \overline{q_\lambda^{\nu+1}\, q_\mu^{\nu+1}}\, (E^\nu)^T{}_{\mu j} \\ &= \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} E^\nu{}_{i\lambda} (H^\nu)^{-1}_{\lambda\mu}\, (E^\nu)^T{}_{\mu j} \quad . \end{aligned} \tag{V-10}$$

It is then clear that the normalized adjustment (or "stretch") in x_i,

$$S_i(x) = \frac{x_i^{\nu+1} - x_i^m}{(x_i^{\nu+1} - x_i^m)_{rms}} = (x_i^{\nu+1} - x_i^m) \left\{ \sum_{\substack{\lambda=1 \\ \mu=1}}^{L} E^\nu{}_{i\lambda} (H^\nu)^{-1}_{\lambda\mu}\, (E^\nu)^T{}_{\mu i} \right\}^{-1/2} , \tag{V-11}$$

should be normally distributed about zero with a standard deviation of 1. The program GUTS computes this set of quantities and they may be used, together with the value of χ^2, to calibrate the empirical errors assigned to each variable as well as to test for systematic effects. Thus a sample of events may be run through the program several times, adjusting the errors until the proper χ^2 distribution is obtained and the distribution of each $S_i(x)$ has the correct width. An asymmetric distribution for any of the $S_i(x)$ may indicate the presence of a systematic error in the measurement of the corresponding variable. It should be noted, however, that such asymmetries may also occur because the input variables are not normally distributed or the constraint functions are not linear over the range of the variables considered.

VI. ADDITIONAL FEATURES

A. Range Energy

In using the program GUTS as presently written, a quantity D must be supplied. This quantity is the distance upstream from the interaction vertex at which the quantities k and δk are considered to have been measured for the first particle. All other quantities must always be specified at the interaction vertex. This feature is included in GUTS to take care of the fact that for very slow tracks the error in $k \approx 1/p$ is very skew. Specifying k and δk a distance D in front of the vertex places the iteration variable in a region where the assigned errors are more nearly Gaussian.

Using this feature, one may analyze connected vertices in succession, propagating errors through the range-energy relation, as described in Paper IV of this series.

For D = 0, the range-energy feature is not used and all variables and errors are taken to be at the vertex. In any case, even for $D \neq 0$, the output

variables and error matrices are all evaluated at the interaction vertex. If the range-energy feature is used, a distinction must be made between k_1, the measured curvature variable of the incident particle, and k_{v_1}, the value of this quantity at the vertex.

The constraint functions F_λ and their derivatives $F_{i\lambda}$ are first evaluated at the vertex and then the $F_{i\lambda}$, which refer to the true iteration variable, are modified according to the formula

$$F_{i\lambda}(x) = \frac{\partial F_\lambda}{\partial x_i} + \frac{\partial F_\lambda}{\partial k_{v_1}} \frac{\partial k_{v_1}}{\partial x_i} . \qquad \text{(VI-1)}$$

Similarly, the output error matrices are first computed in terms of the iteration variable k_1, and are then modified to apply to the vertex according to the formula

$$\overline{\delta k_{v_1} \delta x_i} = \sum_{j=1}^{I} \frac{\partial k_{v_1}}{\partial x_j} \overline{\delta x_j \delta x_i} . \qquad \text{(VI-2)}$$

Since k_{v_1} is a function only of k_1 and $\tan \lambda_1$, the only nonvanishing terms of the form $\partial k_{v_1} / \partial x_i$ are $\partial k_{v_1} / \partial k_1$ and $\partial k_{v_1} / \partial \tan \lambda_1$. These may be evaluated on the assumption that over the length of track under consideration a range-momentum relation of the form

$$p_1 = p_0 R_1^{\alpha} \qquad \text{(VI-3)}$$

is sufficiently accurate, where p_1 is the momentum and R_1 the residual range of the particle, and p_0 and α are constants in the region considered. Using the definitions of the variables, one may then compute

$$\partial k_{v_1} / \partial \tan \lambda_1 = - \left(\frac{D}{R_{v_1}} \right) \frac{\tan \lambda_1}{k_{v_1} p_{v_1}}^2 \qquad \text{(VI-4)}$$

and

$$\partial k_{v_1} / \partial k_1 = R_1 p_1 / R v_1 \, p v_1 \qquad \text{(VI-5)}$$

In neither of these expressions does a or p_0 appear, so that no reference to these constants need be made in the program. It should be noted that the approximation (VI-3) is used only to calculate the derivatives of k_{v_1}, the actual range-momentum calculation being accomplished by a table-look-up operation on R/m versus p/m.

B. Nonanalytic Constraints

To achieve some degree of control over the fitting process, we apply a number of nonanalytic constraints. Most of these are imposed by inquiring, after a step has been taken, whether the new variables $x_i^{\nu+1}$ lie in an unphysical or undesirable region. If they do then the new variables are modified according to the equation

$$x_i^{\nu} + \frac{x_i^{\nu+1} - x_i^{\nu}}{f} \longrightarrow x_i^{\nu+1}, \qquad \text{(VI-6)}$$

where f is an adjustable factor which is at present set equal to 2. This step-cutting procedure is repeated until the new variables are acceptable, a condition that will always be achieved if the previous step lies in an acceptable region. The specific situations in which this cutting-down procedure is used are the following:

1. The input variables k_q are ordinarily specified as positive. If one of the input k_q is given as negative, this is not interpreted as indicating that the corresponding particle was negative, but rather as indicating that the curvature of this particle was measured as having the opposite sign from the charge of the particle. The measured value of this k_q is then left with a negative sign but a small positive value is inserted for the starting value of the fitting procedure. If we start with positive k_q, at no time during the fitting process are any of the k_q allowed to pass through zero and become

negative, and if a step is taken such that any k_q become negative, the cutting-down procedure described above is used to force the k_q to be positive. Note that if any one of the k^m_q are negative, the adjusted value can never agree with the measured value and a large value of χ^2 may result. This procedure essentially imposes conservation of charge on the kinematic analysis.

2. If the range-energy feature is invoked for the incident particle, then at no time is the measured length D allowed to be longer than the residual range of the particle calculated from the range-energy relation by using the momentum of the particle at a distance D back from the vertex. If a step is taken which violates this condition, the cutting-down procedure is used. If this condition is violated by the measured values themselves, the starting value of the momentum of the incident particle is set to give a small positive residual range at the vertex, and the fitting proceeds as before. Again, the measured value of the momentum is untouched and will contribute to the final value of χ^2.

3. It may occur that during the fitting process a step is taken such that in the 3c case E_ℓ, the energy of the particle whose momentum was not measured, or in the 2c case k_ℓ or k_m, the two unmeasured curvatures, become negative or imaginary. In such cases, the cutting-down procedure is used.

4. It is found that the convergence of the iteration is in general accelerated if one also imposes the condition that the quantity $\overline{F^{\nu+1}}$ defined by (IV-16) after each step be smaller than the same quantity $\overline{F^{\nu+1}}$ calculated from the previous step. Again the cutting-down procedure is used to impose this condition. However, this condition is not imposed if $\overline{F^{\nu+1}}$ satisfies the inequality (IV-22), since when $\overline{F^{\nu+1}}$ becomes very small, rounding errors may make it unreasonable to insist on further improvement.

In all the above cases involving nonanalytic constraints, it may happen that the mathematical stationary value of M lies in a region that is forbidden. In this case, it is clearly unreasonable to impose the condition (IV-25) although the condition (IV-22) may still in general be met. Thus, if a large number of cut-downs are required, indicating that the sought-for stationary value lies in a forbidden region, the condition (IV-25) is by-passed by the program. In any case, a maximum number of iterations as well as cut-downs is allowed and if either of these numbers is exceeded, the event is rejected.

VII. EQUATIONS GOVERNING THE PARTICULAR CASES

We have five particular cases, corresponding to L equal to 4, 3, 2, 1, or 0. In each of these cases, we have a different set of constraint functions F_λ as well as a different set of constraint derivative matrices $F_{i\lambda}$. Once we have computed the F_λ, $F_{i\lambda}$, y_a and U_{ia} the calculation proceeds identically for all cases except the L=0 case. We shall thus give only the expressions for these quantities below.

In the case in which the range-energy feature is used, the $F_{i\lambda}$ corresponding to the first (incident) particle (i = 1, 2, 3) must be modified, since the $F_{i\lambda}$ as given are with respect to the variables at the vertex and must be adjusted to refer to the true variables of iteration. This modification has been described in Section IV A. The F_λ are always computed by using the variables at the vertex and need not be modified.

We use the symbols

$$\pi_x = \sum_s (\pm)_s \cos \phi_s / k_s \,, \qquad \text{(VII-1a)}$$

$$\pi_y = \sum_s (\pm)_s \sin \phi_s / k_s \,, \qquad \text{(VII-1b)}$$

$$\pi_z = \sum_s (\pm)_s \tan \lambda_s / k_s \,, \qquad \text{(VII-1c)}$$

$$\epsilon = \sum_s (\pm)_s E_s - m_T \,, \qquad \text{(VII-1d)}$$

where $\sum_s$ denotes the sum over all completely measured particles, $(\pm)_s$ is $+1$ if particle s is outgoing and -1 if particle s is incoming, and m_T is the mass of the target. Note that the following two special cases are allowed:

(a) No incoming particle. This corresponds to assuming that the incident particle stopped, and m_T should be set equal to the target mass plus the mass of the stopped incident particle. Thus if a K^- meson is absorbed in deuterium, for example, a correct treatment of the kinematics will be obtained if m_T is set equal to the mass of the deuteron plus that of the K^- meson, and the incoming particle is otherwise ignored.

(b) $m_T = 0$. This corresponds to a decay.

In addition to using the subscript i, we shall use the subscripts s, l, m, and u as defined in Section II. We shall also use the notation $F_{j\lambda}(S)$, with $j = 1, 2$, or 3, to indicate one of the constraint derivatives for particle s. Thus

$$F_{1\lambda}(S) = \frac{\partial F_\lambda}{\partial x_{3s-2}} = \frac{\partial F_\lambda}{\partial \phi_s}, \qquad \text{(VII-2a)}$$

$$F_{2\lambda}(S) = \frac{\partial F_\lambda}{\partial x_{3s-1}} = \frac{\partial F_\lambda}{\partial \tan \lambda_s}, \qquad \text{(VII-2b)}$$

$$F_{3\lambda}(S) = \frac{\partial F_\lambda}{\partial x_{3s}} = \frac{\partial F_\lambda}{\partial k_s}, \qquad \text{(VII-2c)}$$

and so on. If the particle referred to is not completely measured--i.e. particle ℓ or m--we replace s by ℓ or m as required.

All the variables are measured and we make use of all four constraint equations. We have

$$F_1 = \sum_s (\pm)_s \cos\phi_s/k_s = \pi_x , \qquad \text{(VII-3a)}$$

$$F_2 = \sum_s (\pm)_s \sin\phi_s/k_s = \pi_y , \qquad \text{(VII-3b)}$$

$$F_3 = \sum_s (\pm)_s \tan\lambda_s/k_s = \pi_z , \qquad \text{(VII-3c)}$$

$$F_4 = \sum_s (\pm)_s E_s - m_T = \epsilon . \qquad \text{(VII-3d)}$$

For the $F_{i\lambda}$ we have

$$F_{11}(S) = -(\pm)_s \frac{\sin\phi_s}{k_s} ; \quad F_{12}(S) = (\pm)_s \frac{\cos\phi_s}{k_s} ; \quad F_{13}(S) = 0 ;$$

$$F_{14}(S) = 0; \quad F_{21}(S) = 0; \quad F_{22}(S) = 0 ; \quad F_{23}(S) = (\pm)_s \frac{1}{k_s} ;$$

$$F_{24}(S) = (\pm)_s \frac{\tan\lambda_s}{E_s k_s^2} ; \quad F_{31}(S) = -(\pm)_s \frac{\cos\phi_s}{k_s^2} ; \quad F_{32}(S) = -(\pm)_s \frac{\sin\phi_s}{k_s^2} ;$$

$$F_{33}(S) = -(\pm)_s \frac{\tan\lambda_s}{k_s^2} ; \quad F_{34}(S) = -(\pm)_s \frac{P_s^2}{E_s k_s} .$$

There are P such blocks, since s runs from 1 to P. For $D \neq 0$, this matrix must be modified as described in Section IV A.

B. The 3 c Case

In this case k_ℓ is unmeasured. We solve the energy equation

$$\epsilon = -(\pm)_\ell E_\ell \qquad \text{(VII-4)}$$

for k_ℓ, getting

$$k_\ell = \left\{ \frac{1 + \tan^2\lambda_\ell}{\epsilon^2 - m_\ell^2} \right\}^{1/2} , \qquad \text{(VII-5)}$$

and use this value of k_ℓ in the remaining constraint equations,

$$F_1 = \pi_x + (\pm)_\ell \cos\phi_\ell/k_\ell \ , \tag{VII-6a}$$

$$F_2 = \pi_y + (\pm)_\ell \sin\phi_\ell/k_\ell \ , \tag{VII-6b}$$

$$F_3 = \pi_z + (\pm)_\ell \tan\lambda_\ell/k_\ell \ . \tag{VII-6c}$$

To write the $F_{i\lambda}$ for this case and following cases, we define for convenience

$$V_{j\beta}(s,\ell) = (\pm)_s(\pm)_\ell \ \partial y_\beta/\partial x_j \ . \tag{VII-7}$$

These are simply the $U_{i\alpha}$ defined by (V-7) but with an additional sign convention in order to simplify the formulae that follow. Here J=1, 2, or 3 refers to the measured variables of particle s and β=1, 2, or 3 refers to the unmeasured variables. The index s may be replaced by ℓ or m and the index ℓ may be replaced by m when the corresponding particles are referred to. Since k_ℓ is the only unmeasured variable in the 3c case, we have P-1 blocks of the form

$$V_{11}(s,\ell) = (\pm)_s(\pm)_\ell \ \partial k_\ell/\partial\phi_s = 0 \ , \tag{VII-8a}$$

$$V_{21}(s,\ell) = (\pm)_s(\pm)_\ell \frac{\partial k_\ell}{\partial \tan\lambda_s} = \frac{\tan\lambda_s}{E_s k_s^2} \frac{k_\ell E_\ell}{P_\ell^2} \ , \tag{VII-8b}$$

$$V_{31}(s,\ell) = (\pm)_s(\pm)_\ell \frac{\partial k_\ell}{\partial k_s} = - \frac{P_s^2}{E_s k_s} \frac{k_\ell E_\ell}{P_\ell^2} \ , \tag{VII-8c}$$

and one block of the form

$$V_{11}(\ell,\ell) = \partial k_\ell/\partial\phi_\ell = 0 \ , \tag{VII-9a}$$

$$V_{21}(\ell,\ell) = \partial k_\ell/\partial\tan\lambda_\ell = \frac{\tan\lambda_\ell}{k_\ell P_\ell^2} \ . \tag{VII-9b}$$

We may now write the constraint derivative matrix as P=1 blocks of the form

$$F_{11}(s) = - (\pm)_s \frac{\sin\phi_s}{k_s} \ ; \quad F_{12}(s) = (\pm)_s \frac{\cos\phi_s}{k_s} \ ; \quad F_{13}(s) = 0 \ ;$$

$$F_{21}(s) = - (\pm)_s \frac{\cos\phi_\ell}{k_\ell^2} \ V_{21}(s,\ell) \ ; \quad F_{22}(s) = - (\pm)_s \frac{\sin\phi_\ell}{k_\ell^2} \ V_{21}(s,\ell) \ ;$$

$$F_{23}(s) = (\pm)_s \left[\frac{1}{k_s} - \frac{\tan\lambda_\ell}{k_\ell^2} V_{21}(s,\ell) \right];$$

$$F_{31}(s) = -(\pm)_s \left[\frac{\cos\phi_s}{k_s^2} + \frac{\cos\phi_\ell}{k_\ell^2} V_{31}(s,\ell) \right];$$

$$F_{32}(s) = -(\pm)_s \left[\frac{\sin\phi_s}{k_s^2} + \frac{\sin\phi_\ell}{k_\ell^2} V_{31}(s,\ell) \right];$$

$$F_{33}(s) = -(\pm)_s \left[\frac{\tan\lambda_s}{k_s^2} + \frac{\tan\lambda_\ell}{k_\ell^2} V_{31}(s,\ell) \right];$$

and one block of the form

$$F_{11}(\ell) = -(\pm)_\ell \frac{\sin\phi_\ell}{k_\ell}; \quad F_{12}(\ell) = (\pm)_\ell \frac{\cos\phi_\ell}{k_\ell}; \quad F_{13}(\ell) = 0;$$

$$F_{21}(\ell) = -(\pm)_\ell \frac{\cos\phi_\ell}{k_\ell^2} V_{21}(\ell,\ell); \quad F_{22}(\ell) = -(\pm)_\ell \frac{\sin\phi_\ell}{k_\ell^2} V_{21}(\ell,\ell);$$

$$F_{23}(\ell) = (\pm)_\ell \left[\frac{1}{k_\ell} - \frac{\tan\lambda_\ell}{k_\ell^2} V_{21}(\ell,\ell) \right].$$

For $D \neq 0$, this matrix must be modified as described in Section VIA.

C. The 2c Case

Here k_ℓ and k_m are unmeasured. We solve for these by using the two constraint equations

$$\pi_x + (\pm)_\ell \frac{\cos\phi_\ell}{k_\ell} + (\pm)_m \frac{\cos\phi_m}{k_m} = 0, \qquad \text{(VII-10a)}$$

$$\pi_y + (\pm)_\ell \frac{\sin\phi_\ell}{k_\ell} + (\pm)_m \frac{\sin\phi_m}{k_m} = 0, \qquad \text{(VII-10b)}$$

getting

$$k_\ell = (\pm)_\ell \frac{\sin(\phi_\ell - \phi_m)}{\Sigma(\pm)_s \frac{\sin(\phi_m - \phi_s)}{k_s}}, \qquad \text{(VII-11)}$$

$$k_m = (\pm)_m \frac{\sin(\phi_m - \phi_\ell)}{\sum_s (\pm)_s \frac{\sin(\phi_\ell - \phi_s)}{k_s}} \quad ; \qquad \text{(VII-12)}$$

these two values are used in the two remaining constraint equations,

$$F_1 = \pi_z + (\pm)_\ell \frac{\tan\lambda_\ell}{k_\ell} + (\pm)_m \frac{\tan\lambda_m}{k_m} \quad , \qquad \text{(VII-13a)}$$

$$F_2 = \epsilon + (\pm)_\ell E_\ell + (\pm)_m E_m \, . \qquad \text{(VII-13b)}$$

Using the notation of the previous sections, we have P-2 blocks of the form

$$V_{11}(s,\ell) = (\pm)_s(\pm)_\ell \frac{\partial k_\ell}{\partial \phi_s} = \frac{k_\ell^2}{k_s} \frac{\cos(\phi_m - \phi_s)}{\sin(\phi_\ell - \phi_m)} \quad ;$$

$$V_{12}(s,m) = (\pm)_s(\pm)_m \frac{\partial k_m}{\partial \phi_s} = \frac{k_m^2}{k_s} \frac{\cos(\phi_\ell - \phi_s)}{\sin(\phi_m - \phi_\ell)} \quad ;$$

$$V_{21}(s,\ell) = (\pm)_s(\pm)_\ell \frac{\partial k_\ell}{\partial \tan\lambda_s} = 0; \quad V_{22}(s,m) = (\pm)_s(\pm)_m \frac{\partial k_m}{\partial \tan\lambda_s} = 0;$$

$$V_{31}(s,\ell) = (\pm)_s(\pm)_\ell \frac{\partial k_\ell}{\partial k_s} = \frac{k_\ell^2}{k_s^2} \frac{\sin(\phi_m - \phi_s)}{\sin(\phi_\ell - \phi_m)} \quad ;$$

$$V_{32}(s,m) = (\pm)_s(\pm)_m \frac{\partial k_m}{\partial k_s} = \frac{k_m^2}{k_s^2} \frac{\sin(\phi_\ell - \phi_s)}{\sin(\phi_m - \phi_\ell)} \quad ;$$

one block of the form

$$V_{11}(\ell,\ell) = \frac{\partial k_\ell}{\partial \phi_\ell} = k_\ell \cot(\phi_\ell - \phi_m) \; ; \quad V_{12}(\ell,m) = (\pm)_\ell(\pm)_m \frac{\partial k_m}{\partial \phi_\ell}$$

$$= \frac{k_m^2}{k_\ell} \frac{1}{\sin(\phi_m - \phi_\ell)} ;$$

$$V_{21}(\ell,\ell) = \frac{\partial k_\ell}{\partial \tan\lambda_\ell} = 0; \quad V_{22}(\ell,m) = (\pm)_\ell(\pm)_m \frac{\partial k_m}{\partial \tan\lambda_\ell} = 0; \quad \text{and}$$

one block of the form

$$V_{11}(m,\ell) = (\pm)_m(\pm)_\ell \frac{\partial k_\ell}{\partial \phi_m} = \frac{k_\ell^2}{k_m} \frac{1}{\sin(\phi_\ell - \phi_m)} \; ; \quad V_{12}(m,m) = \frac{\partial k_m}{\partial \phi_m}$$

$$= k_m \cot(\phi_m - \phi_\ell);$$

$$V_{21}(m,\ell) = (\pm)_m(\pm)_\ell \frac{\partial k_\ell}{\partial \tan\lambda_m} = 0; \quad V_{22}(m,m) = \frac{\partial k_m}{\partial \tan\lambda_m} = 0 \, .$$

The constraint derivative matrix may then be written as P-2 blocks of the form

$$F_{11}(s) = -(\pm)_s\left[\frac{\tan\lambda_\ell}{k_\ell^2}\,V_{11}(s,\ell) + \frac{\tan\lambda_m}{k_m^2}\,V_{12}(s,m)\right];$$

$$F_{12}(s) = -(\pm)_s\left[\frac{P_\ell^2}{E_\ell k_\ell}\,V_{11}(s,\ell) + \frac{P_m^2}{E_m k_m}\,V_{12}(s,m)\right];$$

$$F_{21}(s) = (\pm)_s\, 1/k_s\ ;\quad F_{22}(s) = (\pm)_s\,\frac{\tan\lambda_s}{E_s k_s^2}\ ;$$

$$F_{31}(s) = -(\pm)_s\left[\frac{\tan\lambda_s}{k_s^2} + \frac{\tan\lambda_\ell}{k_\ell^2}\,V_{31}(s,\ell) + \frac{\tan\lambda_m}{k_m^2}\,V_{32}(s,m)\right];$$

$$F_{32}(s) = -(\pm)_s\left[\frac{P_s^2}{E_s k_s} + \frac{P_\ell^2}{E_\ell k_\ell}\,V_{31}(s,\ell) + \frac{P_m^2}{E_m k_m}\,V_{32}(s,m)\right];$$

one block of the form

$$F_{11}(\ell) = -(\pm)_\ell\left[\frac{\tan\lambda_\ell}{k_\ell^2}\,V_{11}(\ell,\ell) + \frac{\tan\lambda_m}{k_m^2}\,V_{12}(\ell,m)\right];$$

$$F_{12}(\ell) = -(\pm)_\ell\left[\frac{P_\ell^2}{E_\ell k_\ell}\,V_{11}(\ell,\ell) + \frac{P_m^2}{E_m k_m}\,V_{12}(\ell,m)\right];$$

$$F_{21}(\ell) = (\pm)_\ell\, 1/k_\ell\ ;\quad F_{22}(\ell) = \tan\lambda_\ell/E_\ell k_\ell^2\ ;$$

and one block of the form

$$F_{11}(m) = -(\pm)_m\left[\frac{\tan\lambda_\ell}{k_\ell^2}\,V_{11}(m,\ell) + \frac{\tan\lambda_m}{k_m^2}\,V_{12}(m,m)\right];$$

$$F_{12}(m) = -(\pm)_m\left[\frac{P_\ell^2}{E_\ell k_\ell}\,V_{11}(m,\ell) + \frac{P_m^2}{E_m k_m}\,V_{12}(m,m)\right];$$

$$F_{21}(m) = (\pm)_m\ 1/k_m;\quad F_{22}(m) = \tan\lambda_m/E_m k_m^2\ .$$

For $D \neq 0$, this matrix must be modified as described in Section VIA.

D. The 1 c Case

In this case ϕ_ℓ , $\tan\lambda_\ell$, and k_ℓ are all unmeasured. We may solve the three momentum equations

$$\pi_x + (\pm)_\ell \cos\phi_\ell / k_\ell = 0 , \qquad \text{(VII-14a)}$$

$$\pi_y + (\pm)_\ell \sin\phi_\ell / k_\ell = 0 , \qquad \text{(VII-14b)}$$

$$\pi_z + (\pm)_\ell \tan\lambda_\ell / k_\ell = 0 \qquad \text{(VII-14c)}$$

for the missing variables, getting

$$\phi_\ell = \tan^{-1}(\pi_y / \pi_x) , \qquad \text{(VII-15a)}$$

$$\tan\lambda_\ell = - (\pm)_\ell k_\ell \pi_z , \qquad \text{(VII-15b)}$$

$$k_\ell = \left\{ \pi_x^2 + \pi_y^2 \right\}^{1/2} . \qquad \text{(VII-15c)}$$

The remaining constraint equation is

$$F_1 = \epsilon + (\pm)_\ell E_\ell . \qquad \text{(VII-16)}$$

We have P-1 blocks of the form

$$V_{11}(s,\ell) = \frac{\tan\phi_\ell}{k_\ell} V_{13}(s,\ell) + (\pm)_\ell \frac{\cos\phi_s}{\pi_x k_s} ; \; V_{12}(s,\ell) = \frac{\tan\lambda_\ell}{k_\ell} V_{13}(s,\ell) ;$$

$$V_{13}(s,\ell) = (\pm)_\ell \frac{k_\ell^3}{k_s} \left[\pi_x \sin\phi_s - \pi_y \cos\phi_s \right] ;$$

$$V_{21}(s,\ell) = 0; \; V_{22}(s,\ell) = - k_\ell / k_s ; \; V_{23}(s,\ell) = 0 ;$$

$$V_{31}(s,\ell) = \frac{\tan\phi_\ell}{k_\ell} V_{33}(s,\ell) - (\pm)_\ell \frac{\sin\phi_s}{\pi_x k_s} ;$$

$$V_{32}(s,\ell) = \frac{\tan\lambda_\ell}{k_\ell} V_{33}(s,\ell) + \frac{k_\ell \tan\lambda_s}{k_s^2};$$

$$V_{33}(s,\ell) = (\pm)_\ell \frac{k_\ell^3}{k_s^2} \left[\pi_y \sin\phi_s + \pi_x \cos\phi_s\right].$$

The constraint derivative matrix is then P=1 blocks of the form

$$F_{11}(s) = (\pm)_s \left\{ - \frac{P_\ell^2}{E_\ell k_\ell} V_{13}(s,\ell) + \frac{\tan\lambda_\ell}{E_\ell k_\ell^2} V_{12}(s,\ell) \right\},$$

$$F_{21}(s) = (\pm)_s \left\{ \frac{\tan\lambda_s}{E_s k_s^2} + \frac{\tan\lambda_\ell}{E_\ell k_\ell^2} V_{22}(s,\ell) \right\},$$

$$F_{31}(s) = (\pm)_s \left\{ - \frac{P_s^2}{E_s k_s} - \frac{P_\ell^2}{E_\ell k_\ell} V_{33}(s,\ell) + \frac{\tan\lambda_\ell}{E_\ell k_\ell^2} V_{32}(s,\ell) \right\}.$$

For $D \neq 0$, this matrix must be modified as described in Section VIA.

E. The 0 c Case

In this case k_u as well as ϕ_ℓ, $\tan\lambda_\ell$, and k_ℓ is unmeasured. Note that the k_u belongs to the second-to-last outgoing particle whose momentum is unknown but whose angle variables are known. We write the four constraint equations in the form

$$\pi_x + (\pm)_u \frac{\cos\phi_u}{k_u} + (\pm)_\ell \frac{\cos\phi_\ell}{k_\ell} = 0, \qquad \text{(VII-16a)}$$

$$\pi_y + (\pm)_u \frac{\sin\phi_u}{k_u} + (\pm)_\ell \frac{\sin\phi_\ell}{k_\ell} = 0, \qquad \text{(VII-16b)}$$

$$\pi_z + (\pm)_u \frac{\tan\lambda_u}{k_u} + (\pm)_\ell \frac{\tan\lambda_\ell}{k_\ell} = 0, \qquad \text{(VII-16c)}$$

$$\epsilon + (\pm)_u E_u + (\pm)_\ell E_\ell = 0. \qquad \text{(VII-16d)}$$

Eliminating ϕ_ℓ, $\tan\lambda_\ell$, and k_ℓ from these equations, we get a quadratic equation for k_u,

$$(a^2 - \epsilon^2 m_u^2)\, k_u^2 + 2abk_u + b^2 - \epsilon^2(1 + \tan^2\lambda_u) = 0, \qquad \text{(VII-17a)}$$

where

$$a = 1/2 \left[\pi_x^2 + \pi_y^2 + \pi_z^2 - \epsilon^2 - m_u^2 + m_\ell^2 \right], \qquad \text{(VII-17b)}$$

$$b = (\pm)_u \left[\pi_x \cos\phi_u + \pi_y \sin\phi_u + \pi_z \tan\lambda_u \right]. \qquad \text{(VII-17c)}$$

The quadratic equation (VII-17a) may be immediately solved for k_u once the ambiguity in the choice of roots has been decided. This ambiguity is fundamental to the 0c case, and the sign of the radicand in the solution for k_u must be supplied to the program. Once k_u has been computed, this second-to-last particle may be treated formally as a measured particle and the formulae of the 1c case are used to compute the reamining unmeasured variables of the last particle.

The derivatives of k_u are found by differentiating Eq. (VII-17a) and solving the resultant linear equation,

$$\frac{\partial k_u}{\partial x_i} = - \frac{k_u(ak_u+b)\frac{\partial a}{\partial x_i} + (ak_u + b)\frac{\partial b}{\partial x_i} - \epsilon(1+m_u^2 k_u^2 + \tan^2\lambda_u)\,\partial\epsilon/\partial x_i}{(a^2 - \epsilon^2 m_u^2)\, k_u + ab} \,. \qquad \text{(VII-18)}$$

The detailed derivatives $\partial a/\partial x_i$ and $\partial b/\partial x_i$ are somewhat complicated but may easily be computed by straightforward differentiation. The remaining derivatives required in order to compute the errors involving the computed variables according to Eqs.(V-8) and (V-9) are obtained by considering particle u to be completely measured and again using the formalism of the 1c case. These derivatives are then modified according to the formula

$$\partial y_\ell / \partial x_i + \partial y_\ell / \partial k_\mu \; \partial k_\mu / \partial x_i \longrightarrow \partial y_\ell / \partial x_i \,, \qquad \text{(VII-19)}$$

and Eqs. (V-8) and (V-9) are used to compute the appropriate error matrices.

VIII. CONCLUSION

Further vertex types, such as a different 1c type in which the momenta of three particles are unknown, may easily be included in the above formalism and such inclusions will be made in the near future. In addition, attempts will be made to reduce the percentage of rejected events by finding better starting values and by other methods.

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FOOTNOTE

1. The general method of least squares with constraints was first developed by Gauss, see, e. g., William E. Deming, Statistical Adjustment of Data (John Wiley and Sons, Inc., New York, 1943), 1st ed.