Title
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Monopolies in Two-Sided Markets: Comparative Statics and Identification*

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Abstract

Many models of monopoly in two-sided markets have been proposed (Rochet and Tirole, 2006), but little is known about them. I provide a full set of comparative statics for three models, one that generalizes that of Rochet and Tirole (2003), a second that generalizes Armstrong (2006) and a third that fuses the two models. This answers a number of questions central to the theoretical literature, such as the effects of market power and price controls and the relationship between different models. I also show how, given exogenous cost variations, these models can be almost fully (locally) identified (e.g. market power and predatory pricing), tested and distinguished from one another. I highlight applications of the results to a wide variety of theoretical, empirical and policy questions, including merger analysis.

PRELIMINARY AND INCOMPLETE.

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1 Introduction

Recent regulatory and antitrust issues in the payment card, operating system and internet service provision industry have created challenges and opportunities for economic analysis. These "two-sided markets", where firms connect two distinct sets of consumers, bring interactions between pricing of various services that are different than the extensively theoretically studied (Tirole, 1988) and empirically measured (Pakes, 2008) relations of substitutes and complements. On the other hand, the greater structure natural to these interactions holds the potential of making them more analytically tractable and easier to measure empirically than those in general network industries (Katz and Shapiro, 1985; Economides, 1996). Nonetheless, while several models of two-sided markets have been formulated (Caillaud and Jullien, 2001, 2003; Rochet and Tirole, 2003; Armstrong, 2006), until recently little was understood even under monopoly about what parameters the (policy-relevant) comparative statics of these models depended on, how these could be measured and how the models could be tested or selected for applications.

In Weyl (2008d), I exploit some convenient features of the Rochet and Tirole (2003) (RT2003) model to show that, under monopoly and a simple form of competition considered by RT2003, most questions of policy interest can be answered, and the model can be tested, for general demand functions using exogenous cost variations. Yet this analysis leaves out the other most prominent model of two-sided markets, that of Armstrong (2006), as well as a wide variety of models bridging the two of them in the broader Rochet and Tirole (2006) (RT2006) class. Many of the most basic questions about the Armstrong model have not be answered: for example, what are the welfare and price effects of price controls, costs reductions or reductions in market power on one or both sides of the market? Furthermore, because strategies for identifying the model have not been explored, empirical work on two-sided markets which has largely focused on this model (Rysman, 2004; Kaiser and Wright, 2006) has been forced to rely on strong parametric assumptions. For example, no general empirical approach exists for identifying the degree of market power.

To help clarify policy analysis of two sided markets, I extend my results in Rochet and Tirole (2003) to a much wider class of monopoly models, which I discuss in Section 2. In particular I consider a generalization of the RT2003 model to allow for membership, in addition to usage, costs and benefits for consumers and the platform , a generalization of the Armstrong model to allow usage costs, and a hybrid model with where consumers are Armstrong-like on one side of the market and RT2003-like on the other side. In many markets this last model is likely most realistic.

For each model I provide, in Section 3, a complete set of comparative statics, showing in full generality the parameters of the demand system on which the effects of shocks to market power, demand levels and certain demand parameters. I also provide a simple and extremely general means of determining the effects of price controls. While theory determines the sign of many comparative statics, empirical identification is needed for quantitative results.

When sufficiently detailed micro-level data is available and sufficiently strong structural assumptions are made, direct estimation of the full distribution of demand may be possible (Berry et al., 1995), even in complex two-sided markets (Lee, 2007). Often, however only local (first-order) properties of demand, which exogenous cost variations identify, can reliably be estimated. In Section 4, I show that nonetheless, under reasonable assumptions about cost-observabilty, this is sufficient to identify *without any functional form assumptions* nearly all of the parameters which determine comparative statics of these models. Furthermore these same exogenous cost variations can be used to test the models as a group and against one another.

These results have a wide variety of applications to policy and empirical analysis of two-sided markets, some of which I discuss in Section 5. They show that the welfare economics of the (generalized) Armstrong model are much simpler and more intuitive than those for the (generalized) RT2003 model. They help explain the sources of much-discussed (Caillaud and Jullien, 2003; Bolt and Tieman, 2005; Kaiser and Wright, 2006; Jullien, 2008) price skewness in these models. They allow a straightforward analysis of mergers with non-two-sided firms and extend the analysis of double marginalization in two-sided markets (Weyl, 2008b) to the (generalized) Armstrong model. Despite their focus on monopoly, they can be used to analyze simple forms of competition in both the GRT2003 and GArmstrong models. In the latter, I provide by far the most general analysis of (symmetric, single-homing) competition to date and show that in a wide range of cases a merger will lower demand and welfare on both sides of the market. I also emphasize how the models differ, despite RT2006's effort to unify them, and provide a more intuitive first-order condition (based on primitives rather than fixed points) as well as the first-yet second-order condition for the general RT2006 class. Finally I show how the results can be used to identify the degree of market power, as well as to detect predatory pricing. Section 6 concludes by briefly discussing broader insights based on my results and possible directions for future research.

2 Models

Two-sided markets connect distinct groups of consumers whose utility from using the service depends on the number of consumers participating on the other side of the market. In a monopoly setting, a simple way to capture this feature of two-sided markets (my notation is due to RT2006) is to assume that every consumer (in a continuum) $i \in [0, N^I]$ on each side of the market $I = A, B$ has a utility that is linear in price and the number of consumers participating on the other side of the market

$$u_i^I = B_i^I + (b_i^I - p^I)n^{-I} - P^I \tag{1}$$

where B_i^I is the membership benefit or cost earned by consumer i for participating, b_i^I is the usage benefit derived by consumer i, p^I is the usage (usage) price charged by the monopoly platform to side I, n^{-I} is the number of consumers participating on the other side of the market (which is obviously endogenous) and P^I is the membership (membership) price charged by platform to side I. The profits earned by the monopolist are then

$$(P^A - C^A)n^A + (P^B - C^B)n^B + (p^A + p^B - c)n^A n^B \qquad (2)$$

where C^I is the membership cost to side $I$of serving consumers on side I and c is a usage cost to the platform. Obviously the number of consumers participating on each side will depend (implicitly) on the prices charged to each side, but I defer the explicit discussion of this to the more specialized models below. Note more generally, though, that modulo some technical caveats about strategic complementarity and uniqueness of equilibrium, only the combination of prices $p^I n^{-I} + P^I$ matters to the utility, participation and profits derived from consumers on side I of the market. Therefore p^I and P^I are not really two separate instruments (in the monopoly context) and we can move between them freely as is convenient.

A restricted form of this model of consumer preferences is the foundation of the RT2003 model, the Armstrong model and my hybrid model. It was stated in its fully generality first by RT2006, who derive general first-order conditions for monopoly pricing under it. However the model in its full generality has a number of disadvantages:

1. Despite the elegance of the first-order conditions derived by RT2006, these conditions have essentially no content, in their full generality, for the comparative statics of monopoly pricing. That is, as I show in Section 5, virtually any behavior by a monopolist is consistent with these conditions and therefore they tell us nothing about how to predict or regulate monopoly pricing.

2. The model in its full generality is difficult to analyze, as complex features the joint distribution of B^I and b^i are crucial, as I discuss in Section 5.

3. The model in its full generality is, under reasonable assumptions about observability (see Section 4), under-identified.

For the most part, therefore, I focus in this paper on developing a full understanding of three special cases of this general model which, I will argue below, are each reasonable approximations to different real world markets. Nonetheless there are likely some markets which are reasonably approximated by the broader RT2006 model but by none of the sub-models I consider; I leave the analysis of the broad RT2006 model for the purposes of understanding these markets to future research. However, in Section 5 I will discuss the light that my results shed in general terms on this broader model.

2.1 Generalized RT2003

The RT2003 model is the special case of the RT2006 model where $C^I, B_i^I = 0, \forall i, I$; that is all consumer benefits or costs of the two-sided service come from the value of connection and all firm costs are usage. This model gives rise to very simple and elegant economics, because, as I show in Weyl (2008d), the only cross effects between the monopolist's pricing of the two goods are the cross-subsidies provided to each side provided by the per-transaction price charged to the other. However, this model is often quite unrealistic as both the platform and participants often have membership membership costs of joining the platform and, on occasion, may gain membership benefits from joining regardless of the number of participating partners on the other side of the market.

I therefore here consider a generalized version of the RT2003 model where C^I make take arbitrary value for each I and $B_i^I \equiv B^I$. In other words firms and consumers on each may have arbitrary membership costs (or benefits), but these costs are the same across all consumers on a particular side of the market. As we will see in the next section, this generalization makes little difference to the basic economics of the model: the essential difference between the Armstrong and RT2003 models lies not in their specification of *levels* of costs and benefit, but in the source of the heterogeneity among consumers.

The conventional wisdom in the literature is that the RT2003 model applies best to the credit card industry. However once we allow for membership costs for both consumers and firms, it becomes less clear that that is the only reasonable application of the model. For example, it seems reasonable to suppose that the main source of heterogeneity among consumers using dating websites or visiting clubs is their differing value of finding a mate, rather than differences in their membership cost or value of using the service. Similarly it seems reasonable to suppose that in many forms of commercial intermediation (e.g. supermarkets, E-Bay, stock markets) the membership costs or benefits to participants of using the service are relatively homogenous, while the benefits of interacting with partners may vary widely across consumers. Of course neither of these examples fix obviously better with the GRT2003 framework than one of the other two below, but neither does the GRT2003 model seem obviously ill-suited.

Let F^I be the cumulative distribution function of b_i^I. Assume that $P^I = B^I$. That is the firm absorbs as a membership membership or subsidy the membership cost of joining the platform. I do not assume that membership and usage fees are separately observable and therefore this assumption is without loss of generality. However care is necessary in mapping these prices back to observables. Note that given this assumption by equation 1 consumer i on side i will join the platform if and only if

$$b_i^I > p^I \tag{3}$$

and thus $n^I(p^I, p^{-I}, P^I, P^{-I}) = d^I(p^I) \equiv N^I\big(1 - F[p^I]\big)$ and thus the per-transaction price to each side determines demand on that side. The surplus of consumers on side I of the market takes a simple form: $d^{-I}(p^{-I})v^I(p^I)$ where $v^I(p^I) \equiv \int_{p^I}^{\infty} d^I(p)dp$. The profits of the platform are

$$\pi(p^A, p^B) = \big(p^A + p^B - c - c^A[p^B] - c^B[p^A]\big)d^A(p^A)d^B(p^B) \tag{4}$$

where $c^I(p^{-I}) \equiv \frac{C^I - B^I}{d^{-I}(p^{-I})}$. The firm's first-order conditions can be expressed in two simple ways

$$p^I - \left(c + c^I[p^J] - p^J\right) \quad = \quad m^I(p^I) \quad I = A, B \tag{5}$$

$$m \equiv p^A + p^B - \left(c + c^A[p^A] + c^B[p^B]\right) \quad = \quad m^A(p^A) - c^B(p^A) = m^B(p^A) - c^A(p^B) \tag{6}$$

where the *usage market power* on side I $m^I(p^I) \equiv -\frac{d^I(p^I)}{d^{I'}(p^I)}$. The first expression is perhaps more intuitive than the second. It states that the (usage) mark-up on side I over total side I cost is equated to the (usage) market power on side I. This is exactly the same as a standard market where $p - c = m(p)$ except that

1. Every dollar earned usage on side $-I$ acts as a subsidy to side I, as each additional side I participant brings the firm a profit of p^{-I} from side $-I$.

2. The usage cost to the firm on side I depends on the number of consumers participating on side $-I$, because of the effective two-sided scale economy created by the membership membership costs.

The second expression states that the total mark-up of the *price level* $\overline{p} \equiv p^A + p^B$ over total usage cost is equated to the market power on each side of the market, less the usage cost on the other side of the market. The usage cost on the other side of the market is subtracted from market power as the larger per-transaction costs are on the other side of the market, the more benefit the firm derives from attracting more customers on side I, as this allows them to reduce their effective per-transaction costs on the other side of the market.

The second-order condition for the monopolist's problem is, in addition to the standard condition (Weyl, 2008c) that the slope of market power be less than 1 on both sides of the market,

$$\tilde{r}^A(p^A, p^B)\tilde{r}^B(p^B, p^A) < 1 \tag{7}$$

where $\tilde{r}^I \equiv \frac{r^I(p^A)m}{m^{-I}(p^{-1})}$ is the *pseudo usage pass-through rate* (PUPTR) and $r^I(p^I) =$

$\frac{1}{1-m^{I'}(p^I)}$ is the *usage pass-through rate* (UPTR) on side I. The PUPTR captures the rate at which the cross-subsidies from one side of the market are pass-through to the other side. As long as $B^{-I} < C^{-I}$, $c^{-I} > 0$, $m < m^{-I}$ by equation (6) and $\tilde{r}^I < r^I$. Intuitively, the cross-subsidy of a rise in price on side $-I$ of the market to side I is smaller with membership costs on side I, as this rise reduces the participation on side $-I$, increasing the usage size of the side I membership costs. I consider the case of net membership costs (rather than benefits) to be focal: membership costs are likely more important than membership benefits, as the many benefits from two-sided firms are their two-sided, rather than membership, services. If this holds, condition (7) is less restrictive than the second-order condition in the standard RT2003 model (Weyl, 2008d), as the danger of explosion of cross-subsidies being passed back and forth across the two sides of the market is smaller. I assume that this condition holds globally, to ensure a global first-order solution.

A major focus of literature (Evans, 2003; Wright, 2004; Weyl, 2008d) based on the RT2003 model has been the distinction between the price level and the price structure or *price balance*, the division of the total usage price between consumers on the two sides of the market. At the optimal price level, the price balance is given by $m^A - c^B = m^B - c^A$; when $c^I = 0$ (the RT2003 model) this is $m^A = m^B$, the RT2003 optimal price balance, which holds even away from the optimal price level. However, in the GRT2003 model the formula for the optimal price balance depends on the price level. This is crucial, as we will see in the following section, to the effects of price controls. The optimal price balance given price level $\overline{p}$ is given by

$$\frac{m^A(p^A)}{m^B(p^B)} = \frac{\overline{p} - c^A(p^B)}{\overline{p} - c^B(p^A)} \tag{8}$$

2.2 Generalized Armstrong

In contrast to RT2003, the Armstrong model specializes RT2006 by focusing on membership value/cost heterogeneity for consumer and membership costs for firms. It assumes that

$b_i^I \equiv b^I$ for both I and that $c = 0$. Just as with the RT2003 model, the Armstrong model can be generalized without changing its basic economics by allowing arbitrary values for c. This is useful because many markets to which it is suggested the Armstrong model may apply, such as supermarkets, shopping malls, yellow pages and dating clubs, feature usage as well as membership membership costs.

This generalized Armstrong (GArmstrong) model may provide a reasonable approximation to dating clubs, yellow pages and commercial intermediation, as is often argued. It is not immediately apparent that the main source of heterogeneity among consumers in these sorts of markets is membership for membership (is the main source of difference among men using a dating service their membership costs or benefits of visiting a club?), but neither does it seem excludable *a priori*. Probably the most immediately compelling example is advertising industries. While consumers likely differ in their distaste for advertising, probably the major source of differentiation among them is the differing extent to which they value the newspaper, website or television program carrying the advertisements. Similarly, advertisers probably differ in the value they gain from an advertisement reaching consumers, but it seems plausible that, at least in some cases, the primary source of differentiation among them is the membership cost of creating the advertisement. Of course it is likely that consumers on both sides are differentiated along both dimensions[1], but such a possibility is beyond the scope of this paper.

Let G^I be the cumulative distribution of B_i^I and assume WLOG that $p_i^I = b_i^I$. This is the equivalent assumption in the GArmstrong context to the firm absorbing the membership costs (or benefits) of the consumer in the GRT2003 model: the firm absorbs the (homogeneous) usage costs or benefits of each side of the market. This assumption is the same "hedonic price" trick used by Armstrong. Again this assumption simplifies matters

[1] A simple way to generalize my analysis here without making things too complicated would be to assume that consumers vary along both dimensions, but that B_i is a function of b_i so that this two-dimesional variation is actually one dimensional. This is likely plausible in many cases: for example, consumers who like a newspaper most likely dislike the advertisements most as they are likely to be the wealthier consumers. This fits nicely with the Berry et al. (1995) view of consumer heterogeneity, though obviously in a much simpler context.

as consumers will join the platform if and only if $B_i^I > P^I$ and thus $n^I(p^I, p^{-I}, P^I, P^{-I}) = D^I(P^I) \equiv N^I\big(1 - G[P^I]\big)$. The surplus of consumers is simpler than in the GRT2003 model: $V^I(P^I) = \int_{P^I}^{\infty} D^I(P)dP$. The platform profits are

$$\pi(P^A, P^B) = (P^A - C^A)D^A(P^A) + (P^B - C^B)D^B(P^B) + (b^A + b^B - c)D^A(P^A)D^B(P^B) \quad (9)$$

The monopolist's first-order conditions are therefore

$$P^I - \big(C^I - \alpha D^{-I}[P^{-I}]\big) = M^I(P^I) \quad I = A, B \quad (10)$$

where the *net two-sidedness* $\alpha \equiv b^A + b^B - c$ and the *membership market power* on side I $M^I(P^I) \equiv -\frac{D^I(P^I)}{D^{I\prime}(P^I)}$. I assume that $\alpha > 0$; if not it would be optimal for the firm to separate the two markets, ending the two-sidedness. However it would be trivial to consider the case of $\alpha < 0$. Condition (10) is again a simple variation on the standard monopoly first-order condition for each good independently: here the only difference is that the price on each side of the market is subsidized proportional to the product of the net two-sidedness and the demand on the opposite side of the market.

Note that only the net two-sidedness matters to the membership pricing on each side. Imagine a market where $b^A \gg b^B$; Armstrong and empirical work based on his model (Kaiser and Wright, 2006) have argued that this imbalance in the two-sided benefits may account for observed skewness in prices (i.e. A is charged a high price and B is charged a low price). Given that shifting b^A and b^B while leaving their sum constant has no effect on membership pricing and therefore on demand, it is extremely simple to non-parametrically determine the effect this imbalance has on pricing if b^A and b^B are known. In particular suppose there were no imbalance but the same net two-sidedness: $\tilde{b}^A \equiv \tilde{b}^B \equiv \frac{b^A + b^B}{2}$. Then all that would change is the usage price to each side. Because usage and membership prices are not uniquely defined and we typically observe membership prices in the markets to which Armstrong's model is applied, we would expect the observed membership price on side A to fall by $\frac{b^A - b^B}{2}\hat{D}^B$ where $\hat{D}^B$ is the observed equilibrium demand and the price on side B to rise by $\frac{b^A - b^B}{2}\hat{D}^B$. Of course, b^A and b^B are typically not directly observable, so for the non-parametric approach to succeed we need a means of non-parametrically identifying b^A and b^B; this is supplied in Subsection 4.2 below.

The associated second-order condition, in addition to $M^{I'} < 1$ is

$$\alpha^2 D^{A'} D^{B'} R^A R^B < 1 \tag{11}$$

where the *membership pass-through rate* (MPTR) on side I $R^I(P^I) \equiv \frac{1}{1-M^{I'}(P^I)}$. Intuitively, a decrease in side A price will increase the two-sided cross-subsidy by $D^{A'}\alpha$. This is passed through to side B at a rate R^B and this increases the cross-susidy to side A by $\alpha D^{B'}$. If when this total $\alpha^2 D^{A'} D^{B'} R^B$ is passed-through at rate R^A prices fall by more than in the first round, the problem will explode. Thus the second-order condition, which I again assume holds globally, requires condition (11).

2.3 Hybrid

The final model I consider, one which is novel to this paper, combines the GRT2003 and GArmstrong models by assuming that side A consumers are usage (RT2003) heterogeneous (i.e. $B_i^A \equiv B^A$) and side B consumers are heterogenous in their membership valuations (i.e. $b_i^B \equiv b^B$). This model is of particular interest as it seems a plausible fit to several industries of interest. For example in video games, the usage (sale) benefit of game developers is likely fairly homogenous across developers as game prices tend to be quite similar (Evans et al., 2006; Lee, 2007), but their membership costs of development vary widely across title and developer. On the other hand, the membership costs or benefits of using a video game system are likely quite homogenous across gamers, but their benefit per game available on the system may differ widely. A similar argument fits quite well with the internet service provision industry and in advertising platforms if the many source of heterogeneity among advertisers is the benefits they receive per-view, this model would fit well given my arguments about about the likely predominately membership differentiation of content consumers.

Setting the analysis of this model requires an intuitive combination of the approaches to the the GRT2003 and GArmstrong models. Assume WLOG $P^A = B^A$, $p^B = b^B$, let the partial net two-sidedness $\tilde{\alpha} \equiv b^B - c$ and otherwise maintain *mutatis mutandis* the notation from above. Surplus on side B is, predictably, V^B and on side A is $v^A D^B$. Profits are given by

$$\pi(p^A, P^B) = \Big(P^B - C^B + \big[p^A - c - c^A(P^B)\big] d^A[p^A]\Big) D^B(P^B) \tag{12}$$

Then the first-order conditions for the Hybrid model are

$$\begin{aligned} p^A - \big(c^A[P^B] + c - b^B\big) &= m^A(p^A) \quad (13) \\ P^B - \big(C^B - [\tilde{\alpha} + p^A]d^A[p^A]\big) &= M^B(P^B) \quad (14) \end{aligned}$$

Note the optimal (usage) price on side A is independent of the (membership) price on side B if the net membership cost on side A, $C^A - B^A$, is 0. Furthermore if $c^A = 0$ then the monopolist chooses p^A to maximize $(\tilde{\alpha} + p^A)d^A(p^A)$ and thus, by the envelope theorem, the optimal value of P^B is independent of p^A, at the first order. Thus it is the existence of net membership costs or benefits on side A that link pricing on the two sides of the market. The second order condition is therefore easily satisfied if c^A is small and again I assume it holds globally:

$$\frac{{c^A}^2 r^A R^B d^A}{m^A M^B} < 1 \tag{15}$$

3 Comparative Statics

In this section I consider the comparative statics of pricing, demand and welfare with respect to (multiplicative) shifts in demand, market power (holding demand constant) and homogenous consumer costs or benefits, as well as the effects of imposing a price control. While these comparative statics only directly answer a small number of policy-relevant questions, we will see in Section 5 that many if not most policy-relevant questions about these models depend in simple ways on these comparative statics.

One important comparative static I do note consider here is the effect of cost variations on prices and welfare. These are deliberately deferred to the next section, where they are used as the basis of identification. However, the comparative static effects of homogenous consumer benefits an costs are effectively the same as those of the corresponding costs, as we will see below.

In each model my approach is local, but fully general, determining for arbitrary demand systems assumed to globally satisfy second-order conditions what the effects of changes in certain parameters is on the equilibrium.

3.1 GRT2003

Because usage and membership prices are not uniquely defined in the model, I define two notions which are more likely to correspond to observable prices. The *observable usage price*, *observable price level* and *observable membership* price on side I are

$$\hat{p}^I \equiv p^I + \frac{B^I}{d^J(p^J)} \tag{16}$$

$$\hat{\bar{p}} \equiv p^A + p^B \tag{17}$$

$$\hat{P}^I \equiv B^I + p^I d^{-I}(p^{-I}) = d^{-I}(p^{-I})\hat{p}^I \tag{18}$$

In empirical terms, these correspond, respectively, to the sum of all net charges (in dollars or dollar-equivalent "incentives") charged by the company for its services over some period (say a year) *per participant on the other side of the market* and to the sum of all net charges over that period in gross. Note that if we assume that the match rate is constant (as the model effectively does by imposing linear valuation of participation on the other side), then then usage price multiplied divided by this rate is the *match price*, for example the price per use of a credit card or price per date on a dating website. This match price is emphasized by RT2003, while the usage price is emphasized by RT2006. Of course, the assumption of constant match rate strains credibility, but this is as much a problem with the model overall as it is with this particular interpretation. Throughout this subsection (to establish some comparative statics) I maintain the following assumption:

Assumption 1. *$C^I > 0 > B^I$ for $I = A, B$.*

It is easy to consider the case when some of these inequalities are reversed, but I do not consider this case particularly empirically relevant and so do not address it here. As is typical in multi-variable optimization, all comparative statics are scaled up by the inverse of the stability factor (the degree of obedience of the second-order condition):

$$\sigma_{\text{RT2003}} \equiv \frac{1}{1 - \tilde{r}^A \tilde{r}^B} \tag{19}$$

3.1.1 Demand

A multiplicative shift in demand on on one side of the market has no effect on market power, but it does reduce the per-interaction size of the firm and consumer membership costs and therefore affects optimal pricing. Formally suppose that demand increases by a multiplicative factor δ on side I opposite side J. Then equilibrium is given by

$$p^I - \left(c + c^I[p^J] - p^J\right) = m^I(p^I)$$

$$p^J - \left(c + \frac{c^J[p^I]}{\delta} - p^I\right) = m^J(p^J)$$

$$\hat{p}^I = p^I + \frac{B^I}{d^J(p^J)}$$

$$\hat{p}^J = p^J + \frac{B^J}{d^I(p^I)\delta}$$

$$\hat{P}^I = d^J(p^J)\hat{p}^I$$

$$\hat{P}^J = \delta d^I(p^I)\hat{p}^J$$

$\frac{1}{\sigma_{\mathrm{RT2003}}} \frac{\partial \text{ row}}{\partial \delta}\Big\vert_{\delta=1}$	Magnitude	Sign
p^I	$\tilde{r}^I r^J c^J$	$+$
p^J	$-r^J c^J$	$-$
$\overline{p}$	$-r^J c^J(1-\tilde{r}^I)$	?
I-side demand	$d^{I'}\tilde{r}^I r^J c^J$	$-$
J-side demand	$-d^{J'} r^J c^J$	$+$
I-side welfare	$\left(\frac{\overline{v}^I}{m^I}\frac{m^I}{m^I-c^J} - r^I\right)\frac{d^I d^J r^J c^J (m^J - c^I)}{m^J}$	$+$*
J-side welfare	$\left(1 - \frac{m^I-c^J}{m^I} r^I \frac{\overline{v}^J}{m^J}\right) d^I d^J r^J c^J$	$+$*
Consumer welfare	$\left(1 + \frac{\overline{v}^I}{m^J} - \tilde{r}^I\left[1 + \frac{\overline{v}^J}{m^I}\right]\right) d^I d^J r^J c^J$	$+$*
Profits	$c^J d^I d^J$	$+$*
Social welfare	$\left(\left[1 + \frac{\overline{v}^I}{m^J} - \tilde{r}^I\left(1 + \frac{\overline{v}^J}{m^I}\right)\right] r^J + 1\right) d^I d^J c^J$	$+$*
$\hat{p}^I$	$r^J c^J\left(\tilde{r}^I - \frac{B^I}{d^J m^J}\right)$	$+$
$\hat{p}^J$	$-\left(\frac{B^J}{d^J}\left[1 - \tilde{r}^I\tilde{r}^J - \frac{\tilde{r}^I r^J c^J}{m^I}\right] + r^J c^J\right)$	?
$\hat{\overline{p}}$	$\frac{B^J}{d^J}\left(1 - \tilde{r}^I\tilde{r}^J\right) + r^J c^J\left(\tilde{r}^I\left[1 - \frac{B^J}{d^J m^I}\right] - \frac{B^B}{d^J m^J} - 1\right)$	?
$\hat{P}^I$	$d^J r^J c^J\left(\tilde{r}^I + \frac{p^I}{m^J}\right)$	?
$\hat{P}^J$	$d^I\left(p^J\left[1 - \tilde{r}^I\tilde{r}^J - \frac{\tilde{r}^I r^J c^J}{m^I}\right] - r^J c^J\right)$	?

Table 1: I-side demand comparative statics for GRT2003 model, scaled down by $\frac{1}{\sigma_{\mathrm{RT2003}}}$

Differentiating and solving the resultant system of equations yields the comparative statics in Table 1. These are all of the form $\frac{\partial X}{\partial \delta}$, where X is the row variable. Following the expression is a "+", "−" or "?" indicating whether the comparative static is always positive, always negative or that its sign depends on parameter values. For normative results, some of the signs are marked with a star; this indicates that the guarantee of sign holds under the (broad and influential) Bulow and Pfleiderer (1983) constant pass-through class of demands, but counter-examples exist outside of this class. Also demand on side I and all normative and profit variables are divided by δ, when this shifts away from 1, to avoid counting simple scale effects. This can be viewed as profit per market participant and δ as a scaling up of N^I; it therefore still counts the direct effect of increasing δ on decreasing the usage size of opposite side membership costs. The following proposition expresses these results formally

Proposition 1. *The first-order effect of a change in δ starting at $\delta = 1$ are given in Table 1. They are marked by a "+" when they are always positive, "−" when always negative and "?" when parameter values (obeying second-order conditions and positivity, monotonicity and continuity of demand) exist for which they are either positive or negative. Sign results with a star indicate that the result holds if pass-through is constant, but need not otherwise.*

Proof. The results follow from simple differentiation and solving of the resulting system of two equations. All signed comparative statics are trivial consequences of $r^I, \tilde{r}^I, d^I, c^I > 0 > d^{I'}$ at all prices; all indeterminate ones have simple counter examples obeying second-order conditions drawn from the Bulow and Pfleiderer (1983) constant pass-through demand class. I omit these here, but they are available on request. In the case of normative results, sign guarantees with stars hold under constant pass-through, all following directly from the fact that in this demand class $\frac{\overline{v}}{d} = r$ (Weyl, 2008c), and there are simple counter examples outside this class. ?'s indicate indeterminacy even within this class. □

The only effect of an increase in demand on side I of the market, other than scaling things up, is to reduce the per-interaction (usage) size of the fixed membership costs of consumers and the firm on side J. This reduces prices on side J and therefore reduces the cross-subsidy to side I, raising prices on side I. This is the celebrated *topsy-turvy* effect of RT2003, commonly called the *see-saw* effect (Weyl, 2008d; Lerner and Tirole, 2008). Whether this may lead an increase in demand to raise the price level depends, as usual, on the pass-through rate.

While the rationale for the seesaw effect seems clear in the GRT2003 model, it is actually a bit subtler than it appears. While a rise in the usage price on side I of the market would seem to encourage a reduction of prices on side J, this depends on the fact that increasing demand on side J has no effect on side I demand. In the Armstrong model this will not be the case and the seesaw effect will fail. What really drives the seesaw effect is that the firm internalizes the external usage benefits of the *marginal* consumer, the consumer indifferent about her participation in the system. Because in the GRT2003 model the only source of heterogeneity among consumers is their usage valuation, higher prices inherently mean a higher usage benefit of the marginal consumer. Usage prices on the two sides of the market are therefore *substitutes* for the monopolist. However, in the GArmstrong model all consumers have the same usage benefit and therefore an increase in price has no effect on the marginal consumer's usage benefit and, as we will see below, membership prices to the two sides will therefore be *complements*.

The seesaw effect does not necessarily mean that consumers on side I are harmed by their demand increasing, as they benefit from the increase in J-side demand. Because the firm internalizes only the usage benefits of the *marginal* and not the *average* I-side consumer, much depends on the size of the average (external) usage benefit derived by I-side consumers from the resultant expansion on J-side demand. In fact because pass-through rates are closely tied to average surplus, the benefits to them of this rise in J-side demand scale with the rate at which the fall in J-side prices is passed through to side I. In fact, under constant pass-through, this effect is strong enough to guarantee that the increase in demand on side I always benefits side J and does not harm side I even without membership costs on side I[2] (even when scale effects are ignored). When side I membership costs are added, the benefits of side J rises in demand increase, leading to strictly positive side-I benefits. Of course this depends sensitively on the link between pass-through and average surplus, which is unique to the monopoly setting and constant pass-through demand. I will not discuss extensively

[2] This softens, in a very restricted setting, the counterintuitive possibilities under total cost amplification that I established in Weyl (2008d).

the effect of changes in parameters on observable prices, as we rarely engage in retrospective analysis of shifts in demand, market power and homogenous membership benefits, but these are simple extensions of the reasoning about unobservables.

3.1.2 Market power

On the other hand, an additive shift in market power leaving constant demand (a shift in demand elasticity keeping level constant) directly effect pricing on the side where it takes place and indirectly effects the price on the other side of the market through the seesaw effect. This can be written as

$$p^I - \left(c + c^I[p^J] - p^J\right) = m^I(p^I) + \mu$$

$$p^J - \left(c + c^J[p^I] - p^I\right) = m^J(p^J)$$

where μ is the amount of the shift and the definitions of observables in terms of unobservables stay fixed. The comparative static effects of an increase in μ starting at $\mu = 0$ are expressed in Table 2 in the same format as in Table 1.

Proposition 2. *The first-order effect of a change in μ starting at $\mu = 0$ are given in Table 2, in the same format as in Table 1 for Proposition 1.*

Proof. The proof is exactly as in Proposition 1. □

Like a shift in I-side demand, shift in I-side market power only directly effects pricing on one side of the market: an increase in market power raises prices on side I. Again this is passed through the seesaw effect as a price decrease on side J but, under constant pass-through, it still leads both sides to be worse off.

3.1.3 Homogeneous consumer membership costs/benefits

Finally suppose there is an additive decrease in the consumer membership cost on side I of the market. This is equivalent in its effect on the (unobservable) usage prices to an decrease

$\frac{1}{\sigma_{\text{RT2003}}} \frac{\partial \text{ row}}{\partial \mu}\Big\vert_{\mu=0}$	Magnitude	Sign
p^I	r^I	$+$
p^J	$-\tilde{r}^J r^I$	$-$
$\overline{p}$	$r^I(1-\tilde{r}^J)$	?
I-side demand	$d^{I\prime} r^I$	$-$
J-side demand	$-d^{J\prime}\tilde{r}^J r^I$	$+$
I-side welfare	$-\left(1-\frac{m^J-c^I}{m^J} r^J \frac{\overline{v}^I}{m^I}\right) d^I d^J r^I$	$-*$
J-side welfare	$-\left(\frac{m^J}{m^J-c^I}\frac{\overline{v}^J}{m^J} - r^J\right)\frac{d^I d^J r^I (m^I-c^J)}{m^I}$	$-*$
Consumer welfare	$-\left(1+\frac{\overline{v}^J}{m^I} - \tilde{r}^J\left[1+\frac{\overline{v}^I}{m^J}\right]\right) d^I d^J r^I$	$-*$
Profits	0	0
Social welfare	$-\left(1+\frac{\overline{v}^J}{m^I} - \tilde{r}^J\left[1+\frac{\overline{v}^I}{m^J}\right]\right) d^I d^J r^I$	$-*$
$\hat{p}^I$	$r^I\left(1-\frac{\tilde{r}^J B^I}{d^J m^J}\right)$	$+$
$\hat{p}^J$	$r^I\left(\frac{B^J}{d^I m^I} - \tilde{r}^J\right)$	$-$
$\hat{\overline{p}}$	$r^I\left(m[d^I - \tilde{r}^I d^J] + C^J - \tilde{r}^I C^I\right)$	?
$\hat{P}^I$	$r^I d^J\left(1+\frac{\tilde{r}^J p^I}{m^J}\right)$	?
$\hat{P}^J$	$r^I d^I\left(\frac{p^J}{m^I} - \tilde{r}^J\right)$	?

Table 2: I-side market power comparative statics for GRT2003 model, scaled down by $\frac{1}{\sigma_{\text{RT2003}}}$

in firm membership cost on side I, though it also directly effects the observable prices as well.

$$p^I - \left(c + c^I[p^J] - \frac{\beta}{d^J[p^J]} - p^J\right) = m^I(p^I)$$

$$p^J - \left(c + c^J[p^I] - p^I\right) = m^J(p^J)$$

$$\hat{p}^I = p^I + \frac{B^I + \beta}{d^J(p^J)}$$

$$\hat{p}^J = p^J + \frac{B^J}{d^I(p^I)}$$

$$\hat{P}^I = d^{-I}(p^J)\hat{p}^I,\ I = A, B$$

where β is the amount of the shift. These comparative statics are expressed in Table 3 and in the following proposition.

Proposition 3. *The first-order effect of a change in β starting at $\beta = 0$ are given in Table*

$\frac{1}{\sigma_{\text{RT2003}}}\frac{\partial \text{ row}}{\partial \text{ column}}\Big\vert_{\beta=0}$	Magnitude	Sign
p^I	$-\frac{r^I}{d^J}$	$-$
p^J	$\frac{\tilde{r}^J r^I}{d^J}$	$+$
$\overline{p}$	$-\frac{r^I(1-\tilde{r}^J)}{d^J}$	?
I-side demand	$\frac{-d^I r^I}{d^J}$	$+$
J-side demand	$\frac{-\tilde{r}^J r^I}{m^J}$	$-$
I-side welfare	$\left(1-\frac{m^J-c^I}{m^J}r^J\frac{\overline{v}^I}{m^I}\right)d^I r^I$	$+^*$
J-side welfare	$\left(\frac{m^J}{m^J-c^I}\frac{\overline{v}^J}{m^J}-r^J\right)\frac{d^I r^I(m^I-c^J)}{m^I}$	$+^*$
Consumer welfare	$\left(1+\frac{\overline{v}^J}{m^I}-\tilde{r}^J\left[1+\frac{\overline{v}^I}{m^J}\right]\right)d^I d^J r^I$	$+^*$
Profits	d^I	$+$
Social welfare	$\left(\left[1+\frac{\overline{v}^J}{m^I}-\tilde{r}^J\left(1+\frac{\overline{v}^I}{m^J}\right)\right]r^I+1\right)d^I$	$+^*$
$\hat{p}^I$	$\frac{1-r^I}{d^J}-\frac{r^I\tilde{r}^J B^I}{(d^J)^2 m^J}$	?
$\hat{p}^J$	$\frac{r^I}{d^J}\left(\tilde{r}^J-\frac{B^J}{d^I m^I}\right)$	$+$
$\hat{\overline{p}}$	$\frac{1}{d^J}\left(1+r^I\left[\tilde{r}^J\left(1-\frac{B^I}{d^J m^J}\right)-\frac{B^J}{d^I m^I}\right]\right)$	?
$\hat{P}^I$	$1-r^I\left(1-\frac{\tilde{r}^J p^I}{m^J}\right)$	?
$\hat{P}^J$	$\frac{r^I d^I}{d^J}\left(\tilde{r}^J-\frac{p^J}{m^I}\right)$	?

Table 3: B^I comparative statics in the GRT2003 model (scaled down by $\frac{1}{\sigma_{\text{RT2003}}}$)

3, in the same format as in Table 1 for Proposition 1.

Proof. The proof is exactly as in Proposition 1. □

Just as with shifts in demand and market power, the fall in homogenous costs affects only one side of the market directly, reducing prices on side I and raising them on side J. Thus the comparative static effects of all three shifts are essentially the same in their unobservable price and welfare effects, differing only in scale and their effects on profits. All of these results are simple extensions of the logic of the standard RT2003 model I developed in Weyl (2008d).

3.1.4 Price controls

I now turn to the effects of imposing price controls. Following my first-order approach through out this paper, I will consider two questions. First, starting at unregulated prices, what is the effect on prices of imposing a "just" binding price control? Second, starting at regulated prices, what is the effect of slightly relaxing the price control? Even assuming that a regulator can only enforce controls on observables, there are many sorts of price controls a regulator might impose; in fact there is a very large continuum of such controls as the regulator could require an arbitrary function $\kappa(\hat{p}^A, \hat{p}^B, \tilde{d}^A, \tilde{d}^B) < \overline{\kappa}$, where $\tilde{d}^I$ is the demand on side I, d^I for the GRT2003 model and D^I for the GArmstrong model. In addressing my first problem, of just binding price controls, I will consider general κ. When starting at regulated prices, I will specialize to more.

This first problem, of just binding controls" can be most easily addressed using the following theorem, which is a simple consequence of symmetry. I am nearly certain this theorem is well-known, but I have found no reference for it and therefore include a proof.

Consider the n variable unconstrained optimization problem

$$\mathbf{x}^\star = \text{argmax}_{\mathbf{x}} F(\mathbf{x}) - \sum_{i=1}^{n} w_i x_i \tag{20}$$

where column vectors $\mathbf{x}, \mathbf{w} \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a strictly concave function and x_i, w_i represent the ith entry of the vectors $\mathbf{x}$ and $\mathbf{w}$ respectively. Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be a twice continuously differentiable function and consider the maximization problem in equation (20) subject to the constraint that $K(\mathbf{x}) < \overline{K}$. Assume that $\forall \overline{K} > \tilde{K}$, this program has an interior solution given some $\tilde{K} < K(\mathbf{x}^\star)$ and call this solution to this problem $\overline{\mathbf{x}}^\star$. Let $\Delta F, \Delta K$ be the column vector gradient of F and K respectively and let $H(F)$ be the nxn matrix whose i,j entry is $\frac{\partial^2 F}{\partial x^i \partial x^j}$.

Theorem 1.

$$\left.\frac{\partial \overline{x}_i^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)^-} = \frac{\left.\frac{\partial K(\mathbf{x})}{\partial w_i}\right|_{\mathbf{x}=\mathbf{x}^\star}}{\Delta K(\mathbf{x}^\star)^\top H\left(F[\mathbf{x}^\star]\right) \Delta K(\mathbf{x}^\star)}$$

and therefore

$$sign\left(\left.\frac{\partial \overline{x}_i^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)^-}\right) = -sign\left(\frac{\partial K(x^\star)}{\partial w_i}\right)$$

Proof. See Appendix A. □

Intuitively this theorem is just a simple extension of classical demand symmetry. Transform the problem so that the constrained function K of the choice variables replaces one of the choice variables. Then the result simply states that if an increase in the price of variable A leads to a decrease (increase) in the optimal choice of variable K, then a forced reduction in K will lead to an increase (decrease) in the chosen value of A. A simple corollary of this, the following proposition, links the effect of market power (an opportunity cost having more consumers on one side) on the optimal observable price to the effect of a control on that price on the optimal choice of demand and therefore unobservable prices.

Proposition 4. *Let $\hat{p}^{I^\star}, \tilde{d}^{I^\star}$ be the monopolist's unconstrained optimal choice of observable I-side price and demand. Suppose a price control is placed on some observed membership or usage price $\kappa(\hat{p}^A, \hat{p}^B, \tilde{d}^A, \tilde{d}^B)$ at price $\overline{\kappa}$. Let $\tilde{p}^I$ be the unobservable, welfare-relevant price on side I and let $\pi(\tilde{p}^A, \tilde{p}^B)$ be the monopolist's profits. Let $\kappa(\tilde{p}^A, \tilde{p}^B) \equiv \kappa\left(\hat{p}^A[\tilde{p}^A, \tilde{p}^B], \hat{p}^B[\tilde{p}^B, \tilde{p}^A], \tilde{d}^A[\tilde{p}^A], \tilde{d}^B[\tilde{p}^B]\right)$. Finally let $\tilde{p}^{I^\star}$ be the monopolist's unconstrained optimal choice of I side unobservable welfare-relevant price and let $\tilde{p}^{I^{\star\star}}$ be the same choice subject to $\kappa < \overline{\kappa}$.*

$$sign\left(\left.\frac{\partial \tilde{p}^{I^{\star\star}}}{\partial \overline{\kappa}}\right|_{\overline{\kappa}=\kappa(\hat{p}^{A^\star}, \hat{p}^{B^\star}, \tilde{d}^{A^\star}, \tilde{d}^{B^\star})}\right) = sign\left(\frac{\partial \kappa(\tilde{p}^{A^\star}, \tilde{p}^{B^\star})}{\partial C^I}\right)$$

and in particular

$$\left.\frac{\partial \tilde{p}^{I^{\star\star}}}{\partial \overline{\kappa}}\right|_{\overline{\kappa}=\kappa(\hat{p}^{A^\star}, \hat{p}^{B^\star}, \tilde{d}^{A^\star}, \tilde{d}^{B^\star})} = \left.\frac{\left(\tilde{d}^{A'}\tilde{d}^{B'}\right)^2}{\tilde{d}^{I'}\left(\left[\tilde{d}^{A'}\right]^2\kappa_1^2\pi_{11} + 2\tilde{d}^{A'}\tilde{d}^{B'}\kappa_1\kappa_2\pi_{12} + \left[\tilde{d}^{B'}\right]^2\kappa_2^2\pi_{22}\right)}\right|_{(\tilde{p}^I, \tilde{p}^J)=(\tilde{p}^{I^\star}, \tilde{p}^{J^\star})} \frac{\partial \kappa(\tilde{p}^{A^\star}, \tilde{p}^{B^\star})}{\partial C^I}$$

Proof. See Appendix A. □

This result gives a simple means of computing the effect of a small price control on unobservable prices (and thereby on welfare) on both sides of the market. I omit the actual formulae here for brevity, but they are easy to compute from the comparative statics above. Note that the derivative of κ with respect to a change in membership costs is the same as the derivative of κ with respect to an increase in market power in the in the GArmstrong model and the derivative with respect to an increase in market power, scaled down by demand on the other side of the market, in the GRT2003 model. As I will discuss further below this allows an easy empirical test to determine the effects of a price control based on exogenous shocks to membership prices. If membership price shocks are unavailable and one must rely on usage cost shocks, as in the original RT2003 model discussed in subsection 4.3.

The final draft of this paper may include comparative statics at price controlled levels and tests, at those levels, for the effects of removing price controls.

3.2 GArmstrong

Observable price variables in the Armstrong model are

$$\hat{p}^I = b^I + \frac{P^I}{D^J(P^J)}$$

$$\hat{\overline{p}} = b^A + b^B + \frac{P^A}{D^B(P^B)} + \frac{P^B}{D^A(P^A)}$$

$$\hat{P}^I = b^I D^J(P^J) + P^I$$

The appropriate inverse stability factor in the Armstrong model is

$$\sigma_{GA} = \frac{1}{1 - \alpha^2 D^{A\prime} D^{B\prime} R^A R^B} \tag{21}$$

$\frac{1}{\sigma_{\text{GA}}} \frac{\partial \text{ row}}{\partial \delta}\Big\vert_{\delta=1}$	Magnitude	Sign
P^I	$\alpha^2 D^{J'} D^I R^I R^J$	$-$
P^J	$-\alpha D^I R^J$	$-$
I-side demand	$\alpha^2 D^{J'} D^{I'} D^I R^I R^J$	$+$
J-side demand	$-\alpha D^{J'} D^I R^J$	$+$
I-side welfare	$-\alpha^2 D^{J'} D^{I^2} R^I R^J$	$+$
J-side welfare	$\alpha D^J D^I R^J$	$+$
Consumer welfare	$\alpha D^I D^J R^J \left(\frac{\alpha D^I R^I}{M^J} + 1\right)$	$+$
Profits	0	0
Social welfare	$\alpha D^I D^J R^J \left(\frac{\alpha D^I R^I}{M^J} + 1\right)$	$+$
$\hat{p}^I$	$-\frac{\alpha D^I R^J (P^I + \alpha D^J R^I)}{D^J M^J}$	?
$\hat{p}^J$	$-R^J \left(\alpha + P^J \left[\frac{1}{D^I} - \frac{\alpha^2 D^{J'} R^I}{M^I}(1 - R^J)\right]\right)$	?
$\hat{\overline{p}}$	$-R^J \left(\frac{P^J}{D^I} + \alpha\left[1 + \frac{D^I (P^I + \alpha D^J R^I)}{D^J M^J} - \frac{\alpha D^{J'} R^I}{M^I}(1 - R^J)\right]\right)$	?
$\hat{P}^I$	$\alpha R^J D^{J'} D^I \left(\alpha R^I - b^I\right)$	?
$\hat{P}^J$	$D^I (b^J - R^J \alpha)$	?

Table 4: I-side demand comparative statics for GArmstrong model, scaled down by $\frac{1}{\sigma_{\text{GA}}}$

3.2.1 Demand

As in the GRT2003 model, scaling up demand on side I has no direct effects on pricing side I. Rather its only effect is on the scale of homogenous costs and benefits. In the Armstrong model these are usage costs and benefits; thus an increase demand rather than reducing the per-interaction size of membership costs now increases the per-member size of net two-sidedness. Formally if demand on side I is scaled up by δ equilibrium is given by

$$P^I - C^I - \left(c - \alpha D^J[P^J]\right) = M^I(P^I)$$

$$P^J - C^J - \left(c - \alpha \delta D^I[P^I]\right) = M^J(P^J)$$

$$\hat{p}^J = b^J + \frac{P^J}{\delta D^I(P^I)}$$

$$\hat{\overline{p}} = \hat{p}^J + \hat{p}^I$$

$$\hat{P}^J = \delta b^J D^I(P^I) + \hat{P}^J$$

The definitions of $\hat{p}^I$ and $\hat{P}^I$ in terms of unobservables remain the same. The comparative statics are expressed in Table 4. As in the GRT2003 model, welfare, profits and demand directly involving demand on side I are scaled down by δ to avoid measuring direct scale effects. These results are stated formally in the following proposition:

Proposition 5. *The first-order effect of a change in δ starting at $\delta = 1$ in the GArmstrong model are given in Table 4, in the same format as in Table 1 for Proposition 1. The only change is that now there are no conditional results marked by *s which hold on in the constant pass-through class.*

Proof. The proof is exactly as in Proposition 1, except that the Armstrong first-order conditions are used. □

In the GArmstrong model, increasing demand on side I of the market raises the membership scale of the net two-sidedness (net usage benefits to all participants) to members on side J. Because the monopolist internalizes all net two-sidedness this effectively increases the membership size of the subsidy to them, lowering the membership price they face. This, in turn, increases demand on side J and thereby increase the subsidy to side I, decrease the membership price on side I. Thus membership prices to the two sides are *complements* in the GArmstrong model, not substitutes. The crucial reason is that the usage benefits are the same for all consumers on one side of the market, so raising prices does not raise the usage benefit of the marginal consumer and therefore the subsidy internalized by the monopolist does not rise as prices do.

Furthermore, the welfare economics of the GArmstrong model are much simpler than those of the GRT2003 model. Because all consumer on a given side have the same usage benefit and because this benefit is fully internalized by the monopolist, conditional on the membership price (and a usage price equal to the homogenous usage benefit) the welfare of each side is independent of participation on the other side. Thus, as in a standard market, welfare on side I is determined entirely by P^I and, therefore, because prices are complements, welfare on the two sides move together as well regardless of the shape of demand. Again, I defer the discussion of effects on observables to the following section on identification.

3.2.2 Market power

As in the GRT2003 model, an increase in (membership) market power on side I is equivalent to raising the (membership) costs on that side.

$$P^I - C^I - \big(c - \alpha D^J[P^J]\big) = M^I(P^I) + \mu$$

$$P^J - C^J - \big(c - \alpha D^I[P^I]\big) = M^J(P^J)$$

The definition of all observables in terms of unobservables remain fixed. The comparative static effects of this shift are shown in Table 5 and stated in the following proposition:

Proposition 6. *The first-order effect of a change in δ starting at $\mu = 0$ in the GArmstrong model are given in Table 5, in the same format as in Table 4 for Proposition 5.*

Proof. Same as Proposition 5. □

An increase in market power raises prices on side I and, through complementarity, on side J. This makes both sides worse off, just as a rise in demand on one side makes both sides better off, even beyond the direct scale effect.

3.2.3 Homogenous consumer usage benefits/costs

Unlike every other shift considered above in either model, a rise in the homogenous usage costs on one side of the market directly affects pricing on *both* sides of the market, because the net two-sidedness determines the (internalized) subsidy given to both sides.

$$P^I - \big(c - [\alpha + \beta]D^J[P^J]\big) = M^I(P^I)$$

$$P^J - \big(c - [\alpha + \beta]D^I[P^I]\big) = M^J(P^J)$$

$\frac{1}{\sigma_{\text{GA}}} \frac{\partial \text{ row}}{\partial \mu}\Big\vert_{\mu=0}$	Magnitude	Sign
P^I	R^I	$+$
P^J	$-\alpha D^{I'} R^I R^J$	$+$
I-side demand	$D^{I'} R^I$	$-$
J-side demand	$-\alpha D^{J'} D^{I'} R^I R^J$	$-$
I-side welfare	$-R^I D^I$	$-$
J-side welfare	$\alpha D^J D^{I'} R^I R^J$	$-$
Consumer welfare	$-R^I D^I \left(1 + \frac{D^J R^I R^J}{M^I}\right)$	$-$
Profits	0	0
Social welfare	$-R^I D^I \left(1 + \frac{D^J R^I}{M^I}\right)$	$-$
$\hat{p}^I$	$\frac{R^I}{D^J M^J}(M^J - P^I \alpha D^{I'} R^J)$	?
$\hat{p}^J$	$\frac{R^I}{M^I D^I}(\alpha R^J D^I + P^J)$	?
$\hat{\bar{p}}$	$\frac{R^I}{M^I D^I}(\alpha R^J D^I + P^J) + \frac{R^I}{D^J M^J}(M^J - P^I \alpha D^{I'} R^J)$	?
$\hat{P}^I$	$R^I(1 - b^I \alpha D^{I'} D^{J'} R^J)$	?
$\hat{P}^J$	$-R^I D^{I'}(\alpha R^J - b^J)$	?

Table 5: I-side market power comparative statics for GArmstrong model, scaled down by $\frac{1}{\sigma_{\text{GA}}}$

$$\hat{p}^I = \beta + b^I + \frac{P^I}{D^J(P^J)}$$

$$\hat{\bar{p}} = \hat{p}^I + \hat{p}^J$$

$$\hat{P}^I = \hat{p}^I D^J(P^J)$$

The definitions of $\hat{p}^J, \hat{P}^J$ in terms of unobservables remain fixed. The comparative static effects of an increase (decrease) in I-side usage benefits (costs) are shown in Table 6 and given in the following proposition.

Proposition 7. *The first-order effect of a change in β starting at $\beta = 0$ in the GArmstrong model are given in Table 6, in the same format as in Table 4 for Proposition 5.*

Proof. Same as Proposition 5. □

An increase in the usage benefit on side I benefits both sides for two reasons. First, it increases the size of the net two-sidedness internalized by the firm, increasing the subsidy to each side and directly decreasing membership prices on each side. Second, this effect is

$\frac{1}{\sigma_{\text{GA}}} \frac{\partial \text{ row}}{\partial \beta}\Big\rvert_{\beta=0}$	Magnitude	Sign
P^I	$R^I D^{J'}(\alpha R^J D^I + M^J)$	$-$
P^J	$R^J D^{I'}(\alpha R^I D^J + M^I)$	$-$
I-side demand	$D^{I'} D^{J'} R^I(\alpha R^J D^I + M^J)$	$+$
J-side demand	$R^J D^{J'} D^{I'}(\alpha R^I D^J + M^I)$	$+$
I-side welfare	$-R^I D^I D^{J'}(\alpha R^J D^I + M^J)$	$+$
J-side welfare	$-R^J D^J D^{I'}(\alpha R^I D^J + M^I)$	$+$
Consumer welfare	$D^{I'} D^{J'}\big([R^I + R^J]M^I M^J + \alpha R^I R^J[M^I D^I + M^J D^J]\big)$	$+$
Profits	$D^I D^J$	$+$
Social welfare	$D^{I'} D^{J'}\big([R^I + R^J + 1]M^I M^J + \alpha R^I R^J[M^I D^I + M^J D^J]\big)$	$+$
$\hat{p}^I$	$-\frac{R^I}{M^J}(\alpha R^J D^I + M^J) + \frac{P^I R^J D^{I'}}{D^J M^J}(\alpha R^I D^J + M^I) + 1 - \alpha^2 D^{I'} D^{J'} R^I R^J$	?
$\hat{p}^J$	$-R^J\big(\frac{\alpha R^I D^I}{M^I} + 1\big) + \frac{P^J D^{J'} R^I}{M^I}\big(\alpha R^I + \frac{M^J}{D^I}\big)$	?
$\hat{\bar{p}}$	Sum of above two, omitted for brevity	?
$\hat{P}^I$	$D^J\left(1 - \alpha^2 D^{I'} D^{J'} R^I R^J - \frac{1}{M^J}\left[R^I R^I D^I \alpha\left(1 - \frac{D^J b^I}{M^I}\right) + R^I M^J + R^J M^I D^{I'} b^I\right]\right)$	?
$\hat{P}^J$	$D^{I'}\left(R^I R^I D^J \alpha\left[1 - \frac{D^I b^J}{M^J}\right] + R^J M^I + R^I M^J D^{J'} b^J\right)$	?

Table 6: I-side homogenous usage benefit/cost comparative statics for GArmstrong model, scaled down by $\frac{1}{\sigma_{\text{GA}}}$

reinforced through complementarity. The most important thing to note is that the effect of an increase in the usage benefit has symmetric effects across the two sides on unobservables, in the sense that an increase in b^A has the same effects on membership prices, demand and welfare on both sides of the market as the same increase in b^B. This is a simple consequence of my earlier observation that only *net* two-sidedness is welfare relevant. The only effective difference between an increase in b^A and one in b^B is on observables, namely the welfare-irrelevant usage price will go up to side A rather than B and thus we will observe a larger rise or less of a fall in A's price rather than would have been the case if b^B has risen. This means that in terms of welfare and unobservables, it is simple and easy to ask what the effect of an increase in overall two-sidedness is in the context of the Armstrong model, an issue I focus on in Section 5.

3.3 Hybrid

Observable price variables in the Hybrid model are

$$\hat{p}^A = p^A + \frac{B^A}{D^B(P^B)}$$

$$\hat{p}^B = b^B + \frac{P^B}{d^A(p^A)}$$

$$\hat{\hat{p}} = p^A + b^B + \frac{B^A}{D^B(P^B)} + \frac{P^B}{d^A(p^A)}$$

$$\hat{P}^A = p^A D^B(P^B) + B^A$$

$$\hat{P}^B = P^B + b^B d^A(p^A)$$

The appropriate inverse stability factor in the Hybrid model is

$$\sigma_H = \frac{m^A M^B d^A}{m^A M^B d^A - c^{A2} r^A R^B} \tag{22}$$

3.3.1 Demand

In the hybrid model the two sides of the market are fundamentally asymmetric and therefore we cannot simply consider a shock to, say, demand an arbitrary side of the market I. Instead we must separately consider the effects of a shock to the level demand on side A and on side B. On side A this can be written as

$$p^A - \left(c^A[P^B] + c - b^B\right) = m^A(p^A)$$

$$P^B - \left(C^B - [\tilde{\alpha} + p^A]\delta^A d^A[p^A]\right) = M^B(P^B)$$

where δ^A now specifically is a multiplicative shock to side-A demand. I do not repeat the definitions of observables in terms of unobservables that shift from now on in the interests of brevity, as these replicate the appropriate case from above. The comparative statics effects of this increase in A-side demand are given in Table 7 and the following proposition.

Proposition 8. *The first-order effect of a change in δ^A starting at $\delta^A = 1$ in the Hybrid*

$\frac{1}{\sigma_H} \frac{\partial \text{ row}}{\partial \delta^A}\Big\rvert_{\delta^A=1}$	Magnitude	Sign
p^A	$-\frac{c^A r^A d^A R^B (m^A+c^A)}{M^B}$	$-$
P^B	$-d^A(m^A + c^A)R^B$	$-$
A-side demand	$-\frac{c^A d^{A'} r^A d^A R^B (m^A+c^A)}{M^B}$	$+$
B-side demand	$-D^{B'} d^A (m^A + c^A) R^B$	$+$
A-side welfare	$-d^{A^2} D^{B'} R^B (m^A + c^A)(\overline{v}^A + c^A r^A)$	$+$
B-side welfare	$D^B d^A (m^A + c^A) R^B$	$+$
Consumer welfare	$d^A R^B (m^A + c^A) \left(\frac{c^A r^A d^A}{M^B} + D^B \right)$	$+$
Profits	0	0
Social welfare	$d^A R^B (m^A + c^A) \left(\frac{c^A r^A d^A}{M^B} + D^B \right)$	$+$
$\hat{p}^A$	$-\frac{R^B d^A (m^A+c^A)}{M^B D^B} \left(C^A + B^A [1 - r^A] \right)$	?
$\hat{p}^B$	$-(m^A + c^A)R^B + \frac{P^B D^{B'} (m^A+c^A) R^B}{m^A} - \frac{P^B \left(m^A M^B d^A - c^{A^2} r^A R^B \right)}{m^A M^B d^{A^2}}$	?
$\hat{\overline{p}}$	Sum of above two, omitted for brevity	?
$\hat{P}^A$	$D^{B'} R^B d^A (m^A + c^A) d^A (c^A r^A - p^A)$	?
$\hat{P}^B$	$-d^A(m^A + c^A)R^B - b^B \frac{c^A d^{A'} r^A d^A R^B (m^A+c^A)}{M^B} + b^B \frac{m^A M^B d^A - c^{A^2} r^A R^B}{m^A M^B d^{A^2}}$	?

Table 7: A-side demand comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

model are given in Table 7, in the same format as in Table 4 for Proposition 5.

Proof. The proof is exactly as in Proposition 5, except that the hybrid first-order conditions are used. □

As in the Armstrong model, an increase in demand on side A increases the membership size of the net usage benefits on side B, increasing the subsidy to that side and decreasing its price. Note that this is true even though side A is as in the GRT2003 model: the effect of shifts in demand are determined by the nature of the *other* side of the market, not one's own. The increase in side B demand that results lowers the usage size of the firm and homogenous consumer membership costs on side A. Thus costs and therefore prices fall on side A. The hybrid model therefore exhibits complementarity between the two prices. However, this complementarity is much weaker that the substitutability in the GRT2003 model and the complementarity in the GArmstrong model: it is driven only by the fact that $c^A > 0$. If there were no (net) membership costs or benefits on side A, then pricing on the two sides would be independent and the increase in demand on side A would not affect optimal pricing on side A. If $c^A < 0$ then there would be substitutability. Thus the linkage between the two prices in the Hybrid model is much more fragile than in either of the two pure models.

The reason for this fragility is intuitive. Pricing on side B does not affect the marginal usage benefit on that side (the GRT2003 mechanism for linkage) nor does the size of demand on side B affect the usage pricing on side A (the GArmstrong linkage mechanism), unless their are membership costs, as pricing is all driven by the per-interaction size of the two-sided effects. In the absence of fixed costs on side A, prices there will be chosen to maximize the profits earned *per participant on side* B. Therefore by the envelope theorem, there is no direct linkage between pricing on the two sides at the margin. Thus there is no "natural" source of linkage between pricing on the two-sides: whatever occurs is a result of membership costs (benefits) on side A.

An increase in side A demand benefits both sides: side B directly because the membership price falls on side B and that directly raises welfare on side B. Side A consumers benefit both because their prices fall and because they fall on side B, raising demand and thereby magnifying the two-sided benefits received by side A.

Increases in demand on side B affect side A as they would in the GRT2003 model: they reduce the usage size of the net membership costs on side A:

$$p^A - \left(\frac{c^A[P^B]}{\delta^B} + c - b^B \right) = m^A(p^A)$$

$$P^B - \left(C^B - [\tilde{\alpha} + p^A] d^A[p^A] \right) = M^B(P^B)$$

The comparative static effects of this in the Hybrid model context are given in Table 8 and the following proposition:

Proposition 9. *The first-order effect of a change in δ^B starting at $\delta^B = 1$ in the Hybrid model are given in Table 8, in the same format as in Table 7 for Proposition 8.*

Proof. Same as Proposition 8. □

Just as in the GRT2003 model, an increase in demand on the side opposite side A reduces

$\frac{1}{\sigma_H} \frac{\partial \text{ row}}{\partial \delta^B}\Big\|_{\delta^B=1}$	Magnitude	Sign
p^A	$-r^A c^A$	$-$
P^B	$d^{A\prime} r^A c^{A^2} R^B$	$-$
A-side demand	$-d^{A\prime} r^A c^A$	$+$
B-side demand	$d^{A\prime} D^{B\prime} r^A c^{A^2} R^B$	$+$
A-side welfare	$r^A c^A d^A D^B \left(1 + \frac{\overline{v}^A C^A R^B}{m^A M^B}\right)$	$+$
B-side welfare	$-D^B d^{A\prime} r^A c^{A^2} R^B$	$+$
Consumer welfare	$r^A c^A d^A D^B \left(1 + \frac{R^B c^A}{m^A}\left[1 + \frac{v^A}{M^B}\right]\right)$	$+$
Profits	0	0
Social welfare	$r^A c^A d^A D^B \left(1 + \frac{R^B c^A}{m^A}\left[1 + \frac{v^A}{M^B}\right]\right)$	$+$
$\hat{p}^A$	$-r^A c^A + \frac{B^A d^{A\prime} r^A c^{A^2} R^B}{M^B D^B} - \frac{B^A(m^A M^B d^A - c^{A^2} r^A R^B)}{D^B m^A M^B d^A}$	?
$\hat{p}^B$	$-\frac{r^A c^A}{m^A d^A}\left(C^A R^B + P^B\right)$	?
$\hat{\overline{p}}$	Sum of above two, omitted for brevity	?
$\hat{P}^A$	$-r^A C^A + p^A \left(\frac{m^A M^B d^A - c^{A^2} r^A R^B}{D^B m^A M^B d^A} + d^{A\prime} D^{B\prime} r^A c^{A^2} R^B\right)$	?
$\hat{P}^B$	$d^{A\prime} r^A c^A (c^A R^B - b^B)$	?

Table 8: B-side demand comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

effective usage costs of serving side A, leading the monopolist to reduce the price on side A. By complementarity this lowers prices on side B and makes both groups of consumers better off.

3.3.2 Market power

An increase in (usage) market power on side A can be written as

$$p^A - \left(c^A[P^B] + c - b^B\right) = m^A(p^A) + \mu^A$$

$$P^B - \left(C^B - [\tilde{\alpha} + p^A] d^A[p^A]\right) = M^B(P^B)$$

The comparative static effects of this shift are given in Table 9 and the following proposition

Proposition 10. *The first-order effect of a change in μ^A starting at $\mu^A = 0$ in the Hybrid model are given in Table 9, in the same format as in Table 7 for Proposition 8.*

$\frac{1}{\sigma_H} \frac{\partial \text{ row}}{\partial \mu^B}\Big\vert_{\mu^A=0}$	Magnitude	Sign
p^A	r^A	$+$
P^B	$-d^{A'} r^A c^A R^B$	$+$
A-side demand	$d^{A'} r^A$	$-$
B-side demand	$-d^{A'} D^{B'} r^A c^A R^B$	$-$
A-side welfare	$-r^A d^A D^B \left(1 - \frac{d^{A'} \overline{v}^A c^A R^B}{M^B}\right)$	$-$
B-side welfare	$D^B d^{A'} r^A c^A R^B$	$-$
Consumer welfare	$-r^A d^A D^B \left(1 + \frac{R^B c^A}{m^A}\left[1 + \frac{v^A}{M^B}\right]\right)$	$-$
Profits	0	0
Social welfare	$-r^A d^A D^B \left(1 + \frac{R^B c^A}{m^A}\left[1 + \frac{v^A}{M^B}\right]\right)$	$-$
$\hat{p}^A$	$r^A \left(1 - \frac{B^A d^{A'} c^A R^B}{M^B D^B}\right)$	?
$\hat{p}^B$	$\frac{r^A}{m^A d^A}\left(C^A R^B + P^B\right)$	?
$\hat{\bar{p}}$	$r^A \left(1 - \frac{B^A d^{A'} c^{A^2} R^B}{M^B D^B} + \frac{r^A [C^A R^B + P^B]}{m^A d^A}\right)$	?
$\hat{P}^A$	$r^A D^B \left(1 - \frac{p^A d^{A'} c^A R^B}{M^B}\right)$	?
$\hat{P}^B$	$-d^{A'} r^A (c^A R^B - b^B)$	?

Table 9: A-side market power comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

Proof. Same as Proposition 8. □

As in the GRT2003 model, an increase in (usage) market power on side A increases the (usage) price to side A. By complementarity this increases the membership price to side B and makes both groups of consumers worse off. An increase in (membership) market power on side B is given by

$$p^A - \left(c^A[P^B] + c - b^B\right) = m^A(p^A)$$

$$P^B - \left(C^B - [\tilde{\alpha} + p^A] d^A[p^A]\right) = M^B(P^B) + \mu^B$$

Comparative statics are expressed in Table 10 and the following proposition:

Proposition 11. *The first-order effect of a change in μ^B starting at $\mu^B = 0$ in the Hybrid model are given in Table 10, in the same format as in Table 7 for Proposition 8.*

Proof. Same as Proposition 8. □

$\frac{1}{\sigma_H} \frac{\partial \text{ row}}{\partial \mu^B}\Big\rvert_{\mu^B=0}$	Magnitude	Sign
p^A	$\frac{c^A r^A R^B}{M^B}$	$+$
P^B	R^B	$+$
A-side demand	$\frac{c^A d^{A'} r^A R^B}{M^B}$	$-$
B-side demand	$D^{B'} R^B$	$-$
A-side welfare	$d^A D^{B'} R^B (\overline{v}^A + c^A r^A)$	$-$
B-side welfare	$-D^B R^B$	$-$
Consumer welfare	$D^{B'} R^B \left(v^A + C^A r^A + M^B R^B\right)$	$-$
Profits	0	0
Social welfare	$D^{B'} R^B \left(v^A + C^A r^A + M^B R^B\right)$	$-$
$\hat{p}^A$	$\frac{R^B}{M^B} \left(c^A r^A + \frac{B^A}{D^B}\right)$	?
$\hat{p}^B$	$\frac{R^B}{d^A} \left(1 + \frac{P^B c^A r^A}{m^A M^B}\right)$	?
$\hat{\overline{p}}$	$R^B \left(\frac{1}{M^B}\left[c^A r^A + \frac{B^A}{D^B}\right] + \frac{1}{d^A}\left[1 + \frac{P^B c^A r^A}{m^A M^B}\right]\right)$	?
$\hat{P}^A$	$-D^{B'} R^B (c^A r^A - p^A)$	?
$\hat{P}^B$	$R^B \left(1 + \frac{b^B d^{A'} c^A r^A}{M^B}\right)$	?

Table 10: B-side market power comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

As in the GArmstrong model, an increase in (membership) market power on side B increases the (membership) price to side B. By complementarity this increases the membership price to side A and makes both groups of consumers worse off.

3.3.3 Homogenous consumer benefits/costs

A decrease (increase) in the homogenous consumer membership cost (benefit) on side A can be written as

$$p^A - \left(c^A[P^B] - \frac{\beta^A}{D^B[P^B]} + c - b^B\right) = m^A(p^A)$$

$$P^B - \left(C^B - [\tilde{\alpha} + p^A]d^A[p^A]\right) = M^B(P^B)$$

Its comparative static effects are given by Table 11 and the following proposition:

Proposition 12. *The first-order effect of a change in β^A starting at $\beta^A = 0$ in the Hybrid model are given in Table 11, in the same format as in Table 7 for Proposition 8.*

$\frac{1}{\sigma_H} \frac{\partial \text{ row}}{\partial \beta^A}\Big\vert_{\beta^A=0}$	Magnitude	Sign
p^A	$-\frac{r^A}{D^B}$	$-$
P^B	$\frac{d^{A'} r^A c^A R^B}{D^B}$	$-$
A-side demand	$-\frac{d^{A'} r^A}{D^B}$	$+$
B-side demand	$-\frac{d^{A'} r^A c^A R^B}{M^B}$	$+$
A-side welfare	$r^A d^A \left(1 + \frac{\overline{v}^A C^A R^B}{m^A M^B}\right)$	$+$
B-side welfare	$-d^{A'} r^A c^A R^B$	$+$
Consumer welfare	$r^A d^A \left(1 + \frac{R^B c^A}{m^A} \left[1 + \frac{v^A}{M^B}\right]\right)$	$+$
Profits	d^A	$+$
Social welfare	$d^A \left(1 + r^A \left[1 + \frac{R^B c^A}{m^A} \left(1 + \frac{v^A}{M^B}\right)\right]\right)$	$+$
$\hat{p}^A$	$\frac{1}{D^B} \left(\frac{m^A M^B d^A - c^{A^2} r^A R^B}{m^A M^B d^A} - r^A + \frac{B^A d^{A'} c^{A^2} R^B}{M^B D^B}\right)$	?
$\hat{p}^B$	$-\frac{r^A}{m^A d^A D^B} \left(C^A R^B + P^B\right)$	?
$\hat{\overline{p}}$	$\frac{m^A M^B d^A - c^{A^2} r^A R^B}{m^A M^B D^B d^A} - \frac{r^A}{D^B} \left(1 - \frac{B^A d^{A'} c^{A^2} R^B}{M^B D^B} + \frac{r^A [C^A R^B + P^B]}{m^A d^A}\right)$	?
$\hat{P}^A$	$\frac{m^A M^B d^A - c^{A^2} r^A R^B}{m^A M^B d^A} - r^A + \frac{p^A d^{A'} c^A r^A R^B}{M^B}$	?
$\hat{P}^B$	$d^{A'} r^A (c^A R^B - b^B)$	?

Table 11: A-side homogeneous membership cost/benefit comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

Proof. Same as Proposition 8. $\square$

As in the GRT2003 model, a decrease (increase) in side-A membership costs in the hybrid model reduces the costs of serving side A, encouraging the monopolist to lower the usage price to side A, which in turn decreases prices on side B by complementarity and benefits both groups of consumers. Just a bit more complex is the effect of an increase (decrease) in the side B homogeneous usage benefit (cost). This can be expressed as

$$p^A - \left(c^A[P^B] + c - b^B - \beta^B\right) = m^A(p^A)$$

$$P^B - \left(C^B - [\tilde{\alpha} + \beta^B + p^A]d^A[p^A]\right) = M^B(P^B)$$

The comparative static effects are given in Table 12 and by the following proposition:

Proposition 13. *The first-order effect of a change in β^B starting at $\beta^B = 0$ in the Hybrid*

$\frac{1}{\sigma_H}\frac{\partial \text{ row}}{\partial \beta^B}\Big\vert_{\beta^B=0}$	Magnitude	Sign
p^A	$-r^A\left(1+\frac{R^BC^A}{M^B}\right)$	$-$
P^B	$-R^B(d^A+c^Ar^A)$	$-$
A-side demand	$-d^{A\prime}r^A\left(1+\frac{R^BC^A}{M^B}\right)$	$+$
B-side demand	$-D^{B\prime}R^B(d^A+c^Ar^A)$	$+$
A-side welfare	$-D^{B\prime}d^A\left(M^Br^A\left[1+\frac{R^BC^A}{M^B}\right]+R^B(d^A+c^Ar^A)\overline{v}^A\right)$	$+$
B-side welfare	$D^BR^B(d^A+c^Ar^A)$	$+$
Consumer welfare	$-D^{B\prime}\left(d^A\left[M^Br^A\left(1+\frac{R^BC^A}{M^B}\right)+R^B(d^A+c^Ar^A)\overline{v}^A\right]+M^BR^B[d^A+c^Ar^A]\right)$	$+$
Profits	d^AD^B	$+$
Social welfare	$-D^{B\prime}\left(d^A\left[M^Br^A\left(1+\frac{R^BC^A}{M^B}\right)+R^B(d^A+c^Ar^A)\overline{v}^A\right]+M^B\left[d^A(1+R^B)+R^Bc^Ar^A\right]\right)$	$+$
$\hat{p}^A$	$-r^A\left(1+\frac{R^BC^A}{M^B}\right)-\frac{R^B(d^A+c^Ar^A)}{M^BD^B}$	?
$\hat{p}^B$	$\frac{m^AM^Bd^A-c^{A^2}r^AR^B}{m^AM^Bd^A}-R^B\left(1+\frac{c^Ar^A}{d^A}\right)-\frac{P^Ar^A}{d^Am^A}\left(1+\frac{R^BC^A}{M^B}\right)$	?
$\hat{\overline{p}}$	$\frac{m^AM^Bd^A-c^{A^2}r^AR^B}{m^AM^Bd^A}-r^A\left(1+\frac{R^BC^A}{M^B}\right)\left(1+\frac{P^A}{d^Am^A}\right)-R^B(d^A+c^Ar^A)\left(\frac{1}{M^BD^B}+\frac{1}{d^A}\right)$	?
$\hat{P}^A$	$-D^{B\prime}\left(p^AR^B(d^A+c^Ar^A)-r^AM^B\left[1+\frac{R^BC^A}{M^B}\right]\right)$	?
$\hat{P}^B$	$\frac{m^AM^Bd^A-c^{A^2}r^AR^B}{m^AM^B}-R^B(d^A+c^Ar^A)-d^{A\prime}b^Br^A\left(1+\frac{R^BC^A}{M^B}\right)$	?

Table 12: B-side homogeneous usage benefit/cost comparative statics for Hybrid model, scaled down by $\frac{1}{\sigma_H}$

model are given in Table 12, in the same format as in Table 7 for Proposition 8.

Proof. Same as Proposition 8. □

As in the GArmstrong model, an increase (decrease) in side B homogenous usage benefit (cost) directly affects both sides. On side B it has the same effect as in the GArmstrong model, increasing the net two-sidedness. On side A it has the same effect as an increase in prices on the opposite side of the market would have: it increases the (usage) subsidy to side A. On both sides of the market this directly reduces prices and, by complementarity, indirectly further reduces prices. Thus the welfare of both groups of consumers is improved.

While many of the comparative statics of these models can be signed purely on the basis of theory, to make quantitative statements, as well as qualitative predictions if in the Hybrid model c^A is not know or about the effect of shocks on observable prices, one needs to measure at least some properties of the demand system. This identification is the focus of the following section.

4 Identification and Testing

As always, the properties of the demand system that can be identified and the predictions of the model that are testable depend on what sort of data is available. In this section I consider the parameters that can be identified and tests that can be executed when there is very limited data available. In particular, I will assume that data with exogenous cost variations are available, but all that can be reliably estimated is the first-order effect of cost on price and demand on both sides of the market. In other words, I assume that data on exogenous cost variations is limited enough that only a linear regression of quantities and prices on the cost variation can be reliably estimated. While this approach severely limits the data that can be employed, I will briefly argue below that these limitations are often realistic. Furthermore, in the the particular problem of the models of monopolies in two-sided markets I consider, we will see that the properties of demand that can be identified are typically all of those relevant to determining the magnitude of the (first-order) comparative statics in the previous section. The one exception is the identification of average consumer surplus, but identifying this is a difficult task in indeed, as it requires calculations of deep infra-marginal quantities that nearly any empirical approach will struggle with. Thus, at least in the particular problems I consider, I would argue that the wide applicability[3] of the first-order cost-based approach trumps the limited identification it yields as it does identify those magnitudes crucial to policy analysis which can realistically be identified by any empirical strategy.

[3]Of course, if detailed micro-level data is available and restrictive assumptions are imposed on the shape of demand, alternative identification strategies are available, which, along the lines of Berry et al. (1995), try to directly recover the shape of demand from the distribution of consumer valuations of product characteristics (Rysman, 2004; Kaiser and Wright, 2006; Lee, 2007). Which approach is preferable when such data is available remains an open question. Despite the greater data brought to bear by the Berry et al. (1995) approach, it relies on very strong structural assumptions about distributions to identify the first-order relevant properties of demand in small samples, though asymptotically in can be non-parametrically identified (Berry and Halie, 2008). Hopefully future research will clarify what identifies pass-through and elasticities with reasonable amounts of data in the characteristics approach, both in one- and two-sided markets.

The use of exogenous cost variations to identify demand goes all the way back to Working (1927). These are often used both in competitive and imperfectly competitive models as instruments for price shifts. For this purpose their precise effects on marginal cost need not be known (Berry et al., 1995). However, if we are interested in estimating pass-through rates which are crucial to the comparative statics we are interested in, we must have *quantitatively observable* exogenous cost variations. If the level of costs are observable, then any instrument for cost can be used, such as the firm-specific shocks proposed by Berry et al. (1995), exchange rates, input prices, taxes, etc. If cost levels are unobservable, more care is necessary, but several approaches are available based on taxes (Sidhu, 1971; Sumner, 1981), exchange rates (Menon, 1995; Campa and Goldberg, 2005), input cost shocks (Sijm et al., 2006) and technological shifts (Besank et al., 2005). All require some structural knowledge of how the instrument quantitatively effects marginal cost. Thus while such quantitively observable exogenous cost variations are certainly not universally available, there are many plausible approaches to generating them.

However, it should be remembered that to be exogenous these cost shifts must be uncorrelated with other price shifters in the model, such as demand, other costs and homogenous membership usage benefits/costs. If one truly believes the market is a monopoly, this is not a major problem: as long as sufficiently dimensional exogenous cost shifters are available and their relationship to all costs is understood, recovery of individual costs shifters is trivial. However, if prices in markets excluded from the model affect demand (imperfect monopoly) and are correlated with the "exogenous" cost shock, then the cost shock loses its exogeneity. In this situation the most plausible approach is to explicitly incorporate these related markets into the model, an exercise beyond the scope of this paper. Furthermore, as with much of the recent literature on non-parametric identification, I do not give much attention here to estimation or even to explicit probabilistic formulations.

However in contrast to this literature, I do not assume a large amount of data is available. Instead than assuming that full distributions can be observed (i.e. data is sufficiently rich to precisely estimate arbitrary effects of cost on prices), I instead make the opposite stark assumption that only the first order effects of cost variations on prices and demand can be reliably estimated[4]. Given the short time periods over which demand can reasonably be assumed to be stable and therefore estimable, this seems a reasonable assumption. If, to the contrary sufficiently high order effects of cost on prices and demand could be estimated precisely, then exogenous cost variations even within an arbitrary small price range could be used to identify a demand system (Weyl, 2008a) which seems implausible. Of course, my approach here could easily be extended to second- (or higher-) order identification[5] and, given that demand on each side of the market is assumed independent of prices on the other side, this would lead to substantially stronger identification.

Even given my first-order, cost-based approach, one must still decide which cost variations are observable and which other economic variables are directly observable. Throughout this section I maintain

Assumption 2. *Demand levels on both sides of the market are observable. I assume that the transfers from any consumer on one side of the market to (or from) the firm either per-interaction (usage) or for membership is uniform across consumer (no price discrimination) and that this transfer is observable. This price can be interpreted as either a usage or membership price by simple multiplication or division.*

For the first three subsections, I assume

[4]The precise assumption that first-order effects can be arbitrarily well-estimated and all other effects cannot be estimated at all is obviously too stark and the precise Bayesian justification for the approach here is hazy at best. Of course this criticism could be level at all linear regression models, but it would still be useful to understand more precisely when the approach taken here is roughly sensible. While focus on first-order effects likely embodies a prior assumption that "most uncertainty" is focused on these effects and that higher-order properties have less diffuse priors, part of the justification simply arises from the fact that more data is required to precisely estimate higher order effects, even if all uncertainty is focused on these, than is necessary to identify first-order effects from data based on local variations, because higher-order variation is smaller in local regions.

[5]However, it should be noted that I know of no demand function which allows full variation in both pass-through rates and their slopes (and therefore allows for second-order identification in the same way the constant pass-through class allows for first-order identification). In work in progress jointly with Charles Fefferman (Fabinger and Weyl, 2008) I am trying to formulate a demand system which fills this gap.

Assumption 3. *b^I, B^I, C^I, c are not observable for any I, but expressions of the form $\frac{\partial X}{\partial Y}$ are observable for $X = \hat{P}^I, \hat{p}^I, D^I, d^I$ and $Y = C^I, c$ as appropriate to the model*[6].

but I briefly consider other sets of observables in the final subsection.

4.1 Identification

Because I rely on the effects of cost variations on demand and prices, it is useful to give these mathematical notation. Let $\hat{r}_I \equiv \frac{\partial \hat{p}^I}{\partial c}$, $\hat{R}_I \equiv \frac{\partial \hat{P}^I}{\partial c}$, $\hat{r}_{IJ} \equiv \frac{\partial \hat{p}^I}{\partial C^J}$, $\hat{R}_{IJ} \equiv \frac{\partial \hat{P}^I}{\partial C^J}$, $\eta_I \equiv \frac{\partial d^I}{\partial c}$ or $\frac{\partial D^I}{\partial c}$ (depending on which model is being referred to) and $\eta_{IJ} \equiv \frac{\partial d^I}{\partial C^J}$ or $\frac{\partial D^I}{C^J}$ for $I, J = A, B$. Despite the substantial differences in the models, they can all be identified in essentially the same way; in fact, even the formulas for parameters turn out to be nearly identical. This is particularly evident in the identification of the inverse stability factor.

Proposition 14. *In all three models, the inverse stability factor is equal to*

$$1 - \frac{\eta_{AB}^2}{\eta_{AA}\eta_{BB}}$$

Proof. In the GRT2003 model simple calculations show that

$$\frac{\eta_{II}}{\sigma_{GRT2003}} = \frac{d^{I'} r^I}{d^J}$$

$$\frac{\eta_{AB}}{\sigma_{GRT2003}} = \frac{r^A r^B m}{m^A m^B}$$

In the GArmstrong model

$$\frac{\eta_{II}}{\sigma_{GA}} = D^{I'} R^I$$

$$\frac{\eta_{AB}}{\sigma_{GA}} = -D^{A'} D^{B'} \alpha R^A R^B$$

[6] Note that $\hat{P}^I \equiv d^{-I}\hat{p}^I$ or $\hat{P}^I \equiv D^{-I}\hat{p}^I$ depending on the model, $\frac{\partial \hat{P}^I}{\partial Y}$ and $\frac{\partial \hat{p}^I}{\partial Y}$ are not independent observations.

In the hybrid model

$$\frac{\eta_{AA}}{\sigma_H} = \frac{d^{A'} r^A}{D^B}$$

$$\frac{\eta_{AB}}{\sigma_H} = \frac{d^{A'} r^A R^B c^A}{M^B}$$

$$\frac{\eta_{BB}}{\sigma_H} = R^B D^{B'}$$

From these simple algebra yields the result formulas for the inverse stability factor in each model from above. □

This is a simple consequence of classical demand theory. The classic inverse stability factor (Deaton and Muellbauer, 1980) from demand theory for two-good demand systems, where D^I_J represents the derivative of the demand for product I with respect to the price of product J, is $1 - \frac{(D^2_1)^2}{D^1_1 D^2_2}$. An increase in the membership cost on one side of the market is effectively an increase in the price (opportunity cost) to the firm of participation on that side. Therefore thinking of the firm's decision in terms of demand theory yields the formula in the proposition immediately. The parallels between the models do not end there, however. The formulas for the market power in the models are closely related as well.

Proposition 15. *In the GRT2003 model (usage) pass-through, market power and (homogeneous) membership costs/benefits respectively are*

$$r^I = d^J \left(\hat{r}_{II} - \frac{\eta_{AB}}{\eta_{JJ}} \hat{r}_{JI} \right)$$

$$m^I = \frac{d^I (\eta_{AB} \hat{r}_{JI} - \eta_{JJ} \hat{r}_{II})}{\eta_{AA}\eta_{BB} - \eta^2_{AB}}$$

$$B^I = \frac{d^{J^2} (\eta_{AB} \hat{r}_{II} - \eta_{II} \hat{r}_{JI})}{\eta_{AA}\eta_{BB} - \eta^2_{AB}}$$

In the GArmstrong

$$R^I = \hat{R}_{II} - \frac{\eta_{AB}}{\eta_{JJ}} \hat{R}_{JI}$$

$$M^I = \frac{D^I(\eta_{AB}\hat{R}_{JI} - \eta_{JJ}\hat{R}_{II})}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

$$b^I = \frac{\eta_{II}\hat{R}_{JI} - \eta_{AB}\hat{R}_{II}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the hybrid model, the GRT2003 formulas apply on side A and the GArmstrong formulas on side B.

Proof. In the GRT2003 model simple calculations show

$$\frac{\hat{r}_{II}}{\sigma_{GRT2003}} = \frac{r^I}{d^J}\left(1 - \frac{\tilde{r}^J B^I}{m^J d^J}\right)$$

$$\frac{\hat{r}_{IJ}}{\sigma_{GRT2003}} = \frac{r^I}{d^J}\left(\frac{B^J}{d^I m^I} - \tilde{r}^J\right)$$

In the GArmstrong model

$$\frac{\hat{R}_{II}}{\sigma_{GA}} = R^I\left(1 - \alpha b^I D^{I'} D^{J'} R^J\right)$$

$$\frac{\hat{R}_{IJ}}{\sigma_{GA}} = -D^{I'} R^I(\alpha R^J - b^J)$$

In the Hybrid model

$$\frac{\hat{r}_{AA}}{\sigma_H} = \frac{r^A}{D^B}\left(1 - \frac{d^{A'} B^A c^A r^A R^B}{D^B M^B}\right)$$

$$\frac{\hat{r}_{BA}}{\sigma_H} = \frac{R^B}{M^B}\left(c^A r^A + \frac{B^A}{D^B}\right)$$

$$\frac{\hat{R}_{AB}}{\sigma_H} = -\frac{d^{A'} r^A(R^B c^A - b^B)}{D^B}$$

$$\frac{\hat{R}_{BB}}{\sigma_H} = R^B\left(1 - \frac{b^B d^A c^A r^A}{m^A M^B}\right)$$

From these formulas the quoted results can be derived using simple algebra. □

The formulas are similar across models because they follow a similar identification strategy. Consider pass-through in the GRT2003 model. The basic problem of identification is that observed usage pass-through rates include both the effects of costs on usage prices and the effects of changes in the demand on the opposite side of the market on the usage size of membership prices. To solve this problem we must neutralize the second effect to isolate the first. This is easy to do as $\frac{\eta_{IJ}}{\eta_{JJ}}$ tells us the relative effect that shocks to C_I versus shocks to C_J have on demand on side J. Therefore $\hat{r}_{II} - \frac{\eta_{IJ}}{\eta_{JJ}}\hat{r}_{JI}$ includes no effects mediated through membership prices, isolating the usage price and allowing pass-through to be measured. The same logic applies for the Armstrong model and for the measurement of homogeneous benefits/costs. Once pass-through is determined, market power is easy to estimate as it is simply determined by the elasticity of demand, which can be inferred from the ratio of the effects of cost shocks on demand to the pass-through rate. The identification of costs is less similar across models, as it exploits the linkage between the demands and prices to the two sides, the sources of which differ across models. I express usage costs in all models in terms of observed membership pass-through and membership costs in terms of observed usage pass-through so as to allow comparsion

Proposition 16. *In the GRT2003 model, the usage cost for the firm is given by*

$$c = \frac{\eta_{AB} + \sum_{I=A,B;\ J=-I} \left(\eta_{JJ}\frac{d^I}{d^J} - \eta_{AB}\right)\hat{R}_{II} + \left(\eta_{II} - \eta_{AB}\frac{d^I}{d^J}\right)\hat{R}_{JI}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the GArmstrong model, it is

$$c = \frac{\eta_{AB}(\hat{R}_{AA} + \hat{R}_{BB}) - (\eta_{AA}\hat{R}_{BA} + \eta_{BB}\hat{R}_{AB}) + \eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the Hybrid model, it is

$$c = \frac{\left(\frac{d^A}{D^B}\eta_{BB} - \eta_{AB}\right)\hat{R}_{AA} + \left(\eta_{AA} - \eta_{AB}\frac{d^A}{D^B}\right)\hat{R}_{BA} + \eta_{BB}\hat{R}_{AB} - \eta_{AB}\hat{R}_{BB} + \eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the GRT2003 model the I-side membership cost to the firm is

$$C^I = d^J \frac{(d^I \eta_{AB} - d^J \eta_{II})\hat{r}_{JI} + (d^J \eta_{AB} - d^I \eta_{JJ})\hat{r}_{II} - \eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the GArmstrong model it is

$$C^I = D^J \frac{(D^J \eta_{AB} + D^I \eta_{JJ})\hat{r}_{II} - (D^J \eta_{II} + D^I \eta_{AB})\hat{r}_{JI} - \eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

In the Hybrid model it is

$$C^A = D^B \frac{(D^B \hat{r}_{AA} - 1)\eta_{AB} - D^B \eta_{AA}\hat{r}_{BA}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

$$C^B = d^A \frac{\eta_{AB} + (d^A \eta_{AB} + D^B \eta_{AA})\hat{r}_{BB} - (D^B \eta_{AB} + d^A \eta_{BB})\hat{r}_{AB} + d^A \eta_{AB}\hat{r}_{BA} - d^A \eta_{BB}\hat{r}_{AA}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

Proof. For each model three parameters must be identified. This is done using the identified variables above and the two first-order conditions (which have not yet been used), in conjunction with the following results which follow from simple algebra given the expressions derived in the proofs of the preceding propositions:

1. In the GRT2003 model

$$m = \frac{\eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2} \tag{23}$$

2. In the GArmstrong model

$$\alpha = -\frac{\eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2} \tag{24}$$

3. In the Hybrid model

$$c^A = -\frac{\eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2} \tag{25}$$

$\square$

Thus all parameters relevant to the first-order positive comparatives statics analyzed in the previous section can be identified. On any side of the market where consumers are heterogeneous in their membership costs/benefits alla Armstrong, normative comparative statics are also identified. On the other hand, when consumers are usage heterogeneous alla RT2003, the average surplus which is crucial to the normative comparative statics of that model is unidentified. However, it seems unlikely that other approaches to identification would be very effective at estimating this. Average surplus depends heavily on the size of the tails of the distribution of consumer preferences, which require deep inframarginal data on preferences which strains the credibility of any identification strategy.

If, for simulation purposes, one wants more than identification of first order effects, these results can easily be used to fit a demand function from the Bulow and Pfleiderer (1983) constant pass-through class to each side of the market. This class of demand functions is useful in this context because

1. It allows full variation in all estimated parameters of demand (level, elasticity/market power given price, and pass-through) and therefore can always be made to fit estimated (first-order) parameters. As far as I know it is the only known demand function with this property.

2. All of its parameters can be recovered from knowledge of the current price, elasticity/market power, level of demand and pass-through.

3. It provides a natural extension of the first-order approach to identification (Weyl, 2008c).

4. In this case $\overline{V} = \rho m$, so average surplus is identified in the GRT2003 and hybrid models.

Of course, other parametric demand functions can be fit, though if the estimated parameters are inconsistent with the assumptions about pass-through implicit to the functional forms a more sophisticated fitting procedure than simple method of moments would be needed.

4.2 Testing

Can data help us distinguish which model should be applied to a particular market? While the answer is yes, at least a bit, there are a surprising number of similarities between the predictions made by the various models. These predictions serve as a test of whether the framework assumed throughout this paper is appropriate. I suspect, but have not shown, that they are predictions of the broad RT2006 class to which all the models considered here belong.

Proposition 17. *All three models predict that the following five linearly independent conditions will be satisfied.*

$$\eta_{AB} = \eta_{BA} \tag{26}$$

$$\eta_I = \eta_{II}\tilde{d}^J + \eta_{JI}\tilde{d}^I \quad I = A, B; \; J = -I \tag{27}$$

$$\hat{R}_I = \hat{R}_{II}\tilde{d}^J + \hat{R}_{JI}\tilde{d}^I \quad I = A, B; \; J = -I \tag{28}$$

where $\tilde{d}^I$ is d^I if side I in the model is usage heterogeneous and D^I if membership heterogeneous. Condition (28) is equivalent to

$$\hat{r}_I = \hat{r}_{II}\tilde{d}^J + \hat{r}_{JI}\tilde{d}^I$$

given the other conditions hold and therefore the result can be formulated equivalently in terms of observe usage or membership pass-through. The following four independent inequalities are also satisfied in all three models

$$
\begin{aligned}
\eta_{AA}\eta_{BB} &> \eta_{AB}^2 && (29)\\
\eta_{AB}\hat{R}_{JI} &> \eta_{JJ}\hat{R}_{II} \quad I = A, B;\; J = -I && (30)\\
\eta_{AA} &< 0 && (31)
\end{aligned}
$$

Condition (30) is equivalent to

$$\eta_{AB}\hat{r}_{JI} > \eta_{JJ}\hat{r}_{II} \quad I = A, B;\; J = -I$$

Proof. All of the equality conditions can be checked by simple math based on the equations derived in previous proofs. the equivalence of the condition for $\hat{r}_I$ and $\hat{R}_I$ can be seen by

$$\hat{R}_I = \hat{r}_I\tilde{d}^J + \eta_J\hat{p}^I = \hat{r}_I\tilde{d}^J + (\eta_{II}\tilde{d}^J + \eta_{JI}\tilde{d}^I)\hat{p}^I$$

if and only if the $\hat{r}_I$ condition holds then this is the same as

$$(\hat{r}_{II}\tilde{d}^J + \hat{r}_{JI}\tilde{d}^I)\tilde{d}^J + (\eta_{II}\tilde{d}^J + \eta_{JI}\tilde{d}^I)\hat{p}^I = \hat{R}_{II}d^J + \hat{R}_{JI}d^I$$

The first inequality is a direct consequence of the second-order condition in each model, as this implies that the inverse stability factor is strictly positive. The second inequality is a direct consequence of the fact that demand is decreasing and therefore market power is positive. For some models the second version of this condition is relevant, but it is easy to see these are equivalent:

$$\eta_{AB}\hat{R}_{JI} > \eta_{JJ}\hat{R}_{II} \iff \eta_{AB}(\hat{r}_{JI}d^J + \hat{p}_I\eta_{JJ}) > \eta_{JJ}(\hat{r}_{II}d^J + \hat{p}_I\eta_{AB}) \iff \eta_{AB}\hat{r}_{JI} > \eta_{JJ}\hat{r}_{II}$$

The third inequality is a joint consequence of the second and the fact that the second order conditions are satisfied so pass-through is strictly positive.

The linear independence of the equality conditions follows from the fact that each succeeding condition introduces a new parameter not used in the identification or any preceding equality condition. The independence of the first inequality condition follows from the fact that the magnitude of $\eta_{AA}\eta_{BB}$ was not in anyway restricted by any previous result. The independence of the second follows from the fact that previous results did not restrict the magnitude of observed pass-through rates. The independence of the third follow the fact that any (positive) value of $\eta_{AA}\eta_{BB}$ can be achieved either with both positive or both negative $\square$

The common predictions of the models are simple consequences of their common features. $\eta_{AB} = \eta_{BA}$ and $\eta_{AA}\eta_{BB} > \eta_{AB}^2$ are direct consequences of the symmetry and stability conditions of classical demand theory, as discussed earlier. Conditions (27) and (28) express that the effect of an increase in the usage cost on both prices and demand on both sides of the market is the same as an increase in the membership cost on each side of the market, scaled up by demand on the other side. The conditions on pass-through and market power ensure that the firm obeys the laws of demand in this simple linear setting. These basically test the joint hypothesis that (1) what we are labeling as usage and membership costs really do act as those in the market and (2) that the firm obeys basic principles of optimization (treating all sources of a particular opportunity cost the same). While this does not provide a terribly compelling test of the framework here specifically, or even of the broader RT2006 framework, validation of these conditions at least provides a sanity check on the identification procedure and substantial rejection could serious undermine it.

Restrictions on other demand parameters offer little future help in determining which model should be applied to a given setting. Assumptions that homogeneous membership or usage benefits are signed make the same prediction that an assumption that equilibrium membership or usage fees are signed (or that the full support of heterogeneous benefits lies above or below 0).

Proposition 18. *An assumption that equilibrium (either homogeneous or heterogeneous) membership costs (benefits) are positive on side I implies*

$$\eta_{AB}\hat{r}_{II} > (<)\eta_{II}\hat{r}_{JI}$$

and an assumption that equilibrium usage benefits (costs) are positive on side I implies

$$\eta_{AB}\hat{R}_{II} > (<)\eta_{II}\hat{R}_{JI}$$

Proof. For homogeneous benefits/costs the results follow directly from the formulas derived above. In the case of heterogeneous benefits/costs the same calculations lead to the conclusion that the formulas for homogeneous benefits/costs identify the equilibrium level of those costs/benefits when they are heterogeneous. $\square$

The simplest way to distinguish between the models is based the fact that GRT2003 exhibits substitutability of participation on the two sides, while GArmstrong exhibits complementarity. However, given that there are three models, this cannot provide a complete test between then as shown in the next proposition.

Proposition 19. *The GRT2003 model and the Hybrid model with $c^A < 0$ predict that $\eta_{AB} > 0$. The GArmstrong model and the Hybrid model with $c^A > 0$ predict that $\eta_{AB} < 0$.*

Proof. The prediction in the GRT2003 model follows from equation (23) and the fact that the firm would only operate if their total mark-up $m > 0$. The prediction in the GArmstrong model follows from equation (24) and the fact that net two-sidedness is assumed strictly positive. The prediction for the Hybrid model follows from equation (25). $\square$

The test for substitution v. complementing therefore distinguishes between the GRT2003 and GArmstrong models or identifies the sign of the net membership costs on side A in the Hybrid model, but cannot distinguish between a positive net membership cost Hybrid model and the GArmstrong model or between the GRT2003 model and a negative net membership cost GRT2003 model. Given that we likely often would find it reasonable to assume that net membership costs are positive, this confound is most important in the second case: substitution would seem to be a reasonable indication that the GRT2003 model is the better fit.

A final means for distinguishing more finely between the models is the reasonable assumption that usage and membership costs (for both sides of the market) are weakly positive. This generates different testable inequality predictions for each model, potentially allowing one to empirically distinguish among the models.

Proposition 20. *The assumption that $c, C^I > 0$ for both I implies testable joint restrictions on observables that are neither mutually exclusive nor redundant with one another across models. These are independent of the obedience of all restrictions mentioned above.*

Proof. See Appendix B. □

Because these conditions certainly vary across models and because there are three of them, in many cases they can be used a one means of determining which model is most appropriate to apply to a given setting. In conjunction with the complements v. substitutes test, it seems plausible that in most cases it will be possible to choose one model which is best for any given setting. However, because the predictions are non-exclusive, difficult situations may arise where, say, one condition is satisfied for each model and two are violated for each model. In this case, more sophisticated statistical techniques would have to be invoke, such as Bayesian model selection. In some cases all three conditions may be obeyed for all three models, in which case it is difficult to use the test to favor any model.

Despite this somewhat pessimistic conclusion, it is important to note that for many predictive purposes the fact that identifying which model prevails is irrelevant. For example the sign and even magnitude of the effect of an increase in market power on demand and (observable) prices on both sides of the market can be recovered directly from the effect of changes in membership costs on these variables, as all models imply these effects will be the same. The greatest challenge, therefore, posed by the inability to distinguish among models is normative: the GRT2003 model creates challenging normative ambiguities because the monopolist only internalizes the marginal, rather than the average, two-sided benefit of consumers. No test has enough power to *always* establish or refute the existence of these effects, but between the substitution v. complements test and the cost-based tests this is likely to often be identified. More detailed analysis of the cases when the models can be distinguished and those in which they cannot is an interesting topic for future research.

4.3 Alternative sets of observables

In the previous subsection I maintained a fairly arbitrary set of assumptions about observables. While it is infeasible to consider every permutation, I here briefly consider one alternative: that corresponding to the original versions of the RT2003 and Armstrong models. The RT2003 model specializes the GRT2003 model by assuming that $B^I = C^I = 0$ for both I and the Armstrong model specializes the GArmstrong model by assuming that $c = 0$. In terms of identification, this means that in the RT2003 model $\hat{r}_{IJ}$ and η_{IJ} for $I, J = A, B$ are no longer observable, but B^I, C^I are observable. In the GArmstrong model this means that $\hat{r}_I$ and $\hat{\eta}_I$ are no longer observable for $I = A, B$, but c is observable. While the specific values of 0 at which the restricted variables are observed is not really relevant to the logic of identification, it simplifies calculations to I will maintain it in this subsection.

Proposition 21. *Under the RT2003 model*

$$m^I = -\frac{\eta_I d^I}{\hat{r}_I}$$

$$r^I = \frac{\hat{r}_I}{1-\hat{r}_J}$$

$$c = \hat{p}^A + \hat{p}^B - m^I$$

The model also predicts following equality will be satisfied

$$m^A = m^B$$

As well as the following five independent inequality conditions

$$\begin{aligned} \frac{\eta_I}{\hat{r}_I} &< 0 \quad I = A, B & (32) \\ \frac{\hat{r}_I}{1-\hat{r}_J} &> 0 \quad I = A, B & (33) \\ (1-\hat{r}_A-\hat{r}_B)\hat{r}_A\hat{r}_B &> 0 & (34) \end{aligned}$$

The assumption that $c > 0$ *implies*

$$\hat{p}^A + \hat{p}^B - m^I > 0$$

Proof. Specializing the GRT2003 model in the RT2003 model

$$\frac{\eta_I}{\sigma_{RT2003}} = -\frac{d^I r^I (1-r^J)}{m^I}$$

$$\frac{\hat{r}_I}{\sigma_{RT2003}} = r^I(1-r^J)$$

where $\frac{1}{\sigma_{RT2003}}$ the inverse stability factor for the RT2003 model

$$\frac{1}{\sigma_{RT2003}} = 1 - r^A r^B$$

From these and the first-order conditions

$$p^A + p^B - c = m^A = m^B$$

the identifications, and the consequences of $c > 0$, are simple to derive. Condition (32) arises from the fact that demand is decreasing. Condition (33) is a result of the second-order condition (positive pass-through). Condition (34) is a result of the stability condition $r^A r^B < 1$ which implies that

$$\frac{\hat{r}^A \hat{r}^B}{1 - \hat{r}^A - \hat{r}^B + \hat{r}^A \hat{r}^B} < 1$$

In other words there are two possibilities

1. $$(1 - \hat{r}^A)(1 - \hat{r}^B) > 0$$

   and

   $$1 - \hat{r}^A - \hat{r}^B + \hat{r}^A \hat{r}^B < \hat{r}^A \hat{r}^B \iff \hat{r}^A + \hat{r}^B < 1$$

   By condition (34)

   $$(1 - \hat{r}^A)(1 - \hat{r}^B) > 0 \iff \hat{r}^A \hat{r}^B > 0$$

   so in this case

   $$(1 - \hat{r}_A - \hat{r}_B)\hat{r}_A \hat{r}_B > 0$$

2. $$(1 - \hat{r}^A)(1 - \hat{r}^B) < 0$$

and

$$\hat{r}^A + \hat{r}^B > 1$$

so again

$$(1 - \hat{r}_A - \hat{r}_B)\hat{r}_A\hat{r}_B > 0$$

The independence of the three conditions can be seen by the fact that the ratio of the sign of η_I is independent of the second two conditions. The second two conditions are also independent: $\hat{r}^I = -1$ for both I satisfies the second but not the first, $\hat{r}^A = 2, \hat{r}^B = -2$ satisfies the first but not the second and $\hat{r}^A = .5 = \hat{r}^B$ satisfies both.

$\square$

These results formalize the my informally identification strategy in Weyl (2008d), where I primarily focused on positive and normative comparative statics. Some, but not all, of the tests of the model were discussed there.

Proposition 22. *Under the Armstrong model*

$$R^I = \hat{R}_{II} - \frac{\eta_{AB}}{\eta_{JJ}}\hat{R}_{JI}$$

$$M^I = \frac{D^I(\eta_{AB}\hat{R}_{JI} - \eta_{JJ}\hat{R}_{II})}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

$$b^I = \frac{\eta_{II}\hat{R}_{JI} - \eta_{AB}\hat{R}_{II}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

$$C^I = D^J \frac{(D^J\eta_{AB} + D^I\eta_{JJ})\hat{r}_{II} - (D^J\eta_{II} + D^I\eta_{AB})\hat{r}_{JI} - \eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

The model predicts

$$\eta_{AB} = \eta_{BA}$$

And the following five independent inequalities

$$\eta_{AA}\eta_{BB} > \eta_{AB}^2$$

$$\eta_{II} < 0$$

$$\eta_{AB} < 0$$

$$\eta_{AB}\hat{R}_{JI} > \eta_{JJ}\hat{R}_{II} \quad I = A, B; \;\; J = -I$$

An assumption that $C^I > 0$ implies that

$$(D^J\eta_{AB} + D^I\eta_{JJ})\hat{r}_{II} > (D^J\eta_{II} + D^I\eta_{AB})\hat{r}_{JI} + \eta_{AB}$$

An assumption that $b^I > 0$ implies that

$$\eta_{II}\hat{R}_{JI} > \eta_{AB}\hat{R}_{II}$$

An assumption that equilibrium (either homogeneous or heterogeneous) membership costs (benefits) are positive on side I implies

$$\eta_{AB}\hat{R}_{II} > (<)\eta_{II}\hat{R}_{JI}$$

Proof. The results follow directly from the earlier reasoning about the Armstrong model, as η_I and $\hat{R}_I$ were only invoked for testing. $\square$

Note that because the RT2003 and Armstrong models assume different costs exist, there is no need for sophisticated empirical tests to distinguish among them. Any market where firms have non-zero membership costs does not fit the RT2003 model and any market with non-zero firm usage costs doe not fit the Armstrong model.

5 Applications

The preceding sections have a somewhat abstract focus. Here I turn to some more concrete, policy relevant issues that can be resolved using the results above.

5.1 Comparative statics

I begin by discussing applications based on the comparative static results.

5.1.1 Welfare effects in the Armstrong model

It is well-known that the RT2003 model has counter-intuitive welfare properties: a reduction in market power or cost on one or even both sides of the market may harm one or both groups of consumers. The reason is that the monopolist does not internalize the usage benefit of the average consumer, but only the marginal consumer. Some (e.g. RT2006) have speculated that this may be a general property of two-sided markets. However, the results in Section 3 clearly show that because the monopolist in the Armstrong model internalizes the usage benefit of the average consumer on both sides of the market, no such counter-intutiive results arise. In the Armstrong model prices on both sides, and therefore welfare on both sides, are complements. Therefore "pro-consumer" forces such as cost-reductions and reductions in market power are always beneficial to both sides of the market.

5.1.2 Price skewness

A major topic of interest in the two-sided markets literature from its beginning has been the fact that prices in these markets are often skewed (high for one side and low for the other), even though costs are shared (usage) or similar for the two sides. Which preference parameters lead to price skewness?

In the GRT2003 model, just as in the RT2003 model, a rise in market power on one side of the market will increase (unobserved and observed, if consumer membership costs are weakly positive) usage prices on that side of the market, decreasing usage prices on the other side of the market. Because prices are substitutes in the GRT2003 model, shifts in market power on one side of the market drive not only increases in prices on that side, but decreases on the other side, leading to skewed pricing. On the other hand, shifts in consumer membership costs will not necessarily cause shifts in skewness: while an increase in membership costs on one side of the market causes directly causes a fall in the membership price to that side, it indirectly causes a fall in the usage price and an increase in the usage price on the other side of the market. Therefore its effect on skewness is unclear ex-ante, and identification of parameters, particularly pass-through, is necessary to understand whether this is a source of skewness.

In the GArmstrong model, a common suggestion (Armstrong, 2006; Kaiser and Wright, 2006) has been that imbalanced two-sidedness may account for skewed prices. In one sense this is certainly true: a shift in two-sidedness from being balanced across the two sides of the market to being imbalanced will lead to skewed prices. In fact because membership prices are invariant to shifts in two-sidedness between sides that leave net two-sidedness fixed, it is simple to calculate, in complete generality, how much of price skewness can be accounted for by imbalanced two-sidedness. Suppose that $b^A < b^B$. If the market were instead such that $\tilde{b}^A + \tilde{b}^B = b^A + b^B$ but $\tilde{b}^A = \tilde{b}^B$ then observed membership prices would increase by exactly $\frac{(b^B - b^A)D^B(P^B)}{2}$ to side A and decrease by $\frac{(b^B - b^A)D^A(P^A)}{2}$ to side A.

It should be noted, though, that it is skewed two-sidedness *per se* and not two-sidedness on one side of the market which can be said to account (generally) for skewed pricing. An increase in b^A need not increase prices to side A and decrease them to side B. In fact this will always decrease (unobserved) membership prices to both sides, but its effects on observed membership prices depend on parameter values. Thus a market with skewed two-sidedness will have skewed prices relative to those that would be charged in a market with balanced two-sidedness, but not necessarily relative to one with no two-sidedness.

In the GArmstrong model (as well as in the Armstrong model), shifts in market power do not lead naturally to two-sidedness; the effects on observed membership prices on both sides of the market are ambiguous. In fact because prices are complements, an increase in market power on side A of the market may lead both prices to fall, may lead prices on side A to increase and those on side B to decrease or may lead prices on side A to decrease while those on side B decrease! Thus depending on parameter values differences in market power (elasticity) across sides of the market may lead to skewed pricing in the intuitive or the counter-intuitive direction, but do not systematically do either. The general lesson here is that shifts between the two sides in usage benefits may account for skewness but shifts in membership benefits provide a less persuasive explanation.

5.1.3 Mergers with non-two-sided firms

Competition in two-sided markets is complicated a number issues discussed in the following subsubsection and force me there to consider only simple forms of competition. However, in many instances two-sided firms have horizontal or vertical relations with non-two-sided firms. On the complements side, video game consoles often have add-ons, such a joysticks, produced by separate firms. An example of substitutes is the relationship between one-sided (non-debit) ATM cards and two-sided credit cards or that between a newspaper and a book publisher.

In Weyl (2008b) I extensively treat the effects of a merger in the RT2003 model with a firm producing a good that is perfectly complementary with the two-sided service. I therefore focus here on the Armstrong model, leaving analysis of mergers with non-two-sided firms in the Hybrid model and more general mergers with non-two-sided firms in the GRT2003 model to future research.

Consider the following modified version of the Armstrong model. There are two firms, one is a two-sided platform, one is a standard firm selling to consumers on side A with a cost of production of $\tilde{C}^A$. Consumers on side B are just as in the standard Armstrong model, but consumers on side A are characterized by three heterogeneous membership benefits B_i^A,

$\tilde{B}_i^A$ and $\ddot{B}_i^A$ representing respectively the membership benefit derived if i buys the two-sided service, the non-two-sided service and both services and two homogeneous usage benefits b^I and $\ddot{b}^I$ representing respectively the usage benefit if she buys only two-sided service and if she buys both services respectively. This is quite a general model, encompassing both the newspaper-book case where the goods are substitutes and $\ddot{b}^I = b^I$ and $\tilde{B}_i^A < B_i^A + \ddot{B}_i^A$ (at least for most i) and the joystick case where $\ddot{b}^I > b^I$ and $\tilde{B}_i^A \geq B_i^A + \ddot{B}_i^A$. It also includes, as a very special case, the most general two-good discrete choice models in one-sided markets (Gentzkow, 2007). Let μ be the probability measure jointly over $\tilde{B}^A, B^A$ and $\ddot{B}^A$.

The non-two-sided firm charges a price $\tilde{P}^A$ for its product, while the two-sided firm charges as in the GArmstrong model a usage prices $p^I = b^I$ and a membership price P^I. Consumers i therefore earns utility

$$B_i^A - P^A$$

if she purchases only the two-sided service. She earns utility

$$\tilde{B}_i^A - \tilde{P}^A$$

if she purchases only the non-two-sided service. Finally she earns utility

$$\ddot{B}_i^A + (\ddot{b}^A - b^A)D^B(P^B) - P^A - \tilde{P}^A$$

if she purchases both goods. We can then define the demand functions on side A as

$$D^A\left(P^A, \tilde{P}^A, [\ddot{b}^A - b^A]D^B[P^B]\right) \equiv N^A\mu\left(\max\left\{B^A, \ddot{B}^A + (\ddot{b}^A - b^A)D^B(P^B) - \tilde{P}^A\right\} - P^A > \max\{\tilde{B}^A - \tilde{P}^A, 0\}\right) \tag{35}$$

for the two-sided firm and

$$\tilde{D}^A\left(\tilde{P}^A, P^A, [\ddot{b}^A - b^A]D^B[P^B]\right) = N^A\mu\left(\max\left\{\tilde{B}^A, \ddot{B}^A + (\ddot{b}^A - b^A)D^B(P^B) - P^A\right\} - \tilde{P}^A > \max\{B^A - P^A, 0\}\right) \quad (36)$$

I assume that preferences are sufficiently diffuse that wherever demand is positive $D_2^A, \tilde{D}_2^A, D_3^A, \tilde{D}_3^A > 0 > D_1^A, \tilde{D}_1^A$ holds (with strict inequality). I also assume that there is a unique, globally stable first-order equilibrium or optimum if the firms merge; conditions for this to hold are available on request but I omit them here. The following proposition gives the first-order conditions which uniquely determine the equilibrium given this assumption.

Proposition 23. *Equilibrium when the firms are separate is given the following three first-order conditions*

$$P^A - C^A + \alpha D^B = M^A$$

$$P^B - C^B + M^A(\ddot{b}^A - b^A)D_3^A + \alpha D^A = M^B$$

$$\tilde{P}^A - \tilde{C}^A = \tilde{M}^A$$

where all functions on side B are evaluated at P^B, all functions on side A are evaluated at $P^A, \tilde{P}^A$ and $(\ddot{b}^A - b^A)D^B(P^B)$ and where $M^A \equiv -\frac{D^A}{D_1^A}, \tilde{M}^A \equiv -\frac{\tilde{D}^A}{\tilde{D}_1^A}$, $M^B \equiv -\frac{D^B}{D^{B\prime}}$ and $\alpha \equiv b^A + b^B - c$. When the firms are integrated, the monopoly optimum is given by

$$P^A - C^A + \alpha D^B = \frac{M^A + \tilde{M}^A\tilde{\theta}^A}{1 - \theta^A\tilde{\theta}^A}$$

$$P^B - C^B + \alpha D^A + \frac{(\ddot{b}^A - b^A)\left(M^A\left[D_3^A + \theta^A\tilde{D}_3^A\right] + \tilde{M}^A\left[\tilde{D}_3^A + \tilde{\theta}^A D_3^A\right]\right)}{1 - \theta^A\tilde{\theta}^A} = M^B$$

$$\tilde{P}^A - \tilde{C}^A = \frac{\tilde{M}^A + M^A\theta^A}{1 - \theta^A\tilde{\theta}^A}$$

where $\theta^A = -\frac{D_2^A}{\tilde{D}_1^A}, \tilde{\theta}^A = -\frac{\tilde{D}_2^A}{D_1^A}$

Proof. See Appendix C. □

Analyzing this general model of mergers with non-two-sided firms is fairly straightforward given my results above. However it is quite long and tedious, so I do not develop it here. The proposition above and the results that precede it provide a framework for a wide variety of applications and I consider one simple application here: the Cournot (1838)-Spengler (1950) double marginalization problem. That is I assume that $B_i^A = \tilde{B}_i^A \equiv -\infty$ (ie the goods are perfect complements) for all i and assume that the two-sided firm charges $p^I = \ddot{b}^I$. In this special case things simplify dramatically. We can now write

$$D^A(P^A + \tilde{P}^A) \equiv N^A \mu(\ddot{B}^A > \tilde{P}^A + P^A)$$

Assume $M^{A'} < \frac{1}{2}$ and let $\ddot{R}^A = \frac{1}{1-2M^{A'}}$. Also assume that $1 > \ddot{R}^A R^B \ddot{\alpha}^2 D^{A'} D^{B'}$ where $\ddot{\alpha} \equiv \ddot{b}^A + \ddot{b}^B - c$. Then the equilibrium conditions become a simple two-sided markets version of the classic double marginalization equilibrium conditions (Weyl, 2008c):

$$P^A - C^A + \ddot{\alpha} D^B = M^A$$

$$\tilde{P}^A - \tilde{C}^A = M^A$$

$$P^B - C^B + \ddot{\alpha} D^A = M^B$$

where $\ddot{\alpha} \equiv \ddot{b}^A + b^B - c$. A tidy way to express the first two conditions is

$$\ddot{P}^A - \ddot{C}^A + \ddot{\alpha} D^B = 2M^A$$

where $\ddot{P}^A = P^A + \tilde{P}^A$. On the other hand if the firms are owned jointly then the optimum is given by

$$\ddot{P}^A - \ddot{C}^A + \ddot{\alpha} D^B = M^A$$

$$P^B - C^B + \ddot{\alpha} D^A = M^B$$

Both of these are first-order conditions for standard GArmstrong two-sided markets; the only difference between them is that under separation market power is greater, as is typically the case under double marginalization. Thus my earlier result that an increase in the market power on one side of the market raises prices and harms welfare on both sides in the GArmstrong model almost immediately yields the following proposition.

Proposition 24. *Vertical integration in the GArmstrong double marginalization problem lowers prices and improves welfare on both sides of the market, as well as industry profits.*

Proof. See Appendix C. □

This result is reassuring, as it shows that the basic logic of double marginalization carries over into GArmstrong models of two-sided markets, as I showed in Weyl (2008b) in the RT2003 model.

5.1.4 Simple competition and mergers

This paper focuses overwhelmingly on monopoly models. However the tools developed here can be used to analyze some simple models of competition; despite their simplicity, they more general than some considered in the literature.

First, the approach here can be used to analyze the form of competition proposed by RT2003. In their model of competition, two platforms are symmetrically differentiated for card-holders and identical (at least exogenously) for merchants. Merchants choose whether to accept cards or not and consumers are randomly paired with them. Conditional on the merchants accepting both cards the consumers can choose which to use; conditional on them accepting one or none, the consumer must choose between the other and cash or just use cash. In equilibrium all merchants multihome, but there is competition created for merchants because a firm can persuade them to go exclusive by lowering its price. Competition for consumers arises from the exogenous substitutability of the two cards.

This model can be easily transfered to the GRT2003 model. Some subtleties arise about fixed membership costs, as some consumers may not find it worthwhile to carry a particular card card if they do not expect to use it often. This tends to increase consumer single-homing and reduces (or even eliminates) competition for merchants in the absence of further additions to the model. The resulting analysis, as is or altered to allow competition for merchants, is very similar to that of the RT2003 model presented in Weyl (2008d); I therefore omit it here, but it is available on request.

Second and more novel, we can consider a variant on the model of competition considered by Armstrong. He assumes that firms commit to observed membership prices; the resulting model is sufficiently complex that he is forced to consider only a case of linear (Hötelling) demand and fixed levels of consumer participation in the market as a whole, ruling out welfare consequences of competition and making it difficult to compare the competitive outcome to that which would result under monopoly (as prices would be infinite under monopoly). Here I consider a model, briefly alluded to by Armstrong, where firms commit to unobservable membership prices (what he calls hedonic prices) and usage prices equal to the homogeneous usage benefits. The two platforms are symmetrically differentiated (though perhaps differently across sides) for both sides of the market, but general demand functions are allowed. Only limited results are possible in this general model, so I specialize to a tractable case, which includes markets that are symmetric-across sides (Armstrong's case) and those where $p^I = b^I$.

To give a bit more general sense of how competition works in the Armstrong model, first consider a more general setting. Consumers on each side I have a usage benefit b^I which depends only on which side of the market they are on, not which platform they use nor their identity. Consumers have a membership benefit for each platform 1 and 2, $B_i^{I,1}$ and $B_i^{I,2}$ respectively, drawn from a symmetric density function $g^I : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that each firm i commits to some membership price $P^{I,i}$ and some usage price p^I (symmetric across firms) on side I. Suppose that $N^{I,i}$ consumers participate in platform i on side I. Then a consumer j on side I will have an incentive to participate in platform I if and only if

$$N^{J,i}(b^I - p^I) + B_j^{I,i} - P^{I,i} > \max\left\{N^{J,-i}(b^I - p^I) + B_j^{I,-i} - P^{I,-i}, 0\right\}$$

Thus if we let the marginal density of $B^{I,i}$ be

$$g^{I,i}(B^{I,i}) \equiv \int_{-\infty}^{\infty} g^I(B^{I,i}, B^{I,-i})dB^{I,-i}$$

and the conditional cumulative distribution of $B^{I,i}$ be

$$G^{I,i|-i}(B^{I,i}|B^{I,-i}) \equiv \frac{\int_{-\infty}^{\infty} B^{I,i}g^I(B, B^{I,-i})dB}{g^{I,-i}(B^{I,-i})}$$

Then the the number of participants on side I of platform i is

$$\hat{N}^{I,i}(P^{I,i}, P^{I,-i}, N^{-I,i}, N^{-I,-i}) = D^I\left(P^{I,i} - N^{-I,i}[b^I - p^I], P^{I,-i} - N^{-I,-i}[b^I - p^I]\right)$$

where

$$D^I(h^{I,i}, h^{I,-i}) \equiv \overline{N}^I E_{B^{I,-i}}\left[1 - G^{I,i|-i}\left(\max\left\{h^{I,i} - h^{I,-i} + B^{I,-i}, 0\right\}\middle| B^{I,-i}\right)\right]$$

and $\overline{N}^I$ is the total number of consumers on side I.

Assume[7] that the resulting problem $N^{I,i} = \hat{N}^{I,i}(P^{I,i}, P^{I,i}, N^{-I,i}, N^{-I,-i})$ for $I = A, B$ and $i = 1, 2$ has a unique fixed point. Then we can write demand on each side of the market for each firm as a function of the four membership prices; I assume fixed usage prices, following Armstrong, as each firm is indifferent individually (given the choices of the other firm) as to its mix of usage and membership prices. Let $N^I(P^{I,i}, P^{I,-i}, P^{-I,i}, P^{-I,-i})$ be this solution for each $I = A, B$ and $i = 1, 2$; by symmetry, the function is firm-indepdent. Firm i's profits can then be written as

[7] I do not here explicitly develop the conditions under which this holds.

$$\pi(P^{A,i}, P^{A,-i}, P^{B,i}, P^{B,-i}) =$$

$$(P^{A,i} - C^A)N^A(P^{A,i}, P^{A,-i}, P^{B,i}, P^{B,-i}) + (P^{B,i} - C^B)N^B(P^{B,i}, P^{B,-i}, P^{A,i}, P^{A,-i}) +$$

$$(p^A + p^B - c)N^A(P^{A,i}, P^{A,-i}, P^{B,i}, P^{B,-i})N^B(P^{B,i}, P^{B,-i}, P^{A,i}, P^{A,-i})$$

The conditions for a symmetric equilibrium are given in the following proposition.

Proposition 25. *At a symmetric-across-platforms Nash-Bertrand equilibrium in membership prices* (P^A, P^B) *with usage price* p^I *on side* I

$$P^I - (b^I - p^I)N^J - C^I + \alpha N^J + \gamma^I = M_o^I \tag{37}$$

for both I*,* $J = -I$ *where*

$$\gamma^I \equiv \frac{(b^J - p^J)D_2^I D_2^J \left(D_1^J M_o^I[b^I - p^I] - \left[P^J - \left(b^J - p^J\right) N^I - C^J + \alpha N^I\right]\right)}{D_1^I + (b^I - p^I)\left(b^J - p^J\right) D_1^J \left(\left[D_1^I\right]^2 - D_2^I D_2^J\right)}$$

while D_i^I *is the derivative of* D^I *with respect to the* i*th argument, the* own price market power $M_o^I \equiv -\frac{N^I}{D_1^I}$ *and all expressions are evaluated at symmetric-across-firm equilibrium prices* P^A, P^B*.*

Proof. See Appendix D. □

This first-order condition is a (substantial) generalization of Armstrong (2006)'s equation (11) and follows a similar form. This differs from first-order condition for a GArmstrong monopolist in a few ways:

1. First and least importantly, because the firms do not charge a usage price equal to the usage benefit, their membership price is adjusted up or down to offset this. This is

irrelevant to the actual prices faced by consumers because each firm takes the prices of the other as given and therefore, as a monopolist, charges total prices independent of the mixture between usage and membership prices.

2. The market power is now the ratio of demand to the partial derivative of demand with respect to own price. This is simply the analog of market power in the case when the firms controls only the price of one platform. This adjustment is the standard, one-sided market effect of competition.

3. Prices on side I are subsidized by an amount γ^I. This term is driven by $b^J - p^J$ and is 0 if this is 0, as well as by D_2^I and D_2^J. This term arises because platforms have the capacity to reduce the quality of their competitor's product by stealing consumers on one side of the market from them. This is similar to the incentive a firm has to advertise more if its advertisements not only increase consumer's opinions of their product but also reduce their opinion of their competitor's product. Here this *sabotage* term (if positive) is channelled towards price reduction, however, as Armstrong points out in the case of linear demand, constant total market size on both side of the market and zero usage prices. This is because in this simple model, price reductions on one side of the market are the only means of sabotaging the competitor on the other side.

   However, this term need not necessarily be positive. The reason is that sabotaging your opponent on side A may not necessarily benefit you if are subsidizing the other side of the market so heavily that you actually are harmed by having more consumers on the other side. This somewhat bizarre possibility can arise only if the sabotage motive on the *other* side of the market is sufficiently strong to lead to such *over-subsidization* on the other side of the market (see Appendix D). Intuitively, sabotaging on side B (to lower the quality of your competitor's offering on side A) may generate such great profits on side A that, at the optimal price to side B, the firm are actually unhappy with gaining an extra consumer on side B *even after taking into account the business* *this generates on side A*. The firm does not, however, have an incentive to raise prices on side B because this would send more customer's to your competitor's platform on side B and increase her quality on side A. This counter-intuitive case is crucial to Farhi and Hagiu (2007)'s result that an increase in cost for one firm may actually benefit its competitor firm, if it causes the cost-reducer to steal consumers on the over-subsidized side of the market.

While the possibility of over-subsidizaton, ignored by Armstrong, may be important in some two-sided markets , it substantially complicates the analysis of the effects of competition. I therefore leave the more detailed study of this phenomenon to future research. However, this effect is only possible when $b^I \gg p^I$ and there is substantial asymmetry between sides of the market. This allows me to focus, and thereby obtain sharper results, in a number of special cases where over-subsidization does not arise, as I show below.

To understand the effects of a merger, we can compare the symmetric-across-platforms Bertrand-Nash equilibrium to a symmetric-across-platforms monopoly optimum, assuming one exists[8]. The following proposition gives the intuitive first-order conditions for such a monopoly optimum.

Proposition 26. *At a symmetric-across platforms monopoly optimum, assuming WLOG firms charge a usage price on each platform of* $p^I = b^I$

$$P^I - C^I + \alpha D^J(P^J) = M^I(P^I) \tag{38}$$

where $M^I(P^I) \equiv -\frac{D^I(P^I)}{D^{I\prime}(P^I)} = -\frac{D^I(P^I)}{D_1^I(P^I,P^I)+D_2^I(P^I,P^I)}$ *and* $D^I(P^I) \equiv D^I(P^I, P^I)$.

Proof. WLOG I can assume both firms choose $p^I = b^I$ by the same reasoning as before, given that I am considering only symmetric-across firm optima. Then the derivation follows

[8]Again, I omit conditions for this here, but they are available upon request.

precisely from the derivation of the GArmstrong first order conditions, letting $D^I(P^I) \equiv D^I(P^I, P^I)$. $\square$

The effects of mergers depend on both the classical tendency to increase market power and on the elimination of the sabotage term. When the sabotage term is positive (over-subsidization is not too strong), things are relatively straight-forward. Competition will put downward pricing pressure on both sides of the market, both by reducing market power and by giving incentives for sabotage. Because prices are complementary across the two sides of the market, these effects reinforce one another and lead to lower prices and improved welfare for both groups of consumers. This result is expressed in the following proposition.

Proposition 27. *Suppose that $\gamma^I - M^I \frac{D_2^I}{D_1^I} > 0$ as well as the second-order conditions for the GArmstrong model applied to the monopoly optimum in this context: $M^{I'} < 1$ for both I and $\alpha^2(D_1^A + D_2^A)(D_1^B + D_2^B)R^A R^B < 1$. Then* true membership *prices (i.e. $\hat{P}^I - b^I N^J$) are lower, and welfare and demand higher, on both sides of the market (and for both firms) under symmetric Nash-Bertrand equilibrium than under monopoly.*

Proof. See Appendix D. $\square$

This result contrasts significantly with the RT2003 model, where competition can easily harm one or even both groups of consumers. Of course the assumption of the proof rules out some extreme cases that may allow this to occur in the GArmstrong model. Nonetheless the assumption includes a variety of situations of interest as special cases.

Corollary 1. *If any of the following conditions hold, then the result of Proposition 27 follows:*

1. $\gamma^I \geq 0$

2. *Neither side is over-subsidized and neither has a usage price strictly greater than its homogeneous usage benefit: $P^J - \left(b^J - p^J\right) N^I - C^J + \alpha N^I > 0$ and $b^I \geq p^I$ for both I*

3. *Demand is symmetric across sides of the market*

4. $p^I = b^I$ for both I

Proof. $M^I \frac{D_2^I}{D_1^I} < 0$, so the first condition obviously implies the assumption in Proposition 27. The denominator in the expression for γ^I is negative (see Appendix D), $(b^J - p^J)D_2^I D_2^J$ is weakly positive so long as $b^J \geq p^J$ and $D_1^J M_o^I (b^I - p^I)$ is weakly negative under the same condition. So in this case no over-subsidization implies $\gamma^I > 0$. If demand is symmetric across sides of the market, then neither side can be over-subsidized; if either side were over-subsidized then both would be. But if that were the case then prices would be below cost on both sides of the market and the firms would stop producing. Finally if $p^I = b^I$ then $\gamma^I = 0$ on both sides. □

Thus there are a reasonable, though certainly not exhaustive, set of cases when competition will always be good for both groups of consumers. Despite the specialness of the settings I consider, the symmetry across firms and bounds on the sabotage effect's size, they are far more general than those anywhere else in the literature. Beyond the particular conditions I consider, my general set-up of symmetry across firms is quite limiting as well. For example, the existence a symmetric equilibrium and monopoly depends on the platforms being strongly inherently differentiated for consumers and only considering symmetric monopoly optima excludes the possibility of competition induced coordination failures (Ambrus and Argenziano, Forthcoming). This overlooks coordination issues that are important, especially in immature markets. Furthermore the assumption that $p^I \leq b^I$, while seemingly benign, is actually quite unrealistic in many markets. Finally I do not consider any comparative statics under competition as this is much more complicated that the comparative statics of monopoly; for example it is still unclear whether Armstrong's claim that a fall in p^I generally leads to lower prices to both groups of consumers holds, as a fall in one side's price may reduce sabotage motives on the other side.

Nonetheless, these results demonstrate both the extent to which Armstrong's basic results on the effects of competition substantially generalize. They also demonstrate how an important difference between the GRT2003 model and the GArmstrong model of monopoly generalize to simple forms of competition: where competition in the GRT2003 model may harm one or both sides of the market as it may be that it causes one price to rise by the seesaw effect, because the GArmstrong model exhibits complements between demands and prices competition is always good for all consumers.

Models of competition in the Hybrid model are more complicated, and I leave them to future research.

5.1.5 RT2006 unification

RT2006 presents a unified model of monopolies in two-sided markets which is a superset of all models presented here. They show how the first-order conditions for all models considered in this paper can be written in a more general form. They suggest that this shows a basic similarity among models of two-sided markets. While this is certainly true in a general sense, and their first-order conditions are a useful starting point, the significance of these common first-order conditions is not clear. For example the first-order conditions for any two variable optimization problem over X and Y can be written as

$$X = F(X, Y)$$

$$Y = G(X, Y)$$

But the fact that the first-order conditions for a one period unconstrained investment problem and the pricing condition for a two-product monopolist can both be written in this form says nothing about the similarity between these problems except that they both involve two variables. That is not to say that nothing can be said about the general RT2006. It simply indicates that anything that can be said requires further analysis. In fact, two models within this class, the GArmstrong and the GRT2003 model behave very differently in terms of their normative and positive economics. On the normative side, the GRT2003 model has "weird" effects driven by the fact that only marginal, and not average, externalities are internalized by the monopolist, while the Armstrong model has simple and intuitive welfare economics. On the positive side, the GRT2003 exhibits substitution of the demands and usage prices on the two sides of the market (the seesaw effect), while the GArmstrong model exhibits complementing. This means that nothing generally can be said in the RT2006 class about the sign of the linkage between prices on the two sides, in contrast to RT2006's claim about the generality of their topsy-turvy principle. One simple way to see the sources of differences among these sub-models is to re-write the RT2006 first-order conditions in a form that relates in a simple way to the first-order conditions above. This can be done in two ways. Before doing so, it is useful to introduce a bit of notation.

Let $f^I : \mathbb{R}^2 \to \mathbb{R}^2$ be the joint density of B^I and b^I. Let the marginal density of B^I be

$$f^I_M(B^I) \equiv \int_{-\infty}^{\infty} f^I(B^I, b^I)db^I$$

the marginal density of b^I be

$$f^I_U(b^I) \equiv \int_{-\infty}^{\infty} f^I(B^I, b^I)dB^I$$

the conditional cumulative distribution of B^I be

$$F^I_{M|U}(B^I|b^I) \equiv \frac{\int_{-\infty}^{B^I} f^I(x, b^I)dx}{f^I_U(b^I)}$$

and the conditional cumulative distribution and density of b^I be defined *mutatis mutandis*. If the number of consumers participating on side $J = -I$ is n^J and the monopolist charges a membership price P^I then consumer i on side I will participate if and only if

$$B^I_i + b^I_i n^J \geq P^I$$

Thus we can write the demand on side I as a function of participation on side J and (membership) price on side I:

$$\hat{D}^I(P^I, n^J) = N^I E_{b^I}\left[1 - F^I_{M|U}(P^I - b^I n^J | b^I)\right]$$

where N^I is the total potential consumers on side I. Let the density of *marginal consumers* on side I

$$\hat{\mu}^I(P^I, n^J) \equiv N^I \int_{-\infty}^{\infty} f^I(P^I - b^I n^J, b^I) db^I$$

and the *average usage benefit of marginal consumers* (AUBMC)

$$\hat{\overline{b}}^I(P^I, n^J) \equiv \frac{N^I \int_{-\infty}^{\infty} f^I(P^I - b^I n^J, b^I) b^I db^I}{\hat{\mu}^I(P^I, n^J)}$$

As shown in Appendix E, the simple necessary and sufficient condition for the resulting problem $n^A = \hat{D}^A(P^A, n^B), n^B = \hat{D}^B(P^B, n^A)$ to have a unique fixed point for all values of (P^A, P^B) is that for all P^A, P^B, n^A, n^B

$$\sigma^M_{RT2006}(P^A, P^B, n^A, n^B) \equiv 1 - \hat{\mu}^A(P^A, n^B)\hat{\mu}^B(P^B, n^A)\hat{\overline{b}}^A(P^A, n^B)\hat{\overline{b}}^B(P^B, n^A) > 0$$

Maintaining this assumption we can write demand on side I as a function of prices on the two sides of the market $D^I(P^I, P^J)$, where this function is the unique solution for n^I the fixed point problem. Then the density of marginal consumers becomes $\mu^I_M(P^I, P^J) \equiv \hat{\mu}^I\big(P^I, D^J[P^J, P^I]\big)$ and AUBMC becomes $\overline{b}^I(P^I, P^J) \equiv \hat{\overline{b}}^I\big(P^I, D^J[P^J, P^I]\big)$. I will also refer to the following moments below, where $f^I_{M|U} \equiv F^{I'}_{M|U}$.

$$\overline{\lambda_{f^I_{M|U}}} = \frac{\int_{-\infty}^{\infty} f^I\left(P^I - b^I D^J[P^J, P^I], b^I\right) \frac{f^{I'}_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\right)}{f^I_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\right)} db^I}{\mu^I\left(P^I, P^J\right)}$$

$$\overline{b^I \lambda_{f^I_{M|U}}} = \frac{\int_{-\infty}^{\infty} f^I\left(P^I - b^I D^J[P^J, P^I], b^I\right) b^I \frac{f^{I'}_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\right)}{f^I_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\right)} db^I}{\mu^I\left(P^I, P^J\right)}$$

$$\overline{b^{I^2}\lambda_{f^I_{M|U}}} = \frac{\int_{-\infty}^{\infty} f^I\left(P^I - b^I D^J[P^J, P^I], b^I\right) b^{I^2} \frac{f^{I\prime}_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\ \right)}{f^I_{U|M}\left(P^I - b^I D^J[P^J, P^I] | b^I\ \right)} db^I}{\mu^I\left(P^I, P^J\right)}$$

If instead the firm charges a usage price p^I then consumer i on side I will participate if and only if

$$\frac{B^I_i}{N^J} + b^I_i \geq p^I$$

Thus demand on side I is

$$\hat{d}^I(p^I, n^J) = E_{B^I}\left[1 - F^I_{U|M}\left(p^I - \frac{B^I}{n^J} | B^I\right)\right]$$

I show, again in Appendix E, that the simple necessary and sufficient condition for the resulting problem $n^A = \hat{d}^A(p^A, n^B), n^B = \hat{d}^B(p^B, n^A)$ to have a unique fixed point for all values of (p^I, p^J) is that for all p^I, p^J, n^I, n^J

$$n^{A^2} n^{B^2} > \left(p^A - \overline{b}^A[p^A \hat{n}^B, n^B]\right)\left(p^B - \overline{b}^B[p^B \hat{n}^A, n^A]\right)\hat{\mu}^A(p^A n^B, n^B)\hat{\mu}^B(p^B n^A, n^A)$$

Then, as before, demand $d^I(p^I, p^J)$ is a function of the two prices. With this notation, the following proposition expresses the first-order conditions for the RT2006 model in both membership and usage pricing terms.

Proposition 28. *Supposing that the monopolist sets membership prices P^I, then first-order conditions for the RT2006 model can be written as*

$$P^I - C^I + \left(\overline{b}^J[P^J, P^I] - c\right) D^J(P^J, P^I) = M^I(P^I, P^J) \tag{39}$$

where $M^I \equiv \frac{N^I}{\mu^I_M}$. If the monopolist sets usage prices then the first-order conditions are

$$p^I + \overline{b}^{-I} - \left(c + c^I[p^I, p^J]\right) = m^I(p^I, p^J) \tag{40}$$

where $c^I \equiv \frac{C^I}{d^J}$ *and* $m^I \equiv \frac{n^I}{n^J \mu_U^I}$. *These first-order conditions are sufficient for maximization if the following second order conditions hold globally (for all* P^A, P^B*):*

$$\sigma^I_{RT2006} \equiv 1 + \frac{\mu^I D^J \left(\overline{b^{J2}\lambda_{f^J_{M|U}}} - \bar{b}^J \overline{b^J \lambda_{f^J_{M|U}}}\right) - (\bar{b}^J - c)\mu^I \mu^J \bar{b}^J + M^I \left(\mu^I \mu^J \bar{b}^J \left[\overline{b^I \lambda_{f^I_{M|U}}} - \bar{b}^I \overline{\lambda_{f^I_{M|U}}}\right] - \overline{\lambda_{f^I_{M|U}}}\right) + 1}{\sigma^M_{RT2006}} > 0 \quad I = A, B; \; J = -I \tag{41}$$

$$\sigma^S_{RT2006} \equiv 1 - \chi^A \chi^B > 0$$

where for $I = A, B$ *and* $J = -I$

$$\chi^I \equiv \frac{\mu^J \left(\bar{b}^J - \bar{b}^I - c - M^I \overline{b^I \lambda_{f_{M|U}}}\right) - \left(\overline{b^J \lambda_{f^J_{M|U}}} - \bar{b}^J \overline{\lambda_{f^J_{M|U}}} + \mu^A \mu^B \bar{b}^I \left[\overline{b^{J2}\lambda_{f^J_{M|U}}} - \bar{b}^J \overline{b^J \lambda_{f^J_{M|U}}} + \bar{b}^J \overline{\lambda_{f^J_{M|U}}} - \overline{b^J \lambda_{f^J_{M|U}}}\right]\right) D^J}{2 + \mu^I D^J \left(\overline{b^{J2}\lambda_{f^J_{M|U}}} - \bar{b}^J \overline{b^J \lambda_{f^J_{M|U}}}\right) - (\bar{b}^I + \bar{b}^J - c)\mu^I \mu^J \bar{b}^J + M^I \left(\mu^I \mu^J \bar{b}^J \left[\overline{b^I \lambda_{f^I_{M|U}}} - \bar{b}^I \overline{\lambda_{f^I_{M|U}}}\right] - \overline{\lambda_{f^I_{M|U}}}\right)} \tag{42}$$

Proof. See Appendix E. □

These first-order conditions are effectively the same as those for the GArmstrong and GRT2003 models respectively with two caveats. First the formulas must be slightly adjusted because the firm only charges one price, and therefore the first order conditions do not include any term for homogenous own-sided usage or membership benefits. Second, the *average* usage/membership benefit/cost of the marginal consumer, rather than the homogeneous benefit/cost of *all* consumers is crucial. This makes a big difference to both the positive and normative economics of the problem. On the positive side, because, for example, the AUBMC in condition (39) can vary substantially as prices change, the model could exhibit complementarity or substitutability of prices/demands. In fact, it is possible that AUBMC could actually *decline* as price increases, where it is fixed in the GArmstrong model and increasing in the GRT2003 model. If consumers who have the smallest membership costs also have the smallest usage benefits, raising the price could select these consumers, causing a strong complementarity between demands/prices on the two sides of the market driven by exactly the mechanism that creates substitutability in the GRT2003 model!

Things are equally slippery on the normative side. While the monopolist only internalizes the average usage benefit of the *marginal* consumers, it is the average usage benefit of the *average* participating consumer which should be internalized. In the GRT2003 model this always leads monopolistic prices, even in the absence of market power, to be too high and therefore the optimal price level to be below cost by the amount of average consumer (usage) surplus (Weyl, 2008e). This is because infra-marginal consumers always have higher usage benefit than marginal consumers. But this need not generally be true: if the average usage benefit of the average participating consumer is smaller than that of the average marginal consumer, market-power free prices might be too *low*! I do not explicitly develop examples of these cases here, though they are available on request. My goal is just to make clear what a wide range of cases the RT2006 model accommodates, despite the apparently simple unified framework they can be placed within. Because of this much greater complexity, the identification and testing of the RT2006 model is a substantial challenge for future research.

5.2 Identification

I now turn to applications of my identification results.

5.2.1 Identifying market power

A major focus of the two-sided markets literature has been how to identify market-power, given that observed price-cost margins offer only a vague picture. The basic confound in the GArmstrong model is that two-sidedness creates cross-subsidies, so these must be measured to determine market power. In the GArmstrong model identifying market power is particularly important because it shares with one-sided market models the property that socially optimal prices are those charged by a monopolist with no market power (Gaudeul and Jullien, 2005). This can be accomplished simply using my identification results, using a simple formula involving the effects of exogenous variation in membership costs on the two sides of the market on observed demand and prices on both sides of the market.

In the GRT2003 model the basic confound is that the monopolist compensates (charges) the consumers for their membership costs (benefits). This prevents usage price-cost margins from revealing market power. This too is easily overcome and market power can be measured. However, market power is a much less welfare-relvant indicator in the GRT2003 model; socially optimal prices correspond to charging a price to each side I of the market $\overline{V}^{-I}$ below the zero-market-power monopoly optimal price (Weyl, 2008e). Nonetheless, a measurement of market power may be useful in some policy settings and is easy to obtain based on the results here.

5.2.2 Identifying predatory pricing

It is often argued that below-cost prices in a two-sided market should not be taken as a warning sign of potential anticompetitive behavior, because prices are often below cost because of two-sided externalities. The models I consider in this paper are basically monopoly models and are therefore not really suitable to understanding such behavior. However, in the absence of a better approach, I here propose a simple approach to testing for predatory pricing based on the model here. I outline it for the GArmstrong model, but it may be applied as easily to the GRT2003 and Hybrid models.

Consider a very simplified model of predatory pricing. A firm's profit function is their profits plus a continuation value derived from the probability they are able to drive a competitor out of the market. If this probability is a function of prices charged on the two sides of the market, it can be written as

$$\pi(P^A, P^B) = (P^A - C^A)D^A(P^A) + (P^B - C^B)D^B(P^B) + (b^A + b^B - c)D^A(P^A)D^B(P^B) + \phi(P^A, P^B)$$

If ϕ is linearly separable this can be reduced to

$$\pi(P^A, P^B) = (P^A - C^A)D^A(P^A) + (P^B - C^B)D^B(P^B) + (b^A + b^B - c)D^A(P^A)D^B(P^B) + \phi^A(P^A) + \phi^B(P^B)$$

This gives rise to a first-order condition for predatory pricing

$$P^I - C^I + \alpha D^J(P^J) = M^I(P^I) + \gamma^I(P^I) \tag{43}$$

where $\gamma^I(P^I) = -\frac{\phi^{I\prime}(P^I)}{D^{I\prime}(P^I)}$. Simple math shows that, maintaining my notation from the identification section, here too

$$\alpha = -\frac{\eta_{AB}}{\eta_{AA}\eta_{BB} - \eta_{AB}^2}$$

Once α is recovered, a simple test for predatory pricing, analogous to the "price below cost" test is

$$P^I < C^I - \alpha D^J$$

Of course this requires C^I to be observable, but so does the standard markets test for predatory pricing. If ϕ is not linearly separable, b^I and b^J can be recovered[9] in the same fashion as in the standard GArmstrong model. After this is done, if c is also observable, α can be computed and the test performed. Also, with either separable or non-separable ϕ, several equality and inequality tests for the models remain valid, so the basis for this rather casual approach can be at least partially validated. Extreme caution should certainly be applied in using this approach which has at best the weakest of theoretical foundations. However in the absence of a more sophisticated test, it offers at least some alternative to simple abandonment of concern with predatory pricing in two-sided markets.

[9] A demonstration of this is available upon request.

6 Conclusion

This paper tries to understand the working of monopolies in two-sided markets. I provides a detailed analysis of three models, a generalization of the RT2003 model, a generalization of the Armstrong model and a mixture of the two. While this focuses on monopoly, it extends to mergers with non-two-sided firms and some simple forms of two-sided competition. Many of my results on identification may be general to the broader RT2006 class, for which I provide a new, more fundamental and intuitive first-order condition, as well as the first second-order condition which ensure the first-order condition is sufficient for maximization. My empirical strategy, based on Slutsky symmetry, for identifying the effects of price controls is almost completely general.

The advantage of the approach taken here is that it provides a sufficiently complete analysis of an approach to two-sided markets that most (or at least many) question of policy interest can be answered in substantial generality, in a model which generalizes a pair of standard one-sided markets. This avoids the potentially corrosive incompleteness of the early two-sided markets literature, which emphasized that classic approaches to one-sided markets are inappropriate in two-sided markets, without providing policy makers an alternative approach.

Nonetheless, my analysis has a number of severe limitations. In a future draft, I will likely reformulate it more explicitly around the RT2006 model, specializing only where and when this is necessary to obtain results. But beyond this much work remains to be done. Here I briefly summarize some of the directions I find most promising.

While I provide a strategy for identifying many of the important local properties of the demand system, these seem likely to be insufficient in the broader RT2006 class or when more realistic and unrestricted forms of competition exist. Therefore developing equally robust but more powerful techniques for identification is an important direction for future research. Allowing for more general competition is an important direction as well; no one has even formulated, much less solved, a model of competition allowing heterogeneity along both the membership and usage cost dimension. Such a model if it could be solved at least computationally, would offer a strong foundation for applied structural work. Even simpler general analyses of competition remain an open question: all analysis I am aware of thus far in the GRT2003 and GArmstrong models has been restricted to symmetric-across firm duopolies, which are clearly unrealistic.

More challenging and novel would be an investigation of markets with more than two sides, non-linear valuations of consumers on the other side of the market, crowding network effects on one side of the market (which are clearly important in some two-sided markets like dating websites) and dependence of utility on match quality as well as the number of match. Other important issues, such as coordination problems crucial in immature industries (Caillaud and Jullien, 2001, 2003; Ambrus and Argenziano, Forthcoming) and the effects of direct interactions between the two sides of the market (Guthrie and Wright, 2007; Rochet and Tirole, 2007), have begun to be analyzed. However, this work, like the early work on mature, exogenous demand two-sided markets I analyze here, almost exclusively uses restrictive functional form assumptions that make it inappropriate for empirical work and limit its applicability to policy analysis. I am hopeful that this literature will move beyond raising problems with classical approach to a more comprehensive framework for applied analysis.

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A Proof of Theorem 1 and Proposition 4

I first establish Theorem 1.

Proof. Given the strict global concavity of F, sufficient conditions for the maximization with respect to $\mathbf{x}$ are

$$\Delta F(\mathbf{x}) = \mathbf{w}$$

so by the implicit function theorem, letting $\mathbf{1}_i$ be the column vector with 0 in all entries but the ith and 1 in the ith,

$$\Delta\Delta F(\mathbf{x})\frac{\partial \mathbf{x}}{\partial w_i} = \mathbf{1}_i$$

so

$$\frac{\partial \mathbf{x}^\star}{\partial w_i} = \Delta\Delta F(\mathbf{x}^\star)^{-1}\mathbf{1}_i$$

as $\Delta\Delta F(\mathbf{x})$ is invertible by the strict concavity of F. Therefore

$$\frac{\partial K(\mathbf{x}^\star)}{\partial w_i} = \Delta K(\mathbf{x}^\star)^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\mathbf{1}_i \tag{44}$$

On the other hand the necessary first-order conditions for maximizing subject to $K(\mathbf{x}) \leq \overline{K}$, assuming the constraint is binding, are

$$\Delta F(\mathbf{x}) = \mathbf{w} + \lambda \Delta K(\mathbf{x})$$

$$K(\mathbf{x}) = \overline{K}$$

But the constraint is, of course, binding by the strict global concavity of F. Thus

$$\Delta\Delta F(\overline{\mathbf{x}}^\star)\frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}} = \frac{\partial \lambda^\star}{\partial \overline{K}}\Delta K(\overline{\mathbf{x}}^\star) + \lambda\Delta\Delta K(\overline{\mathbf{x}}^\star)\frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}}$$

$$\Delta K(\overline{\mathbf{x}}^\star)^\top \frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}} = 1$$

Note that this holds for the optimum, so long as it is interior as is assumed in the statement of the proposition, regardless of whether there is a unique solution to the first-order conditions. Also when this is evaluated at $\overline{K} = K(\overline{x}^\star)$, it must be that $\lambda = 0$ so this simplifies to

$$\Delta\Delta F(\mathbf{x}^\star) \left.\frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} = \left.\frac{\partial \lambda^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} \Delta K(\mathbf{x}^\star)$$

$$\Delta K(\overline{\mathbf{x}}^\star)^\top \left.\frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} = 1$$

Therefore

$$\left.\frac{\partial \lambda^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} = \frac{1}{\Delta K\left(\mathbf{x}^\star\right)^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\Delta K\left(\mathbf{x}^\star\right)}$$

This implies

$$\left.\frac{\partial \overline{x}_i^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} = \mathbf{1}_i^\top \left.\frac{\partial \overline{\mathbf{x}}^\star}{\partial \overline{K}}\right|_{\overline{K}=K(\mathbf{x}^\star)} = \frac{\mathbf{1}_i^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\Delta K\left(\mathbf{x}^\star\right)}{\Delta K\left(\mathbf{x}^\star\right)^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\Delta K\left(\mathbf{x}^\star\right)} = \frac{\Delta K\left(\mathbf{x}^\star\right)^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\mathbf{1}_i}{\Delta K\left(\mathbf{x}^\star\right)^\top \Delta\Delta F(\mathbf{x}^\star)^{-1}\Delta K\left(\mathbf{x}^\star\right)}$$

as the matrix is symmetric (singleton). Plugging in equation (44) establishes the result.

$\square$

Proposition 4 is a trivial corollary, plugging in the relevant derivatives where needed.

B Proof of Proposition 20

Establishing the mutual independence of all of these conditions is complex and omitted here, but available upon request. I provide two examples here. Consider a case when demand on the two sides of the market is symmetric. Then the GRT2003 formula for the usage cost, multiplied by $\eta_{II}^2 - \eta_{IJ}^2$ simplifies to

$$\eta_{IJ}(1 - 2\hat{R}_{II} - 2\hat{R}_{IJ}) + 2\eta_{II}(\hat{R}_{II} + \hat{R}_{IJ})$$

while in the GArmstrong model it is

$$\eta_{IJ}(1 - 2\hat{R}_{II}) - 2\eta_{II}\hat{R}_{IJ}$$

The first minus the second is equal to

$$-2\eta_{IJ}\hat{R}_{IJ} + 2\eta_{II}(\hat{R}_{II} + 2\hat{R}_{IJ})$$

which may taken on arbitrary values as $\hat{R}_{IJ}, \hat{R}_{II}$ are generally unrestricted. Thus these two conditions are independent, but not mutually exclusive. The independence of the Hybrid model's rescaled usage cost from those of the GArmstrong and GRT2003 models can be seen from the fact that it is not symmetric in $A = B$, as the other two are.

C Proof of Propositions 23 and 24

First I establish Proposition 23 and then I turn to Proposition 24, which is quite a bit simpler.

Proof. I start with the case in which the firms are separated. The first order conditions on side A follow directly from the derivation of the GArmstrong first-order conditions for the two-sided firm and follow directly from the standard monopolist (oligopolist) first-order competitions in the case of the non-two-sided firm. To see the two-sided firm's B-side first-order condition note that the profit function for the two-sided firm is

$$\pi(P^A, P^B, \tilde{P}^A) = (P^A - C^A)D^A + (P^B - C^B)D^B + (b^A + b^B - c)D^A D^B$$

Taking the derivative with respect to P^B yields

$$D^B(P^A - C^A + \alpha D^B)D_3^A(\ddot{b}^A - b^A)D^{B'} + (P^B - C^B)D^{B'} + \alpha D^A D^{B'}$$

equating this to 0, dividing by $D^{B'}$, plugging in the first-order condition on side A for the two-sided firm and rearranging yields the desired condition.

When the firms are merged things are a bit messier. The profit function is now

$$\pi(P^A, P^B, \tilde{P}^A) = (P^A - C^A)D^A + (P^B - C^B)D^B + (b^A + b^B - c)D^A D^B + (\tilde{P}^A - \tilde{C}^A)\tilde{D}^A(\tilde{P}^A)$$

The first-order conditions with respect to P^A, P^B and $\tilde{P}^A$ respectively are, following the same procedure as above,

$$\begin{aligned} P^A - C^A + \alpha D^B + (\tilde{P}^A - C^A)\tilde{\theta}^A &= M^A && (45) \\ P^B - C^B + \alpha D^A + (\ddot{b}^A - b^A)\left([P^A - C^A + \alpha D^B]D_3^A + [\tilde{P}^A - C^A]\tilde{D}_3^A\right) &= M^B && (46) \\ \tilde{P}^A - \tilde{C}^A + (P^A - C^A + \alpha D^B)\theta^A &= \tilde{M}^A && (47) \end{aligned}$$

Substituting (47) into (45), and vice-versa, yield the stated first-order conditions for the integrated case on side A. Substituting these expressions into (46) yields

$$P^B - C^B + \alpha D^A + \frac{\left(\ddot{b}^A - b^A\right)\left(\left[M^A + \tilde{\theta}^A \tilde{M}^A\right] D_3^A + \left[\tilde{M}^A + \theta^A M^A\right] \tilde{D}_3^A\right)}{1 - \theta^A \tilde{\theta}^A} = M^B$$

Re-arranging this yields the condition for the B-side price and completes the proof.

□

I now establish Proposition 24 .

Proof. By my reasoning in the paper, under double marginalization prices are given by

$$P^B - C^B + \ddot{\alpha} D^A = M^B$$

and

$$\ddot{P}^A - \ddot{C}^A + \ddot{\alpha} D^B = 2M^A$$

Under monopoly the condition for side B is identical but the condition for side A becomes

$$\ddot{P}^A - \ddot{C}^A + \ddot{\alpha} D^B = M^A$$

Consider the system governed by the equations

$$P^B - C^B + \ddot{\alpha} D^A = M^B$$

$$\ddot{P}^A - \ddot{C}^A + \ddot{\alpha} D^B = \mu M^A$$

Note that for $\mu = 1$ we have the monopoly solution and for $\mu = 2$ we have the double marginalization solution. Prices on both sides of the market are increasing in μ as the first-order effect of an increase in μ is M^A times that of in increase in M^A, which leads to an increase in both prices by my reasoning in Proposition 6. This holds uniformly regardless of the value of μ provided $\mu < 2$ and the assumptions I made in the text hold ($M^{A'} < \frac{1}{2}$ and $1 > \ddot{R}^A R^B \ddot{\alpha}^2 D^{A'} D^{B'}$) as these are equivalent to the corresponding assumptions in the GArmstrong monopoly model. Thus prices on both sides are higher, and therefore welfare and demand on both sides are lower, under double marginalization than under monopoly.

$\square$

D Proof of Propositions 25 and 27

I first establish Proposition 25, then Proposition 27.

Proof. Consider either (given symmetry) firm's optimal choice of $\tilde{P}^{I,i}$ where $\tilde{P}^{I,i} = P^{I,i} + (b^I - p^I)N^{J,i}$. That is, consider her choice as if she were charging a usage cost equal to the homogeneous usage benefit. Given that each firm takes the others' prices as fixed, this is a valid approach to the optimization for the exact same reason it was in the monopoly case. Let

$$\tilde{N}^{I,i}(\tilde{P}^{I,i}, P^{I,j}, \tilde{P}^{J,i}, P^{J,j}) \equiv$$

$$D^I\left(\tilde{P}^{I,i}, P^{I,j} - \tilde{N}^{J,j}\left[P^{J,j}, \tilde{P}^{J,i}, P^{I,j}, \tilde{P}^{I,i}\right]\left[b^I - p^I\right]\right)$$

$$\tilde{N}^{I,j}(P^{I,j}, \tilde{P}^{I,i}, P^{J,j}, \tilde{P}^{J,i}) \equiv$$

$$D^I\left(P^{I,j} - \tilde{N}^{J,j}\left[P^{J,j}, \tilde{P}^{J,i}, P^{I,j}, \tilde{P}^{I,i}\right]\left[b^I - p^I\right], \tilde{P}^{I,i}\right)$$

The profit function for firm i can then be written as

$$(\tilde{P}^{A,i} - C^A)\tilde{N}^{A,i}(\tilde{P}^{A,i}, P^{A,j}, \tilde{P}^{B,i}, P^{B,j}) + (\tilde{P}^{B,i} - C^B)\tilde{N}^{B,i}(\tilde{P}^{B,i}, P^{B,j}, \tilde{P}^{A,i}, P^{A,j}) +$$

$$\alpha\tilde{N}^{A,i}(\tilde{P}^{A,i}, P^{A,j}, \tilde{P}^{B,i}, P^{B,j})\tilde{N}^{B,i}(\tilde{P}^{B,i}, P^{B,j}, \tilde{P}^{A,i}, P^{A,j})$$

The first-order conditions for optimization on side I are

$$\left(\tilde{P}^{I,i} - C^I + \alpha N^{J,i}\right) N_1^{I,i} + \left(\tilde{P}^{J,i} - C^J + \alpha N^{I,i}\right) N_3^{J,i} + N^{I,i} = 0$$

or

$$\tilde{P}^{I,i} - C^I + \alpha N^{J,i} + \left(\tilde{P}^{J,i} - C^J + \alpha N^{I,i}\right) \frac{N_3^{J,i}}{N_1^{I,i}} = -\frac{N^{I,i}}{N_1^{I,i}} \tag{48}$$

I therefore need to calculate $\tilde{N}_1^{I,i}$ and $\tilde{N}_3^{I,i}$.

$$\tilde{N}_1^{I,i} = D_1^I - D_2^I(b^I - p^I)\tilde{N}_4^{J,j}$$

$$\tilde{N}_3^{I,i} = -D_2^I(b^I - p^I)\tilde{N}_2^{J,j}$$

and

$$\tilde{N}_2^{I,j} = D_2^I - D_1^I(b^I - p^I)\tilde{N}_4^{J,j}$$

$$\tilde{N}_4^{I,j} = -D_1^I(b^I - p^I)\tilde{N}_2^{J,j}$$

therefore

$$\tilde{N}_2^{I,j} = \frac{D_2^I}{1 - D_1^I D^J - 1\left(b^I - p^I\right)\left(b^J - p^J\right)}$$

$$\tilde{N}_4^{I,j} = -\frac{D_1^I D_2^J (b^I - p^I)}{1 - D_1^I D^J \left(b^I - p^I\right)\left(b^J - p^J\right)}$$

and

$$\tilde{N}_1^{I,i} = D_1^I + \frac{D_1^J \left(D_2^I\right)^2 \left(b^J - p^J\right)}{1 - D_1^I D_1^J \left(b^I - p^I\right)\left(b^J - p^J\right)}$$

$$\tilde{N}_3^{I,i} = -\frac{D_2^I D_2^J}{1 - D_1^I D_1^J \left(b^I - p^I\right)\left(b^J - p^J\right)}$$

This allows equation (48) to be rewritten as

$$\tilde{P}^{I,i} - C^I + \alpha N^{J,i} + \gamma^I = -\frac{N^{I,i}}{D_1^I}$$

where

$$\gamma^I = \left(\tilde{P}^{J,i} - C^J + \alpha N^{I,i}\right)\frac{N_3^{J,i}}{N_1^{I,i}} - \frac{N^{I,i}}{D_1^I}\left(1 - \frac{D_1^I}{N_1^{I,i}}\right) =$$

$$\frac{\left(\left[\tilde{P}^{J,i} - C^J + \alpha N^{I,i}\right] N_3^{J,i} - \frac{N^{I,i}\left[N_1^{I,i} - D_1^I\right]}{D_1^I}\right)\left(1 - D_1^I D_1^J \left[b^I - p^I\right]\left[b^J - p^J\right]\right)}{N_1^{I,i}\left(1 - D_1^I D_1^J \left[b^I - p^I\right]\left[b^J - p^J\right]\right)}$$

Some algebra, which I omit here, yields the (somewhat) simplified form for γ^I in the text.

□

I now prove Proposition 27.

Proof. Rewrite condition (37) as

$$P^I - (b^I - p^I)N^J - C^I + \alpha N^J + \gamma^I = M^I + M^I\left(\frac{M_o^I}{M^I} - 1\right)$$

or

$$\tilde{P}^I - C^I + \alpha D^J(\tilde{P}^J) + \gamma^I - M^I\frac{D_2^I}{D_1^I} = M^I(\tilde{P}^I)$$

where $\tilde{P}^I = P^I - (b^I - p^I)N^J$ for both I where $M^I(\tilde{P}^I) \equiv \frac{D^I(\tilde{P}^I)}{D_1^I(\tilde{P}^I)+D_2^I(\tilde{P}^I)}$.

Consider a symmetric-across-firms Nash-Bertrand equilibrium and let the value at that equilibrium of $\gamma^I - M^I\frac{D_2^I}{D_1^I}$ be $G^I > 0$ by the assumptions in the statement. Now consider the monopoly problem where costs are given by

$$C^I - tG^I$$

for $t \in [0,1]$. Clearly for $t = 0$ the solution is the same as the symmetric-across-firms monopoly optimum (which is unique by the assumed second-order conditions). For $t = 1$, prices must be the same as at the symmetric-across-firms Nash-Bertrand equilibrium. Therefore if I can show that prices on both sides of the market decline monotonically in t on both sides of the market, I have established the results. But we know from my analysis of the GArmstrong model in Sections 3 and 4 that falls in costs on either side lower prices on both sides, so this is clearly true as $G^I > 0$ so costs fall on both sides as t rises. This shows that $\tilde{P}^I$ at the Nash-Bertrand equilibrium is lower than P^I is at the monopoly optimum . But $\tilde{P}^I = P^I - (b^I - p^I)N^J = \hat{P}^I - b^I N^J$ in the Nash-Bertrand equilibrium and $P^I = \hat{P}^I - b^I N^J$ in the monopoly problem, establishing the result.

□

E Results and proofs on the general RT2006 model

In the text I state conditions that I assert are necessary and sufficient for there to be there is unique fixed of the two-sided market given any value of either membership or usage prices chosen. This is formalized in the following proposition. While I do not explicitly formulate the interaction between consumers on the two sides of the market in game theoretic terms, this is essentially a necessary and sufficient condition for uniqueness of equilibrium for all value of prices.

Proposition 29. *Suppose that, for all values of* $(n^A, n^B) \in [0, N^A] \times [0, N^B]$ *and for all* $P^A, P^B \in \mathbb{R}^2$, $\sigma^M_{RT2006}(P^A, P^B, n^A, n^B) > 0$. *Then there is a unique solution for* n^A, n^B *to* $n^A = \hat{D}^A(P^A, n^B), n^B = \hat{D}^B(P^B, n^A)$ *for all* $(P^A, P^B) \in \mathbb{R}^2$. *Similarly suppose that, for all values of* $(n^A, n^B) \in [0, N^A] \times [0, N^B]$ *and for all* $(p^A, p^B) \in \mathbb{R}^2$

$$n^{A^2} n^{B^2} > \left(p^A - \bar{b}^A[p^A \hat{n}^B, n^B]\right)\left(p^B - \bar{b}^B[p^B \hat{n}^A, n^A]\right)\hat{\mu}^A(p^A n^B, n^B)\hat{\mu}^B(p^B n^A, n^A)$$

Then there is a unique solution for n^A, n^B *to* $n^A = \hat{d}^A(p^A, n^B), n^B = \hat{d}^B(p^B, n^A)$ *for all* $p^A, p^B \in \mathbb{R}^2$.

Conversely suppose that there $\exists (n^A, n^B) \in [0, N^A] \times [0, N^B]$ *and* $P^A, P^B \in \mathbb{R}^2$ *such that* $\sigma^M_{RT2006}(P^A, P^B, n^A, n^B) < 0$. *Then there exists some* $(P^A, P^B) \in \mathbb{R}^2$ *for which there is more than one solution to* n^A, n^B *to* $n^A = \hat{D}^A(P^A, n^B), n^B = \hat{D}^B(P^B, n^A)$. *Similarly suppose that* $\exists (n^A, n^B) \in [0, N^A] \times [0, N^B]$ *and* $p^A, p^B \in \mathbb{R}^2$ *such that*

$$n^{A^2} n^{B^2} < \left(p^A - \bar{b}^A[p^A \hat{n}^B, n^B]\right)\left(p^B - \bar{b}^B[p^B \hat{n}^A, n^A]\right)\hat{\mu}^A(p^A n^B, n^B)\hat{\mu}^B(p^B n^A, n^A)$$

Then there exists some $(p^A, p^B) \in \mathbb{R}^2$ *for which there is more than one solution to* n^A, n^B *to* $n^A = \hat{d}^A(p^A, n^B), n^B = \hat{d}^B(p^B, n^A)$.

Proof. There may be a small mistake in the quantifiers of the necessity and sufficiency of the second-order conditions used here. This will be fixed in a future draft and the appropriate proof then inserted here. □

I now provide a proof of the Proposition 28.

Proof. Suppose that the firm chooses membership prices. Its profits are then

$$\pi(P^A, P^B) = (P^A - C^A)D^A(P^A, P^B) + (P^B - C^B)D^B(P^B, P^A) - cD^A(P^A, P^B)D^B(P^B, P^A)$$

Her first-order conditions are therefore

$$(P^I - C^I)D_1^I + D^I + (P^J - C^J)D_2^J - c(D_1^I D^J + D^I D_2^J) = 0$$

where D_i^I is the derivative of D^I with respect to its ith argument. This can be rewritten as

$$P^I - C^I - cD^J = -\frac{D^I}{D_1^I} - \left(P^J - C^J - cD^I\right)\frac{D_2^J}{D_1^I}$$

or

$$P^I - C^I - cD^J = \frac{-\frac{D^I}{D_1^I} + \frac{D^J D_2^J}{D_1^J D_1^I}}{1 - \frac{D_2^I D_2^J}{D_1^I D_1^J}} \tag{49}$$

To simplify this we need to obtain an expression for D_i^I. Recall that by definition

$$D^I(P^I, P^J) = N^I \int_{-\infty}^{\infty} \int_{P^I - b^I D^J(P^J, P^I)}^{\infty} f^I(B^I, b^I) dB^I db^I$$

Therefore

$$D_1^I = -N^I \int_{-\infty}^{\infty} \left(1 - b^I D_2^J\right) f^I(P^I - b^I D^J, b^I) db^I = N^I \mu^I (\bar{b}^I D_2^J - 1)$$

$$D_2^I = N^I \mu^I \bar{b}^I D_1^J$$

Solving this system of equations yields

$$D_1^I = -\frac{N^I \mu^I}{\sigma_{RT2006}^M}$$

$$D_2^I = -\frac{N^I N^J \mu^A \mu^B \overline{b}^I}{\sigma^M_{RT2006}}$$

Using this we can rewrite (49) as

$$P^I - C^I - cD^J = \sigma^M_{RT2006} \frac{M^I - \overline{b}^J D^J}{1 - \mu^A \mu^B \overline{b}^A \overline{b}^B} = M^I - \overline{b}^J D^J$$

establishing the membership pricing version of the first-order condition. Dividing through by D^J the second first-order condition. One classic condition for global optimality of the solution to the first-order conditions in two variable optimization problems is that (a sign-preserving transformation of) the first derivative of each optimized variable be globally decreasing in its argument and that the product of the effects of a change in one variable on the optimal value of the other be less than 1. Here we can write the first condition as, for both I and for all values of P^I, P^J

$$\frac{\partial \left(P^I - C^I - \left[\overline{b}^J - c \right] D^J - M^I \right)}{\partial P^I} > 0$$

and

$$\frac{\partial P^{A\star}}{\partial P^B} \frac{\partial P^{B\star}}{\partial P^A} < 1$$

So it suffices to show that

$$\frac{\partial \left(P^I - C^I + \left[\overline{b}^J - c \right] D^J - M^I \right)}{\partial P^I} = \sigma^I_{RT2006}$$

and

$$\chi^I = \frac{\partial P^{I\star}}{\partial P^J}$$

First,

$$\frac{\partial\left(P^I - C^I + \left[\bar{b}^J - c\right] D^J - M^I\right)}{\partial P^I} = 1 + \bar{b}_2^J D^J + \left(\bar{b}^J - c\right) D_2^J - M_1^I$$

Thus we need to calculate $\bar{b}_i^I$ and M_i^I.

$$\bar{b}^I(P^I, P^J) = \frac{N^I \int_{-\infty}^{\infty} f^I\left(P^I - b^I D^J[P^J, P^I], b^I\right) b^I db^I}{\mu^I(P^I, n^J)} \equiv \frac{\beta^I(P^I, P^J)}{\mu^I(P^I, P^J)}$$

So

$$\bar{b}_i^I = \frac{\beta_i^I \mu^I - \mu_i^I \beta^I}{\mu^{I^2}}$$

and

$$M^I = \frac{D^I}{\mu^I}$$

so

$$M_i^I = \frac{D_i^I \mu^I - \mu_i^I D^I}{\mu^{I^2}}$$

Thus the task is really to calculate μ_i^I and β_i^I.

$$\mu^I(P^I, P^J) = N^I \int_{-\infty}^{\infty} f^I\left(P^I - b^I D^J[P^I, P^J], b^I\right) db^I = N^I \int_{-\infty}^{\infty} f_{M|U}^I\left(P^I - b^I D^J[P^I, P^J] | b^I\right) f_U^I(b^I) db^I$$

So

$$\mu_1^I = N^I \int_{-\infty}^{\infty} \left(1 - b^I D_2^J\right) f_{M|U}^{I'}\left(P^I - b^I D^J[P^J, P^I] | b^I\right) f_U^I(b^I) db^I =$$

$$N^I \int_{-\infty}^{\infty} \left(1 - b^I D_2^J\right) \frac{f_{M|U}^{I'}\left(P^I - b^I D^J[P^J, P^I]|b^I\right)}{f_{M|U}^{I}\left(P^I - b^I D^J[P^J, P^I]|b^I\right)} f_{M|U}^{I}\left(P^I - b^I D^J[P^J, P^I]|b^I\right) f_U^I(b^I) db^I =$$

$$N^I \int_{-\infty}^{\infty} \left(1 - b^I D_2^J\right) \frac{f_{M|U}^{I'}\left(P^I - b^I D^J[P^J, P^I]|b^I\right)}{f_{M|U}^{I}\left(P^I - b^I D^J[P^J, P^I]|b^I\right)} f^I\left(P^I - b^I D^J[P^J, P^I], b^I\right) db^I =$$

$$\mu^I \left(\overline{\lambda_{f_{M|U}^I}} - D_2^J \overline{b^I \lambda_{f_{M|U}^I}}\right)$$

By the same reasoning

$$\mu_2^I = -\mu^I D_1^J \overline{b^I \lambda_{f_{M|U}^I}}$$

$$\beta_1^I = \mu^I \left(\overline{b^I \lambda_{f_{M|U}^I}} - D_2^J \overline{b^{I2} \lambda_{f_{M|U}^I}}\right)$$

$$\beta_2^I = -\mu^I D_1^J \overline{b^{I2} \lambda_{f_{M|U}^I}}$$

Therefore

$$M_1^I = M^I \left(\overline{\lambda_{f_{M|U}^I}} + \frac{\mu^A \mu^B \overline{b}^J \overline{b^I \lambda_{f_{M|U}^I}}}{\sigma_{RT2006}^M}\right) - \frac{1}{\sigma_{RT2006}^M} = \frac{D^I \left(\mu^A \mu^B \overline{b}^J \left[\overline{b}^I \overline{\lambda_{f_{M|U}^I}} - \overline{b^I \lambda_{f_{M|U}^I}}\right] + \overline{\lambda_{f_{M|U}^I}}\right) - \mu^I}{\mu^I \sigma_{RT2006}^M}$$

$$M_2^I = -\frac{\mu^J \overline{b}^I + M^I \mu^J \overline{b^I \lambda_{f_{M|U}}}}{\sigma_{RT2006}^M} = -\frac{\mu^J}{\sigma_{RT2006}^M} \left(\overline{b}^I + M^I \overline{b^I \lambda_{f_{M|U}}}\right)$$

$$\overline{b}_1^I = \left(\overline{b^I \lambda_{f_{M|U}^I}} - D_2^J \overline{b^{I2} \lambda_{f_{M|U}^I}}\right) - \left(\overline{\lambda_{f_{M|U}^I}} - D_2^J \overline{b^I \lambda_{f_{M|U}^I}}\right) \overline{b}^I =$$

$$\left(\overline{b^I\lambda_{f^I_{M|U}}}+\frac{\mu^A\mu^B\bar{b}^I\overline{b^{I^2}\lambda_{f^I_{M|U}}}}{\sigma^M_{RT2006}}\right)-\left(\overline{\lambda_{f^I_{M|U}}}+\frac{\mu^A\mu^B\bar{b}^I\overline{b^I\lambda_{f^I_{M|U}}}}{\sigma^M_{RT2006}}\right)\bar{b}^I=$$

$$\frac{\overline{b^I\lambda_{f^I_{M|U}}}-\bar{b}^I\overline{\lambda_{f^I_{M|U}}}+\mu^A\mu^B\bar{b}^J\left(\overline{b^{I^2}\lambda_{f^I_{M|U}}}-\bar{b}^I\overline{b^I\lambda_{f^I_{M|U}}}+\bar{b}^I\overline{\lambda_{f^I_{M|U}}}-\overline{b^I\lambda_{f^I_{M|U}}}\right)}{\sigma^M_{RT2006}}$$

$$\bar{b}^I_2=\frac{\mu^J\overline{b^{I^2}\lambda_{f^I_{M|U}}}-\mu^J\bar{b}^I\overline{b^I\lambda_{f^I_{M|U}}}}{\sigma^M_{RT2006}}=\frac{\mu^J\left(\overline{b^{I^2}\lambda_{f^I_{M|U}}}-\bar{b}^I\overline{b^I\lambda_{f^I_{M|U}}}\right)}{\sigma^M_{RT2006}}$$

so

$$1-\bar{b}^J_2D^J+\left(\bar{b}^J-c\right)D^J_2-M^I_1=$$

$$1-\frac{M^I\left(\mu^A\mu^B\bar{b}^J\left[\bar{b}^I\overline{\lambda_{f^I_{M|U}}}-\overline{b^I\lambda_{f^I_{M|U}}}\right]+\overline{\lambda_{f^I_{M|U}}}\right)-1}{\sigma^M_{RT2006}}+\frac{\mu^ID^J\left(\overline{b^{J^2}\lambda_{f^J_{M|U}}}-\bar{b}^J\overline{b^J\lambda_{f^J_{M|U}}}\right)}{\sigma^M_{RT2006}}-\frac{\mu^A\mu^B\bar{b}^J\left(\bar{b}^J-c\right)}{\sigma^M_{RT2006}}$$

This is clearly the same as the expression in equation (41). Now I want to show that $\chi^I=\frac{\partial P^{I^\star}}{\partial P^J}$.

$$\frac{\partial P^{I^\star}}{\partial P^J}=\frac{M^I_2-\bar{b}^J_1D^J-\left[\bar{b}^J-c\right]D^J_1}{\sigma^I_{RT2006}}=$$

$$\frac{\mu^J\left(\bar{b}^J-\bar{b}^I-c-M^I\overline{b^I\lambda_{f_{M|U}}}\right)-\left(\overline{b^J\lambda_{f^J_{M|U}}}-\bar{b}^J\overline{\lambda_{f^J_{M|U}}}+\mu^A\mu^B\bar{b}^I\left[\overline{b^{J^2}\lambda_{f^J_{M|U}}}-\bar{b}^J\overline{b^J\lambda_{f^J_{M|U}}}+\bar{b}^J\overline{\lambda_{f^J_{M|U}}}-\overline{b^J\lambda_{f^J_{M|U}}}\right]\right)D^J}{\sigma^I_{RT2006}\sigma^M_{RT2006}}$$

$\square$