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CHOICE-FREE REPRESENTATION OF ORTHOLATTICES

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Abstract. We continue the study of the topological duality for ortholattices started by Goldblatt (1975), comparing it with the choice-free duality sketched in Bezhanishvili and Holliday (2019), which is examined here in detail. In both cases, we characterize the duals of ortholattices and extend the duality categorically with the suitable morphisms. Afterwards, we identify a separate, first-order definable way in which the original ortholattice may be obtained from its choice-free dual. An application of this is a nontrivial characterization of the duals of *orthomodular* lattices.

1. Duals of Ortholattices

Definition 1.1.

(i) A relational structure $(X, \perp)$ with a irreflexive symmetric relation $\perp$, is called an *orthoframe*. The relation $\perp$ is the *orthogonality relation* of the orthoframe.

(ii) Let L be an ortholattice.

(a) The space $X_L^{\pm} = (X_L^{\pm}, \perp)$ of proper filters of L has the topology generated by the sets of the form $\widehat{a} = \{x \in X_L^{\pm} \mid a \in x\}$ and their complements for $a \in L$ with the binary relation defined by

$$x \perp y \iff (\exists a \in L)[a \in x \,\&\, a^{\perp} \in y].$$

This appeared in Goldblatt [3].

(b) The space $X_L^{+} = (X_L^{+}, \perp)$ consists of the same points and the same binary relation, but its topology is generated by sets of the form $\widehat{a}$ only. This was briefly discussed in Holliday and N. Bezhanishvili [1].

Proposition 1.2. Every ortholattice is isomorphic L to $\mathrm{COR}(X_L^+)$, where $\mathrm{COR}(X)$ is the ortholattice of compact open $\perp$-regular subsets of X.

Proof. We show that the image of the ortholattice embedding $\widehat{\cdot}$ from L to the ortholattice of $\perp$-regular subsets of X_L^+ is $\mathrm{COR}(X_L^+)$. Suppose that $A \in \mathrm{COR}(X_L^+)$. Since it is open, it is a union of basic open sets, which are of the form $\widehat{a}$. Since it is compact, it is a finite union of basic open sets: $A = \widehat{a_1} \cup \cdots \cup \widehat{a_n}$ for $a_1, \dots, a_n \in L$. Since it is $\perp$-regular, we have

$$(a_1 \vee \cdots \vee a_n)\widehat{} = (\widehat{a_1} \cup \cdots \cup \widehat{a_n})^{\perp\perp} = A^{\perp\perp} = A,$$

so A is in the image of $\widehat{\cdot}$. □

The following is an analogue of the characterization of the duals of Boolean algebras studied in Bezhanishvili and Holliday [1, § 5].

Proposition 1.3. An orthoframe $(X, \perp)$ with topology is homeomorphic[1] to X_L^+ for some ortholattice L iff all of the following conditions are met:

[1] A homeomorphism between two such structures is a homeomorphism between the two topological spaces that is an isomorphism between their orthoframe reducts.

(1) X is T_0.
(2) $\mathrm{COR}(X)$ is closed under $\cap$ and ${}^\perp$.
(3) $\mathrm{COR}(X)$ is a basis of X.
(4) Every proper filter of $\mathrm{COR}(X)$ is of the form $\mathrm{COR}^X(x)$ for some $x \in X$.
(5) If $x \perp y$, then there is $U \in \mathrm{COR}(X)$ such that $x \in U$ and $y \in U^\perp$.

Here, $\mathrm{COR}^X(x) = \{U \in \mathrm{COR}(X) \mid x \in U\}$.

Proof.
"Only if" direction. Note that $x \leqslant y$ if and only if $x \subseteq y$. Condition (3) is the definition of the topology of X_L^+. We show Condition (1). Suppose that $x \nleqslant y$, i.e., $x \nsubseteq y$. Take then $a \in x \setminus y$. Note that $\widehat{a}$ is $\perp$-regular and open. It can also be shown that $\widehat{a}$ is compact. Indeed, Let $(\widehat{b_i})_{i\in I}$ be a cover of $\widehat{a}$ by basic open sets: $\widehat{a} \subseteq \bigcup_i \widehat{b_i}$. Since the principal filter $\uparrow a$ generated by a is in $\widehat{a}$, so is it in $\bigcup_i \widehat{b_i}$, i.e., for some $i \in I$ we have $\uparrow a \in \widehat{b_i}$. This means that $a \le b_i$ and that $\widehat{a} \subseteq \widehat{b_i}$, the unary union $\widehat{b_i}$ being a finite subcover of $\widehat{a}$. Therefore, $\widehat{a} \in \mathrm{COR}(X_L^+)$, $x \in \widehat{a}$, and $y \notin \widehat{a}$. Condition (2) follows from $\mathrm{COR}(X_L^+)$ being isomorphic to L. For (4), let G be a proper filter of $\mathrm{COR}(X_L^+)$. Let x be the image of G under the isomorphism $\mathrm{COR}(X_L^+) \to L$. Then $x \in X_L^+$. It is easy to see that $G = \mathrm{COR}^{X_L^+}(x)$. Finally, Condition (5) is the definition of $\perp$.

"If" direction. We will show that if the five conditions are met, then there is a homeomorphism $\epsilon : (X, \perp) \to X^+_{\mathrm{COR}(X)}$ given by $x \mapsto \mathrm{COR}^X(x)$. That ϵ is surjective follows from Condition (4). To see ϵ is injective, let $x \neq y$ be in X. By X being T_0, either $x \nleqslant y$ or $y \nleqslant x$. Assume the former (the other case can be addressed in the same manner). By (3), take $U \in \mathrm{COR}(X)$ with $x \in U$ and $y \notin U$. We have $U \in \mathrm{COR}^X(x)$ and $U \notin \mathrm{COR}^X(y)$, and we have established the injectivity of ϵ. For the continuity of ϵ, we show that the inverse image of every basic open set in $X^+_{\mathrm{COR}(X)}$ is open in X. An arbitrary basic open set in $X^+_{\mathrm{COR}(X)}$ is of the form $\widehat{U}$ for $U \in \mathrm{COR}(X)$. We have:

$$\begin{aligned}\epsilon^{-1}(\widehat{U}) &= \{x \in X \mid \mathrm{COR}^X(x) \in \widehat{U}\}\\ &= \{x \in X \mid U \in \mathrm{COR}^X(x)\}\\ &= \{x \in X \mid x \in U\}\\ &= U,\end{aligned}$$

which is open. Now, since $\mathrm{COR}(X)$ is a basis of U, we conclude that ϵ is a homeomorphism. Finally, we show that ϵ is an isomorphism between the orthoframe reduct. We have:

$$\begin{aligned}\mathrm{COR}^X(x) \perp \mathrm{COR}^X(y) &\iff (\exists U \in \mathrm{COR}(X))\, U \in \mathrm{COR}^X(x) \;\&\; U \in \mathrm{COR}^X(y)\\ &\iff (\exists U \in \mathrm{COR}(X))\, x \in U \;\&\; y \in U^\perp\\ &\iff x \perp y,\end{aligned}$$

where the last $\Leftarrow$ is Condition (5), and the $\Rightarrow$ follows from the definition of $(\cdot)^\perp$. $\square$

Proposition 1.4 (AC)**.** A orthoframe $(X, \perp)$ with topology is homeomorphic to $X_L^\pm$ for some ortholattice L iff all of the following conditions are met:

(1) X is T_0 and compact.
(2) $\mathrm{ClopR}(X)$ is closed under $\cap$ and ${}^\perp$.

(3) If $x \neq y$, then there is $U \in \mathrm{ClopR}(X)$ such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.
(4) Every proper filter of $\mathrm{ClopR}(X)$ is of the form $\mathrm{ClopR}(x)$ for some $x \in X$.
(5) If $x \perp y$, then there is $U \in \mathrm{ClopR}(X)$ such that $x \in U$ and $y \in U^{\perp}$.

Here, $\mathrm{ClopR}(X)$ is the lattice of clopen $\perp$-regular subsets of X, and $\mathrm{ClopR}(x) = \{U \in \mathrm{ClopR}(X) \mid x \in U\}$.

Proof.
"Only if" direction. Goldblatt [3] showed that $X_L^{\pm}$ is compact and that $\mathrm{ClopR}(X_L^{\pm}) \cong L$. Since the topology of $X_L^{\pm}$ is finer than that of X_L^{+}, the space $X_L^{\pm}$ is T_0 as well. "If" direction. As before, we will show that $\epsilon : (X, \perp) \to X^{\pm}_{\mathrm{ClopR}(X)}$ given by $x \mapsto \mathrm{ClopR}^X(x)$ is a homeomorphism. The injectivity of ϵ follows from (3), and its surjectivity follows from (4) as before. The continuity of ϵ can be proved in the same manner. Note that $X_L^{\pm}$ is Haudorff. Since ϵ is a continuous map from a compact space to a Hausdorff space, it is a homeomorphism. Finally, it can be shown from (5) that ϵ is an isomorphism between the orthoframe reducts. □

Corollary 1.5 (AC)**.** (1) The preceding proposition obtains if the conditions (1), (2), and (3) are replaced by the following respectively:
(1′) X is compact and Hausdorff.
(2′) $\mathrm{ClopR}(X)$ is closed under $^{\perp}$.
(3′) If $x \not\leq y$, then there is $U \in \mathrm{ClopR}(X)$ such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.
In particular, the space $X_L^{\pm}$ for an ortholattice L is a Stone space, i.e., a compact Hausdorff zero-dimensional space.
(2) The space $(X_L^{\pm}, \not\perp)$ with the complement of $\perp$ is a modal space, i.e., the dual of some BAO (B-algebra).
(3) The clopens $B(L)$ of $X_L^{\pm}$, i.e., the Boolean algebra dual to the space, is generated by $\mathrm{COR}(X_L^{\pm})$.

Proof. (1) Easy.
(2) $(\not\perp)[y] = \{x \mid \forall a \in y\, a' \notin x\} = \bigcap_{a \in L} \complement\phi(a')$, where $\complement$ denotes set-theoretic complement. Note that $\complement\phi(a')$ is clopen.
(3) A clopen U of $X_L^{\pm}$ is compact, and hence a finite union of basic opens, each of which is of intersections of sets of the form either $\phi(a)$ or $\complement\phi(a)$. Note that $\complement$ and $\cup$ are the Boolean complement and join of the said Boolean algebra.

□

2. Morphisms

Definition 2.1. For orthoframes $X = (X, \perp)$ and $X' = (X', \perp')$ where X and X' are also topological spaces, a *p-morphic spectral map* $f : X \to X'$ is a spectral map $X \to X'$ that is also a p-morphism $(X, \not\perp) \to (X', \not\perp')$ between these Kripke frames.

Note that a p-morphic spectral map need not be p-morphic with respect to the specialization order of the topological spaces. This makes an interesting contrast to the situation in [1] where that condition is required to obtain the dual equivalence result, whereas we do not need it here:

Proposition 2.2. The category **UVO** whose objects are of the form X_L^+ for an ortholattice L and whose morphisms are p-morphic spectral maps is dually equivalent to the category of ortholattices and homomorophisms.

Proof. Suppose that $f : (X, \perp) \to (X', \perp')$ is a p-morphic spectral map. Given $U \in \mathrm{COR}(X')$, let $f^+(U)$ be the inverse f-image of U. Since f is a spectral map, $f^+(U)$ is compact open. We see that $f^+(U)$ is $\perp$-regular as well. Indeed, since U is $\perp'$-regular, we have $\Box'\Diamond' U = U^{\perp'\perp'} = U$, where $\Box'$ and $\Diamond'$ are the modal operators corresponding to $\not\perp'$ [2]. Since f is p-morphic with respect to $\not\perp'$, the map f^+ preserves modal operators. Thus we have $f^+(U)^{\perp\perp} = \Box\Diamond f^+(U) = f^+(\Box'\Diamond' U) = f^+(U)$, where $\Box$ and $\Diamond$ are defined from $\not\perp$ likewise. We now have a map $f^+ : \mathrm{COR}(X') \to \mathrm{COR}(X)$. It is easy to see that $\mathrm{COR}(\cdot)$ and $(\cdot)^+$ combined give rise to a functor from **UVO** to the category of ortholattices.

Secondly, suppose that $h : L \to L'$ is an ortholattice homomorphism. Given $u' \in X_{L'}^+$, let $h_+(u) \subseteq L$ be the inverse h-image of $u' \subseteq L'$. It is easy to see that $h_+(u)$ is a proper filter, so $h_+(u) \in X_L^+$. We now have a function $h_+ : X_{L'}^+ \to X_L^+$. We show that h_+ is a p-morphic spectral map. For each $U' \in \mathrm{COR}(X_L^+)$, the inverse h_+-image of U' is compact open; indeed, a routine calculation shows that the inverse h_+-image of $\widehat{a}$ is $\widehat{h(a)}$, which is compact open in $X_{L'}^+$. By Lemma 6.6[2] of [1], the map h_+ is a spectral map. It remains to show that h_+ is p-morphic with respect to the complements of the orthogonality relations. Suppose that $u' \not\perp' v'$ in $X_{L'}^+$, where $\perp'$ is the orthogonality relation of $X_{L'}^+$, and that $h_+(u') \perp h_+(v')$ by way of contradiction. Then there is $a \in L$ such that $a \in h_+(u')$ and $a^\perp \in h_+(v')$. By definition, we have $h(a) \in u'$ and $h(a)^{\perp'} = h(a^\perp) \in v'$, where $\perp'$ is the orthocomplement operation of L'. This is a contradiction. Suppose next that $h_+(u') \not\perp v$ for some $u' \in X_{L'}^+$ and $v \in X_L^+$. Let v' be the filter generated by the h-image of v. It is easy to see that v' is proper and thus in $X_{L'}^+$ and that $h_+(v') = v$. Suppose by way of contradiction that $u' \perp v'$, i.e., there is $a' \in L'$ such that $a' \in v'$ and $a'^{\perp'} \in u'$. By definition, there is $a \in L$ with $h(a) \leq a'$ and $a \in v$. We now have $a'^{\perp'} \leq h(a)^{\perp'} = h(a^\perp) \in u'$. This means that $a^\perp \in h_+(u')$, which contradicts $h_+(u') \not\perp v$. It easy to see that $X_\cdot^+$ and $(\cdot)_+$ combined give rise to a functor from the category of ortholattices to **UVO**.

Finally, it is not hard to show that the two functors consist a dual equivalence of the categories. ∎

Proposition 2.3 (AC)**.** The category of modal spaces of the form $(X_L^\pm, \not\perp)$ for an ortholattice L is dually equivalent to the category of ortholattices.

Proof. This can easily proved via the duality for the category of modal B-algebras that are generated by the fixed points of the composite $\Box\Diamond$ of their modal operators. ∎

3. Logical Considerations

Objects in **UVO** can be regarded as a two-sorted first-order structure $\mathfrak{X} = (X, \mathcal{B}, \perp, \in)$ given a basis $\mathcal{B}$ of X_L^+ in the following manner:

- The first sort of $\mathfrak{X}$ consists of the points of X_L^+.
- The second sort of $\mathfrak{X}$ is $\mathcal{B}$.

[2]The proof of the lemma does not use the fact that the spaces are spectral.

- The binary relation symbol $\perp$ between elements of the first sort is interpreted as the orthogonality relation of X_L^+.
- The binary relation symbol $\in$ between elements of the first sort and the second sort, respectively, is interpreted as the membership relation.

Let $\mathcal{L}$ be the first-order language for such structures. We use lowercase and uppercase variables for the first and the second sort of $\mathcal{L}$, respectively.

We say that an $\mathcal{L}$-formula is *invariant* if for every quantification $\exists U$, atomic formulas of the form $x \in U$ occurs only negatively. Note that $(X, \mathcal{B}, \perp, \in) \models \phi$ if and only if $(X, \mathcal{B}', \perp, \in) \models \phi$ for bases $\mathcal{B}, \mathcal{B}'$ of the same topology on X and an invariant sentence ϕ. Invariant $\mathcal{L}$-formulas are essentially formulas of Ziegler's logic $(\mathcal{L}')^t_{\omega\omega}$ for $\mathcal{L}' = \{\perp\}$ [5]. The idea is that invariant sentences depend only on the intrinsic topological information of objects of **UVO**, not how it is presented.

A *bottomless ortholattice* is a first-order structure of the form L^- in the language of two binary relation symbols, where $L^- = (L \setminus \{0\}, \leq, G^-_\perp)$, the relation $\leq$ is the partial order induced by that of L, and $G^-_\perp$ is the intersection of $(L^-)^2$ and the graph of the orthocomplement of L.

Proposition 3.1. There is an interpretation Γ (in the sense of Hodges [4, 5.4 (b)]) of the class of bottomless ortholattices in **UVO** with the following properties:

(1) The interpretation Γ consists of invariant formulas. Consequently, for two bases $\mathcal{B}, \mathcal{B}'$ of $X \in$ **UVO**, we have $\Gamma((X, \mathcal{B}, \perp, \in)) = \Gamma((X, \mathcal{B}', \perp, \in))$. (We write $\Gamma(X)$ for that ortholattice.)
(2) For $X \in$ **UVO**, each element of the carrier set of $\Gamma(X)$ is a point of X, i.e., an element of the first sort of X as an $\mathcal{L}$-structure.
(3) $\Gamma(X_L^+) \cong L^-$ for every ortholattice L.

Furthermore, for every first-order sentence ϕ in the language of ortholattices, there exists an invariant $\mathcal{L}$-sentence ϕ^* such that for every ortholattice L we have $L \models \phi$ if and only if $X_L^+ \models \phi^*$.

Proof. Now the specialization order $\sqsubseteq$ of an object X in **UVO** is uniformly definable by an invariant $\mathcal{L}$-sentence:

$$x \sqsubseteq y \iff X \models \neg \exists U \ni x [y \notin U].$$

Furthermore, the set of principal filters of X_L^+ is defined by the invariant formula $\phi(x) := \exists U \ni x \forall y \sqsubset x [y \notin U]$, where $\sqsubset = \sqsubseteq \setminus =$. Indeed, it is easy to see that $X_L^+ \models \phi(\uparrow a)$ for every $a \in L$. On the other hand, assume that $u \in X_L^+$ is not principal and that $X_L^+ \models \phi(u)$. Let $U \subseteq X_L^+$ be an open set witnessing $\phi(u)$. Since $\{\widehat{a} \mid a \in L\}$ is a basis for X_L^+, there exists $S \subseteq L$ such that $U = \bigcup_{a \in S} \widehat{a}$. Since U is a neighborhood around u, there exists $b \in S$ such that $u \in \widehat{b}$, i.e., $b \in u$. Let $y = \uparrow b$. Since $y \sqsubset u$, and u is not principal, $y \sqsubset u$. However, we have $y \in \widehat{b} \subseteq U$, which contradicts $\phi(u)$.

Let $\Gamma(X_L^+) = (M, \leq^M, G^-_\perp)$ be defined as follows: $M := \phi(X_L^+)$, i.e., the set of principal filters in X_L^+; the order $\leq^M$ is induced by the order-theoretic dual of the specialization order $\sqsubseteq$ in X_L^+, which can be defined by an invariant formula; and $(x, y) \in G^-_\perp$ if and only if $x \perp y$ and $\forall y' \sqsubset y\, x \not\perp y'$.

It is clear that the interpretation given above satisfies the promised properties. The last claim follows from the fact that L and L^- are bi-interpretable for every ortholattice L. $\square$

This proposition may be used to obtain the characterizations of the duals of many important classes of ortholattices. For instance, let ϕ state orthomodularity in the language of ortholattices. Then, the invariant $\mathcal{L}$-sentence ϕ^* obtained as in the proposition defines the class of the duals of orthomodular lattices in **UVO**.

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