Title
Generating functions for descents and levels over words which avoid a consecutive pattern

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Publication Date
2017

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Generating functions for descents and levels over words which avoid a consecutive pattern

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2017

The dissertation of Luvreet Sangha is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2017

DEDICATION

To Guneet, Mom, Dad, Saatchi, and DRP.

EPIGRAPH

Never eat raspberries.

—Arnold's Grandpa

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ACKNOWLEDGEMENTS

I would like to thank Jeff Remmel for the weekly math talks, the stacks on stacks of notes, and for counting words with me. You were a great advisor. I would like to thank Laura Stevens for the feedback and help during my first foray into lecturing the advice during my job search, and the reassurance when the search looked bleak. It was a pleasure to TA for you.

Thank you Todd Miller for the Athenian years. While taking one of your classes, I realized I wanted to study math in college, and deciding to pursue that helped make attending school for another ten years a little more bearable.

Thank you Mike Wong and Mike Leong for getting me started with the SLC. Thinking back to how nervous I was for my teaching demo at the end of my first semester there, I am incredibly thankful to both of you for helping me grow as a tutor and teacher. Thank you for introducing me to the world of parallel problems and not holding a student's pencil more than they do. Also, thank you for always happily sharing your resources with me even after I left for San Diego. Thank you also to Richard Gibson for the five years of Math 32 and Summer Bridge. Seeing how effortlessly you teach influenced my own teaching style greatly. This was by far my most rewarding teaching experience to date. Thank you for teaching me about the "Weeeeee!!!" Rule and showing up to class late. Thank you also to all of the other tutors and workshop leaders I met along the way for helping make my memories of the SLC so great.

Thank you to Mom and Dad for your unending support and helping me get settled in all of my apartments, all the long drives down, and the birthday dinners. Thank you for putting up with all of my late nights at home in high school doing problems and playing videogames. Guneet, thank you for coming to San Diego to visit acai eateries, beaches, and Eva aunty instead of me. Thanks for ordering our joint gifts for the padres and everyone else all the time. Thanks for following me to Athenian and Berkeley, being my vacation room partner, and walking over to the bushes when we were playing tennis that one time .... Thank you also to Eva aunty and Arvin uncle for always opening your home to me and making me feel like family was never far.

Thank you Saatchi for being a fellow pizza aficionado and admirer of the sweetness of a certain divine figure's cheeks. You complained a lot at practices, and you tried to be funny. Thank you for understanding whenever things got busy and being down to work together. Atlanta and Lexington will be nice.

DRP, thank you for being one of my greatest joys. Thank you for being half of my wardrobe and for making me more familiar with ways of sneaking into Galbraith Hall than a person should be. Thank you to my fallen littles. Thank you for letting me dance and for dancing with me. I will miss you.

To my bhangra friends, thank you for the post-practice dinners, the half-hearted gig preparations, the bus ride boliyan that were just thirty minutes of yelling, and too much drama. I am most thankful I got involved in dance because I met you all. Hook it up with a performance when I turn fifty.

Chapter 1, in part, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 2, in particular the proofs of Theorems 1, 2, 3, and 4, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 3, in full, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 4, in part, is currently being prepared for submission for publication of the material. Sangha, Luvreet; Remmel, Jeffrey. The dissertation author was the primary author of this material.

Chapter 5, in part, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

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ABSTRACT OF THE DISSERTATION

Generating functions for descents and levels over words which avoid a consecutive pattern

by

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Doctor of Philosophy in Mathematics

University of California, San Diego, 2017

Professor Jeffrey Remmel, Chair

In this thesis, we extend the reciprocity method introduced by Jones and Remmel [26, 27] to study the distributions of descents and levels over words which have no u-matches for words u that have at most one descent or at most one level, respectively.

Chapter 1

Introduction

1.1 Basic Definitions and Background

The study of patterns in permutations and words which is generally referred to as *permutation patterns* has been an extremely active area of research in recent years. The study of permutation patterns has it roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. The first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt which appeared in 1985. The field has experienced explosive growth since 1992. The notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various sorting algorithms including sorting stacks and sortable permutations. There are two recents books on the subject: *Combinatorics of Compositions and Words* by Heubach and Mansour which studies patterns in words and *Patterns in Permutations and Words* by Kitaev which mainly focuses on patterns in permutations.

My work has focused on applications of the so-called reciprocity method of Jones and Remmel to find generating functions for patterns in words. To motivate my results, we will start with some basic definitions of patterns in permutations. Let S_n denote the set of permutations of $[n] = \{1, \ldots, n\}$. Given any sequence of distinct integers $a_1 \cdots a_j$, we let $red(a_1 \cdots a_j)$ denote the permutation of S_j whose elements have the same relative order as $a_1 \cdots a_j$. For example, $red(5\ 3\ 7\ 2) = 3\ 2\ 4\ 1$. We then say that a permutation $\tau \in S_j$ *occurs* in a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$ and there is a *τ-match starting at position i* if $\text{red}(\sigma_i \sigma_{i+1} \dots \sigma_{i+j-1}) = \tau$. We say σ *avoids* τ if τ does not occur in σ. We let $\tau\text{-mch}(\sigma)$ denote the number of τ-matches in σ. For example, $\pi = 3142$ occurs in $\sigma = 5137642$ as is shown by the bolded subsequences: **51**3**7**6**4**2, **51**37**64**2, **51**3**7**64**2**, **51**37**6**4**2**, **5**1**37**6**4**2, or **5**1**3**7**64**2. Since σ contains no increasing subsequence of length 4, σ avoids 1234.

We let $\mathcal{S}_n(\tau)$ denote the set of permutations of S_n which avoid τ and $\mathcal{NM}_n(\tau)$ denote the set of permutations of S_n which have no τ-matches. We let $S_n(\tau) = |\mathcal{S}_n(\tau)|$ and $NM_n(\tau) = |\mathcal{NM}_n(\tau)|$. If α and β are elements of S_j, then we say that α and β are *Wilf-equivalent* if $S_n(\alpha) = S_n(\beta)$ for all $n \geq 1$, and we say that α and β are *c-Wilf-equivalent* if $NM_n(\alpha) = NM_n(\beta)$ for all n. For any permutation τ, let

$$A_\tau(x,t) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}.$$

Thus, α and β are c-Wilf equivalent if and only if $A_\alpha(0,t) = A_\beta(0,t)$. We say that α is *strongly c-Wilf-equivalent* to β if $A_\alpha(x,t) = A_\beta(x,t)$. There is an outstanding conjecture of Nakamura [45] that α and β are c-Wilf equivalent if and only if α and β are strongly c-Wilf equivalent.

One can make similar definitions for words, except that in the case of words, we have two different ways to match patterns.

Let $\mathbb{P} = \{1, 2, \dots\}$ denote the set of positive integers and for any $k \in \mathbb{P}$, let $[k] = \{1, \dots, k\}$. We let $\mathbb{P}^*$ ($[k]^*$) denote the set of all words over the alphabet $\mathbb{P}$ ($[k]$). If $u = u_1 \dots u_j \in \mathbb{P}^j$, then we let $|u| = j$ denote the length of u. We let ϵ denote the empty word and we say ϵ has length 0. We let $\mathbb{P}^+ = \mathbb{P}^* - \{\epsilon\}$ and $[k]^+ = [k]^* - \{\epsilon\}$.

If $u = u_1 \dots u_j$ and $v = v_1 \dots v_i$ are words in $\mathbb{P}^*$, we let $uv = u_1 \dots u_j v_1 \dots v_i$ denote the concatenation of u and v. Suppose that we fix $j \geq 1$. Then for any word $w = w_1 \dots w_n$, we say that a word $u = u_1 \dots u_j$ is a *prefix* of a word w if there is a word v such that $uv = w$, is a *suffix* of w if there is a word v such that $vu = w$, and is a *factor* of w if there are words f and v such that $fuv = w$.

We let $\text{red}(w)$ denote the word that results from w by replacing all occurrences of the ith smallest letter in w by i. For example, if $w = 44537792$, then $\text{red}(w) = 33425561$. Let $u = u_1 \ldots u_j \in \mathbb{P}^j$ and $w = w_1 \ldots w_n \in \mathbb{P}^n$. Then if $\text{red}(u) = u$, a u-match in w is a factor of v of w such that $\text{red}(v) = u$. An exact u-match in w is a factor v of w such that $v = u$. We let $u\text{mch}(w)$ denote the number of u-matches in w if $\text{red}(u) = u$ and $eu\text{mch}(w)$ denote the number of exact u-matches in w. For example, if $w = 31442521337792$ and $u = 213$, then w has three u-matches, namely 314, 425, and 213, but only one exact u-match. Thus $u\text{mch}(w) = 3$ and $eu\text{mch}(w) = 1$. For any word $w \in \mathbb{P}^*$ and $i, j \in \mathbb{P}$, we let $\mathbf{ij}(w)$ denote the number of exact matches of ij in w. If $\Gamma \subseteq \mathbb{P}^*$ ($[k]^*$) is a finite family of words such that $\text{red}(v) = v$ for all $v \in \Gamma$ and $w \in \mathbb{P}^*$ ($[k]^*$), then there is a Γ-match in w if there is a factor u of w such that $\text{red}(u) \in \Gamma$. In such a situation, we let $\Gamma\text{-mch}(w)$ denote the number of Γ-matches in w. Similarly, if $\Gamma \subseteq \mathbb{P}^*$ ($[k]^*$) is a finite family of words and $w \in \mathbb{P}^*$ ($[k]^*$), then we say that there is an exact Γ-match in w if there is a factor u of w such that $u \in \Gamma$. In such a situation, we let $\text{e}\Gamma\text{-mch}(w)$ denote the number of exact Γ-matches in w.

Let $\mathbf{z}_k = z_1, \ldots, z_k$, $\mathbf{z}_\infty = z_1, z_2, \ldots$, and if $w = w_1, \ldots w_n$, let $\overline{z}^w = z_{w_1} z_{w_2} \ldots z_{w_n}$. Suppose that Γ and Δ are finite sets of words such that for all $v \in \Gamma \cup \Delta$, $\text{red}(v) = v$, and Ω is a finite set of words. The main goal of this thesis is to develop methods to compute generating functions of the following form:

$$F_\Delta^{\mathbb{P},\Gamma}(x, \mathbf{z}_\infty, t) = \sum_{w \in \mathbb{P}^*, \Delta\text{-mch}(w)=0} x^{\Gamma\text{-mch}(w)+1} \overline{z}^w t^{|w|}, \tag{1.1}$$

$$F_\Delta^{k,\Gamma}(x, \mathbf{z_k}, t) = \sum_{w \in [k]^*, \Delta\text{-mch}(w)=0} x^{\Gamma\text{-mch}(w)+1} \overline{z}^w t^{|w|}, \tag{1.2}$$

$$EF_\Omega^{\mathbb{P},\Gamma}(x, \mathbf{z}_\infty, t) = \sum_{w \in \mathbb{P}^*, \text{e}\Omega-\text{mch}(w)=0} x^{\Gamma\text{-mch}(w)+1} \overline{z}^w t^{|w|}, \text{ and} \tag{1.3}$$

$$EF_\Omega^{k,\Gamma}(x, \mathbf{z}_\infty, t) = \sum_{w \in [k]^*, \text{e}\Omega-\text{mch}(w)=0} x^{\Gamma\text{-mch}(w)+1} \overline{z}^w t^{|w|}. \tag{1.4}$$

In this thesis, we will focus on the cases where Γ is one of the sets $\{12\}$, $\{11\}$, $\{21\}$, $\{11, 21\}$, or $\{11, 12\}$. If $w_1 \ldots w_n$ is a word in $\mathbb{P}^*$, then

1. a $\{12\}$-match in w corresponds to a pair of letters $w_i w_{i+1}$ such that $w_i < w_{i+1}$ which is called a *rise*,

2. a $\{11\}$-match in w corresponds to a pair of letters w_iw_{i+1} such that $w_i = w_{i+1}$ which is called a *level*,

3. a $\{21\}$-match in w corresponds to a pair of letters w_iw_{i+1} such that $w_i > w_{i+1}$ which is called a *descent*,

4. a $\{11, 12\}$-match in w corresponds to a pair of letters w_iw_{i+1} such that $w_i \leq w_{i+1}$ which is called a *weak rise*, and

5. a $\{11, 21\}$-match in w corresponds to a pair of letters w_iw_{i+1} such that $w_i \geq w_{i+1}$ which is called a *weak descent*.

Thus we make the following definitions. Suppose that $n \geq 1$ and $w = w_1 \dots w_n \in \mathbb{P}^n$. We let

$$\begin{array}{ll} Des(w) = \{i : w_i > w_{i+1}\} & WDes(w) = \{i : w_i \geq w_{i+1}\} \\ Rise(w) = \{i : w_i < w_{i+1}\} & WRise(w) = \{i : w_i \leq w_{i+1}\} \\ \text{des}(w) = |Des(w)| & \text{wdes}(w) = |WDes(w)| \\ \text{rise}(w) = |Rise(w)| & \text{wrise}(w) = |WRis(w)| \\ Lev(w) = \{i : w_i = w_{i+1}\} & \text{lev}(w) = |Lev(w)|. \end{array}$$

We shall refer to elements of $Des(w)$, $WDes(w)$, $Rise(w)$, $WRise(w)$, and $Lev(w)$ as descents, weak descents, rises, weak rises, and levels of w, respectively.

For any word $u = u_1 \dots u_j \in [k]^j$ such that $\text{red}(u) = u$, let $St^{(\mathbb{P})}(u)$ ($St^{([k])}(u)$) equal the set of $1 < s \leq j$ such that there exists a word $w = w_1 \dots w_{s+j-1}$ in $\mathbb{P}^*$ ($[k]^*$) such that $\text{red}(w_1 \dots w_j) = u$ and $\text{red}(w_s \dots w_{s+j-1}) = u$. That is, $St^{(\mathbb{P})}(u)$ ($St^{([k])}(u)$) is the set of positions $1 < s \leq j$ such that there is a word w in $\mathbb{P}^*$ ($[k]^*$) in which there is a pair of overlapping u-matches such that the first u-match starts at position 1 and the second u-match starts at position s. We say that u is $\mathbb{P}$-*minimal overlapping* ($[k]$-*minimal overlapping*) if $St^{(\mathbb{P})}(u) = \{j\}$ ($St^{([k])}(u) = \{j\}$). Thus u is $\mathbb{P}$-minimal overlapping if any two consecutive u-matches can share at most one letter which must be the last letter of the first u-match and the first letter of the second u-match. We say that u has the $\mathbb{P}$-*weakly decreasing ($\mathbb{P}$-weakly increasing, $\mathbb{P}$-level)* overlapping property if $s \in St^{(\mathbb{P})}(u)$ implies that $u_1 \geq u_s$ ($u_1 \leq u_s$, $u_1 = u_s$). We say that u has the $[k]$-*weakly decreasing*

($[k]$-weakly increasing, $[k]$-level) overlapping property if $s \in St^{([k])}(u)$ implies that $u_1 \geq u_s$ ($u_1 \leq u_s$, $u_1 = u_s$). If $k \geq 2$, then we say that u is $[k]$-*non-overlapping* if $St^{([k])}(u) = \emptyset$. For example, suppose that $u = 123234$. Then

1. $w^{(1)} = 123234345$ witnesses that $4 \in St^{(\mathbb{P})}(u)$,
2. $w^{(2)} = 1232345456$ witnesses that $5 \in St^{(\mathbb{P})}(u)$, and
3. $w^{(3)} = 12323456567$ witnesses that $6 \in St^{(\mathbb{P})}(u)$.

It is easy to see that in each case, $w^{(i)}$ uses the smallest alphabet possible. Clearly 2 and 3 are not in $St^{(\mathbb{P})}(u)$ or $St^{([k])}(u)$ for any $[k]$. It thus follows that $St^{(\mathbb{P})}(u) = St^{([k])}(u) = \{4,5,6\}$ for any $k \geq 7$. However, $St^{([4])}(u) = \emptyset$ so that u is [4]-non-overlapping, $St^{([5])}(u) = \{4\}$, and $St^{([6])}(u) = \{4,5\}$. Note that u has the $\mathbb{P}$-weakly increasing overlapping property and the $[k]$-weakly increasing overlapping property for any $k \geq 5$. Next suppose that $v = 345123$. Then

1. $w^{(4)} = 567345123$ witnesses that $4 \in St^{(\mathbb{P})}(v)$,
2. $w^{(5)} = 4561345123$ witnesses that $5 \in St^{(\mathbb{P})}(v)$, and
3. $w^{(6)} = 34512345123$ witnesses that $6 \in St^{(\mathbb{P})}(v)$.

Again it is easy to see that in each case, $w^{(i)}$ uses the smallest alphabet possible and 2 and 3 are not in $St^{(\mathbb{P})}(u)$ or $St^{([k])}(v)$ for any $[k]$. It follows that $St^{(\mathbb{P})}(v) = St^{([k])}(v) = \{4,5,6\}$ for any $k \geq 7$. However, $St^{([5])}(u) = \{6\}$ so that u is [5]-minimal overlapping and $St^{([6])}(u) = \{5,6\}$. Note that u has the $\mathbb{P}$-weakly decreasing overlapping property and the $[k]$-weakly decreasing overlapping property for any $k \geq 5$ but that it also has the [5]-weakly increasing overlapping property and the [5]-level overlapping property.

We can also make similar definitions for exact matchings. That is, for $u = u_1 \ldots u_j \in [k]^j$, let $ESt^{(\mathbb{P})}(u)$ ($ESt^{([k])}(u)$) equal the set of $1 < s \leq j$ such that there exists a word $w = w_1 \ldots w_{s+j-1}$ in $\mathbb{P}^*$ ($[k]^*$) such that $w_1 \ldots w_j = u$ and $w_s \ldots w_{s+j-1} = u$. Thus, $ESt^{(\mathbb{P})}(u)$ ($ESt^{([k])}(u)$) is the set of positions $1 < s \leq j$ such that there is a word w in $\mathbb{P}^*$ ($[k]^*$) in which there is a pair of overlapping exact u-matches such that the first exact u-match starts at position 1 and the second exact u-match starts at position s. We say that u is *exact* $\mathbb{P}$*-minimal overlapping* (*exact* $[k]$*-minimal overlapping* if $ESt^{(\mathbb{P})}(u) = \{j\}$ ($ESt^{([k])}(u) = \{j\}$). Thus u is exact $\mathbb{P}$-minimal overlapping if any two consecutive exact u-matches can share at most one letter which must be the last letter of the first exact u-match and the first letter of the second exact u-match. For example $u = 131$ is a word that has the exact $\mathbb{P}$-minimal overlapping property. We say that u is *exact* $\mathbb{P}$*-non-overlapping* (*exact* $[k]$*-non-overlapping* if $ESt^{(\mathbb{P})}(u) = \emptyset$ ($ESt^{([k])}(u) = \emptyset$). For example $u = 132$ is a word that has the exact $\mathbb{P}$-non-overlapping property.

Let $\mathbf{z}_k = z_1, \ldots, z_k$ and $\mathbf{z}_\infty = z_1, z_2, \ldots$. Then for any $u \in [k]^j$, we let

$$EN_{n,u}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, eumch(w)=0} x^{\mathrm{des}(w)+1} \overline{z}^w,$$

$$EN_{n,u}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, eumch(w)=0} x^{\mathrm{des}(w)+1} \overline{z}^w,$$

$$LEN_{u,n}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, eumch(w)=0} x^{\mathrm{lev}(w)+1} \overline{z}^w, \text{ and}$$

$$LEN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, eumch(w)=0} x^{\mathrm{lev}(w)+1} \overline{z}^w.$$

Similarly for $u \in [k]^j$ such that $\mathrm{red}(u) = u$, we let

$$N_{n,u}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, \mathrm{umch}(w)=0} x^{\mathrm{des}(w)+1} \overline{z}^w,$$

$$N_{n,u}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, \mathrm{umch}(w)=0} x^{\mathrm{des}(w)+1} \overline{z}^w,$$

$$LN_{u,n}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, \mathrm{umch}(w)=0} x^{\mathrm{lev}(w)+1} \overline{z}^w \text{ and}$$

$$LN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, \mathrm{umch}(w)=0} x^{\mathrm{lev}(w)+1} \overline{z}^w.$$

The main goal of this thesis is to study the generating functions

$$\mathcal{EN}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} EN_{n,u}^{(k)}(x,\mathbf{z})t^n,$$
$$\mathcal{EN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} EN_{n,u}^{(\mathbb{P})}(x,\mathbf{z})t^n,$$
$$\mathcal{N}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} N_{n,u}^{(k)}(x,\mathbf{z})t^n, \text{ and}$$
$$\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} N_{n,u}^{(\mathbb{P})}(x,\mathbf{z})t^n$$

in the case where $\text{red}(u) = u$ and $\text{des}(u) \leq 1$ and the generating functions,

$$\mathcal{LEN}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} LEN_{u,n}^{(k)}(x,\mathbf{z}_k)t^n,$$
$$\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} LEN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n,$$
$$\mathcal{LN}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} LN_{u,n}^{(k)}(x,\mathbf{z}_k)t^n, \text{ and}$$
$$\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} LN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n$$

in the case where $\text{red}(u) = u$ and $\text{lev}(u) \leq 1$.

When k and $|u|$ are small, there are well-known recursive methods to compute $N_{n,u}^{(k)}(x,\mathbf{z}_k)$ or $EN_{n,u}^{(k)}(x,\mathbf{z}_k)$. That is, suppose that $|u| = r$. For any word $v \in [k]^*$, we let $\mathcal{B}_v^{(k)} = \{w \in [k]^* : v \text{ is a prefix of } w\}$ and

$$\mathcal{N}_{u,v}^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} t^n \sum_{w\in\mathcal{B}_v^{(k)}\cap[k]^n, \text{umch}(w)=0} x^{\text{des}(w)+1}\overline{z}^w.$$

For example, if $k = 3$, $u = 123$, and $v = 12$, then the words in $\mathcal{B}_{12}^{(3)}$ are of the form 12 or 1 concatenated with either a word in $\mathcal{B}_{21}^{(3)}$, $\mathcal{B}_{22}^{(3)}$, or $\mathcal{B}_{23}^{(3)}$. Words of the form $1 * \mathcal{B}_{23}^{(3)}$ cannot contribute to $\mathcal{N}_{u,v}^{(k)}(x,\mathbf{z}_k,t)$ since they all start with a 123-match. It follows that

$$\mathcal{N}_{u,12}^{(3)}(x,\mathbf{z}_3,t) = xz_1z_2t^2 + z_1t\mathcal{N}_{u,21}^{(3)}(x,\mathbf{z}_3,t) + z_1t\mathcal{N}_{u,22}^{(3)}(x,\mathbf{z}_3,t).$$

In this way, we can show that the functions $\mathcal{N}^{(3)}_{u,v}(x,\mathbf{z}_3,t)$ where $|v|=|u|-1$ satisfy simple recursions. Bringing the terms that do not involve the generating functions to one side, one can rewrite these equations in the form

$$\vec{v} = M \begin{pmatrix} \mathcal{N}^{(3)}_{u,11}(x,\mathbf{z},t) \\ \mathcal{N}^{(3)}_{u,12}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,13}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,21}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,22}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,23}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,31}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,32}(x,\mathbf{z}_3,t) \\ \mathcal{N}^{(3)}_{u,33}(x,\mathbf{z}_3,t). \end{pmatrix}.$$

Then if one can invert the matrix M, one can solve for the generating functions $\mathcal{N}^{(3)}_{u,ij}(x,\mathbf{z}_3,t)$ from which one can easily recover the desired generating function $\mathcal{N}^{(3)}_{u}(x,\mathbf{z}_3,t)$. More details on the method can be found in [23]. The problem with this method is that it requires us to invert a $k^{|u|-1}$ matrix with multivariable entries which is impractical to compute as k and $|u|$ get large.

The method that we will employ is what Jones and Remmel [24,26–28] call the *reciprocity method*. Jones and Remmel developed the reciprocity method to compute the generating functions of the form

$$\mathrm{NM}_\tau(t,x,y) = 1 + \sum_{\sigma \in \mathcal{NM}(\tau)} y^{\mathrm{des}(\sigma)+1} x^{\mathrm{LRmin}(\sigma)} \frac{t^n}{n!}$$

where $\tau \in \mathcal{S}_j$ and for $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$, $\mathrm{LRmin}(\sigma)$ is the number of left-to-right minima in σ which is the number of $1 \leq j \leq n$ such that $\sigma_i > \sigma_j$ for all $i < j$. For example, if $\sigma = 3425167$, $\sigma_1 = 3$, $\sigma_3 = 2$ and $\sigma_5 = 1$ are the set of left-to-right minima of σ. They were able to compute $\mathrm{NM}_\tau(t,x,y)$ for certain families of permutations τ in which τ starts with 1 and $\mathrm{des}(\tau) = 1$. The basic idea of their approach is as follows. If τ starts with 1, then the results in [24] allow us to write $\mathrm{NM}_\tau(t,x,y)$ in the form

$$\mathrm{NM}_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x$$

where $U_\tau(t,y) = \sum_{n\geq 0} U_{\tau,n}(y)\frac{t^n}{n!}$.

Next one writes

$$U_\tau(t,y) = \frac{1}{1+\sum_{n\geq 1} \mathrm{NM}_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{1.5}$$

One can then use the homomorphism method to give a combinatorial interpretation of the right-hand side of (1.5) which can be used to find simple recursions for the coefficients $U_{\tau,n}(y)$. This homomorphism method was first introduced by Brenti [11] and later developed by Remmel and his students which is the subject of the book "Counting with Symmetric Functions" by Mendes and Remmel [43]. The so-called homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities such as

$$H(t) = 1/E(-t), \tag{1.6}$$

where $H(t)$ and $E(t)$ are the generating functions for the homogeneous and elementary symmetric functions, respectively:

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}\frac{1}{1-x_i t} \text{ and } E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}(1+x_i t). \tag{1.7}$$

In their case, Jones and Remmel defined a homomorphism θ_τ on Λ by setting

$$\theta_\tau(e_n) = \frac{(-1)^n}{n!}\mathrm{NM}_{\tau,n}(1,y).$$

Then

$$\theta_\tau(E(-t)) = \sum_{n\geq 0}\mathrm{NM}_{\tau,n}(1,y)\frac{t^n}{n!} = \frac{1}{U_\tau(t,y)}.$$

Hence

$$U_\tau(t,y) = \frac{1}{\theta_\tau(E(-t))} = \theta_\tau(H(t)),$$

which implies that

$$n!\theta_\tau(h_n) = U_{\tau,n}(y). \tag{1.8}$$

Thus, if we can compute $n!\theta_\tau(h_n)$ for all $n \geq 1$, then we can compute the polynomials $U_{\tau,n}(y)$ and the generating function $U_\tau(t,y)$, which in turn allows us to compute the generating function $\mathrm{NM}_\tau(t,x,y)$. Jones and Remmel [27, 28] showed that one can interpret $n!\theta_\tau(h_n)$ as a certain signed sum of the weights of filled, labeled brick tabloids when τ starts with 1 and $\mathrm{des}(\tau) = 1$. They then defined a weight-preserving, sign-reversing involution I on the set of such filled, labeled brick tabloids which allowed them to give a relatively simple combinatorial interpretation for $n!\theta_\tau(n_n)$. Consequently, they showed how such a combinatorial interpretation allowed them to prove that for certain families of such permutations τ, the polynomials $U_{\tau,n}(y)$ satisfy simple recursions.

In [3], Remmel and Bach extended the reciprocity method to study the polynomials $U_{\Gamma,n}(y)$ where

$$U_\Gamma(t,y) = 1 + \sum_{n\geq 1} U_{\Gamma,n}(y)\frac{t^n}{n!} = \frac{1}{1+\sum_{n\geq 1}\mathrm{NM}_{\Gamma,n}(1,y)\frac{t^n}{n!}}$$

in the case where Γ is a set of permutations such that for all $\tau \in \Gamma$, τ starts with 1 and $\mathrm{des}(\tau) \leq 1$. Specifically, they studied the case where

$$\begin{aligned}\Gamma_{k_1,k_2} = \{\sigma \in S_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, &\sigma_1 < \sigma_2 < \cdots < \sigma_{k_1}, \\ &\sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}.\end{aligned}$$

That is, Γ_{k_1,k_2} consists of all permutations σ of length p where 1 is in position 1, 2 is in position k_1+1, and σ consists of two increasing sequences, one starting at 1 and the other starting at 2. In certain cases, they were able to obtain explicit formulas for the polynomials $U_{\Gamma_{k_1,k_2,s},n}(y)$ for certain values of k_1, k_2, and s. For instance, if $\Gamma = \{1324, 123\}$, then they obtained the following result for the polynomials $U_{\Gamma,n}(y)$'s. For all $n \geq 0$,

$$\begin{aligned}U_{\Gamma,2n}(y) &= \sum_{k=0}^{n} \frac{(2k+1)\binom{2n}{n-k}}{n+k+1}(-y)^{n+k+1} \text{ and} \\ U_{\Gamma,2n+1}(y) &= \sum_{k=0}^{n} \frac{2(k+1)\binom{2n+1}{n-k}}{n+k+2}(-y)^{n+k}.\end{aligned}$$

Another example in [3] where they could find an explicit formula is the case $\Gamma_{2,2,s} = \{1324, 1342, 123\}$ where they showed that $U_{\Gamma_{2,2,s},1}(y) = -y$, and for $n \geq 2$, the polynomials $U_{\Gamma_{2,2,s},n}(y)$'s satisfy the recursion

$$U_{\Gamma_{2,2,s},n}(y) = -yU_{\Gamma_{2,2,s},n-1}(y) - \sum_{k=0}^{s-2} \left((n-k-1)yU_{\Gamma_{2,2,s},n-k-2}(y) + (n-k-2)y^2 U_{\Gamma_{2,2,s},n-k-3}(y)\right). \tag{1.9}$$

Bach and Remmel further extended the reciprocity method to study the generating functions $\mathrm{NM}_\Gamma(t,x,y)$ where all the permutations Γ start with 1 but they did not put any conditions on the number of descents in a permutations in Γ. While the basic concepts of the reciprocity method still hold, the involution defined by Jones and Remmel no longer works. Thus, they defined a new sign-reversing, weight-preserving mapping J_Γ and, under this new involution, they were able to compute the recursion for the polynomials $U_{\Gamma,n}(y)$ for the special cases where $\tau \in \Gamma$ such that $\mathrm{des}(\tau) = j \geq 1$ and the bottom elements of these descents are $2, \ldots, j+1$ when reading from left to right. In most of the cases here, the analysis of the fixed points of the involution J_Γ can be associated with counting the number of linear extensions for certain Hasse diagrams.

We apply the reciprocity method of Jones and Remmel for our problem and assume that we can write the generating function $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ as

$$\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)) = \frac{1}{U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)} \text{ where } U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = 1 + \sum_{n \geq 1} U_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) t^n. \tag{1.10}$$

Thus

$$U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 + \sum_{n \geq 1} N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) t^n}. \tag{1.11}$$

One can then use the homomorphism method to give a combinatorial interpretation to the right-hand side of (1.11) which can be used to find a combinatorial interpretation for $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$. In our case, we define a homomorphism Θ_u on Λ by setting

$$\Theta_u(e_n) = (-1)^n N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$$

Then

$$\Theta_u(E(-t)) = \sum_{n \geq 0} N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) t^n = \frac{1}{U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)}.$$

Hence

$$U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1}{\Theta_u(E(-t))} = \Theta_u(H(t))$$

which implies that

$$\Theta_u(h_n) = U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty). \tag{1.12}$$

Thus if we can compute $\Theta_u(h_n)$ for all $n \geq 1$, then we can compute the polynomials $U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ and the generating function $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ which in turn allows us to compute the generating function $\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$. The same method can be applied to find combinatorial interpretations for $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $EU_u^{(k)}(x,\mathbf{z}_k,t)$, $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$ where

$$\begin{aligned}
\mathcal{N}_u^{(k)}(x,\mathbf{z}_k,t)) &= \frac{1}{U_u^{(k)}(x,\mathbf{z}_k,t)}, \\
\mathcal{EN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)) &= \frac{1}{EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)}, \\
\mathcal{EN}_u^{(k)}(x,\mathbf{z}_k,t)) &= \frac{1}{EU_u^{(k)}(x,\mathbf{z}_k,t)}, \\
\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)) &= \frac{1}{LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)}, \\
\mathcal{LN}_u^{(k)}(x,\mathbf{z}_k,t)) &= \frac{1}{LU_u^{(k)}(x,\mathbf{z}_k,t)}, \\
\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)) &= \frac{1}{LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)}, \text{ and} \\
\mathcal{LEN}_u^{(k)}(x,\mathbf{z}_k,t)) &= \frac{1}{LEU_u^{(k)}(x,\mathbf{z}_k,t)}.
\end{aligned}$$

The final steps of the reciprocity method that we employ will be different from the ones used by Jones and Remmel [26, 27] for permutations. For the generating function for permutations that they studied, Jones and Remmel used the combinatorial interpretation that arose from the analog of $\Theta_u(h_n)$ to obtain simple recursions satisfied by their analog of $U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$. In our case, we shall use the combinatorial interpretation of $\Theta_u(h_n)$ that comes out of the homomorphism method plus a map which we call the "collapse map" to show that we can obtain a closed expression for the generating functions $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ or $U_u^{([k])}(x,\mathbf{z}_k,t)$ by an appropriate substitution in certain other generating functions for words.

The generating functions that we will substitute into will depend on the relative order of u_1 and u_j where $u = u_1 \dots u_j$. In each case our generating function will be over the variables x_{ij} where $i, j \in \mathbb{P}$, the variables z_i where $i \in \mathbb{P}$, and t which we denote as $(\mathbf{x}_\infty, \mathbf{z}_\infty, t)$. In the case where $u_1 > u_j$, $\text{red}(u) = u$, and $\text{des}(u) = 1$, our final expression for our desired generating functions $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ or $U_u^{([k])}(x, \mathbf{z}_k, t)$ will be a substitution into the generating function

$$\mathcal{D}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \sum_{w \in \mathbb{P}^*} t^{|w|} \overline{z}^w \prod_{i<j} x_{ji}^{\mathbf{ji}(w)}.$$

In the case where $u_1 < u_j$, $\text{red}(u) = u$, $\text{des}(u) = 1$, and u has the $\mathbb{P}$-weakly increasing overlapping property ($[k]$-weakly increasing overlapping property), our final expression for our desired generating function $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ ($U_u^{([k])}(x, \mathbf{z}_k, t)$) will be a substitution into the generating function

$$\mathcal{R}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \sum_{w \in w_1 \le w_2 \le \cdots \le w_n \in \mathbb{P}^*} t^{|w|} \overline{z}^w \prod_{i<j} x_{ij}^{\mathbf{ij}(w)}.$$

In the case were $u_1 = u_j$, $\text{red}(u) = u$, $\text{des}(u) = 1$, and u has the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property), our final expression for our desired generating function $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ ($U_u^{([k])}(x, \mathbf{z}_k, t)$) will be a substitution into the generating function

$$\mathcal{L}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \sum_{w = w_1 \le w_2 \le \cdots w_n \in \mathbb{P}^*} t^{|w|} \overline{z}^w \prod_{i} x_{ii}^{\mathbf{ii}(w)}.$$

If u does not have the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property), it will still be the case that u has the $\mathbb{P}$-weakly decreasing overlapping property ($[k]$-weakly decreasing overlapping property). In such a case, our final expression for our desired generating function $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ ($U_u^{([k])}(x, \mathbf{z}_\infty, t)$) will be a substitution into the generating function

$$\mathcal{WD}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \sum_{w \in \mathbb{P}^*} t^{|w|} \overline{z}^w \prod_{i \le j} x_{ji}^{\mathbf{ji}(w)}.$$

In the case where $u_1 > u_j$, $\text{red}(u) = u$, $\text{lev}(u) = 1$, u has the $\mathbb{P}$-weakly decreasing overlapping property ($[k]$-weakly decreasing overlapping property), our final expression for our desired generating functions $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ or $LU_u^{([k])}(x,\mathbf{z}_k,t)$ will be a substitution into the generating function

$$\mathcal{H}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)=\sum_{w\in\mathbb{P}^*,\mathrm{lev}(w)=0} t^{|w|}\overline{z}^w\prod_{i<j} x_{ji}^{\mathbf{ji}(w)}.$$

In the case where $u_1<u_j$, $\mathrm{red}(u)=u$, $\mathrm{lev}(u)=1$, and u has the $\mathbb{P}$-weakly increasing overlapping property ($[k]$-weakly increasing overlapping property), our final expression for our desired generating function $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ ($LU_u^{([k])}(x,\mathbf{z}_k,t)$) will be a substitution into the generating function

$$\mathcal{G}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)=\sum_{w\in\mathbb{P}^*,\mathrm{lev}(w)=0} t^{|w|}\overline{z}^w\prod_{i>j} x_{\mathbf{ji}}^{\mathbf{ji}(w)}.$$

In this case we will also need to consider the following analog to $\mathcal{D}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$:

$$\mathcal{E}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)=\sum_{w\in\mathbb{P}^*} t^{|w|}\overline{z}^w\prod_{i>j} x_{ji}^{\mathrm{ji}(w)}.$$

In the case where $u_1=u_j$, $\mathrm{red}(u)=u$, $\mathrm{lev}(u)=1$, and u has the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property), our final expression for our desired generating function $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ ($LU_u^{([k])}(x,\mathbf{z}_k,t)$) will be a substitution into the generating function

$$\mathcal{WD}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)=\sum_{w\in\mathbb{P}^*} t^{|w|}\overline{z}^w\prod_{i\leq j} x_{ji}^{\mathbf{ji}(w)}.$$

We will prove the following theorems for the generating functions $\mathcal{D}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, $\mathcal{L}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, $\mathcal{R}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, $\mathcal{WD}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, $\mathcal{H}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, $\mathcal{G}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$, and $\mathcal{E}^{\mathbb{P}}(\mathbf{x}_\infty,\mathbf{z}_\infty,t)$. Given a nonempty set $S\subseteq\mathbb{P}$, we let

$$DXZ(S)=\begin{cases} z_j & \text{if } S=\{j\},\ \text{and}\\ z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-1) & \text{if } S=\{j_1<\cdots<j_k\} \text{ where } k\geq 2.\end{cases} \tag{1.13}$$

$$RXZ(S)=\begin{cases} \frac{z_j}{1-z_jt} & \text{if } S=\{j\},\ \text{and}\\ \left(\prod_{i=1}^{k}\frac{z_{j_i}}{1-z_{j_i}t}\right)\prod_{i=1}^{k-1}x_{j_ij_{i+1}} & \text{if } S=\{j_1<\cdots<j_k\} \text{ where } k\geq 2.\end{cases} \tag{1.14}$$

$$EYZ(S) = \begin{cases} z_j & \text{if } S = \{j\}, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1}(x_{j_i j_{i+1}} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2. \end{cases} \tag{1.15}$$

Let $WD\mathbb{P}^*$ ($WD[k]^*$) denote the set of all weakly decreasing words in $\mathbb{P}^*$ ($[k]^*$). Given a nonempty word v in $WD\mathbb{P}^*$, we let

$$WDXZ(v) = \begin{cases} z_j & \text{if } v = j, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1}(x_{j_i j_{i+1}} - 1) & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2. \end{cases} \tag{1.16}$$

Theorem 1.

$$\mathcal{D}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} DXZ(S)}. \tag{1.17}$$

Theorem 2.

$$\mathcal{WD}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)}. \tag{1.18}$$

Theorem 3.

$$\mathcal{R}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = 1 + \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} RXZ(S). \tag{1.19}$$

Theorem 4.

$$\mathcal{L}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \prod_{i \geq 1} \left(1 + \frac{z_i t}{1 - x_{ii} z_i t}\right) \tag{1.20}$$

Theorem 5.

$$\mathcal{H}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \mathcal{D}^{\mathbb{P}}(\mathbf{x}_\infty, \frac{\mathbf{z}_\infty}{1 + \mathbf{z}_\infty t}, t) \tag{1.21}$$

Theorem 6.

$$\mathcal{G}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \mathcal{E}^{\mathbb{P}}(\mathbf{x}_\infty, \frac{\mathbf{z}_\infty}{1 + \mathbf{z}_\infty t}, t) \tag{1.22}$$

Theorem 7.

$$\mathcal{E}^{\mathbb{P}}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} EYZ(S)}. \tag{1.23}$$

The main advantage of our approach is that we obtain a uniform way to find expressions for the generating functions $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $EU_u^{(k)}(x,\mathbf{z}_k,t)$, $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$ which are independent of the length of u as long as $\mathrm{des}(u)=1$ or $\mathrm{lev}(u)=1$ and u satisfies the appropriate overlapping conditions. In fact our general methods can be applied even in cases where $\mathrm{des}(u)>1$ or $\mathrm{lev}(u)>1$. However in such cases the combinatorial interpretation of $\Theta_u(h_n)$ that comes out the homomorphism method is significantly more complicated so that we will not pursue such results in this thesis.

The outline of this thesis is as follows. In section 1.2, we give an overview of symmetric functions. In section 1.3, we outline the reciprocal method and the collapse map by looking at an example in the case where $u=u_1\ldots u_j$, $\mathrm{des}(u)=1$, and $u_1>u_j$. In Chapter 2, we will give the proofs of theorems 1, 2, 3, 4, 5, 6, and 7.

In section 3.1, we will prove the results about the involution I_u that we mentioned earlier. In section 3.2, we shall show how to use Theorem 1 to find expressions for $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $EU_u^{(k)}(x,\mathbf{z}_k,t)$ in the case where $u=u_1\ldots u_j$, $u_1>u_j$, and $\mathrm{des}(u)=1$. In section 3.3, we shall show how to use Theorem 3 to find expressions for $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $EU_u^{(k)}(x,\mathbf{z}_k,t)$ in the case where $u=u_1\ldots u_j$, $u_1<u_j$, $\mathrm{des}(u)=1$, and u has the $\mathbb{P}$-weakly increasing overlapping property or $[k]$-weakly increasing overlapping property. In section 3.4, we shall show how to use Theorems 2 and 4 to find expressions for $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $EU_u^{(k)}(x,\mathbf{z}_k,t)$ in the case where $u=u_1\ldots u_j$, $u_1=u_j$, $\mathrm{des}(u)\leq 1$, and u has the $\mathbb{P}$-level overlapping property or $[k]$-level overlapping property.

In section 4.2, we shall show how to use Theorem 5, to find expressions for $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$ in the case where $u=u_1\ldots u_j$, $u_1>u_j$, $\mathrm{lev}(u)=1$, and u has the $\mathbb{P}$-weakly decreasing overlapping property or $[k]$-weakly decreasing overlapping property. In section 4.3, we shall show how to use Theorems 6 and 7, to find expressions for $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$ in the case where $u = u_1 \ldots u_j$, $u_1 < u_j$, $\mathrm{lev}(u) = 1$, and u has the $\mathbb{P}$-weakly increasing overlapping property or $[k]$-weakly increasing overlapping property. In section 4.4, we shall how to use Theorem 2, to find expressions for $LU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $LU_u^{(k)}(x, \mathbf{z}_k, t)$, $LEU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $LEU_u^{(k)}(x, \mathbf{z}_k, t)$ in the case where $u = u_1 \ldots u_j$, $u_1 = u_j$, $\mathrm{lev}(u) = 1$, and u has the $\mathbb{P}$-level overlapping property or $[k]$-level overlapping property.

Finally, in Chapter 5 we will discuss some further extensions of our methods. For example, we will discuss how we can extend our methods to handle cases where u has more than one descent, and we will discuss how we can replace the statistics $\mathrm{des}(u)$ or $\mathrm{lev}(u)$ which are essentially the patterns 21 and 11, respectively, with patterns of length $k > 2$.

1.2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs. We shall consider the ring of symmetric functions, Λ, over infinitely many variables $x_1, x_2, \ldots$. The homogeneous symmetric functions, $h_n \in \Lambda$, and elementary symmetric functions, $e_n \in \Lambda$, are defined by the generating functions

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \text{ and } E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t).$$

The n-th power symmetric function, $p_n \in \Lambda$, is defined as $p_n = \sum_{i=1}^{\infty} x_i^n$.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition; that is, λ is a finite sequence of weakly increasing non-negative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions, see [37], says that $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n, the space of symmetric functions which are homogeneous of degree n. Equivalently, the fundamental theorem of symmetric functions states that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for the ring Λ. It follows that one can completely specify a ring homomorphism $\Gamma : \Lambda \to R$ from Λ into a ring R by giving the values of $\Gamma(e_n)$ for $n \geq 0$.

Next we give combinatorial interpretations to the expansion of h_μ in terms of the elementary symmetric functions. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and $\mu \vdash n$, a *λ-brick tabloid of shape μ* is a filling of the Ferrers diagram of shape μ with bricks of size $\lambda_1, \ldots, \lambda_\ell$ such that each brick lies in a single row and no two bricks overlap. For example, Figure 1.1 shows all the λ-brick tabloids of shape μ where $\lambda = (1, 1, 2, 2)$ and $\mu = (2, 4)$.

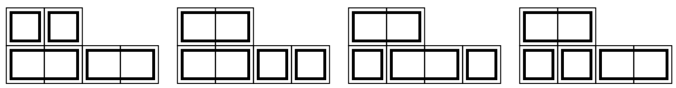

Figure 1.1: The four $(1, 1, 2, 2)$-brick tabloids of shape $(2, 4)$.

If T is a brick tabloid of shape (n) such that the lengths of the bricks, reading from left to right, are $b_1, \ldots, b_\ell$, then we shall write $T = (b_1, \ldots, b_\ell)$. For example, the brick tabloid $T = (2, 3, 1, 4, 2)$ is pictured in Figure 1.2.

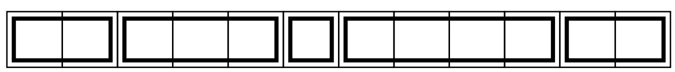

Figure 1.2: The brick tabloid $T = (2, 3, 1, 4, 2)$.

Let $\mathcal{B}_{\lambda,\mu}$ denote the set of all λ-brick tabloids of shape μ and let $B_{\lambda,\mu} = |\mathcal{B}_{\lambda,\mu}|$. Eğecioğlu and Remmel proved in [16] that

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda. \quad (1.24)$$

1.3 Outline of Reciprocal Method, Collapse Map, and a few results

In this section, we shall apply the reciprocal method to give combinatorial interpretations to $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $U_u^{(k)}(x, \mathbf{z}_k, t)$, $EU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $EU_u^{(k)}(x, \mathbf{z}_k, t)$.

Fix a word u such that $\mathrm{des}(u) \leq 1$. In this introduction, we will only consider $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ but the other cases are similar. Recall that

$$U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 + \sum_{n \geq 1} N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) t^n}. \tag{1.25}$$

Thus if we let $\Theta_u(e_n) = (-1)^n N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$ and $\Theta_u(e_0) = 1$, we see that

$$\begin{aligned} \Theta_u(H(t)) &= 1 + \sum_{n \geq 1} \Theta_u(h_n) \\ &= \Theta_u\left(\frac{1}{E(-t))}\right) = \frac{1}{1 + \sum_{n \geq 1} (-1)^n \Theta_u(e_n)} \\ &= \frac{1}{1 + \sum_{n \geq 1} N_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) t^n} = U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t). \end{aligned}$$

Thus it follows that $\Theta_u(h_n) = U_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$.

By (1.24), we have that

$$\begin{aligned} \Theta_u(h_n) &= \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} B_{\lambda,n}\ \Theta_u(e_\lambda) \\ &= \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \sum_{(b_1, \ldots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} N_{u,b_i}^{(\mathbb{P})}(x, \mathbf{z}_\infty) \\ &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \ldots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} N_{u,b_i}^{(\mathbb{P})}(x, \mathbf{z}_\infty). \end{aligned} \tag{1.26}$$

Our next goal is to give a combinatorial interpretation to the right-hand side of (1.26). Fix a partition λ of n and a λ-brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$. We will interpret $\prod_{i=1}^{\ell(\lambda)} N_{u,b_i}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ as the number of ways of picking words $(w^{(1)}, \ldots, w^{(\ell(\lambda))})$ such that for each i, $w^{(i)} \in \mathbb{P}^{b_i}$ is a word such that $u\mathrm{mch}(w) = 0$ and assigning a weight to this $\ell(\lambda)$-tuple to be $\prod_{i=1}^{\ell(\lambda)} x^{\mathrm{des}(w^{(i)})+1} \overline{z}^{w^{(i)}}$.

We can then use the pair $\langle B, (w^{(1)}, \ldots, w^{(\ell(\lambda))}) \rangle$ to construct a filled-labeled-brick tabloid $O_{\langle B, (w^{(1)}, \ldots, w^{(\ell(\lambda))}) \rangle}$ as follows. First for each brick b_i, we place the word $w^{(i)}$ in the cells of the brick, reading from left to right. Then we label each cell of b_i that starts a descent of $w^{(i)}$ with a x and we also label the last cell of b_i with x. This accounts for the factor $x^{\mathrm{des}(w^{(i)})+1}$. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from x to $-x$. For example, suppose $n = 17$,

$u = 312$, $B = (3, 7, 4, 3)$ $w^{(1)} = 1\ 1\ 7$, $w^{(2)} = 3\ 6\ 6\ 5\ 2\ 5\ 1$, $w^{(3)} = 3\ 4\ 7\ 6$, and $w^{(4)} = 2\ 5\ 2$. Then we have pictured the filled-labeled brick tabloid $O_{\langle B,(w^{(1)},\ldots,w^{(4)})\rangle}$ constructed from the pair $\langle B, (w^{(1)}, \ldots, w^{(4)})\rangle$ in Figure 1.3.

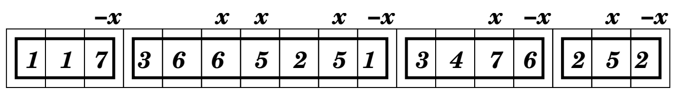

Figure 1.3: The construction of a filled-labeled-brick tabloid.

Clearly, we can recover the pair $\langle B, (w^{(1)}, \ldots, w^{(\ell(\lambda))})\rangle$ and the labels on the cells from B and the word w which is obtained by reading the elements in the cells of $O_{\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle}$ from left to right. Thus we shall specify the filled-labeled-brick tabloid $O_{\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle}$ by (B, w). We let $\mathcal{O}_{u,n}^{(\mathbb{P})}$ denote the set of all filled-labeled-brick tabloids constructed in this way. That is, $\mathcal{O}_{u,n}^{(\mathbb{P})}$ consists of all pairs $O = (B, w)$ where

1. $B = (b_1, \ldots, b_{\ell(\lambda)})$ is brick tabloid of shape (n),
2. $w = w_1 \ldots w_n \in \mathbb{P}^n$ such that there is no u-match of σ which is entirely contained in a single brick of B, and
3. if there is a cell c such that a brick b_i contains both cells c and $c+1$ and $w_c > w_{c+1}$, then cell c is labeled with a x and the last cell of any brick is labeled with $-x$.

The sign of O, sgn(O), is $(-1)^{\ell(\lambda)}$ and the weight of O, $wt(O)$, is $x^{\ell(\lambda)+\text{intdes}(\sigma)}\overline{z}^{w}$ where intdes(w) denotes the number of i such that $w_i > w_{i+1}$ and w_i and w_{i+1} lie in the same brick. We shall refer to such i as an *internal descent* of O. Note that the labels on O are completely determined by the underlying brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$ and the underlying word w. Thus the filled-labeled-brick tabloid O pictured in Figure 1.3 equals $((3, 7, 4, 3), 1\ 1\ 7\ 3\ 6\ 6\ 5\ 2\ 5\ 1\ 3\ 4\ 7\ 6\ 2\ 5\ 2)$.

It follows that

$$\Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}} sgn(O)wt(O). \tag{1.27}$$

Next we define a weight-preserving, sign-reversing involution I_u on $\mathcal{O}_{u,n}^{(\mathbb{P})}$. Given an element $O = (B, w) \in \mathcal{O}_{u,n}^{(\mathbb{P})}$ where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$, scan the cells of O from left to right looking for the first cell c such that either

(i) c is labeled with a x or

(ii) c is a cell at the end of a brick b_i, $w_c > w_{c+1}$, and there is no u-match of w that lies entirely in the cells of bricks b_i and b_{i+1}.

In case (i), if c is a cell in brick b_j, then we split b_j into two bricks b_j' and b_j'' where b_j' contains all the cells of b_j up to and including cell c and b_j'' consists of the remaining cells of b_j and we change the label on cell c from x to $-x$. In case (ii), we combine the two bricks b_i and b_{i+1} into a single brick b and change the label on cell c from $-x$ to x. If neither case (i) nor case (ii) applies, then we define $I_u(O) = O$. For example, consider the element $O \in \mathcal{O}_{312,17}^{(\mathbb{P})}$ pictured in Figure 1.3. Note that even though the number in the last cell of brick 1 is greater than the number in the first cell of brick 2, we can not combine these two bricks because 7 3 6 would be a 312-match. Thus the first place that we can apply the involution is on cell 6 which is labeled with an x so that $I_u(O)$ is the object pictured in Figure 1.4.

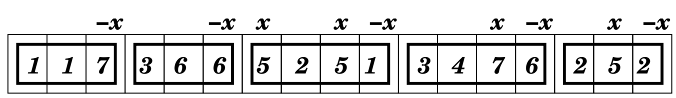

Figure 1.4: $I_u(O)$ for O in Figure 1.3.

We claim that whenever u is a word such that $\mathrm{red}(u) = u$ and $\mathrm{des}(u) \leq 1$, I_u is an involution, i.e. I_u^2 is the identity. We will prove this in a later section, but for now we will accept this is true and examine the fixed points of I_u. So assume that (B, w) is a fixed point of I_u.

There are two cases to consider.

Case 1. $\mathrm{des}(u) = 0$.

Suppose that (B,w) is a fixed point where $B = (b_1,\ldots,b_k)$ and $w = w_1\ldots w_n$. We will prove in a later section that (B,w) is a fixed point if and only if w is a weakly increasing word such that w has no u-match that lies entirely within one of the bricks of B.

Case 2. $\mathrm{des}(u) = 1$.

We will prove in a later section that $O = (B,w)$ where $B = (b_1,\ldots,b_k)$ and $w = w_1\ldots w_n$ is a fixed point if and only if

1. there are no cells labeled with x in O, i.e., the elements of w in each brick of O are weakly increasing and

2. if b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is a weak increase between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^i |b_j|} \leq w_{1+\sum_{j=1}^i |b_j|}$, or (b) there is a decrease between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^i |b_j|} > w_{1+\sum_{j=1}^i |b_j|}$, but there is a u-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $w_{\sum_{j=1}^i |b_j|}$ and $w_{1+\sum_{j=1}^i |b_j|}$.

It follows that

$$\Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}, I_u(O)=O} sgn(O)wt(O). \tag{1.28}$$

We conclude the introduction by briefly describing how to compute the generating function $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ for $u = u_1\ldots u_j$ such that $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, and $u_1 > u_j$.

We start by considering a special class of words $u = u_1\ldots u_j$ which have the $\mathbb{P}$-minimal overlapping property ($[k]$-minimal overlapping property). This means that any two consecutive u-matches can share at most one letter. For example $u = 2341$ has the $\mathbb{P}$-minimal overlapping property while $u = 3412$ does not have the $\mathbb{P}$-minimal overlapping property since in the word $w = 563412$, the u-matches 5634 and 3412 share two letters.

Thus assume that $u = u_1\ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 > u_j$, and u has the $\mathbb{P}$-minimal overlapping property. First we introduce what we shall call the *collapse map* which maps fixed points of I_u to a certain subset of words in $\mathbb{P}^*$. This is best explained through an example. Suppose that $u = 2341$ and we want to compute $U_{2341}^{(7)}(x, \mathbf{z}_7, t)$. By (1.28), we know that

$$U_{u,n}^{(7)}(x, \mathbf{z}_7) = \sum_{O \in \mathcal{O}_{u,n}^{(7)}, I_u(O)=O} sgn(O)wt(O). \tag{1.29}$$

Now suppose that we are given a fixed point (B, w) of I_u where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$ such as the one pictured in Figure 1.5. We know that to be a fixed point of I_u, w must be weakly increasing within bricks of B and that for any $i < k$, if c is last cell in brick b_i and $w_c > w_{c+1}$, then there must be a u-match in w which is contained in the cells of b_i and b_{i+1}. In our particular example, since $u = 2341$ has a single descent, this match must involve the last three cells of b_i and the first cell of b_{i+1}. In Figure 1.5, we have indicated the two such 2341-matches in our example by placing stars below the cells in the 2341-matches. In this case the collapse map just maps (B, w) to the word $v = C(B, w, u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 1.5 where again we have starred the elements in $C(B, w, u)$ that remain from the original 2341-matches in w. What makes the case where u has the minimal overlapping property easier is that, since any two consecutive u-matches can share at most letter, there is no possibility that an end point of a u-match in w occurs in the middle of another u-match in w so that the letters that we remove from w for any pair of u-matches are disjoint from each other.

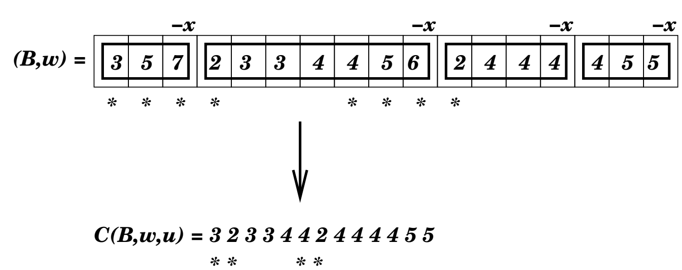

Figure 1.5: A fixed point of I_{2341}.

The next question that we want to consider is how can we construct all the fixed points of (B,w) of I_u such that $C(B,w,v)$ is equal to a given word $v = v_1 \ldots v_n$. First it easy to see that the only descents that appear in a word $C(B,w,u)$ must come from 2341-matches that straddled two bricks in B. Thus if $v_s > v_{s+1}$, then v_s must have played the role of 2 in the original 2341-match and v_{s+1} must have played the role of 1 in the original 2341-match. Such a requirement rules out certain words from being in the range of the collapse map C. For example, suppose that the underlying alphabet is [7]. Then if $v_s = 6$ and $v_{s+1} = i$ where $i < 6$, then v could not have come from the collapse of 2341-match because we can not add two letters which could play the role of 3 and 4 in the 2341-match. If we consider the first descent 32 in the $C(B,w,u)$ of Figure 1.5, then we see there are many ways that we could add the two middle letters. That is, the original 2341-match could have been any $3cd2$ where $c < d$ and $c, d \in \{4,5,6,7\}$. It follows that the extra weight from these possibilities that is not included in $\overline{z}^{C(B,w,u)} t^{|C(B,w,u)|}$ in this case would be $-xt^2 \sum_{4 \leq c < d \leq 7} z_c z_d$. Here the $-x$ comes from the fact that we know that the original match straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. On the other hand, if $v_s \leq v_{s+1}$, then we have only two choices. That is, either cell s was the end of a brick or cell s was an internal cell of a brick. This implies that each weak rise in v contributes a factor of $(1-x)$ since if s is at the end of a brick, there is a weight of $-x$ associated with the last cell of a brick. In this way, we can associate a weight with each weak rise or descent of v which will allow us to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w) wt(B,w).$$

In our case where $u = 2341$ and $k = 7$, the weights associated with the descents are given in table 1.1.

However, if $u = 2341$ and we want to compute $U_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$, the weights for any descent ji would be $-xt^2 \sum_{j<c<d} z_c z_d$ which is an infinite sum.

Going back to our example where $u = 2341$ and $k = 7$, it follows that for any $v \in [7]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w) wt(B,w) =$$

Table 1.1: The weights $wt_{2341,7}(ji)$

Descents	$wt_{2341,7}(ji)$
$7i$ $(i<7)$	0
$6i$ $(i<6)$	0
$5i$ $(i<5)$	$-xz_6z_7t^2$
$4i$ $(i<4)$	$-x(z_5z_6+z_5z_7+z_6z_7)t^2$
$3i$ $(i<3)$	$-x(z_4z_5+z_4z_6+z_4z_7+z_5z_6+z_5z_7+z_6z_7)t^2$
21	$-x(z_3(z_4+z_5+z_6+z_7)+z_4(z_5+z_6+z_7)+z_5(z_6+z_7)+z_6z_7)t^2$

$$- x\overline{z}^v t^{|v|}(1-x)^{\text{wrise}(v)} \prod_{s\in Des(v)} wt_{2341,7}(v_sv_{s+1}). \tag{1.30}$$

Here the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. It follows that

$$\begin{aligned} U_{2341}^{(7)}(x,\mathbf{z}_7,t) &= 1+\sum_{n\geq 1} U_{2341,n}^{(7)}(x,\mathbf{z}_7)t^n \\ &= 1+\sum_{v\in[7]^+} -x(1-x)^{\text{wrise}(v)}\overline{z}^v t^{|v|} \prod_{s\in Des(v)} wt_{2341,7}(v_sv_{s+1}). \end{aligned} \tag{1.31}$$

Hence we could compute $\mathcal{N}_{2341,n}^{(7)}(x,\mathbf{z}_7,t) = \dfrac{1}{U_{2341,n}^{(7)}(x,\mathbf{z}_7,t)}$ if we can compute the right-hand side of (1.31).

What we need to be able to compute the right-hand side of (1.31) is the generating function over all words $v\in\mathbb{P}^*$ where we not only keep track of the descents of P but also of type of descents of P.

By Theorem 1, we know that

$$\frac{1}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)} = 1+\sum_{w=w_1\ldots w_n\in\mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}}. \tag{1.32}$$

Hence

$$\begin{aligned} \sum_{w=w_1\ldots w_n\in\mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}} &= \left(\frac{1}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)}\right)-1 \\ &= \frac{\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)}. \end{aligned} \tag{1.33}$$

Next, if we replace t by $(1-x)t$ and x_{ij} by $\frac{wt_u(ij)}{y}$ and perform some algebraic simplifications we obtain the following theorem:

Theorem 8. *Suppose that* $u = u_1 \dots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 > u_j$. *Then*

$$\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1 - (1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P}, |S|=n} DXTZ_u(S)}{1 - \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P}, |S|=n} DXTZ_u(S)} \tag{1.34}$$

where

$$DXTZ_u(S) = \begin{cases} z_j & \text{if } S = \{j\}, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1}(wt_u(j_{i+1}j_i) + x - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{1.35}$$

where $k \geq 2$.

Chapter 1, in part, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 2

The proofs of Theorems 1, 2, 3, 4, 5, 6, and 7

In this section, we shall prove Theorems 1, 2, 3, 4, 5, 6, and 7.

We start with the proof of Theorem 1.

Proof of Theorem 1.

Recall that given a set $S \subseteq \mathbb{P}$, we let

$$DXZ(S) = \begin{cases} z_j & \text{if } S = \{j\}, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1}j_i} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2. \end{cases} \tag{2.1}$$

Define a ring homomorphism $\Gamma : \Lambda \to Q[\mathbf{x}, \mathbf{z}]$ by defining $\Gamma(e_0) = 1$ and, for $n \geq 1$,

$$\Gamma(e_n) = (-1)^{n-1} \sum_{S \subseteq \mathbb{P}, |S|=n} DXZ(S). \tag{2.2}$$

Then we claim that

$$\Gamma(h_n) = \sum_{w \in \mathbb{P}^n} \overline{z}^w \prod_{i<j} x_{ji}^{\mathbf{ji}(w)}. \tag{2.3}$$

That is,

$$\begin{aligned}\Gamma(h_n) &= \sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,(n)}\Gamma(e_\mu)\\ &= \sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n)}}\prod_{j=1}^{\ell(\mu)}(-1)^{b_j-1}\sum_{S_j\subseteq\mathbb{P},|S_j|=b_j}DXZ(S_j)\\ &= \sum_{\mu\vdash n}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n)}}\prod_{j=1}^{\ell(\mu)}\sum_{S_j\subseteq\mathbb{P},|S_j|=b_j}DXZ(S_j) \qquad (2.4)\end{aligned}$$

Next we want to give a combinatorial interpretation to (2.4). First we pick a brick tabloid $B=(b_1,\ldots,b_k)$ of length n. Then we interpret $\prod_{j=1}^{\ell(\mu)}\sum_{S_j\subseteq\mathbb{P},|S_j|=b_j}DXZ(S_j)$ as picking a sequence of subsets of $\mathbb{P}$, $(S_1,\ldots,S_{\ell(\mu)})$, such that S_j has size b_j and placing the elements of S_j in the cells of b_j in decreasing order for $j=1,\ldots,\ell(\mu)$. If $S_j=\{a_1>\cdots>a_{b_j}\}$, then we interpret the factor $DXZ(S_j)=z_{a_1}\cdots z_{a_{b_j}}\prod_{i=1}^{b_j-1}(x_{a_ia_{i+1}}-1)$ as the ways of labeling the cells of b_j that contain a_i where $i<b_j$ with either $z_{a_i}x_{a_i,a_{i+1}}$ or with $-z_{a_i}$ and labeling the last of cell b_j with $z_{a_{b_j}}$. We shall call all such objects created in this way *filled labeled brick tabloids* and let $\mathcal{H}_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus $\mathcal{H}_n$ consists of all triples (B,w,L) such that

1. $B=(b_1,\ldots,b_k)$ is a brick tabloid of length n,
2. $w=w_1\ldots w_n$ is a word in $\mathbb{P}^n$ such that w is strictly decreasing in each brick, and
3. L is a labeling of the cells of B such that $L(i)$ is equal to z_a if i is the last cell of some brick b_j which contains a and $L(i)=-z_a$ or $L(i)=x_{ab}z_a$ if i is not the last cell of a brick, cell i contains a and cell $i+1$ contains b.

We then define the weight of (B,w,L), $wt(B,w,L)$, to be the product of all the x_{ab} and z_a labels in L and the sign of (B,w,L), $sgn(B,w,L)$, to be the product of all the -1 factors in the labels in L. This process is illustrated in Figure 2.1 to construct an element (B,w,L) of H_{12} such that

$$wt(B,w,L)=z_1^3z_2^2z_3^2z_4^2z_5z_6^2x_{54}x_{41}x_{64}x_{32}x_{63}$$

and $sgn(B, w, L) = -1$.

Thus

$$\Gamma(h_n) = \sum_{(B,w,L)\in\mathcal{H}_n} sgn(B, w, L)wt(B, w, L). \tag{2.5}$$

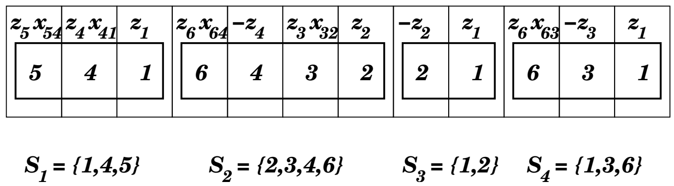

Figure 2.1: An element $(B, w, L) \in \mathcal{H}_{12}$.

Next we define a weight-preserving sign-reversing involution $I : \mathcal{H}_n \to \mathcal{H}_n$. To define $I(C)$, we scan the cells of $C = (B, w, L)$ from left to right looking for the leftmost cell, t, such that either (i) t is labeled with $-z_{w_t}$ or (ii) t is at the end of a brick, b_j, there is a brick b_{j+1} immediately following b_j, and $w_t > w_{t+1}$. In case (i), $I(C) = (B, w', L')$ where B' is the result of replacing the brick b in B containing t by two bricks b^* and b^{**}, where b^* contains all the cells of b weakly to the left of cell t and b^{**} contains all the cells of b strictly to the right of t, $w' = w$, and L' is the labeling that results from L by changing the label of cell t from $-z_{w_t}$ to z_{w_t}. In case (ii), $I(C) = (B', w', L')$ where B is the result of replacing the bricks b_j and b_{j+1} in B by a single brick b, $w' = w$, and L' is the labeling that results from L by changing the label of cell t from z_{w_t} to $-z_{w_t}$. If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if C is the element of $\mathcal{H}_{12}$ pictured in Figure 2.1, then $I(C)$ is pictured in Figure 2.2.

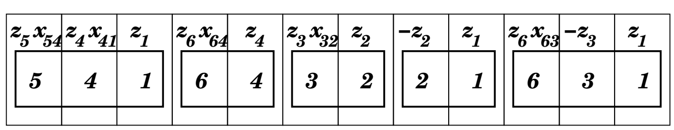

Figure 2.2: $I(C)$ for C in Figure 2.1.

It is easy to see that $I^2(C) = C$ for all $C \in \mathcal{H}_n$ and that if $I(C) \neq C$, then $sgn(C)w(C) = -sgn(I(C))w(I(C))$. Hence I is a weight-preserving and sign-reversing involution that shows

$$\Gamma(h_n) = \sum_{C\in\mathcal{H}_n, I(C)=C} sgn(C)w(C). \tag{2.6}$$

Thus, we must examine the fixed points, $C = (B, w, L)$, of I. First, there can be no $-z_a$ labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in B and t is the last cell of b_j, then it cannot be the case that $w_t > w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. Thus for each cell t such that $w_t > w_{t+1}$, it must be the case that cells t and $t+1$ lie in the same brick and, hence, cell t is labeled with $z_{w_t}x_{w_t w_{t+1}}$.

It follows that $sgn(C)w(C) = \overline{z}^w \prod_{1\leq i<j} x_{ji}^{\mathbf{ji}(w)}$. For example, Figure 2.3 shows a fixed point of I in H_{12}.

Vice versa, if $w \in \mathbb{P}^n$, then we can create a fixed point, $C = (B, w, L)$, by having the bricks of B end at cells t such that either $w_t \leq w_{t+1}$ or $t = n$, labeling each cell t such that $w_t > w_{t+1}$ with $z_{w_t}x_{w_t w_{t+1}}$ and labeling the remaining cells t with z_{w_t}. Thus we have shown that

$$\Gamma(h_n) = \sum_{w\in\mathbb{P}^n} \overline{z}^w \prod_{i<j} x_{ji}^{\mathbf{ji}(w)}.$$

as desired.

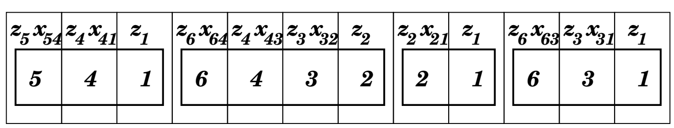

Figure 2.3: A fixed point of I.

Applying Γ to the identity $H(t) = \frac{1}{E(-t)}$, we get

$$\begin{aligned}
\sum_{n\geq 0}\Gamma(h_n)t^n &= 1 + \sum_{n\geq 1} t^n \sum_{w\in\mathbb{P}^n} \overline{z}^w \prod_{i<j} x_{ji}^{\mathbf{ji}(w)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n\Gamma(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-1)^n t^n (-1)^{n-1}\sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)} \\
&= \frac{1}{1-\sum_{n\geq 1}t^n\sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)}
\end{aligned}$$

which proves (1.17).

Proof of Theorem 2.

One can easily modify the proof of Theorem 1 to prove Theorem 2.

Recall that given a weakly decreasing word w from $\mathbb{P}^*$, we let

$$WDXZ(w) = \begin{cases} z_j & \text{if } w = j, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1}(x_{j_i j_{i+1}} - 1) & \text{if } w = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2. \end{cases} \tag{2.7}$$

Define a ring homomorphism $\Gamma_w : \Lambda \to Q[\mathbf{x}, \mathbf{z}]$ by defining $\Gamma(e_0) = 1$ and, for $n \geq 1$,

$$\Gamma_w(e_n) = (-1)^{n-1} \sum_{w \in WD\mathbb{P}^*, |w|=n} WDXZ(w). \tag{2.8}$$

Then we claim that

$$\Gamma_w(h_n) = \sum_{w \in \mathbb{P}^n} \overline{z}^w \prod_{i \leq j} x_{ji}^{\mathbf{ji}(w)}. \tag{2.9}$$

That is,

$$\begin{aligned} \Gamma_w(h_n) &= \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma_w(e_\mu) \\ &= \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j - 1} \sum_{w_j \subseteq WD\mathbb{P}^*, |w_j| = b_j} WDXZ(w_j) \\ &= \sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} \sum_{w_j \in WD\mathbb{P}^*, |w_j| = b_j} WDXZ(w_j) \end{aligned} \tag{2.10}$$

Next we want to give a combinatorial interpretation to (2.10). First we pick a brick tabloid $B = (b_1, \ldots, b_k)$ of length n. Then we interpret $\prod_{j=1}^{\ell(\mu)} \sum_{w_j \subseteq WD\mathbb{P}^*, |w_j| = b_j} WDXZ(w_j)$ as picking a sequence of weakly descreasing words in $WD\mathbb{P}^*$, $(w_1, \ldots, w_{\ell(\mu)})$, such that w_j has length b_j and placing the elements of w_j in the cells of b_j in $j = 1, \ldots, \ell(\mu)$. If $w_j = a_1 \geq \cdots \geq a_{b_j}$, then we interpret the factor $WDXZ(w_j) = z_{a_1} \cdots z_{a_{b_j}} \prod_{i=1}^{b_j - 1}(x_{a_i a_{i+1}} - 1)$ as the ways of labeling the cells of b_j that contain a_i where $i < b_j$ with either $z_{a_i}x_{a_ia_{i+1}}$ or with $-z_{a_i}$ and labeling the last of cell b_j with $z_{a_{b_j}}$. We shall call all such objects created in this way *filled labeled brick tabloids* and let $\mathcal{WDH}_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus $\mathcal{WDH}_n$ consists of all triples (B, w, L) such that

1. $B = (b_1, \ldots, b_k)$ is a brick tabloid of length n,
2. $w = w_1 \ldots w_n$ is a word in $\mathbb{P}^n$ such that w is weakly decreasing in each brick, and
3. L is a labeling of the cells of B such that $L(i)$ is equal to z_a if i is the last cell of some brick b_j which contains a and $L(i) = -z_a$ or $L(i) = x_{ab}z_a$ if i is not the last cell of a brick, cell i contains a and cell $i+1$ contains b.

We then define the weight of (B, w, L), $wt(B, w, L)$, to be the product of all the x_{ab} and z_a labels in L and the sign of (B, w, L), $sgn(B, w, L)$, to be the product of all the -1 factors in the labels in L. This process is illustrated in Figure 2.4 to construct an element (B, w, L) of H_{12} such that

$$wt(B, w, L) = z_1 z_2^3 z_3 z_4^3 z_5 z_6^3 x_{54} x_{44} x_{64} x_{32} x_{66}$$

and $sgn(B, w, L) = -1$.

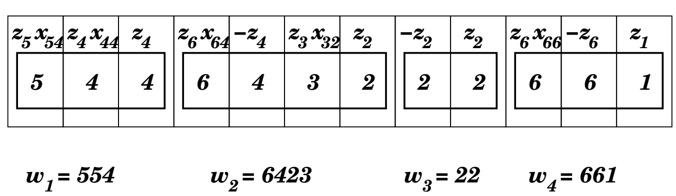

Figure 2.4: A element $(B, w, L) \in \mathcal{WDH}_{12}$.

At this point, the only difference in the proof is that we are dealing with filled brick tabloids which have weakly decreasing sequences in the bricks rather than strictly decreasing sequences in the bricks. This means that we can modify the involution I of Theorem 1 by splitting bricks at cells labeled with $-z_i$ or combining two bricks such that the elements in the two bricks form a weakly decreasing sequence. Then essentially the same proof will show that (2.9) holds.

Proof of Theorem 3.

Given any weakly increasing word $w = w_1 \dots w_n$, we let $S(w)$ denote the set of letters that appear in W. For example, if $w = 11123555$, then $S(w) = \{1,2,3,5\}$. We claim that for any non-empty set $S = \{j_1 < \cdots < j_k\}$ contained in $\mathbb{P}$,

$$\sum_{w \in \mathbb{P}^+, S(w)=S} t^{|w|} \overline{z}^w \prod_{i<j} x_{ij}^{\mathbf{ij}(w)} = t^{|S|} RXZ(S).$$

That is, if $S = \{j\}$, then w must be of the for j^k for some $k \geq 0$ so that in this case

$$\sum_{w \in \mathbb{P}^+, S(w)=S} t^{|w|} \overline{z}^w \prod_{i<j} x_{ij}^{\mathbf{ij}(w)} = \frac{z_j t}{1 - z_j t} = t^{|S|} \frac{z_j}{1 - z_j t} = t^{|S|} RXZ(S).$$

If $S(w) = \{j_1 < \cdots < j_k\}$ where $k \geq 2$, then w must be of the form $w = j_1^{a_1} j_2^{a_2} \dots j_k^{a_k}$ where $a_i \geq 1$ for $i = 1, \dots, k$. For any such word, it is easy to see that

$$\prod_{i<j} x_{ij}^{\mathbf{ij}(w)} = \prod_{i=1}^{k-1} x_{j_i j_{i+1}}.$$

Hence,

$$\begin{aligned}
\sum_{w \in \mathbb{P}^+, S(w)=S} t^{|w|} \overline{z}^w \prod_{i<j} x_{ij}^{\mathbf{ij}(w)} &= \left(\prod_{i=1}^{k} \frac{z_{j_i} t}{1 - z_{j_i} t} \right) \prod_{i=1}^{k-1} x_{j_i j_{i+1}} \\
&= t^{|S|} \left(\prod_{i=1}^{k} \frac{z_{j_i}}{1 - z_{j_i} t} \right) \prod_{i=1}^{k-1} x_{j_i j_{i+1}} \\
&= t^{|S|} RXZ(S).
\end{aligned}$$

Thus

$$\mathcal{R}(\mathbf{x}_\infty, \mathbf{z}_\infty, t) = 1 + \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} RXZ(S).$$

Proof of Theorem 4.

Consider a factor $\left(1+\frac{z_i t}{1-x_{ii}z_i t}\right)$. One can think of the choice of 1 in that factor as not choosing i to occur in the word where as the factor $\frac{z_i t}{1-x_{ii}z_i t}$ corresponds to choosing one of $i, ii, iii, iiii, \ldots$ in word. Equation (1.20) easily follows.

Proof of Theorem 5.

We wish to show the following

$$\mathcal{H}^{\mathbb{P}}(\mathbf{x},\mathbf{z},t) = \mathcal{D}^{\mathbb{P}}(\mathbf{x}, \frac{\mathbf{z}}{1+\mathbf{z}t}, t) \tag{2.11}$$

or equivalently

$$\mathcal{D}^{\mathbb{P}}(\mathbf{x},\mathbf{z},t) = \mathcal{H}^{\mathbb{P}}(\mathbf{x}, \frac{\mathbf{z}}{1-\mathbf{z}t}, t) \tag{2.12}$$

We will show (2.12), and then obtain (2.11) by back substitution. Notice that the function $\mathcal{H}^{\mathbb{P}}(\mathbf{x},\mathbf{z},t)$ sums over words with no levels, and it keeps track of the types of descents that occur. If we substitute $\frac{\mathbf{z}}{1-\mathbf{z}t}$ for $\mathbf{z}$ and use the geometric series expansion, it will replace each letter z_i by the expression $z_i + z_i^2 t + z_i^3 t^2 + \ldots$. The first term in this sum, z_i, corresponds to replacing z_i with z_i. The second term in this sum, $z_i^2 t$, corresponds to replacing z_i with two z_i's, so we need an extra t for this extra letter. The third term corresponds to replacing z_i with three z_i's, so we need t^2 for the two extra letters. In this way, we have not changed any of the types of rises or the descents that occur, but this substitution transforms our function so that we are summing over all words as desired.

To get (2.11) we back substitute. Let

$$\begin{aligned} u_i &= \frac{z_i}{1-z_i t} \\ u_i - u_i z_i t &= z_i \\ u_i &= z_i(1+u_i t) \\ \frac{u_i}{1+u_i t} &= z_i \end{aligned}$$

Replacing u_i with z_i on the left-hand side we get (2.11).

Proof of Theorem 6.

We wish to show the following

$$\mathcal{G}^{\mathbb{P}}(\mathbf{x}, \mathbf{z}, t) = \mathcal{E}^{\mathbb{P}}(\mathbf{x}, \frac{\mathbf{z}}{1 + \mathbf{z}t}, t) \tag{2.13}$$

or equivalently

$$\mathcal{G}^{\mathbb{P}}(\mathbf{x}, \mathbf{z}, t) = \mathcal{E}^{\mathbb{P}}(\mathbf{x}, \frac{\mathbf{z}}{1 - \mathbf{z}t}, t) \tag{2.14}$$

We will show (2.14), and then obtain (2.13) by back substitution as we did in the previous proof. Notice that the function $\mathcal{G}^{\mathbb{P}}(\mathbf{x}, \mathbf{z}, t)$ sums over words with no levels, and it keeps track of the types of rises that occur. If we substitute $\frac{\mathbf{z}}{1-\mathbf{z}t}$ for $\mathbf{z}$ and use the geometric series expansion, it will replace each letter z_i by the expression $z_i + z_i^2 t + z_i^3 t^2 + \ldots$. The first term in this sum, z_i, corresponds to replacing z_i with z_i. The second term in this sum, $z_i^2 t$, corresponds to replacing z_i with two z_i's, so we need an extra t for this extra letter. The third term corresponds to replacing z_i with three z_i's, so we need t^2 for the two extra letters. In this way, we have not changed any of the types of rises or the descents that occur, but this substitution transforms our function so that we are summing over all words as desired.

To get (2.13) we back substitute as we did above. Let

$$\begin{aligned} u_i &= \frac{z_i}{1 - z_i t} \\ u_i - u_i z_i t &= z_i \\ u_i &= z_i(1 + u_i t) \\ \frac{u_i}{1 + u_i t} &= z_i \end{aligned}$$

Replacing u_i with z_i on the left-hand side we get (2.13).

Proof of Theorem 7.

The proof of Theorem 7 is analogous to the proof of Theorem 1. We replace $DXZ(S)$ with $EXZ(S)$, and the cells in the bricks of our filled labelled brick tabloids will be strictly increasing instead of strictly decreasing. We can then modify the involution I of Theorem 1 by splitting at bricks labeled with $-z_i$ or combining two bricks such that the elements in the two bricks form a strictly increasing sequence. From there essentially the same proof completes the argument.

Chapter 2, in particular the proofs of Theorems 1, 2, 3, and 4, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 3

Descents: Results when $\mathrm{des}(u) = 1$

3.1 Introduction

In the introduction, we outlined how whenever u is a word such that $\mathrm{red}(u) = u$ and $\mathrm{des}(u) \leq 1$, I_u is an involution, i.e. I_u^2 is the identity. We begin this section by proving this claim and carefully examining the fixed points of I_u. Then we will show how we can define a similar involution J_u in the case of exact u-matches and examine its fixed points.

We begin by proving I_u is an involution. First we consider the case where $\mathrm{des}(u) = 1$. Now suppose that we are in case (i) where we split a brick b_j at cell c which is labeled with a x. In that case, we let a be the number in cell c and a' be the number in cell $c+1$ which must also be in brick b_j. It must be the case that there is no cell labeled x before cell c since otherwise we would not use cell c to define the involution. However, we have to consider the possibility that when we split b_j into b_j' and b_j'', we might then be able to combine the brick b_{j-1} with b_j' because the number in that last cell of b_{j-1} is greater than the number in the first cell of b_j' and there is no u-match in the cells of b_{j-1} and b_j'. Since we always take an action on the left most cell possible when defining $I_u(O)$, we know that we cannot combine b_{j-1} and b_j so that there must be a u-match in the cells of b_{j-1} and b_j. Clearly, that u-match must have involved the number a' and the number in cell d which is the last cell in brick b_{j-1}. But that is impossible because then there would be two descents among the numbers between cell d and cell $c+1$

which would violate our assumption that u has only one descent. Thus whenever we apply case (i) to define $I_u(O)$, the first action that we can take is to combine bricks b'_j and b''_j so that $I_u^2(O) = O$.

If we are in case (ii), then again we can assume that there are no cells labeled x that occur before cell c. When we combine bricks b_i and b_{i+1}, then we will label cell c with a x. It is clear that combining the cells of b_i and b_{i+1} cannot help us combine the resulting brick b with an earlier brick since it will be harder to have no u-matches with the larger brick b. Thus the first place cell c where we can apply the involution to $I_u(O)$ will again be cell c which is now labeled with a x so that $I_u^2(O) = O$ if we are in case (ii).

The case where $\text{des}(u) = 0$ is even easier. Suppose that a is number in the the last cell of b_j and a' is the number in the first cell of b_{j+1} and $a > a'$. Then there can be no u-match of w that is contained in the cells of b_j and b_{j+1} because by our definitions there is no u-match in the cells of b_j and there is no u-match in the cells of b_{j+1} so that the only possible u-match in the cells of b_j and b_{j+1} would have to involve a and a' which is impossible if $\text{des}(u) = 0$. It easily follows that we will apply the involution to the first possible cell c which is labeled with either x or $-x$ and what ever action we take at cell c to create $I_u(O)$, we will come back to cell c to undo that action to define $I^2(O)$.

Our definitions ensure that if $I_u(O) \neq O$, then $sgn(O)wt(O) = -sgn(I_u(O))wt(I_u(O))$. Hence, if we let $\mathcal{IO}_{u,n}^{(\mathbb{P})}$ denote the set of all $O = (B, w) \in \mathcal{O}_{u,n}^{(\mathbb{P})}$ such that $I_u(O) = O$, then

$$\Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}} sgn(O)wt(O) = \sum_{O \in \mathcal{IO}_{u,n}^{(\mathbb{P})}} sgn(O)wt(O). \tag{3.1}$$

Next we provide a rigorous examination of the fixed points of I_u that we omitted in the introduction. So assume that (B, w) is a fixed point of I_u. There are two cases to consider.

Case 1. $\text{des}(u) = 0$.

Suppose that $(B, w) \in \mathcal{IO}_{u,n}$ where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$.

There can be no cell c which is labeled with x in (B,w) since we could use such a cell to define I_u which would violate our assumption that (B,w) is a fixed point of I_u. Similarly there can be no cell c which is at the end of a brick b_j such that $w_c > w_{c+1}$ since again we could use such a cell to define $I_u(O)$. This means that w must be weakly increasing within any brick and if c is a cell at the end of brick b_j which is followed by another brick b_{j+1}, then $w_c \leq w_{c+1}$. Thus (B,w) is a fixed point if and only if w is a weakly increasing word such that w has no u-match that lies entirely within one of the brick of B. If B has k bricks, then then weight of (B,w) is just $(-x)^k \overline{z}^w$. We let $\mathcal{WIO}_{u,n} = \{(B,w) \in \mathcal{IO}_{u,n}^{(\mathbb{P})} : w_1 \leq w_2 \leq \cdots \leq w_n\}$ denote the set of elements of $\mathcal{IO}_{u,n}^{(\mathbb{P})}$ where w is weakly increasing. Then we have the following lemma. Let $\mathbb{Q}(x,\mathbf{z}_\infty)$ be the set of rational functions in the variables x and $\mathbf{z}_\infty$ over the rationals $\mathbb{Q}$.

Lemma 1. *Suppose that u is a word in $\mathbb{P}^+$ such that* $\mathrm{red}(u) = u$ *and* $\mathrm{des}(u) = 0$. *Let $\Theta_u : \Lambda \to \mathbb{Q}(x,\mathbf{z}_\infty)$ be the ring homomorphism defined by setting $\Theta_u(e_0) = 1$ and $\Theta_u(e_n) = (-1)^n N_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty) = \Theta_u(h_n) = \sum_{((b_1,\ldots,b_k),w) \in \mathcal{WIO}_{u,n}} (-x)^k \overline{z}^w. \tag{3.2}$$

Case 2. $\mathrm{des}(u) = 1$.

First it is easy to see that there can be no cells which are labeled with x so that numbers in each brick of O must be weakly increasing. Second we cannot combine two consecutive bricks b_i and b_{i+1} in O which means that either there is an weak increase between the bricks b_i and b_{i+1} or there is a decrease between the bricks b_i and b_{i+1}, but there is a u-match in the cells of the bricks b_i and b_{i+1}. Thus we have proved the following.

Lemma 2. *Suppose that $u \in \mathbb{P}^+$,* $\mathrm{red}(u) = u$, *and* $\mathrm{des}(u) = 1$. *Let $\Theta_u : \Lambda \to \mathbb{Q}(x,\mathbf{z}_\infty)$ be the ring homomorphism defined by setting $\Theta_u(e_0) = 1$ and $\Theta_u(e_n) = (-1)^n N_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty) = \Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}, I_u(O) = O} sgn(O) wt(O) \tag{3.3}$$

where $\mathcal{O}_{u,n}^{(\mathbb{P})}$ *is the set of objects and* I_u *is the involution defined above. Moreover* $O = (B, w)$*, where* $B = (b_1, \ldots, b_k)$ *and* $w = w_1 \ldots w_n$*, is a fixed point of* I_u *if and only if it has the following two properties:*

1. *there are no cells labeled with* x *in* O*, i.e., the elements of* w *in each brick of* O *are weakly increasing and*

2. *if* b_i *and* b_{i+1} *are two consecutive bricks in* O*, then either (a) there is a weak increase between* b_i *and* b_{i+1}*, i.e.,* $w_{\sum_{j=1}^{i}|b_j|} \leq w_{1+\sum_{j=1}^{i}|b_j|}$*, or (b) there is a decrease between* b_i *and* b_{i+1}*, i.e.,* $w_{\sum_{j=1}^{i}|b_j|} > w_{1+\sum_{j=1}^{i}|b_j|}$*, but there is a* u*-match contained in the elements of the cells of* b_i *and* b_{i+1} *which must necessarily involve* $w_{\sum_{j=1}^{i}|b_j|}$ *and* $w_{1+\sum_{j=1}^{i}|b_j|}$*.*

Clearly, if we restrict to the alphabet $[k]$ instead of $\mathbb{P}$, we will get the same two lemmas except that the words all have to be in $[k]^*$ rather than in $\mathbb{P}^*$.

Next we want to consider what happens when we replace u-matches by exact u-matches. We can follow the same steps to interpret $EU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$. That is,

$$EU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 + \sum_{n\geq 1} EN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)t^n}. \tag{3.4}$$

Thus if we let $\Gamma_u(e_n) = (-1)^n EN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$ and $\Gamma_u(e_0) = 1$, we see that

$$\begin{aligned}
\Gamma_u(H(t)) &= 1 + \sum_{n\geq 1} \Gamma_u(h_n) \\
&= \Gamma_u\left(\frac{1}{E(-t)}\right) = \frac{1}{1 + \sum_{n\geq 1}(-1)^n \Gamma_u(e_n)} \\
&= \frac{1}{1 + \sum_{n\geq 1} EN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)t^n} = EU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t).
\end{aligned}$$

Thus it follows that $\Gamma_u(h_n) = EU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$.

By (1.24), we have that

$$\begin{aligned}\Gamma_u(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n}\ \Gamma_u(e_\lambda) \\ &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} EN^{(\mathbb{P})}_{u,b_i}(x, \mathbf{z}_\infty) \\ &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} EN^{(\mathbb{P})}_{u,b_i}(x, \mathbf{z}_\infty) \qquad (3.5)\end{aligned}$$

Again we can give a combinatorial interpretation to the right-hand side of (3.5). Fix a partition λ of n and a λ-brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$. We will interpret $\prod_{i=1}^{\ell(\lambda)} EN^{(\mathbb{P})}_{u,b_i}(x, \mathbf{z}_\infty)$ as the number of ways of picking words $(w^{(1)}, \ldots, w^{(\ell(\lambda))})$ such that for each i, $w^{(i)} \in \mathbb{P}^{b_i}$ is a word such that $eu\text{mch}(w) = 0$ and assigning a weight to this $\ell(\lambda)$-tuple to be $\prod_{i=1}^{\ell(\lambda)} x^{\text{des}(w^{(i)})+1} \overline{z}^{w^{(i)}}$.

Following the same steps that we did to interpret $\Theta_u(h_n)$, we let $\mathcal{EO}^{(\mathbb{P})}_{u,n}$ denote the set of all filled-labeled-brick tabloids constructed in this way. That is, $\mathcal{EO}^{(\mathbb{P})}_{u,n}$ consists of all pairs $O = (B, w)$ where

1. $B = (b_1, \ldots, b_{\ell(\lambda)})$ is brick tabloid of shape (n),
2. $w = w_1 \ldots w_n \in \mathbb{P}^n$ such that there is no exact u-match of σ which is entirely contained in a single brick of B, and
3. if there is a cell c such that a brick b_i contains both cells c and $c+1$ and $w_c > w_{c+1}$, then cell c is labeled with a x and the last cell of any brick is labeled with $-x$.

The sign of O, sgn(O), is $(-1)^{\ell(\lambda)}$ and the weight of O, $wt(O)$, is $x^{\ell(\lambda)+\text{intdes}(\sigma)} \overline{z}^{w}$. Then as before we can conclude

$$\Gamma_u(h_n) = \sum_{O \in \mathcal{EO}^{(\mathbb{P})}_{u,n}} sgn(O) wt(O). \qquad (3.6)$$

At this point, we can define an involution J_u exactly as we did for I_u except replace u-match by exact u-matches in the definitions. This will allow us to prove the following two lemmas.

Lemma 3. *Suppose that u is a word in $\mathbb{P}^+$ such that* $\mathrm{des}(u) = 0$. *Let $\Gamma_u : \Lambda \to \mathbb{Q}(x)$ be the ring homomorphism defined by setting $\Gamma_u(e_0) = 1$ and $\Gamma_u(e_n) = (-1)^n EN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$EU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \theta_u(h_n) = \sum_{((b_1,\ldots,b_k),w)\in \mathcal{WIEO}_{u,n}} (-x)^k \overline{z}^w \tag{3.7}$$

where $\mathcal{WIEO}_{u,n}$ is the set of all $(B, w) \in \mathcal{EO}_{u,n}$ such that $J_u(B, w) = (B, w)$ and w is weakly increasing.

Lemma 4. *Suppose that $u \in \mathbb{P}^+$ and* $\mathrm{des}(u) = 1$. *Let $\Gamma_u : \Lambda \to \mathbb{Q}(y)$ be the ring homomorphism defined by setting $\Gamma_u(e_0) = 1$ and $\Gamma_u(e_n) = (-1)^n EN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$EU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \Gamma_u(h_n) = \sum_{O\in \mathcal{EO}_{u,n}^{(\mathbb{P})}, J_u(O)=O} sgn(O)wt(O) \tag{3.8}$$

where $\mathcal{EO}_{u,n}^{(\mathbb{P})}$ is the set of objects and J_u is the involution defined above. Moreover $O = (B, w)$, where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$, is a fixed point of J_u if and only if it has the following two properties:

1. *there are no cells labeled with x in O, i.e., the elements of w in each brick of O are weakly increasing and*

2. *if b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is a weak increase between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^{i}|b_j|} \leq w_{1+\sum_{j=1}^{i}|b_j|}$, or (b) there is a decrease between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^{i}|b_j|} > w_{1+\sum_{j=1}^{i}|b_j|}$, but there is an exact u-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $w_{\sum_{j=1}^{i}|b_j|}$ and $w_{1+\sum_{j=1}^{i}|b_j|}$.*

3.2 The case $u = u_1 \ldots u_j$, $\mathrm{des}(u) = 1$, and $u_1 > u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$, $\mathcal{EN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\mathrm{des}(u) = 1$, and $u_1 > u_j$.

Now suppose that $u = u_1 \dots u_j$, $\text{red}(u) = u$, $u_1 > u_j$, and $\text{des}(u) = 1$. Let $1 \leq s < j$ be the position such that $u_s > u_{s+1}$ so that $u_1 \leq \cdots \leq u_s > u_{s+1} \leq \cdots \leq u_j$. Then $St^{(\mathbb{P})}(u) \subseteq \{s+1, \dots, j\}$ since if we try to start a match at one of the positions $2, \dots, s$, the descent in the second match would not be in the right place. It follows that u automatically has the $\mathbb{P}$-weakly decreasing overlapping property and the $[k]$-weakly decreasing overlapping property for any $k \geq 2$.

We start by considering a special class of words $u = u_1 \dots u_j$ which have the $\mathbb{P}$-minimal overlapping property ($[k]$-minimal overlapping property). This means that any two consecutive u-matches can share at most one letter. For example $u = 2341$ has the $\mathbb{P}$-minimal overlapping property while $u = 3412$ does not have the $\mathbb{P}$-minimal overlapping property since in the word $w = 563412$, the u-matches 5634 and 3412 share two letters.

Thus assume that $u = u_1 \dots u_j$, $\text{red}(u) = u$, $\text{des}(u) = 1$, $u_1 > u_j$, and u has the $\mathbb{P}$-minimal overlapping property. First we reintroduce the *collapse map* that we mentioned in the introduction which maps fixed points of I_u or J_u to a certain subset of words in $\mathbb{P}^*$. This is best explained through an example. Suppose that $u = 2341$ and we want to compute $U^{(7)}_{2341}(x, \mathbf{z}_7, t)$. By (3.3), we know that

$$U^{(7)}_{u,n}(x, \mathbf{z}_7) = \sum_{O \in \mathcal{O}^{(7)}_{u,n}, I_u(O) = O} sgn(O) wt(O). \tag{3.9}$$

Now suppose that we are given a fixed point (B, w) of I_u where $B = (b_1, \dots, b_k)$ and $w = w_1 \dots w_n$ such as the one pictured in Figure 3.1. We know that to be a fixed point of I_u, w must be weakly increasing within bricks of B and that for any $i < k$, if c is last cell in brick b_i and $w_c > w_{c+1}$, then there must be a u-match in w which is contained in the cells of b_i and b_{i+1}. In our particular example, since $u = 2341$ has a single descent, this match must involve the last three cells of b_i and the first cell of b_{i+1}. In Figure 3.1, we have indicated the two such 2341-matches in our example by placing stars below the cells in the 2341-matches. In this case the collapse map just maps (B, w) to the word $v = C(B, w, u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 3.1 where again we have starred the elements in $C(B, w, u)$ that remain from the original 2341-matches in

w. What makes the case where u has the minimal overlapping property easier is that, since any two consecutive u-matches can share at most letter, there is no possibility that an end point of u-match in w occurs in the middle of another u-match in w so that the letters that we remove from w for any pair of u-matches are disjoint from each other.

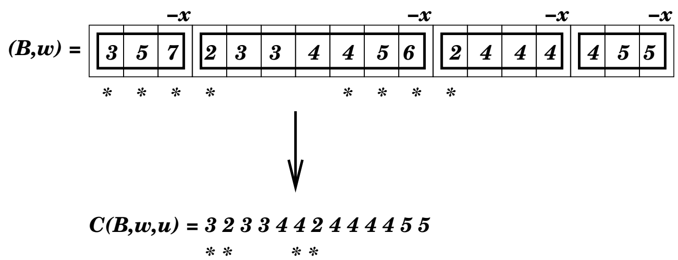

Figure 3.1: A fixed point of I_{2341}.

The next question that we want to consider is how can we construct all the fixed points of (B, w) of I_u such that $C(B, w, v)$ is equal to a given word $v = v_1 \ldots v_n$. First it easy to see that the only descents that appear in a word $C(B, w, u)$ must come from 2341-matches that straddled two bricks in B. Thus if $v_s > v_{s+1}$, then v_s must have played the role of 2 in the original 2341-match and v_{s+1} must have played the role of 1 in the original 2341-match. Such a requirement rules out certain words from being in the range of the collapse map C. For example, suppose that the underlying alphabet is [7]. Then if $v_s = 6$ and $v_{s+1} = i$ where $i < 6$, then v could not have come from the collapse of 2341-match because we can not add two letters which could play the role of 3 and 4 in the 2341-match. If we consider the first descent 32 in the $C(B, w, u)$ of Figure 3.1, then we see there are many ways that we could add the two middle letters. That is, the original 2341-match could have been any $3cd2$ where $c < d$ and $c, d \in \{4, 5, 6, 7\}$. It follows that the extra weight from these possibilities that is not included in $\overline{z}^{C(B,w,u)} t^{|C(B,w,u)|}$ in this case would be $-xt^2 \sum_{4 \leq c < d \leq 7} z_c z_d$. Here the $-x$ comes from the fact that we know that the original match straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. On the other hand, if $v_s \leq v_{s+1}$, then we have only two choices. That is, either cell s was the end of a

Table 3.1: The weights $wt_{2341,7}(ji)$

Descents	$wt_{2341,7}(ji)$
$7i$ $(i<7)$	0
$6i$ $(i<6)$	0
$5i$ $(i<5)$	$-xz_6z_7t^2$
$4i$ $(i<4)$	$-x(z_5z_6+z_5z_7+z_6z_7)t^2$
$3i$ $(i<3)$	$-x(z_4z_5+z_4z_6+z_4z_7+z_5z_6+z_5z_7+z_6z_7)t^2$
21	$-x(z_3(z_4+z_5+z_6+z_7)+z_4(z_5+z_6+z_7)+z_5(z_6+z_7)+z_6z_7)t^2$

brick or cell s was an internal cell of a brick. This implies that each weak rise in v contributes a factor of $(1-x)$ since if s is at the end of a brick, there is a weight of $-x$ associated with the last cell of a brick. In this way, we can associate a weight with each weak rise or descent of v which will allow us to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

In our case where $u=2341$ and $k=7$, the weights associated with the descents are given in the table 3.1.

However, if $u=2341$ and we want to compute $U^{(\mathbb{P})}_{u,n}(x,\mathbf{z}_\infty)$, the weights for any descent ji would be $-xt^2\sum_{j<c<d} z_cz_d$ which is an infinite sum.

Going back to our example where $u=2341$ and $k=7$, it follows that for any $v\in[7]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w) = \\ -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} wt_{2341,7}(v_sv_{s+1}). \tag{3.10}$$

Here the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. It follows that

$$U^{(7)}_{2341}(x,\mathbf{z}_7,t) = 1+\sum_{n\geq 1} U^{(7)}_{2341,n}(x,\mathbf{z}_7)t^n \\ = 1+\sum_{v\in[7]^+} -x(1-x)^{\mathrm{wrise}(v)}\overline{z}^v t^{|v|}\prod_{s\in Des(v)} wt_{2341,7}(v_sv_{s+1}). \tag{3.11}$$

Table 3.2: The weights $ewt_{2341}(ji)$

Descents	weight $ewt_{2341,\mathbb{P}}(ji)$
ji where either $j \neq 2$ or $i \neq 1$	0
21	$-xz_3z_4t^2$

Hence we could compute $\mathcal{N}^{(7)}_{2341,n}(x,\mathbf{z}_7,t) = \dfrac{1}{U^{(7)}_{2341,n}(x,\mathbf{z}_7,t)}$ if we can compute the right-hand side of (3.11).

The case of exact matches is even simpler. In that case, we want to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } J_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

Going back to our example of $u = 2341$ over the alphabet [7], we see the only descents that appear in a word $v = C(B,w,u)$ must come from exact 2341-matches that straddled two bricks in B. Thus if $v_s > v_{s+1}$, then it must be the case that $v_s = 2$, $v_{s+1} = 1$ and we must have eliminated a 3 and 4 from w. Thus if we want to compute $EU^{(\mathbb{P})}_{2341,n}(x,\mathbf{z}_\infty)$ or $EU^{(k)}_{2341,n}(x,\mathbf{z}_k)$ for $k \geq 4$, we must compute the corresponding weights which are listed in table 3.2.

It follows that for any $v \in \mathbb{P}^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } J_{2341} \\ C(B,w,2341)=v}} sgn(B,w)wt(B,w) = - x\overline{z}^v t^{|v|}(1-x)^{\text{wrise}(v)} \prod_{s \in Des(v)} ewt_{2341,\mathbb{P}}(v_s v_{s+1}). \tag{3.12}$$

and

$$\begin{aligned} EU^{(\mathbb{P})}_{2341,n}(x,\mathbf{z}_\infty,t) &= 1 + \sum_{n\geq 1} EU^{(\mathbb{P})}_{2341,n}(x,\mathbf{z}_\infty)t^n \\ &= 1 + \sum_{v \in \mathbb{P}^+} -x\overline{z}^v t^{|v|}(1-x)^{\text{wrise}(v)} \prod_{s \in Des(v)} ewt_{2341,\mathbb{P}}(v_s v_{s+1}). \end{aligned} \tag{3.13}$$

In our case, $\prod_{s \in Des(v)} ewt_{2341,\mathbb{P}}(v_s v_{s+1}) = 0$ unless the only descents in v are of the from 21. It follows that the only nonempty words v that can contribute to

(3.13) are words v of the form w or of the form $1^{a_1}2^{b_1}211^{a_2}2^{b_2}21\ldots 1^{a_r}2^{b_r}21w$ for some $r \geq 1$ where w is a weakly increasing word. Let

$$W(x,\mathbf{z}_\infty,t) := \prod_{i=1}^{\infty} \frac{1}{(1-(1-x)z_it)}.$$

The generating function of $-x\overline{z}^v t^{|v|}(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} ewt_{2341,7}(v_sv_{s+1})$ over all nonempty weakly increasing words is just

$$\frac{-x}{(1-x)}\left(-1+W(x,\mathbf{z}_\infty,t)\right). \tag{3.14}$$

The generating function of $-x\overline{z}^v t^{|v|}(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} ewt_{2341,7}(v_sv_{s+1})$ over all words v of the form $1^{a_1}2^{b_1}2$ is

$$\frac{z_2t}{(1-(1-x)z_1t)(1-(1-x)z_2t)}.$$

The generating function of $-x\overline{z}^v t^{|v|}(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} ewt_{2341,7}(v_sv_{s+1})$ over all words v of the form $1^{a_1}2^{b_1}211^{a_2}2^{b_2}21\ldots 1^{a_r}2^{b_r}21w$ where w is weakly increasing is

$$-xW(x,\mathbf{z}_\infty,t)\left(\frac{z_2t}{(1-(1-x)z_1t)(1-(1-x)z_2t)}\right)^r(-xz_1z_3z_4t^3)^r(1-x)^{r-1}.$$

Here the term $(-xz_1z_3z_4t^3)^r$ comes from the weights $ewt_{2341,\mathbb{P}}(21)$ that arise from the descents 21 and $(1-x)^{r-1}$ comes from the weights of the rises coming from the first $r-1$ 1s which are the second elements of the descents 21. It follows that the generating function of $-x\overline{z}^v t^{|v|}(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} ewt_{2341,7}(v_sv_{s+1})$ over all v such that v is of the form $1^{a_1}2^{b_1}211^{a_2}2^{b_2}21\ldots 1^{a_r}2^{b_r}21w$ for some $r \geq 1$ where w is a weakly increasing word is equal to

$$\begin{aligned}&-xW(x,\mathbf{z}_\infty,t)\sum_{r\geq 1}\left(\frac{-xz_1z_2z_3z_4t^4}{(1-(1-x)z_1t)(1-(1-x)z_2t)}\right)^r(1-x)^{r-1} = \\ &\frac{-xW(x,\mathbf{z}_\infty,t)}{(1-x)}\left(-1+\frac{1}{1-\frac{-xz_1z_2z_3z_4t^4(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}}\right)\end{aligned} \tag{3.15}$$

Putting (3.14) and (3.15) together we see that

$$\begin{aligned}EU_{2341,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) &= 1+\frac{-x}{(1-x)}\left(-1+W(x,\mathbf{z}_\infty,t)\right)+\\ &\quad\frac{-xW(x,\mathbf{z}_\infty,t)}{(1-x)}\left(-1+\frac{1}{1-\frac{-xz_1z_2z_3z_4t^4(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}}\right)\\ &= 1+\frac{x}{(1-x)}-\frac{xW(x,\mathbf{z}_\infty,t)}{(1-x)}\frac{1}{1+\frac{xz_1z_2z_3z_4t^4(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}}.\end{aligned} \tag{3.16}$$

Thus

$$\mathcal{EN}^{(\mathbb{P})}_{2341}(x,\mathbf{z}_\infty,t) = \frac{1}{1+\frac{x}{(1-x)} - \frac{xW(x,\mathbf{z}_\infty,t)}{(1-x)}\frac{1}{1+\frac{xz_1z_2z_3z_4t^4(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}}}. \tag{3.17}$$

It should be clear from our arguments that the only role that the 2 and 3 played in the final form $\mathcal{EN}^{(\mathbb{P})}_{2341}(x,\mathbf{z}_\infty,t)$ was to contribute a factor of $z_3z_4t^2$ to the expression $\frac{xz_1z_2z_3z_4t^4(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}$ on the right hand side of (3.17). Thus our arguments show that if $u = 2\alpha 1$ where α is non-empty weakly increasing word in $\{2,3,\ldots\}^*$, then we have the following theorem.

Theorem 9. *Let $u = 2\alpha 1$ where α is non-empty weakly increasing word in $\{2,3,\ldots\}^*$. Then*

$$\mathcal{EN}^{(\mathbb{P})}_{2\alpha 1}(x,\mathbf{z}_\infty,t) = \frac{1}{1+\frac{x}{(1-x)} - \frac{xW(x,\mathbf{z}_\infty,t)}{(1-x)}\frac{1}{1+\frac{xz_1z_2\overline{z}^{\alpha}t^{2+|\alpha|}(1-x)}{(1-(1-x)z_1t)(1-(1-x)z_2t)}}}. \tag{3.18}$$

Other examples where the weights $wt_{u,\mathbb{P}}(ij)$ are easy to compute are words of the form $u = 2^r1$ or $u = 21^r$ where $r \geq 2$. It is easy to see that both 2^r1 and 21^r have the minimal overlapping property. In this case, the only u-matches are of the form b^ra where $b > a \geq 1$ if $u = 2^r1$ or ba^r where $b > a \geq 1$ if $u = 21^r$. For example, suppose that $u = 2^31$. Then in Figure 3.2, we have pictured a fixed point of I_u where we have indicated the two 2^31-matches in our example by placing stars below the cells in the 2^31-matches. In the case the collapse map just maps (B,w) to the word $v = C(B,w,u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 3.2 where again we have starred the elements in $C(B,w,u)$ that remain from the original 2^31-matches in w.

In this case, if we have a descent ji, then $wt_{2^31,k}(ji) = wt_{2221,\mathbb{P}}(ji) = -xz_j^2t^2$ since we will always add back two js for each descent of of the form ji. Thus if $u = 2^31$ it follows that for any $v \in \mathbb{P}^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{2221} \\ C(B,w,2221)=v}} sgn(B,w)wt(B,w) = -x(1-x)^{\mathrm{wrise}(v)} \prod_{s \in Des(v)} -xz_{v_s}^2t^2. \tag{3.19}$$

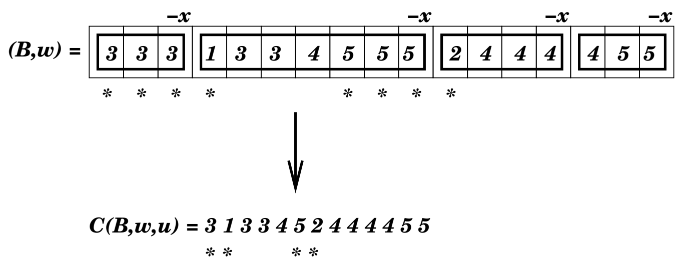

Figure 3.2: A fixed point of I_{2221}.

and

$$\begin{aligned} U^{(\mathbb{P})}_{2221,n}(x,\mathbf{z}_\infty,t) &= 1+\sum_{n\geq 1} U^{(\mathbb{P})}_{2221,n}(x,\mathbf{z}_\infty)t^n \\ &= 1+\sum_{v\in\mathbb{P}^+} -x(1-x)^{\mathrm{wrise}(v)} \prod_{s\in Des(v)} -xz_{v_s}^2 t^2. \end{aligned} \tag{3.20}$$

Hence we could compute $\mathcal{N}^{(\mathbb{P})}_{2221,n}(x,\mathbf{z}_\infty,t) = \dfrac{1}{U^{(\mathbb{P})}_{2221,n}(x,\mathbf{z}_\infty,t)}$ if we can compute the right-hand side of (3.20).

When u does not have the minimal overlapping property, we can obtain similar results but the collapse maps and the weights $wt_u(ji)$ are more complicated. Again this is best explained through an example. Suppose that $u = 3412$ and $k = 8$.

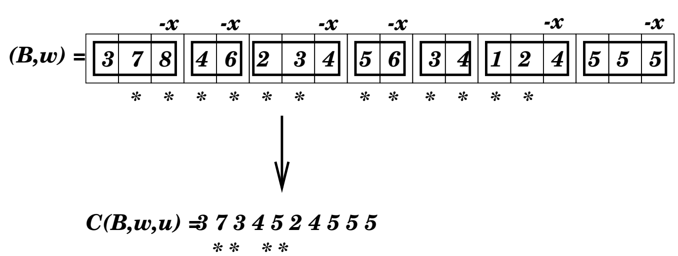

Figure 3.3: A fixed point of I_{3412}.

When u does not have the minimal overlapping property, then we can have a situation such as the one pictured in Figure 3.3. If we look at the descents between bricks 1 and 2 which correspond to the u-match 7846, we see that we would like to eliminate the 8 and 4. However, this u-match overlaps the u-match associated with the descent between bricks 2 and 3 which is 4623. Thus we would also like to eliminate the 6 and 2. We will say that two such matches are *linked* if one of the end points of first match is one of middle elements of the second match. Depending on the pattern we could have a series of u-matches in a fixed point of (B, w) which are linked. The fact that we are assuming that $u = u_1 \dots u_j$ where $u_1 > u_j$ and $\mathrm{des}(u) = 1$ implies that u has the $\mathbb{P}$-weakly decreasing overlapping property or $[k]$-weakly decreasing overlapping property. It is then easy to see that if $w = w_1 \dots w_n$ is a word such that there is a u-match starting position 1 and a u-match ending at position n and any two consecutive u-matches in w are linked, then $w_1 > w_n$. In such a situation, the collapse map will eliminate all the symbols except for the first element of the first match and last element of the last match in a maximal sequence of linked u-matches which will result in a descent. This is illustrated in Figure 3.3 where we have two maximal blocks of linked 3412-matches. Thus in the linked 3412-matches in cells 2 through 7, we keep only the 7 and the 3 and in the linked matches in cells 9 through 14, we keep only the 5 and the 2. Because we are assuming that $u_1 > u_j$, we know that maximal blocks of linked u-matches must be finite since the end point of such matches must strictly decrease. When we see a descent ji in a word $C(B, w, u)$, the weight associated with such a decent is now more complicated. For example, in our case where $u = 3412$ and $k = 8$, a decent of the form 72 can correspond to a single 3412-match which would have to be of the form 7812, it could correspond to a maximum block with 2 linked 3412-matches in which case it must be of the form $78cd12$ where $3 \leq c < d \leq 6$, or it could correspond to a maximum block with 3 linked 3412-matches in which case it must be 78563412. Thus

$$wt_{3412}(72) = -xz_1z_8t^2 + x^2t^4z_1z_8 \left(\sum_{3 \leq c < d \leq 6} z_c z_d \right) - x^3t^6 z_1z_3z_4z_5z_6z_8$$

On the other hand a descent of the form ji where $j - i \leq 2$ can only correspond to a single 3412-match so that $wt_{3412}(ji) = -xt^2(\sum_{j<s\leq 8} z_s)(\sum_{1 \leq t < i} z_t)$.

We give the weights associated with the descents for $u = 3412$ and $k = 8$ in the table 3.3.

Table 3.3: The weights $wt_{3412,8}(ji)$

Descents	$wt_{3412,8}(ji)$
$8i$ $(i<8)$	0
$j1$ $(j>2)$	0
ji $(j>i$ & $j-i\leq 2$	$-xt^2(\sum_{j<s\leq 8} z_s)(\sum_{1\leq t<i} z_t)$
72	$-xz_1z_8t^2+x^2t^4z_1z_8(\sum_{3\leq c<d\leq 6} z_cz_d)-x^3t^6z_1z_3z_4z_5z_5z_6z_8$
62	$-xt^2(z_7+z_8)z_1+x^2t^4(z_7+z_8)z_1(\sum_{3\leq c<d\leq 5} z_cz_d)$
52	$-xt^2(z_6+z_7+z_8)z_1+x^2t^4(z_6+z_7+z_8)z_1z_3z_4$
73	$-xt^2z_8(z_1+z_2)+x^2t^4z_8(z_1+z_2)+$ $x^2t^4z_8(z_1+z_2)(\sum_{4\leq c<d\leq 6} z_cz_d)$
63	$-xt^2(z_7+z_8)(z_1+z_2)+x^2t^4(z_7+z_8)(z_1+z_2)z_4z_5$
74	$-xt^2z_8(z_1+z_2+z_3)+x^2t^4z_8(z_1+z_2+z_3)z_5z_6$

It follows that for any $v\in[8]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{3412} \\ C(B,w,3412)=v}} sgn(B,w)wt_{3412}(B,w) =$$

$$-\,x\overline{z}^v t^{|v|}(1-x)^{\text{wrise}(v)} \prod_{s\in Des(v)} wt_{3412,8}(v_sv_{s+1}). \quad (3.21)$$

and

$$\begin{aligned} U^{(8)}_{3412,n}(x,\mathbf{z}_8,t) &= 1+\sum_{n\geq 1} U^{(8)}_{3412,n}(x,\mathbf{z}_8)t^n \\ &= 1+\sum_{v\in[8]^+} -x\overline{z}^v t^{|v|}(1-x)^{\text{wrise}(v)} \prod_{s\in Des(v)} wt_{3412,8}(v_sv_{s+1}). \quad (3.22) \end{aligned}$$

What we need to be able to compute the right-hand sides of either (3.11), (3.13), (3.20), or (3.22) is the generating functions over all words $v\in\mathbb{P}^*$ where we not only keep track of the descents of P but also of type of descents of P.

By Theorem 1, we know that

$$\frac{1}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} DXZ(S)} = 1+\sum_{w=w_1\dots w_n\in\mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}}. \quad (3.23)$$

Hence

$$\sum_{w=w_1\dots w_n\in\mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}} = \left(\frac{1}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXZ(S)}\right) - 1$$
$$= \frac{\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXZ(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXZ(S)}. \qquad (3.24)$$

Next suppose that we replace t by yt and x_{ij} by $\frac{x_{ij}}{y}$. Under this substitution the left-hand side in (3.24) becomes

$$\sum_{w=w_1\dots w_n\in\mathbb{P}^+} t^{|w|}y^{\mathrm{wrise}(w)+1}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}}.$$

Note that for $S = \{j_1 < \cdots < j_k\}$ where $k \geq 2$, our substitution replaces $t^k DXZ(S)$ by

$$y^k t^k z_{j_1}\cdots z_{j_k} \prod_{i=1}^{k-1}\left(\frac{x_{j_{i+1}j_i}}{y} - 1\right) = yt^k z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i} - y).$$

Thus if we let

$$DXYZ(S) = \begin{cases} z_j & \text{if } S=\{j\}, \text{ and} \\ z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-y) & \text{if } S=\{j_1<\cdots<j_k\} \text{ where } k\geq 2, \end{cases} \qquad (3.25)$$

then we see that right-hand side of (3.24) becomes

$$\frac{y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}.$$

It follows that

$$-x\sum_{w=w_1\dots w_n\in\mathbb{P}^+} t^{|w|}y^{\mathrm{wrise}(w)}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}} =$$
$$\frac{-x\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}.$$

Thus

$$1-x\sum_{w=w_1\dots w_n\in\mathbb{P}^+} t^{|w|}y^{\mathrm{wrise}(w)}\overline{z}^w \prod_{i\in Des(w)} x_{w_iw_{i+1}} =$$

$$\frac{1-(x+y)\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXYZ(S)}. \tag{3.26}$$

By setting $z_i = 0$ for $i > k$, we also obtain that

$$1 - x \sum_{w=w_1\dots w_n\in[k]^+} t^{|w|} y^{\mathrm{wrise}(w)} \overline{z}^w \prod_{i\in Des(w)} x_{w_i w_{i+1}} =$$

$$\frac{1-(x+y)\sum_{n=1}^{k} t^n \sum_{S\subseteq [k],|S|=n} DXYZ(S)}{1-y\sum_{n=1}^{k} t^n \sum_{S\subseteq [k],|S|=n} DXYZ(S)}. \tag{3.27}$$

Note that if we replace y by $(1-x)$ and x_{ji} by $wt_u(ji)$, the left-hand side of (3.26) becomes $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (3.27) becomes $U_u^{(k)}(x,\mathbf{z}_k,t)$. Similarly, if we replace y by $(1-x)$ and x_{ji} by $ewt_u(ji)$, the left-hand side of (3.26) becomes $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (3.27) becomes $EU_u^{(k)}(x,\mathbf{z}_k,t)$. Then using the fact that $\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and that $\mathcal{EN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, we have the following theorem.

Theorem 10. *Suppose that* $u = u_1\dots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 > u_j$. *Then*

$$\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXTZ_u(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} DXTZ_u(S)} \tag{3.28}$$

and

$$\mathcal{EN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} EDXTZ_u(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} EDXTZ_u(S)} \tag{3.29}$$

where

$$DXTZ_u(S) = \begin{cases} z_j & \text{if } S=\{j\}, \text{ and} \\ z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(wt_u(j_{i+1}j_i)+x-1) & \text{if } S=\{j_1<\cdots<j_k\} \end{cases} \tag{3.30}$$

where $k \geq 2$ *and*

$$EDXTZ_u(S) = \begin{cases} z_j & \text{if } S=\{j\}, \text{ and} \\ z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(ewt_u(j_{i+1}j_i)+x-1) & \text{if } S=\{j_1<\cdots<j_k\} \end{cases} \tag{3.31}$$

where $k \geq 2$.

If we set $z_i = 0$ for all $i > k$, then we obtain the following theorem.

Theorem 11. *Now suppose that* $u = u_1 \ldots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 > u_j$. *Then*

$$\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1 - (1-x)\sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} DXTZ_u(S)}{1 - \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} DXTZ_u(S)} \tag{3.32}$$

and

$$\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1 - (1-x)\sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} EDXTZ_u(S)}{1 - \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} EDXTZ_u(S)}. \tag{3.33}$$

It follows from Theorem 11 that to compute the generating function $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$, we need only compute sums of the form

$$P_{n,u}(x,t) = \sum_{S \subseteq [k], |S|=n} DXTZ_u(S)$$

for $1 \leq n \leq k$. As an example, suppose that $k = 5$ and we want to compute $\mathcal{N}_{2341}^{(5)}(x, \mathbf{z}_5, t)$ where we set $z_i = 1$ for all i. Then with this specialization, it is easy to see that

1. $wt_{2341}(21) = -3xt^2$,
2. $wt_{2341}(3i) = -xt^2$ for all $i < 3$,
3. $wt_{2341}(4i) = 0$ for all $i < 4$, and
4. $wt_{2341}(5i) = 0$ for all $i < 5$.

It follows that that

1. $DXTZ_{2341}(\{1,2\})|_{z_i=1} = -3xt^2 + x - 1$,
2. $DXTZ_{2341}(\{i,3\})|_{z_i=1} = -xt^2 + x - 1$ for all $i < 3$,
3. $DXTZ_{2341}(\{i,4\})|_{z_i=1} = x - 1$ for all $i < 4$, and
4. $DXTZ_{2341}(\{i,5\})|_{z_i=1} = x - 1$ for all $i < 5$.

One can then compute that

1. $P_{1,2341}(x,t) = 5$,

2. $P_{2,2341}(x,t) = -10 + 10x - 5xt^2$,

3. $P_{3,2341}(x,t) = 10 - 20x + 14t^2x + 10x^2 - 14t^2x^2 + 3t^4x^2$,

4. $P_{4,2341}(x,t) = -5 + 15x - 13t^2x - 15x^2 + 26t^2x^2 - 6t^4x^2 + 5x^3 - 13t^2x^3 + 6t^4x^3$, and

5. $P_{5,2341}(x,t) = (-3xt^2 + x - 1)(-xt^2 + x - 1)(x-1)^2$.

Thus

$$\sum_{w \in [5]^*, 2341\text{-mch}(w)=0} x^{\text{des}(w)+1} = \frac{1-(1-x)(\sum_{n=1}^{5} P_{n,2341}(x,t)t^n)}{1-\sum_{n=1}^{5} P_{n,2341}(x,t)t^n}. \tag{3.34}$$

We have computed that the initial terms of this series are

$$\begin{aligned}
&1 + 5xt + 5(3x + 2x^2)t^2 + 5(7x + 16x^2 + 2x^3)t^3 + 5(14x + 72x^2 + 37x^3 + x^4)t^4 + \\
&(126x + 1210x^2 + 1492x^3 + 246x^4 + x^5)t^5 + \\
&(210x + 3387x^2 + 7921x^3 + 3522x^4 + 210x^5)t^6 + \\
&(330x + 8344x^2 + 32461x^3 + 28902x^4 + 5471x^5 + 120x^6)t^7 + \cdots.
\end{aligned}$$

One can obtain several interesting generating functions from $\mathcal{N}_{2341}^{(5)}(x, \mathbf{z}_5, t)$. For example setting $x = 0$ in $\frac{1}{2}\frac{\partial^2}{\partial x^2}\mathcal{N}_{2341}^{(5)}(x, \mathbf{z}_5, t)$, one finds that the generating function for the number of words w in $[5]^*$ such that $\text{des}(w) = 1$ and $2341\text{-mch}(w) = 0$ is

$$\frac{t^2(10 - 20t + 10t^2 + 10t^3 - 13t^4 + 4t^5)}{(1-t)^{10}}.$$

Similarly setting $x = 0$ in $\frac{1}{6}\frac{\partial^3}{\partial x^3}\mathcal{N}_{2341}^{(5)}(x, \mathbf{z}_5, t)$, one finds that the generating function for the number of words w in $[5]^*$ such that $\text{des}(w) = 2$ and $2341\text{-mch}(w) = 0$ is

$$\frac{t^3 Q(t)}{(1-t)^{15}}$$

where

$$\begin{aligned}
Q(t) = {}&10 + 35t - 233t^2 + 416t^3 - 219t^4 - 266t^5 + 458t^6 - 167t^7 - 161t^8 + 198t^9 \\
&- 83t^{10} + 13t^{11}.
\end{aligned}$$

If $u = 2^s1$ where $s \geq 2$ and we set $z_i = 1$ for all i, then it is easy to see that $wt_{2^s1}(ji) = -xt^{s-1}$ for all $j > i$ and that $DXTZ(S)|_{z_i=1} = (-xt^{s-1} + 1 - x)^{|S|-1}$ for all $S \subseteq [k]$ where $|S| \geq 1$. It then easily follows from Theorem 11

$$\sum_{w \in [k]^*, 2^s1\text{-mch}(w)=0} x^{\text{des}(w)+1} = \frac{1-(1-x)(\sum_{n=1}^{k} \binom{k}{n}(-xt^{s-1}+1-x)^{n-1}t^n)}{1-\sum_{n=1}^{k}\binom{k}{n}(-xt^{s-1}+1-x)^{n-1}t^n}. \tag{3.35}$$

As an example,

$$\sum_{w \in [5]^*, 2^31\text{-mch}(w)=0} x^{\text{des}(w)+1} = \frac{1-(1-x)(\sum_{n=1}^{5} \binom{5}{n}(-xt^{2}+x-1)^{n-1}t^n)}{1-\sum_{n=1}^{k}\binom{5}{n}(-xt^{2}+x-1)^{n-1}t^n}. \tag{3.36}$$

We have computed that the initial terms of this series are

$$1 + 5xt + 5(3x + 2x^2)t^2 + 5(7x + 16x^2 + 2x^3)t^3 + 5(14x + 71x^2 + 37x^3 + x^4)t^4+$$
$$(126x + 1166x^2 + 1486x^3 + 246x^4 + x^5)t^5+$$
$$5(42x + 634x^2 + 1553x^3 + 704x^4 + 42x^5)t^6+$$
$$(330x + 7554x^2 + 30998x^3 + 28662x^4 + 5471x^5 + 120x^6)t^7 + O[t]^8.$$

3.3 The case $u = u_1 \ldots u_j$, $\text{des}(u) = 1$, and $u_1 < u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$, $\mathcal{EN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\text{des}(u) = 1$, $u_1 < u_j$, and u has the $\mathbb{P}$-weakly increasing overlapping property ($[k]$-weakly increasing overlapping property).

Again the simplest case is when u has the $\mathbb{P}$-minimal overlapping property in which case u automatically has the $\mathbb{P}$-weakly increasing overlapping property. For example, suppose that $u = 12433$. Now suppose that we are given a fixed point (B, w) of I_u, where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$, such as the one pictured in Figure 3.4. We know that to be a fixed point of I_u, w must be weakly increasing within bricks of B and that for any $i < k$, if c is last cell in brick b_i and $w_c > w_{c+1}$, then there must be a u-match in w which is contained in the cells of b_i and b_{i+1}. In our particular example, since $u = 12433$ has a single descent, this match must involve the last three cells of b_i and the first two cells of b_{i+1}. In Figure 3.4, we have indicated the three such matches in our example by placing stars below the cells in the 12433-matches. In this case, the collapse map just maps (B, w) to the word $v = C(B, w, u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 3.4 where again we have starred the elements in $C(B, w, u)$ that remain from the original 12433-matches in w. In this case, the resulting word $C(B, w, u)$ must be weakly increasing.

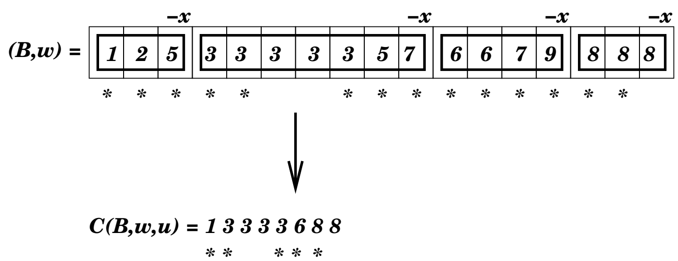

Figure 3.4: A fixed point of I_{12433}.

As in the previous section, we want to construct the set of fixed points of (B, w) of I_u such that $C(B, w, v)$ is equal to a given word $v = v_1 \dots v_n$ where $v_1 \leq \cdots \leq v_n$.

If $v_s < v_{s+1}$, then we have three possibilities: (i) $v_s v_{s+1}$ could lie in the same brick b_i of B, (ii) v_s could end a brick b_i and v_{s+1} could start the brick b_{i+1} in B, or (iii) $v_s v_{s+1}$ arose from a collapse across two bricks b_i and b_{i+1} where there was a decrease between bricks b_i and b_{i+1} and v_s played the role of 1 in the u-match and v_{s+1} plays the role of second 3 in the u-match that must cross the bricks b_i and b_{i+1}. For example, suppose that the underlying alphabet is [9]. If $v_s = 8$ and $v_{s+1} = 9$, then v could not have come from the collapse of 12433-match because we can not add a letter which could play the role of 4 in the 12433-match. Hence, the weight associated to a rise 89 is just $1 - x$. If we consider the first rise 13 in the $C(B, w, u)$ of Figure 3.4, then we see there are many ways that we could add the three letters middle letters. That is, the original 12433-match could have been any $12c33$ where $c \in \{4, 5, 6, 7, 8, 9\}$. It follows that the extra weight from these possibilities that is not included in $\overline{z}^{C(B,w,u)}t^{|C(B,w,u)|}$ is $-xt^3z_2z_3\sum_{4\leq c\leq 9} z_c$. Here the $-x$ comes from the fact that we know that the original match straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. Thus the weight associated with the rise 13 is $1-x-xt^3z_2z_3\sum_{4\leq c\leq 9} z_c$. If we consider the second rise 36 in the $C(B,w,u)$ of Figure 3.4, then we see there are many ways that we could add the three letters middle letters. That is, the original 12433-match could have been any $3cd66$ where $c \in \{4,5\}$ and $d \in \{7,8,9\}$. It follows that the extra weight from these possibilities that is not included in $\overline{z}^{C(B,w,u)}t^{|C(B,w,u)|}$ in this case is $-xt^3z_6(z_4+z_5)(z_7+z_8+z_9)$. Thus the weight associated with the rise 36 is $1-x-xt^3z_6(z_4+z_5)(z_7+z_8+z_9)$. Finally, we consider the third rise 68 in the $C(B,w,u)$ of Figure 3.4, then there is only one way to add the three middle letters. That is, the original 12433-match must have been 67988. It follows that the extra weight in this case that is not included in $\overline{z}^{C(B,w,u)}t^{|C(B,w,u)|}$ would be $-xt^3z_7z_8z_9$. Thus the weight associated with the final rise 68 is $1-x-xt^3z_7z_8z_9$. On the other hand, if $v_s = v_{s+1}$, then we have only two choices. That is, either cell s was the end of a brick or cell s was an internal cell of a brick. This implies that each level in v contributes a factor of $(1-x)$ since if s is at the end of a brick, there is a weight of $-x$ associated with the last cell of a brick. In this way, we can associate a weight with each level or rise of v which will allows to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

In our case where $u = 12433$ and $k = 9$, the weights associated with the rises are given in table 3.4.

However, if $u = 12433$ and we want to compute $U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$, the weights for any rise ij where $i+1 < j$ would be $1-x-xt^3z_j(\sum_{i<s<j} z_s)(\sum_{j<d} z_d)$ which is an infinite sum.

Going back to our example where $u = 12433$ and $k = 9$, it follows that for any $v \in [9]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w) =$$

Table 3.4: The weights $wt_{12433,9}(ij)$

Rises	$wt_{12433,9}(ij)$
$i9$ $(i \le 7)$ or $i(i+1)$	$1-x$
$i8$ $(i \le 6)$	$1-x-xt^3 z_8 z_9(\sum_{i<j<8} z_j)$
$i7$ $(i \le 5)$	$1-x-xt^3 z_7(z_8+z_9)(\sum_{i<j<7} z_j)$
$i6$ $(i \le 4)$	$1-x-xt^3 z_6(z_7+z_8+z_9)(\sum_{i<j<6} z_j)$
$i5$ $(i \le 3)$	$1-x-xt^3 z_5(z_6+z_7+z_8+z_9)(\sum_{i<j<5} z_j)$
$i4$ $(i \le 2)$	$1-x-xt^3 z_4(\sum_{4<s\le 9} z_s)(\sum_{i<j<4} z_j)$
13	$1-x-xt^3 z_2 z_3(\sum_{3<s\le 9} z_s)$

$$- x\overline{z}^v t^{|v|}(1-x)^{\mathrm{lev}(v)} \prod_{s\in Rise(v)} wt_{12433,9}(v_s v_{s+1}). \tag{3.37}$$

As in the previous section, the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. But then we know that

$$\begin{aligned} U^{(9)}_{12433}(x,\mathbf{z}_9,t) &= 1+\sum_{n\ge 1} U^{(9)}_{12433,n}(x,\mathbf{z}_9)t^n \\ &= 1+\sum_{v\in[9]^+,\mathrm{des}(v)=0} -x(1-x)^{\mathrm{lev}(v)}\overline{z}^v t^{|v|} \prod_{s\in Rise(v)} wt_{12433,9}(v_s v_{s+1}). \end{aligned} \tag{3.38}$$

Hence we could compute $\mathcal{N}^{(9)}_{12433}(x,\mathbf{z}_9,t) = \dfrac{1}{U^{(9)}_{12433}(x,\mathbf{z}_9,t)}$ if we can compute the right-hand side of (3.38)

As in the previous section, the case of exact matches is much simpler. In that case, we want to compute

$$\sum_{\substack{(B,w)\text{ is a fixed point of } J_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

Going back to our example of $u = 12433$ over the alphabet $[9]$, we see that the weight associated to a rise $v_s < v_{s+1}$ is $1-x$ unless $v_s = 1$, $v_{s+1} = 3$. If $v_s = 1$, $v_{s+1} = 3$, then we must have eliminated a 243 from w. Thus if we want to compute

Table 3.5: The weights $ewt_{12433}(ij)$

Rise	weight $ewt_{12433,\mathbb{P}}(ij)$
ij where either $i \neq 1$ or $j \neq 3$	$1-x$
13	$1-x-xz_2z_3z_4t^3$

$EU^{(\mathbb{P})}_{12433,n}(x,\mathbf{z}_\infty)$ or $EU^{(k)}_{12433,n}(x,\mathbf{z}_k)$ for $k \geq 4$, the weights associated to rises are given in table 3.5. It follows that for any $v \in [9]^+$,

$$\sum_{\substack{(B,v) \text{ is a fixed point of } J_{12433} \\ C(B,w,12433)=v}} sgn(B,w)wt(B,w) = \\ -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{lev}(v)} \prod_{s \in Rise(v)} ewt_{12433,9}(v_s v_{s+1}) \tag{3.39}$$

and

$$\begin{aligned} EU^{(9)}_{12433}(x,\mathbf{z}_9,t) &= 1 + \sum_{n\geq 1} EU^{(9)}_{12433,n}(x,\mathbf{z}_9)t^n \\ &= 1 + \sum_{v\in[9]^+,\mathrm{des}(v)=0} -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{lev}(v)} \prod_{s\in Rise(v)} ewt_{12433,9}(v_s v_{s+1}). \end{aligned} \tag{3.40}$$

When u does not have the minimal overlapping property, we can obtain similar results if u has the $\mathbb{P}$-weakly increasing overlapping property or the $[k]$-weakly increasing overlapping property. For example suppose that $u = u_1, \ldots, u_j$, $\mathrm{des}(u) = 1$, $u_1 < u_j$, and u has the $\mathbb{P}$-weakly increasing overlapping property. Now suppose that $w = w_1 \ldots w_n$ is a maximal sequence of linked u-matches. That is, we assume w starts and ends with a u-match and any two consecutive u-matches share at least two letters. Then if the u-matches in w start at positions $1 = i_1 < i_2 < \cdots < i_k$, then the $\mathbb{P}$-weakly increasing overlapping property in w ensures that $w_1 = w_{i_1} \leq \cdots \leq w_{i_k} < w_n$. Thus in a collapse map, if we eliminate $w_2 \ldots w_{n-1}$ we will be left with a rise $w_1 w_n$. This may not happen if u does not have the $\mathbb{P}$-weakly increasing overlapping property. For example, suppose $u = 2413$, then the words $w^{(1)} = 472613$, $w^{(2)} = 472614$, and $w^{(3)} = 472615$ have u-matches starting at positions 1 and 3. Thus in such a case, we have no control over the relationship between first and last letter of a maximal sequence of linked u-matches.

Thus assume that $u = u_1 \dots u_j$, $\mathrm{des}(u) = 1$, $u_1 < u_j$ and u has the $\mathbb{P}$-weakly increasing overlapping property. Then we shall see that the collapse map still works but the weight function $wt_u(ij)$ is more complicated. As we saw in the previous section, we must pay attention to overlapping u-matches that share more than one letter. We will consider the example where $u = 11124333$ and $k = 7$. Clearly u has the weakly increasing overlapping property. In this case, u-matches can overlap in either one, two, or three letters. As in the previous section, the collapse map will keep only the first and last letters of a consecutive sequence of u-matches such that each consecutive pair share at least two letters. For example, at the top of Figure 3.5, we have given an example where two consecutive u-matches share 3 letters and at the bottom of Figure 3.5, we have given an example where two consecutive u-matches share 2 letters.

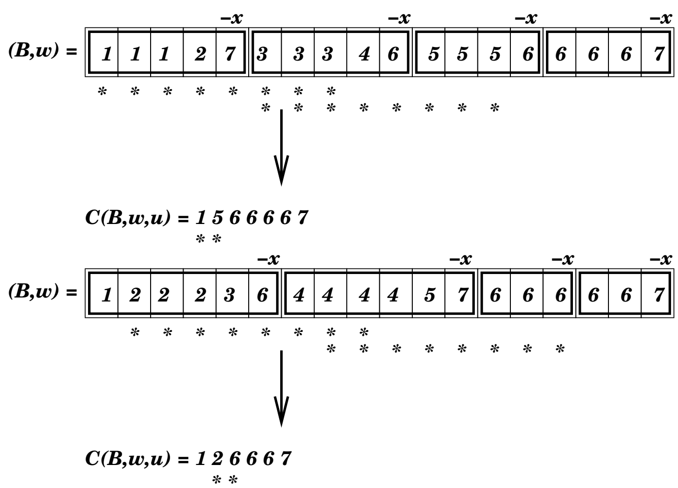

Figure 3.5: A fixed point of $I_{11124333}$.

As before, if we are given a weakly increasing word $v = v_1 \dots v_n \in [7]^+$, we want to find the sum of the weights of all fixed points (B, w) of I_u such that $C(B, w, u) = v$. Now if $v_s = v_{s+1}$, then either $v_s v_{s+1}$ lie in the same brick which contributes a factor of 1 or $v_s v_{s+1}$ lie in different bricks which contributes a factor of $-x$ for the brick that ends at v_s. Thus we obtain a factor of $1 - x$ for each level of v. For the rises of v, we should observe that the start and the end of any two consecutive u-matches which share more than one letter must differ by at least 4. Similarly, the start and the end of any three consecutive u-matches in which each two consecutive u-matches share more than one letter must differ by at least 6. Hence, for $k = 7$, we can not have three consecutive u-matches in which each two consecutive u-matches share more that one letter because the smallest starting point is 1 the smallest ending point is 7 which leaves no room for a letter which is larger than the last three letters in such a sequence. For each pair, $v_s < v_{s+1}$ which occurs in v, we get a factor of $1-x$ as we did for levels. However in this case, we must also consider the possible collapses that could give rise to $v_s v_{s+1}$. These are as follows.

1. Rises of the form $i(i+1)$ or $i7$ where $1 \leq i \leq 5$ can not arise from the collapse map in our case so that $wt_{11124333,7}(v_s v_{s+1}) = 1-x$ in these cases.

2. $v_s v_{s+1} = 13$. In this case, a u-match that could give rise to 13 under the collapse map must be of the form $1112a333$ where $a \in \{4,5,6,7\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_1^2 z_2 (z_4 + z_5 + z_6 + z_7) z_3^2.$$

3. $v_s v_{s+1} = 14$. In this case, a u-match that could give rise to 14 under the collapse map must be of the form $111ab444$ where $a \in \{2,3\}$ and $b \in \{5,6,7\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_1^2 (z_2 + z_3)(z_5 + z_6 + z_7) z_4^2.$$

4. $v_s v_{s+1} = 15$. In this case, a single u-match that could give rise to 15 under the collapse map must be of the form $111ab555$ where $a \in \{2,3,4\}$ and $b \in \{6,7\}$. There are also two possibilities for linked u-matches that could give rise to 15 under the collapse map, namely, (i) $1112a3334b555$ or (ii) $1112a33334b555$ where $a \in \{4,5,6,7\}$ and $b \in \{6,7\}$. Thus

$$\begin{aligned} wt_{11124333,7}(v_s v_{s+1}) &= 1 - x - xt^6 z_1^2 (z_2 + z_3 + z_4)(z_6 + z_7) z_5^2 - \\ &\quad xt^{11} z_1^2 z_2 (z_4 + z_5 + z_6 + z_7) z_3^3 z_4 (z_6 + z_7) z_5^2 - \\ &\quad xt^{12} z_1^2 z_2 (z_4 + z_5 + z_6 + z_7) z_3^4 z_4 (z_6 + z_7) z_5^2. \end{aligned}$$

5. $v_s v_{s+1} = 16$. In this case, a single u-match that could give rise to 16 under the collapse map must be of the form $111a7666$ where $a \in \{2,3,4,5\}$. There are also four possibilities for linked u-matches that could give rise to 16 under the collapse map, namely,
(i) $1112a333b7666$ $a \in \{4,5,6,7\}$ and $b \in \{4,5\}$, (ii) $1112a3333b7666$ where $a \in \{4,5,6,7\}$ and $b \in \{4,5\}$, (iii) $111ab44457666$ $a \in \{2,3\}$ and $b \in \{5,6,7\}$, or (iv) $111ab444457666$ where $a \in \{2,3\}$ and $b \in \{5,6,7\}$. Thus

$$\begin{aligned} wt_{11124333,7}(v_s v_{s+1}) &= 1 - x - xt^6 z_1^2(z_2+z_3+z_4+z_5)z_7 z_6^2 - \\ &\quad xt^{11} z_1^2 z_2(z_4+z_5+z_6+z_7)z_3^3(z_4+z_5)z_7 z_6^2 - \\ &\quad xt^{12} z_1^2 z_2(z_4+z_5+z_6+z_7)z_3^4(z_4+z_5)z_7 z_6^2 - \\ &\quad xt^{11} z_1^2(z_2+z_3)(z_5+z_6+z_7)z_4^3 z_5 z_7 z_6^2 - \\ &\quad xt^{12} z_1^2(z_2+z_3)(z_5+z_6+z_7)z_4^4 z_5 z_7 z_6^2. \end{aligned}$$

6. $v_s v_{s+1} = 24$. In this case, a u-match that could give rise to 24 under the collapse map must be of the form $2223a444$ where $a \in \{5,6,7\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_2^2 z_3(z_5+z_6+z_7)z_4^2.$$

7. $v_s v_{s+1} = 25$. In this case, a u-match that could give rise to 25 under the collapse map must be of the form $222ab555$ where $a \in \{3,4\}$ and $b \in \{6,7\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_2^2(z_3+z_4)(z_6+z_7)z_5^2.$$

8. $v_s v_{s+1} = 26$. In this case, a single u-match that could give rise to 26 under the collapse map must be of the form $222a7666$ where $a \in \{3,4,5\}$. There are also two possibilities for linked u-matches that could give rise to 26 under the collapse map, namely, (i) $2223a44457666$ or (ii) $2223a444457666$ where $a \in \{5,6,7\}$. Thus

$$\begin{aligned} wt_{11124333,7}(v_s v_{s+1}) &= 1 - x - xt^6 z_2^2(z_3+z_4+z_5)z_7 z_6^2 - \\ &\quad xt^{11} z_2^2 z_3(z_5+z_6+z_7)z_4^3 z_5 z_7 z_6^2 - \\ &\quad xt^{12} z_2^2 z_3(z_5+z_6+z_7)z_4^4 z_5 z_7 z_6^2. \end{aligned}$$

9. $v_s v_{s+1} = 35$. In this case, a u-match that could give rise to 35 under the collapse map must be of the form $3334a555$ where $a \in \{6,7\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_3^2 z_4 (z_6 + z_7) z_5^2.$$

10. $v_s v_{s+1} = 36$. In this case, a u-match that could give rise to 36 under the collapse map must be of the form $333a7666$ where $a \in \{4,5\}$. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_3^2 (z_4 + z_5) z_7 z_6^2.$$

11. $v_s v_{s+1} = 46$. In this case, a u-match that could give rise to 46 under the collapse map must be of the form 44457666. Thus

$$wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_4^2 z_5 z_7 z_6^2.$$

It follows that for any $v \in [7]^+$ such that v is weakly increasing,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{11124333} \\ C(B,w,11124333)=v}} sgn(B,w) wt_{11124333,7}(B,w) = \\ - x\overline{z}^v t^{|v|} (1-x)^{\mathrm{lev}(v)} \prod_{s \in Rise(v)} wt_{11124333,7}(v_s v_{s+1}). \quad (3.41)$$

and

$$U^{(7)}_{11124333}(x, \mathbf{z}_7, t) = 1 + \sum_{n \geq 1} U^{(7)}_{11124333,n}(x, \mathbf{z}_7) t^n = \\ 1 + \sum_{v \in [7]^+, \mathrm{des}(v)=0} -x\overline{z}^v t^{|v|} (1-x)^{\mathrm{lev}(v)} \prod_{s \in Rise(v)} wt_{11124333,7}(v_s v_{s+1}). \quad (3.42)$$

What we need to be able to compute the right-hand sides of either (3.38), (3.40), or (3.42), is the generating function over all weakly increasing words $v \in \mathbb{P}^*$ where we not only keep track of the rises of P but also the type of rises.

By Theorem 3, we know that

$$\sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} RXZ(S) = \sum_{w = w_1 \leq \cdots \leq w_n \in \mathbb{P}^+} t^{|w|} \overline{z}^w \prod_{i \in Rise(w)} x_{w_i w_{i+1}}. \quad (3.43)$$

If we first replace t by yt and x_{ij} by x_{ij}/y in (3.43) and then divide by y, the right-hand side (3.43) becomes

$$\sum_{w=w_1\leq\cdots\leq w_n\in\mathbb{P}^+} t^{|w|}\overline{z}^w y^{\mathrm{lev}(w)} \prod_{i\in Rise(w)} x_{w_iw_{i+1}}$$

and the left-hand side becomes

$$\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} RXYZ(S)$$

where

$$RXYZ(S) = \begin{cases} \frac{z_j}{1-z_jyt} & \text{if } S=\{j\}, \text{ and} \\ \left(\prod_{i=1}^k \frac{z_{j_i}}{1-z_{j_i}yt}\right)\prod_{i=1}^{k-1} x_{j_ij_{i+1}} & \text{if } S=\{j_1<\cdots<j_k\} \text{ where } k\geq 2. \end{cases} \tag{3.44}$$

Hence

$$1 - x\sum_{w=w_1\leq\cdots\leq w_n\in\mathbb{P}^+} t^{|w|}y^{\mathrm{lev}(w)}\overline{z}^w \prod_{i\in Rise(w)} x_{w_iw_{i+1}} = 1 - x\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} RXYZ(S). \tag{3.45}$$

If we set $z_i = 0$ for $i > k$, then we obtain that

$$1 - x\sum_{w=w_1\leq\cdots\leq w_n\in[k]^+} t^{|w|}y^{\mathrm{lev}(w)}\overline{z}^w \prod_{i\in Rise(w)} x_{w_iw_{i+1}} = 1 - x\sum_{n=1}^{k} t^n \sum_{S\subseteq[k],|S|=n} RXYZ(S). \tag{3.46}$$

Note that if we replace y by $(1-x)$ and x_{ij} by $wt_u(ij)$, the left-hand side of (3.45) becomes $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (3.46) becomes $U_u^{(k)}(x,\mathbf{z}_k,t)$. Similarly, if we replace y by $(1-x)$ and x_{ij} by $ewt_u(ij)$, the left-hand side of (3.45) becomes $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (3.46) becomes $EU_u^{(k)}(x,\mathbf{z}_k,t)$. Then using the fact that $\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and that $\mathcal{EN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ $= 1/EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, we have the following theorem.

Theorem 12. *Suppose that* $u = u_1 \ldots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 < u_j$, *and* u *has the* $\mathbb{P}$*-weakly increasing overlapping property. Then*

$$\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 - x\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P}, |S|=n} RXTZ(S)} \tag{3.47}$$

and

$$\mathcal{E}\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 - x\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P}, |S|=n} ERXTZ(S)} \tag{3.48}$$

where

$$RXTZ_u(S) = \begin{cases} \frac{z_j}{1-(1-x)z_j t} & \text{if } S = \{j\}, \text{ and} \\ \left(\prod_{i=1}^{k} \frac{z_{j_i}}{1-(1-x)z_{j_i} t}\right) \prod_{i=1}^{k-1}(wt_u(j_i j_{i+1})) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{3.49}$$

where $k \geq 2$ *and*

$$ERXTZ_u(S) = \begin{cases} \frac{z_j}{1-(1-x)z_j t} & \text{if } S = \{j\}, \text{ and} \\ \left(\prod_{i=1}^{k} \frac{z_{j_i}}{1-(1-x)z_{j_i} t}\right) \prod_{i=1}^{k-1} ewt_u(j_i j_{i+1}) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{3.50}$$

where $k \geq 2$.

If we specialize the variables so that $z_i = 0$ for all $i > k$, then we have the following theorem.

Theorem 13. *Suppose that* $u = u_1 \ldots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 < u_j$, *and* u *has the* $[k]$*-weakly increasing overlapping property. Then*

$$\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1}{1 - x\sum_{n=1}^{k} t^n \sum_{S\subseteq [k], |S|=n} RXTZ(S)} \tag{3.51}$$

and

$$\mathcal{E}\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1}{1 - x\sum_{n=1}^{k} t^n \sum_{S\subseteq [k], |S|=n} ERXTZ(S)}. \tag{3.52}$$

It follows from Theorem 13 that to compute the generating function we need to $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$, we need only compute sums of the form

$$P_{n,u}(x,t) = \sum_{S\subseteq [k], |S|=n} RXTZ_u(S)$$

for $1 \leq n \leq k$ and that to compute the generating function we need to $\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t)$, we need only compute sums of the form

$$P_{n,u}(x,t) = \sum_{S \subseteq [k], |S|=n} ERXTZ_u(S)$$

for $1 \leq n \leq k$.

For example, suppose that we want to compute $\mathcal{EN}_u^{(9)}(x, \mathbf{z}_9, t)$ where $u = 12433$ and we set $z_i = 1$ for $i = 1, \ldots, 9$. For each set singleton $S = \{j\}$, $ERXTZ_u(S) = \frac{1}{(1-(1-x)t)}$. For sets S of cardinality greater than 2, there are two types of sets $S = \{j_1 < j_2 < \ldots j_n\}$ to consider, namely, those where $j_1 = 1$ and $j_2 = 3$ and those sets where it is not the case that $j_1 = 1$ and $j_2 = 3$. If $S = \{j_1 < j_2 < \ldots j_n\}$ where it is not the case that $j_1 = 1$ and $j_2 = 3$, then we know that $ERXTZ_u(S) = \frac{(1-x)^{k-1}}{(1-(1-x)t)^k}$. If S is of the form $\{1, 3\} \cup T$ where $T \subseteq \{4, 5, 6, 7, 8, 9\}$, then

$$ERXTZ_u(S) = (1 - x - xt^3)(1 - x)^{|T|} \frac{1}{(1 - (1 - x)t)^{|T|+2}}.$$

If follows that

$$\begin{aligned}
\sum_{n=1}^{9} t^n \sum_{S \subseteq [9], |S|=n} ERXTZ(S) &= \sum_{n=1}^{p} \binom{9}{k} \frac{t^k (1-x)^{k-1}}{(1-(1-x)t)^k} - \\
&\quad \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2} (1-x)^{j+1}}{(1-(1-x)t)^{j+2}} + \\
&\quad \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2} (1-x-xt^3)(1-x)^{j}}{(1-(1-x)t)^{j+2}} \\
&= \sum_{n=1}^{p} \binom{9}{k} \frac{t^k (1-x)^{k-1}}{(1-(1-x)t)^k} - \\
&\quad \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2} (xt^3)(1-x)^{j}}{(1-(1-x)t)^{j+2}}.
\end{aligned}$$

Thus if we let

$$A_{12433,9}(x,t) = 1 - x \left(\sum_{k=1}^{p} \binom{9}{k} \frac{t^k (1-x)^{k-1}}{(1-(1-x)t)^k} - \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2} (xt^3)(1-x)^{j}}{(1-(1-x)t)^{j+2}} \right),$$

then

$$\mathcal{EN}^{(9)}_{12433}(x,\mathbf{z}_9,t)|_{z_i=1} = \frac{1}{A_{12433,9}(x,t)}. \tag{3.53}$$

We have used (3.53) to compute the first few terms in the series of $\mathcal{EN}^{(9)}_{12433}(x,\mathbf{z}_\infty,t)|_{z_i=1}$.

$$\begin{aligned}
&\mathcal{EN}^{(9)}_{12433}(x,\mathbf{z}_9,t)|_{z_i=1} = \\
&1+9tx+t^2\left(45x+36x^2\right)+t^3\left(165x+480x^2+84x^3\right)+ \\
&t^4\left(495x+3510x^2+2430x^3+126x^4\right)+ \\
&t^5\left(1287x+18612x^2+31212x^3+7812x^4+126x^5\right)+ \\
&t^6\left(3003x+79925x^2+262626x^3+167826x^4+17976x^5+84x^6\right)+ \\
&t^7\left(6435x+294616x^2+1683386x^3+2132496x^4+634446x^5+31536x^6+\right. \\
&\left.36x^7\right)+ \\
&t^8\left(12870x+965709x^2+8885187x^3+19458252x^4+11854197x^5+\right. \\
&\left.1826577x^6+43677x^7+9x^8\right)+ \\
&t^9\left(24310x+2881330x^2+40454572x^3+140542120x^4+149803150x^5+\right. \\
&\left.49462810x^6+4200670x^7+48610x^8+x^9\right)+\cdots.
\end{aligned}$$

We end this section with a remark about the case where $u = u_1 \dots u_j$, $\mathrm{des}(u) = 1$, $u_1 < u_j$, and u does not have the weakly increasing overlapping property. There are two problems in this case. First, as we saw earlier, it is possible that the end points of collapse u-match in a fixed (B,w) point of I_u can lead to a rise, a level, or a descent in $C(B,w,u)$. This means that the weights $w_{u,\mathbb{P}}(ij)$ or $w_{u,[k]}(ij)$ are much more complicated. The second problem is to find $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, we would need to substitute in a generating function of the form

$$1+\sum_{n\geq 1} t^n \sum_{w=w_1\dots w_n\in\mathbb{P}^n} \prod_{i=1}^{n-1} x_{w_i w_{i+1}} \tag{3.54}$$

and we do not know of any way to find a compact form for such a generating function.

3.4 The case $u = u_1 \ldots u_j$, $\mathrm{des}(u) = 1$, and $u_1 = u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$, $\mathcal{EN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\mathrm{des}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-level (or $[k]$-level) property.

As in the previous sections, we need to compute $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $U_u^{(k)}(x, \mathbf{z}_k, t)$, $EU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $EU_u^{(k)}(x, \mathbf{z}_k, t)$. To compute these generating functions, we use Theorem 2 or 4 plus the collapse map.

First assume that $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$- minimal overlapping property. We can define the collapse map to fixed points of I_u or J_u exactly as in the previous sections. For example, suppose that $u = 12311$ and we want to compute $U_{12311}^{(7)}(x, \mathbf{z}_7, t)$. By (3.3), we know that

$$U_{12311,n}^{(7)}(x, \mathbf{z}_7) = \sum_{O \in \mathcal{O}_{12311,n}^{(k)}, I_{12311}(O) = O} sgn(O) wt(O). \tag{3.55}$$

As before, we know that if (B, w) is a fixed point of I_{12311}, then elements in the bricks are weakly increasing and if there is a decrease between two brick b_i and b_{i+1}, there must be a 12311-match that involves the last 3 cells of b_i and the first three cells of b_{i+1}. We have pictured such a fixed point in Figure 3.6.

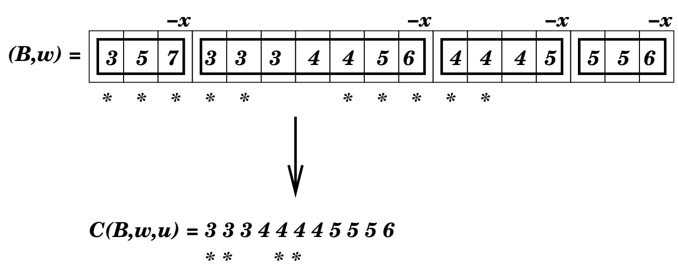

Figure 3.6: A fixed point of I_{12311}.

The difference between this case and the previous case where $u_1 > u_j$ is that a 12311-match of the form $ijkii$ will just be replaced by ii so that only factors of the form ii could have come from a 12311-match in the collapse of a fixed point of I_{12311}. The fact that 12311 has the $\mathbb{P}$-minimal overlapping property ensures that any two such 12311-matches can only intersect at the right-hand endpoint of the

Table 3.6: The weights $wt_{12311,7}(ii)$

Levels	$wt_{12311,7}(ii)$
77	$1-x$
66	$1-x$
55	$1-x-xt^3z_5z_6z_7$
44	$1-x-xt^3z_4(\sum_{4<c<d\leq 7} z_cz_d)$
33	$1-x-xt^3z_3(\sum_{3<c<d\leq 7} z_cz_d)$
22	$1-x-xt^3z_2(\sum_{2<c<d\leq 7} z_cz_d)$
11	$1-x-xt^3z_1(\sum_{1<c<d\leq 7} z_cz_d)$

first match and left-hand endpoint of the second match. It follows that $C(B,w,u)$ will always be a weakly increasing word. We claim that in this case a factor of the form ii must have weight $1-x-xt^3z_i\sum_{i<c<d\leq k} z_cz_d$ if we are computing $U^{(k)}_{12311,n}(x,\mathbf{z}_k)$ and $1-x-xt^3z_i\sum_{i<c<d} z_cz_d$ if we are computing $U^{(\mathbb{P})}_{12311,n}(x,\mathbf{z}_\infty)$. That is, the 1 corresponds to the case where ii are in the same brick, the $-x$ corresponds to the case where the first i is in last cell of some brick b_j and the second i is in the first cell of the next brick, and the third term corresponds to the cases where we have a decrease between two consecutive bricks and we deleted the second, third, and fourth elements of the 12311-match between the two bricks. In our example, the weight of the levels for computing $U^{(7)}_{12311,n}(x,\mathbf{z}_7)$ are listed in table 3.6.

In this case, rises in $C(B,w,12311)$ of the form ij where $i<j$ correspond to a factor of $1-x$ where the 1 comes from the case where ij are in the same brick and the $-x$ corresponds to the case where i and j are in different bricks.

It follows that for any $v\in[7]^+$ which is weakly increasing,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{12311} \\ C(B,w,12311)=v}} sgn(B,w)wt_{12311}(B,w) =$$

$$-x\overline{z}^v t^{|v|}(1-x)^{\mathrm{rise}(v)} \prod_{s\in Lev(v)} wt_{12311,7}(v_sv_{s+1}). \tag{3.56}$$

and

$$U_{12311}^{(7)}(x,\mathbf{z}_7,t) = 1+\sum_{n\geq 1} U_{12311,n}^{(7)}(x,\mathbf{z}_7)t^n$$
$$= 1+\sum_{v\in[7]^+,\mathrm{des}(v)=0} -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{rise}(v)} \prod_{s\in Lev(v)} wt_{12311,7}(v_s v_{s+1}). \tag{3.57}$$

Next suppose that $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-level overlapping property or the $[k]$-level overlapping property, but u does not have the $\mathbb{P}$-minimal overlapping property. The fact that u has the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property) ensures that if $w = w_1 \ldots w_n$ is word which starts and ends with a u-match and any two consecutive u-matches in w share at least two letters, then it must be the case that $w_1 = w_n$. Thus under the collapse map, any collapse will end up with a level of the form ii. The main difference in this case is that it is possible to have the weights $wt_{u,k}(ii)$ or $wt_{u,\mathbb{P}}(ii)$ correspond to infinite families of words of different lengths even in the case where the alphabet is finite. For example, suppose that $u = 11211$. Then it is possible that in a fixed point (B, w) of I_{11211}, w has a factor where consecutive occurrences of the pattern 11211 are linked of the form $iiy_1iiy_2iiy_3ii\ldots iiy_nii$ where $y_1,\ldots,y_n > i$ like those that occur in the first 14 cells of the fixed point pictured in Figure 3.7. For each given maximal sequence of this type, the collapse map would eliminate all the symbols between the first and the last i. In such a case, the weight corresponding to the symbols that are eliminated for such a string in the collapse map would be $(-x)^n z_i^{2n} z_{y_1}\cdots z_{y_n} t^{3n}$. It would follow that if we are working in $\mathbb{P}^*$, then

$$wt_{11211,\mathbb{P}}(ii) = 1-x+\frac{-xz_i^2\left(\sum_{s>i} z_s\right)t^3}{1+xz_i^2\left(\sum_{s>i} z_s\right)t^3}$$

while if we are working in $[k]^*$, then for $1\leq i<k$,

$$wt_{11211,k}(ii) = 1-x+\frac{-xz_i^2\left(\sum_{s=i+1}^{k} z_s\right)t^3}{1+xz_i^2\left(\sum_{s=i+1}^{k} z_s\right)t^3}$$

and

$$wt_{11211,k}(kk) = 1-x.$$

That is, in each of these expressions the 1 corresponds to the case where both is are part of the same brick, the $-x$ corresponds to the case where the two is are the last and first elements of two consecutive bricks, and the series $\frac{-xz_i^2\left(\sum_{s>i} z_s\right)t^3}{1+xz_i^2\left(\sum_{s>i} z_s\right)t^3}$ corresponds the fact that we could have eliminated sequences of the form $iy_1iiy_2iiy_3ii\ldots iiy_ni$ for any $n \geq 1$ between the two is.

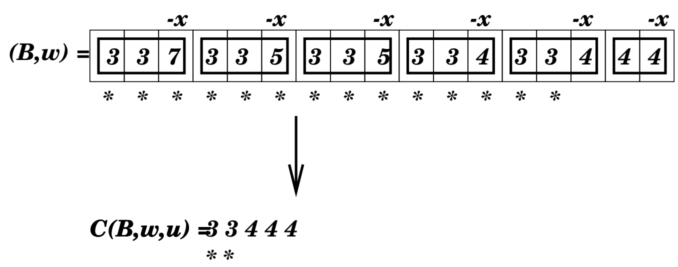

Figure 3.7: A fixed point of I_{11211}.

Nevertheless, we can still apply the same reasoning as above to prove that for any $v \in [7]^+$ which is weakly increasing,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{11211} \\ C(B,w,11211)=v}} sgn(B,w)wt_{11211}(B,w) = \\ -x\overline{z}^v t^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in Lev(v)} wt_{11211,7}(v_s v_{s+1}). \tag{3.58}$$

and

$$\begin{aligned} U_{11211}^{(7)}(x, \mathbf{z}_7, t) &= 1 + \sum_{n \geq 1} U_{11211,n}^{(7)}(x, \mathbf{z}_7)t^n \\ &= 1 + \sum_{v \in [7]^+, \text{des}(v)=0} -x\overline{z}^v t^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in Lev(v)} wt_{11211,7}(v_s v_{s+1}). \end{aligned} \tag{3.59}$$

We should note that as patterns get more complicated, it becomes increasingly difficult to compute $wt_{u,\mathbb{P}}(ii)$ or $wt_k(ii)$. For example, suppose $u = 3^5451 23^5$. Then linked patterns can overlap at either 1,2,3,4, or 5 symbols.

It follows from Theorem 4 that

$$\sum_{v \in \mathbb{P}^+, \text{des}(v)=0} \overline{z}^v t^{|v|} \prod_{i \in Lev(v)} x_{v_i v_i} = -1 + \prod_{i \geq 1} \left(1 + \frac{z_i t}{1 - x_{ii} z_i t}\right). \tag{3.60}$$

Replacing t by yt and x_{ij} by x_{ii}/y, we see that

$$\sum_{\substack{v=v_1\ldots v_n\in\mathbb{P}^+\\ v_1\le v_2\le\cdots\le v_n}} \overline{z}^v t^{|v|} y^{\mathrm{rise}(v)+1} \prod_{i\in Lev(v)} x_{v_iv_i} = -1 + \prod_{i\geq 1}\left(1+\frac{yz_it}{1-x_{ii}z_it}\right). \tag{3.61}$$

Thus

$$1+\sum_{\substack{v=v_1\ldots v_n\in\mathbb{P}^+\\ v_1\le v_2\le\cdots\le v_n}} -x\overline{z}^v t^{|v|} y^{\mathrm{rise}(v)} \prod_{i\in Lev(v)} x_{v_iv_i} = 1+\frac{-x}{y}\left(-1+\prod_{i\geq 1}\left(1+\frac{yz_it}{1-x_{ii}z_it}\right)\right). \tag{3.62}$$

and

$$1+\sum_{\substack{v=v_1\ldots v_n\in[k]^+\\ v_1\le v_2\le\cdots\le v_n}} -x\overline{z}^v t^{|v|} y^{\mathrm{rise}(v)} \prod_{i\in Lev(v)} x_{v_iv_i} = 1+\frac{-x}{y}\left(-1+\prod_{i=1}^{k}\left(1+\frac{yz_it}{1-x_{ii}z_it}\right)\right). \tag{3.63}$$

But then it follows that if $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-level overlapping property, then

$$\begin{aligned} U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) &= 1+\sum_{v\in\mathbb{P}^+,\mathrm{des}(v)=0} -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{rise}(v)} \prod_{i\in Lev(v)} wt_{u,\mathbb{P}}(v_iv_i) \\ &= 1+\frac{-x}{1-x}\left(-1+\prod_{i\geq 1}\left(1+\frac{((1-x)z_it}{1-wt_{u,\mathbb{P}}(ii)z_it}\right)\right) \end{aligned}$$

and, for all $k \geq 1$, if $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, and u has the $[k]$-level overlapping property, then

$$\begin{aligned} U_u^{(k)}(x,\mathbf{z}_k,t) &= 1+\sum_{v\in[k]^+,\mathrm{des}(v)=0} -x\overline{z}^v t^{|v|}(1-x)^{\mathrm{rise}(v)} \prod_{i\in Lev(v)} wt_{u,k}(v_iv_i) \\ &= 1+\frac{-x}{1-x}\left(-1+\prod_{i=1}^{k}\left(1+\frac{((1-x)z_it}{1-wt_{u,k}(ii)z_it}\right)\right). \end{aligned}$$

Thus we have the following theorem.

Theorem 14. *If* $u = u_1 \ldots u_j \in \mathbb{P}^*$ *is such that* $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, *and* u *has the* $\mathbb{P}$*-level overlapping property, then*

$$\mathcal{N}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1}{1-\frac{x}{1-x}\left(-1+\prod_{i\geq 1}\left(1+\frac{(1-x)z_it}{1-wt_{u,\mathbb{P}}(ii)z_it}\right)\right)}. \tag{3.64}$$

If $u = u_1 \dots u_j \in [k]^*$ *is such that* $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, *and* u *has the* $[k]$*-level overlapping property, then*

$$\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1}{1 - \frac{x}{1-x}\left(-1 + \prod_{i=1}^{k}\left(1 + \frac{(1-x)z_i t}{1-wt_{u,k}(ii)z_i t}\right)\right)} \tag{3.65}$$

Note that if $u = u_1 \dots u_j$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, then u automatically has the exact $\mathbb{P}$-level overlapping property (exact $[k]$-level overlapping property).

Theorem 15. *If* $u = u_1 \dots u_j \in \mathbb{P}^*$ *is such that* $\mathrm{des}(u) = 1$ *and* $u_1 = u_j$, *then*

$$\mathcal{EN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1}{1 - \frac{x}{1-x}\left(-1 + \prod_{i \geq 1}\left(1 + \frac{(1-x)z_i t}{1-ewt_{u,\mathbb{P}}(ii)z_i t}\right)\right)}. \tag{3.66}$$

and if $u = u_1 \dots u_j \in [k]^*$ *is such that* $\mathrm{des}(u) = 1$ *and* $u_1 = u_j$, *then*

$$\mathcal{EN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1}{1 - \frac{x}{1-x}\left(-1 + \prod_{i=1}^{k}\left(1 + \frac{(1-x)z_i t}{1-ewt_{u,k}(ii)z_i t}\right)\right)} \tag{3.67}$$

For example, suppose we want to compute $\mathcal{N}_{12311}^{(7)}(x, \mathbf{z}_7, t)$ where we set $z_i = 1$ for all i. It follows from (3.65) that

$$\mathcal{N}_{12311}^{(7)}(x, \mathbf{z}_7, t) = \frac{1}{1 - \frac{x}{(1-x)}(-1 + \prod_{i=1}^{7} Q_i(x,t))} \tag{3.68}$$

where

1. $Q_1(x,t) = 1 + \frac{(1-x)t}{1-(1-x-15xt^3)t}$,
2. $Q_2(x,t) = 1 + \frac{(1-x)t}{1-(1-x-10xt^3)t}$,
3. $Q_3(x,t) = 1 + \frac{(1-x)t}{1-(1-x-6xt^3)t}$,
4. $Q_4(x,t) = 1 + \frac{(1-x)t}{1-(1-x-3xt^3)t}$,
5. $Q_5(x,t) = 1 + \frac{(1-x)t}{1-(1-x-xt^3)t}$,
6. $Q_6(x,t) = 1 + \frac{(1-x)t}{1-(1-x)t}$, and
7. $Q_7(x,t) = 1 + \frac{(1-x)t}{1-(1-x)t}$.

We have computed that

$$\mathcal{N}^{(7)}_{12311}(x, \mathbf{z_7}, \mathbf{t}) =$$
$$1 + 7xt + 7(4x + 3x^2)t^2 + 7(12x + 32x^2 + 5x^3)t^3+$$
$$7(30x + 190x^2 + 118x^3 + 5x^3)t^4 + 7(66x + 823x^2 + 1236x^3 + 268x^3 + 3x^5)t^5+$$
$$7(132x + 2912x^2 + 8500x^3 + 4770x^4 + 422x^5 + x^6)t^6+$$
$$(1716x + 62532x^2 + 312558x^3 + 349315x^4 + 88852x^4 + 3424x^6 + x^7)t^7+$$
$$7(429x + 24609x^2 + 194029x^3 + 374249x^4 + 197729x^5 25209x^6 + 429x^7)t^8 + \cdots .$$

Finally we shall consider the case where $u = u_1 \dots u_j$, $\text{red}(u) = u$, $\text{des}(u) = 1$, $u_1 = u_j$ and u does not have the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property). Given such a u, let s be the position such that $u_s > u_{s+1}$. Then we must have that $u_{s+1} \leq \cdots \leq u_j = u_1$ and $St^{(\mathbb{P})}(u) \subset \{s+1, \dots, j\}$ ($St^{([k])}(u) \subset \{s+1, \dots, j\}$). This means that u automatically has the $\mathbb{P}$-weakly decreasing overlapping property ($[k]$-weakly decreasing overlapping property) and u is not $\mathbb{P}$-minimal overlapping ($[k]$-minimal overlapping). Now suppose that $w = w_1 \dots w_n$ is a maximal sequence of linked u-matches. That is, we assume w starts and ends with a u-match and any two consecutive u-matches share at least two letters. Then if the u-matches in w start at positions $1 = i_1 < i_2 < \cdots < i_k$, then the $\mathbb{P}$-weakly decreasing overlapping property ensures that $w_1 = w_{i_1} \geq \cdots \geq w_{i_k} = w_n$. Thus in a collapse map, if we eliminate $w_2 \dots w_{n-1}$, then we will be left with a weak descent $w_1 w_n$. Thus we must figure out the weights $wt_u(ji)$ for $j \geq i$.

To illustrate the process, we will consider the example where $u = 2312$ and the alphabet is $[4]$. If $w = w_1 w_2 w_3 w_4 \in [4]^*$ and $\text{red}(w) = 2312$, then clearly w must start with either 2 or 3 since those are the only letters a which have at least one letter in $[4]$ bigger than a and one letter in $[4]$ which is less than a. It follows that $wt_{2312,4}(44) = wt_{2312,4}(11) = 1 - x$. Also $wt_{2312,4}(4i) = 0$ for $i = 1, 2, 3$ and $wt_{2312,4}(j1) = 0$ for $j = 2, 3, 4$.

Next consider $wt_{2312,4}(22)$. There are only two possible words in $[4]^4$ that reduce to u, namely, $w = 2312$ and $v = 2412$. Since there is no u-match that can start with 1, there cannot be a pair of linked u-matches that start with either w or v. Thus there can be no maximal sequences of linked u-matches that start and

Table 3.7: The weights $wt_{2312,4}(ji)$

Weak Descents	$wt_{2312,4}(ji)$
44	$1-x$
$4i$ $(i<4)$	0
33	$1-x-xz_4(z_1+z_2)t^2$
32	$x^2z_1z_2z_3z_4t^4$
31	0
22	$1-x-xz_1(z_3+z_4)t^2$
21	0
11	$1-x$

end with 2. This means that when we collapsed to 22, either we started with 2312 and eliminated 31 or we started with 2412 and we eliminated 41. It follows that $wt_{2312,4}(22) = 1-x-xz_1(z_3+z_4)t^2$.

Next consider $wt_{2312,4}(33)$. There are only two possible words in $[4]^4$ that reduce to u, namely, $w = 3413$ and $v = 3423$. Since there is no u-match that can start with 1, there cannot a be pair of linked u-matches that start with w. There is a pair of linked u matches that start with v, namely, 342312. However this pair can not be extended. Thus there can be no maximal sequences of linked u-matches that start and end with 3. This means that when we collapsed to 33 either we started with 3413 and eliminated 41 or we started with 3423 and eliminated 42. It follows that $wt_{2312,4}(33) = 1-x-xz_4(z_1+z_2)t^2$.

Finally we consider $wt_{2312,4}(32)$. In this case, the only possible way to have a maximal sequence w of linked u-matches starting with a u-match whose first letter is 3 and ending with a u-match whose last letter is 2 is $w = 342312$. Since in the fixed points of I_{2312}, the sequences in the bricks are weakly increasing, the only way that 32 occurs in the collapse of fixed point (B,w) of I_{2312} is if we started with 342312, which means that a brick ended after 4 and a brick ended after the second 3, and eliminated 4231. Hence $wt_{2312,4}(32) = x^2z_1z_2z_3z_4t^4$.

We list all the weights $wt_{2312,4}(ji)$ in table 3.7.

It follows that for any $v \in [4]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } I_{2312} \\ C(B,w,2312)=v}} sgn(B,w)wt(B,w) =$$

$$- x\overline{z}^v t^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in WDes(v)} wt_{2312,4}(v_s v_{s+1}). \tag{3.69}$$

Here the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. It follows that

$$\begin{aligned} U^{(4)}_{2312}(x,\mathbf{z}_4,t) &= 1 + \sum_{n \geq 1} U^{(4)}_{2312,n}(x,\mathbf{z}_4)t^n \\ &= 1 + \sum_{v \in [4]^+} -x(1-x)^{\text{rise}(v)}\overline{z}^v t^{|v|} \prod_{s \in WDes(v)} wt_{2312,4}(v_s v_{s+1}). \end{aligned} \tag{3.70}$$

Hence we could compute $\mathcal{N}^{(4)}_{2312}(x,\mathbf{z}_4,t) = \dfrac{1}{U^{(4)}_{2312}(x,\mathbf{z}_4,t)}$ if we can compute the right-hand side of (3.70)

What we need to be able to compute the right-hand side of (3.70) is the generating function over all words $v \in \mathbb{P}^*$ where we not only keep track of the weak descents of P but also of type of weak descents of P.

By Theorem 2, we know that

$$\frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)} = 1 + \sum_{w = w_1 \ldots w_n \in \mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i \in WDes(w)} x_{w_i w_{i+1}}. \tag{3.71}$$

Hence

$$\begin{aligned} &\sum_{w = w_1 \ldots w_n \in \mathbb{P}^+} t^{|w|}\overline{z}^w \prod_{i \in WDes(w)} x_{w_i w_{i+1}} \\ &\qquad = \left(\frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)} \right) - 1 \\ &\qquad = \frac{\sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)}{1 - \sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)}. \end{aligned} \tag{3.72}$$

Next suppose that we replace t by yt and x_{ij} by $\frac{x_{ij}}{y}$. Under this substitution the left-hand side in (3.72) becomes

$$\sum_{w=w_1\ldots w_n\in\mathbb{P}^+} t^{|w|}y^{\text{rise}(w)+1}\overline{z}^w \prod_{i\in WDes(w)} x_{w_iw_{i+1}}.$$

Note that for $v = j_1 \geq \cdots \geq j_k$ where $k \geq 2$, our substitution replaces $t^kWDXZ(v)$ by

$$y^kt^kz_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}\left(\frac{x_{j_{i+1}j_i}}{y}-1\right) = yt^kz_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-y).$$

Thus if we let

$$WDXYZ(v) = \begin{cases} z_j & \text{if } v = j, \text{ and} \\ z_{j_1}\cdots z_{j_k}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-y) & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2, \end{cases} \tag{3.73}$$

then we see that the right-hand side of (3.72) becomes

$$\frac{y\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P}^*,|v|=n}WDXYZ(v)}{1-y\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P}^*,|v|=n}WDXYZ(v)}.$$

It follows that

$$-x\sum_{w=w_1\ldots w_n\in\mathbb{P}^+} t^{|w|}y^{\text{rise}(w)}\overline{z}^w \prod_{i\in WDes(w)} x_{w_iw_{i+1}} =$$

$$\frac{-x\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P}^*,|v|=n}WDXYZ(v)}{1-y\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P}^*,|v|=n}WDXYZ(v)}.$$

Thus

$$1-x\sum_{w=w_1\ldots w_n\in\mathbb{P}^+} t^{|w|}y^{rise(w)}\overline{z}^w \prod_{i\in WDes(w)} x_{w_iw_{i+1}} = \frac{1-(x+y)\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P}^*,|v|=n}WDXYZ(v)}{1-y\sum_{n\geq 1}t^n\sum_{v\in WD\mathbb{P},|v|=n}WDXYZ(v)}. \tag{3.74}$$

By setting $z_i = 0$ for $i > k$, we also obtain that

$$1-x\sum_{w=w_1\ldots w_n\in[k]^+} t^{|w|}y^{\text{rise}(w)}\overline{z}^w \prod_{i\in WDes(w)} x_{w_iw_{i+1}} = \frac{1-(x+y)\sum_{n=1}^k t^n\sum_{v\in WD[k]^*,|v|=n}WDXYZ(v)}{1-y\sum_{n=1}^k t^n\sum_{v\in WD[k]^*,|v|=n}WDXYZ(v)}. \tag{3.75}$$

Note that if we replace y by $(1-x)$ and x_{ji} by $wt_u(ji)$, the left-hand side of (3.74) becomes $U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ and the left-hand side of (3.75) becomes $U_u^{(k)}(x, \mathbf{z}_k, t)$. Then using the fact that $\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = 1/U_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, we have the following theorem.

Theorem 16. *Suppose that* $u = u_1 \dots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, *and* u *does not have the* $\mathbb{P}$*-level overlapping property (so it automatically has the* $\mathbb{P}$*-weakly decreasing property). Then*

$$\mathcal{N}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXTZ_u(v)}{1-\sum_{n\geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXTZ_u(v)} \tag{3.76}$$

where

$$WDXTZ_u(v) = \begin{cases} z_j & \text{if } v = j, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1}(wt_u(j_{i+1}j_i) + x - 1) & \text{if } v = j_1 \geq \cdots \geq j_k \end{cases} \tag{3.77}$$

where $k \geq 2$.

If set $z_i = 0$ for all $i > k$, then we obtain the following theorem.

Theorem 17. *Now suppose that* $u = u_1 \dots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{des}(u) = 1$, $u_1 = u_j$, *and* u *does not have the* $[k]$*-level overlapping property (so it automatically has the* $[k]$*-weakly decreasing property). Then*

$$\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1-(1-x)\sum_{n=1}^{k} t^n \sum_{v \in WD[k]^*, |v|=n} WDXTZ_u(v)}{1-\sum_{n=1}^{k} t^n \sum_{v \in WD[k]^*, |v|=n} WDXTZ_u(v)}. \tag{3.78}$$

The key to be able to compute $\mathcal{N}_u^{(k)}(x, \mathbf{z}_k, t)$ in the case of Theorem 17 is to be able to compute $\sum_{n\geq 1} t^n \sum_{v \in WD[k]^*, |v|=n} WDXTZ_u(v)$. This is often complicated because of the large number of weakly decreasing words in $WD[k]^*$, but for certain patterns we can compute it. For example, consider the case where $u = 2312$, $k = 4$, and we set $z_i = 1$ for $i = 1, \dots, 4$. With this substitution, we list the weights $WDXTX_{2312}(ij)$ in table 3.8.

Because $WDXTZ_{2312}(44) = WDXTZ_{2312}(11) = 0$, it follows that the only words that we have to consider are 1, 4, 41 and words in $(\epsilon + 4)(\{2\}^+ + \{3\}^+ +$

Table 3.8: The weights $WDXTZ_{2312}(ij)$ in the case $z_i = 0$ for $i = 1, \ldots, 4$

Weak Descents	$WDXTX_{2312}(ij)$
44	0
$4i$ $(i < 4)$	$x - 1$
33	$-2xt^2$
32	$x - 1 + x^2t^4$
31	$x - 1$
22	$-2xt^2$
21	$x - 1$
11	0

$\{3\}^*32\{2\}^*)(\epsilon + 1)$. It is easy to see that

$$\sum_{n\geq 1} t^n \sum_{v\in\{3\}^+, |v|=n} WDXTZ_u(v) = \frac{t}{1+2xt^3}.$$

That is, the first 3 gives a factor of t and each additional 3 gives a factor of $-2xt^3$. Similarly,

$$\sum_{n\geq 1} t^n \sum_{v\in\{2\}^+, |v|=n} WDXTZ_u(v) = \frac{t}{1+2xt^3}.$$

When considering words in $\{3\}^*32\{2\}^*$, the 32 gives a factor of $(x-1)t^2 + x^2t^6$ and each additional 3 to the left gives a factor of $-2xt^3$ and each additional 2 to the right gives a factor of $-2xt^3$. Thus

$$\sum_{n\geq 1} t^n \sum_{v\in\{3\}^*32\{2\}^*, |v|=n} WDXTZ_u(v) = \frac{(x-1)t^2 + x^2t^6}{(1+2xt^3)^2}.$$

Thus

$$\sum_{n\geq 1} t^n \sum_{v\in\{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*, |v|=n} WDXTZ_u(v) = \frac{2t + 4xt^4 + (x-1)t^2 + x^2t^6}{(1+2xt^3)^2}.$$

Hence if $E = (\epsilon + 4)(\{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*)(\epsilon + 1)$, it follows that

$$\sum_{n\geq 1} t^n \sum_{v\in E, |v|=n} WDXTZ_{2312}(v) = \frac{(1+(x-1)t)^2(2t + 4xt^4 + (x-1)t^2 + x^2t^6)}{(1+2xt^3)^2}$$

since adding a 4 to the left of a word $w \in \{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*$ gives rise to a factor of $(x-1)t$ and adding a 1 to the right of a word $w \in \{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*$ gives rise to a factor of $(x-1)t$. It follows that

$$\sum_{n \geq 1} t^n \sum_{v \in WD[4]^*, |v|=n} WDXTZ_{2312}(v) =$$

$$2t + (x-1)^t + \frac{(1+(x-1)t)^2(2t + 4xt^4 + (x-1)t^2 + x^2t^6)}{(1+2xt^3)^2} = \frac{P(x,t)}{(1+2xt^3)^2}.$$

where

$$\begin{aligned} P(x,t) &= 4t + (-6+6x)t^2 + \left(4 - 8x + 4x^2\right)t^3 + \left(-1 + 15x - 3x^2 + x^3\right)t^4 + \\ &\quad \left(-12x + 12x^2\right)t^5 + \left(4x - 7x^2 + 4x^3\right)t^6 + \\ &\quad \left(6x^2 + 2x^3\right)t^7 + \left(-3x^2 + 2x^3 + x^4\right)t^8. \end{aligned}$$

Thus

$$\mathcal{N}_{2312}^{(4)}(x,1,1,1,1,t) = \frac{1 - (x-1)\frac{P(x,t)}{(1+2xt^3)^2}}{1 - \frac{P(x,t)}{(1+2xt^3)^2}}. \tag{3.79}$$

We have used Mathematica to compute the first few terms in this series:

$$\begin{aligned} &1 + 4xt + 2\left(5x + 3x^2\right)t^2 + 4\left(5x + 10x^2 + x^3\right)t^3 + \\ &\left(35x + 151x^2 + 65x^3 + x^4\right)t^4 + \\ &4\left(14x + 109x^2 + 111x^3 + 14x^4\right)t^5 + \\ &\left(84x + 1068x^2 + 2009x^3 + 716x^4 + 28x^5\right)t^6 + \\ &2\left(60x + 1166x^2 + 3561x^3 + 2535x^4 + 362x^5 + 4x^6\right)t^7 + \\ &\left(165x + 4670x^2 + 21400x^3 + 25650x^4 + 8172x^5 + 486x^6 + x^7\right)t^8 + \cdots. \end{aligned}$$

Chapter 3, in full, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

Chapter 4

Levels: Results when $\text{lev}(u) = 1$

4.1 Introduction

In this Chapter we examine how our results change when we change the statistic $\text{des}(u)$ with $\text{lev}(u)$. We will apply the reciprocal method to obtain filled-labelled-brick tabloids as we did in Chapter 3. However, now we will label levels within bricks with an x instead of descents. We will define a similar involution L_u; however, our fixed points will not have any levels within bricks instead of not having any descents within bricks as in Chapter 3. The collapse map will work as it did in the earlier sections, but our final results will require substituting into a different set of auxiliary generating functions.

Let $\mathbf{z}_k = z_1, \ldots, z_k$ and $\mathbf{z}_\infty = z_1, z_2, \ldots$. Then for any $u \in [k]^j$, we let

$$LEN_{u,n}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, eumch(w)=0} x^{\text{lev}(w)+1} \overline{z}^w \text{ and}$$

$$LEN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, eumch(w)=0} x^{\text{lev}(w)+1} \overline{z}^w.$$

Similarly for $u \in [k]^j$ such that $\text{red}(u) = u$, we let

$$LN_{u,n}^{(k)}(x, \mathbf{z}_k) = \sum_{w \in [k]^n, \text{umch}(w)=0} x^{\text{lev}(w)+1} \overline{z}^w \text{ and}$$

$$LN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \sum_{w \in \mathbb{P}^n, \text{umch}(w)=0} x^{\text{lev}(w)+1} \overline{z}^w.$$

The main goal of this chapter is to study the generating functions

$$\mathcal{LEN}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} LEN_{u,n}^{(k)}(x,\mathbf{z}_k)t^n \text{ and}$$
$$\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} LEN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n,$$

in the case where u is a word with $\text{lev}(u) \leq 1$, and the generating functions

$$\mathcal{LN}_u^{(k)}(x,\mathbf{z}_k,t) = 1 + \sum_{n\geq 1} LN_{u,n}^{(k)}(x,\mathbf{z}_k)t^n \text{ and}$$
$$\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} LN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n,$$

in the case where $\text{red}(u) = u$ and $\text{lev}(u) \leq 1$.

We start by assuming that

$$\begin{aligned}
\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) &= \frac{1}{LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)}, \\
\mathcal{LN}_u^{k)}(x,\mathbf{z}_\infty,t) &= \frac{1}{LU_u^{(k)}(x,\mathbf{z}_\infty,t)}, \\
\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) &= \frac{1}{LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)}, \text{and} \\
\mathcal{LN}_u^{(k)}(x,\mathbf{z}_\infty,t) &= \frac{1}{LEU_u^{(k)}(x,\mathbf{z}_\infty,t)}.
\end{aligned}$$

Fix a word u such that $\text{lev}(u) \leq 1$. Following the ideas of the previous chapter, we shall show how to compute $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$.

We will start out by considering how to compute

$$LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1 + \sum_{n\geq 1} LU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n.$$

in the case where $u \in \mathbb{P}^j$ and $\text{red}(u) = u$. In this case,

$$LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1}{1 + \sum_{n\geq 1} LN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n}. \tag{4.1}$$

Thus if we let $\Theta_u(e_n) = (-1)^n LN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ for $n \geq 1$ and $\Theta_u(e_0) = 1$, we see that

$$\begin{aligned}\Theta_u(H(t)) &= 1 + \sum_{n\geq 1} \Theta_u(h_n) \\ &= \Theta_u\left(\frac{1}{E(-t))}\right) = \frac{1}{1+\sum_{n\geq 1}(-1)^n \Theta_u(e_n)} \\ &= \frac{1}{1+\sum_{n\geq 1} LN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n} = LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t).\end{aligned}$$

Thus it follows that $\Theta_u(h_n) = LU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$.

By (1.24), we have that

$$\begin{aligned}\Theta_u(h_n) &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} B_{\lambda,n}\ \Theta_u(e_\lambda) \\ &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} LN_{u,b_i}^{(\mathbb{P})}(x,\mathbf{z}_\infty) \\ &= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} LN_{u,b_i}^{(\mathbb{P})}(x,\mathbf{z}_\infty) \qquad (4.2)\end{aligned}$$

Our next goal is to give a combinatorial interpretation to the right-hand side of (4.2). Fix a partition λ of n and a λ-brick tabloid $B = (b_1,\ldots,b_{\ell(\lambda)})$. We will interpret $\prod_{i=1}^{\ell(\lambda)} LN_{u,b_i}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ as the number of ways of picking words $(w^{(1)},\ldots,w^{(\ell(\lambda))})$ such that for each i, $w^{(i)} \in \mathbb{P}^{b_i}$ is a word such that $u\text{mch}(w) = 0$ and assigning a weight to this $\ell(\lambda)$-tuple to be $\prod_{i=1}^{\ell(\lambda)} x^{\text{lev}(w^{(i)})+1} \bar{z}^{w^{(i)}}$.

We can then use the pair $\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle$ to construct a filled-labeled-brick tabloid $O_{\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle}$ as follows. First for each brick b_i, we place the word $w^{(i)}$ in the cells of the brick, reading from left to right. Then we label each cell of b_i that starts a level of $w^{(i)}$ with a x and we also label the last cell of b_i with x. This accounts for the factor $x^{\text{lev}(w^{(i)})+1}$. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from x to $-x$. For example, suppose $n = 17$, $u = 3221$, $B = (3,7,4,3)$ $w^{(1)} = 4\ 2\ 2$, $w^{(2)} = 1\ 3\ 2\ 2\ 5\ 7\ 2$, $w^{(3)} = 6\ 6\ 3\ 1$, and $w^{(4)} = 2\ 4\ 7$. Then we have pictured the filled-labeled-brick tabloid $O_{\langle B,(w^{(1)},\ldots,w^{(4)})\rangle}$ constructed from the pair $\langle B,(w^{(1)},\ldots,w^{(4)})\rangle$ in Figure 4.1.

Clearly, we can recover the pair $\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle$ and the labels on the cells from B and the word w which is obtained by reading the elements in

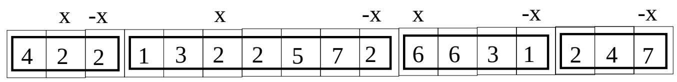

Figure 4.1: The construction of a filled-labeled-brick tabloid.

the cells of $O_{\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle}$ from left to right. Thus we shall specify the filled-labeled-brick tabloid $O_{\langle B,(w^{(1)},\ldots,w^{(\ell(\lambda))})\rangle}$ by (B,w). We let $\mathcal{O}_{u,n}^{(\mathbb{P})}$ denote the set of all filled-labeled-brick tabloids constructed in this way. That is, $\mathcal{O}_{u,n}^{(\mathbb{P})}$ consists of all pairs $O=(B,w)$ where

1. $B=(b_1,\ldots,b_{\ell(\lambda)})$ is brick tabloid of shape (n),
2. $w=w_1\ldots w_n\in\mathbb{P}^n$ such that there is no u-match of σ which is entirely contained in a single brick of B, and
3. If there is a cell c such that a brick b_i contains both cells c and $c+1$ and $w_c=w_{c+1}$, then cell c is labeled with a x and the last cell of any brick is labeled with $-x$.

The sign of O, $\mathrm{sgn}(O)$, is $(-1)^{\ell(\lambda)}$ and the weight of O, $wt(O)$, is $x^{\ell(\lambda)+\mathrm{intlev}(\sigma)}\bar{z}^w$ where $\mathrm{intlev}(w)$ denotes the number of i such that $w_i=w_{i+1}$ and w_i and w_{i+1} lie in the same brick. We shall refer to such i as an *internal level* of O. Note that the labels on O are completely determined by the underlying brick tabloid $B=(b_1,\ldots,b_{\ell(\lambda)})$ and the underlying word w. Thus the filled-labeled-brick tabloid O pictured in Figure 4.1 equals $((3,7,4,3), 4\ 2\ 2\ 1\ 3\ 2\ 2\ 5\ 7\ 2\ 6\ 6\ 3\ 1\ 2\ 4\ 7)$.

It follows that

$$\Theta_u(h_n)=\sum_{O\in\mathcal{O}_{\tau,n}^{(\mathbb{P})}} sgn(O)wt(O). \tag{4.3}$$

Next we define a weight-preserving sign-reversing involution L_u on $\mathcal{O}_{u,n}^{(\mathbb{P})}$. Given an element $O=(B,w)\in\mathcal{O}_{u,n}^{(\mathbb{P})}$ where $B=(b_1,\ldots,b_k)$ and $w=w_1\ldots w_n$, scan the cells of O from left to right looking for the first cell c such that either

(i) c is labeled with a x or

(ii) c is a cell at the end of a brick b_i, $w_c = w_{c+1}$, and there is no u-match of w that lies entirely in the cells of bricks b_i and b_{i+1}.

In case (i), if c is a cell in brick b_j, then we split b_j into two bricks b_j' and b_j'' where b_j' contains all the cells of b_j up to an including cell c and b_j'' consists of the remaining cells of b_j and we change the label on cell c from x to $-x$. In case (ii), we combine the two bricks b_i and b_{i+1} into a single brick b and change the label on cell c from $-x$ to x. For example, consider the element $O \in \mathcal{O}^{(\mathbb{P})}_{3221,17}$ pictured in Figure 4.1. The first place that we can apply the involution is on cell 2 which is labeled with an x so that $L_u(O)$ is the object pictured in Figure 4.2. Finally, if neither case (i) or case (ii) applies, then we define $L_u(O) = O$.

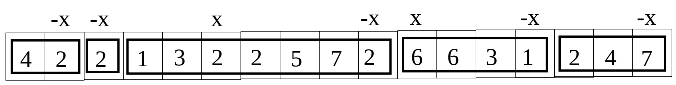

Figure 4.2: $L_u(O)$ for O in Figure 4.1.

We claim that whenever u is a word such that $\text{red}(u) = u$ and $\text{lev}(u) \leq 1$, L_u is an involution, i.e. L_u^2 is the identity. First we consider the case where $\text{lev}(u) = 1$. Now suppose that we are in case (i) where we split a brick b_j at cell c which is labeled with a x. In that case, we let a be the number in cell c and a' be the number in cell $c+1$ which must also be in brick b_j. It must be the case that there is no cell labeled x before cell c since otherwise we would not use cell c to define the involution. However, we have to consider the possibility that when we split b_j into b_j' and b_j'' that we might then be able to combine the brick b_{j-1} with b_j' because the number in that last cell of b_{j-1} is equal to the number in the first cell of b_j' and there is no u-match in the cells of b_{j-1} and b_j'. Since we always take an action on the left most cell possible when defining $L_u(O)$, we know that we cannot combine b_{j-1} and b_j so that there must be a u-match in the cells of b_{j-1} and b_j. Clearly, that u-match must have involved the number a' and the number in cell d which is the last cell in brick b_{j-1}. But that is impossible because then there would be two levels among the numbers between cell d and cell $c+1$ which would violate our assumption that u has only one level. Thus whenever we apply case (i) to define $L_u(O)$, the first action that we can take is to combine bricks b'_j and b''_j so that $L_u^2(O) = O$.

If we are in case (ii), then again we can assume that there are no cells labeled x that occur before cell c. When we combine brick b_i and b_{i+1}, then we will label cell c with a x. It is clear that combining the cells of b_i and b_{i+1} cannot help us combine the resulting brick b with an earlier brick since it will be harder to have no u-matches with the larger brick b. Thus the first place cell c where we can apply the involution will again be cell c which is now labeled with a x so that $L_u^2(O) = O$ if we are in case (ii).

The case where $\text{lev}(u) = 0$ is even easier. Suppose that a is number in the the last cell of b_j and a' is the number in the first cell of b_{j+1} and $a = a'$. Then there can be no u-match of w that is contained in the cells of b_j and b_{j+1} because by our definitions there is no u-match in the cells of b_j and there is no u-match in the cells of b_{j+1} so that the only possible u-match in the cells of b_j and b_{j+1} would have to involve a and a' if $\text{lev}(u) = 0$. It easily follows that we will apply the involution to the first possible cell c which is labled with either x or $-x$ and what ever action we take at cell c to create $L_u(O)$, we will come back to cell c to undo that action to define $L^2(O)$.

It is clear from our definitions that if $L_u(O) \neq O$, then $sgn(O)wt(O) = -sgn(L_u(O))wt(L_u(O))$. Hence, if we let $\mathcal{LO}_{u,n}^{(\mathbb{P})}$ denote the set of all $(B, w) \in \mathcal{O}_{u,n}^{(\mathbb{P})}$ such that $L_u(O) = O$, then (4.3) implies that

$$\Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}} sgn(O)wt(O) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}, L_u(O)=O} sgn(O)wt(O). \tag{4.4}$$

Thus we must examine the fixed points of L_u. So assume that O is a fixed point of L_u. There are two cases to consider.

Case 1. $\text{lev}(u) = 0$.

First of all, there can be no cells which are labeled with x since we can take a possible action to define $L_u(O)$ at such a cell. Similarly there can be no cell c which is at the end of brick b_j such that $w_c = w_{c+1}$ since again we can take a possible action to define $L_u(O)$ at such a cell. This means that w must have no levels within any brick and if c is a cell at then end of brick b_j which is followed by another brick b_{j+1}, then $w_c \neq w_{c+1}$. Thus (B,w) is a fixed point if and only if w is word with no levels and w has no u-match that lies entirely within one of the brick of B. If B has k bricks, then then weight of (B,w) is just $(-x)^k \bar{z}^w$. We let $\mathcal{LIO}_{u,n} = \{(B,w) \in \mathcal{LO}_{u,n}^{(\mathbb{P})} : w_1 \neq w_2 \neq \cdots \neq w_n\}$ denote the set of elements of $\mathcal{LO}_{u,n}^{(\mathbb{P})}$ where w has no levels. Then we have the following lemma. Let $\mathbb{Q}(x, \mathbf{z}_\infty)$ be the set of rational functions in the variables x and $\mathbf{z}_\infty$ over the rationals $\mathbb{Q}$.

Lemma 5. *Suppose that u is a word in $\mathbb{P}^+$ such that* $\text{red}(u) = u$ *and* $\text{lev}(u) = 0$. *Let $\Theta_u : \Lambda \to \mathbb{Q}(x, \mathbf{z}_\infty)$ be the ring homomorphism defined by setting $\Theta_u(e_0) = 1$ and $\Theta_u(e_n) = (-1)^n LN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$LU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \Theta_u(h_n) = \sum_{((b_1,\ldots,b_k),w) \in \mathcal{LIO}_{u,n}} (-x)^k \bar{z}^w. \tag{4.5}$$

Case 2. $\text{lev}(u) = 1$.

First it is easy to see that there can be no cells which are labeled with x so that there are no levels in each brick of O. Second we cannot combine two consecutive bricks b_i and b_{i+1} in O which means that either there is no level between the last cell of b_i and the first cell of b_{i+1} or there is a level between the bricks b_i and b_{i+1}, but there is a u-match in the cells of the bricks b_i and b_{i+1}. Thus we have proved the following.

Lemma 6. *Suppose that $u \in \mathbb{P}^+$,* $\text{red}(u) = u$, *and* $\text{lev}(u) = 1$. *Let $\Theta_u : \Lambda \to \mathbb{Q}(x, \mathbf{z}_\infty)$ be the ring homomorphism defined by setting $\Theta_u(e_0) = 1$ and $\Theta_u(e_n) = (-1)^n LN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$LU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(\mathbb{P})}, L_u(O) = O} sgn(O) wt(O) \tag{4.6}$$

where $\mathcal{O}_{u,n}^{(\mathbb{P})}$ is the set of objects and L_u is the involution defined above. Moreover $O = (B,w)$ where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$ is a fixed point of L_u if and only if it has the following two properties:

1. *there are no cells labeled with x in O, i.e., the elements of w in each brick of O have no levels and*

2. *if b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is no level between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^{i}|b_j|} \neq w_{1+\sum_{j=1}^{i}|b_j|}$, or (b) there is a level between b_i and b_{i+1}, i.e., $w_{\sum_{j=1}^{i}|b_j|} = w_{1+\sum_{j=1}^{i}|b_j|}$, but there is u-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $w_{\sum_{j=1}^{i}|b_j|}$ and $w_{1+\sum_{j=1}^{i}|b_j|}$.*

Clearly, if we restrict to the alphabet $[k]$ instead of $\mathbb{P}$, we will get the same two lemmas except that the words all have to be in $[k]^*$ rather than $\mathbb{P}^*$.

Next we want to consider what happens when we replace u-matches by exact u-matches. We can follow the same steps to interpret $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$. That is,

$$LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1}{1+\sum_{n\geq 1} LEN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n}. \tag{4.7}$$

Thus if we let $\Gamma_u(e_n) = (-1)^n LEN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$ for $n \geq 1$ and $\Gamma_u(e_0) = 1$, we see that

$$\begin{aligned}
\Gamma_u(H(t)) &= 1 + \sum_{n\geq 1} \Gamma_u(h_n) \\
&= \Gamma_u\left(\frac{1}{E(-t)}\right) = \frac{1}{1+\sum_{n\geq 1}(-1)^n \Gamma_u(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1} LEN_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)t^n} = LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t).
\end{aligned}$$

Thus it follows that $\Gamma_u(h_n) = LEU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$.

By (1.24), we have that

$$\begin{aligned}
\Gamma_u(h_n) &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} B_{\lambda,n}\ \Gamma_u(e_\lambda) \\
&= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} LEN_{u,b_i}^{(\mathbb{P})}(x,\mathbf{z}_\infty) \\
&= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} LEN_{u,b_i}^{(\mathbb{P})}(x,\mathbf{z}_\infty)
\end{aligned} \tag{4.8}$$

Again we can give a combinatorial interpretation to the right-hand side of (4.8). Fix a partition λ of n and a λ-brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$. We will interpret $\prod_{i=1}^{\ell(\lambda)} LEN_{u,b_i}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ as the number of ways of picking words $(w^{(1)}, \ldots, w^{(\ell(\lambda))})$ such that for each i, $w^{(i)} \in \mathbb{P}^{b_i}$ is a word such that $eu\text{mch}(w) = 0$ and assigning a weight to this $\ell(\lambda)$-tuple to be $\prod_{i=1}^{\ell(\lambda)} x^{\text{lev}(w^{(i)})+1} \bar{z}^{w^{(i)}}$.

Following the same steps that we did to interpret $\Theta_u(h_n)$, we let $\mathcal{EO}_{u,n}^{(\mathbb{P})}$ denote the set of all filled-labeled-brick tabloids constructed in this way. That is, $\mathcal{EO}_{u,n}^{(\mathbb{P})}$ consists of all pairs $O = (B, w)$ where

1. $B = (b_1, \ldots, b_{\ell(\lambda)})$ is brick tabloid of shape (n),
2. $w = w_1 \ldots w_n \in \mathbb{P}^n$ such that there is no exact u-match of σ which is entirely contained in a single brick of B, and
3. if there is a cell c such that a brick b_i contains both cells c and $c+1$ and $w_c = w_{c+1}$, then cell c is labeled with a x and the last cell of any brick is labeled with $-x$.

The sign of O, $\text{sgn}(O)$, is $(-1)^{\ell(\lambda)}$ and the weight of O, $wt(O)$, is $x^{\ell(\lambda)+\text{intlev}(\sigma)} \bar{z}^w$ where $\text{intlev}(w)$ denotes the number of i such that $w_i = w_{i+1}$ and w_i and w_{i+1} lie in the same brick. Then as before we can conclude

$$\Gamma_u(h_n) = \sum_{O \in \mathcal{EO}_{u,n}^{(\mathbb{P})}} sgn(O) wt(O). \tag{4.9}$$

At this point, we can define an involution K_u exactly as we did for L_u except we replace u-match by exact u-matches in the definitions. This will allow us to prove the following two lemmas.

Lemma 7. *Suppose that u is a word in $\mathbb{P}^+$ such that* $\text{lev}(u) = 0$. *Let $\Gamma_u : \Lambda \to \mathbb{Q}(x)$ be the ring homomorphism defined by setting $\Gamma_u(e_0) = 1$ and $\Gamma_u(e_n) = (-1)^n LEN_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ for $n \geq 1$. Then*

$$LEU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty) = \theta_u(h_n) = \sum_{((b_1,\ldots,b_k),w) \in \mathcal{IEO}_{u,n}} (-x)^k \bar{z}^w \tag{4.10}$$

where $\mathcal{IEO}_{u,n}$ is the set of all $(B, w) \in \mathcal{EO}_{u,n}$ such that $K_u(B, w) = (B, w)$ and w has no levels.

Lemma 8. *Suppose that* $u \in \mathbb{P}^+$ *and* $\mathrm{lev}(u) = 1$*. Let* $\Gamma_u : \Lambda \to \mathbb{Q}(x)$ *be the ring homomorphism defined by setting* $\Gamma_u(e_0) = 1$ *and* $\Gamma_u(e_n) = (-1)^n LEN^{(\mathbb{P})}_{u,n}(x, \mathbf{z}_\infty)$ *for* $n \geq 1$*. Then*

$$LEU^{(\mathbb{P})}_{u,n}(x, \mathbf{z}_\infty) = \Gamma_u(h_n) = \sum_{O \in \mathcal{EO}^{(\mathbb{P})}_{u,n}, K_u(O)=O} sgn(O)wt(O) \tag{4.11}$$

where $\mathcal{EO}^{(\mathbb{P})}_{u,n}$ *is the set of objects and* K_u *is the involution defined above. Moreover* $O = (B, w)$ *where* $B = (b_1, \ldots, b_k)$ *and* $w = w_1 \ldots w_n$ *is a fixed point of* K_u *if and only if it has the following two properties:*

1. *there are no cells labeled with* x *in* O*, i.e., the elements of* w *in each brick of* O *have no levels and*

2. *if* b_i *and* b_{i+1} *are two consecutive bricks in* O*, then either (a) there is no level between* b_i *and* b_{i+1}*, i.e.,* $w_{\sum_{j=1}^{i} |b_j|} \neq w_{1+\sum_{j=1}^{i} |b_j|}$*, or (b) there is a level between* b_i *and* b_{i+1}*, i.e.,* $w_{\sum_{j=1}^{i} |b_j|} = w_{1+\sum_{j=1}^{i} |b_j|}$*, but there is an exact* u*-match contained in the elements of the cells of* b_i *and* b_{i+1} *which must necessarily involve* $w_{\sum_{j=1}^{i} |b_j|}$ *and* $w_{1+\sum_{j=1}^{i} |b_j|}$*.*

4.2 The case $u = u_1 \ldots u_j$, $\mathrm{lev}(u) = 1$, and $u_1 > u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{LN}^{(\mathbb{P})}_u(x, \mathbf{z}_\infty, t)$, $\mathcal{LN}^{(k)}_u(x, \mathbf{z}_k, t)$, $\mathcal{LEN}^{(\mathbb{P})}_u(x, \mathbf{z}_\infty, t)$, and $\mathcal{LEN}^{(k)}_u(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\mathrm{lev}(u) = 1$, $u_1 > u_j$, and u has the $\mathbb{P}$-weakly decreasing property (or $[k]$-weakly decreasing property).

First assume that $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, $u_1 > u_j$, and u has the $\mathbb{P}$-minimal overlapping property. As before, we define the *collapse map* which maps fixed points of J_u or K_u to a certain subset of words in $\mathbb{P}^*$. This is best explained through an example. Suppose that $u = 2231$ and we want to compute $LU^{(7)}_{2231}(x, \mathbf{z}_7, t)$. By (4.6), we know that

$$LU^{(k)}_{u,n}(x, \mathbf{z}_k) = \sum_{O \in \mathcal{O}^{(k)}_{u,n}, L_u(O)=O} sgn(O)wt(O). \tag{4.12}$$

Now suppose that we are given a fixed point (B, w) of L_u where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$ such as the one pictured in Figure 4.3. We know that to be a fixed point of L_u, w must have no levels within bricks of B and that for any $i < k$, if c is last cell in brick b_i and $w_c = w_{c+1}$, then there must be a u-match in w which is contained in the cells of b_i and b_{i+1}. In our particular example, since $u = 2231$ has a single level, this match must involve the last cell of b_i and the first three cells of b_{i+1}. In Figure 4.3, we have indicated the two such matches in our example by placing stars below the cells in the 2231-matches. In this case the collapse map just maps (B, w) to the word $v = C(B, w, u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 4.3 where again we have starred the elements in $C(B, w, u)$ that remain from the original 2231-matches in w. What makes the case where u has the minimal overlapping property easier is that, since any two consecutive u-matches can share at most letter, there is no possibility that an end point of a u-match in w occurs in the middle of another u-match in w so that the letters that we remove from w for any pair of u-matches are disjoint from each other.

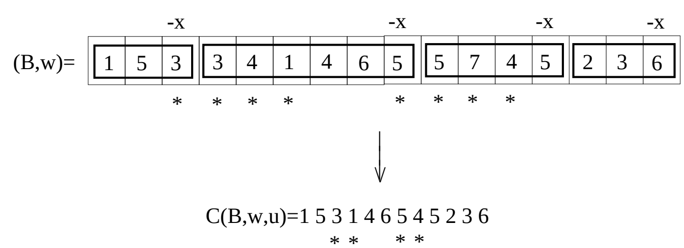

Figure 4.3: A fixed point of L_{2231}.

The question that we want to ask ourselves is given a $v = v_1 \ldots v_s$, how can we construct all the fixed points of (B, w) of L_u such that $C(B, w, v)$ is equal to v. First, it is easy to see that v has no levels. If $v_s < v_{s+1}$, then we have two choices. That is, either cell s was the end of a brick or cell s was an internal cell of a brick. This implies that each rise in v contributes a factor of $(1 - x)$ since if s is at the end of a brick, there is a weight of $-x$ associated with the last cell of

Table 4.1: The weights $wt_{2231,7}(ji)$

Descents	$wt_{2231,7}(ji)$
$7i$ $(i<7)$	$1-x$
$6i$ $(i<6)$	$1-x-xz_6z_7t^2$
$5i$ $(i<5)$	$1-x-xz_5(z_6+z_7)t^2$
$4i$ $(i<4)$	$1-x-xz_4(z_5+z_6+z_7)t^2$
$3i$ $(i<3)$	$1-x-xz_3(z_4+z_5+z_6+z_7)t^2$
21	$1-x-xz_2(z_3+z_4+z_5+z_6+z_7)t^2$

a brick. If $v_s > v_{s+1}$, then we have three choices. Either cell s was the end of a brick, cell s was an internal cell of a brick, or this descent came from a 2231-match that straddled two bricks in B. Thus if $v_s > v_{s+1}$, then v_s must have played the role of 2 in the original 2231-match and v_{s+1} must have played the role of 1 in the original 2231-match. If we consider the first descent 31 in the $C(B,w,u)$ of Figure 4.3, then we see there are many ways that we could add the two middle letters. That is, the original 2231-match could have been any $33c1$ where $c \in \{4,5,6,7\}$. It follows that the extra weight from these possibilities that is not included in $\bar{z}^{C(B,w,u)}t^{|C(B,w,u)|}$ in this case would be $-xt^2\sum_{4\leq c\leq 7} z_3z_c$. Here the $-x$ comes from the fact that we know that the original match straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. In this way, we can associate a weight with each rise or descent of v which will allow us to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

In our case where $u = 2231$ and $k = 7$, the weights associated with the descents are given in table 4.1.

However, if $u = 2231$ and we want to compute $LU^{(\mathbb{P})}_{u,n}(x,\mathbf{z}_\infty)$, the weights for any descent ji would be $-x\sum_{j<c} z_jz_c$ which is an infinite sum.

Going back to our example where $u = 2231$ and $k = 7$, it follows that for any $v \in [7]^+$ with no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w) =$$

$$- x(1-x)^{\text{rise}(v)} z_v t^{|v|} \prod_{s \in Des(v)} wt_{2231,7}(v_s v_{s+1}). \quad (4.13)$$

Here the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. But then we know that

$$\begin{aligned} LU_{2231}^{(7)}(x, \mathbf{z}_7, t) &= 1 + \sum_{n \geq 1} LU_{2231,n}^{(7)}(x, \mathbf{z}_7) t^n \\ &= 1 + \sum_{\substack{v \in [7]^+ \\ \text{lev}(v)=0}} -x(1-x)^{\text{rise}(v)} z_v t^{|v|} \prod_{s \in Des(v)} wt_{2231,7}(v_s v_{s+1}). \quad (4.14) \end{aligned}$$

Hence we could compute $\mathcal{LN}_{2231,n}^{(7)}(x, \mathbf{z}_7, t) = \frac{1}{LU_{2231,n}^{(7)}(x, \mathbf{z}_7, t)}$ if we can compute the right-hand side of (4.14).

The case of exact matches is even simpler. In that case, we want to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } K_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

Going back to our example of $u = 2231$ over the alphabet [7], if $v_s < v_{s+1}$ then either v_s and v_{s+1} were internal to a brick or v_s was at the end of a brick. This implies each rise in v contributes a factor of $1-x$. We see the only descents that appear in a word $v = C(B,w,u)$ could appear in three ways. If $v_s > v_{s+1}$, then v_s could be internal to a brick, v_s could be at the end of a brick, or v_s and v_{s+1} could have been part of an exact 2231-match that straddled two bricks in B. In the last scenario, it must be the case that $v_s = 2$, $v_{s+1} = 1$ and we must have eliminated a 2 and 3 from w. Thus we want to compute $LEU_{2231,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$ or $LEU_{2231,n}^{(k)}(x, \mathbf{z}_k)$ for $k \geq 4$, the weights would be the following. It follows that for any $v \in [7]^+$ with no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } K_{2231} \\ C(B,w,2231)=v}} sgn(B,w)wt(B,w) =$$

Table 4.2: The weights $ewt_{2231}(ji)$

Descents	weight $ewt_{2231,\mathbb{P}}(ji)$
ji where either $j \neq 2$ or $i \neq 1$	$1 - x$
21	$1 - x - xz_2z_3t^2$

$$- xz_v(1-x)^{\text{rise}(v)} \prod_{s \in Des(v)} ewt_{2231,7}(v_s v_{s+1}). \quad (4.15)$$

and

$$\begin{aligned} LEU^{(7)}_{2231,n}(x, \mathbf{z}_7, t) &= 1 + \sum_{n \geq 1} LEU^{(7)}_{2231,n}(x, \mathbf{z}_7)t^n \\ &= 1 + \sum_{\substack{v \in [7]^+ \\ \text{lev}(v)=0}} -xz_v(1-x)^{\text{rise}(v)} \prod_{s \in Des(v)} ewt_{2231,7}(v_s v_{s+1}). \quad (4.16) \end{aligned}$$

When u does not have the minimal overlapping property but u has the $\mathbb{P}$-weakly decreasing (or $[k]$-weakly decreasing property), we can obtain similar results but the collapse maps and the weight $wt_u(ji)$ are more complicated. Again this is best explained through an example. Suppose that $u = 3221$ and $k = 8$.

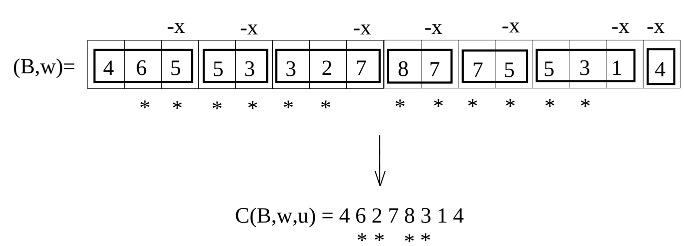

Figure 4.4: A fixed point of L_{3221}.

When u does not have the $\mathbb{P}$-minimal overlapping property, then we can have a situation such as the one pictured in Figure 4.4. If we look at the descents between bricks 1 and 2 which correspond to the u-match 6553, we see that we would like to eliminate the 5 and 5. However, this u-match overlaps the u-match associated with the descent between bricks 2 and 3 which is 5332. Thus we would also like to eliminate the 3 and 3. We will say that two such matches are *linked* if one of the end points of first match is one of middle elements of the second match. Depending on the pattern we could have a series of u-matches in a fixed point of (B, w) which are linked. In such a situation, the collapse map will eliminate all the symbols except for the first element of the first match and last element of the last match in a maximal sequence of linked u-matches. This is illustrated in Figure 4.4 where we have two maximal blocks of linked 3221-matches. Thus in the linked 3221-matches in cells 2 through 7, we keep only the 6 and the 2 and in the linked matches in cells 9 through 14, we keep only the 8 and the 3. Because we are assuming that $u_1 > u_j$, we know that maximal blocks of linked u-matches must be finite since the end point of such matches must strictly decrease. When we see a descent ji in a word $C(B, w, u)$, the weight associated with such a decent is now more complicated. For example, in our case where $u = 3221$ and $k = 8$, a decent of the form 73 can correspond to a single 3221-match which would have to be of the form $7aa3$ where $7 > a > 3$, it could correspond to a maximum block with 2 linked 3221-matches in which case it must be of the form $7ccdd3$ where $3 < d < c < 7$, or it could correspond to a maximum block with 3 linked 3221-matches in which case it must be 76655443. Thus

$$wt_{3221}(73) = 1-x-xt^2\left(\sum_{3<a<7} z_a\right)+x^2t^4\left(\sum_{3<d<c<7}(z_c)^2(z_d)^2\right)-x^3t^6(z_6)^2(z_5)^2(z_4)^2$$

On the other hand a descent of the form ji where $j - i = 2$ can only correspond to single 3221-match so that $wt_{3221}(ji) = 1 - x - xt^2(z_{i+1})^2$ since $i + 1$ plays the role of the 2 in the 3221-match. Finally, a descent of the form ji where $j - i = 1$ can not correspond to a 3221-match, so cell j is either internal to a brick or at the end of a brick. Then, $wt_{3221}(ji) = 1 - x$ if $j - i = 1$.

We give the weights associated with the descents for $u = 3221$ and $k = 5$ in table 4.3. Notice that the weights quickly grow complicated which is why we have chosen to list them for $k = 5$ rather than the example we are considering with $k = 8$.

It follows that for any $v \in [8]^+$ with no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_{3221} \\ C(B,w,3221)=v}} sgn(B, w)wt_{3221}(B, w) =$$

Table 4.3: The weights $wt_{3221,5}(ji)$

Descents	$wt_{3221,5}(ji)$
ji $(j > i)$ & $j - i = 1$	$1 - x$
ji $(j > i)$ & $j - i = 2$	$1 - x - xt^2 z_{i+1}$
ji $(j > i)$ & $j - i = 3$	$1 - x - xt^2((z_{i+1})^2 + (z_{i+2})^2) + x^2t^4(z_{i+1})^2(z_{i+2})^2$
ji $(j > i)$ & $j - i = 4$	$1 - x - xt^2(\sum_{i<s<j}(z_s)^2) + x^2t^4(\sum_{i<a<b<j}(z_a)^2(z_b)^2)$ $- x^3t^6(\prod_{i<s<j}(z_s)^2)$

$$- xz_v(1-x)^{\text{rise}(v)}t^{|v|} \prod_{s \in Des(v)} wt_{3221,8}(v_s v_{s+1}). \tag{4.17}$$

and

$$\begin{aligned} LU^{(8)}_{3221,n}(x, \mathbf{z}_8, t) &= 1 + \sum_{n \geq 1} LU^{(8)}_{3221,n}(x, \mathbf{z}_8) t^n \\ &= 1 + \sum_{\substack{v \in [8]^+ \\ \text{lev}(v)=0}} -xz_v(1-x)^{\text{rise}(v)}t^{|v|} \prod_{s \in Des(v)} wt_{3221,8}(v_s v_{s+1}). \quad (4.18) \end{aligned}$$

What we need to be able to compute the right-hand sides of either (4.14), (4.16), or (4.18) is the generating function over all words $v \in \mathbb{P}^*$ with no levels where we not only keep track of the descents of P but also the type of descents of P. We do this by substituting into an auxiliary generating function. This is the following:

By Theorem 5, we know that

$$1 + \sum_{w \in \mathbb{P}^+, \text{lev}(w)=0} t^{|w|} \bar{z}^w \prod_{i<j} x_{ji}^{\text{ji}(w)} = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} AXZ(S)}$$

where

$$AXZ(S) = \begin{cases} \frac{z_j}{1+z_j t} & \text{if } S = \{j\}, \text{ and} \\ \frac{z_{j_1}}{1+z_{j_1} t} \cdots \frac{z_{j_k}}{1+z_{j_k} t} \prod_{i=1}^{k-1}(x_{j_{i+1} j_i} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.19}$$

where $k \geq 2$.

Hence

$$\sum_{w\in\mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|}\bar{z}^w \prod_{i<j} x_{ji}^{\mathrm{ji}(w)} = \frac{1}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} AXZ(S)} - 1$$
$$= \frac{\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} AXZ(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} AXZ(S)} \tag{4.20}$$

If we replace t by ty and x_{ji} by $\frac{x_{ji}}{y}$, the left-hand side of (4.20) becomes

$$\sum_{w\in\mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|}y^{\mathrm{rise}(w)+1}\bar{z}^w \prod_{i<j} x_{ji}^{\mathrm{ji}(w)}$$

Note that for $S=\{j_1<\cdots<j_k\}$ where $k\geq 2$, our substitution replaces $t^k AXZ(S)$ by

$$y^k t^k \frac{z_{j_1}}{1+z_{j_1}ty}\cdots\frac{z_{j_k}}{1+z_{j_k}ty}\prod_{i=1}^{k-1}\left(\frac{x_{j_{i+1}j_i}}{y}-1\right) =$$
$$yt^k \frac{z_{j_1}}{1+z_{j_1}ty}\cdots\frac{z_{j_k}}{1+z_{j_k}ty}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-y)$$

Thus if we let

$$BXZ(S) = \begin{cases} \frac{z_j}{1+z_j ty} & \text{if } S=\{j\}, \text{ and} \\ \frac{z_{j_1}}{1+z_{j_1}ty}\cdots\frac{z_{j_k}}{1+z_{j_k}ty}\prod_{i=1}^{k-1}(x_{j_{i+1}j_i}-y) & \text{if } S=\{j_1<\cdots<j_k\} \end{cases} \tag{4.21}$$

where $k\geq 2$, then the right-hand side of (4.20) becomes

$$\frac{y\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}. \tag{4.22}$$

It follows that

$$-x\sum_{w\in\mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|}y^{\mathrm{rise}(w)}\bar{z}^w \prod_{i<j} x_{ji}^{\mathrm{ji}(w)} = \frac{-x\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}. \tag{4.23}$$

Thus

$$1-x \sum_{w\in\mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{rise}(w)} \bar{z}^w \prod_{i<j} x_{ji}^{\mathrm{ji}(w)} = \frac{1-(x+y)\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}{1-y\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BXZ(S)}. \tag{4.24}$$

By setting $z_i = 0$ for $i > k$, we also obtain that

$$1-x \sum_{w\in[k]^+,\mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{rise}(w)} \bar{z}^w \prod_{i<j} x_{ji}^{\mathrm{ji}(w)} = \frac{1-(x+y)\sum_{n=1}^{k} t^n \sum_{S\subseteq[k],|S|=n} BXZ(S)}{1-y\sum_{n=1}^{k} t^n \sum_{S\subseteq[k],|S|=n} BXZ(S)}. \tag{4.25}$$

Note that if we replace y by $(1-x)$ and x_{ji} by $wt_u(ji)$, the left-hand side of (4.24) becomes $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (4.25) becomes $LU_u^{(k)}(x,\mathbf{z}_k,t)$. Similarly, if we replace y by $(1-x)$ and x_{ji} by $ewt_u(ji)$, the left-hand side of (4.24) becomes $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (4.25) becomes $LEU_u^{(k)}(x,\mathbf{z}_k,t)$. Then using the fact that $\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and that $\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, we have the following theorem.

Theorem 18. *Suppose that* $u = u_1 \dots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, *and* $u_1 > u_j$. *Then*

$$\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BRZ_u(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} BRZ_u(S)} \tag{4.26}$$

and

$$\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} EBRZ_u(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq\mathbb{P},|S|=n} EBRZ_u(S)} \tag{4.27}$$

where

$$BRZ_u(S) = \begin{cases} \frac{z_j}{1+z_j t(1-x)} & \text{if } S = \{j\}, \text{ and} \\ \dfrac{z_{j_1}}{1+z_{j_1}t(1-x)} \cdots \dfrac{z_{j_k}}{1+z_{j_k}t(1-x)} & \\ \prod_{i=1}^{k-1}(wt_u(j_{i+1}j_i)+x-1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.28}$$

where $k \geq 2$ *and*

$$EBRZ_u(S) = \begin{cases} \frac{z_j}{1+z_j t(1-x)} & \text{if } S = \{j\}, \text{ and} \\ \dfrac{z_{j_1}}{1+z_{j_1}t(1-x)} \cdots \dfrac{z_{j_k}}{1+z_{j_k}t(1-x)} & \\ \displaystyle\prod_{i=1}^{k-1}(ewt_u(j_{i+1}j_i)+x-1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.29}$$

where $k \geq 2$.

If we specialize the variables so that $z_i = 0$ for all $i > k$, then we have the following theorem.

Theorem 19. *Suppose that* $u = u_1 \ldots u_j \in [k]^*$, $\text{red}(u) = u$, $\text{lev}(u) = 1$, *and* $u_1 > u_j$. *Then*

$$\mathcal{LN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1-(1-x)\sum_{n=1}^k t^n \sum_{S\subseteq[k],|S|=n} BRZ_u(S)}{1-\sum_{n=1}^k t^n \sum_{S\subseteq[k],|S|=n} BRZ_u(S)} \tag{4.30}$$

and

$$\mathcal{LEN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1-(1-x)\sum_{n\geq 1} t^n \sum_{S\subseteq[k],|S|=n} EBRZ_u(S)}{1-\sum_{n\geq 1} t^n \sum_{S\subseteq[k],|S|=n} EBRZ_u(S)} \tag{4.31}$$

We end this section by computing one example. Suppose that $u = 44321$ and $k = 7$ and set $z_j = 1$ for $j = 1, \ldots, 7$. Note that in this case $wt_{u,7}(ji)$ depends only on $j - i$. That is, under the collapse map, a descent ji that is the result of a collapse must have come from a sequence $jjabi$ where $j > a > b > i$. It is then easy to see that we obtain table 4.4 for $wt_{u,7}(ji)$ where $7 \geq j > i \geq 1$. Then we list the weights $wt_{44321,7}(ji) + x - 1$ in table 4.5.

To compute $\mathcal{LN}_{44321}^{(7)}(x, 1, 1, 1, 1, 1, 1, 1, t)$, we must compute the polynomials

$$P_n(x,t) = \sum_{S\subseteq[7],|S|=n} BRZ_{44321}(S)$$

when $z_i = 1$ for all i. Now if $S = \{j_1 < \cdots j_n\}$ and $j_{i+1} - j_i \leq 2$ for some $1 \leq i \leq n-1$, then we know that $BRZ_{44321}(S) = 0$. It is easy to see that if $|S| \geq 4$,

Table 4.4: The weights $wt_{u,7}(ji)$

$wt_{44321,7}(ji)$	Descent condition
$1-x$	if $\lvert j-i\rvert \leq 2$
$1-x-xt^3$	if $\lvert j-i\rvert = 3$
$1-x-\binom{3}{2}xt^3$	if $\lvert j-i\rvert = 4$
$1-x-\binom{4}{2}xt^3$	if $\lvert j-i\rvert = 5$
$1-x-\binom{5}{2}xt^3$	if $\lvert j-i\rvert = 6$

Table 4.5: The weights $wt_{44321,7}(ji)+x-1$

$wt_{44321,7}(ji)+x-1$	Descent condition
0	if $\lvert j-i\rvert \leq 2$
$-xt^3$	if $\lvert j-i\rvert = 3$
$-3xt^3$	if $\lvert j-i\rvert = 4$
$-6xt^3$	if $\lvert j-i\rvert = 5$
$-10xt^3$	if $\lvert j-i\rvert = 6$

there will always be such an i. $P_4(x,t) = P_5(x,t) = P_6(x,t) = P_7(x,t) = 0$. The only set of size 3 that does not have such an i is $S = \{1,4,7\}$. For this set

$$\begin{aligned} BRZ(S) &= \frac{1}{(1+t(1-x))^3}(wt_{44321,7}(74)+x-1)(wt_{44321,7}(41)+x-1) \\ &= \frac{1x^2t^6}{(1+t(1-x))^3} \end{aligned}$$

so that

$$P_3(x,t) = \frac{x^2t^6}{(1+t(1-x))^3}.$$

The only sets of size 2 that do not have such an i are the sets $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{1,7\}$, $\{2,5\}$, $\{2,6\}$, $\{2,7\}$, $\{3,6\}$, $\{3,7\}$, and $\{4,7\}$ so that

$$P_2(x,t) = \frac{-35t^3}{(1+t(1-x))^2}.$$

Finally, the contribution from the sets of size 1 gives that

$$P_1(x,t) = \frac{7}{1+t(1-x)}.$$

. Thus

$$\mathcal{LN}^{(7)}_{44321}(x,1,1,1,1,1,1,1,t) = \frac{1-(1-x)\sum_{k=1}^{3} t^k P_k(x,t)}{1-\sum_{k=1}^{3} t^k P_k(x,t)}.$$

We used this formula to compute the first terms of the series $\mathcal{LN}^{(7)}_{44321}(x,1,1,1,1,1,1,1,t)$:

$$\begin{aligned} &1 + 7xt + (42x+7x^2)t^2 + (252x+84x^2+7x^3) + \\ &(1512+756x^2+126x^{+}7x^4)t^4 + \\ &(9072x+6013x^2+1512x^3+168x^4+7x^5)t^5 + \\ &(54432x+44940x^2+15050x^3+2520x^4+210x^5+7x^6)t^6 + \\ &(326592x+322812x^2+134820x^3+30135x^4+3780x^5+252x^6+7x^7)t^7 + \cdots \end{aligned}$$

We note that if one wanted to compute the same generating function using the matrix inversion method described in the introduction, one would have to invert a $7^4 \times 7^4$ matrix in the variables x and t which is infeasible to even write down much less compute.

4.3 The case $u = u_1 \ldots u_j$, $\text{lev}(u) = 1$, and $u_1 < u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $\mathcal{LN}_u^{(k)}(x, \mathbf{z}_k, t)$, $\mathcal{LEN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $\mathcal{LEN}_u^{(k)}(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\text{lev}(u) = 1$, $u_1 < u_j$, and u has the $\mathbb{P}$-weakly increasing property (or $[k]$-weakly increasing property).

This case is similar to the case where $u = u_1 \ldots u_j$, $\text{lev}(u) = 1$, and $u_1 > u_j$. Again the simplest case is when u has the $\mathbb{P}$-minimal overlapping property. For example, suppose that $u = 21334$ and we want to compute $LU_{21334}^{(8)}(x, \mathbf{z}_8, t)$. Then consider the following figure:

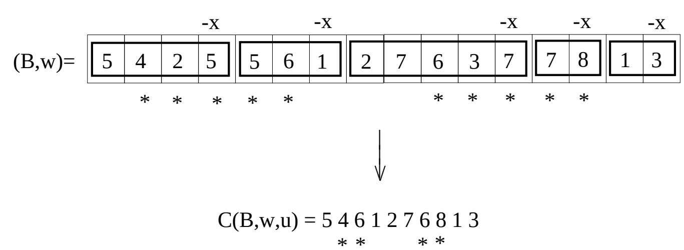

Figure 4.5: A fixed point of L_{21334}.

If we are given a fixed point (B, w) of L_u where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$ such as the one pictured in Figure 4.5. We know that to be a fixed point of L_u, w must be have no levels within bricks of B and that for any $i < k$, if c is last cell in brick b_i and $w_c = w_{c+1}$, then there must be a u-match in w which is contained in the cells of b_i and b_{i+1}. In our particular example, since $u = 21334$ has a single level, this match must involve the last three cells of b_i and the first two cells of b_{i+1}. In Figure 4.5, we have indicated the two such matches in our example by placing stars below the cells in the 21334-matches. In this case, the collapse map just maps (B, w) to the word $v = C(B, w, u)$ which is the result of starting with w and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 4.5 where again we have starred the elements in $C(B, w, u)$ that remain from the original 21334-matches in w. In this case, the resulting word $C(B, w, u)$ must have no levels.

As in the previous section, we want to construct the set of fixed points of (B,w) of L_u such that $C(B,w,v)$ is equal to a given word $v = v_1 \dots v_n$ where $v_1 \neq \cdots \neq v_n$.

When we see a descent ji in a word $C(B,w,u)$, its associated weight is $1-x$ since j may have been internal to a brick or at the end of a brick. If we see a rise ji in a word $C(B,w,u)$, then there are three possibilities: j may have been internal to a brick, j may have been at the end of a brick, or there may have been a 21334-match straddling the bricks with j and i. For example, in our case where $u = 21334$ and $k = 8$ if we see a rise of the form 47, it could correspond to a 21334-match of the form $4abb7$ where $1 \leq a < 4$ and $4 < b < 7$. Thus

$$wt_{21334}(47) = 1 - x - xt^3(z_1 + z_2 + z_3)(z_5^2 + z_6^2)$$

There are some rises that can not have come from 21334-matches. For example, rises of the form $1a$ where $1 < a$ can not have come from a 21334-match since there is no number that could take the role of the 1 in the 21334-match. Thus if $1 < a$, then $wt_{21334}(1a) = 1-x$. Also rises of the form ji where $i - j = 1$ cannot have come from a 21334-match since there would be no number that could take the role of the 3 in the 21334-match. Thus if $i-j=1$, then $wt_{21334}(ji) = 1-x$. In this way, we can associate a weight with each descent or rise of v which will allow us to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w)$$

In our case where $u = 21334$ and $k = 8$, the weights associated with the rises are given in table 4.6:

If $u = 21334$ and we want to compute $LU_{u,n}^{(\mathbb{P})}(x, \mathbf{z}_\infty)$, the weights for any rise ij where $i+1 < j$ would be $1 - x - xt^3(\sum_{s<i} z_s)(\sum_{i<d<j} z_d^2)$.

It follows that for any $v \in [8]^+$ with no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_{21334} \\ C(B,w,21334)=v}} sgn(B,w)wt_{21334}(B,w) =$$

$$-xz_v(1-x)^{\text{des}(v)}t^{|v|} \prod_{s \in Rise(v)} wt_{21334,8}(v_s v_{s+1}). \tag{4.32}$$

Table 4.6: The weights $wt_{21334,8}(ij)$

Rises	$wt_{21334,8}(ij)$
68	$1-x-xt^3(\sum_{s<6} z_s)(z_7)^2$
$5j$ $(j \geq 7)$	$1-x-xt^3(\sum_{s<5} z_s)(\sum_{5<t<j}(z_j)^2)$
$4j$ $(j \geq 6)$	$1-x-xt^3(\sum_{s<4} z_s)(\sum_{4<t<j}(z_j)^2)$
$3j$ $(j \geq 5)$	$1-x-xt^3(\sum_{s<3} z_s)(\sum_{3<t<j}(z_j)^2)$
$2j$ $(j \geq 4)$	$1-x-xt^3 z_1(\sum_{2<t<j}(z_j)^2)$
$1j$ $(j \geq 3)$ or $j(j+1)$	$1-x$

As in the previous section, the initial $-x$ comes from the fact that the last cell of (B,w) always contributes a $-x$ since the last cell is at the end of a brick. But then we know that

$$\begin{aligned} LU^{(8)}_{21334,n}(x,\mathbf{z}_8,t) &= 1+\sum_{n\geq 1} LU^{(8)}_{21334,n}(x,\mathbf{z}_8)t^n \\ &= 1+\sum_{\substack{v\in[8]^+ \\ \mathrm{lev}(v)=0}} -xz_v(1-x)^{\mathrm{des}(v)}t^{|v|}\prod_{s\in Rise(v)} wt_{21334,8}(v_s v_{s+1}). \end{aligned} \tag{4.33}$$

Hence we could compute $\mathcal{LN}^{(8)}_{21334}(x,\mathbf{z}_8,t)=\dfrac{1}{LU^{(8)}_{21334,n}(x,\mathbf{z}_8,t)}$ if we can compute the right-hand side of (4.33)

As in the previous section, the case of exact matches is much simpler. In that case, we want to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } K_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

Going back to our example of $u=21334$ over the alphabet [8], we see that the weight associated to a descent is $1-x$ since v_s could either be internal to a brick which contributes a factor of 1 or at the end of a brick which contributes a factor of $-x$. The weight associated to a rise $v_s<v_{s+1}$ is $1-x$ unless $v_s=2$, $v_{s+1}=4$. If $v_s=2$, $v_{s+1}=4$, then there are 3 possibilities: 2 and 4 were internal to a brick, 2 was the last cell of a brick and 4 was the first cell of the next brick, or 2 and

Table 4.7: The weights $ewt_{12433}(ij)$

Rise	weight $ewt_{21334,\mathbb{P}}(ij)$
ij where either $i \neq 2$ or $j \neq 4$	$1-x$
24	$1-x-xz_1z_3^2t^3$

4 straddled two bricks and there was an exact 21334-match between those two bricks. In the last case, we must have eliminated a 133 from w. Thus if we want to compute $LEU^{(\mathbb{P})}_{21334,n}(x,\mathbf{z}_\infty)$ or $LEU^{(k)}_{21334,n}(x,\mathbf{z}_k)$ for $k \geq 4$, the weights associated to rises are given in table 4.7. It follows that for any $v \in [8]^+$ with no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } K_{21334} \\ C(B,w,21334)=v}} sgn(B,w)wt(B,w) =$$

$$-x\overline{z}^v(1-x)^{\mathrm{des}(v)}t^{|v|} \prod_{s \in Rise(v)} ewt_{21334,8}(v_s v_{s+1}) \tag{4.34}$$

and

$$\begin{aligned} LEU^{(8)}_{21334,n}(x,\mathbf{z}_8,t) &= 1 + \sum_{n\geq 1} LEU^{(8)}_{21334,n}(x,\mathbf{z}_8)t^n \\ &= 1 + \sum_{\substack{v\in[8]^+ \\ \mathrm{lev}(v)=0}} -x\overline{z}^v(1-x)^{\mathrm{des}(v)}t^{|v|} \prod_{s\in Rise(v)} ewt_{21334,8}(v_s v_{s+1}). \end{aligned} \tag{4.35}$$

When u does not have the $\mathbb{P}$-minimal overlapping property but u has the $\mathbb{P}$-weakly increasing (or $[k]$-weakly increasing) property, we can obtain similar results but the collapse maps and the weight function $wt_u(ij)$ are more complicated. As we saw in the previous section, we must pay attention to overlapping u-matches that share more than one letter.

We will consider the example where $u = 123345$ and $k = 9$. In this case, u-matches can overlap in either one, two, or three letters. As in the previous section, the collapse map will keep only the first and last letters of a consecutive sequence of u-matches such that each consecutive pair share at least two letters. For example, at the top of Figure 4.6, we have given an example where two consecutive u-matches share 3 letters and at the bottom of Figure 4.6, we have given an example where two consecutive u-matches share 2 letters.

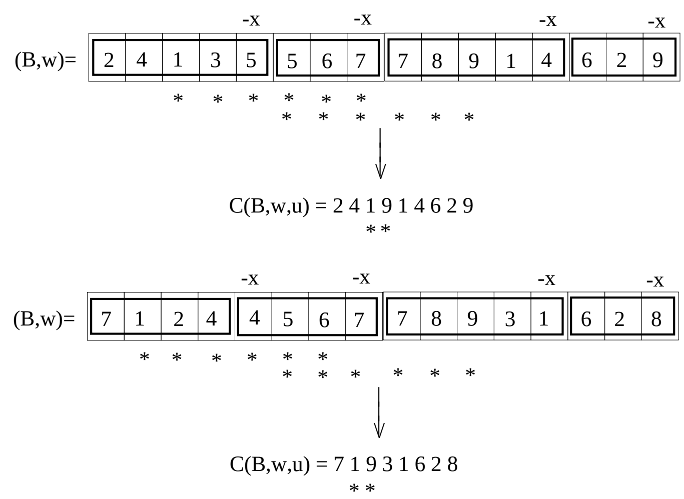

Figure 4.6: A fixed point of L_{123345}.

As before, if we are given $v = v_1 \ldots v_n \in [9]^+$, we want to find the sum of the weights of all fixed points (B, w) of L_u where w is has no levels and $C(B, w, u) = v$. Now if $v_s > v_{s+1}$, then either $v_s v_{s+1}$ lie in the same brick which contributes a factor of 1 or $v_s v_{s+1}$ lie in different bricks which contributes a factor of $-x$ for the brick that ends at v_s. Thus we obtain a factor of $1 - x$ for each descent of v. For the rises of v, we should observe that the start and the end of any two consecutive u-matches which share more than one letter must differ by at least 6. Similarly, the start and the end of any three consecutive u-matches in which each two consecutive u-matches share more that one letter must differ by at least 8. Hence, for $k = 9$, we can have at most three consecutive u-matches in which each two consecutive u-matches share more than one letter. There is one such word, namely, 123345567789. For each pair, $v_s < v_{s+1}$ which occurs in v, we get a factor of $1 - x$ as we did for descents. However in this case, we must also consider the possible collapses that could give rise to $v_s v_{s+1}$. These are as follows.

1. Rises of the form $i(i+1)$, $i(i+2)$, or $i(i+3)$ cannot arise from the collapse map in our case so that $wt_{123345,9}(v_s v_{s+1}) = 1 - x$ in these cases.

2. $v_s v_{s+1} = 15$. In this case, a u-match that could give rise to 15 under the collapse map must be of the form 123345. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 z_2 z_3^2 z_4.$$

3. $v_s v_{s+1} = 16$. In this case, a u-match that could give rise to 16 under the collapse map must be of the form $1abbc6$ where $1 < a < b < c < 6$. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{1<a<b<c<6} z_a z_b^2 z_c.$$

4. $v_s v_{s+1} = 17$. In this case, a single u-match that could give rise to 17 under the collapse map must be of the form $1abbc7$ where $1 < a < b < c < 7$. There is also one possibility for a linked u-match that could give rise to 17 under the collapse map, namely, 123345567. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - \left(xt^4 \sum_{1<a<b<c<7} z_a z_b^2 z_c \right) + xt^7 z_2 z_3^2 z_4 z_5^2 z_6$$

5. $v_s v_{s+1} = 18$. In this case, a single u-match that could give rise to 18 under the collapse map must be of the form $1abbc8$ where $1 < a < b < c < 8$. A linked u-match that could give rise to 18 under the collapse map must be of the form $1abbcdde8$ where $1 < a < b < c < d < e < 8$. Thus

$$wt_{123345,9}(v_s v_{s+1}) =$$
$$1 - x - \left(xt^4 \sum_{1<a<b<c<8} z_a z_b^2 z_c \right) + xt^7 \sum_{1<a<b<c<d<e<8} z_a z_b^2 z_c z_d^2 z_e$$

6. $v_s v_{s+1} = 19$. In this case, a single u-match that could give rise to 19 under the collapse map must be of the form $1abbc9$ where $a \in \{5, 6, 7\}$. Two linked u-matches that could give rise to 19 under the collapse map must be of the form $1abbcdde9$. Finally, there is exactly one way 3 linked u-matches could give rise to 19 under the collapse map, namely, 123345567789. Thus

$$\begin{aligned} wt_{123345,9}(v_s v_{s+1}) &= 1 - x - xt^4 \sum_{1<a<b<c<9} z_a z_b^2 z_c + \\ & xt^7 \sum_{1<a<b<c<d<e<9} z_a z_b^2 z_c z_d^2 z_e - \\ & xt^{10} z_2 z_3^2 z_4 z_5^2 z_6 z_7^2 z_8. \end{aligned}$$

7. $v_s v_{s+1} = 26$. In this case, a single u-match that could give rise to 26 under the collapse map must be of the form 234456. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 z_3 z_4^2 z_5.$$

8. $v_s v_{s+1} = 27$. In this case, a single u-match that could give rise to 27 under the collapse map must be of the form $2abbc7$. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{2<a<b<c<7} z_a z_b^2 z_c$$

9. $v_s v_{s+1} = 28$. In this case, a single u-match that could give rise to 28 under the collapse map must be of the form $2abbc8$. There is exactly one way two linked u-matches could give rise to 28 under the collapse map, namely, 234456678. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{2<a<b<c<8} z_a z_b^2 z_c + xt^7 z_3 z_4^2 z_5 z_6^2 z_7$$

10. $v_s v_{s+1} = 29$. In this case, a single u-match that could give rise to 29 under the collapse map must be of the form $2abbc9$. Two linked u-matches that give rise to 29 under the collapse map must be of the form $2abbcdde9$. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{2<a<b<c<9} z_a z_b^2 z_c + xt^7 \sum_{2<a<b<c<d<e<9} z_a z_b^2 z_c z_d^2 z_e$$

11. $v_s v_{s+1} = 37$. In this case, a single u-match that could give rise to 37 under the collapse map must be of the form 345567. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 z_4 z_5^2 z_6.$$

12. $v_s v_{s+1} = 38$. In this case, a single u-match that could give rise to 38 under the collapse map must be of the form $3abbc8$. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{3<a<b<c<8} z_a z_b^2 z_c.$$

13. $v_s v_{s+1} = 39$. In this case, a single u-match that could give rise to 39 under the collapse map must be of the form $3abbc9$. There is exactly one way two linked u-matches could give rise to 39 under the collapse map, namely, 345567789. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{3<a<b<c<9} z_a z_b^2 z_c + xt^7 z_4 z_5^2 z_6 z_7^2 z_8$$

14. $v_s v_{s+1} = 48$. In this case, a single u-match that could give rise to 48 under the collapse map must be of the form 456678. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 z_5 z_6^2 z_7.$$

15. $v_s v_{s+1} = 49$. In this case, a single u-match that could give rise to 49 under the collapse map must be of the form $4abbc9$. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 \sum_{4<a<b<c<9} z_a z_b^2 z_c.$$

16. $v_s v_{s+1} = 59$. In this case, a single u-match that could give rise to 59 under the collapse map must be of the form 567789. Thus

$$wt_{123345,9}(v_s v_{s+1}) = 1 - x - xt^4 z_6 z_7^2 z_8.$$

It follows that for any $v \in [9]^+$ such that v has no levels,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_{123345} \\ C(B,w,123345)=v}} sgn(B,w) wt_{123345,9}(B,w) =$$
$$- x\overline{z}^v (1-x)^{\mathrm{des}(v)} t^{|v|} \prod_{s \in Rise(v)} wt_{123345,9}(v_s v_{s+1}). \quad (4.36)$$

and

$$LU_{123345}^{(9)}(x, \mathbf{z}_9, t) = 1 + \sum_{n \geq 1} LU_{123345,n}^{(9)}(x, \mathbf{z}_9) t^n =$$
$$1 + \sum_{v \in [9]^+, \mathrm{lev}(v)=0} -x\overline{z}^v (1-x)^{\mathrm{des}(v)} t^{|v|} \prod_{s \in Rise(v)} wt_{123345,9}(v_s v_{s+1}). \quad (4.37)$$

What we need to be able to compute the right-hand sides of (4.33), (4.35), or (4.37) is the generating function over all words $v \in \mathbb{P}^*$ with no levels where we not only keep track of the rises of P but also the type of rises of P. We do this by substituting into an auxiliary generating function. This is the following:

By Theorem 6, we know that

$$1 + \sum_{w \in \mathbb{P}^+, \mathrm{lev}(w)=0} t^{|w|} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)} = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} AYZ(S)}$$

where

$$AYZ(S) = \begin{cases} \frac{z_j}{1+z_j t} & \text{if } S = \{j\}, \text{ and} \\ \frac{z_{j_1}}{1+z_{j_1} t} \cdots \frac{z_{j_k}}{1+z_{j_k} t} \prod_{i=1}^{k-1} (x_{j_i j_{i+1}} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.38}$$

where $k \geq 2$. Hence

$$\begin{aligned} \sum_{w \in \mathbb{P}^+, \mathrm{lev}(w)=0} t^{|w|} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)} &= \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} AYZ(S)} - 1 \\ &= \frac{\sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} AYZ(S)}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} AYZ(S)} \end{aligned} \tag{4.39}$$

If we replace t by ty and x_{ji} by $\frac{x_{ji}}{y}$, the left-hand side of (4.39) becomes

$$\sum_{w \in \mathbb{P}^+, \mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{des}(w)+1} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)}$$

Note that for $S = \{j_1 < \cdots < j_k\}$ where $k \geq 2$, our substitution replaces $t^k AYZ(S)$ by

$$y^k t^k \frac{z_{j_1}}{1 + z_{j_1} ty} \cdots \frac{z_{j_k}}{1 + z_{j_k} ty} \prod_{i=1}^{k-1} \left(\frac{x_{j_i j_{i+1}}}{y} - 1 \right) =$$

$$y t^k \frac{z_{j_1}}{1 + z_{j_1} ty} \cdots \frac{z_{j_k}}{1 + z_{j_k} ty} \prod_{i=1}^{k-1} (x_{j_i j_{i+1}} - y)$$

Thus if we let

$$BYZ(S) = \begin{cases} \frac{z_j}{1+z_j ty} & \text{if } S = \{j\}, \text{ and} \\ \frac{z_{j_1}}{1+z_{j_1} ty} \cdots \frac{z_{j_k}}{1+z_{j_k} ty} \prod_{i=1}^{k-1} (x_{j_i j_{i+1}} - y) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.40}$$

where $k \geq 2$, then the right-hand side of (4.39) becomes

$$\frac{y \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}{1 - y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}. \tag{4.41}$$

It follows that

$$-x \sum_{w\in \mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{des}(w)} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)} = \frac{-x \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}{1 - y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}. \tag{4.42}$$

Thus

$$1-x \sum_{w\in \mathbb{P}^+,\mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{des}(w)} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)} = \frac{1-(x+y) \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}{1 - y\sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BYZ(S)}. \tag{4.43}$$

By setting $z_i = 0$ for $i > k$, we also obtain that

$$\begin{aligned} &1-x \sum_{w\in [k]^+,\mathrm{lev}(w)=0} t^{|w|} y^{\mathrm{des}(w)} \bar{z}^w \prod_{i>j} x_{ji}^{\mathrm{ji}(w)} = \\ &\frac{1-(x+y) \sum_{n=1}^{k} t^n \sum_{S\subseteq [k],|S|=n} BYZ(S)}{1 - y\sum_{n=1}^{k} t^n \sum_{S\subseteq [k],|S|=n} BYZ(S)}. \end{aligned} \tag{4.44}$$

Note that if we replace y by $(1-x)$ and x_{ji} by $wt_u(ji)$, the left-hand side of (4.43) becomes $LU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ and the left-hand side of (4.44) becomes $LU_u^{(k)}(x, \mathbf{z}_k, t)$. Similarly, if we replace y by $(1-x)$ and x_{ji} by $ewt_u(ji)$, the left-hand side of (4.43) becomes $LEU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ and the left-hand side of (4.44) becomes $LEU_u^{(k)}(x, \mathbf{z}_k, t)$. Then using the fact that $\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = 1/LU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$ and that $\mathcal{LEN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = 1/LEU_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, we have the following theorem.

Theorem 20. *Suppose that* $u = u_1 \ldots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, *and* $u_1 < u_j$. *Then*

$$\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1-(1-x) \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BTZ_u(S)}{1 - \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} BTZ_u(S)} \tag{4.45}$$

and

$$\mathcal{LEN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1-(1-x) \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} EBTZ_u(S)}{1 - \sum_{n\geq 1} t^n \sum_{S\subseteq \mathbb{P},|S|=n} EBTZ_u(S)} \tag{4.46}$$

where

$$BTZ_u(S) = \begin{cases} \frac{z_j}{1+z_j t(1-x)} & \text{if } S = \{j\}, \text{ and} \\ \dfrac{z_{j_1}}{1+z_{j_1}t(1-x)} \cdots \dfrac{z_{j_k}}{1+z_{j_k}t(1-x)} & \\ \displaystyle\prod_{i=1}^{k-1}(wt_u(j_i j_{i+1}) + x - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.47}$$

where $k \geq 2$ *and*

$$EBTZ_u(S) = \begin{cases} \frac{z_j}{1+z_j t(1-x)} & \text{if } S = \{j\}, \text{ and} \\ \dfrac{z_{j_1}}{1+z_{j_1}t(1-x)} \cdots \dfrac{z_{j_k}}{1+z_{j_k}t(1-x)} & \\ \displaystyle\prod_{i=1}^{k-1}(ewt_u(j_i j_{i+1}) + x - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \end{cases} \tag{4.48}$$

where $k \geq 2$.

If we specialize the variables so that $z_i = 0$ for all $i > k$, then we have the following theorem.

Theorem 21. *Suppose that* $u = u_1 \ldots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, *and* $u_1 < u_j$. *Then*

$$\mathcal{LN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1 - (1-x)\sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} BTZ_u(S)}{1 - \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} BTZ_u(S)} \tag{4.49}$$

and

$$\mathcal{LEN}_u^{(k)}(x, \mathbf{z}_k, t) = \frac{1 - (1-x)\sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} EBTZ_u(S)}{1 - \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S|=n} EBTZ_u(S)} \tag{4.50}$$

4.4 The case $u = u_1 \ldots u_j$, $\mathrm{lev}(u) = 1$, and $u_1 = u_j$

In this section, we shall consider the problem of computing the generating functions $\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, $\mathcal{LN}_u^{(k)}(x, \mathbf{z}_k, t)$, $\mathcal{LEN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t)$, and $\mathcal{LEN}_u^{(k)}(x, \mathbf{z}_k, t)$ for $u = u_1 \ldots u_j$ such that $\mathrm{lev}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-level overlapping property (or $[k]$-level overlapping property).

As in the previous sections, we need to compute $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_u^{(k)}(x,\mathbf{z}_k,t)$, $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_u^{(k)}(x,\mathbf{z}_k,t)$. To compute these generating functions, we use Theorem 2 plus the collapse map.

First assume that $u = u_1 \ldots u_j$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-minimal overlapping property. We can define the collapse map to fixed points of L_u or K_u exactly as in the previous sections. For example, suppose that $u = 244132$ and we want to compute $LU_{244132}^{(7)}(x,\mathbf{z}_7,t)$. By (4.6), we know that

$$LU_{244132,n}^{(7)}(x,\mathbf{z}_7) = \sum_{O \in \mathcal{O}_{244132,n}^{(k)}, L_{244132}(O)=O} sgn(O)wt(O). \tag{4.51}$$

As before, we know that if (B,w) is a fixed point of L_{244132}, then elements in the bricks have no levels and if there is a level between two bricks b_i and b_{i+1}, there must be a 244132-match that involves the last 2 cells of b_i and the first four cells of b_{i+1}. We have pictured such a fixed point in Figure 4.7.

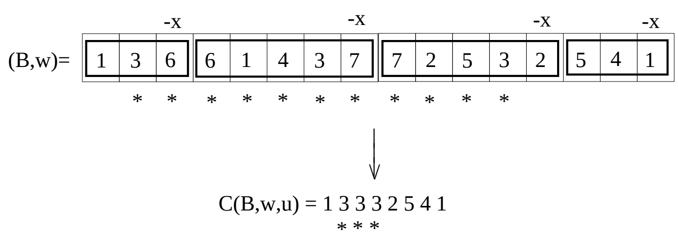

Figure 4.7: A fixed point of L_{244132}.

The difference between this case and the previous case where $u_1 > u_j$ is that a 244132-match of the form $ijjcli$ will just be replaced by ii so that the collapse map will produce words with levels in this case. Moreover, if a word $v = C(B,w,u)$ has a factor of the form ii it must have come from a 244132-match in the collapse of a fixed point of L_{244132}. The fact that 244132 has the minimal overlapping property ensures that any two such 244132-matches can only intersect at the right-hand endpoint of the first match and left-hand endpoint of the second match. In this case a factor of the form ii must have weight $-xt^4 \sum_{i<l<j\leq k} z_l z_j^2 \sum_{1\leq c<i} z_c$ if we are computing $LU_{244132,n}^{(k)}(x,\mathbf{z}_k)$ and $-xt^4 \sum_{i<l<j} z_l z_j^2 \sum_{1\leq c<i} z_c$ if we are computing $LU_{244132,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty)$. That is, the weight corresponds to the case where we have a level

Table 4.8: The weights $wt_{244132,7}(ii)$

Levels	$wt_{244132,7}(ii)$
77	0
66	0
55	$-xt^4 z_6 z_7^2(z_1+z_2+z_3+z_4)$
44	$-xt^4(\sum_{4<c<d\leq 7} z_c z_d^2)(z_1+z_2+z_3)$
33	$-xt^4(\sum_{3<c<d\leq 7} z_c z_d^2)(z_1+z_2)$
22	$-xt^4(\sum_{2<c<d\leq 7} z_c z_d^2)z_1$
11	0

between two consecutive bricks and we deleted the second, third, fourth, and fifth elements of the 244132-match between the two bricks. In our example, the weights of the levels for computing $LU^{(7)}_{244132,n}(x,\mathbf{z}_7)$ are listed in table 4.8.

In this case, factors in $C(B,w,244132)$ of the form ij where $i \neq j$ correspond to a factor of $1-x$ where the 1 comes from the case where ij are in the same brick and the $-x$ corresponds to the case where i and j are in different bricks.

It follows that for any $v \in [7]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_{244132} \\ C(B,w,244132)=v}} sgn(B,w)wt_{244132}(B,w) = \\ -x\overline{z}^v(1-x)^{\mathrm{des}(v)+\mathrm{rise}(v)}t^{|v|}\prod_{s\in Lev(v)} wt_{244132,7}(v_s v_{s+1}). \tag{4.52}$$

and

$$\begin{aligned} LU^{(7)}_{244132}(x,\mathbf{z}_7,t) &= 1+\sum_{n\geq 1} LU^{(7)}_{244132,n}(x,\mathbf{z}_7)t^n \\ &= 1+\sum_{v=v_1\ldots v_n\in[7]^+} -x\overline{z}^v(1-x)^{\mathrm{des}(v)+\mathrm{rise}(v)}t^{|v|}\prod_{s\in Lev(v)} wt_{244132,7}(v_s v_{s+1}). \end{aligned} \tag{4.53}$$

As in the previous section, the case of exact matches is much simpler. In

Table 4.9: The weights $ewt_{244132}(ij)$

Factor	weight $ewt_{244132,7}(ij)$
ij where $i \neq j$	$1-x$
ii where $i \neq 2$	0
22	$-xz_1z_3z_4^2t^4$

that case, we want to compute

$$\sum_{\substack{(B,w) \text{ is a fixed point of } K_u \\ C(B,w,u)=v}} sgn(B,w)wt(B,w).$$

Going back to our example of $u = 244132$ over the alphabet $[7]$, we see that factors of the form ij with $i \neq j$ correspond to a factor of $1-x$ where the 1 comes from the case where ij are in the same brick and the $-x$ comes from the case where i and j are in different bricks. The weight associated to a level $v_s = v_{s+1}$ is 0 unless $v_s = 2$ and $v_{s+1} = 2$. If $v_s = 2$, $v_{s+1} = 2$, then we must have eliminated a 4413 from w. Thus if we want to compute $LEU^{(\mathbb{P})}_{244132,n}(x, \mathbf{z}_\infty)$ or $LEU^{(k)}_{244132,n}(x, \mathbf{z}_k)$ for $k \geq 4$, the weights associated to factors ij are given in table 4.9. It follows that for any $v \in [7]^+$,

$$\begin{aligned} &\sum_{\substack{(B,w) \text{ is a fixed point of } K_{244132} \\ C(B,w,244132)=v}} sgn(B,w)wt(B,w) = \\ &- x\overline{z}^v (1-x)^{\text{des}(v)+\text{rise}(v)} t^{|v|} \prod_{s \in Lev(v)} ewt_{244132,7}(v_s v_{s+1}) \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} LEU^{(7)}_{244132,n}(x, \mathbf{z}_7, t) &= 1 + \sum_{n \geq 1} LEU^{(7)}_{244132,n}(x, \mathbf{z}_7) t^n \\ &= 1 + \sum_{v \in [7]^+} -x\overline{z}^v (1-x)^{\text{des}(v)+\text{rise}(v)} t^{|v|} \prod_{s \in Lev(v)} ewt_{244132,7}(v_s v_{s+1}). \end{aligned} \tag{4.55}$$

Next suppose that $u = u_1 \ldots u_j$, $\text{red}(u) = u$, $\text{lev}(u) = 1$, $u_1 = u_j$, and u has the $\mathbb{P}$-level overlapping property or the $[k]$-level overlapping property, but u

does not have the $\mathbb{P}$-minimal overlapping property. The fact that u has the $\mathbb{P}$-level overlapping property ($[k]$-level overlapping property) ensures that if $w = w_1 \dots w_n$ is word which starts and ends with a u-match and any two consecutive u-matches in w share at least two letters, then it must case that $w_1 = w_n$. Thus under the collapse map, any collapse will end up with a level of the form ii. The main difference in this case is that it is possible to have the weights $wt_{u,k}(ii)$ or $wt_{u,\mathbb{P}}(ii)$ correspond to infinite families of words of different lengths even in the case where the alphabet is finite. For example, suppose that $u = 12133121$. Then it is possible that in a fixed point (B, w) of $L_{12133121}$, w has a factor where consecutive occurrences of the pattern 12133121 are linked of the form $ijiy_1y_1ijiy_2y_2ijiy_3iji\dots ijiy_niji$ where $y_1, \dots, y_n > j > i$ like those that occur in the first 18 cells of the fixed point pictured in Figure 4.8. For each given maximal sequence of this type, the collapse map would eliminate all the symbols between the first and the last i. In such a case, the weight corresponding to the symbols that are eliminated for such a string in the collapse map would be $(-x)^n z_i^{2n} z_j^{n+1} z_{y_1}^2 \cdots z_{y_n}^2 t^{5n+1} = (z_j t)(-x)^n z_i^{2n} z_j^n z_{y_1}^2 \cdots z_{y_n}^2 t^{5n}$. It would follow that if we are working in $\mathbb{P}^*$, then

$$wt_{12133121,\mathbb{P}}(ii) = \frac{-xz_i^2 z_j^2 \left(\sum_{s>i} z_s^2\right) t^6}{1 + xz_i^2 z_j \left(\sum_{s>i} z_s^2\right) t^5}$$

while if we are working in $[k]^*$, then for $1 \leq i < k$,

$$wt_{12133121,\mathbb{P}}(ii) = \frac{-xz_i^2 z_j^2 \left(\sum_{s=i+1}^{k} z_s^2\right) t^6}{1 + xz_i^2 z_j \left(\sum_{s=i+1}^{k} z_s^2\right) t^5}$$

and

$$wt_{112133121,k}(kk) = 0.$$

That is, in each of these expressions, the series $\frac{-xz_i^2 z_j^2 \left(\sum_{s>i} z_s^2\right) t^6}{1+xz_i^2 z_j \left(\sum_{s>i} z_s^2\right) t^5}$ corresponds the fact that we could have eliminated sequences of the form $ijiy_1y_1ijiy_2y_2ijiy_3y_3iji\dots ijiy_ny_niji$ for any $n \geq 1$ between the two is.

Nevertheless, we can still apply the same reasoning as above to prove that for any $v \in [7]^+$,

$$\sum_{\substack{(B,w) \text{ is a fixed point of } L_{12133121} \\ C(B,w,12133121)=v}} sgn(B, w) wt_{12133121}(B, w) =$$

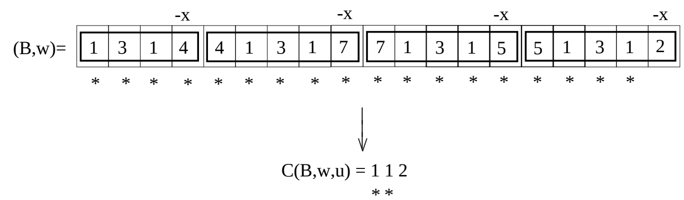

Figure 4.8: A fixed point of $L_{12133121}$.

$$- x\overline{z}^v (1-x)^{\text{des}(v)+\text{rise}(v)} t^{|v|} \prod_{s \in Lev(v)} wt_{12133121,7}(v_s v_{s+1}). \tag{4.56}$$

and

$$\begin{aligned} LU^{(7)}_{12133121}(x, \mathbf{z}_7, t) &= 1 + \sum_{n \geq 1} LU^{(7)}_{12133121,n}(x, \mathbf{z}_7) t^n \\ &= 1 + \sum_{v = v_1 \ldots v_n \in [7]^+} -x\overline{z}^v (1-x)^{\text{des}(v)+\text{rise}(v)} t^{|v|} \prod_{s \in Lev(v)} wt_{12133121,7}(v_s v_{s+1}) \\ &= 1 + \sum_{v = v_1 \ldots v_n \in [7]^+} -x\overline{z}^v (1-x)^{|v|-\text{lev}(v)-1} t^{|v|} \prod_{s \in Lev(v)} wt_{12133121,7}(v_s v_{s+1}). \end{aligned} \tag{4.57}$$

We should note that as patterns get more complicated, it becomes increasingly difficult to compute $wt_{u,\mathbb{P}}(ii)$ or $wt_k(ii)$. For example, suppose $u = (121)^5 33 (121)^5$. Then linked patterns can overlap at either 1,3,5,7, or 9 symbols.

What we need to be able to compute the right-hand sides of (4.53), (4.55), or (4.57) is the generating function over all words $v \in \mathbb{P}^*$ where we not only keep track of the levels of P but also the type of levels of P. We do this by specializing one of our auxiliary generating functions from Chapter 2 and then substituting into that resulting generating function. This is the following:

By Theorem 2, we know that

$$1 + \sum_{w \in \mathbb{P}^+} t^{|w|} \overline{z}^w \prod_{i \leq j} x_{ji}^{\text{ji}(w)} = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v \in WD\mathbb{P}^*, |v|=n} WDXZ(v)}$$

where

$$WDXZ(v) = \begin{cases} z_j & \text{if } v = j, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_i j_{i+1}} - 1) & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2. \end{cases} \tag{4.58}$$

If we specialize this generating function and let $x_{ji} = 1$ if $i < j$, then we get

$$1 + \sum_{w \in \mathbb{P}^+} t^{|w|} \bar{z}^w \prod_{i \geq 1} x_{ii}^{\mathrm{ii}(w)} = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} ADXZ(v)}$$

where

$$ADXZ(v) = \begin{cases} z_i & \text{if } v = i, \text{ and} \\ (z_i)^k (x_{ii} - 1)^{k-1} & \text{if } v = ii\ldots i \text{ where } |v| = k \geq 2. \end{cases} \tag{4.59}$$

Hence

$$\begin{aligned} \sum_{w \in \mathbb{P}^+} t^{|w|} \bar{z}^w \prod_{i \geq 1} x_{ii}^{\mathrm{ii}(w)} &= \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} ADXZ(v)} - 1 \\ &= \frac{\sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} ADXZ(v)}{1 - \sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} ADXZ(v)} \end{aligned} \tag{4.60}$$

If we replace t by ty and x_{ii} by $\frac{x_{ii}}{y}$, the left-hand side of (4.60) becomes

$$\sum_{w \in \mathbb{P}^+} t^{|w|} \bar{z}^w y^{|w| - \mathrm{lev}(w)} \prod_{i \geq 1} x_{ii}^{\mathrm{ii}(w)}$$

Note that for $v = ii\ldots i$ where $|v| = k \geq 2$, our substitution replaces $t^k ADXZ(v)$ by

$$y^k t^k (z_i)^k \left(\frac{x_{ii}}{y} - 1 \right)^{k-1} = y t^k (z_i)^k (x_{ii} - y)^{k-1}$$

Thus if we let

$$BDXZ(v) = \begin{cases} z_i & \text{if } v = i, \text{ and} \\ (z_i)^k (x_{ii} - y)^{k-1} & \text{if } v = ii\ldots i \text{ where } |v| = k \geq 2. \end{cases} \tag{4.61}$$

then the right-hand side of (4.60) becomes

$$\frac{y \sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} BDXZ(v)}{1 - y \sum_{n \geq 1} t^n \sum_{v=ii\ldots i, |v|=n} BDXZ(v)}. \tag{4.62}$$

It follows that

$$-x\sum_{w\in\mathbb{P}^+} t^{|w|}y^{|w|-\mathrm{lev}(w)-1}\bar{z}^w\prod_{i\geq 1}x_{ii}^{\mathrm{ii}(w)} = \frac{-x\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}{1-y\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}. \tag{4.63}$$

Thus

$$1-x\sum_{w\in\mathbb{P}^+} t^{|w|}y^{|w|-\mathrm{lev}(w)-1}\bar{z}^w\prod_{i\geq 1}x_{ii}^{\mathrm{ii}(w)} = \frac{1-(x+y)\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}{1-y\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}. \tag{4.64}$$

By setting $z_i = 0$ for $i > k$, we also obtain that

$$1-x\sum_{w\in[k]^+} t^{|w|}y^{|w|-\mathrm{lev}(w)-1}\bar{z}^w\prod_{i=1}^{k}x_{ii}^{\mathrm{ii}(w)} = \frac{1-(x+y)\sum_{n=1}^{k}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}{1-y\sum_{n=1}^{k}t^n\sum_{v=ii\ldots i,|v|=n}BDXZ(v)}. \tag{4.65}$$

Note that if we replace y by $(1-x)$ and x_{ii} by $wt_u(ii)$, the left-hand side of (4.64) becomes $LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (4.65) becomes $LU_u^{(k)}(x,\mathbf{z}_k,t)$. Similarly, if we replace y by $(1-x)$ and x_{ii} by $ewt_u(ii)$, the left-hand side of (4.64) becomes $LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and the left-hand side of (4.65) becomes $LEU_u^{(k)}(x,\mathbf{z}_k,t)$. Then using the fact that $\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/LU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$ and that $\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = 1/LEU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, we have the following theorem.

Theorem 22. *Suppose that* $u = u_1\ldots u_j \in \mathbb{P}^*$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, *and* $u_1 = u_j$. *Then*

$$\mathcal{LN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDTZ_u(v)}{1-\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}BDTZ_u(v)} \tag{4.66}$$

and

$$\mathcal{LEN}_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t) = \frac{1-(1-x)\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}EBDTZ_u(v)}{1-\sum_{n\geq 1}t^n\sum_{v=ii\ldots i,|v|=n}EBDTZ_u(v)} \tag{4.67}$$

where

$$BDTZ_u(v) = \begin{cases} z_i & \text{if } v = i, \text{ and} \\ (z_i)^k (wt_u(ii) + x - 1)^{k-1} & \text{if } v = ii\ldots i \text{ where } |v| = k \geq 2. \end{cases} \tag{4.68}$$

and

$$EBDTZ_u(v) = \begin{cases} z_i & \text{if } v = i, \text{ and} \\ (z_i)^k (ewt_u(ii) + x - 1)^{k-1} & \text{if } v = ii\ldots i \text{ where } |v| = k \geq 2. \end{cases} \tag{4.69}$$

If we specialize the variables so that $z_i = 0$ for all $i > k$, then we have the following theorem.

Theorem 23. *Suppose that* $u = u_1 \ldots u_j \in [k]^*$, $\mathrm{red}(u) = u$, $\mathrm{lev}(u) = 1$, *and* $u_1 = u_j$. *Then*

$$\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1 - (1-x)\sum_{n=1}^k t^n \sum_{v=ii\ldots i,|v|=n} BDTZ_u(v)}{1 - \sum_{n=1}^k t^n \sum_{v=ii\ldots i,|v|=n} BDTZ_u(v)} \tag{4.70}$$

and

$$\mathcal{LN}_u^{(\mathbb{P})}(x, \mathbf{z}_\infty, t) = \frac{1 - (1-x)\sum_{n=1}^k t^n \sum_{v=ii\ldots i,|v|=n} EBDTZ_u(v)}{1 - \sum_{n=1}^k t^n \sum_{v=ii\ldots i,|v|=n} EBDTZ_u(v)} \tag{4.71}$$

Chapter 4, in part, is currently being prepared for submission for publication of the material. Sangha, Luvreet; Remmel, Jeffrey. The dissertation author was the primary author of this material.

Chapter 5

Possible extensions

The methods that we have used in this paper can be modified to find generating functions of the form

$$\sum_{w \in \mathbb{P}^*, umch(w)=0} t^{|w|} x^{\mathrm{wdes}(w)+1} \overline{z}^w, \qquad \sum_{w \in [k]^*, umch(w)=0} t^{|w|} x^{\mathrm{wdes}(w)+1} \overline{z}^w,$$
$$\sum_{w \in \mathbb{P}^*, eumch(w)=0} t^{|w|} x^{\mathrm{wdes}(w)+1} \overline{z}^w, \text{ and } \sum_{w \in [k]^*, eumch(w)=0} t^{|w|} x^{\mathrm{wdes}(w)+1} \overline{z}^w.$$

in the case where $\mathrm{wdes}(u) = 1$. The idea is that one can modify the reciprocal method presented in Section 3 to replace the statistic $\mathrm{des}(w) + 1$ or $\mathrm{lev}(w) + 1$ by $\mathrm{wdes}(w) + 1$. Then one can modify the collapse map appropriately. Finally, one needs appropriate modifications of Theorems 1, 2, 3, and 4 to produce generating functions which keep track of labeled rises, levels, or descents that can be specialized to compute the generating functions of interest.

By the isomorphism which sends a word $w = w_1 \dots w_n$ to its reverse, $w^r = w_n \dots w_1$, one can automatically produce similar generating functions where the statistics $\mathrm{des}(w)+1$ and $\mathrm{wdes}(w)+1$ are replaced by $\mathrm{wrise}(w)+1$ and $\mathrm{rise}(w)+1$, respectively.

One can also easily modify the methods to keep track of restricted sets of descents. For example, given a word $w = w_1 \dots w_n \in \mathbb{P}^*$, let $\mathrm{edes}(w) = |\{i : w_i > w_{i+1} \text{ and } w_i \text{ is even}\}|$. Then the techniques of this thesis can be easily modified to find closed expressions for

$$\sum_{w\in\mathbb{P}^*,u\text{mch}(w)=0} t^{|w|}x^{\text{edes}(w)+1}\overline{z}^w, \qquad \sum_{w\in[k]^*,u\text{mch}(w)=0} t^{|w|}x^{\text{edes}(w)+1}\overline{z}^w,$$

$$\sum_{w\in\mathbb{P}^*,eu\text{mch}(w)=0} t^{|w|}x^{\text{edes}(w)+1}\overline{z}^w, \text{ and } \sum_{w\in[k]^*,eu\text{mch}(w)=0} t^{|w|}x^{\text{edes}(w)+1}\overline{z}^w$$

in the case were $\text{edes}(u) = 1$.

Finally, one can extend the reciprocal methods in this thesis to give a combinatorial interpretation of $U_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_{u,n}^{(k)}(x,\mathbf{z}_k,t)$, $EU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $EU_{u,n}^{(k)}(x,\mathbf{z}_k,t)$ in the case where $\text{des}(u) > 1$ or of $LU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $LU_{u,n}^{(k)}(x,\mathbf{z}_k,t)$, $LEU_{u,n}^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $LEU_{u,n}^{(k)}(x,\mathbf{z}_k,t)$ in the case where $\text{lev}(u) > 1$. Basically one has to modify the involution I_u presented in the introduction appropriately. This has been done in the case of permutations by Quang Bach and Jeffrey Remmel in the case of permutations [3,4]. However, in the case where $\text{des}(u) > 1$, for example, the corresponding set of fixed points are much more complicated. In particular, in the case where $\text{des}(u) > 1$, it will no longer be the case that in fixed points of the modified version of I_u that the underlying word will be weakly increasing in bricks. These more complicated fixed points then require a more complicated version of the collapse map. Nevertheless, one can still come up with closed formulas for the generating functions $U_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, $U_u^{(k)}(x,\mathbf{z}_k,t)$, $EU_u^{(\mathbb{P})}(x,\mathbf{z}_\infty,t)$, and $EU_u^{(k)}(x,\mathbf{z}_k,t)$. This work will appear in a subsequent paper.

Chapter 5, in part, has been submitted for publication as it may appear in Generating Functions for Descents over Words which Avoid a Consecutive Pattern, 2017, Remmel, Jeffrey; Sangha, Luvreet, Electronic Journal of Combinatorics, 2017, arXiv:1612.04900. The dissertation author was the secondary author of this work.

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