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The Limits Of Logic

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The dissertation of Tomoya Sato is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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DEDICATION

To Professor Kazuhisa Shinozawa

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I would like to express my deepest gratitude to my two supervisors.

Our lives are changed by the people we meet. If we did not meet people we meet in this world, we would live in another possible world, which might be completely different from the actual one. If I had not met Professor Kazuhisa Shinozawa, and if he had not accepted me as his student, I would not have known the delight of philosophy and would have not been able to come to the United States as a graduate student. He is the person whom I want to show the progress I made through my doctoral study. I dedicate this dissertation to him.

When I came to San Diego seven years ago, I had some knowledge of logic and thought I knew what it was. But I now think that I did not. Logic is more complicated than I understood in some aspects and is simpler in others. A solid notion of logic is necessary for the philosophical research of logic, and no new important result can be produced without it. I have developed my own notion under Professor Gila Sher's supervision. Without her insight, advice, encouragement, and patience, this research project could never have been completed. Her dissertation is titled *The Bounds of Logic* and mine is tiled *The Limits of Logic*; the similarity is no accident.

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ABSTRACT OF THE DISSERTATION

The Limits Of Logic

by

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Doctor of Philosophy in Philosophy

University of California, San Diego, 2016

Professor Gila Sher, Chair

Logical validity is relative to logical systems. Some arguments are logically valid in one logic but logically invalid in another logic. There are various logical systems, each of which has been developed based on some notion of what logic is or should be. Whether an argument is logically valid or not depends on one's notion of logic. The main purpose of my dissertation is to establish a new notion of logic and propose a new characterization of logical validity. The new notion, which I call the *minimal notion*, is that logical validity is the validity grounded in a special kind of formal law. Using such special formal laws, I identify the logical systems that validate all and only arguments whose validity can be justified from the minimalist's point of view.

1 Introduction

1.1 The Main Problem

The Problem

Logic plays a special role in our inferential activity in two respects. Logic, on the one hand, provides us with the most certain forms of inferences. The argument

$$\begin{array}{rl} & \text{All men are mortal.} \\ & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

is logically valid; the conclusion follows from the premises by virtue of their logical forms. Provided that all men are mortal and that Socrates is a man, we can infer his mortality with certainty. Inferences based on logically valid arguments are absolutely solid.[1] On the other hand, logic rejects particular inferences as impossible. We cannot infer that Socrates is *not* mortal from the same premises,

[1] What I mean here is that if an argument is *actually* logically valid, then we can infer its conclusion from its premises with certainty. There are arguments that are of the same form as those of logically valid arguments but in fact not valid. Consider, for example, the following argument of the form of modus ponens (McGee[69], p. 462):

> Opinion polls taken just before the 1980 election showed the Republican Ronald Reagan decisively ahead of the Democrat Jimmy Carter, with the other Republican in the race, John Anderson, a distant third. Those apprised of the poll results believed, with good reason:
>
> > If a Republican wins the election, then if it's not Reagan who wins it will be Anderson.
> > A Republican will win the election.
>
> Yet they did not have reason to believe
>
> > If it's not Reagan who wins it will be Anderson.

because such inference is inconsistent with the logical validity of the argument. Inferences violating logical validity do not make sense.

Although other types of validity similarly provide other types of certainty and impossibility, those of logical validity differ from them in kind and degree. Logical validity is the validity that holds by virtue of logical forms of arguments, while other types of validity are not. Consider the following argument:

$$\begin{array}{rl} & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

This argument is valid in that it is impossible from a biological point of view that the premise is true and the conclusion is false. Its validity is based on the biological principle that all men are mortal. The argument may be described as "biologically valid." However, it cannot be regarded as logically valid, because the validity is not by virtue of its logical form. There are many invalid arguments whose logical form is the same as that of the argument above. An example of such invalid arguments is

$$\begin{array}{rl} & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is quadrupedal.} \end{array}$$

Only arguments whose validity can be determined by their logical forms deserve the label "logical," and in this respect, i.e., with respect to their kind, logically valid arguments can be distinguished from other types of valid arguments.

Logical validity can also be distinguished from other validity in the degrees of certainty and impossibility. Consider again the biologically valid argument above (the man-mortal argument). The argument has some sort of certainty: given the knowledge of biology, it is certain that Socrates will die some day in the future.

The belief they would have is that if it's not Reagan who wins, it will be Carter. Therefore, the argument is not valid.

What this example shows is not that modus ponens is not logically valid. Notice that the conditional used in the first premise as the main connective is not the material conditional. In fact, if it was the material conditional, the first premise would be false; its antecedent "a Republican wins the election" is true but its consequent "if it's not Reagan who wins it will be Anderson" is false. What can be concluded from this example is that there is a conditional, which is different from the one of classical logic, such that some argument containing it of the form of modus ponens is not valid. In order to claim that the argument above is a counterexample to modus ponens, one has to show that the conditional is actually a logical constant. A similar example, which is intended to be a counterexample to modus tollens, can be found in Yalcin[136].

On the other hand, it is biologically impossible that he will live forever. Deriving his immortality from his manhood is nonsense in most scientific contexts today. Biological validity, like logical validity, provides its own certainty and its own impossibility.

Nonetheless, logical certainty is more certain than biological certainty and logical impossibility is more impossible than biological impossibility. This is because logical validity is the validity based solely on logical forms of arguments. If an argument about biological statements is logically valid, then it is also biologically valid. But the opposite direction is not true. No matter how strong the inferential relationship between the premises and the conclusion of a biologically valid augment is, there is some biologically invalid argument whose logical form is identical to that of the argument. Therefore, the biologically valid argument is not logically valid. Likewise, if an argument about biological statements is logically impossible to hold, then it is also biologically impossible to hold. But, again, the opposite direction is not true. No matter how weak the inferential relationship between the premises and the conclusion of a biologically impossible argument is, there is some biologically valid argument that has the same logical form as the biologically impossible argument.

With respect to the certainty and impossibility of arguments, thus, the two extremes are those of logical validity (I suppose that similar arguments can be given for the validity of physics, the validity of metaphysics, and so on). One extreme is represented by arguments such that their conclusions logically follow from their premises, and the other is represented by arguments such that the negations of their conclusions logically follow from their premises. Every other argument is located between these two kinds of arguments with respect to the degrees of certainty and impossibility. The man-mortal argument is less certain than any logically valid arguments and less impossible than any logically impossible arguments. And the same can be said for any other arguments. We cannot reject logically valid arguments and also cannot accept logically impossible arguments. The limits of our inferential activity thus are drawn by logic. Any inference is possible only within these limits, and anything outside of them is meaningless.

The main problem of this dissertation is to identify where the limits are drawn. The limits are delineated by the collection of logically valid arguments. Thus, the problem can be expressed as a simple question: What arguments are logically valid? This problem of logical validity is one of the most important problems in the philosophy of logic. Logicians and philosophers have approached the problem from various aspects, and consequently, several characterizations of logical validity have been proposed in the literature. In this dissertation, I propose another characterization. Based on the new characterization, I identify logically valid arguments that define the bounds of every possible meaningful argument.

Diversity of Notions of Logical Validity

Whether or not a given argument is logically valid depends on one's notion of what logic is and what it is for. For an argument, the simple question "Does the conclusion follow logically from the premises?" does not make sense unless some notion of logical validity is set up in advance. An argument can be logically valid under some notion of logical validity, but the same argument can be logically invalid under another notion. For the main purpose of the dissertation, i.e., a characterization of logically valid arguments, therefore, a solid notion of logical validity needs to be defined.

There are a variety of logical systems.[2] Classical logic (the standard first-order logic) is just one among many, and there are various non-classical logics such as modal logic, intuitionistic logic, and relevant logic. Logical systems are diverse. One thing that the diversity of logical systems means is the diversity of notions of logical validity: there are various notions of logical validity. Most of them share the fundamental principle that logical validity is the validity that holds by virtue of logical forms. But, they differ in other aspects. Here, we see two examples.

For a mathematical sentence φ_0 (for instance, Fermat's Last Theorem), consider the following argument:

[2]Throughout the dissertation, I will use the term "logical system" to refer to a logic as a formal system of formal arguments in a formal language. To refer to a particular logical system, I will use its name such as "second-order logic," "modal logic," and "intuitionistic logic."

$$\frac{\neg\neg\varphi_0.}{\therefore \quad \varphi_0.}$$

Here, "$\neg$" is the negation symbol and "$\neg\neg\varphi_0$" means that it is not the case that φ_0 does not hold. This argument is valid in classical logic, because, in classical logic, for any sentence, we can always assign the truth value True ("T") to either the sentence or its negation. If the sentence $\neg\neg\varphi_0$ is true, then $\neg\varphi_0$ is false, and therefore, φ_0 is true. The argument thus holds.

The same argument, however, is not valid in intuitionistic logic. According to intuitionism—the philosophical view based on which intuitionistic logic has been developed—a mathematical sentence can be assigned T only when we have a constructive proof of it. For a mathematical statement is to be confirmed by a mental activity in our mind, not by a mathematical fact or principle that is supposed to objectively exist in the reality. Thus, at a time t when we do not have a constructive proof of φ_0 or a constructive proof of $\neg\varphi_0$, neither of these sentences is true. However, once we obtain a constructive proof of φ_0 at a later time $t + t'$, φ_0 will become true, and the sentence $\neg\neg\varphi_0$ will become true at t retroactively. At t, thus, $\neg\neg\varphi_0$ is true but φ_0 is not. Hence, the argument is not valid.

The notion of logical validity behind intuitionistic logic consists of several related notions. The central one is the anti-realistic notion of the truth of mathematical statements that a mathematical sentence can be regarded as true only when a constructive proof of it is available. In addition to this, the notion of the intuitionistic logical validity presupposes that (possible) worlds are *connected*. In the example above, the world at t and the world $t+t'$ are supposed to be connected (the world at t is *accessible to* the world at $t + t'$), and truth values of sentences in one world and those in other worlds connected to it are affected by each other. The logical validity of intuitionistic logic is based on this notion of truth and this notion of connections of worlds. On the other hand, in the notion of logical validity of classical logic, the existence of constructive proofs and the connection between worlds do not play any role. As mentioned, any sentence and its negation always have opposite truth values. With respect to truth values of sentences, each domain is independent of each other.

Another example that shows the multiplicity of notions of logical validity is arguments involving irrelevancy: arguments containing conditional sentences whose antecedents and consequents are irrelevant, and arguments whose premises and conclusions are irrelevant. Consider the conditional sentence "If Socrates is mortal and is not mortal, then San Diego is in California." Socrates and his mortality have nothing to do with San Diego's being in California. The antecedent and the consequent are totally irrelevant. Nonetheless, the sentence is a logical truth in classical logic, because the antecedent is always false independently of how the world is or could be. In relevant logic, however, counterexamples can be made to the sentence.[3]

An example of arguments whose premises and conclusions are irrelevant, which is valid in classical logic, is:

$$\begin{array}{ll} & \text{Barack Obama is the President of the United States in 2015.} \\ \hline \therefore & \varphi_{ss}. \end{array}$$

where φ_{ss} is the conditional sentence above, "If Socrates is mortal and is not mortal, then San Diego is in California." Since the conclusion φ_{ss} is a logical truth in classical logic, the argument is valid, no matter how weak the relationship between the premise and the consequence is. However, a counterexample to this argument can be made in relevant logic.[4]

In everyday life including academic life, we do not assert any conditional statement whose antecedent is irrelevant to its consequence. Nor we make an argument whose premises are irrelevant to its conclusion. After all, any sentence does

[3] A counterexample to the sentence $(P \wedge \neg P) \to Q$ in the relevant logic system B (the "basic" logical system of relevant logic, Priest[84], p. 189) can be made using a propositional structure of relevant logic $\langle \mathfrak{W}, n, \mathsf{R}, \star, v \rangle$, where $\mathfrak{W} = \{n, w\}$ is the set of worlds including the normal world n, R is a ternary relation to define the truth condition of conditional sentences, $\star$ is the Routley Star to define the truth condition of negation sentences, and v is a truth assignment function. We suppose that $\mathsf{R}nww$, $v_w(P) = \mathsf{T}$, $v_{w^\star}(P) = \mathsf{F}$, and $v_w(Q) = \mathsf{F}$. By the definition of the negation of the relevant logic system B, $v_w(\neg P) = \mathsf{T}$, and therefore $v_w(P \wedge \neg P) = \mathsf{T}$. Hence, we have $v_n((P \wedge \neg P) \to Q) = \mathsf{F}$.

[4] A counterexample to the argument

$$\begin{array}{ll} & R. \\ \hline \therefore & (P \wedge \neg P) \to Q. \end{array}$$

is the propositional structure of relevant logic above $\langle \mathfrak{W}, n, \mathsf{R}, \star, v \rangle$. We here suppose that $v_n(R) = \mathsf{T}$. Then, the premise is true in the normal world n but the conclusion is false. Hence, the argument is not valid.

not *follow from* (in a normal sense) irrelevant sentences. The idea behind relevant logic is that logical validity should reflect this standard concept of implication. For an argument, the relevancy among its components is necessary for it to be regarded as logically valid. Under this notion of logical validity, logical valid arguments are restricted to the ones that faithfully reflect this common practice of our everyday inferential activity.

Where there are multiple options, in many cases, the choice does matter. There is the correct option or the best option for our purpose, and we seek it. For notions of logical validity, however, this is not true. There are at least two (and more) equally good notions of logical validity. It is not the case that there is a unique right notion. Multiple notions can be equally appropriate. Some people suppose that (possible) worlds are connected and logical validity can be characterized in such a way that the connection plays an essential role in determining truth values of sentences and the logical validity of arguments. But some other people suppose that each world is completely independent of other worlds. Some people suppose that in a valid argument, the premises and conclusion have to be relevant to each other. But some other people suppose that the relevancy is not necessary for logical validity. These views are all suitable for an appropriate notion of logical validity, and we cannot say that one is correct and the others are wrong. The two contradicting views on logic are both appropriate, because a notion of logical validity is not that which can be *discovered* by scientific or philosophical research but rather that which is *established* through one's education, experience, and investigation. Founders of logical systems, I believe, have repeatedly asked themselves "What is logic for?" and "What should logic be?" Based on justifiable motivations, they have developed various notions of logical validity and created their systems.

Of course, I do not intend to claim that any arbitrary notion of logical validity is legitimate. Some notions that have been proposed in the literature are not appropriate, an example of which is Rudolf Carnap's view of logic based on so called "Principle of Tolerance":

> *In logic, there are no morals.* Everyone is at liberty to build his own logic, i.e. his own form of language, as he wishes. All that is required

> of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical argument. (Carnap[20], p. 52)

A logic can be defined by specifying a language and there is no constraint on the specification. If you want a logic, you may construct it by introducing syntactical rules of a language. No philosophical justification is necessary.

If this tolerant view is correct, we would be allowed to introduce the connective defined by the following introduction and elimination rules, which is called "*tonk*," to our new logical system:

$$\frac{\varphi}{\varphi \ \textit{tonk} \ \psi}\ \textit{tonk}\text{-I} \qquad\qquad \frac{\varphi \ \textit{tonk} \ \psi}{\psi}\ \textit{tonk}\text{-E}$$

Normally, the logical consequence relation is supposed to be a transitive relation.[5] That is, if the argument $\langle\Gamma, \psi\rangle$ deriving the sentence ψ from the set Γ of sentences is logically valid for all $\psi \in \Sigma$, and if the argument $\langle\Sigma, \varphi\rangle$ deriving φ from Σ is also logically valid, then the argument $\langle\Gamma, \varphi\rangle$ is logically valid. Then, any argument with some premises would come out valid in the new logic. However, no such "logic" can exist, because the argument

$$\begin{array}{ll} & \text{Socrates is mortal.} \\ \hline \therefore & \text{San Diego is in California.} \end{array}$$

is not logically valid in any sense. Hence, Carnap's notion of logical validity is not appropriate.

1.2 The Formal-Structural Notion and the Minimal Notion

Notions of logical validity are pluralistic, and validity can be characterized under each notion. Different notions of logical validity have different views on inferences and their related aspects. Intuitionistic logic is supposed to capture the intuitionist's notion concerning truth, while relevant logic attempts to reflect our

[5] There are logical system whose logical validity is not transitive. Example of non-transitive logical systems are given, for example, in Cook[22] and Frankowski[34].

inferential practice regarding relevancy. In a formal notation: For a legitimate notion N_x of logical validity, there is a collection of arguments LV_x such that an argument is logically valid under the notion N_x if and only if the argument is a member of LV_x. LV_x varies according to N_x.

The notion of logical validity that is appropriate for the characterization of logical validity that draw the limits of our inferential activity, i.e., the notion that I will investigate in this dissertation, is a version of the *formal-structural notion* of logical validity. The formal-structural notion, which has been developed and established in a series of works by Gila Sher (Sher[105], [106], [111], [116], [118], and [121]), is based on a view concerning what validates logically valid arguments, i.e., the source of logical validity. According to the notion, logical validity is the validity grounded in *formal laws.*

Sher's Formal-Structural Notion

A logically valid argument, by definition, is an argument whose conclusion follows from its premises by virtue of their logical forms. The argument

$$\begin{array}{rl} & \text{Some red roses are fragrant.} \\ \hline \therefore & \text{Some roses are fragrant.} \end{array}$$

is logically valid and always holds. This is due to the logical forms of its premise and conclusion. Any argument of the same logical form—an argument deriving $\exists x[Q(x) \wedge R(x)]$ from $\exists x[P(x) \wedge Q(x) \wedge R(x)]$—is logically valid, and its validity does not depend on anything else. One important question that can be raised is about the relationship between the logical form and the logical validity. Why do all the arguments of this form always hold? An answer to this question forms a view of logic.

Sher's answer to the question is as follows. The premise says that the intersection of three sets (the set of red objects, the set of roses, and the set of fragrant objects) is not empty. The conclusion says that the intersection of two sets of these three sets (the set of roses and the set of fragrant objects) is not empty. And there is a *formal law* governing intersections, which is that if the intersection of three sets of objects is not empty, then the intersection of any two of them is not empty. By virtue of this law, the argument and any other arguments of the same logical form hold.

Similar justifications can be given to other logically valid arguments as well. Consider the following argument:

$$\begin{array}{rl} & \text{All men are mortal.} \\ & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

The first premise says that a set is included in another set (the set of all men is included in the set of all mortal objects). The second premise says that an object (Socrates) is a member of the included set. The conclusion says that the object is a member of the including set. And there is a formal law that validates this argument, which is that a member of a set is also a member of another set that includes the set.

What is a formal law that logical validity is to be grounded in? A formal law, according to Sher, is a law governing *formal operators* representing *formal properties* (Sher[111], p. 245). Let me first explain formal properties. Generally, individual objects in a domain (a universe of discourse) satisfy a variety of properties. Socrates in the domain of animals satisfies, for example, the biological property of being male and the physical property of weighing a certain kilogram. Number 7 in the domain of natural numbers satisfies the mathematical property of being odd and the mathematical property of being prime. Sets of individual objects also have various properties. The set of all philosophers satisfies the property of being composed of human beings and the property of containing finite objects.

A formal property is a special kind of property in the sense that its bearers satisfy it independently of what they are. Whether or not Socrates has the property of being male depends on what he is. Whether or not number 7 satisfies the property of being odd also depends on what the number is. These are not formal properties. Consider, however, the property of being identical to itself. Socrates satisfies this property; Socrates is identical to himself. And importantly, even if he was not a philosopher, or even if he was not a teacher of Plato, he would still satisfy this property. Whether or not Socrates satisfies this property is completely independent of what he is. This is a formal property.

A mathematical characterization of formal properties can be given in terms of the concept of *isomorphism* and the concept of *invariance*. Let P be a property of individual objects defined all domains (a domain is a non-empty set of individual objects). For each domain $\mathcal{D}$, P has a subset of the domain as its extension (i.e., the set of all objects satisfying P). If no object in $\mathcal{D}$ satisfies P, its extension $\mathsf{Ext}_{\mathcal{D}}(P)$ on $\mathcal{D}$ is the empty set $\emptyset$.

Consider pairs $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', a'\rangle$, where $\mathcal{D}$ and $\mathcal{D}'$ are domains, $a \in \mathcal{D}$, and $a' \in \mathcal{D}'$. We say that $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', a'\rangle$ are *isomorphic* if there is a bijection $\eta : \mathcal{D} \to \mathcal{D}'$ (a one-to-one and onto function from $\mathcal{D}$ to $\mathcal{D}'$ of the same cardinality) such that $\eta(a) = a'$. We call η an *isomorphism* between $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', a'\rangle$. Note that an bijection $\eta : \mathcal{D} \to \mathcal{D}'$ is an isomorphism between various pairs: for any $a \in \mathcal{D}$, η is an isomorphism between $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', \eta(a)\rangle$. Note also that if $\mathcal{D}$ and $\mathcal{D}'$ are of the same cardinality, then for any $a \in \mathcal{D}$, and for any $a' \in \mathcal{D}'$, the pairs $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', a'\rangle$ are isomorphic by some bijection.

For an isomorphism η, we say that a property P is *invariant under* η if for any isomorphic pair $\langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}', \eta(a)\rangle$, a satisfies P if and only if $\eta(a)$ satisfies P, that is, $\eta(\mathsf{Ext}_{\mathcal{D}}(P)) = \mathsf{Ext}_{\mathcal{D}'}(P)$. Then, a formal property can be defined as follows:

A property P is *formal* if it is invariant under all isomorphisms.

In other words, a formal property P is a property such that any bijection $\eta : \mathcal{D} \to \mathcal{D}'$ maps the extension $\mathsf{Ext}_{\mathcal{D}}(P)$ of P on $\mathcal{D}$ to the extension $\mathsf{Ext}_{\mathcal{D}'}(P)$ of P on $\mathcal{D}'$.

For instance, the property $P_{identity}$ of being identical to itself, as mentioned above, is a formal property. The extension of the property on $\mathcal{D}$, $\{x \in \mathcal{D} : x \text{ is identical to itself}\}$, is $\mathcal{D}$ itself, because every object in $\mathcal{D}$ is identical to itself. Therefore, for any bijection $\eta : \mathcal{D} \to \mathcal{D}'$, we have $\eta(\mathsf{Ext}_{\mathcal{D}}(P_{identity})) = \mathsf{Ext}_{\mathcal{D}'}(P_{identity})$. The property is invariant under all isomorphisms.

The property P_{red} of being red whose extension on $\mathcal{D}$ is $\{x \in \mathcal{D} : x \text{ is red}\}$ is not formal; for some bijection $\eta' : \mathcal{D}_{fruits} \to \mathcal{D}_{fruits}$ (actually a permutation on the domain of fruits) that maps a tomato to a banana, we have $\eta'(\mathsf{Ext}_{\mathcal{D}}(P_{red})) = \{\eta'(x) : x \text{ is red}\} \neq \{x : x \text{ is red}\} = \mathsf{Ext}_{\mathcal{D}}(P_{red})$.

Why can a property be regarded as a "formal" property if it is invariant under all isomorphisms? How does the invariance method capture the notion of formality? The idea behind the definition is that a formal property is a property that disregards differences between individual objects (in other words, as said before, a formal property is a property such that its bearers satisfy it independently of what they are). A tomato and a banana differ in many aspects. A formal property is insensitive to such differences. The property of being red is not a formal property, because it is sensitive to the difference between the two fruits: a tomato satisfies it, while a banana does not. The property of being identical to itself is a formal property; any object, whatever it is, satisfies it.

Under this idea, no two objects $a \in \mathcal{D}$ and $a' \in \mathcal{D}'$ can be distinguished from one another by formal properties. It cannot happen that a satisfies a formal property P and a' does not. That is to say, it must hold that $a \in \mathsf{Ext}_{\mathcal{D}}(P)$ if and only if $a' \in \mathsf{Ext}_{\mathcal{D}'}(P)$. Consider then a bijection $\eta : \mathcal{D} \to \mathcal{D}'$. For any $a \in \mathcal{D}$, then it must hold that $a \in \mathsf{Ext}_{\mathcal{D}}(P)$ if and only if $\eta(a) \in \mathsf{Ext}_{\mathcal{D}'}(P)$. In our words, P has to be invariant under η. Moreover, this must hold for any bijections. Hence, P has to be invariant under all isomorphisms.

Note that although the definition above is applied to properties of individual objects, the definition can also be applicable to sets and more generally any set-theoretic constructs such as sets of sets and Cartesian products of sets. Consider, for example, a property Q that is applied to sets, whose extension on $\mathcal{D}$ is a subset of the power set $\wp(\mathcal{D})$. Consider pairs $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}', X' \rangle$, where X is a subset of $\mathcal{D}$ and X' is a subset of $\mathcal{D}'$. We say that $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}', X' \rangle$ are *isomorphic* if there is a bijection $\eta : \mathcal{D} \to \mathcal{D}'$ such that $\eta(X) \overset{def}{:=} \{\eta(a) : a \in X\} = X'$. We call η an *isomorphism* between $\langle \mathcal{D}, a \rangle$ and $\langle \mathcal{D}', a' \rangle$. For an isomorphism η, we say that a property Q is *invariant under* η if for any isomorphic pair $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}', \eta(X) \rangle$, X satisfies Q if and only if $\eta(X)$ satisfies Q. That is, $\eta(\mathsf{Ext}_{\mathcal{D}}(Q)) = \mathsf{Ext}_{\mathcal{D}'}(Q)$. Then, we say that Q is *formal* if it is invariant under all isomorphisms. Formal properties of other set-theoretic constructs can be defined in the same way.

According to this definition, for any cardinality κ, the property of being of cardinality κ is a formal property of sets; for any bijection $\eta : \mathcal{D} \to \mathcal{D}'$, a set X of

$\mathcal{D}$ contains κ-many objects if and only if $\eta(X)$ contains κ-many objects. Another example of formal properties is the identity property that holds between identical objects. The extension of the property on $\mathcal{D}$ is $\{\langle a, b\rangle \in \mathcal{D} \times \mathcal{D} : a = b\}$. For any bijection $\eta : \mathcal{D} \to \mathcal{D}'$, we have that $a = b$ if and only if $\eta(a) = \eta(b)$.

There are many non-formal properties. For example, the property $P_{human\,set}$ of being composed of human beings, which is applicable to sets, is not formal. Its extension $\mathsf{Ext}_{\mathcal{D}}(P_{human\,set})$ is $\{X \in \wp(\mathcal{D}) : X$ is composed of human beings$\}$. For some bijection $\eta' : \mathcal{D}_{animals} \to \mathcal{D}_{animals}$ (actually a permutation on the domain of animals) that maps a human being to a gorilla, we have $\eta'(\mathsf{Ext}_{\mathcal{D}_{animals}}(P_{human\,set})) \neq \mathsf{Ext}_{\mathcal{D}_{animals}}(P_{human\,set})$.

The formal-structural notion of logical validity is the notion that logical validity is the validity grounded in formal laws. And a formal law is a law governing formal operators representing formal properties. The *formal operator* representing a formal property is the characteristic function of it. The formal operator O_P of a formal property P thus can be identified with the collection of pairs $\langle \mathcal{D}, O_{\mathcal{D}}(P)\rangle$, where $O_{\mathcal{D}}(P)$ is a function such that for any set-theoretic construct X that P can be applied to, $O_{\mathcal{D}}(P)(X) = \mathsf{T}$ if and only if $X \in \mathsf{Ext}_{\mathcal{D}}(P)$.

Let P and Q be formal properties that can be applied to set-theoretic constructs of the same type. For example, we can take as P the property of sets of containing three objects and as Q the property of sets of containing finite objects. Obviously, the set $\{1, 2, 3\}$ satisfies P and *therefore* it also satisfies Q. We can think that this particular fact holds because of the following general law regarding P and Q:

If a set satisfies P, then it also satisfies Q.

Generally, for formal properties $P_1, P_2, \ldots, P_C$ that can be applied to set-theoretic constructs of the same type, a formal law governing formal operators representing $P_1, P_2, \ldots, P_C$ is a law that can be stated in the following form:

If a set-theoretic construct satisfies the formal properties $P_1, P_2, \ldots,$
then it also satisfies the formal property P_C.

Recall the law governing intersections that if the intersection of three sets of objects is not empty, then the intersection of any two of them is not empty.

This law, according to Sher, is the source of the validity of the argument

$$\begin{array}{ll} & \text{Some red roses are fragrant.} \\ \hline \therefore & \text{Some roses are fragrant.} \end{array}$$

The law governs two formal properties of quadruples $\langle \mathcal{D}, X, Y, Z \rangle$. One is the property P_1 that the intersection of subsets X, Y, Z of a domain $\mathcal{D}$ is not empty, and the other is the property P_C that the intersection of any two of X, Y and Z is not empty. It can be easily observed that these are actually formal properties. The law can be expressed as follows:

> If $\langle \mathcal{D}, X, Y, Z \rangle$ satisfies the formal property P_1, then it also satisfies the formal property P_C.

The formal-structural notion of logical validity says that the logical validity of the red-rose-fragrant argument is grounded in this law.

Recall another formal law that we have considered: a member of a set is also a member of another set that includes the set. This law governs three properties that can be applied to quadruples $\langle \mathcal{D}, a, X, Y \rangle$, where a is an object in $\mathcal{D}$ and X, Y are subsets of $\mathcal{D}$. The first formal property P'_1 is that $X \subseteq Y$. The second formal property P'_2 is that $a \in X$. The third formal property P'_C is that $a \in Y$. Then, the law can be stated as:

> If $\langle \mathcal{D}, a, X, Y \rangle$ satisfies the formal properties P'_1 and P'_2, then it also satisfies the formal property P'_C.

By virtue of this formal law, the Socrates-man-mortal argument

$$\begin{array}{ll} & \text{All men are mortal.} \\ & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

holds and becomes a logically valid argument.

The idea behind the formal-structural notion of logical validity is that the validity of a subject can be characterized by specifying the properties that it takes into account (Sher[111], p. 244). Biology takes into account biological properties of objects such as the properties of being a man and of being mortal. A biologically valid argument is an argument that holds by virtue of laws governing biological properties. The argument

$$\frac{\text{Socrates is a man.}}{\therefore \quad \text{Socrates is mortal.}}$$

is biologically valid. What validates this is a biological law of the relationship between the property of being a man and the property of being mortal, which can be stated as follows:

> If an individual object satisfies the biological property of being a man, then it also satisfies the biological property of being mortal.

To characterize physical validity, physical properties of objects need to be identified. A physically valid argument can be defined as an argument that holds by virtue of laws of physical properties. Similarly, in order to characterize logical validity, we need to specify the properties that logic has to take into account. According to the formal-structural notion of logical validity, it is formal properties.

The formal-structural notion, as opposed to the anti-realistic notion of intuitionistic logic, is a realistic notion with respect to formal laws. Formal laws are suppose to exist independently of our constructive mental activity, as scientific laws are normally so supposed. And an argument can be validated by formal laws independently of whether or not we have constructive proofs of them. Also, the formal-structural notion is distinguished from the notion that relevant logic is based on. It does not take into account the relevancy between premises and conclusions. Conventions of our inferential activity have nothing to do with whether or not formal laws hold.

I agree with the fundamental idea of the formal-structural notion of logical validity. There are biological properties of objects, and there are formal properties of objects. There is a kind of validity that holds by virtue of biological properties, biological operators, and biological laws, which is biological validity. Similarly, there is a kind of validity that holds by virtue of formal properties, formal operators, and formal laws, which is logical validity.

Moreover, the formal-structural notion of logical validity fits the purpose of this dissertation research, i.e., the characterization of the logical validity that draws the limits of our inferential activity. If an argument $\langle \Gamma, \varphi \rangle$ deriving the conclusion φ from the set Γ of premises can be validated by some formal laws, then φ follows from Γ with certainty; the certainty of formal laws is most certain among the certainty of laws of other kinds such as physical laws and biological laws.[6] If the intersection of three sets of objects is not empty, then it certainly holds that the intersection of any two of them is not empty. Also, for an logically valid argument $\langle\Gamma, \varphi\rangle$, the argument $\langle\Gamma, \neg\varphi\rangle$ deriving the negation of φ from the same set Γ is impossible to hold; for it to hold, it has to violate the formal laws that makes $\langle\Gamma, \varphi\rangle$ logically valid.

The Minimal Notion of Logical Validity

However, the notion of logical validity that I will employ for this research is not Sher's original version of the formal-structural notion: I will hold another version of the original one. There are two possible positions of the formal-structural notion of logical validity with respect to "available" formal properties. Sher's original version supposes that all formal laws are available. There are various formal properties and various formal laws. Some are about non-emptiness and some are about cardinalities. They are all entitled to validate arguments under the original formal-structural notion. The other possible position can be obtained by restricting available formal properties and formal laws to particular ones. Under this restricted position, not all formal properties or formal laws are allowed to use. Only selected ones can be regarded as legitimate formal properties and legitimate formal laws.

There are various possible choices with respect to the selected available formal properties and formal laws. Different sets of available formal properties and formal laws produce different versions of the formal-structural notion. The version that I take is what I call the *minimal notion* of logical validity. Sher's formal-structural notion of logical validity can be seen as a "maximal" notion: the

[6]According to the formal-structural notion, the certainty of logically valid arguments is by virtue of the certainty of formal laws. Thus, an account of the certainty of logical valid arguments can be reduced to an account of the certainty of formal laws. Why do formal laws certainly hold? In terms of what notions can the certainty of formal laws be explained? Could some formal law be violated? In order to specify the source of the certainty of logically valid arguments, these questions need to be answered. The problem of how the certainty of logical validity, and other related characteristic features of logic as well, can be justified has been crucially important in the philosophy of logic. I call this problem of logical validity the *justification problem*. I will provide a further explanation of the problem and its importance in the next chapter (pp. 49–52).

collection of available formal laws are maximally large; every formal properties and formal laws are usable. The minimal notion, as opposed to the maximal notion, is the notion under which available formal properties and formal laws are minimally small.

There are many formal properties and formal laws that are available under the maximal notion but unavailable under the minimal notion. Examples of such formal laws are laws governing formal properties of cardinalities (I will explain why they count as unavailable under the minimal notion later). For any cardinality κ, being of cardinality κ is a formal property, because it is invariant under all isomorphisms. According to the maximal notion, therefore, the following argument is valid:

$$\begin{array}{rl} & \text{There are } \aleph_1\text{-many roses.} \\ \hline \therefore & \text{There are } \aleph_0\text{-many roses.} \end{array}$$

This is because it is validated by a formal law that a set of cardinality $\aleph_1$ contains a set of cardinality $\aleph_0$ as a subset. This argument, however, is not valid under the minimal notion; the cardinality properties of being $\aleph_1$ and of being $\aleph_0$ are not available and therefore laws governing them are not allowed to validate arguments. Besides this, there are many arguments that are "logically valid" under the maximal notion but "logically invalid" under the minimal notion.

Why are these cardinality properties not available under the minimal notion? Are there any other unavailable properties? More generally, what formal laws are in or out of the minimally small collection of available formal laws? How can such special formal properties and formal laws be characterized? How can the choice be justified? Answering these question requires detailed considerations of several aspects of formal properties and formal laws. I will address them in later chapters. Here, let me explain the basic idea behind the minimal notion.

Consider two properties: the property of being male and the property of being of cardinality $\aleph_0$. They are different in many aspects. First, they differ in the categories of their bearers. The male-property is applicable to individual objects, while the $\aleph_0$-property can be applied to sets. Second, they differ in the range of their applicability. The male-property is applicable to individual objects and only to living objects. The sentence "Number 7 is male" does not make sense. But, the

$\aleph_0$-property can be applied to any sets. Third, the male-property is a biological property, while the $\aleph_0$-property is a formal property. The male-property is not invariant under all isomorphisms, but the $\aleph_0$-property is.

There is, however, an important similarity between these properties: they both constitute what their bearers are. Socrates satisfies the male-property. Being male is a property that constitutes what Socrates is in the sense that any individual cannot be identical to Socrates without having the property. Analogously, the $\aleph_0$-property is a property that partially determines the identity of the set $\mathbb{N}$ of natural numbers. Without containing $\aleph_0$-many objects, any set cannot be identical to $\mathbb{N}$.

The logical validity of an argument does not depend on any properties of individual objects that constitute what their bearers are, or so logic has been characterized. For an argument about Socrates, whether or not it is logically valid is independent of his maleness. Indeed, if the argument is logically valid, then the argument obtained from it by replacing "Socrates" in it with "Xanthippe" will still hold:

$$\begin{array}{ll} & \text{Socrates is a male philosopher.} \\ \hline \therefore & \text{Some philosopher is male.} \end{array}$$

$$\begin{array}{ll} & \text{Xanthippe is a male philosopher.} \\ \hline \therefore & \text{Some philosopher is male.} \end{array}$$

The basic idea of the minimal notion is to extend this principle to sets and more generally any set-theoretic constructs. The minimal notion of logical validity claims that the logical validity of an argument should not depend on most properties of sets that constitute what their bearers are and so logic should be characterized.

For example, logical validity should be supposed to be insensitive to the $\aleph_0$-property and the $\aleph_1$-property. And therefore, the $\aleph_1$-$\aleph_0$-roses argument

$$\begin{array}{ll} & \text{There are } \aleph_1\text{-many roses.} \\ \hline \therefore & \text{There are } \aleph_0\text{-many roses.} \end{array}$$

should not count as logically valid. We deny that Socrates's maleness plays any role for the logical validity of arguments, since it constitutes what he is, and logic should disregard such a biological property of individuals. It then seems to be consistent to deny for the same reason that the $\aleph_0$-property and the $\aleph_1$-property can have any effect on logical validity. The minimal notion proposes, for this reason, that as many formal properties as possible should be excluded from the list of available properties, and consequently, that as many formal laws as possible should be taken as unavailable laws.

The difference between the original maximal formal-structural notion and the minimal notion lies in whether or not the extension of the following principle can be admitted:

> The logical validity of an argument should not depend on most properties that constitute what their bearers are.

The maximal notion emphasizes the formality of properties and laws and allows all formal laws to validate logically valid arguments. The minimal notion takes into account the fact that there are many formal properties that constitutes what their bearers are and allows only a few formal laws to justify the logical validity of arguments. Which notion one takes depends on one's pre-theoretic thought on what logic is. Sher thinks that a law can logically validate an argument as long as it is formal and takes the maximal notion. I think that not every formal law is allowed to validate argument and take the minimal notion.

A legitimate notion N_x of logical validity determines the collection LV_x of arguments that are logically valid under the notion N_x. And so does the minimal notion. There is a collection of arguments that can be regarded as logically valid according to the minimal notion. I call the logical validity under the minimal notion *prime logical validity*. The characterization of prime logical validity is the main objective in the following chapters.

1.3 Restrictions

The problem of logical validity, and the problem of prime logical validity as well, is huge. I thus impose five restrictions on the research.

Restriction 1

I will only consider the logical validity in formal languages but not the logical validity in natural languages. Thus, what I mean by "prime logical validity" is one kind of the logical validity in formal languages. Arguments in formal languages represent logical forms of arguments in natural languages. A sentence "Pc" represents the logical form of the sentence "Socrates is a man." A valid argument in a logical system represents a logical form of logically valid arguments in a natural language. The pair $\langle \{\forall x[Px \to Qx], Pc\}, Qc\rangle$ represents the logical form of the Socrates-man-mortal argument:

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

We can thus expect that the logical variety of many arguments in natural languages will be determined by identifying logically valid arguments in formal languages.

Restriction 2

I will only consider the logical validity in formal languages whose terms are *rigid*. What is meant by "rigidity" is that the meanings of terms are "completely exhausted by their semantic definitions" (Sher[105], p. 56). Consider, for example, a binary sentential connective $\circledcirc$ defined as follows:

$$\text{“}\varphi \circledcirc \psi\text{''} \text{ is } \begin{cases} \text{true} & \text{if both “}\varphi\text{”and“}\psi\text{” are true,} \\ & \text{and if Galileo believed that the earth moves;} \\ \text{false} & \text{otherwise.} \end{cases}$$

The standard conjunction $\wedge$ of classical logic and this connective $\circledcirc$ have the same truth function as their semantic values: they are extensionally equivalent to each other. This is because Galileo actually believed that the earth moves. By this restriction, I suppose that formal languages we will consider do not contain terms like $\circledcirc$, i.e., terms whose meanings are affected by external factors such as Galieo's belief. Our formal languages contain only terms whose meanings are determined by and identified with their extension.[7]

Restriction 3

I will only consider the logical validity in first-order formal languages, second-order formal languages, and formal language with modal operators. I suppose that our languages contains, as extra-logical terms, denumerably many constant symbols and relation symbols. For simplicity, I suppose that our language does not contain function symbols.

Restriction 4

I will only consider arguments whose components are sentences, i.e., formulas with no free variables. When I discuss an argument $\langle \Gamma, \varphi \rangle$ deriving φ from Γ, I always suppose that all elements in Γ and φ are sentences.

Restriction 5

I will only consider logical systems with two truth values, True ("T") and False ("F"). The reason for this restriction is that arguments/theories of logical systems with two truth values that I will develop in this dissertation can be transformed into arguments/theories of "standard" many-valued logical systems.[8]

[7]In the study of logical constants, it is often discussed that whether or not a term is a logical constant is not determined based solely on its semantic value. $\circledcirc$ is an example of such problematic terms. The standard conjunction $\wedge$ normally counts as logical, while $\circledcirc$ intuitively does not. This is because if $\circledcirc$ was a legitimate logical connective, there would be logically valid arguments whose validity is not necessary, an example of which is the argument deriving $\varphi \circledcirc \psi$ from the two premises φ and ψ; the truth value of $\varphi \circledcirc \psi$ depends on Galileo's belief. Timothy McCarthy says, "the logical status of an expression is not settled by the functions it introduces, independently of how those functions are *specified*" (McCarthy[67], p. 516). By this restriction, I suppose that we can avoid this problem. For more on this problem, see, for example, McCarthy[67] (pp. 514–516), Sher[105] (pp. 64–65), and MacFarlane[61] (pp. 188–192).

[8]What I mean by a "standard" many-valued logical system is a logical system with n-many truth value whose set of designated values (truth values to be preserved from the premises to the conclusion) is constant. In such a system, a set V of n-many truth values and a set D of designated values, which is a subset of V, are given. An argument $\langle \Gamma, \varphi \rangle$ is said to be (semantically) valid if in all structures in which all the sentences in Γ are assigned some designated value in D, φ is also assigned some value in D.

There are n-valued logical systems whose validity cannot be defined in this standard way. Such n-valued logical systems can be found, for example, in Malinowski[63] and Frankowski[34]. In these logical systems, the designated values that are to be assigned to premises and those to

1.4 Prospectus

The main purpose of this dissertation is to characterize logical validity as the limits of our inferential activity, i.e., prime logical validity. For the characterization of prime logical validity, three problems have to be answered:

(i) In what way should prime logical validity be characterized?

(ii) What terms are logical constants?

(iii) Among various arguments, what arguments are logically valid under the minimal notion?

I will address these problems in the following chapters.

Chapter 2

Characterizing prime logical validity in a formal language means identifying a particular set of arguments, namely, the set of all logically valid arguments under the minimal notion. For the identification, a principled method is required. Logically valid arguments cannot be chosen arbitrarily. But rather, they have to be identified in such a way that the logical validity of an argument can be explained based on a certain justifiable principle.

There are two major methods to characterize logical validity: the *semantic method* and the *proof-theoretic method*. In the semantic system of a logic, legitimate ways/cases of assigning truth values to sentences are given (e.g., truth-assignment functions of sentential logic and structures of first-order logic). Logical validity is defined in terms of the concept of *truth-preserving*: An argument $\langle \Gamma, \varphi \rangle$ deriving φ from the set Γ of sentences is *semantically valid* if for any ways/cases of assigning truth values, if all sentences in Γ are true, φ is also true. In a semantically valid

conclusions are not identical. In their semantic system, the set V of n-many truth values and two subsets D_p and D_c are given. The relationship between D_p and D_c varies according to logical systems. For example, in the logical system introduced in Malinowski[63], it is supposed that $D_p \supseteq D_c$, while in the logical system introduced in Frankowski[34], it is required that $D_p \subseteq D_c$. Then, an argument is said to be valid if it satisfies the following condition:

> In all structures in which all the sentences in Γ are assigned some value in D_p, φ is assigned some value in D_c.

In such logical systems, there is an asymmetry between the truth values that the premises of a valid argument has to have and the truth values that the conclusion has to have.

argument, truth is preserved from its premises to its conclusion.

In the proof-theoretic method, on the other hand, logical validity is defined in terms of derivability. In the proof-theoretic system of a logical system (a Hilbert-style proof system), a set of sentences that have a special status (i.e., *axioms*) and a set of inference rules are given. A *formal proof* of an argument $\langle \Gamma, \varphi \rangle$ is a finite sequence $\langle \varphi_1, \ldots, \varphi_n, \varphi_{n+1} \rangle$ such that φ_{n+1} is φ and such that for all $i \leq n+1$, one of the following holds:

(i) φ_i is a sentence in Γ

(ii) φ_i is an axiom;

(iii) φ_i is obtained by applying an inference rule to some of $\varphi_1, \ldots, \psi_{i-1}$.

An argument $\langle \Gamma, \varphi \rangle$ is *proof-theoretically valid* if there is a formal proof of $\langle \Gamma, \varphi \rangle$. In a proof-theoretically valid argument, its conclusion is derivable from its premises.

Which method, the semantic method or the proof-theoretic method, is suitable for the characterization of prime logical validity? In Chapter 2, I will address this problem. In particular, I will focus on the model-theoretic semantic system and the Hilbert-style proof system and argue that the former is appropriate but the latter is not.

Chapter 3

For the characterization of prime logical validity, a demarcation between logical and extra-logical terms is necessary. Without knowing whether a given term is logical or extra-logical, we cannot determine whether a given argument is logically valid or not.[9] Tarski, the founder of the model-theoretic semantics, was

[9]Consider an argument $\langle \{\forall x Px\}, \exists x Px \rangle$ deriving $\exists x Px$ from $\forall x Px$. If the universal quantifier ($\forall$) and the existential quantifier ($\exists$) are both logical constants, then the argument is valid (in the standard first-order logic), because the conclusion $\exists x Px$ is true in every structure in which the premise $\forall x Px$ is true. Assume, however, that these quantifiers are not logical terms, contrary to our normal understanding of logicality based on classical logic. Then, the argument is no longer valid. Since $\forall$ and $\exists$ are extra-logical terms, they can be assigned different meanings in different structures. In some structure, for example, $\forall$ is assigned the usual meaning of $\exists$, and $\exists$ is assigned the usual meaning of $\forall$. And also the predicate P is assigned a non-empty set that is not identical to its domain. In such a structure, the premise "$\forall x Px$" is true, because it means $\exists x Px$ in the standard reading, while the conclusion "$\exists x Px$" is false, because it means $\forall x Px$. Therefore, according to definition of the model-theoretic validity, the argument $\langle \{\forall x Px\}, \exists x Px \rangle$ is not valid. The logical validity of an argument thus varies according to a boundary between logical and extra-logical vocabularies.

aware of the problem:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage (Tarski[125], p. 418).

In the contemporary model-theoretic approach to logic, logical constants have been characterized using the concepts of *invariance* and of *similarity relation*:

> A term is logical if its characteristic function is invariant under "appropriate" similarity relations between structures.

Regarding what similarity relations are appropriate, several candidates have been proposed, and as a result, there are several theories available. In Chapter 3, I propose another theory of logical constants based on another similarity relation. I first characterize logical terms of classical logic using a similarity relation that is different from the ones employed by existing theories. I then show that the characterization can be extended to several non-classical logics such as modal logic, intuitionistic logic, and relevant logic.

Chapter 4

As will be seen, many terms can be sanctioned by the theory of logical constants I propose in Chapter 3. Examples of the sanctioned logical terms are the standard truth-functional logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$), the standard first-order and second-order quantifiers ($\forall, \exists$), and the modal operators of the standard systems of modal logic ($\Box, \Diamond$). In formal languages containing these terms, various sentences and various arguments can be expressed.

Remember that under the minimal notion, only selected formal laws governing selected formal properties are allowed to validate arguments. In order for an argument to be logically valid under the notion, it has to be validated by the selected formal laws. Thus, even if an argument contains only sentences whose terms are all logical, and even if it can be validated by some formal laws, it does not necessarily mean that it can also be regarded as logically valid under the minimal notion.

In Chapter 4, I will address two problems:

(i) What formal properties and formal laws are legitimate under the minimal notion?

(ii) What arguments are logically valid under the minimal notion?

The problem (ii) is is the main problem of the dissertation, and answering problem (i) is necessary for answering (ii). For the identification of the logically valid arguments, I will define the legitimate formal laws that hold in structures of classical logic and the legitimate formal laws that hold in structures of non-classical logics. And for each legitimate formal laws, I will introduce a logical system to characterize prime logical validity.

Chapter 5

In Chapter 5, I summarize the main results of the dissertation.

2 Semantics and Proof-Theory

The main objective of this dissertation is to characterize prime logical validity, i.e., the logical validity under the minimal notion. To define prime logical validity, a correct method is necessary. Two major methods to characterize logical validity are the semantic method and the proof-theoretic method. In this chapter, I will address the problem of which method is appropriate for the characterization of prime logical validity.

I will first examine the semantic method, in particular, the model-theoretic method. I will show that if an argument is model-theoretically valid, then there exists a formal law that validates the argument. This can be interpreted as showing that the model-theoretic method captures the basic idea underlying the minimal notion that a logically valid argument is to be validated by a certain formal law. I will also show that the model-theoretic method captures two characteristic properties of logical validity—formality and necessity. Based on these facts, I will conclude that the model-theoretic method is suitable for the characterization of prime logical validity.

I will then examine the proof-theoretic method, in particular, the Hilbert-style proof-theoretic method. I will show that, as is the case with the model-theoretic method, if an argument is proof-theoretically valid, then there is a formal law that validates the argument. I will argue, however, that the proof-theoretic method is problematic for two reasons (at least for the characterization of prime logical validity).

2.1 Model-Theoretic Validity

Definitions

Model-theoretic semantics is a mathematical apparatus to identify a set of particular arguments, i.e., the set of model-theoretically valid arguments. Let L be a standard first-order language or a standard second-order language. In the model-theoretic semantics for L, each sentence in L is assigned a truth value in a *structure*. A structure is a pair $\langle \mathcal{D}, I \rangle$ of a non-empty set $\mathcal{D}$ and an interpretation function I for extra-logical terms of L.[1] An interpretation function I assigns an extra-logical term a set-theoretic construct of the corresponding type. For example, I assigns an individual constant symbol c an object in $\mathcal{D}$, a unary relation symbol P a set of objects, and an n-ary relation symbol R a set of n-tuples of objects. By I, each sentence is given its "meaning" in $\mathcal{D}$. For a set Γ of sentence, a *model* of Γ is a structure in which all the sentences in Γ are true. An argument $\langle \Gamma, \varphi \rangle$ is *model-theoretically valid* if every model of Γ is also a model of $\{\varphi\}$.

Model-Theoretic Validity and Formal Laws

Is the model-theoretic method an appropriate method to characterize prime logical validity? Can a model-theoretically valid argument be regarded as logically valid under the minimal notion? The first thing to be done to give affirmative answers to these questions is to show that for a model-theoretically valid argument, there is a formal law that validates it. Since the formal law might not be on the list of available formal laws under the minimal notion, showing that such a formal law exists is not sufficient to claim that the model-theoretic method actually captures the minimal notion. However, it is necessary, because the minimal notion is a version of the original formal-structural notion, which requires the existence of formal laws for an argument to be regarded as logically valid.

In order to see the idea to show the existence of the formal law that vali-

[1]Throughout this chapter, I suppose that a demarcation between logical terms and extra-logical terms of L is given. The problem of the characterization of logical constants—What terms are logical constants?—will be addressed in Chapter 3.

dates a model-theoretically valid argument,[2] let us consider the following simple argument:

$$\begin{array}{ll} & \forall x[Px \to Qx]. \\ & Pc. \\ \hline \therefore & Qc. \end{array}$$

This argument is model-theoretically valid in classical first-order logic; in any structure $\langle \mathcal{D}, I \rangle$, if the two premises are true, then the conclusion is also true.

Now consider the following three properties R_1, R_2, R_C that are applicable to quadruples $\langle \mathcal{D}, a, X, Y \rangle$, where a is an individual object in $\mathcal{D}$ and X, Y are subsets of $\mathcal{D}$:

(1) $\langle \mathcal{D}, a, X, Y \rangle$ satisfies R_1 if $X \subseteq Y$.

(2) $\langle \mathcal{D}, a, X, Y \rangle$ satisfies R_2 if $a \in X$.

(C) $\langle \mathcal{D}, a, X, Y \rangle$ satisfies R_C if $a \in Y$.

R_1 can be seen as describing the truth condition of the first premise $\forall x[Px \to Qx]$; the first premise is true if $I(P) \subseteq I(Q)$. Similarly, R_2 and R_C can be seen as describing the truth conditions of the second premise and the conclusion respectively: the second premise is true if $I(c) \in I(P)$; the conclusion is true if $I(c) \in I(Q)$. It can be easily observed that these properties are all formal properties. Let $\mathcal{D}$ and $\mathcal{D}'$ be domains of the same cardinality. For any bijection $\eta : \mathcal{D} \to \mathcal{D}'$, and for subsets X, Y of $\mathcal{D}$, we have that $\langle \mathcal{D}, a, X, Y \rangle$ satisfies R_1 if and only if $\langle \mathcal{D}', \eta(a), \eta(X), \eta(Y) \rangle$ satisfies R_1. Thus, R_1 is invariant under all isomorphisms and therefore a formal properties. Similar arguments can be given for R_2 and R_C.

The formal law that validates the argument above can be stated in terms of these formal properties:

> If $\langle \mathcal{D}, a, X, Y \rangle$ satisfies the formal properties R_1, R_2, then it also satisfies the formal property R_C.

Suppose that the premises $\forall x[Px \to Qx]$ and Pa are both true in a structure $\langle \mathcal{D}, I \rangle$. Then, the quadruple $\langle \mathcal{D}, I(c), I(P), I(Q) \rangle$ satisfies the formal properties

[2] Recall the definition of formal laws: for formal properties $P_1, P_2, \ldots, P_C$ that can be applied to set-theoretic constructs of the same type, a formal law governing formal operators representing $P_1, P_2, \ldots, P_C$ is a law that can be stated in the following form:

> If a set-theoretic construct satisfies the formal properties $P_1, P_2, \ldots$,
> then it also satisfies the formal property P_C.

R_1 and R_2. By the formal law, $\langle \mathcal{D}, I(c), I(P), I(Q) \rangle$ also satisfies R_C. Therefore, the conclusion Qc is true in the structure.

In a similar way, we can show that for any model-theoretically valid argument, there is a formal law that validates it. For simplicity, we here give a proof for a first-order language containing only individual constant symbols and unary relation symbols as extra-logical terms. Let $\langle \Gamma, \varphi \rangle$ be a model-theoretically valid argument, where $\Gamma = \{\psi_0, \psi_1, \ldots\}$. Let $c_0, c_1, \ldots$ be individual constant symbols appearing in some of $\psi_0, \psi_1, \ldots, \varphi$. Also, let $P_0, P_1, \ldots$ be unary relation symbols in $\psi_0, \psi_1, \ldots, \varphi$. We define properties $R_1, R_2, \ldots, R_C$ that can be applicable to infinite tuples $\langle \mathcal{D}, a_0, a_1, \ldots, X_0, X_1, \ldots \rangle$ as the ones describing the truth conditions of $\psi_0, \psi_1, \ldots, \varphi$ respectively in such a way that meets the following condition:

> $\langle \mathcal{D}, a_0, a_1, \ldots, X_0, X_1, \ldots \rangle$ satisfies R_i (or R_C) if and only if φ_i (or φ) is true in any structure $\langle \mathcal{D}, I \rangle$ such that $I(c_j) = a_j$ and $I(P_k) = X_k$ for all j, k.

$R_1, R_2, \ldots, R_C$ are formal properties, because they are invariant under all isomorphisms.[3] The formal law that validates the argument $\langle \Gamma, \varphi \rangle$ is the following:

> If $\langle \mathcal{D}, a_0, a_1, \ldots, X_0, X_1, \ldots \rangle$ satisfies the formal properties $R_1, R_2, \ldots$, then it also satisfies the formal property R_C.

The proof that the argument $\langle \Gamma, \varphi \rangle$ holds by virtue of this law goes in the same way as above. Suppose that all the premises in Γ are true in a structure $\langle \mathcal{D}, I \rangle$. Then, the infinite tuple $\langle \mathcal{D}, I(c_0), I(c_1), \ldots, I(P_0), I(P_1) \ldots \rangle$ satisfies all $R_1, R_2, \ldots$. By the formal law, the infinite tuple also satisfies R_C. Therefore, the conclusion φ is true in the structure. Similar arguments can be given for other formal languages such as first-order language containing n-ary relation symbols and second-order languages.

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[3] Let $\mathcal{D}$ and $\mathcal{D}'$ be domains of the same cardinality. Then, for any bijection $\eta : \mathcal{D} \to \mathcal{D}'$, we have that:

$$\begin{aligned} & \langle \mathcal{D}, a_0, a_1, \ldots, X_0, X_1, \ldots \rangle \text{ satisfies } R_i \text{ (or } R_C). \\ \overset{iff}{\Longleftrightarrow}\ & \varphi_i (\text{or } R_C) \text{ is true in any structure } \langle \mathcal{D}, I \rangle \text{ such that } I(c_j) = a_j \text{ and } I(P_k) = X_k. \\ \overset{iff}{\Longleftrightarrow}\ & \varphi_i (\text{or } R_C) \text{ is true in any structure } \langle \mathcal{D}', I' \rangle \text{ such that } I'(c_j) = \eta(a_j) \text{ and } I(P_k) = \eta(X_k). \\ \overset{iff}{\Longleftrightarrow}\ & \langle \mathcal{D}', \eta(a_0), \eta(a_1), \ldots, \eta(X_0), \eta(X_1), \ldots \rangle \text{ satisfies } R_i \text{ (or } R_C). \end{aligned}$$

The equivalence between the second and third lines can be obtained by the fact that $a_j \in X_k$ if and only if $\eta(a_j) \in \eta(X_k)$.

2.2 Formality and Necessity

If an argument holds by virtue of a formal law, it can be regarded as absolutely certain; the formal law itself can be regarded as absolutely certain. We have shown above that for any model-theoretically valid argument, there is a formal law that validates it. Thus, if an argument is model-theoretically valid, it holds with certainty. The model-theoretic method, in this respect, can be taken as a promising method for the characterization of prime logical validity.

However, certainty is just one properties that logical validity has been supposed to possess. There are many other characteristic properties that have been attributed to logical validity. Examples of such properties are necessity, formality generality, apriority, analyticity, topic neutrality, and normativity. Among these characteristic properties, necessity and formality are the two properties that Alfred Tarski, the founder of model-theoretic semantics,[4] supposed the model theoretic

[4]Although Tarski is the founder of the model theoretic semantic system, there are several issues concerning his original definition of logical consequence that have been widely discussed. Two major issues are the followings. First, whether or not Tarski's definition is identical to the modern definition of model-theoretic validity is not obvious. In the standard model-theoretic definition we use today, domains vary, and as a result, there are a variety of structures. However, some philosophers have provided the interpretation to the effect that Tarski's definition is the fixed-domain definition. That is, there is only one domain and all structures have it as its domain. If the interpretation is correct, then there seems to be a serious flaw in Tarski's definition. Suppose that the domain contains at least two objects. Then, the following argument can be validated:

$$\begin{array}{rl} & \text{Some object exists.} \\ \hline \therefore & \text{Two objects exist.} \end{array}$$

However, we can reasonably suppose that a domain $\mathcal{D}_0$ with only one object is possible, and the argument does not hold in the structure whose domain is $\mathcal{D}_0$. The fixed-domain definition thus validates an argument whose consequence relation seems to be not necessary. This contradicts the necessity requirement that Tarski himself sets out.

The second issue is about the ω-rule. Tarski's motivation for the development of model-theoretic semantics is that he thinks the proof-theoretic characterization of logical consequence—the characterization that logicians in his time believed grasps "almost exactly the content of the common concept of consequence" or rather define "a new concept which coincided in extent with the common one" (Tarski[125], p. 409)—is a wrong characterization, because it fails to validate some arguments that should count as logically valid. An example that shows the inappropriateness of the proof-theoretic characterization is an ω-incomplete theory, in which the statements

A_0. 0 *possesses the given property* P,
A_1. 1 *possesses the given property* P,
$\vdots$
A_n. n *possesses the given property* P,

validity to satisfy:

> Certain consideration of an intuitive nature will form our starting-point. Consider any class K of sentences and a sentence X which follows from the sentences of the class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, since we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects spoken about in the sentence X or the sentences of the class K. The consequence relation cannot be destroyed by replacing the designation of the objects referred to in these sentences by the designation of any other objects. (Tarski[125], pp. 414–415).

The logical consequence relation between the premises and the conclusion of an argument is a necessary relation in that it is impossible that all the sentences in K are true and X is false. If it were possible, we would not call the relation between the premises and the conclusion a "consequence" relation. The term "consequence" implies that the relation necessarily holds. Necessity, however, is not an exclusive feature of logical validity; any other kinds of validity possess necessity of some degree. What makes logical validity special among other kinds of validity is its "formal" necessity. A logical consequence relation has to be a formal relation in that it is independent of properties of objects that are known on the basis of experience. Tarski says that he has a proof that model-theoretic validity has the property of the necessity and formality,[5] but he does not provide it.

$\vdots$

are contained as its theorem, but the general statement

A. Every natural number possesses the given property P,

cannot be proven using the normal rules of inferences. He claims that the sentence A seems to be a logical consequence of $A_0, A_1, \ldots$, because "[p]rovided all these sentences are true, the sentence A must also be true" (ibid., p. 411). However, generally, the ω-rule does not hold in a first-order theory. That is to say, this argument is invalid not only in the proof-theory of first-order logic but also in his model-theoretic semantics. Tarski's motivation thus seems not to be reflected even in the model-theoretic definition of the first-order logical validity.

I will not address these issues in this dissertation. Rather, I will focus on the problem of whether or not the contemporary notion of model-theoretic validity actually captures the two natures of logical consequence, i.e., necessity and formality. For these issues, see Bays[3], Etchemendy[28], [29], Gómez-Torrente[39], [42], Ray[90], and Sher[106].

[5] "In particular, it can be proved, on the basis of this definition, that every consequence of

Formality of Model-Theoretic Validity

The formality of model theoretic validity in Tarski's sense can be verified without difficulty. If an argument is model-theoretically valid, it holds no matter how extra-logical terms are interpreted. From the following model-theoretically valid argument about Socrates

All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

we can obtain a variety of arguments by replacing the name "Socrates" with another proper name or by replacing predicates "is a man" or "is mortal" with another predicate. And every obtained argument will still be model-theoretically valid. In this sense, model-theoretic validity does not depend on contents of arguments. Whether or not an argument is model-theoretically valid can be determined based solely on its logical form.

Necessity of Model-Theoretic Validity

How can the necessity of model-theoretic validity be shown? A possible argument for the necessity would be the following:

(P1) Structures $\langle \mathcal{D}, I \rangle$ cover all formally possible situations.

(P2) Provided that structures $\langle \mathcal{D}, I \rangle$ cover all formally possible situations, if an argument holds in any structures, then it is impossible that all the premises are true and the conclusion is false.

(P3) An argument is model-theoretically valid if and only if an argument holds in any structures.

From these three premises, the following conclusion can be derived:

(C) If an argument is model-theoretically valid, then it is impossible that all the premises are true and the conclusion is false. (That is to say, the model-theoretically valid argument necessarily holds.)

true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences" (Tarski[125], p. 417).

The premise (P2) is based on a pre-theoretic view of necessity that if an argument holds in every formally possible situation, then it holds necessarily. I suppose that this view is acceptable and can be taken as true. The premise (P3) is the definition of the model theoretic validity, and thus can also be taken as true.

According to this argument, therefore, the problem of the necessity can be reduced to the problem of whether the premise (P1) is true or false—i.e., whether or not structures $\langle \mathcal{D}, I \rangle$ actually cover all formally possible situations. The premise (P1) is crucially important. The main role that a structure plays with respect to the necessity of logical validity is to reject some arguments as logically invalid. Consider, for example, the following argument whose consequence relation is not necessary:

$$\begin{array}{rl} & \text{There are two objects satisfying the property } P. \\ \hline \therefore & \text{There are three objects satisfying the property } P. \end{array}$$

In order to invalidate this argument in the model-theoretic framework, structures whose domains contain exactly two objects satisfying P must be available; only those structures can be a counterexample to the argument. Without such structures, the argument could be wrongly regarded as valid. Generally, to reject an argument, there has to be some structure available that represents the situation in which the premises are true and the conclusion is false. To reject all arguments that hold contingently, the collection of model-theoretic structures has to cover all formally possible situations. That is, the premise (P1) has to be true.

Intuitively, there seems to be sufficiently many structures in model-theoretic semantics to represent all possible situations. Any possible set can be the domain of a structure and any possible interpretation of extra-logical terms can be realized in some structure. No formally possible situation would be missing. This intuition, however, needs careful consideration. In fact, the premise (P1) has been a major point of contention in the literature. Some philosophers have found the premise problematic: there is or could be missing structures that should be available for the model-theoretic semantics. As we have seen, model-theoretic validity captures formality. However, to show the appropriateness of the model-theoretic method for the characterization of prime logical validity, its necessity also has to be verified.

In what follows, I will consider two major criticisms of the premise (P1). For each criticism, I will provide a solution to defend the model-theoretic method.

Dependence on the Background Set Theory

The first criticism is that the set theory behind model-theoretic semantics could be another one.[6] A structure $\langle \mathcal{D}, I \rangle$ is composed of a non-empty set $\mathcal{D}$ and an interpretation function I. What structures there are depends on what sets there are. Wherever there is a set, there is a structure. Where there is not a certain set, there is not a structure corresponding to it. When we approach logic using the model-theoretic method, a set theory is employed. The employed set theory determines the collection of all possible sets and therefore all possible structures.

Let T and T' be two set theories that produce different collections of sets. There is a set X whose existence is guaranteed in T but not in T'. And there is a structure $S = \langle X, I \rangle$ with the domain X that is available in the model-theoretic semantics based on T but not in the model-theoretic semantics based on T'. Then, there might be an argument that can be invalidated in S but not in any structures the model-theoretic semantics based on T'. If our current set theory is T', then that our current model-theoretic semantics might fail to cover all formally possible situations. As a result, it might fail to invalidate all the argument that should be rejected from a logical point of view.

To illustrate the point, consider an extreme case of finitists (Etchemendy[29], pp. 119–120). The finitist thinks that there are only finitely many objects. There are only finite sets, and therefore there are only structures whose domains are finite. Now, let $\varphi_{<\infty}$ be the following sentence sentence:

$$[\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz) \land \forall x \neg Rxx] \to \exists x \forall y \neg Ryx,$$

where R is a binary relation symbol. $\varphi_{<\infty}$ is true in every finite model but false in some infinite models.[7] Thus, under the model-theoretic account of logical validity

[6] This criticism has been developed, for example, in Etchemendy[29] and McGee[70]. MacFarlane[62], Sher[105], [106] have provided arguments to defend model-theoretic validity from their criticisms.

[7] Let $\mathcal{D}_0$ be a finite domain and $\langle \mathcal{D}_0, I_0 \rangle$ be a structure. If the antecedent of $\varphi_{<\infty}$ is true by I_0, that is, if R is transitive and irreflexive (no loop occurs), then, there is a "starting point" of R. Thus, $\varphi_{<\infty}$ is true.

based on the finitist's set theory, $\varphi_{<\infty}$ is a logical truth and any argument whose conclusion is $\varphi_{<\infty}$ is logically valid.

However, there is a sense in which $\varphi_{<\infty}$ should not count as a logical truth. Although, as the finitist says, there are actually only finitely many objects, there *could have been* infinitely many objects, and there *could have been* infinite sets and structures with infinite domains. Even though there is no "actual" structure in which $\varphi_{<\infty}$ is false, there could be "possible" structures that falsify $\varphi_{<\infty}$. It seems that there is no reason that we have to limit ourselves to actual structures (with finite domains), rather than considering all possible structures (including the ones with infinite structures). Therefore, φ_{∞} is not a logical truth regardless of how many objects there are actually.

The problem here is not whether there are finitely many objects or infinitely many objects. The problem is that, under the modal-theoretic method, the logical validity of an argument varies according to the background set theory. Two different set-theories T and T' might prove the existence of different sets, and as a result there might be some argument that is valid in the model-theoretic semantic system based on one set-theory T' and invalid in the model-theoretic semantic system based on the other T. If our current theory is T', then our model-theoretic method will wrongly sanction the argument whose consequence relation is not a necessary relation.

In addition to this difficulty, there is another, but similar, difficulty, which is related to the formal-structural notion of logical validity. Under the formal-structural notion, the logical validity is validated by formal laws. A formal law is a law that can be applied to set-theoretic constructs. Thus, a formal law can be seen as a theorem of a set theory. Behind formal laws, there is a set theory, and it is the set theory that validates formal laws. Two different set-theories entail two different collections of theorems. Then, it might happen that an argument can be validated by a formal law that is in one set theory but not in the other. Logical validity, under the formal-structural notion and the minimal notion, depends on

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Let us now consider a structure $\langle \mathbb{Z}, I_{\mathbb{Z}} \rangle$, where $\mathbb{Z}$ is the set of integers. We suppose that $I_{\mathbb{Z}}(R) = \{\langle z_1, z_2 \rangle : z_1 < z_2\}$. In the domain, $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$ and $\forall x \neg Rxx$ are true, while $\exists x \forall y \neg Ryx$ is false. Thus, $\varphi_{<\infty}$ is false.

the background set theory.

If different arguments are sanctioned in different model-theoretic semantics based on different set-theories, model-theoretic validity cannot be regarded as producing a necessary consequence relation. No matter what our current set theory is, it seems to be conceivable that another set theory could govern the world. There are at least two legitimate set-theories that could be a base of the model-theoretic method. Our current set theory holds contingently but not necessarily. The contingency of our current set theory implies the contingency of model-theoretic validity. The model-theoretic method thus fails to capture one of the essential properties of logical validity, namely, necessity.

I think that this dependence problem can be (partially) solved by weakening one assumption behind the arguments above. What is assumed in the arguments is a one-to-one correspondence between the model-theoretic method and set theory. For the model-theoretic method, there is only one correct set theory that determines what sets exist, what structures there are, and what formal laws hold. In order to solve the dependence problem of necessity, I propose a one-to-many relation between the model-theoretic method and set-theories. Instead of assuming one correct set theory, we can suppose that there are plural legitimate set-theories, which can work behind the model-theoretic method together.

Let $T_0, T_1, \ldots, T_n$ be legitimate set-theories. Each T_i produces a collection D_{T_i} of sets whose existence is guaranteed in T_i. Then, we define a structure as a pair $\langle \mathcal{D}, I \rangle$ where $\mathcal{D}$ is in some D_{T_i}. Under this pluralistic position, available sets and available structures increase, and a model-theoretically valid argument is required to hold in all the available structures. In other words, if an argument can be rejected in a model-theoretic semantic system based on some legitimate set-theory, it is an invalid argument under the new definition of model-theoretic validity. Only arguments that are valid in every possible semantic system are taken to be model-theoretically valid. Two contradicting theories T_i and T_j can be employed as legitimate set-theories. The point is that we can avoid missing structures and missing formal laws by allowing to use multiple different theories.

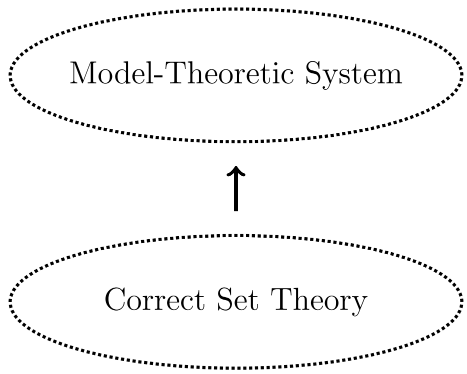

Figure 2.1: One-to-one correspondence

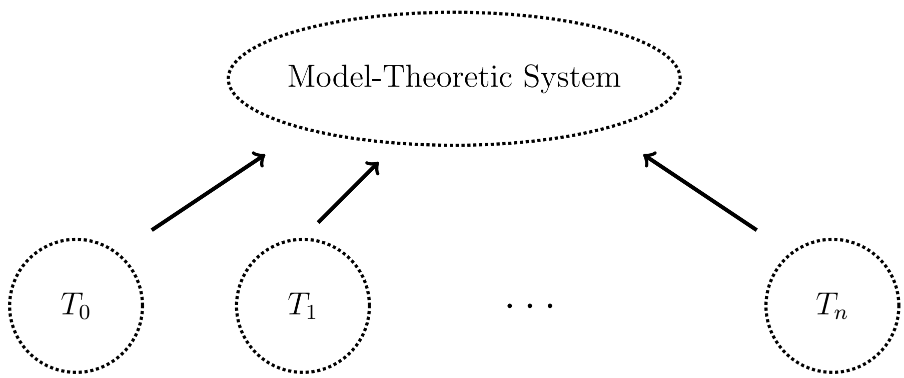

Figure 2.2: One-to-many correspondence

It is easy to see how the dependence problem can be overcome under this pluralistic position on the background set-theories. Let T be the set theory that correctly describes what sets are and correctly derives laws of sets that exist. Also, let T' be a set theory that correctly describes what sets could be and laws of sets that could exist. We suppose that there is a set X that is not in T but in T' and that there is a structure $\langle X, I_X \rangle$ and an argument $\langle \Gamma, \varphi \rangle$ such that $\langle \Gamma, \varphi \rangle$ holds in any structures of the model-theoretic semantic system based on T but not hold in $\langle X, I_X \rangle$. Under the monistic position on the background set-theory that admits T as the correct set theory, the argument $\langle \Gamma, \varphi \rangle$ can be wrongly regarded as model-theoretically valid, because there is no available structure in which all the premises in Γ are true and the conclusion φ is false. Under the pluralistic position, however, it is not a valid argument; it can be correctly invalidated in the structure $\langle X, I_X \rangle$.

The problem of the contingency of formal laws can be answered in a similar way. If an argument is valid under the new definition of model-theoretic validity based on the pluralistic position, then it is validated by a formal law that holds in any possible legitimate set-theories. Thus, we can regard the consequence relation between its premises and conclusion as a necessary relation. Under the new position, an argument holds by virtue of some formal law, which themselves holds necessarily.

In this way, the dependence problem can be solved by accepting multiple background set-theories. It has to be recognized, however, that the solution provided above is a partial solution. The solution will be regarded as complete after legitimate set theories are identified based on a principled characterization. I believe that ZFC, which is the set theory that is commonly supposed to the background set theory of the standard model-theoretic semantics, can count as legitimate. However, what other theories about sets can be taken as legitimate are not known: a necessary and sufficient condition for a theory of sets to be a set-theory that can be employed in the model-theoretic method has not been obtained yet. I will not go into this problem in this dissertation, because in order to answer the problem, a comprehensive philosophical theory of what sets are is necessary.

Proper Class Domain

The second criticism of the necessity of the model-theoretic semantics is that there might be arguments such that in order to validate or invalidate them, considering structures with a "proper class domain" is necessary.[8] Consider the following argument:

All sets are mathematical objects.
All mathematical objects are abstract objects.
∴ All sets are abstract objects.

This argument is about sets. In order to determine the truth value of the conclusion under the two assumption that the premises are both true, variable of the universal quantifier "all" has to range over collections containing all sets. Thus, to judge

[8] The criticism is discussed, for example, in Etchemendy[29], Field[33], Hanson[47], [48], Sher[105], [106].

the logical validity of the argument, we need check if it holds in structures whose "domains" contain all sets. As is well known, however, a collection containing all sets itself is not a set. Therefore, in the model-theoretic method, the logical validity of the argument cannot be determined, because such a collection is not available as the domain of a structure (remember that a domain is a non-empty "set"). There might be an argument that holds in every structure whose domain is a set but can be invalidated in some structure whose domain is a proper class. If that is the case, the model-theoretic consequence relation of the argument will turn out to be a relation that is not necessary.

To claim that the model-theoretic method only sanction necessary consequence relations, we have to argue that if an argument $\langle \Gamma, \varphi \rangle$ is model-theoretically valid, then there is no structure whose domain is a proper class in which the premises in Γ are true and the conclusion φ is false. This can be actually shown for arguments in a first-order language (Kreisel[57] and Sher[105]). Let $\langle \Gamma, \varphi \rangle$ be a model-theoretically valid argument of first-order logic. By the completeness theorem of first-order logic, there is a formal proof $\langle \varphi_0, \varphi_1, \ldots, \varphi \rangle$ of the argument $\langle \Gamma, \varphi \rangle$. Consider then a standard proof-theoretic system of first-order logic. It seems that every axiom of the system are necessarily hold regardless of the size of a domain. For example, sentences of the form "$\forall x \varphi(x) \rightarrow \varphi(t)$"[9] seem to be true in a structure whose domain is the collection of all set, because if something is true for any set, it has to be true for a particular set. Inference rules seem to hold necessarily as well. Modus ponens as an inference rule is independent of the size of a domain. And the inference rule deriving $\psi \rightarrow \psi'$ from $\psi \rightarrow \forall x \psi'$, where x does not occur freely in ψ', is necessarily truth-preserving, because ψ' is a sentence and therefore the attached quantifier part "$\forall x$" in "$\forall x \psi'$" does not have any effect on the truth value of $\forall x \psi'$. It then can be concluded that there is no structure, whatever its domain size is, in which all sentences in Γ is true but φ is false. Therefore, $\langle \Gamma, \varphi \rangle$ hold necessarily.[10]

The arguments above consists in two statements:

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[9] t is a term containing no variable. Remember our Restriction 4, p. 21, that we do not deal with open formulas.

[10] Sher's formulation of the argument takes a slightly different form the one given here, but they are essentially the same.

(i) If an argument $\langle \Gamma, \varphi \rangle$ is model-theoretically valid, then it is proof-theoretically valid.

(ii) If an argument $\langle \Gamma, \varphi \rangle$ is proof-theoretically valid, then it does necessarily hold.

The statement (i) is the completeness theorem. The correctness of the statement (ii) is based on our intuitive sense that all axioms and inference rules do necessarily hold.

Notice that the argument above, though it is about sets, can be expressed in a first-order language:

$$\begin{array}{rl} & \forall x[Px \to Qx]. \\ & \forall x[Qx \to Rx]. \\ \hline \therefore & \forall x[Px \to Rx]. \end{array}$$

Thus, according to the proof above, any structures including structures whose domains are proper classes cannot be a counterexample to the argument. The consequence relation between its premises and conclusion is necessary. The proper-class-domain problem can be solved for arguments in a first-order language.

The problem, however, still remains. Although the completeness theorem holds in first-order logic, it does not for higher-order logics, in particular, for second-order logic with standard semantics.[11] Thus, we cannot apply the proof above to second-order logic. For a proof-theoretic system of second-order logic, there are infinitely many sentences that are true in all standard structures but cannot be derived by using its inference rules. Even if we can reasonably assume that all the axioms are necessarily true and all the inference rules are necessarily truth-preserving, there will still be many valid arguments whose necessity needs to be shown.

> Let $\mathcal{L}$ be a standard first-order system, L the language of $\mathcal{L}$, K a set of sentences of L and X a sentence of L. Suppose it is intuitively possible that all the member of K are true and X is false. Then, if we presume that the rule of inference of standard first-order logic are necessarily truth-preserving, $K \cup \{\sim\text{X}\}$ is intuitively consistent in the proof-theoretic sense: for no first-order sentence Y are both Y and $\sim$Y provable from $K \cup \{\sim\text{X}\}$. It follows from the completeness theorem for first-order logic that there is a model for $\mathcal{L}$ in which all the sentences of K are true and X is false. (Sher[105], p. 42)

[11] Recall our Restriction 3 at p. 21, that we restrict our concern to logical systems for first- or second-order languages.

What the incompleteness of second-order logic implies is the possibility that the model-theoretic method, as it stands, might fail to sanction only necessary consequence relations. There might be some argument in a second-order language that holds in any set domains but not in some proper class domain. Such an argument is validated in the standard model-theoretic semantics of second-order logic but cannot be thought of as necessary. The cause of the problem is obvious. The domain of a structure, in the standard definition of structures, is supposed to be a non-empty-set but not a non-empty class.

The solution is also obvious, however: to extend the definition, from the one that only accept structures whose domains are non-empty sets to the one that accept structures whose domains are non-empty classes. There are theories about classes that are available for the solution. An example is Von Neumann-Bernays-Gödel set theory. In the theory, classes are taken as the basic objects defined by axioms, and a set is defined as a class that is a member of some other class. Let us T be a theory of classes. We can then define a structure as a pair $\langle \mathcal{C}, I \rangle$ where $\mathcal{C}$ is a class whose existence can be proved in T and I is an interpretation function I for extra-logical terms. A model-theoretically valid argument is defined as an argument that hold not only in structures whose domains are sets but also in structures whose domains are proper classes. model-theoretic validity thus can be extended in such a way that its necessity are inherently embedded in it.

Is Von Neumann-Bernays-Gödel set theory really appropriate for the extension of the definition of structures? What is a necessary and sufficient condition for a theory of classes to be employed in the model-theoretic method? These questions have to be answered for the obvious solution above to be accepted. (And these questions can be seen as analogous to the question we asked when we discussed the first criticism of the model-theoretic approach to logic. “What is a legitimate theory of sets?” p. 38) However, what is important for the problem of the necessity of the model-theoretic method (and what is more important than discussions of what a set is or what a class is) is the possibility of the extension. Whether a counter-example to an argument can be found in a set-domain structure or in a proper-domain structure does not really matter. What matters is that we can define a model-theoretic semantics, at least in principle, in such a way that the counter-example can actually be found in it. We can suppose that the model-theoretic method with an appropriate background theory of classes validates only necessary consequence relations. If an argument $\langle \Gamma, \varphi \rangle$ is model-theoretically valid, the consequence relation between Γ and φ necessarily holds.

Counterexamples

The two criticisms of the model-theoretic method above—the criticism concerning the background set theory and the criticism concerning proper class domains—are directed toward the claim that structures $\langle \mathcal{D}, I \rangle$ cover all formally possible situations. If there is some structure that is not represented in the model-theoretic framework, we cannot suppose that model-theoretic validity captures the necessity of logic. We have argued above that the criticisms can be overcome.

There is another kind of criticisms of the model-theoretic method: there are actually arguments that are model-theoretically valid but not necessary. That is, counterexamples to the necessity of model-theoretic validity exist. We will examine two examples given in Zalta[137]:

(i) $\mathcal{A}\varphi \to \varphi$;

(ii) $P(\iota x)Qx \to \exists y Q y$.

Here, "$\mathcal{A}$" in (i) is a unary modal operator in a sentential modal language, which is intended to mean "It is actually the case that," in other words, "It is true at the actual world that." Thus, the sentence (i) means that if φ is true at the actual world then φ is true. "$(\iota x)Qx$" in (ii) is a definite description phrase, which is intended to mean "*the* x that uniquely satisfies Q at the actual world." Thus, $P(\iota x)Qx$ means that the x that uniquely satisfies Q at the actual world also satisfies P, and the whole sentence (ii) says that if the x that satisfies Q at the actual world also satisfies P, then there is an object y that satisfies Q. Zalta claims that these sentences are true in any structures but their model-theoretic validity is not necessary.

Let us first consider the sentence $\mathcal{A}\varphi \to \varphi$. Let L be a sentential modal language. A *propositional modal structure*, is a four-tuple $\langle W, R, w_0, v \rangle$, where W

is a non-empty set of possible worlds, R is a binary relation on W (an accessibility relation between possible worlds), w_0 is a member of W (the actual world of the structure), and v is a truth-assignment function that assigns a truth value, T or F, to pairs $\langle w, p\rangle$ of a possible world $w \in W$ and an atomic sentence p. v can be extended, in a standard way, to a function $\bar{v}$ that assigns a truth value to pairs of $\langle w, \psi\rangle$, where ψ is any sentence (atomic or complex). The truth condition of a sentence $\mathcal{A}\psi$ is the following: $\bar{v}(\langle w, \mathcal{A}\psi\rangle) = \mathsf{T}$ if $\bar{v}(\langle w_0, \psi\rangle) = \mathsf{T}$. That is to say, $\mathcal{A}\psi$ is true at a world w if ψ is true at the actual world w_0.

Zalta says that a sentence ψ is *logically true* if for every propositional modal structure $\langle W, R, w_0, v\rangle$, we have $\bar{v}(\langle w_0, \psi\rangle) = \mathsf{T}$. That is, a logical truth is a sentence that is true at the actual world of any propositional modal structures. Note that this definition of logical truth is different from the definition of logical truth that is employed in the contemporary study of modal logic.[12] In the standard framework of modal logic, a sentence is said to be logical true if it is true at any world (actual or possible) in any propositional modal structure. Zalta's definition of logical truth and the standard definition of logical truth are different, and this is a key assumption to show the existence of model-theoretically valid, but contingently true, sentences.

It can be easily observed that $\mathcal{A}\varphi \to \varphi$ is a logical truth in Zalta's sense. Suppose $\mathcal{A}\varphi$ is true at w_0. Then by the definition of $\mathcal{A}$, we have φ is true at w_0. Thus, the whole sentence $\mathcal{A}\varphi \to \varphi$ is true at w_0. Therefore, it is logically true.

However, Zalta says, $\mathcal{A}\varphi \to \varphi$ is not a necessary truth, because $\Box(\mathcal{A}\varphi \to \varphi)$ is not logically true. Consider a propositional modal structure $\langle W', R', w'_0, v'\rangle$ such that $W' = \{w'_0, w'_1\}$ and $R' = \{\langle w'_0, w'_1\rangle\}$. We suppose that for an atomic sentence

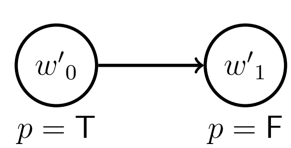

Figure 2.3: Propositional modal structure $\langle W', R', w'_0, v'\rangle$

[12]Zalta, however, points out that his definition is identical to the one Kripke gave in his original work on modal logic.

p, $v(\langle w'_0, p\rangle) = \mathsf{T}$ and $v(\langle w'_1, p\rangle) = \mathsf{F}$. $\mathcal{A}p$ is true at w'_1, but p is false at w'_1. Therefore, $\mathcal{A}p \to p$ is not true at w'_1 and $\Box(\mathcal{A}p \to p)$ is not logically true. Zalta concludes from this that $\mathcal{A}p \to p$ is not a necessary truth.

The logicality of the second sentence $P(\iota x)Qx \to \exists yQy$ and its contingency can be shown in the same way. An *objectual modal structure* is a four-tuples $\langle D, R, d_0, I\rangle$, where D is a non-empty set of possible domains, R is an accessibility relation between possible domains, d_0 is a member of D (the actual world domain of the structure), and I be an interpretation function for extra-logical terms. I assigns a set-theoretic construct to a extra-logical term at each possible domain. Suppose that $P(\iota x)Qx$ is true at d_0. Then, there is an object o that uniquely satisfies Q at d_0. o also satisfies P. Because of the existence of o, the consequent $\exists yQy$ is true at d_0. Consequently, $P(\iota x)Qx \to \exists yQy$ is logically true.

However, it is not necessarily true. Consider an objectual modal structure $\langle D', R', d'_0, I'\rangle$ such that $D' = \{d'_0, d'_1\}$ and $R' = \{\langle d'_0, d'_1\rangle\}$. We also suppose that there is an object o' such that o' uniquely satisfies Q at d'_0 and satisfies P at d'_1. In addition, we suppose that no object in d'_1 satisfies Q. In this objectual modal structure, the antecedent $P(\iota x)Qx$ is true at d'_1 is true but $\exists yQy$ is false at d'_1. Therefore, $\Box(P(\iota x)Qx \to \exists yQy)$ is false. $P(\iota x)Qx \to \exists yQy$ is not a necessary truth.

An objection against Zalta's arguments can be immediately raised: his definition of logical truth is inappropriate. The standard definition of logical truth is that a logical truth is a sentence that is true at every world in every modal structure. Being true at the actual world is just a necessary condition. For a sentence to be regarded as logically true, it has to hold not only at the actual world but also at any other possible world. If a sentence φ is true at any possible world in any modal structure, the sentence $\Box\varphi$ is also true at any possible world in any modal structure. Hence, we can conclude that if a sentence is logically true, then it is necessarily true.

Which definition of logical truth is correct? Zalta claims, as a matter of course, that his definition is the right one. According to him, "the most important semantic definition for a language is the definition of truth under an interpretation"

(Zalta[137], p. 66). "An interpretation" here means a modal structure in our terminology. That is, the most important definition of a semantic system for a modal language is the definition of truth in a modal structure. And using the definition, a logical truth should be defined as a sentence that is true in every modal structure. In his definition, the notion of truth in a modal structure can be defined as a truth at the actual world. As a result, a logical truth is defined as a sentence that is true at the actual world of every modal structure. Having a distinguished world (namely, having the actual world) in a modal structure is a necessary condition for the modal semantic system to be able to define a logical truth.

This cannot be done in the "standard" definition we normally use in the contemporary study, because in the standard definition, no world is distinguished as the actual world from other possible worlds. If no world in a modal structure has a special status, we cannot define the notion of truth in a modal structure. In such a case, it does not make sense to say that the sentence φ is true in a modal structure. Consequently, the standard definition does not properly capture the nature of logical truth.

This Zalta's argument for his definition is weak. Although we can agree with his definition that a sentence is said to be true in a modal structure if it is true at the actual world, there seems to be no reason to accept his definition of logical truth. We can define a truth in a modal structure as he does on the one hand, we can define a logical truth, as opposed to his definition, as a sentence that is true not only at the actually world but also at any possible worlds in any modal structures on the other. A logical truth has been suppose to be true at any possible situations. The actual world is just one of the possible situations, and there is no distinction among them from a logical point of view. It then seems that a sentence that holds at any possible worlds in any modal structures is more appropriate for the label "logical truth" than a sentence that holds just at the actual world in any modal structure. The set of all sentences of the former kind is a proper subset of the set of all sentences of the latter kind. Why can't logicality be attributed only to the sentences of the former kind? Why should the sentences of the latter also be taken as logically true? Zalta fails to provide a convincing argument that the definition of logical truth has to be his but not the standard one.

Moreover, Zalta's argument is based on another controversial assumption, which is that the modal operator $\mathcal{A}$ and the definite description operator ιx are logical constants. If they are not logical constants, then their meanings can vary according to modal structures. For example, in one propositional modal structure, "$\mathcal{A}$" can mean the possibility operator $\Diamond$, and in another propositional structure, it can mean the necessity operator $\Box$. Then, if no restriction is imposed on the accessibility relation, $\mathcal{A}\varphi \to \varphi$ will no longer be a logical truth even in his definition, because $\Diamond\varphi \to \varphi$ and $\Box\varphi \to \varphi$ are not true at the actual worlds in some propositional modal structures such as the following one $\langle W'', R'', w''_0, v'' \rangle$ with the actual world w''_0:

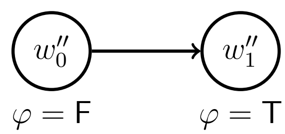

Figure 2.4: Propositional modal structure $\langle W'', R'', w''_0, v'' \rangle$

Showing the logicality of the terms $\mathcal{A}$ and ιx, therefore, is necessary for his claim of the existence of an unnecessary logical truth.

Zalta points out that part of the traditional conception of logical constants are that a logical constant is a term that is "evaluated in the recursive clause of the definition of truth" (Zalta[137], p. 67). The truth of standard sentential connectives of classical logic $(\neg, \wedge, \vee, \to, \leftrightarrow)$ depends only on the truth of their component sentences. Likewise, the truth of the sentence $\mathcal{A}\varphi$ depends only on the truth of the component sentence φ, i.e., whether or not φ is true at the actual world. With respect to this particular aspect, there is no crucial difference between the way of assigning a truth value to the sentence $\mathcal{A}\varphi$ and the way of assigning a truth value to the sentence, say, $\neg\varphi$. The negation $\neg$ has been taken as a logical constant. Then, it seems reasonable to suppose that the operator $\mathcal{A}$ is also a logical constant.

As Zalta says, the possibility of being evaluated in the recursive clause of the definition of truth might be one possible condition for a term to be regarded as logical. However, the characterization of logical constants cannot be given in terms of the possibility alone. True, many terms that have been taken as logical meet the condition. But, for the characterization, one has to be able to explain why the possibility guarantees its logicality. Why can a term be regarded as a logical constant if its truth condition evaluated in the recursive clause of the definition of truth? What is the relationship between the special status of a term and the special way of giving its truth condition? The possibility itself cannot be a basis of the logicality. The possibility is to be derived from a philosophically justifiable idea on what logicality is.

What is lacking in the Zalta's argument is this fundamental idea. Without a solid idea of logicality, an appropriate characterization of logical constants will not be obtained. In Chapter 3, I will propose a new idea of logicality and a new characterization of logical constants. The new criterion is based on the minimal notion of logical validity. Using it, I will show that the modal operator $\mathcal{A}$ and the definite description operator ι are not logical constants. If the criterion is correct, we can conclude that the two sentences $\mathcal{A}\varphi \rightarrow \varphi$ and $P(\iota x)Qx \rightarrow \exists yQy$, as opposed to his claim, are not logical truths.

Other Properties of Logical Validity

So far, we have discussed the appropriateness of the model-theoretic method for the characterization of prime logical validity. We have shown that any model-theoretically valid argument can be grounded in some formal laws. We have also shown that model-theoretic validity is formal validity. Then, we have examined three criticisms of the necessity of model-theoretic validity. We have argued that all of the criticisms can be answered. By the arguments, I conclude that we can suppose that the consequence relation between the premises and conclusion of any model-theoretically valid argument is necessary.

Formality and necessity are two properties among many characteristic properties of logical validity. Thus, some might rightly point out that, in order to claim that the model-theoretic validity is appropriate for the characterization of prime logical validity, we need to also show that model-theoretically valid arguments satisfy other properties as well. Example of the properties that have been traditionally attributed to logically valid arguments are generality, apriority, analyticity, topic neutrality, and normativity.

There are arguments that some of these properties can be explained in terms of formality and necessity.[13] These two properties can be thought of as fundamental properties of logical validity and other properties are derivable from them. For example, a model-theoretic valid argument can be thought of as general and also topic-neutral due to its formality: the argument holds independent of any particular fact or any particular principle of the world other than formal aspects of it. In addition, an inference based on a model-theoretic valid argument is normative, primarily because the conclusion necessarily holds ("necessarily" in the strongest sense), provided that the premises are actually absolutely certain. The negation of the conclusion cannot be true without violating a formal law. Thus, inferring the negation of the conclusion does not make sense.

Sher claims that analyticity is incompatible with her formal-structural notion. Model-theoretically valid arguments hold by virtue of formal laws, and formal laws themselves hold by virtue of the formal structure of the world but not by virtue of the meanings of terms expressing them. Formal laws stand in the reality as physical laws do, but not in the language use. According to her, a change in formal laws might happen. That is, some formal laws that are believed today might turn out to be false in the future. Of course, formal laws are not affected by most discoveries in scientific research. But, she points out a possible scenario that some scientific observations might find certain peculiar behaviors of objects (or physical states) which are radically different from what we can find elsewhere. In such a scenario, we might have good reason to believe that formal laws concerning those

[13] See, for example, Sher[111] (p. 259), [116] (pp. 316–318) and [118] (p. 362).

behaviors need to be corrected to the right ones.[14] If the scenario is possible as she supposes, then formal laws can no longer be justified in terms of the meanings of terms. In addition, they cannot be seen as (fully) apriori, because some (correct) formal laws might be obtained based on empirical evidence. Instead, she says that model-theoretic validity is *quasi*-apriori.

The Justification Problem

Although I agree with the fundamental ideal of the formal-structural notion that logically valid arguments hold by virtue of formal laws, I think that the truth of these Sher's claims should be judged based on further consideration of formal laws. In fact, whether or not a model-theoretically valid argument holds apriori and analytically is a part of a fundamental problem of logic: How can the logical validity of an argument be justified? If an argument is of a certain logical form, it holds certainly and necessarily, its consequence relation can be known a priori (at lease in most cases), and deriving the conclusion from the premises can be thought of as normative. Just having a certain form of an argument guarantees its having special characteristic properties. How can this fact be explained? By virtue of what does an augment of a certain logical form have a special type of validity? I call this problem of logical validity the *justification problem*. The justification problem has been crucially important in the philosophy of logic. Under the formal-structural notion, the justification of a logically valid argument can be reduced to the justification of the formal laws that validates it.

Regarding the justification of formal laws, there are three possible views: (i) the "logic-in-the-world" view; (ii) the "logic-in-the-brain" view; (iii) the "logic-in-the-meaning" view. Sher advocates the "logic-in-the-world" view that formal laws are in the world. As the world has the physical structure (or at least we normally

[14] Thus, formal laws are subject to revision. Sher shares the idea of the revision of logically valid arguments with Quine:

> Revision even of the logical law of the excluded middle has been proposed as a means of simplifying quantum mechanics; and what difference is there in principle between such a shift and the shift whereby Kepler superseded Ptolemy, or Einstein Newton, or Darwin Aristotle? (Quine[86], p. 40)

so suppose), the world has the formal structure. And as the physical structure does not depend on our way of thinking or conceiving the world, the formal structure is independent of us. Formal laws hold due to this formal structure in the same sense that physical laws holds due to the physical structure of the world (but in different degree). The formal law of the form of modus ponens holds, because the world is/must be so structured that whenever the premises $\varphi \to \psi$ and φ are true, so is the conclusion ψ. The world forms formal laws.

The second possible view of the justification of formal laws is the logic-in-the-brain view, which is that formal laws are grounded in facts about the structure and functions of the human brain (more generally, of the brains of rational animals). Consider the formal law that if the intersection of two sets X and Y is nonempty, then X is nonempty. According to the logic-in-the brain view, our brain is so configured and functioning that whenever it cognizes the nonemptiness of the intersection of X and Y, it also cognizes the nonemptiness of X. In other words, the certainty of the formal law can be explained in principle in terms of physical features of our brain. Seemingly, the world appears to own the formal law. But, this is not the case. It is the structure of our brain that gives rise to the strong connection between the two nonemptinesses.

Robert Hanna holds this view. Formal laws are justified not by the world but by the brain. Some physical features of our brain guarantee the following-from relation between the nonemptiness of the intersection X and Y and the nonemptiness of X. The justification of the law by the brain structure is one consequence from Hanna's whole theory of logic called *logical cognitivism*, one of the central claims of which is that logic is cognitively constructed by rational animals[15]:

[15] Logical cognitivism, as Hanna defines it, is the view expressed by the conjunction of two claims. One of them is the one about the logical faculty in the human brain mentioned here and the other is the following claim about the connection between logic and rationality:

> *[R]ational human animals are essentially logical animals*, in the sense that a rational human animal is defined by its being an animal with an innate constructive modular capacity for cognizing logic, a competent cognizer of natural language, a real-world logical reasoner, a competent follower of logical rules, a knower of necessary logical truths by means of logical intuition, and a logical moralist. (Hanna[46], xviii)

> To say that logic is cognitively constructed by rational animals[16] is to say that rational animals—including all rational humans—possess a cognitive faculty that is innately configured for representing logic and is the means by which all actual and possible logical systems are constructed. (Hanna[46], p. 25)

According to logical cognitivism, we, rational humans, have a special cognitive faculty. The cognitive faculty is presupposed by our language faculty and is innately set up for representing our logical system.

The third possible view is the logic-in-the-meaning view that formal laws are true by virtue of meanings of words expressing them. The formal law that if the intersection of X and Y is nonempty then X is nonempty can be applied to triples $\langle \mathcal{D}, X, Y \rangle$. Then, the law can be expressed in terms of the triples as the following:

If some object in $\mathcal{D}$ is in X *and* the object is Y, *then* some object in $\mathcal{D}$ is in X.

The law holds mainly due to the meanings of the terms "If-then" and "and". A similar analysis can be given for any formal law. Consider, for instance, a logical truth "$\forall x P(x) \rightarrow P(c)$" can be validated by a formal law that can be applied to triples $\langle \mathcal{D}, a, X \rangle$:

If all objects in $\mathcal{D}$ is in X, *then* $a \in \mathcal{D}$ is in X.

The law holds mainly by virtue of the meaning of "If-then" and "all".

Where is a formal law? Is it in the world, in the brain, or in the meaning of terms? Or, can we say that it is both in the world and the brain or in another combination of them? In order to obtain a deep understanding of the nature of logic, we have to identify where formal laws are located. Indeed, the three views differ in terms of analyticity and apriority of logical validity. As Sher claims, under the logic-in-the-world view, logically valid arguments cannot be regarded as analytic or fully apriori. The logic-in-the-mind view holds, as the logic-in-the-

[16] Hanna characterizes rational animals as "conscious, rule-following, intentional (that is, possessing capacities for object-directed cognition and purposive action), volitional (possessing a capacity for willing), self-evaluating, self-justifying, self-legislating, reasons-giving, reasons-sensitive, and reflectively self-conscious—or, for short, 'normative-reflective'—animals, whose inner and outer lives alike are sharply constrained by their possession of concepts expressing strict modality" (Hanna[46], xv).

world view does, that logical validity is not analytic validity, but it implies that the consequence relation of a logically valid argument is apriori. We experience the world in such a way that logically valid arguments hold: logical validity is prior to our experience. According to the logic-in-the-meaning view, logically valid arguments are analytic and therefore are independent of experience, i.e., apriori. Characteristic properties that can be attributed to logical validity vary according to how formal laws are justified.

The justification problem is a huge problem, as huge as the characterization problem of logically valid arguments (the main problem of the dissertation research). To answer the justification problem, a philosophical investigation of laws in general, a scientific research of human brain, and a consideration of analyticity are necessary, which are beyond the scope of this research project. Thus, I will not address the justification problem in this dissertation. Throughout the dissertation, I will just suppose that formal laws exist and that they are always true without clarifying the source of their certainty.

2.3 Proof-Theoretic Validity

We now turn to the examination of proof-theory. In the Hilbert-style proof system of a logical system, validity is defined in terms of the concepts of derivability: an argument $\langle \Gamma, \varphi \rangle$ is *proof-theoretically valid* if φ is derivable from Γ using axioms and inference rules, in other words, there is a formal proof of it. The problem is whether or not the proof-theoretic method is appropriate for the characterization of prime logical validity.

Soundness

What argument is a proof-theoretically valid depends on our choice of axioms and inference rules. Generally, we cannot introduce axioms and inference rules into our proof-theoretic system as we like. If we employed the inference tules of the problematic connective tonk defined by the following inference rules,

$$\frac{\varphi}{\varphi \ tonk \ \psi}\ tonk\text{-I} \qquad\qquad \frac{\varphi \ tonk \ \psi}{\psi}\ tonk\text{-E}$$

our system would be able to validate any arguments with some premises, provided that validity is a transitive consequence relation.

Axioms and inference rules are normally required to be truth-preserving so that the proof-theoretic method only validates truth-preserving arguments, namely, model-theoretically valid arguments. Every axiom is a sentence that is true in all structures, and every inference rule is of the form of some model-theoretically valid argument. This required property is called *soundness*: a proof-theoretic system is *sound* with respect to a model-theoretic system if every proof-theoretically valid argument is model-theoretically valid.

Formal Laws and Characteristic Properties

Whether or not the proof-theoretic method is appropriate for the characterization of prime logical validity has to be determined based on the same standards as the ones used for the examination of the model-theoretic method. We asked whether or not a model-theoretically valid argument can be validated by some formal laws, and obtained an affirmative answer. We also asked whether or not model-theoretic validity possess the properties that have been attributed to logical validity such as formality, necessity generality, apriority, analyticity, topic neutrality, and normativity. We have argued that model-theoretically valid arguments are formal, necessary, general, topic neutral, and normative and that we can suppose that they are apriori and analytical under some view of formal laws. Do proof-theoretically valid arguments, like model-theoretically valid arguments, hold by virtue of formal laws? Do they satisfy the characteristic properties? Affirmative answers to these questions are necessary for the appropriateness of the proof-theoretic method.

The answers can be easily obtained due to the soundness. If an argument is proof-theoretically valid, then it is model-theoretically valid. Therefore, if an argument is proof-theoretically valid, then there is a formal law that validates it. Also, if an argument is proof-theoretically valid, it satisfies the characteristic properties because of its model-theoretic validity.

Under the formal-structural notion and the minimal notion, the proof-theoretic method can be seen as the second method. The existence of formal laws that validate proof-theoretically valid arguments is reduced to the existence of formal laws that validate model-theoretically valid arguments. This would not be surprising if we understood that formal laws are inherently semantical. A formal law is a law governing formal operators representing formal properties. A formal operator representing a formal property is the characteristic function of the property, which maps a set-theoretic construct to the truth value T if it satisfies the property and to F otherwise. A formal law thus can be defined in terms of semantic concepts such as satisfaction and truth. The sentence $\forall x P(x) \rightarrow P(c)$ is model-theoretically true, and it is easy to identify the formal law justifying its validity, which is the formal law applicable to $\langle \mathcal{D}, a, X \rangle$ that if X contains every object in $\mathcal{D}$, then $a \in \mathcal{D}$ is a member of X. However, it seems to be difficult to find the law without knowing or using its model-theoretic validity. This fact, I think, can be one reason for thinking that the model-theoretic method is more appropriate for the characterization of prime logical validity than the proof-theoretic method.[17]

2.4 Two Criticisms of the Proof-Theoretic Approach

There are two other reasons that the proof-theoretic method is not as appropriate as the model-theoretic method for the characterization of prime logical validity. Even though proof-theoretic validity is validated by formal laws, and even though it captures various properties that are supposed to be satisfied by logical

[17] I do not deny that the proof-theoretic method can be the primal method under another notion of logical validity. *Inferentialism*, for example, is a philosophical position that claims that logical notions such as logical constancy, logical truth, and logically valid augment should be defined in terms of inference. One fact that inferentialists emphasize is that inference are our activity. We *do* inference and our inference goes through step by step. In one step, we infer a statement, and in another step, we infer another statement. One inference always finishes in finite steps. Through finite steps and finite statements, we reach the conclusion. Logical validity should reflect this aspect of our intellectual activity. Under this notion, the existence of formal laws is just one necessary condition for an argument to be logically valid. The more important condition is the existence of finite processes to get to the conclusion, namely, the existence of a formal proof.

validity, the proof-theoretic approach is not adequate for our purpose.

Inapplicability to Other Kinds of Validity

The first reason is that the proof-theoretic method is not available for characterizing other kinds of validity than logical validity. The main objective of this dissertation is the characterization of prime logical validity. Prime logical validity is one kind of logical validity, which is the logical validity based on the minimal notion (a version of the formal-structural notion). Under the formal-structural notion, logical validity itself can be seen as one special kind of validity among various kinds of validity such as biological validity and physical validity. Biological validity is a kind of validity that holds by virtue of biological laws, and physical validity is a kind of validity that holds by virtue of physical laws. Similarly, logical validity is a kind of validity that holds by virtue of formal laws.

This similarity has to be reflected in the method of characterizing each validity. If a method is applicable to a characterization of logical validity, it also has to be applicable to those of other kinds of validity. If the proof-theoretic characterization of logical validity is appropriate, then biological validity and physical validity have to be able to be characterized within the proof-theoretic framework as well.

Indeed, the model-theoretic characterization can be applied to other kinds of validity. Consider, again, the following argument:

$$\begin{array}{ll} & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

This argument is biologically valid; it is biologically impossible that the premise is true and the conclusion is false. The biological validity of the argument can be explained in the model-theoretic framework as follows. Let L be a language that is designed to develop biology, and lets us suppose that a demarcation between biological terms and others is given. A biological term is a term that expresses a biological property such as manhood and mortality. The unary predicates "is a man," "is mortal," and the binary predicate "is the male genitor (the biological father) of" count as biological terms, while the terms "is a philosopher," "is rich" and "is a spouse of" do not. A *biological structure* is a pair $\langle \mathcal{D}, I \rangle$ of a non-empty set $\mathcal{D}$ of concrete objects and an interpretation function I such that:

(i) I assigns a biological term in L a set-theoretic construct satisfying the biological property expressed by the term:

(ii) I assigns an extra-biological term in L some set-theoretic construct of the corresponding type.

For example, in any structure $\langle \mathcal{D}, I \rangle$, I assigns the unary predicate "is a man" the set of all men in $\mathcal{D}$, and the binary predicate "is the male genitor of" the set of all pairs $\langle a, b \rangle$ of objects a, b such that a is the male genitor of b. The assignment for biological terms is constant. On the other hand, the assignment for extra-biological terms varies from domain to domain. The predicate "is a philosopher" is assigned the set of baseball players in some biological structure and the set of dogs in another biological structure. Then, biological validity can be semantically characterized as follows:

> An argument is *biologically valid* if its conclusion is true in every biological structure in which all its premises are true.

The definition of biological validity in terms of the semantic notion of truth-preservation is thus possible.

A problem with the proof-theoretic method is that the characterization of biological validity seems to be impossible using the concept of derivability. Let $\mathcal{P}$ be a proof-theoretic system of the language L above in which a demarcation between biological terms and others is given. $\mathcal{P}$ has its own axioms and inference rules. For example, the sentence "a living being is mortal" might be an axiom of $\mathcal{P}$, and the rule deriving the sentence "the living being is a mammal" from another sentence "the living being is a man" might count as a legitimate inference rule. The proof-theoretic characterization then says that an argument $\langle \Gamma, \varphi \rangle$ is biologically valid if φ can be derived from the sentences in Γ and axioms by applying inference rules.

This proof-theoretic characterization of biological validity, which might look like well-defined, involves a conceptual mistake: biological validity, in its essence, has nothing to do with the forms of arguments, but rather it has to do with the biologically properties spoken in them. The following man-mortal argument has a formal proof in $\mathcal{P}$:

$$\frac{\text{Socrates is a man.}}{\therefore\ \text{Socrates is a mammal.}}$$

However, its validity is not by virtue of the form of this argument: this argument holds because of the relationship between manhood and mammality (the former entails the latter). The opposite argument

$$\frac{\text{Socrates is a mammal.}}{\therefore\ \text{Socrates is a man.}}$$

does not have a formal proof in $\mathcal{P}$. Its invalidity is not due to the form of this argument: this argument does not hold because what it states violates the relationship between manhood and mammality. A biological consequence relation is a posteori and synthetic. It is not something that can be captured by particular forms of arguments.

There are two kinds of validity. One is a kind of validity such that an argument can be regarded as valid if it holds by virtue of its form, and the other is a kind of validity such that an argument can be regarded as valid if it holds by virtue of its content. Logical validity is an example of the former, while biological validity is an example of the latter. True, we can characterize logical validity in the proof-theoretic framework. However, the point here is that the proof-theoretic method is not allowed to be used for the characterization of logical validity under the minimal notion, namely, the characterization of prime logical validity. For, under the minimal notion, in which logical validity can be taken as one special kind of validity among others, if logical validity can be defined by a method, the method has to be usable for characterizations of other kinds of validity. If the proof-theoretic method cannot characterize biological validity, logical validity cannot be defined by that method. I do not deny that the proof-theoretic method is available or even should count as the primary method for the characterization of logical validity under some notion of logical validity. However, under the minimal notion, logical validity has to be characterized without using the concept of derivability.

Impossibility of the Characterization of Logical Constants

The second reason that the proof-theoretic method is not appropriate for the characterization of prime logical validity is that it seems to be impossible to determine the logicality of some significant terms in the proof-theoretic framework. As we have argued, the logical validity of an augment varies according to a demarcation between logical constants and other terms. Some argument is logically valid under one demarcation, but the same argument is not logically valid under another demarcation. We have seen an example that the validity of the argument $\langle\{\forall xPx\}, \exists xPx\rangle$ deriving $\exists xPx$ from $\forall xPx$ changes according to whether the universal quantifier ($\forall$) and the existential quantifier ($\exists$) can count as logical constants (p. 23). Moreover, we have seen, in the last section, that the demarcation will have a serious impact on the necessity of logical validity. If the actual modal operator $\mathcal{A}$ and the definite description phrase ιx are logical terms, then it is possible under a certain definition of logical truth to derive the controversial claim that there is a logical truth that is not necessary. Thus, the identification of logical constants is one of the most important parts of the characterization of logical validity. However, the proof-theoretic method cannot provide a complete list of logical constants.

Some philosophers claim that logical constants can be and should be defined in the proof-theoretic way. Their account, which was first developed by Prawitz and Dummett, has been referred to as *proof-theoretic semantics*. Proof-theoretic semantics is a formal semantics, which intends to provide an account of logical constants primarily in the proof-theoretic framework and ultimately an account of logical consequence as well. The basic idea about logical constants behind proof-theoretic semantics is that the meaning of a logical constant is defined in terms of inferential rules:

> The meaning of each [logical] constant is to be given by specifying, for any sentence in which that constant is the main operator, what is to count as a proof of that sentence, it being assumed that we already know what is to count as a proof of any of the constituents. (Dummett[26], p. 8)

For each logical connective, for example, there is a truth table that defines its truth condition and there are inferential rules that identify its proof condition. According to the account based on proof-theoretic semantics, the meaning of a logical connective can be specified by the proof condition rather than the truth condition.

Not every rule can characterize logical constants. As we have seen, inference rules for logical constants cannot be defined arbitrarily (remember Prior's connective tonk, p. 8 and p. 52). Some conditions have to be imposed on inference rules that determine the meanings of logical constants. One condition that has been often discussed in the proof-theoretic approach to logic is *conservativeness* proposed in Belnap[6]. Let $\mathcal{L}$ be a logical system for a language L and $\mathcal{L}'$ be a logical system for a language L'. The logical system $\mathcal{L}'$ is said to be an *extension of* the logical system $\mathcal{L}$ if they satisfy the following conditions:

(i) The vocabulary of L' contains the vocabulary of L. In particular, a constant term in L is also a constant term of L'. Also, a sentence in L is a sentence L';

(ii) If an argument $\langle \Gamma, \varphi \rangle$ in L is valid in $\mathcal{L}$, then it is also valid in $\mathcal{L}'$.

The extension of $\mathcal{L}$ is said to be *conservative* if for an argument $\langle \Gamma, \varphi \rangle$ in L, if it is valid in $\mathcal{L}'$, then it is also valid in $\mathcal{L}$. In other words, a conservative extension is an extension that preserves the validity of $\mathcal{L}$ and that do not expand it in the system $\mathcal{L}'$. If an argument is valid in $\mathcal{L}'$ but not valid in $\mathcal{L}$, then it involves some terms that are in L' but not in L. It can be easily observed that the extension of sentential logic by *tonk* is not conservative; the argument $\langle \{P\}, Q \rangle$ is valid in the extended logical system but invalid in sentential logic.

Is conservativeness a necessary and sufficient condition for a term to be regarded as a logical constant? Are there other conditions to be imposed on inference rules that we have to take into account for the proof-theoretic characterization of logical constants? These are important problems, in particular, for the proof-theoretic approach to logic. However, whatever the correct answers to them are, and whatever the necessary and sufficient condition is, the proof-theoretic method is inappropriate for the characterization of logical constants under the minimal notion. The reason is that there are uncountably many candidates for logical constants but there are only countably many inference rules.

An inference rule is composed of two parts: sentences to derive from and a sentence to be derived. Since there are countably many sentence in a formal language (remember Restriction 3, p. 21), there are countably many inference rules to characterize logical constants. Applying the possible necessary and sufficient condition to terms described by inference rules, we would be able to identify their logicality or non-logicality. However, there are a variety of, uncountably many, other terms whose logicality has to be examined. Examples of terms whose logicality or non-logicality is crucially important for the characterization of prime logical validity are quantifier terms "there are κ-many objects such that."

Consider, for example, the quantifier term $Q_{2^{\aleph_0}}$ "there are $2^{\aleph_0}$-many objects such that" and the quantifier term $Q_{\aleph_1}$ "there are $\aleph_1$-many objects such that." If they are both logical constants, and if the identity relation "=" is a logical constant, then the Continuum Hypothesis can be expressed only using logical terms:

$$Q_{2^{\aleph_0}}x(x = x) \leftrightarrow Q_{\aleph_1}x(x = x).$$

It then turns out that the truth of the Continuum Hypothesis can be determined based solely on the logical basis. However, this is controversial; the hypothesis is about particular sets, and therefore can be seen as belonging to set-theory not to logic. Is the Continuum Hypothesis a logical claim or a set-theoretical claim? In order to answer this question, it is necessary to examine the logical statuses of $Q_{2^{\aleph_0}}$ and $Q_{\aleph_1}$.

Since there are uncountably many cardinalities, there are uncountably many quantifier terms. The meanings of most quantifier terms cannot be specified by any inference rule. Therefore, their logical statuses cannot be determined by the proof-theoretic method alone, because no proof-theoretic necessary and sufficient condition for logical constants can be applied to them. As long as we are concerned with terms used in standard logical systems (e.g., negation, conjunction, disjunction, conditional, universal and existential quantifiers), the proof-theoretic characterization would produce the right result with respect to their logicality. However, for the characterization of prime logical validity, a complete list of logical constants is necessary, which cannot be obtained by examining inference rules.

Some logical constant might be absent from the list created in the proof-theoretic way.

Those who advocate the proof-theoretic method might then ask if the model-theoretic method could generate a complete list. Can the model-theoretic method provide a criterion that determines the logicality of any term? If it cannot, then the model-theoretic approach too has to be criticized for the same reason. We would have to conclude that the model-theoretic method is not appropriate for the characterization of prime logical validity. Some philosophers doubt the possibility of such a criterion. They have argued that the characterization of logical constants is impossible. Hanson, for example, thinks that the choice of logical constants depends on the goals we bring to logical inquiry (Hanson[47], p. 377). If whether or not a given term is a logical constant varies based on our purpose of a philosophical investigation using a logical system, there would be no logical constant simpliciter. Etchemendy claims that for some language, the demarcation between logical and non-logical terms cannot be successfully drawn (Etchemendy[29], p. 134). If this is correct, the logicality of certain terms, in principle, is undecidable.

However, it is also the fact that several criteria of logical constants that can be defined in terms of semantic notions have been proposed in the literature. And I believe that a criterion is actually possible (at least for the characterization of prime logical validity). In the next chapter, I will introduce a new criterion of logical constants and show that the model-theoretic method can solve the problem that the proof-theoretic method cannot.

3 Logical Constant

A characterization of prime logical validity is the main goal of this dissertation research. What arguments are logically valid or invalid under the minimal notion is the problem of greatest concern. As we have seen (p. 23), the logical validity of an argument varies according to a demarcation of logical constants. For our purpose, therefore, we have to identify all logical constants. A characterization of logical constants is the main objective of this chapter.

3.1 Logical Operators

A logical constant is a term that is assigned the same meaning in all structures, while an extra-logical term is a term whose meaning varies structure by structure. In any structure, the universal quantifier phrase "$\forall x$" has the same meaning "for all objects in the domain." But, a unary predicate "P", an extra-logical term, means "is a man" in one structure and "is mortal" in another structure.

In the contemporary model-theoretic study of logic, a characterization of *logical operators* has been supposed to be necessary for the characterization of logical constants. An *operator*, in the study, is a function that assigns truth values (T or F) to set-theoretic constructs (e.g., objects, sets of objects, and Cartesian products of sets) or tuples of propositions with truth values (all operators we will consider in this paper are finitary operators). For example, the function defined on a domain $\mathcal{D}$ that assigns an object T if it is red and F otherwise is an operator that is applied to objects. The function that assigns a set of objects T if it contains only finite objects and F otherwise is an operator that is applied to sets. Also, functions that can be applied to pairs of proposition are also operators. For example, the function that assigns a pair of propositions T if both of the propositions are true and F otherwise is an operator (a truth-functional operator).

The reason that the characterization of logical operators is fundamental for the characterization of logical constants is that in order for a term to be logical, it is necessary for it to have a logical operator as its semantic value. Among various operators, there are operators that can be regarded as clearly not logical. The red-operator above is a typical example; a logical operator, however it is defined, should be insensitive to whether an object is red or not. If a term has such a non-logical operator as its semantic value, it will not count as a logical constant. If a term is logical, it's semantic function will have to be defined by a logical operator. Prior to characterizing logical constants, therefore, we need to characterize logical operators.

Is a characterization of logical operators not only necessary but also sufficient for the characterization of logical constants? If the semantic value of a term is on the list of logical operators, can we count the term as a logical constant? In general, the answer is no. We have seen a term such that its semantic function is the same as that of the connective conjunction but it should not be regarded as a logical constant (p. 20):

$$\text{“}\varphi \circledcirc \psi'' \text{ is } \begin{cases} \text{true} & \text{if both “}\varphi\text{”and“}\psi\text{” are true,} \\ & \text{and Galileo believed that the earth moves;} \\ \text{false} & \text{otherwise.} \end{cases}$$

Remember, however, that we imposed a restriction on formal languages (Restriction 2, p. 20): we deal only with formal languages whose terms are rigid and the meaning of our terms are identified with their extension and therefore with their characteristic functions. Under this restriction, I believe, the sufficiency is actually true. For there is a one-to-one correspondence between terms and operators, and there seems to be no justifiable reason to reject the logicality of a term if it has a logical operator as its semantic value. I thus suppose that the following equivalence holds:

A term is a logical constant if and only if it has a logical operator as its semantic value.

This equivalence guarantees that we can characterize logical constants if we can characterize logical operators.

In the model-theoretic study of logic, the logicality of operators has been characterized in terms of the concepts of *invariance* and *similarity relation*: an operator is logical if it is invariant under "appropriate" similarity relations between structures. Several candidates have been proposed as appropriate similarity relations for the definition of logicality; as a result, there are several theories of logicality on the table. I agree that the concepts of invariance and similarity relation are essential for the characterization of logical constants. However, those existing theories have been established based on different notions of logical validity than ours, namely, the minimal notion). In what follows, I will provide another theory of logicality that reflects the minimal notion. In the process, I will explain why the existing theories are unsatisfactory from the point of view of the minimal notion.

3.2 Classical Logical Operators

There are two kinds of logical constants: (i) logical constants whose operators can be defined on domains of classical logic; (ii) logical constants whose operators can be defined domains of non-classical logic. The operators of terms used in a standard first-order or a standard second-order language are defined on domains that are independent of each other. As opposed to this, the operators of terms used in a modal language are defined on domains that are connected by accessibility relations. For these different kinds on operators defined on different types of domains, different approaches are inevitable. We call operators of the former kind *classical logical operators* and operators of the latter kind *non-classical logical operators*. Characterizing logical operators means characterizing classical and non-classical logical operators in a unified manner. They are both essential parts of a theory of logical constants. We will first define classical logical operator in this section. Non-classical logical operators will be identified in the next section.

The Isomorphism Invariance Criterion

The characterization of classical logicality that has been a basis for other proposed characterizations in the contemporary study is the characterization based on *the isomorphism invariance criterion*.[1] To precisely describe the criterion, we will define several concepts. An *objectual structure* of a domain $\mathcal{D}$ (a non-empty set of objects) is a tuple $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ where X_i is an object of $\mathcal{D}$ of a finite relational type.[2] A *similarity relation* is a collection of pairs of objectual structures. For domains $\mathcal{D}$ and $\mathcal{D}'$, if two structures $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ and $\langle \mathcal{D}', Y_1, \ldots, Y_m \rangle$ are similar with respect to a similarity relation S, we say that they are S-*similar* and write this as:

$$\langle \mathcal{D}, X_1, \ldots, X_n \rangle \; S \; \langle \mathcal{D}', Y_1, \ldots, Y_m \rangle.$$

Also, we say that n-tuples $\langle X_1, \ldots, X_n \rangle$ and $\langle Y_1, \ldots, Y_m \rangle$ are S-similar if this is not misleading (in particular, we often say objects X and Y are S-similar if $\langle \mathcal{D}, X \rangle \; S \; \langle \mathcal{D}', Y \rangle$). For an operator O defined over all domains, we will write the corresponding operator acting on $\mathcal{D}$ as "$O_{\mathcal{D}}$". An operator defined across domains thus can be identified with a collection of the pairs $\langle \mathcal{D}, O_{\mathcal{D}} \rangle$ of a domain and an operator defined on that domain. An operator O is said to be S-*invariant* if we have $O_{\mathcal{D}}(X_1, \ldots, X_n) = O_{\mathcal{D}'}(Y_1, \ldots, Y_n)$ for all S-similar objectual structures $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ and $\langle \mathcal{D}', Y_1, \ldots, Y_m \rangle$ such that $O_{\mathcal{D}}$ is applied to $\langle X_1, \ldots, X_n \rangle$ and $O_{\mathcal{D}'}$ to $\langle Y_1, \ldots, Y_m \rangle$.

A bijection η between two domains $\mathcal{D}$ and $\mathcal{D}'$ of the same cardinality determines a similarity relation S_η:

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[1]Another characterization that has played an important role in the study of logicality is Tarski's *permutation invariance criterion* (Tarski[127]). Though the permutation invariance criterion can be seen as more basic than the isomorphism invariance criterion in that the latter can be obtained by extending the former, I think that the permutation invariance criterion, strictly speaking, cannot be regarded as a theory of classical logicality proper. Rather, it is a theory of the classical logicality restricted to a single domain.

[2]A *finite relational type* is obtained by the following inductive rule:

(i) The basic type 0 is a finite relational type;

(ii) If $\tau_1, \ldots, \tau_m$ are finite relational types, then so is $(\tau_1, \ldots, \tau_m)$.

Each finite relational type is associated with a set. The basic type 0 is associated with $\mathcal{D}$, and $(\tau_1, \ldots, \tau_m)$ with the power set $\wp(\mathcal{D}_{\tau_1} \times \cdots \times \mathcal{D}_{\tau_m})$ where $\mathcal{D}_{\tau_i}$ is the set associated with type τ_i. An object of a finite relational type of $\mathcal{D}$ is an object in the set associated with that type.

$$\langle \mathcal{D}, X_1, \ldots, X_n \rangle \ S_\eta \ \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle \quad \stackrel{def}{\Longleftrightarrow} \quad Y_i = \eta(X_i) \text{ for all } i \in \{1, \ldots, n\},$$

where $\eta(X_i)$ is an object obtained by replacing $a \in \mathcal{D}$ occurring in X_i with $\eta(a)$. The collection of all bijections between domains also determines a similarity relation S_{bi}: two structures are S_{bi}-*similar* if there is some bijection η such that they are S_η-similar. S_{bi} is the union of all S_η. The isomorphism invariance criterion then can be stated as follows:

An operator O is logical if and only if O is S_{bi}-invariant.

In the study of logicality, two structures are said to be *isomorphic* if they are S_{bi}-similar. "Isomorphism invariance" comes from this convention.

Consider, for example, the self-identity operator O_I such that $O_{I\mathcal{D}}(x) = \mathsf{T}$ if and only if $x \in \mathcal{D}$ is identical to itself. Since any object is identical to itself, we have $O_{I\mathcal{D}}(a) = \mathsf{T}$ for any $a \in \mathcal{D}$. It then follows that for any S_{bi}-similar objectual structures $\langle \mathcal{D}, a \rangle$ and $\langle \mathcal{D}', a' \rangle$, it holds that $O_{I\mathcal{D}}(a) = O_{I\mathcal{D}}(a') = \mathsf{T}$. Hence O_I is an S_{bi}-invariant operator, and therefore a logical operator. The self-identity predicate "x is identical to itself" can be regarded as a logical constant.

Consider another example: the red operator O_R such that $O_{R\mathcal{D}}(a) = \mathsf{T}$ if and only if a is red. For the domain $\mathcal{D}_{fruits}$ of fruits, the two objectual structures $\langle \mathcal{D}_{fruits}, \text{tomato} \rangle$ and $\langle \mathcal{D}_{fruits}, \text{banana} \rangle$ are S_{bi}-similar by some bijection $\eta : \mathcal{D}_{fruits} \to \mathcal{D}_{fruits}$ (actually a permutation on $\mathcal{D}_{fruits}$) such that $\eta(\text{tomato}) = \text{banana}$. However, $O_{R\mathcal{D}}(\text{tomato}) \neq O_{R\mathcal{D}}(banana)$, because a tomato is red (i.e, $O_{R\mathcal{D}}(\text{tomato}) = \mathsf{T}$) while a banana is yellow (i.e, $O_{R\mathcal{D}}(\text{banana}) = \mathsf{F}$). Hence, the red operator O_R is not S_{bi}-invariant and not logical. The predicate "x is red", therefore, is not a logical constant.

How can the isomorphism invariance criterion be justified? There are two assumptions underlying the identity between classical logicality and S_{bi}-invariance. The first assumption is that a logical operator is an operator that assigns the same truth value to tuples $\langle X_1, \ldots, X_n \rangle$ and $\langle Y_1, \ldots, Y_n \rangle$ that are *logically similar* (or *logically indistinguishable*). If we have $O_{\mathcal{D}}(X_1, \ldots, X_n) = O_{\mathcal{D}'}(Y_1, \ldots, Y_n)$ for any two structures $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ and $\langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$ that are logically similar in some sense, then O is a logical operator. Any theories of logicality have to begin with some concept of what logical operators are and this is the one that most theories[3] are, implicitly or explicitly, based on. Characterizing the logicality of operators using the concept of invariance (normally) means characterizing logical operators by this "definition."

The definition, however, just gives a form; its content varies according to what the logical similarity is. The second assumption is about this point. The isomorphism invariance criterion implies that the logical similarity relation is the similarity relation S_{bi}. That is to say, $\langle X_1, \ldots, X_n \rangle$ and $\langle Y_1, \ldots, Y_n \rangle$ are logically similar if and only if there is a bijection $\eta : \mathcal{D} \to \mathcal{D}'$ such that $Y_i = \eta(X_i)$. Let us write the logical similarity relation of classical logicality "$\equiv_C$" (although we have not identified it yet). Then, the claim about the second assumption is that $\equiv_C$-similarity is S_{bi}-similarity:

$$\langle \mathcal{D}, X_1, \ldots, X_n \rangle \ \equiv_C \ \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$$
$$\text{if and only if}$$
$$\langle \mathcal{D}, X_1, \ldots, X_n \rangle \ S_{bi} \ \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle.$$

According to the equivalence, for example, for $a, b, c \in \mathcal{D}$, two sets $\{a, b\}$ and $\{b, c\}$ are logically similar since these are S_{bi}-similar: there is a bijection $\eta : \mathcal{D} \to \mathcal{D}$ (actually a permutation) that maps $\{a, b\}$ to $\{b, c\}$. $\{a, b\}$ and $\{a, b, c\}$, on the other hand, are not logically similar; there is no bijection whose image of $\{a, b\}$ is $\{a, b, c\}$ and hence these are not S_{bi}-similar.

Why does the equivalence between the logical similarity and S_{bi}-similarity hold? Let us quote from Gila Sher, one of the founders of the isomorphism invariance criterion:

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[3]Sher's and Bonnay's theories (Sher[105] and Bonnay[10]) are examples established under this assumption. Feferman's theory (Feferman[31]) is partially based on the assumption: not all operators that are invariant under the similarity relation that his theory employs count as logical operators. Casanovas's theory uses a different characterization of the logicality of operators (Casanovas[21], p. 41). In his theory, a logical operator O is defined, in our terminology, as an operator that satisfies the following condition:

> If $\langle X_1, \ldots, X_n \rangle$ is logically similar to $\langle Y_1, \ldots, Y_n \rangle$ and if $O_{\mathcal{D}}(X_1, \ldots, X_n) = \mathsf{T}$, then $O_{\mathcal{D}'}(Y_1, \ldots, Y_n) = \mathsf{T}$.

One noteworthy characteristic of Casanovas's theory is that logical operators do not have to assign the truth value F to logically similar objects. Even if $O_{\mathcal{D}}(X_1, \ldots, X_n) = \mathsf{F}$, it does not necessarily hold that $O_{\mathcal{D}'}(Y_1, \ldots, Y_n) = \mathsf{F}$. The operator of the identity relation is an example of such "logical" operators.

> There are terms that take the identity of objects into account and terms that do not. Terms underlying logical consequence must be of the second kind. That is to say, logical terms should not distinguish the identity of objects in the universe of any model. (By "identity of an object" I here mean the features that make an object what it is, the properties that single it out.) (Sher[105], p. 43)[4]

An object in a domain satisfies a variety of properties that constitute its identity, and they make it different from others. For example, Sher's being female and her being a philosopher are among such properties that make her what she is and distinguish her from Barack Obama. Such properties of objects of the basic type can be used to make distinctions between objects of any finite relational types. The difference between the two sets {Sher, Tarski} and {Obama, Tarski} is made by the difference between Sher and Obama, which is based on the difference between the properties they possess. Logical terms, and hence logical operators as their semantic values, should be insensitive to any such distinctions.

The reason why logic should disregard the identity of an object of the basic type is that logic is formal. Logic reckons with formal aspects of objects of a finite relational type. If two objects of a finite relational type share the same formal aspects, then they should be treated as logically similar. If they differ in some formal aspects, they are logically distinct objects. According to Sher, the formality of logic can be characterized by S_{bi}-similarity: "being formal is being invariant under isomorphic structures" (ibid., p. 53) (Remember that "isomorphic structures" means S_{bi}-invariant structures). That is, a formal aspect of an object is an aspect that the object has in common with its S_{bi}-invariant objects. We then have two equivalence relations: the logical similarity is the formal similarity; and the formal similarity is S_{bi}-similarity. As a consequence, the equivalence between the logical similarity and S_{bi}-similarity can be obtained.

The isomorphism invariance criterion thus consists of two claims: (i) a logical operator is an operator that assigns the same truth value to objects that are logically similar; (ii) two objects of a finite relational type are logically similar if and only if they are S_{bi}-similar. Supposing that the first claim provides a suitable

[4]Sher says that this idea that logical terms should be insensitive to the identity of objects was inspired by Andrzej Mostowski (Mostowski[73], p. 13).

form of the definition of logical operators, the second claim is the main thesis of the isomorphism invariance criterion. In general, providing a theory of logicality is characterizing the logical similarity relation, i.e., identifying what structures are logically similar/dissimilar (or logically indistinguishable/distinguishable). The theory is judged based on one aspect: the adequacy of the logically similarity relation it employs. If a theory appropriately defines the logical similarity relation, then the correct set of logical operators will be obtained by the definition (i). Otherwise, the theory will sanction non-logical operators or some logical operators will be missing. Does the isomorphism invariance criterion properly capture the nature of classical logicality? The answer is determined by whether or not S_{bi}-similarity is suitable as the logical similarity.[5]

Criticism of the Isomorphism Invariance Criterion

In the isomorphism invariance criterion, the logical similarity/dissimilarity between objects of a finite relational type is equivalent to the formal similarity/dissimilarity between them (which can be characterized as the S_{bi}-similarity/ dissimilarity). Cardinality is a formal property of sets, and two sets can be regarded as logically distinct sets if they are of different cardinalities. The property of being well-ordered of a relation between objects and the property of being the second-order membership relation between objects and sets are other examples of formal properties, which are entitled to make a distinction between objects that these properties can be applied to. The idea underlying this entitlement is that logic takes into account *all and only* formal aspects of objects of a finite relational type. Considering that formality is an essence of logic, the "only" part is obvious: if there is a logical distinction between two objects, then there has to be some formal property that is possessed by one but not by the other.

[5]In the literature, it has been pointed out that, in addition to the operators of the standard logical constants of first-order logic with identity, the isomorphism invariance criterion sanctions certain operators whose logical status is disputable as logical. McGee shows that an operator O_D acting on a domain D is invariant under permutations if and only if O_D is descried by some formula of the formal language $L_{\infty\infty}$ (McGee[71], p. 572). The view that the criterion overgenerates "logical" operators has motivated several logicians and philosophers to pursue another theory of classical logicality (See, for example, Feferman[31], pp. 37–39).

The "all" part, however, seems far from trivial. Why does logic have to allow for *all* formal properties of objects? Why can a logical distinction between objects be made by *any* formal properties? The equivalence between logicality and formality comes from Sher's formal structural notion of logic: "[L]ogic is a theory of reasoning based on the formal (structural) laws governing our thinking on one hand and reality on the other" (Sher[116], p. 307). Remember that according to the formal structural notion, any formal laws of any formal properties of objects can validate arguments as logical. Under the notion, any formal distinctions between objects can be thought of as distinctions that logic has to be sensitive to. The answers to the questions above then become trivial. Logic *is* a theory that allows for all formal properties of objects, and a logical distinction between objects *is* a distinction that can be made by any formal property.

The formal structural notion is not widely held, however. Critics of the isomorphism invariance criterion do not accept the equivalence between logicality and formality. Denis Bonnay, for example, says, "Formality is a property of logic that is shared by set theory and other branches of mathematics" (Bonnay[10], p. 38). This implies that every logical property of objects is a formal property of them, but not vice versa. The reason why logicality is a proper part of formality is that logic should be free from certain "formal contents" of arguments as well as their "empirical contents." The logical validity of an argument is supposed to be independent of its empirical content, which is composed in part of properties that objects appearing in it satisfy. Being female is an example of such properties, and whether or not an argument about Gila Sher is logically valid does not depend on her femaleness. We thus deny that the property of being female is entitled to logically distinguish her from Obama. Arguments in set theory have "set-theoretic contents," whose truth can be confirmed based on some facts about formal properties of sets such as the property of cardinality $\aleph_1$. It then seems to be consistent to deny, for the same reason, that a set's being of cardinality $\aleph_1$ is entitled to logically distinguish it from the set of all natural numbers. The property of cardinality $\aleph_1$ is located within the realm of formality—particularly, within the realm of set theory—but outside of the realm of logicality. Set theory deals with set-theoretic properties of sets. Logic, on the other hand, should be "even more 'content-free' than set theory" (ibid., p. 38).

Here, two views on what logic is and what logic should be conflict with each other. One says that logical properties are formal properties but the other says that logical properties are a special type of formal property. If the former is correct, then we will have the equivalence between the logical similarity and S_{bi}-similarity, provided that a formal property is a property that is shared by S_{bi}-invariant structures. The isomorphism invariance criterion then can be taken as an appropriate characterization of logicality. If the latter is correct, then the equivalence will not hold, and as a result it can be concluded that the isomorphism invariance criterion does not capture the nature of logicality. The problem is to what extent logic should be formal. In other words, the problem is what formal properties logic should allow for to determine the logical similarity relation. Sher thinks that the answer is "all," while critics of the isomorphism invariance criterion think that it is "not all."

I share the idea of logical properties with those critics. In particular, as have been argued so far, I accept the minimal notion of logic, under which logic should take into account as fewer formal properties as possible and the logical validity of an argument can be justified by some formal laws of those selected formal properties. Sher's formal structural notion of logic can be thought of as a "maximal" notion with respect to formal properties (we have argued this in Chapter 1, pp.16–17). All formal properties are logical properties, and therefore the collection of logical properties and the collection of available formal laws are maximally large. Instead of the maximal notion, I hold the minimal notion of logical properties.

Among various possible notions of logic, why should we hold the minimal notion? A justification for the notion can be given in relation to another characteristic property of logic: generality. The reason that logical distinctions among objects of the basic type cannot be made by any properties they have is that logic is formal on the one hand and logic is general on the other. Logic is not concerned with particular objects, and thus disregards any properties that make them what they are. If logic had to take into account, for example, the property of being female, it would become less general than it is supposed to be. The property of being of a certain cardinality, though it is a formal property, is a property of sets that constitutes what they are: if there is no one-to-one correspondence between elements in two sets, they cannot be identical. In this respect, the property of being female and the property of being of a certain cardinality are similar: they both make their bearers particular. Thus, the generality of logic with respect to objects of the basic type can be naturally extended to the generality with respect to sets of objects. It then follows that if logic had to take into account, for example, the property of cardinality 3, logic would become less general than it can be. For logic to be as general as possible, it has to be insensitive to as many properties of sets as possible. As a result, the collection of logical properties and the collection of available formal laws have to be minimally small.

What if logic is supposed to disregard any formal properties? In this extreme notion of logic, any structures $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ and $\langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$ such that X_i and Y_i are of the same type τ_i are logically similar; there is nothing able to make a logical distinction between them. The result obtained from this notion of logic—the set of logical operators—is minimum. For types $\tau_1, \ldots, \tau_n$, there are only two logical operators that can be applied to tuples $\langle X_1, \ldots, X_n \rangle$: (i) the operator that assigns T to all $\langle X_1, \ldots, X_n \rangle$; (ii) the operator that assigns F to all $\langle X_1, \ldots, X_n \rangle$. Not surprisingly, no contemporary theory of logicality endorses this extreme notion. In this notion, operators sanctioned as logical are operators representing terms that an object belongs to (or does not belong to) a certain semantic type (e.g., "is an object of the basic type," "is a property of objects of the basic type," "is an n-ary relation of objects of the basic type," and so on). Sher correctly points out that a theory that uses only such operators is not logic as "a theory of *inference*" but "a theory of semantic types" (Sher[116], p. 307). Logic should disregard as many formal properties as possible for the generality-property but cannot disregard all formal properties. The problem then is what formal properties remain on the list of logical properties.

A New Theory of Classical Logicality

For our theory of logicality, we need to establish the logical similarity relation for the minimal notion, which is different from the S_{bi}-similarity relation. A logical operator can be defined as an operator that assigns the same truth value to objects that are in the new similarity relation. And a logical constant can be defined as a term whose semantic value can be specified by a logical operator. We will define the logical similarity relation among objects of each type, from simpler types to more complex types in a recursive manner: the logical similarity relation among objects of more complex types will be defined using the logical similarity relation among objects of simpler types.

First, objects of the basic type. We call them *zeroth-order objects*. Regarding zeroth-order objects, I agree with Sher's notion of the formality of logic that a zeroth-order object cannot be distinguished from another by any property that it satisfies. According to the isomorphism invariance criterion, however, for $a \in \mathcal{D}$ and $b \in \mathcal{D}'$, the objectual structures $\langle \mathcal{D}, a \rangle$ and $\langle \mathcal{D}', b \rangle$ are dissimilar if $\mathcal{D}$ and $\mathcal{D}'$ are of different cardinalities. This is because under Sher's maximalist notion of logic, logical distinctions among objectual structures can be made by any formal properties including cardinality properties. Instead of the maximal notion, we seek a minimal notion of logic, according to which logical distinctions should be made by as few formal properties as possible and so objectual structures cannot be distinguished by the cardinalities of their domains. Any zeroth-order objects a and b are supposed to be logically similar, and any domains $\mathcal{D}$ and $\mathcal{D}'$ are also supposed to be logically similar. Consequently, we suppose that any objectual structures $\langle \mathcal{D}, a \rangle$ and $\langle \mathcal{D}', b \rangle$ are logically similar. That is to say, for any $a \in \mathcal{D}$ and any $b \in \mathcal{D}'$,

$$\langle \mathcal{D}, a \rangle \equiv_C \langle \mathcal{D}', b \rangle.$$

There are two unary logical operators of zeroth-order objects (i.e., $\equiv_C$-invariant operators): (i) O^* such that $O^*_{\mathcal{D}}(a) = \mathsf{T}$ for all $\mathcal{D}$ and for all $a \in \mathcal{D}$; (ii) O^{**} such that $O^{**}_{\mathcal{D}}(a) = \mathsf{F}$ for all $\mathcal{D}$ and for all $a \in \mathcal{D}$. This similarity relation among zeroth-order objects is the base of our recursive definition of the whole logical similarity relation.

Next, sets of zeroth-order objects. We call a set of zeroth-order objects (an element in the power set $\wp(\mathcal{D})$) a *first-order object*. Unlike the logical similarity relation among zeroth-order objects, the logical similarity among first-order objects is not obvious. Some would think that the set of even numbers of the domain of natural numbers and the set of irrational numbers of the domain of real numbers should be logically similar, and some would think they should not. According to the isomorphism invariance criterion, they are not similar, because the domains, and the sets to be compared as well, are of different cardinalities, and therefore there is no isomorphism between them. As mentioned, however, not everyone accepts the similarity relation made by isomorphisms.

Some might think that the logical similarity relation could be defined based on common ideas on what quantifiers have been regarded as logical. The first-order universal quantifier is normally taken as a logical constant. In order for the operator $O^{\forall}$ of the universal quantifier to be a logical operator, the structures $\langle \mathcal{D}, \mathcal{D} \rangle$ and $\langle \mathcal{D}, X \rangle$ cannot be logically similar if X is a proper subset of $\mathcal{D}$, and $\langle \mathcal{D}, \mathcal{D} \rangle$ can only be logically similar to $\langle \mathcal{D}', \mathcal{D}' \rangle$. Similarly, for the operator $O^{\exists}$ of the first-order existential quantifier to be logical, $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}, \emptyset \rangle$ cannot be logically similar unless X is the empty set. $\langle \mathcal{D}, X \rangle$ such that $X \neq \emptyset$ can only be logically similar to $\langle \mathcal{D}', X' \rangle$ such that $X' \neq \emptyset$.

This "backward" strategy, however, contradicts the idea behind the characterization of logical operators in terms of the concepts of invariance and similarity relation. The idea is that a characterization of the logical similarity relation precedes a characterization of logical operators and a characterization of logical terms. What operators are logical, and what terms are logical, can be determined by what first-order objects are logically similar or dissimilar, but not the other way around. It is begging the question to advocate the logicality of an operator based on the assumption that it is invariant under a similarity relation which is established based on the logicality itself. The logical similarity relation cannot be defined so that some particular operators become logical operators. The logical similarity relation has to be defined using some concept that does not depend on the logicality of any terms in a standard first-order language.

To define the logical similarity relation among first-order objects, we make use of the concept of identity. The concept of identity of first-order objects is well-established compared to the concept of similarity. Different people have different feelings on whether given first-order objects should be regarded as logically similar or not. But, we all have the same answer as to whether given first order objects are identical or not. According to the standard definition, first-order objects X and Y in $\wp(\mathcal{D})$ are *identical* if they satisfy the following condition:

(I) For any zeroth-order object $a \in \mathcal{D}$, $a \in X$ if and only if $a \in Y$.

That is, first-order objects are identical if they are composed of the same zeroth-order objects.

The condition (I) is equivalent to the conjunction of the following two conditions:

(I-1) For any zeroth-order object $a \in \mathcal{D}$, there exists $b \in \mathcal{D}$ such that b is identical to a and such that $a \in X$ if and only if $b \in Y$;

(I-2) For any zeroth-order object $b \in \mathcal{D}$, there exists $a \in \mathcal{D}$ such that a is identical to b and such that $a \in X$ if and only if $b \in Y$.

(Actually, the three conditions (I), (I-1), and (I-2) are equivalent to each other, since (I-1) and (I-2) are equivalent.) By replacing the term "identical" in these conditions with the term "logically similar," we can obtain a definition of logical similarity. For first-order objects $X \in \wp(\mathcal{D})$ and $Y \in \wp(\mathcal{D}')$, we say that objectual structures $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}', Y \rangle$ are *logically similar* (written as "$\langle \mathcal{D}, X \rangle \equiv_C \langle \mathcal{D}', Y \rangle$") if they satisfy the following conditions:

(S-1) For any zeroth-order object $a \in \mathcal{D}$, there exists $b \in \mathcal{D}'$ such that b is logically similar to a and such that $a \in X$ if and only if $b \in Y$;

(S-2) For any zeroth-order object $b \in \mathcal{D}'$, there exists $a \in \mathcal{D}$ such that a is logically similar to b and such that $a \in X$ if and only if $b \in Y$.

This definition says that X and Y are logically similar if X and Y, and also their complements in the domains, are composed of logically similar zeroth-order objects. Since any zeroth-order objects in $\mathcal{D}$ and $\mathcal{D}'$ are logically similar, (S-1) and (S-2) are equivalent to the following conditions:

(S-1)′ For any zeroth-order object $a \in \mathcal{D}$, there exists $b \in \mathcal{D}'$ such that $a \in X$ if and only if $b \in Y$;

(S-2)′ For any zeroth-order object $b \in \mathcal{D}'$, there exists $a \in \mathcal{D}$ such that $a \in X$ if and only if $b \in Y$.

According to this definition, $\langle \mathcal{D}, \mathcal{D} \rangle$ is only logically similar to $\langle \mathcal{D}', \mathcal{D}' \rangle$, and $\langle \mathcal{D}, \emptyset \rangle$ is only logically similar to $\langle \mathcal{D}', \emptyset \rangle$. For any other first-order objects $X \in \wp(\mathcal{D})$ and $Y \in \wp(\mathcal{D}')$, $\langle \mathcal{D}, X \rangle$ is logically similar to $\langle \mathcal{D}', Y \rangle$ regardless of the cardinalities of X and Y.[6]

A logical operator of first-order objects is an operator that is invariant under the similarity relation $\equiv_C$ above. There are eight logical operators of first-order objects:

(i) O_1 such that $O_{1\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D})$;
(ii) O_2 such that $O_{2\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\mathcal{D}\}$;
(iii) O_3 such that $O_{3\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\mathcal{D}, \emptyset\}$;
(iv) O_4 such that $O_{4\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\emptyset\}$;
(v) O_5 such that $O_{5\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\mathcal{D}\}$;
(vi) O_6 such that $O_{6\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\mathcal{D}, \emptyset\}$;
(vii) O_7 such that $O_{7\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\emptyset\}$;
(viii) O_8 such that $O_{8\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \emptyset$.

The operators of the first-order universal and existential quantifiers—O_5 and O_4, respectively—are logical, while the operators of cardinality quantifiers are not logical.

Our definition of the logical similarity relation can be justified based on at least two considerations. First, our definition is consistent with the result regarding logical operators of zeroth-order objects. Remember that there are two logical

[6]Note that objectual structures $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}', Y \rangle$ can be logically dissimilar even if X and Y are identical first-order objects. Consider the domains $\mathbb{N}$ of natural numbers and $\mathbb{R}$ of real numbers. According to the definition, we have $\langle \mathbb{N}, \mathbb{N} \rangle \equiv_C \langle \mathbb{R}, \mathbb{R} \rangle$. However, $\langle \mathbb{N}, \mathbb{N} \rangle$ and $\langle \mathbb{R}, \mathbb{N} \rangle$ are not logically similar; the complement of $\mathbb{N}$ in the domain $\mathbb{N}$ is empty, while the complement of $\mathbb{N}$ in the domain $\mathbb{R}$ is not.

operators of zeroth-order objects: (i) O^* such that $O^*_{\mathcal{D}}(a) = \mathsf{T}$ for all $\mathcal{D}$ and for all $a \in \mathcal{D}$; (ii) O^{**} such that $O^{**}_{\mathcal{D}}(a) = \mathsf{F}$ for all $\mathcal{D}$ and for all $a \in \mathcal{D}$. The extension of $O^*_{\mathcal{D}}$ is $\mathcal{D}$ and the extension of $O^{**}_{\mathcal{D}}$ is $\emptyset$. The operators O^* and O^{**} are logically distinguished from other operators. It then seems that their extensions should also be logically distinguished from other first-order objects. Our definition meets this condition. Indeed, for any non-empty proper subset X of $\mathcal{D}$, $\langle \mathcal{D}, X \rangle$ is not logically similar to $\langle \mathcal{D}, \mathcal{D} \rangle$ or $\langle \mathcal{D}, \emptyset \rangle$.

How can the logical similarity among all other first-order objects be explained? Why do they have to be logically similar regardless of their cardinality? Assume that a first-order object X of cardinality κ_X and another first-order object Y of another cardinality κ_Y could be logically distinguished because of the difference between their cardinalities. Let Z be a first-order object of an infinite cardinality κ_Z. We suppose that κ_Z is strictly larger than κ_X and κ_Y. Then, consider the domain $\mathcal{D} = X \cup Y \cup Z$. And also consider structures $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}, Y \rangle$, and structures $\langle \mathcal{D}, Y \cup Z \rangle$ and $\langle \mathcal{D}, X \cup Z \rangle$. By our assumption, $\langle \mathcal{D}, X \rangle$ and $\langle \mathcal{D}, Y \rangle$ are logically dissimilar. It seems, however, that $\langle \mathcal{D}, Y \cup Z \rangle$ and $\langle \mathcal{D}, X \cup Z \rangle$ should be regarded as logically similar, because the cardinality of $Y \cup Z$ is the same as the cardinality $X \cup Z$. It then follows that first-order objects $Y \cup Z$ and $X \cup Z$ are logically similar, while their complements X and Y are logically dissimilar. If first-order objects of a domain are identical, then their complements in the domain are also identical. Analogously, it can be supposed that if first-order objects are logically similar, then their complements in their domains are also logically similar. Distinguishing first-order objects by their cardinalities generates some "similar" structures that violates this principle.

Second, our definition complies with the minimal notion of logic. The idea behind the minimal notion is that the more formal properties logic has to take into account, the less general it becomes (pp. 71–72). Under the minimal notion, therefore, the logical similarity relation has to be made by as few formal properties as possible. According to our definition, whether first-order objects are logically similar or dissimilar is determined by two properties: (i) the property of containing all zeroth-order objects in the domain; (ii) the property of containing no zeroth-order object in the domain. Could it be possible that the logical similarity/dissimilarity can be made by only one property? It seems that it could not be. Assume that there exists a property P that determines the logical similarity relation. First-order objects X and Y of the domain $\mathcal{D}$ are logically similar, provided that X satisfies P if and only if Y satisfies P. Consider the domain $\mathcal{D}_0 = \{a, b\}$. As argued above, since $\mathcal{D}_0$ and the empty set $\emptyset$ are the extensions of distinct logical operators of zeroth-order objects on the domain $\mathcal{D}_0$, they are supposed to be logically distinguished from each other. That is, they can be distinguished by P. Let us suppose that $\mathcal{D}_0$ satisfies P and $\emptyset$ does not. The two first-order objects $\{a\}$ and $\{b\}$ are also supposed to be logically similar, because their components a and b are logically similar zeroth-order objects. That is to say, either both $\{a\}$ and $\{b\}$ satisfy P, or neither $\{a\}$ nor $\{b\}$ satisfies P. If both $\{a\}$ and $\{b\}$ satisfy P, then $\mathcal{D}$ and $\{a\}$ are logically similar and their complements $\emptyset$ and $\{b\}$ are not logically similar. This contradicts the principle mentioned above. Similarly, if neither $\{a\}$ nor $\{b\}$ satisfies P, then $\emptyset$ and $\{a\}$ are logically similar and their complements $\mathcal{D}$ and $\{b\}$ are not logically similar. This also contradicts the principle. Hence, it can be concluded that such a property P does not exist.

Let us move on to the definition of the logical similarity relation among *higher-order objects*. We call a set of first-order objects (i.e., an object in $\wp^2(\mathcal{D}) = \wp(\wp(\mathcal{D})))$ a *second-order object*. More generally, we call an object in $\wp^n(\mathcal{D}) = \wp(\wp^{n-1}(\mathcal{D}))$ an *n-th-order object*. First-order objects are said to be logically similar if they, and also their complements, are composed of logically similar zeroth-order objects. The logical similarity among them is determined based on the logical similarity among lower-order objects. We apply this idea of logical similarity to n-th-order objects. For n-th-order objects $X^{(n)} \in \wp^n(\mathcal{D})$ and $Y^{(n)} \in \wp^n(\mathcal{D}')$, we say that objectual structures $\langle \mathcal{D}, X^{(n)} \rangle$ and $\langle \mathcal{D}', Y^{(n)} \rangle$ are *logically similar* (written as "$\langle \mathcal{D}, X^{(n)} \rangle \equiv_C \langle \mathcal{D}', Y^{(n)} \rangle$") if they, and also their complements, are composed of logically similar $(n-1)$-th-order objects, that is to say, if they satisfy the following conditions:

(S-1) For any $(n-1)$-th-order object $X^{(n-1)} \in \wp^{n-1}(\mathcal{D})$, there exists $(n-1)$-th-order object $Y^{(n-1)} \in \wp^{n-1}(\mathcal{D}')$ such that $\langle \mathcal{D}, X^{(n-1)} \rangle \equiv_C \langle \mathcal{D}', Y^{(n-1)} \rangle$ and such that $X^{(n-1)} \in X^{(n)}$ if and only if $Y^{(n-1)} \in Y^{(n)}$.

(S-2) For any $(n-1)$-th-order object $Y^{(n-1)} \in \wp^{n-1}(\mathcal{D}')$, there exists $(n-1)$-th-order object $X^{(n-1)} \in \wp^{n-1}(\mathcal{D})$ such that $\langle \mathcal{D}, X^{(n-1)} \rangle \equiv_C \langle \mathcal{D}', Y^{(n-1)} \rangle$ and such that $X^{(n-1)} \in X^{(n)}$ if and only if $Y^{(n-1)} \in Y^{(n)}$.

Note that if we substitute 1 for n, we will obtain the definition of the logical similarity relation among first-order objects. Whether higher-order objects are logically similar or dissimilar can be determined based on the logical similarity relation among zeroth-order objects and these conditions.

The logical similarity relation among n-tuples of objects can be defined using the definition of the identity of n-tuples. Recall that n-tuples $\langle X_1, \ldots, X_n \rangle$ and $\langle Y_1, \ldots, Y_n \rangle$ are identical if X_i and Y_i are identical for all $i \in \{1, \ldots, n\}$. The identity between n-tuples can be reduced to the identity between the corresponding components. The idea of reduction can be used for the logical similarity relation. We say that objectual structures $\langle \mathcal{D}, X_1, \ldots, X_n \rangle$ and $\langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$ are *logically similar* (written as "$\langle \mathcal{D}, X_1, \ldots, X_n \rangle \equiv_C \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$") if X_i and Y_i are logically similar, i.e., $\langle \mathcal{D}, X_i \rangle \equiv_C \langle \mathcal{D}', Y_i \rangle$, for all $i \in \{1, \ldots, n\}$.

Finally, the logical similarity relation among objects of a general finite relational type. Let Z be an object of type $(\tau_1, \ldots, \tau_n)$ of $\mathcal{D}$. Z is a set of n-tuples $\langle X_1, \ldots, X_n \rangle$ where X_i is an object of type τ_i of $\mathcal{D}$. Also, let Z' be an object of the same type $(\tau_1, \ldots, \tau_n)$ of $\mathcal{D}'$. We say that $\langle \mathcal{D}, Z \rangle$ and $\langle \mathcal{D}', Z' \rangle$ are *canonically similar* if they satisfy the following conditions:

(C-1) For any n-tuple $\langle X_1, \ldots, X_n \rangle$ of $\mathcal{D}$, there exists n-tuples $\langle Y_1, \ldots, Y_n \rangle$ of $\mathcal{D}'$ such that $\langle \mathcal{D}, X_1, \ldots, X_n \rangle \equiv_C \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$ and such that $\langle X_1, \ldots, X_n \rangle \in Z$ if and only if $\langle Y_1, \ldots, Y_n \rangle \in Z'$.

(C-2) For any n-tuple $\langle Y_1, \ldots, Y_n \rangle$ of $\mathcal{D}'$, there exists n-tuples $\langle X_1, \ldots, X_n \rangle$ of $\mathcal{D}$ such that $\langle \mathcal{D}, X_1, \ldots, X_n \rangle \equiv_C \langle \mathcal{D}', Y_1, \ldots, Y_n \rangle$ and such that $\langle X_1, \ldots, X_n \rangle \in Z$ if and only if $\langle Y_1, \ldots, Y_n \rangle \in Z'$.

This definition says that Z and Z' are canonically similar if they, and also their complements, are composed of logically similar objects. The canonical similarity is a natural extension of the logical similarity of n-th-order objects.

We will define the logical similarity relation among objects of type $(\tau_1 \dots \tau_n)$ as a similarity relation that is contained in the canonical similarity relation as its proper part, that is, in such a way that the logical similarity relation can be seen as a special type of the canonical similarity relation. The reason that the canonical similarity relation cannot be taken as an appropriate logical similarity relation is that it fails to take into account an important aspect of objects of type $(\tau_1, \dots, \tau_n)$: an object of type $(\tau_1, \dots, \tau_n)$ is not just a "set" in $\wp(\mathcal{D}_{\tau_1} \times \cdots \times \mathcal{D}_{\tau_m})$, where $\mathcal{D}_{\tau_i}$ is a set associated with type τ_i, but also an n-ary "relation" over $\mathcal{D}_{\tau_1}, \dots, \mathcal{D}_{\tau_m}$. As sets, $Z \in \wp(\mathcal{D}_{\tau_1} \times \cdots \times \mathcal{D}_{\tau_m})$ and $Z' \in \wp(\mathcal{D'}_{\tau_1} \times \cdots \times \mathcal{D'}_{\tau_m})$ can be regarded as logically similar if they, and also their complements, are composed of logically similar objects. However, whether or not Z and Z' are logically similar as relations should not be determined by the same criterion.

Consider two binary relations $R_0 = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle\}$ and $R'_0 = \{\langle a, a\rangle, \langle a, b\rangle\}$ defined on $\mathcal{D} = \{a, b\}$.

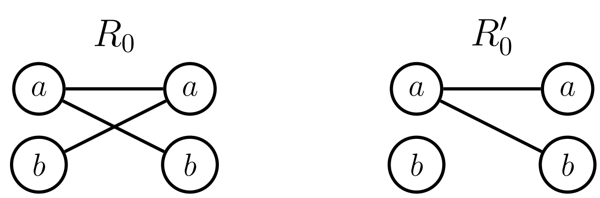

Figure 3.1: Binary relations R_0 and R'_0

Since any pairs of zeroth-order objects are logically similar, R_0 and R'_0 are composed of logically similar pairs. In addition, their complements in $\wp(\mathcal{D} \times \mathcal{D})$, $R_0^{\complement} = \{\langle b, b\rangle\}$ and $R_0'^{\complement} = \{\langle b, a\rangle, \langle b, b\rangle\}$, are also composed of logically similar pairs. Therefore, R_0 and R'_0 are similar as sets. It seems, however, that there is a sense in which they should not count as similar as binary relations. The zeroth-order object b as the first component of a pair is related to some zeroth-order object (namely, a) in R_0, while it is not in R'_0. The set of zeroth-order objects related to the b in R_0 (i.e., the image of b under R_0) is $\{a\}$, and that in R'_0 is the empty set $\emptyset$. According to our definition, they are not logically similar first-order objects. Thus, R_0 and R'_0 should not be regarded as logically similar as binary relations.

We propose that Z and Z' of type $(\tau_1, \ldots, \tau_n)$ should be regarded as logically similar as n-ary relations only if they are composed of logically similar *images* of objects of type τ_i. For an object X of type τ_i, let $Z_i(X)$ denote the set of all $(n-1)$-tuples $\langle X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_n \rangle$ such that $\langle X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots X_n \rangle \in Z$. We call $Z_i(X)$ the *image* of X under Z. The image of X under Z is the set of all $(n-1)$-tuples that are related to X under Z. $Z_i(X)$ is the empty set $\emptyset$ if there is no related $(n-1)$-tuple. We say that $\langle \mathcal{D}, Z \rangle$ and $\langle \mathcal{D}', Z' \rangle$ are *logically similar* (written as "$\langle \mathcal{D}, Z \rangle \equiv_C \langle \mathcal{D}', Z' \rangle$") if they satisfy the following conditions for all i:

(S-1)^i For any X of type τ_i of $\mathcal{D}$, there exists Y of the same type τ_i of $\mathcal{D}'$ such that $\langle \mathcal{D}, X \rangle \equiv_C \langle \mathcal{D}', Y \rangle$ and such that $\langle \mathcal{D}, Z_i(X) \rangle \equiv_C \langle \mathcal{D}', Z'_i(Y) \rangle$;

(S-2)^i For any Y of type τ_i of $\mathcal{D}'$, there exists X of the same type τ_i of $\mathcal{D}$ such that $\langle \mathcal{D}, X \rangle \equiv_C \langle \mathcal{D}', Y \rangle$ and such that $\langle \mathcal{D}, Z_i(X) \rangle \equiv_C \langle \mathcal{D}', Z'_i(Y) \rangle$.

Note that this is a recursive definition. For $m < n$, the logical similarity among objects of type $(\tau_1, \ldots, \tau_n)$ is defined using the logical similarity among objects of type $(\tau_1, \ldots, \tau_m)$.

Let us see two examples. Let $\mathcal{D} = \{a, b, c\}$ and $\mathcal{D}' = \{u, v, w\}$. And let $R_1 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle\}$ and $R'_1 = \{\langle u, v \rangle, \langle v, w \rangle, \langle w, u \rangle\}$. R_1 and R'_1 are logically similar.

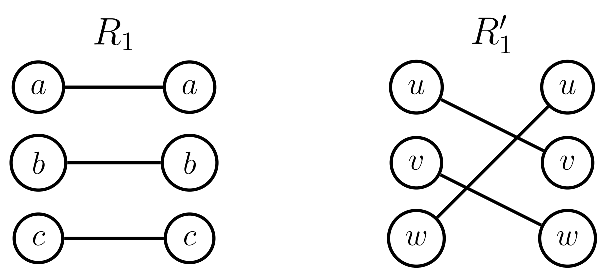

Figure 3.2: Binary relations R_1 and R'_1

For $i = 1, 2$, and for any zeroth-order object $x \in \mathcal{D}$, the image $R_{1i}(x)$ is a non-empty proper subset of $\mathcal{D}$. Also for any zeroth-order object $y \in \mathcal{D}'$, $R'_{1i}(y)$ is a non-empty proper subset of $\mathcal{D}'$. Thus, $\langle \mathcal{D}, R_{1i}(x) \rangle \equiv_C \langle \mathcal{D}', R'_{1i}(y) \rangle$. Therefore, (S-1)^1, (S-2)^1, (S-1)^2, and (S-2)^2 are all satisfied.

Let $R_2 = \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle\}$ and $R_2' = \{\langle u,u\rangle, \langle u,v\rangle, \langle v,w\rangle\}$. R_2 and R_2' are not logically similar.

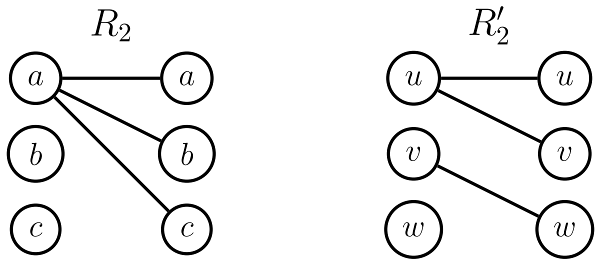

Figure 3.3: Binary relations R_2 and R_2'

We have that $R_{21}(a) = \{a,b,c\} = \mathcal{D}$. But, $R'_{21}(y) \neq \mathcal{D}'$ for any $y \in \mathcal{D}'$. This means that there does not exist y such that the image $R'_{21}(y)$ of y under R_2' is logically similar to the image $R_{21}(a)$ of a under R_2. Therefore, (S-1)[1] is not satisfied.

Note that if an object Z of type $(\tau_1, \ldots, \tau_n)$ of $\mathcal{D}$ and an object Z of type $(\tau_1, \ldots, \tau_n)$ of $\mathcal{D}'$ are logically similar, then they are canonically similar.[7] However, the opposite is not true. R_2 and R_2' above are canonically similar but not logically similar.

Since the logical similarity relation is a special type of the canonical similarity relation, an operator that is invariant under the canonical similarity relation

[7]Proof by induction. The base case: Let $Z \in \wp(\mathcal{D}_{\tau_1} \times \mathcal{D}_{\tau_2})$ be a binary relation over $\mathcal{D}_{\tau_1}$ and $\mathcal{D}_{\tau_2}$, where $\mathcal{D}_{\tau_1} = \wp^m(\mathcal{D})$ and $\mathcal{D}_{\tau_2} = \wp^n(\mathcal{D})$. Also, let $Z' \in \wp(\mathcal{D}'_{\tau_1} \times \mathcal{D}'_{\tau_2})$ be a binary relation over $\mathcal{D}'_{\tau_1}$ and $\mathcal{D}'_{\tau_2}$, where $\mathcal{D}'_{\tau_1} = \wp^m(\mathcal{D}')$ and $\mathcal{D}'_{\tau_2} = \wp^n(\mathcal{D}')$. We suppose that Z and Z' are logically similar. Let $\langle X_1, X_2\rangle \in \mathcal{D}_{\tau_1} \times \mathcal{D}_{\tau_2}$. By the definition of the logical similarity, there exists $Y_1 \in \mathcal{D}'_{\tau_1}$ such that X_1 is logically similar to Y_1 and such that $Z_1(X_1)$ and $Z'_1(Y_1)$ are logically similar. If $\langle X_1, X_2\rangle \in Z$, then from this logical similarity between the images of X_1 and Y_1, it follows that there exists $Y_2 \in Z'_1(Y_1)$ that is logically similar to X_2. Therefore, we have that $\langle Y_1, Y_2\rangle$ is logically similar to $\langle X_1, X_2\rangle$ and that $\langle Y_1, Y_2\rangle \in Z'$. If $\langle X_1, X_2\rangle \notin Z$, then both $Z_1(X_1)$ and $Z'_1(Y_1)$ are empty. Then, for any Y_2 of type $\mathcal{D}'_{\tau_2}$ that is logically similar to X_2, it holds that $\langle Y_1, Y_2\rangle$ is logically similar to $\langle X_1, X_2\rangle$ and that $\langle Y_1, Y_2\rangle \notin Z'$. Thus, the condition (C-1) is satisfied. We can show that the condition (C-2) can also be satisfied in the same way. Hence, Z and Z' are canonically similar. The inductive step: The canonical similarity among logically similar objects of a general type $(\tau_1, \ldots, \tau_n)$ can be shown in a similar way. Let $Z \in \wp(\mathcal{D}_{\tau_1} \times \cdots \times \mathcal{D}_{\tau_m})$ and $Z' \in \wp(\mathcal{D}'_{\tau_1} \times \cdots \times \mathcal{D}'_{\tau_m})$ be logically similar. By the logical similarity, for any n-tuple $\langle X_1, \ldots, X_n\rangle \in \mathcal{D}_{\tau_1} \times \cdots \times \mathcal{D}_{\tau_m}$, there exists $Y_1 \in \mathcal{D}'_{\tau_1}$ such that X_1 is logically similar to Y_1 and such that $Z_1(X_1)$ and $Z'_1(Y_1)$ are logically similar. By the induction hypothesis, $Z_1(X_1)$ and $Z'_1(Y_1)$ are canonically similar. By this canonical similarity, it can be shown that there exists $\langle Y_2, \ldots, Y_n\rangle$ such that $\langle X_1, \ldots, X_n\rangle$ is logically similar to $\langle Y_1, \ldots, Y_n\rangle$ and such that $\langle X_1, \ldots, X_n\rangle \in Z$ if and only if $\langle Y_1, \ldots, Y_n\rangle \in Z'$.

is also invariant under the logical similarity relation: that is, it is a logical operator. However, not all logical operators are invariant under the canonical similarity relation. Consider the operator $O^{\exists\forall}$ defined as follows: for $Z \in \wp(\mathcal{D} \times \mathcal{D})$,

$$O^{\exists\forall}_{\mathcal{D}}(Z) \quad = \quad \begin{cases} \mathsf{T} & \text{if there exists } x \in \mathcal{D} \text{ such that for all } y \in \mathcal{D}, \langle x, y\rangle \in Z; \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

Let $R \in \wp(\mathcal{D} \times \mathcal{D})$ and $R' \in \wp(\mathcal{D}' \times \mathcal{D}')$ be logically similar. By the logical similarity between R and R', there exists $x \in \mathcal{D}$ such that for all $y \in \mathcal{D}$ it holds that $\langle x, y\rangle \in Z$, if and only if there exists $x' \in \mathcal{D}'$ such that for all $y' \in \mathcal{D}'$ it holds that $\langle x', y'\rangle \in Z'$. Therefore, $O^{\exists\forall}_{\mathcal{D}}(R) = \mathsf{T}$ if and only if $O^{\exists\forall}_{\mathcal{D}}(R') = \mathsf{T}$. Hence, $O^{\exists\forall}$ is a logical operator. However, $O^{\exists\forall}$ is not invariant under the canonical similarity relation. In fact, for R_2 and R'_2 above, which are canonically similar, $O^{\exists\forall}_{\mathcal{D}}(R_2) = \mathsf{T}$ and $O^{\exists\forall}_{\mathcal{D}}(R'_2) = \mathsf{F}$.

There are four noteworthy results of our theory of classical logicality. First, the operators of the universal and existential quantifiers of second-order logic are logical. The extension of the former on $\mathcal{D}$ is $\{\wp(\mathcal{D})\}$ and that of the latter is $\{X \in \wp(\wp(\mathcal{D})) : X \neq \emptyset\}$. Let $O^{\forall^2}$ be the operators of the second-order universal quantifier. Suppose that second-order objects X and Y are logically similar. If $O^{\forall^2}_{\mathcal{D}}(X) = \mathsf{T}$, then $X = \wp(\mathcal{D})$. Since the only second-order object of $\mathcal{D}'$ that is logically similar to $\wp(\mathcal{D})$ is $\wp(\mathcal{D}')$, we have $Y = \wp(\mathcal{D}')$. Therefore, $O^{\forall^2}_{\mathcal{D}'}(Y) = \mathsf{T}$. On the other hand, if $O^{\forall^2}_{\mathcal{D}}(X) = \mathsf{F}$, then $X \neq \wp(\mathcal{D})$ and therefore $Y \neq \wp(\mathcal{D}')$. Thus, $O^{\forall^2}_{\mathcal{D}'}(Y) = \mathsf{F}$. Hence, $O^{\forall^2}$ is a logical operator. A similar proof can be given to show the logicality of the operator of the second-order existential quantifier.

Some might wonder if our notion of logical similarity properly captures classical logicality. It has been controversial whether or not second-order logic can be regarded as logic proper because of, for example, its incompleteness and rich expressive power. If one thinks that second-order logic crosses the boundaries of logic, then he or she might think that our logical similarity relation is deficient.

However, this is mistaken. True, second-order logic *with standard semantics* is incomplete and various sentences expressing facts about sets (e.g., the sentence expressing Cantor's theorem) can be validated in it. But, standard semantics is not the only semantic system available. Henkin semantics is another semantic system for a second-order language and has the completeness property. Also, many sentences expressing facts about sets can be invalidated in it. The point here is this: If the logical status of second-order logic is controversial because of its incompleteness, what is problematic is the semantic system employed (i.e., standard semantics) but not the use of the second-order quantifiers. If some sentences are "wrongly" validated, the problem lies with the semantic system that validates them but not the use of the second-order quantifiers. The logicality of the operators of second-order logic is neutral with respect to these issues. If they assign the same truth value to logically similar second-order objects, they should count as logical.

Second, the operator of the identity relation between zeroth-order objects is not logical. By the definition, we have $\langle \mathcal{D}, a, b\rangle \equiv_C \langle \mathcal{D}, a, a\rangle$ for any $a, b \in \mathcal{D}$, because $\langle \mathcal{D}, a\rangle \equiv_C \langle \mathcal{D}, a\rangle$ and $\langle \mathcal{D}, a\rangle \equiv_C \langle \mathcal{D}, b\rangle$. The identity operator $O^{=}_{\mathcal{D}}$ assigns different truth values to logically similar pairs $\langle a, b\rangle$ and $\langle a, a\rangle$ if $a \neq b$: $O^{=}_{\mathcal{D}}(\langle a, b\rangle) = \mathsf{F}$ but $O^{=}_{\mathcal{D}}(\langle a, a\rangle) = \mathsf{T}$. Hence, $O^{=}$ is not logical. Why is the operator of the identity relation not logical? This result would be contrary to the expectations of many people. What needs to be shown to justify the result is that the pairs $\langle a, b\rangle$ and $\langle a, a\rangle$ cannot be logically distinguished. Remember that the pair $\langle a, b\rangle$ can be identified with (sometimes defined as) the second-order object $\{\{a\}, \{a, b\}\}$. Also, $\langle a, a\rangle$ can be identified with $\{\{a\}\}$. According to our definition of the logical similarity relation between second-order objects, in a domain $\mathcal{D}$ such that $\mathcal{D} \neq \{a, b\}$, we have $\langle \mathcal{D}, \{\{a\}, \{a, b\}\}\rangle \equiv_C \langle \mathcal{D}, \{\{a\}\}\rangle$. Therefore, $\langle a, b\rangle$ and $\langle a, a\rangle$ can be regarded as logically similar objects. When we think that $\langle a, b\rangle$ and $\langle a, a\rangle$ differ, we pay attention to one formal aspect: whether or not the components are identical. The argument above shows that this aspect is an aspect that logic should not take into account.

Third, by our characterization of classical logicality, "split" operators—operators that behave differently on different domains—are removed from the list of logical operators. Consider an operator O^S such that $O^S_{\mathcal{D}}$ is the operator of the first-order universal quantifier if $\mathcal{D}$ is a finite domain and the operator of the first-order existential quantifier if $\mathcal{D}$ is an infinite domain. Let $\mathbb{N}_3$ be a finite domain $\{1,2,3\}$. We have $\langle \mathbb{N}_3, \{1,2\}\rangle \equiv_C \langle \mathbb{N}, \{1,2\}\rangle$. Then, $O^S_{\mathbb{N}_3}(\{1,2\}) = \mathsf{F}$ while $O^S_{\mathbb{N}}(\{1,2\}) = \mathsf{T}$. Hence, O^S is not $\equiv_C$-invariant and therefore is not a logical operator. Similar arguments can be given for rejecting the logicality of any split operators.

Finally, operators of first-order quantifier terms in a standard first-order language are all logical. We have shown above that the operator $O^{\exists\forall}$, which is the semantic value of the quantifier term "$\exists x \forall y$", is logical. Besides $O^{\exists\forall}$, we can show that the operators $O^{\forall\forall}$, $O^{\forall\exists}$, and $O^{\exists\exists}$ are logical as well. And more generally, for any quantifier phrase "$Q_1 x_1 \cdots Q_n x_n$" where "Q_i" is either "$\forall$" or "$\exists$", the operator $O^{Q_1 \cdots Q_n}$, which is an operator applied to objects of type (0^n),[8] is a logical operator.[9]

Conversely, for any logical operator O of objects of type (0^n), is there a quantifier term "$Q_1 x_1 \cdots Q_n x_n$" such that O is identical to $O^{Q_1 \cdots Q_n}$? This is not true. A counter-example is the following operator $O^{\exists\wedge\exists\neg}$ applied to first-order objects: for $X \in \wp(\mathcal{D})$

$$O^{\exists\wedge\exists\neg}_{\mathcal{D}}(X) \quad = \quad \begin{cases} \mathsf{T} & \text{if } X \neq \mathcal{D} \text{ and } X \neq \emptyset; \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

Although $O^{\exists\wedge\exists\neg}$ is a logical operator, it is neither $O^{\forall}$ nor $O^{\exists}$.

However, $O^{\exists\wedge\exists\neg}$ is identical to the operator described by the sentence $\varphi_0 \overset{def}{:=} \exists x P x \wedge \exists y \neg P y$. The sentence φ_0 is true in any model in which P is assigned a non-empty proper subset of the domain. More generally, it can be shown that for any logical operator O applied to objects of type (0^n), there exists some sentence φ (not necessarily connective-free) such that O is identical to the operator O^{φ} described

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[8] $(0^n) \overset{def}{:=} (\overbrace{0, \ldots, 0}^{n})$. An object of type (0^n) is an element in $\wp(\mathcal{D}^n)$.

[9] We prove the logicality of $O^{Q_1 \cdots Q_n}$ by induction on n. Clearly, $O^{\forall}$ and $O^{\exists}$ are logical operators. Consider then the operator $O^{\forall Q_2 \cdots Q_{n+1}}$ ("Q_1" is replaced by "$\forall$"). Let $Z \in \wp(\mathcal{D}^{n+1})$ and $Z' \in \wp(\mathcal{D}'^{n+1})$ be logically similar. Suppose that $O^{\forall Q_2 \cdots Q_{n+1}}_{\mathcal{D}}(Z) = \mathsf{T}$. Then, for any $a \in \mathcal{D}$, $O^{Q_2 \cdots Q_{n+1}}_{\mathcal{D}}(Z_1(a)) = \mathsf{T}$. By the definition of the logical similarity, for all $a' \in \mathcal{D}'$, there exists $a \in \mathcal{D}$ such that $Z_1(a)$ is logically similar to $Z'_1(a')$. By the induction hypothesis, thus, $O^{Q_2 \cdots Q_{n+1}}_{\mathcal{D}'}(Z'_1(a')) = \mathsf{T}$, and therefore $O^{\forall Q_2 \cdots Q_{n+1}}_{\mathcal{D}'}(Z') = \mathsf{T}$. It can be shown in the same way that if $O^{\forall Q_2 \cdots Q_{n+1}}_{\mathcal{D}'}(Z') = \mathsf{T}$, then $O^{\forall Q_2 \cdots Q_{n+1}}_{\mathcal{D}}(Z) = \mathsf{T}$. Hence, $O^{\forall Q_2 \cdots Q_{n+1}}$ is a logical operator. We can prove the logicality of the operator $O^{\exists Q_2 \cdots Q_n}$ in a similar way.

by φ (the proof will be given bellow). There are various logical operators applied to objects of type (0^n), and therefore there are various logical terms corresponding to them. Even if such terms are added to a standard first-order language without identity, the expressive power of the language obtained is essentially the same as the original language; for any sentence ψ in the language obtained, there exists a sentence ψ' in the original language such that ψ is equivalent to ψ'.

Proof (pp. 86–89): We will prove that for any logical operator O applied to objects of type (0^n), there exists some sentence φ in a first-order language such that O is identical to the operator O^φ described by φ. The proof is by induction on n of (0^n). Consider first the base case ($n = 1$). There are eight logical operators applied to first-order objects. This is because since there are three logically distinguished first-order objects of $\mathcal{D}$ ($\mathcal{D}$, $\emptyset$, and others), there are eight ways of assigning truth values to each logically similar first-order objects. For our claim, we need to find sentences that describe the eight operators.

The sentences describing three simple logical operators can be easily identified. First, the logical operator assigning T only to $\mathcal{D}$ can be described by the sentence $\forall xPx$. Second, the logical operator assigning T only to $\emptyset$ can be described by the sentence $\forall x\neg Px$. Third, the logical operator assigning T only to all other first-order objects can be described by the sentence $\exists xPx \wedge \exists x\neg Px$.

The sentences describing the other five logical operators can be expressed using these three sentences and the contradiction sentence $\bot = \forall xPx \wedge \neg\forall xPx$. For a logical operator O, we define a disjunction sentence $\psi = \psi_1 \vee \psi_2 \vee \psi_3$ containing three disjuncts as follows. If O assigns T to $\mathcal{D}$, then ψ_1 is the corresponding sentence $\forall xPx$. If O assigns F to $\mathcal{D}$, then ψ_1 is $\bot$. If O assigns T to $\emptyset$, then we define ψ_2 as the corresponding sentence $\forall x\neg Px$. If O assigns F to $\emptyset$, then ψ_2 is $\bot$. Similarity, if O assigns T to all other first-order objects, then ψ_3 is the corresponding sentence $\exists xPx \wedge \exists x\neg Px$. If O assigns F to all other first-order objects, then ψ_3 is $\bot$. For example, for the logical operator assigning T to $\mathcal{D}$, F to $\emptyset$, and T to other first objects, ψ is defined as

$$\forall xPx \ \vee \ \bot \ \vee \ [\exists xPx \wedge \exists x\neg Px].$$

For the logical operator assigning T to $\mathcal{D}$ and $\emptyset$ and F to other first objects, ψ is defined as

$$\forall x Px \ \vee \ \forall x \neg Px \ \vee \ \bot.$$

It can be easily observed that O is identical to the operator O^{ψ} described by ψ defined as above.

The idea for the general case of (0^n) is the same as that for the base case. For each equivalence class of logically similar objects of type (0^n), we identify a sentence φ that describes the operator assigning T only to the objects in the class. Any logical operator then can be described by a sentence that is a disjunction of such φs and $\bot$. We only prove the case of type $(0,0)$ $(n=2)$ and omit a detailed proof for the general case of (0^n), although a brief sketch will be given. This is because the general case can be proven in the same way as the case of $(0,0)$.

Let R be an object of type $(0,0)$, i.e., a binary relation on $\mathcal{D}$. The sentence φ_R describing the operator that assigns T only to logically similar objects to R can be expressed as a conjunction sentence $\varphi_R = \varphi_1 \wedge \cdots \wedge \varphi_6$ containing six conjuncts. We will explain how to define each component φ_i below.

Let Qxy be an atomic formula. For any $a \in \mathcal{D}$, the images $R_1(a)$ and $R_2(a)$ of a under R are first-order objects. Thus, they are $\mathcal{D}$, $\emptyset$, or a first order object X such that $\emptyset \subsetneq X \subsetneq \mathcal{D}$.

(i) If there exists $a \in \mathcal{D}$ such that $R_1(a) = \mathcal{D}$, then φ_1 is $\exists x \forall y Qxy$. If there does not exist such a, then φ_1 is $\neg\exists x \forall y Qxy$;

(ii) If there exists $a \in \mathcal{D}$ such that $R_1(a) = \emptyset$, then φ_2 is $\exists x \forall y \neg Qxy$. If there does not exist such a, then φ_2 is $\neg\exists x \forall y \neg Qxy$;

(iii) If there exists $a \in \mathcal{D}$ such that $R_1(a) = X$, then φ_3 is $\exists x(\exists y Qxy \wedge \exists z \neg Qxz)$. If there does not exist such a, then φ_3 is $\neg\exists x(\exists y Qxy \wedge \exists z \neg Qxz)$;

(iv) If there exists $a \in \mathcal{D}$ such that $R_2(a) = \mathcal{D}$, then φ_4 is $\exists y \forall x Qxy$. If there does not exist such a, then φ_4 is $\neg\exists y \forall x Qxy$;

(v) If there exists $a \in \mathcal{D}$ such that $R_2(a) = \emptyset$, then φ_5 is $\exists y \forall x \neg Qxy$. If there does not exist such a, then φ_5 is $\neg\exists y \forall x \neg Qxy$;

(vi) If there exists $a \in \mathcal{D}$ such that $R_2(a) = X$, then φ_6 is $\exists y(\exists x Qxy \wedge \exists z \neg Qzy)$. If there does not exist such a, then φ_6 is $\neg\exists y(\exists x Qxy \wedge \exists z \neg Qzy)$.

It can be easily seen that for $\varphi_R = \varphi_1 \wedge \cdots \wedge \varphi_6$, $O^{\varphi_R}_{\mathcal{D}}(R) = \mathsf{T}$. That is, the operator O^{φ_R} described by φ_R assigns T to R. Also, for any object $R' \in \wp(\mathcal{D}' \times \mathcal{D}')$ of type $(0, 0)$, it holds that $O^{\varphi_R}_{\mathcal{D}'}(R') = \mathsf{T}$ if and only if R' is logically similar to R. Therefore, O^{φ_R} is a logical operator that assigns T only to logically similar objects to R.

Any object of $(0, 0)$ can be divided into a finite number of equivalence classes with respect to the logically similar relation. Let $R_1 \in \wp(\mathcal{D}_1 \times \mathcal{D}_1), \ldots, R_m \in \wp(\mathcal{D}_m \times \mathcal{D}_m)$ be objects representing those classes. Then, for each R_i, we can define the sentence φ_{R_i} that describes the logical operator that assigning T only to logically similar objects to R in the same way as above.

Let O be a logical operator applied to objects of type $(0, 0)$. We define a disjunction sentence $\psi = \psi_1 \vee \cdots \vee \psi_m$ as follows: if O assigns T to R_i, then ψ_i is φ_{R_i}; if O assigns F to R_i, then ψ_i is $\perp$. We show that O is identical to O^{ψ}, that is to say, $O_{\mathcal{D}}(R) = O^{\psi}_{\mathcal{D}}(R)$ for any $R \in \wp(\mathcal{D} \times \mathcal{D})$.

Suppose that $R \in \wp(\mathcal{D} \times \mathcal{D})$ is logically equivalent to $R_i \in \wp(\mathcal{D}_i \times \mathcal{D}_i)$. Then, we have that $O_{\mathcal{D}}(R) = O_{\mathcal{D}_i}(R_i)$, because R and R_i are logically similar and O is a logical operator. Also, we have that $O_{\mathcal{D}_i}(R_i) = O^{\psi_i}_{\mathcal{D}_i}(R_i)$. This is because if $O_{\mathcal{D}_i}(R_i) = \mathsf{T}$ then $O^{\psi_i}_{\mathcal{D}_i}(R_i) = O^{\varphi_{R_i}}_{\mathcal{D}_i}(R_i) = T$, while if $O_{\mathcal{D}_i}(R_i) = \mathsf{F}$ then $O^{\psi_i}_{\mathcal{D}_i}(R_i) = O^{\perp}_{\mathcal{D}_i}(R_i) = \mathsf{F}$. We also have that $O^{\psi_i}_{\mathcal{D}_i}(R_i) = O^{\psi}_{\mathcal{D}_i}(R_i)$, because $O^{\psi_j}_{\mathcal{D}_i}(R_i) = \mathsf{F}$ if $j \neq i$. Finally, $O^{\psi}_{\mathcal{D}_i}(R_i) = O^{\psi}_{\mathcal{D}}(R)$, because each O^{ψ_i} is a logical operator and as a result O^{ψ} is also a logical operator: O^{ψ} assigns the same truth value to logically similar objects R and R_i. Hence, it holds that $O_{\mathcal{D}}(R) = O_{\mathcal{D}_i}(R_i) = O^{\psi_i}_{\mathcal{D}_i}(R_i) = O^{\psi}_{\mathcal{D}_i}(R_i) = O^{\psi}_{\mathcal{D}}(R)$.

A brief sketch of the proof for the general case of (0^n) is the following. Any object of (0^n) can be divided into a finite number of equivalence classes with respect to the logically similar relation. Let $Z_1 \in \wp({\mathcal{D}_1}^n), \ldots, Z_l \in \wp({\mathcal{D}_l}^n)$ be objects representing the equivalence classes.

(i) First, for each $Z_i \in \wp({\mathcal{D}_i}^n)$, we construct a sentence φ_{Z_i} that describes the logical operator assigning T only to objects which are logically similar to Z_i. In particular, φ_{Z_i} can be expressed as a conjunction sentence $\varphi_1 \wedge \cdots \wedge \varphi_m$. Each φ_j is an existential sentence or the negation of an existential sentence, which is related to a condition of the logical similarity relation among objects of type (0^n);

(ii) Second, for a logical operator O applied to objects of type (0^n), we construct a disjunction sentence $\psi = \psi_1 \vee \cdots \psi_l$ that is supposed to describe O. Each ψ_i is defined as follows: if $O_{\mathcal{D}_i}(Z_i) = \mathsf{T}$, then ψ_i is φ_{Z_i}; if $O_{\mathcal{D}_i}(Z_i) = \mathsf{F}$, then ψ_i is $\bot$;

(iii) Finally, we prove that O is identical to O^ψ, that is, $O_{\mathcal{D}}(Z) = O^\psi_{\mathcal{D}}(Z)$ for any $Z \in \wp(\mathcal{D}^n)$.

Other Theories of Logicality

If our theory of classical logicality is correct, in other words, if our definition of the logical similarity relation $\equiv_C$ is appropriate, that would mean that any theories that employ different similarity relations are deficient. Before moving on, here we will critically observe two other theories of logicality.

Solomon Feferman's theory (Feferman[31]) establishes a similarity relation using surjections. A surjection $h : D \to D'$ determines a similarity relation between zeroth-order objects: for $a \in D$ and $b \in D'$, $\langle D, a\rangle$ and $\langle D', b\rangle$ are similar if $h(a) = b$. Also, for first-order objects $X \in \wp(D)$ and $Y \in \wp(D')$, $\langle D, X\rangle$ and $\langle D', Y\rangle$ are similar if for any $x \in D$, we have $x \in X$ if and only if $h(x) \in Y$. The similarity of higher-order objects and of objects of other types with respect to h can be defined in the same manner. Generally, $\langle D, X_1, \ldots, X_n\rangle$ and $\langle D', Y_1, \ldots, Y_n\rangle$ are said to be similar if there exists some surjection h such that $\langle D, X_1, \ldots, X_n\rangle$ and $\langle D', Y_1, \ldots, Y_n\rangle$ are similar by h. Such h is called a *homomorphism* from $\langle D, X_1, \ldots, X_n\rangle$ to $\langle D', Y_1, \ldots, Y_n\rangle$.

One motivation for choosing homomorphisms is to get rid of problematic cardinality operators, an example of which is the operator $O^{\aleph_1}$ such that for any first-order object $X \in \wp(D)$, $O^{\aleph_1}_D(X) = \mathsf{T}$ if and only if X is of cardinality $\aleph_1$. Since a set of cardinality $\aleph_1$ in a domain can be mapped to another set of a smaller size in a smaller domain by some homomorphism, $O^{\aleph_1}$ is excluded from Feferman's list of logical operators.

Along surjections, non-empty first-order objects can be "shrunk" to smaller first-order objects, and their cardinalities are changed into smaller ones (Feferman [31], p. 39). In the similarity relation made by homomorphisms, as in our logical similarity relation, there are two first-order objects that can be distinguished from others: the domain D and the empty set $\emptyset$. The domain D can be distinguished from other first-order objects, because it can be shrunk only to another (smaller) domain. The empty set $\emptyset$ is a special first-order object in that it cannot be shrunk further. With respect to the similarity relation among first-order objects, Feferman's and ours are quite similar (although they are not identical, because our similarity relation is an equivalence relation, while Feferman's is not).

In Feferman's theory, however, there are some similarity relations between higher-order objects that are disputable. Consider two finite domains $D_0 = \{a, b, c\}$ and $D'_0 = \{u, v\}$, and a surjection $h_0 : D_0 \to D'_0$ such that $h_0(a) = h_0(b) = u$ and $h_0(c) = v$. In his theory, the structures $\langle D_0, \{\emptyset, \{a, b\}, \{c\}, D_0\}\rangle$ and $\langle D'_0, \{\emptyset, \{u\}, \{v\}, D'{}_0\}\rangle$ are similar, because we have the following four similarity relations: (i) $\langle D_0, \emptyset\rangle$ and $\langle D'_0, \emptyset\rangle$; (ii) $\langle D_0, \{a, b\}\rangle$ and $\langle D'_0, \{u\}\rangle$; (iii) $\langle D_0, \{c\}\rangle$ and $\langle D'_0, \{v\}\rangle$; (iv) $\langle D_0, D_0\rangle$ and $\langle D'_0, D'{}_0\rangle$. However, their complements in their domains are not similar. The complement of $\{\emptyset, \{a, b\}, \{c\}, D_0\}$ in $\wp(D_0)$ is $\{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ and the complement of $\{\emptyset, \{u\}, \{v\}, D'{}_0\}$ in $\wp(D'_0)$ is $\emptyset$. $\{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ and $\emptyset$ are not similar in Feferman's theory.

The dissimilarity between the complements is a problem, because, as we have argued, if two second-order objects are logically similar, their complements are supposed to be logically similar as well. But, the more important problem is that the concept of logical similarity that Feferman's theory is based on is unclear. Why do the second-order objects above have to be regarded as logically similar? He might reply that two second-order objects can be regarded as logically similar if one is composed of first-order objects that can be shrunk to component first-order objects of the other. In fact, every component of the second-order object $\{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$ can be shrunk to some component of $\{\emptyset, \{d\}, \{e\}, \{d, e\}\}$ by h_0. Therefore, they have to be taken as logically similar.

I think that this answer is not clear enough. At least three questions on this concept of logical similarity can be raised. First, how could this concept of logical similarity be obtained? It seems that this concept is not widely held. Second, why does the similarity relation have to be made by surjections but not by any other kind of functions? By a surjection, a domain is always shrunk to another domain. But, by some general function (not a surjection), a domain can be shrunk to a proper subset of another domain. Why could not we suppose that the similarity relation can be made by any functions? Third, why does the similarity relation among second-order objects have to be made by surjections from D to D' but not by surjections from $\wp(D)$ to $\wp(D')$? Surjections from $\wp(D)$ to $\wp(D')$ produce a different similarity relation from the similarity relation by surjections from D to D'. Why is the former allowed to be used but the latter is not? Without convincing answers to these questions, we would have to say that his characterization of logicality is questionable.

Bonnay's theory (Bonnay[10]) is another theory that we can doubt. His theory employs *potential isomorphisms* to make a similarity relation. A potential isomorphism I between two structures $\langle D, X_1, \ldots, X_n\rangle$ and $\langle D', Y_1, \ldots, Y_n\rangle$ is a non-empty set of partial isomorphisms[10] $f : D \to D'$ satisfying the following condition:

> For all $f \in I$ and $a \in D$ (respectively, $b \in D'$), there is a $g \in I$ with $f \subseteq g$ and $a \in \text{dom}(g)$ (respectively, $b \in \text{rng}(g)$).

If there is a potential isomorphism, Bonnay thinks, two structures $\langle D, X_1, \ldots, X_n\rangle$ and $\langle D', Y_1, \ldots, Y_n\rangle$ are similar. The idea behind potential isomorphisms is that if two structures are "logically" similar, then they have isomorphic substructures and the partial isomorphisms between them can be *infinitely* extended if their domains are infinite.

Bonnay provides two justifications for his choice of potential isomorphism. The first justification is based on what he calls *the principle of closure under definability*, according to which "[a]n interpreted symbol definable only by means of logical constants is a logical constant" (Bonnay[10], p. 50). For example, if

[10]A function $f : D \to D'$ is a *partial isomorphism* between $\langle D, X_1, \ldots, X_n\rangle$ and $\langle D', Y_1, \ldots, Y_n\rangle$ if there exist a substructure of $\langle D, X_1, \ldots, X_n\rangle$ and a substructure of $\langle D, Y_1, \ldots, Y_n\rangle$ such that f is an isomorphism between them.

a collection S of operators contains the operator associated with the existential quantifier ($\exists$) and the operator associated with the classical negation ($\neg$), then the operator of the quantifier "$\neg\exists$" also has to be in S. He proves that among various possible similarity relations, the similarity relation defined by potential isomorphisms is the largest similarity relation satisfying the principle of closure under definability under which the operators of all standard logical components of first-order logic are invariant (ibid., p. 51, Theorem 3.10).

The second justification is given in relation to the notion of absoluteness. Bonnay claims that "if the only difference between two formally identical structures is set-theoretically problematic, these two structures should be logically similar" (ibid., p. 56). Logic is formal, and therefore logical operators have to be formal. An operator is applied to objects of a finite relational type, and generally objects have a variety of differences. For an operator to be formal, it has to be insensitive to many differences among them. He particularly thinks that a logical operator should be insensitive to set-theoretically problematic differences. What he means by a "set-theoretically problematic difference" is a difference made by a property that is not absolute. For example, the property of non-emptiness is an absolute property, while the property of uncountability is not. The logical similarity relation thus can be made based on the former but not on the latter. He then introduces a theorem that the similarity relation of potential isomorphisms is the smallest similarity relation satisfying the absoluteness condition (ibid., p. 57, Theorem 3.19).

The principle of closure under definability is convincing. As we have shown, in our theory too, the principle holds at least for operators of first-order quantifiers. However, the problem with Bonnay's theory is that the relationship between the notion of absoluteness and the notion of logical similarity is not obvious. In his theory, for example, for $D_0 = \{a, b, c\}$ and $D'_0 = \{u, v\}$, we have that $\langle D_0, \{a, b\}\rangle$ and $\langle D'_0, \{u\}\rangle$ are not similar; there is no potential isomorphism between them. Thus, $\{a, b\} \in \wp(D_0)$ and $\{u\} \in \wp(D'_0)$ are distinguished. The reason for the distinction between these first-order objects is that it can be made by an absolute property, namely, the property of cardinality of at least two.

However, it is unclear why a property can make a logical distinction among first-order objects if it is absolute. Contrary to Bonnay's view, we could suppose that any cardinality property, absolute or not, cannot make a logical distinction. The property of containing at least two objects is absolute, while the property of containing uncountably many objects is not absolute. But, both of them are about the sizes of first-order objects. If a logical distinction cannot be made by the latter, it could be supposed, based on their similarity, that a logical distinction cannot be made by the former either.

Any two first-order objects are similar in some aspects and dissimilar in others, and therefore any property P can make a similarity relation among them. For the characterization of logical operators, however, the problem is, in what sense two first-order objects X and Y can be regarded as logically dissimilar when one satisfies P and the other does not. In our theory, logical distinctions can be made by two properties: (i) the property of containing all zeroth-order objects in the domain; (ii) the property of containing no zeroth-order object in the domain. The reason that X and Y can be logically distinguished by, for example, the property (i) is that their complements are not composed of logically similar zeroth-order objects if X has the property and Y does not. For his choice of potential isomorphisms, Bonnay has to be able to provide an explanation of the same form. In order to claim that logical operators are operators invariant under potential isomorphisms, he needs further arguments to fill in the gap between the notion of absoluteness and the notion of logical similarity.

Logicality of Propositional Operators

In classical sentential logic, "logical connectives" refer to truth-functional connectives. All logical connectives are truth-functional, and only truth-functional connectives can be proper components of classical logic. Our next concern is the logicality of propositional operators. In what sense are the operators of truth-functional connectives logical?

Our approach to the logicality of propositional operators is the same as that to the logicality of objectual operators. We will establish a suitable logical similarity relation and characterize a logical operator as an operator that is invariant under the similarity relation. There are two ways to define propositional logical operators. One way is to use the same framework as the one for the characterization of objectual logical operators. By identifying 0-ary relations with the truth values T or F, an n-ary propositional operator can be identifies with a collection of objectual structures of the form $\langle D, v_1, \ldots v_n \rangle$, where v_i is T or F. The logicality of a propositional operator then can be determined based on whether or not the propositional operator is invariant under the similarity relation among such structures $\langle D, v_1, \ldots v_n \rangle$.

The other way is to use another but similar framework in which a domain W is a non-empty set of propositions with truth values. For a domain W, an n-ary *propositional operator* O_W acting on W is defined as a function $O_W : W^n \to \{\mathsf{T}, \mathsf{F}\}$. Also, for propositions $p_1, \ldots, p_n$ in W, a *propositional structure* of W is defined as an $(n+1)$-tuple $\langle W, p_1, \ldots, p_n \rangle$. By constructing the logical similarity relation among propositional structures, propositional logical operators can be characterized.

We will take the second way, because I think that a propositional operator is primarily applied to propositions but not 0-ary relations: the term "and" is primarily applied to sentences, and a propositional operator as its semantic value is applied to propositions expressed by the sentences. Both the first and second ways are supposed to produce the same results. I think, however, that the second way is more appropriate than the former from a philosophical point of view.

Let p be a proposition in a domain W and q a proposition in a domain W'. Considering that logic does not concern itself with the content of propositions, p and q can be distinguished only by their truth values. Then there are three possible choices about the logical similarity relation between p and q:

(i) p and q are logically similar if and only if they are assigned the same truth value in their domains;

(ii) p and q are logically similar if and only if they are assigned different truth values in their domains;

(iii) p and q are logically similar whatever their truth values are.

Clearly, the second option has to be ruled out; under that option, a proposition with a truth value will turn out to be logically dissimilar to itself.

The first option is our choice since the third option makes nonsense of logicality. I agree with John MacFarlane when he says, "there must be a distinction between designated and undesignated values in [the set of truth values]... And there must be a relation ... on multivalues by means of which implication can be defined" (MacFarlane[61], p. 227). To characterize the implication relation as a truth-preserving relation between premises and conclusions, the difference between the truth values is necessary: one is to be preserved and the other is not to be preserved. If there is no such difference, we would not be able to make a distinction, for example, between truth-preserving arguments and falsity-preserving arguments. In this case, we cannot define validity. If there is no validity, there can be no logical validity. For the notion of logicality to make sense, therefore, the distinction between the truth values must be presupposed.

Following the first option, we define the logical similarity relation between propositional structures as follows:

(i) $\langle W, p\rangle \equiv_C \langle W', q\rangle \overset{def}{\Longleftrightarrow}$ p and q are assigned the same truth value in W and W' respectively;

(ii) $\langle W, p_1, \ldots, p_n\rangle \equiv_C \langle W', q_1, \ldots, q_n\rangle \overset{def}{\Longleftrightarrow} \langle W, p_i\rangle \equiv_C \langle W', q_i\rangle$ for all $i \in \{1, \ldots, n\}$.

We say that a propositional operator O defined on all domains is *logical* if it is $\equiv_C$-invariant. It can be easily shown that all and only operators of truth-functional connectives are logical.

3.3 Non-Classical Logical Operators

A characterization of non-classicality is our concern in the rest of the present chapter. For our theory of logical operators to be comprehensive, it has to be able to explain not only the logicality of operators associated with terms used in classical logic but also the logicality of operators associated with terms used in non-classical logics. There is another reason that motivates us to pursue non-classical logicality: *logical pluralism*. The logical pluralism proposed by JC. Beall and Greg Restall (Beall and Restall[4]) claims that there is more than one genuine deductive consequence relation, examples of which are those of classical logic, intuitionistic logic, and relevant logic.[11] If these non-classical logics are genuine logics, how can their logicality be characterized? If one is a logical pluralist of the Beall-Restall type, it is necessary to answer this question. Even if one is not, it is still fruitful to understand how the characterization goes, since one would not be able to reject the genuineness of the non-classical logics without knowing in what sense they are logical.

Among various non-classical logics, we will mainly discuss the logicality of intuitionistic logic, which we will call "intuitionistic logicality." There are two reasons for our choice. First, intuitionistic logic, as mentioned above, is one of the genuine logics according to the Beall-Restall type of logical pluralism. Second, the characterization of intuitionistic logicality can be straightforwardly applied to those of relevant logic and modal logic. Intuitionistic logicality can serve as a useful base for defining the logicality of other non-classical logics.

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[11] According to the logical pluralism, a genuine logic is given by an admissible instance of what they call the *Generalised Tarski Thesis*:

> An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion (ibid., p. 29).

The Generalised Tarski Thesis is "a recipe for specific accounts of consequence" (ibid., p. 29). By replacing "case$_x$" with an appropriate concept of admissible cases, we will obtain the settled core of the consequence relation of a logic. They then argue that classical logic, intuitionistic logic, and relevant logic can be obtained from the Generalised Tarski Thesis. If we substitute Tarskian model for case$_x$, we have,

> An argument is valid if and only if, in every Tarskian model in which the premises are true, so is the conclusion,

which is identical to Tarski's characterization of the classical logical consequence relation. In addition to Tarskian models, there are other kinds of admissible cases that can properly fill in the Generalised Tarski Thesis to produce the core of a consequence relation: "stages" for intuitionistic logic, and "situations" for relevant logic, where "[s]tages can be thought of as steps in a process of construction or verification" (ibid., p. 62), and "[s]ituations are simply parts of the world" (ibid., p. 49). A distinctive feature of stages is that some are *incomplete*. A stage is incomplete if there is a sentence φ such that both φ and its negation, $\neg\varphi$, are false in the stage. A characteristic of situations is that some are incomplete and some are *inconsistent* in that in each of those situations, there is a sentence ψ such that both ψ and $\neg\psi$ are true. These admissible cases yield different instances of the Generalised Tarski Thesis, and therefore, different genuine logics. Hence, there is more than one genuine logic.

What is needed for the characterization of intuitionistic logicality is, as in the case of classical logicality, a logical similarity relation. For intuitionistic logicality, however, some adjustments are inevitable because of differences between intuitionistic logic and classical logic. The most significant difference between these logical systems for our discussion is that the domains of classical logic are independent of one another, while those of intuitionistic logic are connected by the accessibility relation. And the negation, conditional, and universal quantifier of intuitionistic logic are defined using the accessibility relation. As opposed to the operators of classical logic, the truth values that the operators of these logical constants assign are not determined within individual domains: they are determined in relation to domains accessible from those domains. This difference will affect our approach to intuitionistic logicality. In order to set out the logical similarity relation of intuitionistic logicality, we have to allow for the role that the accessibility relation plays in intuitionistic logic.

Intuitionistic Logicality of Propositional Operators

In intuitionistic logic, domains are associated with each other by the accessibility relation in a *model*. A *propositional model* of intuitionistic logic is a pair $\langle \mathfrak{W}, \sqsubseteq \rangle$ such that:

(i) $\mathfrak{W}$ is a non-empty set of domains;

(ii) A domain $W \in \mathfrak{W}$ is a non-empty set of propositions with truth values;

(iii) $\sqsubseteq$ is a partial order on $\mathfrak{W}$, i.e. $\sqsubseteq$ is a reflexive, antisymmetric, and transitive binary relation on $\mathfrak{W}$;

(iv) Propositions are preserved under $\sqsubseteq$. For a proposition p, if p is in W, and if $W \sqsubseteq W'$, then p is also in W';

(v) The truth of a proposition is taken over under $\sqsubseteq$. That is, if p is true in W, and if $W \sqsubseteq W'$, then p is also true in W.

For a propositional model $M = \langle \mathfrak{W}, \sqsubseteq \rangle$ and for $p_1, \ldots, p_n$ in $W \in \mathfrak{W}$, we define a *propositional structure* of intuitionistic logic as a tuple $\langle M, W, p_1, \ldots, p_n \rangle$. The logical similarity relation to be established for the intuitionistic logicality of propositional operators is a relation between propositional structures.

In order to determine the logical similarity of classical logicality, we had to only compare one domain with another domain, and one proposition with another proposition. However, the present situation is more complex. Let us take two models as an example:

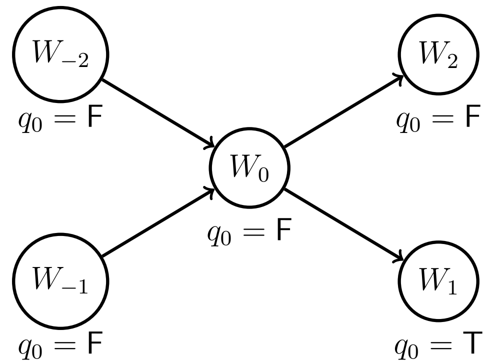

Figure 3.4: Model M_0

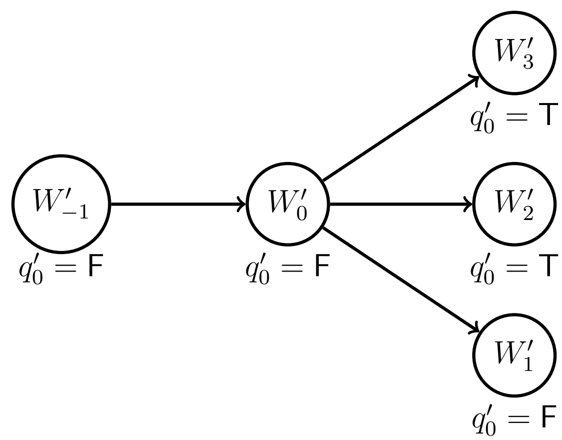

Figure 3.5: Model M'_0

In the figures above, the arrows expresses the accessibility relations. Since the accessibility relation of intuitionistic logic is reflexive and transitive, arrows should also have been drawn from a domain to itself and between two domains such that there is a path representing the accessibility relation between them. However, such arrows are omitted for clarity of illustration. "$q_0 = \mathsf{T}$" (or "$q_0 = \mathsf{F}$") underneath domains means that q_0 is true (or false) in those domains.

Are $\langle M_0, W_0, q_0\rangle$ and $\langle M'_0, W'_0, q'_0\rangle$ logically similar? Or, can they be logically distinguished? The models M_0 and M'_0 contain different domains composed of different propositions. Those different domains are connected in different ways, and q_0 and q'_0 are given different truth values within them. There are various differences between the two models. Of these differences, some are essential to intuitionistic logicality, while others can be ignored. In order to establish the logical similarity relation of intuitionistic logic, we must determine which aspects of the models are eligible to create a logical distinction between them.

Notice that the model M_0 is divided into three parts: left, center, and right. The center part is the set composed of a single domain W_0. The left part is the set of domains accessible *to* W_0 ($\{W_{-1}, W_{-2}\}$). The right part is the set of domains accessible *from* W_0 ($\{W_1, W_2\}$). The same division is applicable to the model M'_0 and to any other models. Generally, for a model $M = \langle \mathfrak{W}, \sqsubseteq\rangle$ and for a domain $W \in \mathfrak{W}$, the domains around W are classified into three kinds on the same basis. We will write the singleton set of the center part "$\mathfrak{W}_W$", the set of the left-hand domains "$\mathfrak{W}_{\sqsubset W}$" and the set of the right-hand domains "$\mathfrak{W}_{W\sqsubset}$":

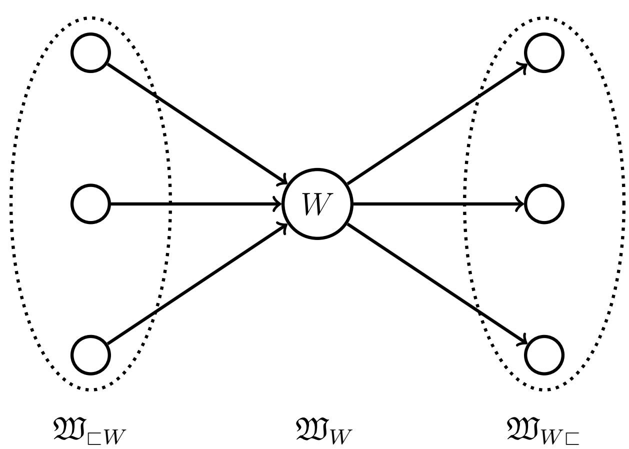

Figure 3.6: Three kinds of domains around W

$$\begin{aligned}
\mathfrak{W}_W &\overset{def}{:=} \{W\};\\
\mathfrak{W}_{\sqsubset W} &\overset{def}{:=} \{W' \in \mathfrak{W} : W' \sqsubseteq W, W' \neq W\};\\
\mathfrak{W}_{W\sqsubset} &\overset{def}{:=} \{W'' \in \mathfrak{W} : W \sqsubseteq W'', W'' \neq W\}.
\end{aligned}$$

One approach to the logical similarity relation of intuitionistic logicality is to compare the "logicality" with respect to each part. For two propositional structures, we will consider three aspects:

(i) Whether or not the propositional structures are logically similar in the center part;

(ii) Whether or not the propositional structures are logically similar in the left part;

(iii) Whether or not the propositional structures are logically similar in the right part.

For the propositional structures to be regarded as logically similar, it is necessary that they are logically similar in all the parts. If they are not logically similar in some parts, they can be logically distinguished.

In order to define the partial logicality restricted to each part, we will introduce two new similarity relations. Let V be a domain of propositions with truth values and let p be a proposition. V may or may not contain p. If V contains p, p is assigned a truth value in V. Then, pairs $\langle V, p\rangle$ fall into three types:

(i) Type 1 $\langle V_1, p_1\rangle$: V_1 contains p_1, and p_1 is true in V_1;

(ii) Type 2 $\langle V_2, p_2\rangle$: V_2 contains p_2, and p_2 is false in V_2;

(iii) Type 3 $\langle V_3, p_3\rangle$: V_3 does not contain p_3.

For example, in the model M_0 above, the pair $\langle W_0, q_0\rangle$ is classified as type 2, while the pair $\langle W_1, q_0\rangle$ is of type 1. For any $i \in \{-2, -1, 0, 1, 2\}$, the pair $\langle W_i, q_0\rangle$ does not belong to type 3 because every W_i contains q_0. These three types are distinguished from each other from a logical point of view in the following sense. Type 3 is distinguished from types 1 and 2 because V_3 does not contain p_3 while V_1 and V_2 do contain p_1 and p_2 respectively. Types 1 and 2 are different from one another. p_1 and p_2 are assigned different truth values in their domains: in other words, $\langle V_1, p_1\rangle$ and $\langle V_2, p_2\rangle$ are not $\equiv_C$-similar.

For tuples $\langle V, p_1, \dots, p_n\rangle$ and $\langle V', p'_1, \dots, p'_n\rangle$, we define a new similarity relation $\equiv_{C\subseteq}$ as follows:

(i) $\langle V, p_i\rangle \equiv_{C\subseteq} \langle V', p'_i\rangle \overset{def}{\iff} \langle V, p_i\rangle$ and $\langle V', p'_i\rangle$ are of the same type;

(ii) $\langle V, p_1, \dots, p_n\rangle \equiv_{C\subseteq} \langle V', p'_1, \dots, p'_n\rangle \overset{def}{\iff} \langle V, p_i\rangle \equiv_{C\subseteq} \langle V', p'_i\rangle$ for all $i \in \{1, \dots, n\}$.

Given that V contains all $p_1, \ldots, p_n$ and also V' contains all $p'_1, \ldots, p'_n$, we have:

$$\langle V, p_1, \ldots, p_n \rangle \equiv_{C\subseteq} \langle V', p'_1, \ldots, p'_n \rangle$$
$$\text{if and only if}$$
$$\langle V, p_1, \ldots, p_n \rangle \equiv_{C} \langle V', p'_1, \ldots, p'_n \rangle.$$

Thus, the new similarity relation $\equiv_{C\subseteq}$ can be regarded as an extension of the logical similarity relation of classical logicality $\equiv_C$.

Another new similarity relation that we will introduce is a relation between tuples $\langle \mathfrak{X}, p_1, \ldots, p_n \rangle$ and $\langle \mathfrak{X}', p'_1, \ldots, p'_n \rangle$ where $\mathfrak{X}$ and $\mathfrak{X}'$ are sets of domains of propositions with truth values. We say that $\langle \mathfrak{X}, p_1, \ldots, p_n \rangle$ and $\langle \mathfrak{X}', p'_1, \ldots, p'_n \rangle$ are *P-similar* if the following conditions are satisfied:

(i) For all $V_1 \in \mathfrak{X}$, there is $V'_1 \in \mathfrak{X}'$ such that $\langle V_1, p_1, \ldots, p_n \rangle \equiv_{C\subseteq} \langle V'_1, p'_1, \ldots, p'_n \rangle$;

(ii) For all $V'_2 \in \mathfrak{X}'$, there is $V_2 \in \mathfrak{X}$ such that $\langle V_2, p_1, \ldots, p_n \rangle \equiv_{C\subseteq} \langle V'_2, p'_1, \ldots, p'_n \rangle$.

Briefly speaking, the P-similarity between $\langle \mathfrak{X}, p_1, \ldots, p_n \rangle$ and $\langle \mathfrak{X}', p'_1, \ldots, p'_n \rangle$ means that $\mathfrak{X}$ and $\mathfrak{X}'$ are composed of domains with respect to which $\langle p_1, \ldots, p_n \rangle$ and $\langle p'_1, \ldots, p'_n \rangle$ are $\equiv_{C\subseteq}$-similar respectively. Thus, if, on the other hand, $\langle \mathfrak{X}, p_1, \ldots, p_n \rangle$ and $\langle \mathfrak{X}', p'_1, \ldots, p'_n \rangle$ are not P-similar, it means that they are composed of logically distinguishable domains. Note that the correspondence between domains of $\mathfrak{X}$ and of $\mathfrak{X}'$ does not need to be one-to-one.

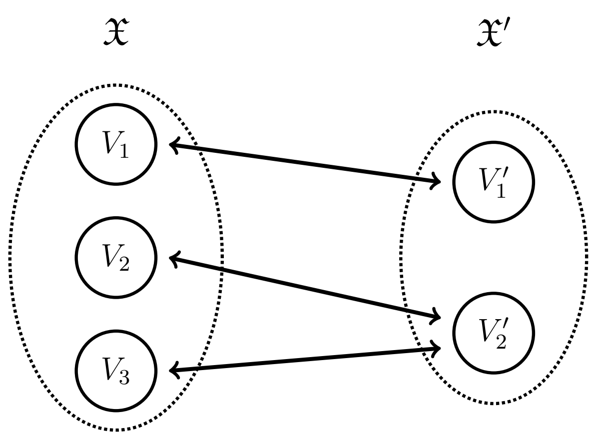

Figure 3.7: $\mathfrak{X}$ and $\mathfrak{X}'$ of different cardinalities

In the example of the models M_0 and M'_0, $\langle \mathfrak{W}_{0\sqsubset W_0}, q_0 \rangle$ and $\langle \mathfrak{W}'_{0\sqsubset W'_0}, q'_0 \rangle$ are P-similar; we have $\langle W_{-1}, q_0 \rangle \equiv_C \langle W'_{-1}, q'_0 \rangle$ and $\langle W_{-2}, q_0 \rangle \equiv_C \langle W'_{-1}, q'_0 \rangle$. Also, it can be easily observed that $\langle \mathfrak{W}_{0W_0}, q_0\rangle$ and $\langle \mathfrak{W}'_{0W'_0}, q'_0\rangle$ are P-similar and that $\langle \mathfrak{W}_{0W_0\sqsubset}, q_0\rangle$ and $\langle \mathfrak{W}'_{0W'_0\sqsubset}, q'_0\rangle$ are P-similar.

Using the notion of P-similarity, I propose the definition of the logical similarity relation of intuitionistic logicality as follows. We say that two propositional structures $\langle M, W, p_1, \ldots, p_n\rangle$ and $\langle M', W', p'_1, \ldots, p'_n\rangle$ are *quasi-logically similar* if they satisfy the following three conditions:

(i) $\langle \mathfrak{W}_W, p_1, \ldots, p_n\rangle$ and $\langle \mathfrak{W}'_{W'}, p'_1, \ldots, p'_n\rangle$ are P-similar;
(ii) $\langle \mathfrak{W}_{\sqsubset W}, p_1, \ldots, p_n\rangle$ and $\langle \mathfrak{W}'_{\sqsubset W'}, p'_1, \ldots, p'_n\rangle$ are P-similar;
(iii) $\langle \mathfrak{W}_{W\sqsubset}, p_1, \ldots, p_n\rangle$ and $\langle \mathfrak{W}'_{W'\sqsubset}, p'_1, \ldots, p'_n\rangle$ are P-similar.

The quasi-logical similarity means that the two propositional structures are P-similar in all the three parts.

For a model $M = \langle \mathfrak{W}, \sqsubseteq\rangle$, let $\langle W_1, \ldots, W_l\rangle$ be an n-tuple of domains such that $W_i \sqsubseteq W_{i+1}$ for all i. We call such an n-tuple a "connected path" from W_1 to W_l. If some W_i is W, we call it a connected path through W. We say that $\langle M, W, p_1, \ldots, p_n\rangle$ and $\langle M', W', p'_1, \ldots, p'_n\rangle$ are *logically similar* if they satisfy the following conditions:

(i) $\langle M, W, p_1, \ldots, p_n\rangle$ and $\langle M', W', p'_1, \ldots, p'_n\rangle$ are quasi-logically similar;
(ii) For any connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l\rangle$, there exists some connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l\rangle$, such that $\langle M, W_i, p_1, \ldots, p_n\rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n\rangle$ are quasi-logically similar for any i;
(iii) For any connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l\rangle$, there exists some connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l\rangle$, such that $\langle M, W_i, p_1, \ldots, p_n\rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n\rangle$ are quasi-logically similar for any i.

What the conditions (ii) and (iii) mean is the correspondence between connected paths through W and connected paths through W'. The condition (ii) says that, for any connected path through W, the existence of some connected path through W' such that their corresponding domains are quasi-logically similar is necessary for the two structures to be regarded as logically similar. And the condition (iii) states the same condition of the opposite direction.

Consider the following two models:

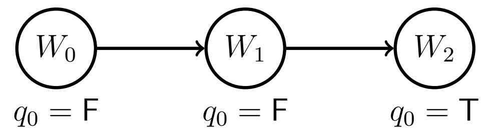

Figure 3.8: Model M_0

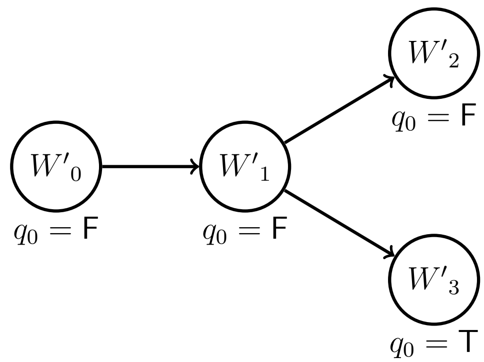

Figure 3.9: Model M_0'

The two propositional structures $\langle M_0, W_0, q_0 \rangle$ and $\langle M'_0, W'_0, q_0 \rangle$ are quasi-logically similar, because their center parts ($\{W_0\}$ and $\{W'_0\}$) and their right parts ($\{W_1, W_2\}$ and $\{W'_1, W'_2, W'_3\}$) are P-similar respectively. However, they are not logically similar. Since $\langle M_0, W_2, q_0 \rangle$ and $\langle M'_0, W'_2, q_0 \rangle$ are not quasi-logically similar, there is no path through W_0 that corresponds to the path through W_0', $\langle W_0', W_1', W_2' \rangle$.

We write $\langle M, W, p_1, \ldots, p_n \rangle \equiv_I \langle M', W', p_1', \ldots, p_n' \rangle$ if they are logically similar. We say that a propositional operator O defined on any domains of any models is *intuitionistically logical* if O is $\equiv_I$-invariant.

The operator O^N of the intuitionistic negation is defined as follows:

$$O_W^N(p) \quad = \quad \begin{cases} \mathsf{T} & \text{if } p \text{ is false in all } W' \text{ such that } W \sqsubseteq W' \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

O^N is $\equiv_I$-invariant. Suppose that $\langle M, W, p \rangle \equiv_I \langle M', W', p' \rangle$. Then, we have $\langle \mathfrak{W}_W, p \rangle$ and $\langle \mathfrak{W}'_{W'}, p' \rangle$ are P-similar, and also $\langle \mathfrak{W}_{W\sqsubset}, p \rangle$ and $\langle \mathfrak{W}'_{W'\sqsubset}, p' \rangle$ are P-similar. We need to show that $O_W^N(p) = O_{W'}^N(p')$. If $O_W^N(p) = \mathsf{T}$, then, by the definition of P-similarity, for all $W'_0 \in \mathfrak{W}'$ such that $W' \sqsubseteq W'_0$, there exists $W_0 \in \mathfrak{W}$ such that $W \sqsubseteq W_0$ and such that $\langle W_0, p\rangle \equiv_{C\subseteq} \langle W'_0, p'\rangle$. Since p is false in W_0, it holds that p' is also false in W'_0. Therefore, $O^N_{W'}(p') = \mathsf{T}$. If, on the other hand, $O^N_W(p) = \mathsf{F}$, then, there exists $W_1 \in \mathfrak{W}$ such that $W \sqsubseteq W_1$ and such that p is true in W_1. By the definition of P-Similarity, there exists $W'_1 \in \mathfrak{W}'$ such that $W' \sqsubseteq W'_1$ and $\langle W_1, p\rangle \equiv_{C\subseteq} \langle W'_1, p'\rangle$. Since p is true in W_1, we have that p' is also true in W'_1. Thus, $O^N_{W'}(p') = \mathsf{F}$. Therefore, we have $O^N_W(p) = O^N_{W'}(p')$, and, as a consequence, O^N is an intuitionistically logical operator. Similar proofs can be given to show the logicality of the operators of the conditional, the conjunction, and the disjunction of intuitionistic logic.[12]

In addition to the operators of the standard logical connectives, there are a variety of intuitionistically logical operators, some examples of which are as follows:

(i) $O^{(1)}_W(p) = \mathsf{T}$ if there exists W' containing p such that $W' \sqsubseteq W$ and p is true in W'; otherwise $O^{(1)}_W(p) = \mathsf{F}$.

(ii) $O^{(2)}_W(p_1, p_2) = \mathsf{T}$ if p_1 is true in all W' such that $W \sqsubseteq W'$ and if there exists W'' containing p_2 such that $W'' \sqsubseteq W$ and p_2 is false in W''; otherwise $O^{(2)}_W(p_1, p_2) = \mathsf{F}$.

Intuitionistic Logicality of Objectual Operators

The logicality of objectual operators of intuitionistic logic can be characterized in the same way as that of propositional operators. We will define objectual models, objectual structures, and the logical similarity relation between them. Although we can identify intuitionistically logical operators of any finite relational types, we here restrict our attention to the intuitionistic logicality of operators of unary first-order quantifiers. The reason for the restriction is twofold. First, I

[12] The operators of the conditional, the conjunction, and the disjunction of intuitionistic logic (written O^{cond}, O^{conj}, and $O^{\mathcal{D}isj}$ respectively) are defined as follows:

(i) $O^{cond}_W(p_1, p_2) = \mathsf{T}$ if p_1 is false or p_2 is true in all W' such that $W \sqsubseteq W'$; otherwise $O^{cond}_W(p_1, p_2) = \mathsf{F}$.

(ii) $O^{conj}_W(p_1, p_2) = \mathsf{T}$ if p_1 and p_2 are true in W; otherwise $O^{conj}_W(p_1, p_2) = \mathsf{F}$.

(iii) $O^{Disj}_W(p_1, p_2) = \mathsf{T}$ if p_1 or p_2 is true in W; otherwise $O^{\mathcal{D}isj}_W(p_1, p_2) = \mathsf{F}$.

suppose that most logicians and philosophers are primarily interested in how the logicality of operators of the universal and existential quantifiers of intuitionistic logic can be obtained. Second, the idea of how to characterize the intuitionistic logicality of unary first-order quantifiers can be straightforwardly applied to that of any other objectual operators.

An *objectual model* of intuitionistic logic is a triple $\langle \mathfrak{D}, \mathcal{D}^* \sqsubseteq \rangle$ such that:

(i) $\mathfrak{D}$ is a non-empty set of domains;

(ii) $\mathcal{D}^*$ is a non-empty set of objects[13];

(iii) A domain $\mathcal{D} \in \mathfrak{D}$ is a set of objects such that $\mathcal{D} \subseteq \mathcal{D}^*$;

(iv) $\sqsubseteq$ is a partial order on $\mathfrak{D}$;

(v) Objects in domains are preserved under $\sqsubseteq$. That is to say, if $\mathcal{D} \sqsubseteq \mathcal{D}'$ then $\mathcal{D} \subseteq \mathcal{D}'$.

For an objectual model $M = \langle \mathfrak{D}, \mathcal{D}^*, \sqsubseteq \rangle$, and for $\mathcal{D} \in \mathfrak{D}$, we define a *first-order objectual structure* of intuitionistic logic as a triple $\langle M, \mathcal{D}, U \rangle$ where U is a set of pairs $\langle \mathcal{D}', X' \rangle$ such that:

(i) $\mathcal{D}' \in \mathfrak{D}$;

(ii) $X' \subseteq \mathcal{D}'$;

(iii) For $\langle \mathcal{D}', X' \rangle$ and $\langle \mathcal{D}'', X'' \rangle$, if $\mathcal{D}' \sqsubseteq \mathcal{D}''$, then $X' \subseteq X''$.

The pair $\langle \mathcal{D}', X' \rangle$ is intended to represent the pair of a domain and the extension of a unary predicate of zeroth-order objects on that domain. U can be seen as the collection of such pairs. For example, the predicate "is an apple" has a subset in a domain as its extension. For $M_0 = \langle \mathfrak{D}_0, \mathcal{D}_0^*, \sqsubseteq \rangle$, and for $\mathcal{D}_0 \in \mathfrak{D}_0$, the predicate determines one first-order objectual structure $\langle M_0, \mathcal{D}_0, U_0 \rangle$ where $U_0 = \{\langle \mathcal{D}_0', X_0' \rangle :$ X_0' is the set of all apples in $\mathcal{D}_0' \in \mathfrak{D}_0\}$.

As in the case of propositional structure of intuitionistic logic, for a model $M = \langle \mathfrak{D}, \mathcal{D}^*, \sqsubseteq \rangle$ and for a domain $\mathcal{D} \in \mathfrak{D}$, the domains around $\mathcal{D}$ are divided into three parts:

$$\begin{aligned} \mathfrak{D}_{\mathcal{D}} &\overset{def}{:=} \{\mathcal{D}\}; \\ \mathfrak{D}_{\sqsubset \mathcal{D}} &\overset{def}{:=} \{\mathcal{D}' \in \mathfrak{D} : \mathcal{D}' \sqsubseteq \mathcal{D}, \mathcal{D}' \neq \mathcal{D}\}; \\ \mathfrak{D}_{\mathcal{D} \sqsubset} &\overset{def}{:=} \{\mathcal{D}'' \in \mathfrak{D} : \mathcal{D} \sqsubseteq \mathcal{D}'', \mathcal{D}'' \neq \mathcal{D}\}. \end{aligned}$$

[13] $\mathcal{D}^*$ is intended to represent the set of all objects that can possibly be constructed at some domains in the objectual model. See Priest[84], pp. 421–423.

Also, as before, we will introduce another similarity relation for the partial logicality with respect to each part. Let E be a domain and let Z be a subset of E. Then, pairs $\langle E, Z\rangle$ are divided into three types:

(i) Type 1 $\langle E_1, Z_1\rangle$: $Z_1 = E_1$;

(ii) Type 2 $\langle E_2, Z_2\rangle$: $Z_2 = \emptyset$;

(iii) Type 3 $\langle E_3, Z_3\rangle$: $\emptyset \subsetneq Z_3 \subsetneq E_3$.

These three types are logically distinguishable; for $i, j \in \{1, 2, 3\}$ with $i \neq j$, $\langle E_i, Z_i\rangle$ and $\langle E_j, Z_j\rangle$ are not $\equiv_C$-similar.

Let $\mathfrak{Y}$ be a non-empty set of pairs $\langle E, Z\rangle$ of a domain E and a subset Z of E. Also, let $\mathfrak{Y}'$ be a non-empty set of pairs $\langle E', Z'\rangle$ of a domain E' and a subset Z' of E'. We say that $\mathfrak{Y}$ and $\mathfrak{Y}'$ are *O-similar* if the following conditions are satisfied:

(i) For all $\langle E_1, Z_1\rangle \in \mathfrak{Y}$, there exists $\langle E_1', Z_1'\rangle \in \mathfrak{Y}'$ such that $\langle E_1, Z_1\rangle$ and $\langle E_1', Z_1'\rangle$ are of the same type;

(ii) For all $\langle E_2', Z_2'\rangle \in \mathfrak{Y}'$, there exists $\langle E_2, Z_2\rangle \in \mathfrak{Y}$ such that $\langle E_2, Z_2\rangle$ and $\langle E_2', Z_2'\rangle$ are of the same type.

The O-similarity between $\mathfrak{Y}$ and $\mathfrak{Y}'$, as with the P-similarity, means that $\mathfrak{Y}$ and $\mathfrak{Y}'$ are composed of logically similar pairs.

For first-order objectual structures $\langle M, \mathcal{D}, U\rangle$ and $\langle M', \mathcal{D}', U'\rangle$, we say that they are *quasi-logically similar* if they satisfy the following conditions:

(i) $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}}\}$ and $\{\langle \mathcal{D}_0', X_0'\rangle \in U' : \mathcal{D}_0' \in \mathfrak{D}'_{\mathcal{D}'}\}$ are O-similar;

(ii) $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\sqsubset\mathcal{D}}\}$ and $\{\langle \mathcal{D}_0', X_0'\rangle \in U' : \mathcal{D}_0' \in \mathfrak{D}'_{\sqsubset\mathcal{D}'}\}$ are O-similar;

(iii) $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}\sqsubset}\}$ and $\{\langle \mathcal{D}_0', X_0'\rangle \in U' : \mathcal{D}_0' \in \mathfrak{D}'_{\mathcal{D}'\sqsubset}\}$ are O-similar.

Also, we say that they are *logically similar* if the following conditions are met (written as "$\langle M, \mathcal{D}, U\rangle \equiv_I \langle M', \mathcal{D}', U'\rangle$"):

(i) $\langle M, \mathcal{D}, U\rangle$ and $\langle M', \mathcal{D}', U'\rangle$ are quasi-logically similar;

(ii) For any connected path thorough $\mathcal{D}$, $\langle \mathcal{D}_1, \ldots \mathcal{D}_{j-1}, \mathcal{D}, \mathcal{D}_{j+1}, \ldots \mathcal{D}_l\rangle$, there exists some connected path thorough $\mathcal{D}'$, $\langle \mathcal{D}_1', \ldots \mathcal{D}_{j-1}', \mathcal{D}', \mathcal{D}_{j+1}', \ldots \mathcal{D}_l'\rangle$, such that $\langle M, \mathcal{D}_i, U\rangle$ and $\langle M', \mathcal{D}_i', U'\rangle$ are quasi-logically similar for any i;

(iii) For any connected path thorough $\mathcal{D}'$, $\langle \mathcal{D}_1', \ldots \mathcal{D}_{j-1}', \mathcal{D}', \mathcal{D}_{j+1}', \ldots \mathcal{D}_l'\rangle$, there exists some connected path thorough $\mathcal{D}$, $\langle \mathcal{D}_1, \ldots \mathcal{D}_{j-1}, \mathcal{D}, \mathcal{D}_{j+1}, \ldots \mathcal{D}_l\rangle$, such that $\langle M, \mathcal{D}_i, U\rangle$ and $\langle M', \mathcal{D}_i', U'\rangle$ are quasi-logically similar for any i;

An objectual operator O is said to be *intuitionistically logical* if it is $\equiv_I$-invariant.

The operators of the universal and existential quantifiers of intuitionistic logic, which are defined as follows, are $\equiv_I$-invariant: for a first-order objectual structure $\langle M, \mathcal{D}, U\rangle$,

(i) $O^{\forall}_{\mathcal{D}}(U) = \mathsf{T}$ if for all $\langle \mathcal{D}', X'\rangle \in U$ such that $\mathcal{D} \sqsubseteq \mathcal{D}'$, it holds that $X' = \mathcal{D}'$; otherwise $O^{\forall}_{\mathcal{D}}(U) = \mathsf{F}$.

(ii) $O^{\exists}_{\mathcal{D}}(U) = \mathsf{T}$ if for $\langle \mathcal{D}, X\rangle \in U$, it holds that $X \neq \emptyset$; otherwise $O^{\exists}_{\mathcal{D}}(U) = \mathsf{F}$.

Suppose that $\langle M, \mathcal{D}, U\rangle \equiv_I \langle M', \mathcal{D}', U'\rangle$. Then, it holds that $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}}\}$ and $\{\langle \mathcal{D}'_0, X'_0\rangle \in U' : \mathcal{D}'_0 \in \mathfrak{D}'_{\mathcal{D}'}\}$ are O-similar. Also, $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}\sqsubset}\}$ and $\{\langle \mathcal{D}'_0, X'_0\rangle \in U' : \mathcal{D}'_0 \in \mathfrak{D}'_{\mathcal{D}'\sqsubset}\}$ are O-similar. By the definition of $O^{\forall}$, $O^{\forall}_{\mathcal{D}}(U) = \mathsf{T}$ if and only if $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}}\}$ and $\{\langle \mathcal{D}_0, X_0\rangle \in U : \mathcal{D}_0 \in \mathfrak{D}_{\mathcal{D}\sqsubset}\}$ contain only pairs of type 1 above. And $O^{\forall}_{\mathcal{D}'}(U') = \mathsf{T}$ if and only if $\{\langle \mathcal{D}'_0, X'_0\rangle \in U' : \mathcal{D}'_0 \in \mathfrak{D}'_{\mathcal{D}'}\}$ and $\{\langle \mathcal{D}'_0, X'_0\rangle \in U' : \mathcal{D}'_0 \in \mathfrak{D}'_{\mathcal{D}'\sqsubset}\}$ contain only pairs that belong to type 1. By the definition of O-similarity, therefore, we have $O^{\forall}_{\mathcal{D}}(U) = O^{\forall}_{\mathcal{D}'}(U')$. Hence, the operator $O^{\forall}$ is intuitionistically logical. The logicality of $O^{\exists}$ can be shown in the same way.

Logicality of Other Non-Classical Logics

Relevant logic is another non-classical logic that the Beall-Restall type of logical pluralism claims as a genuine logic. The characterization of *relevant logicality* is more complex than that of intuitionistic logicality because there are three kinds of relations between domains of relevant logic. However, the process of characterizing relevant logicality is similar to that of characterizing intuitionistic logicality.

A *propositional model* of relevant logic[14] is a 5-tuple $\langle \mathfrak{W}, \mathfrak{N}, \sqsubseteq, \mathsf{R}, \mathsf{C}\rangle$ where

(i) $\mathfrak{W}$ is a non-empty set of domains;

(ii) A domain W is a set of propositions with truth values;

(iii) $\mathfrak{N} \subseteq \mathfrak{W}$ (an element of $\mathfrak{N}$ is a normal domain of W);

(iv) $\sqsubseteq$ is a partial order on $\mathfrak{W}$;

(v) Propositions are preserved under $\sqsubseteq$;

[14]We will formulate propositional models of relevant logic based on Restall[96].

(vi) The truth of a proposition is taken over under $\sqsubseteq$;

(vii) R is a ternary relation on $\mathfrak{W}$ satisfying:

(a) For all $N \in \mathfrak{N}$, $\langle N, W, W' \rangle \in \mathsf{R}$ if and only if $W = W'$;

(b) if $\langle W_1, W_2, W_3 \rangle \in \mathsf{R}$, and also if, for $W_0', W_1', W_2' \in \mathfrak{W}$, we have $W_0' \sqsubseteq W_0, W_1' \sqsubseteq W_1$ and $W_2' \sqsupseteq W_2$, then $\langle W_0', W_1', W_2' \rangle \in \mathsf{R}$;

(viii) C is a binary relation on $\mathfrak{W}$ satisfying:

$$\text{if } \langle W_0, W_1 \rangle \in \mathsf{C},\ W_0' \sqsubseteq W_0, \text{ and } W_1' \sqsubseteq W_1, \text{ then } \langle W_0', W_1' \rangle \in \mathsf{C}.$$

For a propositional model $M = \langle \mathfrak{W}, \mathfrak{N}, \sqsubseteq, \mathsf{R}, \mathsf{C} \rangle$ and for propositions $p_1, \ldots, p_n$ in $W \in \mathfrak{W}$, we define a *propositional structure* of relevant logic as a tuple $\langle M, W, p_1, \ldots, p_n \rangle$.

For a model $M = \langle \mathfrak{W}, \mathfrak{N}, \sqsubseteq, \mathsf{R}, \mathsf{C} \rangle$ and for a domain $W \in \mathfrak{W}$, the domains around W and their pairs are divided into the following parts with respect to the three relations $\sqsubseteq$, R, and C:

$$\begin{aligned}
\mathfrak{W}_W &\overset{def}{:=} \{W\}; \\
\mathfrak{W}_{\sqsubset W} &\overset{def}{:=} \{W' \in \mathfrak{W} : W' \sqsubseteq W, W' \neq W\}; \\
\mathfrak{W}_{W \sqsubset} &\overset{def}{:=} \{W'' \in \mathfrak{W} : W \sqsubseteq W'', W'' \neq W\}; \\
\mathfrak{W}_{\mathsf{R}_1 W} &\overset{def}{:=} \{\langle W', W'' \rangle \in \mathfrak{W} \times \mathfrak{W} : \langle W, W', W'' \rangle \in \mathsf{R}\}; \\
\mathfrak{W}_{\mathsf{R}_2 W} &\overset{def}{:=} \{\langle W', W'' \rangle \in \mathfrak{W} \times \mathfrak{W} : \langle W', W, W'' \rangle \in \mathsf{R}\}; \\
\mathfrak{W}_{\mathsf{R}_3 W} &\overset{def}{:=} \{\langle W', W'' \rangle \in \mathfrak{W} \times \mathfrak{W} : \langle W', W'', W \rangle \in \mathsf{R}\}; \\
\mathfrak{W}_{\mathsf{C} W} &\overset{def}{:=} \{W' \in \mathfrak{W} : \langle W', W \rangle \in \mathsf{C}, W' \neq W\}; \\
\mathfrak{W}_{W \mathsf{C}} &\overset{def}{:=} \{W'' \in \mathfrak{W} : \langle W, W'' \rangle \in \mathsf{C}, W'' \neq W\}.
\end{aligned}$$

For the tuples $\langle \mathfrak{W}_W, p_1, \ldots, p_n \rangle, \langle \mathfrak{W}_{\sqsubset W}, p_1, \ldots, p_n \rangle$, and $\langle \mathfrak{W}_{W \sqsubset}, p_1, \ldots, p_n \rangle$, the concept of P-similarity, which was introduced to characterize intuitionistic logicality, can be used to define the partial logicality of each. For tuples $\langle \mathfrak{W}_{\mathsf{C} W}, p_1, \ldots, p_n \rangle$ and $\langle \mathfrak{W}_{W \mathsf{C}}, p_1, \ldots, p_n \rangle$ also, the notion of P-similarity is applicable, since C, like $\sqsubseteq$, is a binary relation on $\mathfrak{W}$.

For the partial logicality with respect to the new parts—$\mathfrak{W}_{\mathsf{R}_1 W}, \mathfrak{W}_{\mathsf{R}_2 W}$, and $\mathfrak{W}_{\mathsf{R}_3 W}$ —we will define a new similarity relation, which we will symbolize as

$\equiv_{C\subseteq^2}$, using the similarity relation $\equiv_{C\subseteq}$ (Remember that $\equiv_{C\subseteq}$ was the similarity relation introduced to define P-similarity). Let V, V', U, and U' be domains of propositions with truth values, and let $p_1,\ldots,p_n,p'_1,\ldots,p'_n$ be propositions. For tuples $\langle V,U,p_1,\ldots,p_n\rangle$ and $\langle V',U',p'_1,\ldots,p'_n\rangle$,

$$\langle V,U,p_1,\ldots,p_n\rangle \equiv_{C\subseteq^2} \langle V',U',p'_1,\ldots,p'_n\rangle \quad \overset{def}{\Longleftrightarrow}$$

$$\begin{cases} \langle V,p_1,\ldots,p_n\rangle \equiv_{C\subseteq} \langle V',p'_1,\ldots,p'_n\rangle, \\ \langle U,p_1,\ldots,p_n\rangle \equiv_{C\subseteq} \langle U',p'_1,\ldots,p'_n\rangle. \end{cases}$$

Let $\mathfrak{Z}$ and $\mathfrak{Z}'$ be non-empty sets of pairs of domains. For tuples $\langle \mathfrak{Z},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{Z}',p'_1,\ldots,p'_n\rangle$, we say that these are *P^2-similar* if the following conditions are satisfied:

(i) For all $\langle V_1,U_1\rangle \in \mathfrak{Z}$, there exists $\langle V'_1,U'_1\rangle \in \mathfrak{Z}'$ such that $\langle V_1,U_1,p_1,\ldots,p_n\rangle \equiv_{C\subseteq^2} \langle V'_1,U'_1,p'_1,\ldots,p'_n\rangle$;

(ii) For all $\langle V'_2,U'_2\rangle \in \mathfrak{Z}'$, there exists $\langle V_2,U_2\rangle \in \mathfrak{Z}$ such that $\langle V_2,U_2,p_1,\ldots,p_n\rangle \equiv_{C\subseteq^2} \langle V'_2,U'_2,p'_1,\ldots,p'_n\rangle$.

The quasi-logical similarity relation of relevant logic can be defined in terms of P-similarity, and the logical similarity relation can be defined in terms of the quasi-similarity relation. We say that propositional structures $\langle M,W,p_1,\ldots,p_n\rangle$ and $\langle M',W',p'_1,\ldots,p'_n\rangle$ are *quasi-logically similar* if they satisfy the following conditions:

(i) $\{\langle \mathfrak{W}_W,p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{W'},p'_1,\ldots,p'_n\rangle$ are P-similar;

(ii) $\{\langle \mathfrak{W}_{\sqsubset W},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{\sqsubset W'},p'_1,\ldots,p'_n\rangle$ are P-similar;

(iii) $\{\langle \mathfrak{W}_{W\sqsubset},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{W'\sqsubset},p'_1,\ldots,p'_n\rangle$ are P-similar;

(iv) $\{\langle \mathfrak{W}_{\mathsf{R}_1 W},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{\mathsf{R}'_1 W'},p'_1,\ldots,p'_n\rangle$ are P^2-similar;

(v) $\{\langle \mathfrak{W}_{\mathsf{R}_2 W},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{\mathsf{R}'_2 W'},p'_1,\ldots,p'_n\rangle$ are P^2-similar;

(vi) $\{\langle \mathfrak{W}_{\mathsf{R}_3 W},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{\mathsf{R}'_3 W'},p'_1,\ldots,p'_n\rangle$ are P^2-similar;

(vii) $\{\langle \mathfrak{W}_{\mathsf{C} W},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{\mathsf{C} W'},p'_1,\ldots,p'_n\rangle$ are P-similar;

(viii) $\{\langle \mathfrak{W}_{W\mathsf{C}},p_1,\ldots,p_n\rangle$ and $\langle \mathfrak{W}'_{W'\mathsf{C}},p'_1,\ldots,p'_n\rangle$ are P-similar.

Also, we say that they are *logically similar* if the following conditions are met (written as "$\langle M,W,p_1,\ldots,p_n\rangle \equiv_R \langle M',W',p'_1,\ldots,p'_n\rangle$"):

(i) $\langle M, W, p_1, \ldots, p_n \rangle$ and $\langle M', W', p'_1, \ldots, p'_n \rangle$ are quasi-logically similar;

(ii) For any connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l \rangle$ such that $W_i \sqsubseteq W_{i+1}$ for all i, there exists some connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l \rangle$, such that $W'_i \sqsubseteq W'_{i+1}$ for all i and such that $\langle M, W_i, p_1, \ldots, p_n \rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n \rangle$ are quasi-logically similar for any i;

(iii) For any connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l \rangle$, such that $W'_i \sqsubseteq W'_{i+1}$ for all i, there exists some connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l \rangle$ such that $W_i \sqsubseteq W_{i+1}$ for all i, and such that $\langle M, W_i, p_1, \ldots, p_n \rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n \rangle$ are quasi-logically similar for any i;

(iv) Let $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l \rangle$ be l-tuple such that, for some $Y \in \mathfrak{W}$, $\langle W_i, W_{i+1}, Y \rangle \in \mathsf{R}$ or $\langle W_i, Y, W_{i+1}, \rangle \in \mathsf{R}$. For any such connected path through W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l \rangle$, there exists some connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l \rangle$ that satisfies the following conditions:

   (a) If $\langle W_i, W_{i+1}, Y \rangle \in \mathsf{R}$, then there exists $Y' \in \mathfrak{W}'$ such that $\langle W'_i, W'_{i+1}, Y' \rangle \in \mathsf{R}'$.
   (b) If $\langle W_i, Y, W_{i+1} \rangle \in \mathsf{R}$, then there exists $Y' \in \mathfrak{W}'$ such that $\langle W'_i, Y', W'_{i+1} \rangle \in \mathsf{R}'$.
   (c) $\langle M, W_i, p_1, \ldots, p_n \rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n \rangle$ are quasi-logically similar for any i.

(v) Let $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l \rangle$ be l-tuple such that, for some $Y' \in \mathfrak{W}'$, $\langle W'_i, W'_{i+1}, Y' \rangle \in \mathsf{R}'$ or $\langle W'_i, Y', W'_{i+1}, \rangle \in \mathsf{R}'$. For any such connected path through W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l \rangle$, there exists some connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l \rangle$ that satisfies the following conditions:

   (a) If $\langle W'_i, W'_{i+1}, Y' \rangle \in \mathsf{R}'$, then there exists $Y \in \mathfrak{W}$ such that $\langle W_i, W_{i+1}, Y \rangle \in \mathsf{R}$.
   (b) If $\langle W'_i, Y', W'_{i+1} \rangle \in \mathsf{R}'$, then there exists $Y \in \mathfrak{W}$ such that $\langle W_i, Y, W_{i+1} \rangle \in \mathsf{R}$.

(c) $\langle M, W_i, p_1, \ldots, p_n\rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n\rangle$ are quasi-logically similar for any i.

(vi) For any connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l\rangle$ such that $\langle W_i, W_{i+1}\rangle \in \mathsf{C}$ for all i, there exists some connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l\rangle$, such that $\langle W'_i, W'_{i+1}\rangle \in \mathsf{C}$ for all i and such that $\langle M, W_i, p_1, \ldots, p_n\rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n\rangle$ are quasi-logically similar for any i;

(vii) For any connected path thorough W', $\langle W'_1, \ldots W'_{j-1}, W', W'_{j+1}, \ldots W'_l\rangle$, such that $\langle W'_i, W'_{i+1}\rangle \in \mathsf{C}$ for all i, there exists some connected path thorough W, $\langle W_1, \ldots W_{j-1}, W, W_{j+1}, \ldots W_l\rangle$ such that $\langle W_i, W_{i+1}\rangle \in \mathsf{C}$ for all i, and such that $\langle M, W_i, p_1, \ldots, p_n\rangle$ and $\langle M', W'_i, p'_1, \ldots, p'_n\rangle$ are quasi-logically similar for any i.

For a propositional operator O defined on any domain of any model, we say that O is *relevantly logical* if O is $\equiv_R$-invariant.

It can be shown that the operators of logical connectives of relevant logic are all relevantly logical. Here, we will only prove the logicality of the relevant conditional O^{cond},[15] which is defined as follows:

$$O^{cond}_W(p_1, p_2) \quad = \quad \begin{cases} \mathsf{T} & \text{if } p_1 \text{ is false in } W_1 \text{ or if } p_2 \text{ is true in } W_2, \\ & \text{for all } W_1 \text{ and } W_2 \text{ such that } \langle W, W_1, W_2\rangle \in \mathsf{R} \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

Suppose that $\langle M, W, p_1, p_2\rangle \equiv_R \langle M', W', p'_1, p'_2\rangle$. Then, it holds that $\langle \mathfrak{W}_{\mathsf{R}_1 W}, p_1, p_2\rangle$ and $\langle \mathfrak{W}'_{\mathsf{R}_1 W'}, p'_1, p'_2\rangle$ are P^2-similar. If $O^{cond}_W(p_1, p_2) = \mathsf{T}$, then by the definition of O^{cond}, $\mathfrak{W}_{\mathsf{R}_1 W}$ contains only pairs $\langle W_1, W_2\rangle$ such that p_1 is false in W_1 or p_2 is true in W_2. By the definitions of P^2-similarity and the relation $\equiv_{C\subseteq^2}$, for all $\langle W'_1, W'_2\rangle \in \mathfrak{W}'_{\mathsf{R}_1 W'}$, there exists $\langle W^*_1, W^*_2\rangle \in \mathfrak{W}_{\mathsf{R}_1 W}$ such that $\langle W^*_1, p_1\rangle \equiv_{C\subseteq}$

[15]The operators of the negation, the conjunction, and the disjunction of relevant logic (written O^N, O^{conj}, and $O^{\mathcal{D}isj}$ respectively) are defined as follows:

1. $O^N_W(p) = \mathsf{T}$ if p is false in all W' such that $\langle W, W'\rangle \in \mathsf{C}$; otherwise $O^N_W(p) = \mathsf{F}$;
2. $O^{conj}_W(p_1, p_2) = \mathsf{T}$ if p_1 and p_2 are true in W; otherwise $O^{conj}_W(p_1, p_2) = \mathsf{F}$.
3. $O^{\mathcal{D}isj}_W(p_1, p_2) = \mathsf{T}$ if p_1 or p_2 are true in W; otherwise $O^{\mathcal{D}isj}_W(p_1, p_2) = \mathsf{F}$.

Their logicality can be proven in an analogous way.

$\langle W_1', p_1' \rangle$ and such that $\langle W_2^*, p_2 \rangle \equiv_{C\subseteq} \langle W_2', p_2' \rangle$. Thus, p_1' is false in W_1' or p_2' is true in W_2'. Consequently, we have $O_{W'}^{cond}(p_1', p_2') = \mathsf{T}$. By a similar argument, we can show the opposite direction. That is, we can prove that if $O_{W'}^{cond}(p_1', p_2') = \mathsf{T}$ then $O_W^{cond}(p_1, p_2) = \mathsf{T}$. Therefore, $O_W^{cond}(p_1, p_2) = O_{W'}^{cond}(p_1', p_2')$. Hence, O^{cond} is a relevantly logical operator.

The process for identifying the logicality of objectual operators of unary first-order quantifiers in relevant logic is the same as that for first-order intuitionist logic. We define an *objectual model*, a *first-order objectual structure*, a *classification of domains*, and *O-similarity*. Then, we can show that the operators of the universal and existential quantifiers of relevant logic are invariant under that logical similarity relation.

Generally, the procedure for identifying the logicality of operators of a logical system is as follows:

(i) First, identify the "semantic unit" of the logical system, and define it as a model;

(ii) Second, define a structure associated with operators of that logical system;

(iii) Third, characterize the logical similarity between structures;

(iv) Finally, check if a given operator is invariant under the logical similarity relation.

This "recipe" can be used for standard modal logics such as K, T, B, $S4$, and $S5$. These all resemble intuitionistic logic in that they each employ a single accessibility relation. We can thus reuse the method to characterize the logicality of intuitionistic logic for identifying their logicality. The logicality of the necessity and possibility operators of each modal logic ("$\Box$" and "$\Diamond$"), and also the logicality of any operator obtained by some combination of them (e.g., "$\Diamond\Box$" and "$\Box\Diamond\Box$") can be shown in an analogous way.

Zalta's Modal Operators

In Chapter 2, we have examined Zalta's claim to the effect that there is a logical truth that is not necessary (pp. 42–47). We have argued that, for his claim to be true, the modal operator $\mathcal{A}$ and the definite description operator ι have to be logical constants. Here, we will show that is not the case.

First, in order to deny the logicality of the modal operator $\mathcal{A}$, consider the following two propositional modal models:

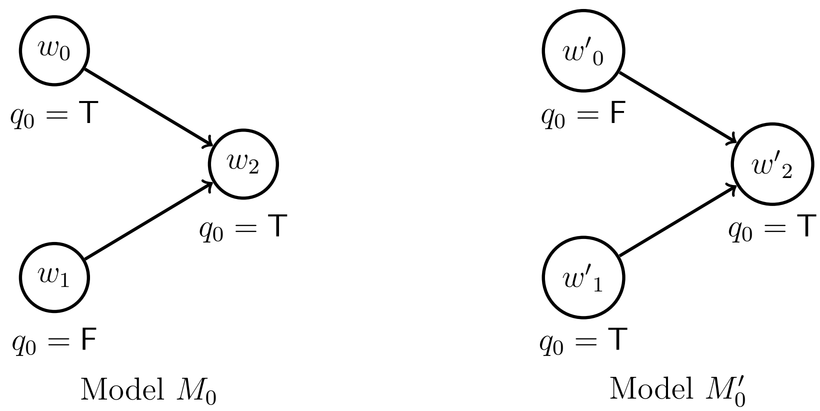

Figure 3.10: Models M_0 and M'_0

Here, w_0 is the actual world of M_0 and w'_0 is the actual world of M'_0. The difference between M_0 and M'_0 is the truth values of q_0 in w_0 and w'_0 and the truth values in w_0 and w'_0. The propositional structures $\langle M_0, w_2, q_0\rangle$ and $\langle M_0, w_2, q_0\rangle$ are logically similar; w_2 and w'_2 are P-similar in the center parts (i.e., $\{w_2\}$ and $\{w'_2\}$) and the left parts (i.e., $\{w_0, w_1\}$ and $\{w'_0, w'_1\}$), and also the connected path $\langle w_0, w_2\rangle$ corresponds to $\langle w'_1, w'_2\rangle$, and $\langle w_1, w_2\rangle$ corresponds to $\langle w'_0, w'_2\rangle$. However, $\mathcal{A}(q_0) = \mathsf{T}$ in w_2, while $\mathcal{A}(q_0) = \mathsf{F}$ in w'_2. Thus, the characteristic function of $\mathcal{A}$ is not invariant under the logical similarity relation. Hence, $\mathcal{A}$ is not a logical constant.

Next, the definite description operator ι. The characteristic function of ι is a binary operator that assigns a truth value to a pair $\langle X, Y\rangle$ of two sets: $P(\iota x)Qx$ is true in the domain $\mathcal{D}$, if the set X that is assigned to Q in the actual world domain $\mathcal{D}_0$ is a singleton set and also if the only object in X is a member of the set Y that is assigned to P in $\mathcal{D}$.

Let $\mathcal{D}_0$ be a domain containing three objects a, b, c, and let M_0 be an objectual modal model that contains only one domain which is $\mathcal{D}_0$. $\mathcal{D}_0$ is the actual world domain of M_0. Consider then two objectual structures, $\langle M_0, \mathcal{D}_0, U\rangle$ and $\langle M_0, \mathcal{D}_0, U'\rangle$, where $U = \{\langle \mathcal{D}_0, \langle\{a\}, \{a, b\}\rangle\rangle\}$ and $U = \{\langle \mathcal{D}_0, \langle\{c\}, \{a, b\}\rangle\rangle\}$. The former structure represents an interpretation function that assigns $\{a\}$ to, for example, a unary predicate Q and $\{a, b\}$ to another unary predicate P. The latter structure represents another interpretation function that assigns $\{c\}$ to Q and $\{a, b\}$ to P. These two structures are logically similar, because $\langle \mathcal{D}_0, \langle \{a\}, \{a, b\} \rangle \rangle$ and $\langle \mathcal{D}_0, \langle \{c\}, \{a, b\} \rangle \rangle$ are logically similar. However, the characteristic function of ι assigns T to the former and F to the latter: in fact, $P(\iota x)Qx$ is true under the interpretation of the former but false under that of the latter. The operator ι, therefore, is not a logical constant. Zalta's argument for unnecessary logical truths thus can be rejected.

Chapter 3, in part, has been submitted for publication of the material as it may appear in Synthese, Springer, 2016. The dissertation author was the sole investigator and author of this paper.

4 Prime Logical Validity

We have so far shown two things:

(i) The model-theoretic method is appropriate for the characterization of prime logical validity. However, the proof-theoretic method is not appropriate for two reasons (Chapter 2);

(ii) A logical constant under the minimal notion is a term whose semantic value is specified by a logical operator, i.e., an operator that is invariant under the logical similarity relation (Chapter 3). Most terms used in major logical systems (classical or non-classical) are logical constants. The identity relation is not a logical constant.

We now have an appropriate method to approach logical validity and formal languages with a clear demarcation between logical terms and extra-logical terms.

The main objective of this chapter is to identify arguments expressed in the formal languages that can be regarded as logically valid under the minimal notion, namely, the characterization of prime logical validity. The minimal notion says that an argument is logically valid if it holds by virtue of certain (selected) formal laws. We call the formal laws that are entitled to validate arguments under the minimal notion *prime formal laws*. For the characterization of prime logical validity, prime formal laws has to be defined.

There are two kinds of formal laws: formal laws that hold in structures of classical logic (*classical structures*) and formal laws that hold in structures of non-classical logics (*non-classical structures*). The characteristic distinction between structures of the two kinds is that a classical structure contain only one domain, while a non-classical structure can contain multiple domains that are connected by accessibility relations (we will consider only non-classical logics with a Kripke semantics). This distinction affects the way to define prime formal laws in each kind of structures. In what follows, I will first characterize prime formal laws of classical structures, and then move on to the consideration on prime formal laws of non-classical structures.

4.1 Prime Logical Validity of Classical Structures

The objective of this section is to characterize the prime logical validity of classical structures. Beyond the first-order level, we will consider what arguments in a second-order language can be regarded as logically valid under the minimal notion. Since the validity of arguments in a first-order language can be taken as a part of that in the second-order language, the characterization of the first-order prime logical validity can be given within the characterization of the second-order prime logical validity.

There are two major semantic systems of second-order logic: standard semantics and Henkin semantics. I will argue that neither of them can capture the prime logical validity. Instead of them, I will propose a version of Henkin semantics (not the original one) as a correct semantic system to define the prime logical validity.

Preliminaries

Throughout this section, we fix a formal language L^2, which is specified by symbols of a first-order language without equality and by second-order variables for relations between objects. For simplicity, we suppose that L^2 contains only unary and binary relation symbols and variables for them. We do not deal with n-ary relations for $n \geq 3$. We also suppose that function symbols are not in the vocabulary of L^2. Sentences in L^2 thus are composed of the following components: (i) constant symbols for objects in domains; (ii) unary relation symbols; (iii) binary relation symbols; (iv) first-order variables; (v) second-order variables for unary relations; (vi) second-order variables for binary relations; (vii) logical connectives $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$; (viii) first-order universal quantifier and second-order quantifier

$(\forall, \exists)$; (ix) second-order universal quantifier and second-order quantifier (we use the same symbols, $\forall$ and $\exists$, as the first-order quantifiers). Due to this restriction, our discussion will be able to avoid complexity that I think is not necessary for understanding the essential idea underlying the characterization of second-order prime logical validity.

A *standard structure* for L^2 is a pair $\langle \mathcal{D}, I \rangle$ of a domain $\mathcal{D}$ and an interpretation function I for extra-logical terms. The range of n-ary relation variables ($n = 1$ or 2) is the power set $\wp(\mathcal{D}^n)$. Standard semantics for L^2, in this respect, is a natural extension of the standard semantic system of first-order logic. As first-order variables range over all objects in $\mathcal{D}$, second-order variables for unary relations range over $\wp(\mathcal{D})$ and second-order variables for binary relations range over $\wp(\mathcal{D} \times \mathcal{D})$. In standard semantics of second-order logic, a valid argument is defined as an argument that holds in all standard structures.

Henkin semantics is a semantic system that can be obtained from standard semantics by a generalization with respect to second-order variable ranges. A *Henkin structure* for L^2 is a quadruple $\langle \mathcal{D}, D(1), D(2), I \rangle$, where a non-empty set $D(n) \subseteq \wp(\mathcal{D}^n)$ is the range of n-ary relation variables. $D(n)$ is not necessarily identical to $\wp(\mathcal{D}^n)$. As opposed to standard semantics, in a Henkin structure, second-order variables for unary relations range over a subset $D(1)$ of $\wp(\mathcal{D})$ and second-order variables for binary relations range over a subset $D(2)$ of $\wp(\mathcal{D} \times \mathcal{D})$. For example, for a second-order variable Z for n-ary relations, a universal sentence $\forall Z \varphi(Z)$ is true in a Henkin structure $\langle \mathcal{D}, D(1), D(2), I \rangle$, if $\varphi(X)$ holds for all $X \in D(n)$. If $D(n)$ of a Henkin structure is a proper subset of $\wp(\mathcal{D}^n)$, the sentence can be true in the structure even if it is false for some set that is in $\wp(\mathcal{D}^n)$ but not in $D(n)$. In Henkin semantics of second-order logic, an argument is said to be valid if it holds in all Henkin structures.

A standard structure $\langle \mathcal{D}, I \rangle$ can be naturally transformed into a Henkin structure, which is $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), I \rangle$. A standard structure can be seen as a special type of Henkin structure. Thus, if an argument is valid in Henkin semantics, then it is also valid in standard semantics. The opposite direction does not hold: there are arguments that are valid in standard semantics but invalid in Henkin semantics. We will see several examples below.

Formal Law of Second-order Logic

According to the minimal notion, a logical valid argument holds by virtue of prime formal laws, which themselves are laws governing formal properties. In Chapter 1, we defined formal properties and formal laws (pp. 10–13). For formal properties $P_1, P_2, \ldots, P_C$ that can be applied to set-theoretic constructs of the same type, a formal law governing formal operators representing $P_1, P_2, \ldots, P_C$ is a law that can be stated in the following form:

> If a set-theoretic construct satisfies the formal properties $P_1, P_2, \ldots$,
> then it also satisfies the formal property P_C.

There, we have seen several examples of formal laws that validate arguments in a first-order language.

There are also formal laws that validate arguments in L^2. Let us see examples. Consider a sentence $\exists Z \forall x Zx$, where Z is a second-order variable for unary relations and x is a first-order variable. The sentence says that there is a unary relation that every object in the domain satisfies. This sentence is true in all standard structures (but false in some Henkin structures[1]). Thus, the sentence is a logical truth of standard semantics and the argument $\langle \emptyset, \exists Z \forall x Zx \rangle$ is valid. Consider now a formal property P_C applied to pairs $\langle \mathcal{D}, \wp(\mathcal{D}) \rangle$ that there is a set X in $\wp(\mathcal{D})$ (a subset X of $\mathcal{D}$) that contains all objects in $\mathcal{D}$ (of course, the set is $\mathcal{D}$). This property is satisfied by any pairs $\langle \mathcal{D}, \wp(\mathcal{D}) \rangle$, and therefore the following statements is a formal law:

Any pairs $\langle \mathcal{D}, \wp(\mathcal{D}) \rangle$ satisfies the formal property P_C.

The validity of $\langle \emptyset, \exists Z \forall x Zx \rangle$ is based on this formal law.

Consider another argument:

$$\begin{array}{ll} & \exists x Qx. \\ \hline \therefore & \exists Z[\forall x(Zx \to Qx) \wedge \neg \forall y(Qy \to Zy)]. \end{array}$$

This argument holds in all standard structures and therefore is valid in standard semantics. There are two formal properties concerning the validity, which can be

[1] In a Henkin structure $\langle \mathcal{D}, D(1), D(2), I \rangle$ such that $D(1)$ does not contain the domain set $\mathcal{D}$, the sentence $\exists Z \forall x Zx$ is false.

applied to triples $\langle \mathcal{D}, \wp(\mathcal{D}), X \rangle$:

(i) P_1: There is an object that is a member of X;

(ii) P_C: There is a set Y in $\wp(\mathcal{D})$ such that Y is a proper subset of X.

The formal law that validates the argument is:

If a triple $\langle \mathcal{D}, \wp(\mathcal{D}), X \rangle$ satisfies P_1, then it also satisfies P_C.

For any set X that satisfies P_1, P_C is satisfied by the empty set.

The validity of arguments containing sentences with binary relation symbols can be explained in a similar way.

$$\begin{array}{rl} & \forall x Rxx. \\ \hline \therefore & \exists Z[\forall x \exists y Zxy]. \end{array}$$

The argument says that if a relation expressed by a binary relation symbol R is reflexive, then there is a binary relation satisfying the condition that for any object $a \in \mathcal{D}$ there is some object $b \in \mathcal{D}$ such that the pair $\langle a, b \rangle$ stands in the relation. This argument is valid in standard semantics. The two formal properties concerning the validity is the ones that can be applied to triples $\langle \mathcal{D}, \wp(\mathcal{D} \times \mathcal{D}), X \rangle$:

(i) P_1: For any $a \in \mathcal{D}$, the pair $\langle a, a \rangle$ is a member of the set $X \in \wp(\mathcal{D} \times \mathcal{D})$;

(ii) P_C: There is a set $Y \in \wp(\mathcal{D} \times \mathcal{D})$ such that for any object $a \in \mathcal{D}$ there is some object $b \in \mathcal{D}$ such that the pair $\langle a, b \rangle$ is a member of Y.

The formal law that justifies the argument is: if a triple $\langle \mathcal{D}, \wp(\mathcal{D} \times \mathcal{D}), X \rangle$ satisfies P_1, then it also satisfies P_C.

Various formal laws validate arguments in L^2, and they can be applied to set-theoretical constructs of various forms. The formal law of the first example above can be applied to pairs of the form $\langle \mathcal{D}, \wp(\mathcal{D}) \rangle$, the formal law of the second example to triples $\langle \mathcal{D}, \wp(\mathcal{D}), X \rangle$, and the formal law of the third example to triples $\langle \mathcal{D}, \wp(\mathcal{D} \times \mathcal{D}), X \rangle$. In order to standardize them, consider quadruples of the following form:

$$\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle.$$

Here, $\mathcal{S}$ is a triple $\langle \langle a_1, a_2, \ldots \rangle, \langle X_1, X_2, \ldots \rangle, \langle Y_1, Y_2, \ldots \rangle \rangle$, where $a_i \in \mathcal{D}$, $X_j \in \wp(\mathcal{D})$, and $Y_k \in \wp(\mathcal{D} \times \mathcal{D})$. We call a quadruple $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle$ a *standard formal-property bearer*. Each standard structure $\langle \mathcal{D}, I \rangle$ for L^2 has a corresponding standard formal-property bearer. Let $c_1, c_2, \ldots$ be constant symbols of L^2. Also, let $Q_1, Q_2, \ldots$ be unary relation symbols, and let $R_1, R_2, \ldots$ be binary relation symbols.

The standard formal-property bearer that corresponds to $\langle \mathcal{D}, I\rangle$ is $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times \mathcal{D}), \mathcal{S}_I\rangle$ such that $\mathcal{S}_I = \langle\langle I(c_1), I(c_2), \ldots\rangle, \langle I(Q_1), I(Q_2), \ldots\rangle, \langle I(R_1), I(R_2), \ldots\rangle\rangle$. We call a formal law a *standard formal law* if it can be applied to standard formal-property bearers and can be satisfied by all of them. If an argument in L^2 is valid in standard semantics, then there is a standard formal law that validates it.

Although a standard formal-property bearer is composed of infinitely many components, not every standard formal law involves its all components. Consider the following standard formal law that validates the sentence $\exists Z\forall xZx$:

> For any standard formal-property bearer $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S}\rangle$, there is a set X in $\wp(\mathcal{D})$ that contains all objects in $\mathcal{D}$

This law states a fact about the first and second components of standard formal-property bearers (i.e., $\mathcal{D}$ and $\wp(\mathcal{D})$), and the third and forth components (i.e., $\wp(\mathcal{D} \times \mathcal{D})$ and $\mathcal{S}$) are irrelevant to it. Generally, if a valid argument does not contain some constant symbols or some relation symbols, then the truth of the standard formal law that validates it has nothing to do with what their corresponding components of standard formal-property bearers are.

A formal law used for validating arguments of second-order logic with Henkin semantics is a special type of standard formal law. If an argument $\langle\Gamma, \varphi\rangle$ is valid in Henkin semantics, $\langle\Gamma, \varphi\rangle$ holds in all Henkin structures. Since a standard structure is a Henkin structure, there is some standard formal law such that $\langle\Gamma, \varphi\rangle$ holds by virtue of it. What is special with the standard formal law is that it can be satisfied not only by standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S}\rangle$ but also by set-theoretic constructs of a more general type. Consider quadruples $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$ such that $\mathcal{D}$ is a domain, $\mathcal{X}(1) \subseteq \wp(\mathcal{D})$, and $\mathcal{X}(2) \subseteq \wp(\mathcal{D} \times \mathcal{D})$. $\mathcal{X}(i)$ may or may not be identical to $\wp(\mathcal{D}^i)$ for $i = 1$ or 2. $\mathcal{S}$ is a triple $\langle\langle a_1, a_2, \ldots\rangle, \langle X_1, X_2, \ldots\rangle, \langle Y_1, Y_2, \ldots\rangle\rangle$, where $a_i \in \mathcal{D}$, $X_j \in \mathcal{X}(1)$, and $Y_k \in \mathcal{X}(2)$. We call quadruples of this form *Henkin formal-property bearers*. By definition, a standard formal-property bearer is a Henkin formal-property bearer. Each Henkin structure $\langle \mathcal{D}, D(1), D(2), I\rangle$ for L^2 has a corresponding Henkin formal-property bearer. The correspondence can be obtained in the same way as the correspondence between standard structures and Standard formal-property bearers above.

Consider the following argument:

$$\begin{array}{rl} & \exists Z(\forall x Zx)\ . \\ \hline \therefore & \exists Z(\exists x Zx). \end{array}$$

This argument is valid not only in standard semantics but also in Henkin semantics. The standard formal law that validates the argument is a formal law governing two formal properties P_1 and P_C that can be applied to standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$:

(P_1) There exists a set $X \in \wp(\mathcal{D})$ that contains all objects in $\mathcal{D}$;

(P_C) There exists a set $X \in \wp(\mathcal{D})$ that contains some object in $\mathcal{D}$.

The standard formal law is:

> If a standard formal-property bearer $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$ satisfies P_1, then it also satisfies P_C.

This standard formal law does hold for any Henkin formal-property bearers as well. That is to say, we have:

> For a Henkin formal-property bearer $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$, if there exists a set in $\mathcal{X}(1)$ that contains all objects in $\mathcal{D}$ (the property P_1), then there exists a set in $\mathcal{X}(1)$ that contains some object in $\mathcal{D}$ (the property P_C).

If $\mathcal{D} \in \mathcal{X}(1)$, then $\mathcal{X}(1)$ contains a non-empty set (namely, $\mathcal{D}$).

We call a standard formal law a *Henkin formal law* if it can be satisfied not only by all standard formal-property bearers but also by all Henkin formal-property bearers. For any valid argument in Henkin semantics, there is a Henkin formal law that validates the argument. A Henkin formal law, by definition, is a standard formal law, but not every standard formal law is a Henkin formal law.

Standard semantics is the semantic system to capture the model-theoretic validity based on standard formal laws, while Henkin semantics is the semantic system to validate arguments justified by Henkin formal laws. If standard formal laws are prime formal laws, that is, if they can be regarded as appropriate for the characterization of logical validity under the minimal notion, then the prime logical validity of classical logic is the model-theoretic validity of standard semantics. In that case, all and only arguments that hold valid in all standard structures are logically valid under the minimal notion. If, on the other hand, Henkin formal laws are prime formal laws, then the model-theoretic validity of Henkin semantics is prime logical validity. In that case, all and only arguments that hold valid in all Henkin structures are logically valid under the minimal notion.

I will argue below that not all standard formal laws can be regarded as prime formal laws and also that some standard formal laws that are not Henkin formal laws can be thought of as prime formal laws. The bounds of prime formal laws are drawn between the bounds of standard formal laws and the bounds of Henkin formal laws. Neither standard semantics nor Henkin semantics is an appropriate

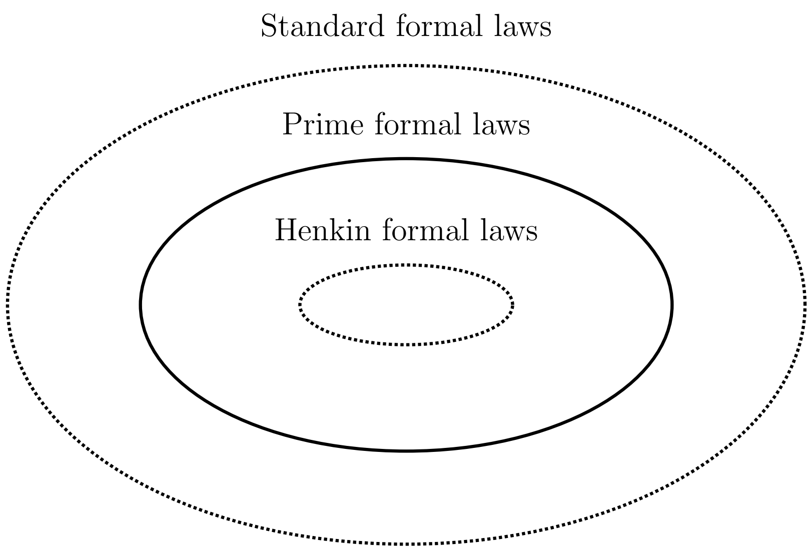

Figure 4.1: The inclusion relationship among three kinds of formal laws

semantic system for the characterization of prime logical validity. Prime logical validity can be defined in another logical system.

Problem of the Standard Semantics

Standard semantics, as its name indicates, can be seen as the "standard" model-theoretic system to validate arguments; it is a natural extension of the model-theoretic semantics for first-order language. However, second-order logic with standard semantics has been controversial in the philosophy of logic. One major criticism that have been directed towards the logical system is that it is incomplete. The completeness of a logical system has been thought of as a desirable property of the system. Leslie Tharp, for example, describes the property as an epistemological advantage, which is not substitutable with other features:

> In general, if completeness fails there is no algorithm to list the valid formulas; so one can expect many of the principles of the logic to be unknowable, or determinable only by means of *ad hoc* or inconclusive arguments. Clearly one will hesitate to substitute other desirable features for completeness in a theory of deduction. The negative evidence, together with the epistemological appeal of the completeness condition, make it seem reasonable to suppose that completeness is essential to an important sense of logic. (Italics original, Tharp[130], p. 7).

Second-order logic with standard semantics is not axiomatizable, thus inherently fails to satisfy this desirable property.[2]

Another major criticism of second-order logic with standard semantics is that the logical system has strong ontological commitments to sets: in Quine's famous terminology, second-order logic with standard semantics is "set theory in sheep's clothing." In a domain $\mathcal{D}$, second-order variables for n-ary relations range over the power set $\wp(\mathcal{D}^n)$. Standard semantics are ontologically committed to sets in the sense that, for (many but not all) sentences in a second-order to be true, the ranges of second-order variables have to be supposed to contain certain sets satisfying certain properties.

The reason that second-order logic with standard semantics is not appropriate for the characterization of prime logical validity is related to this second criticism. The problem with standard semantics is that it validates some arguments that hold by virtue of standard formal laws governing some particular properties of particular sets. Let us see two examples. Let $\varphi_{2\leq}$ be the sentence $\exists x \exists y[\exists X(Xx \wedge \neg Xy)]$, where x and y are first-order variables and X is a second-order variable for unary relations. $\varphi_{2\leq}$ is true in any standard structure whose domain contains at least two objects and false in any standard structure whose domain contains only one object. Let $\varphi_{n\leq}$ (for $n \geq 3$) be a sentence, which is written in a similar manner, that is true in and only in standard structures whose domains contain at least n objects. Also, let φ_{∞} be the sentence

[2] Some philosophers, however, have argued that a logical system should not be rejected based solely on its incompleteness. See, for example, Bueno[17] and Rossberg[98].

$$\exists Z[\forall x\forall y\forall z(Zxy \to (Zyz \to Zxz)) \wedge \forall x(\neg Zxx) \wedge \forall x\exists y(Zxy)],$$

where Z is a second-order variable for binary relations. φ_∞ is true in any standard structure whose domain is an infinite set and false in any standard structure whose domain is a finite set. Consider then the valid argument $\langle\{\varphi_{2\leq}, \varphi_{3\leq}, \ldots\}, \varphi_\infty\rangle$ deriving φ_∞ from $\varphi_{2\leq}, \varphi_{3\leq}, \ldots$. This argument is valid in standard semantics, since φ_∞ is true in every standard structure in which $\varphi_{2\leq}, \varphi_{3\leq}, \ldots$ are true.

The validity of this argument depends on a standard formal law governing particular properties that can be applied to standard formal-property bearers $\langle\mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$. Let $P_{2\leq}$ be a formal property that $\mathcal{D}$ contains at least two objects, and more generally, let $P_{n\leq}$ be a formal property that $\mathcal{D}$ contains at least n objects. Also, let P_C be the formal property that there is a binary relation in $\wp(\mathcal{D} \times \mathcal{D})$ that is transitive, irreflexive, and serial. The standard formal law that validates the argument above is:

> If a standard formal-property bearer $\langle\mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S}\rangle$ satisfies the formal properties $P_{2\leq}, P_{3\leq}, \ldots$, then it also satisfies the formal property P_C.

This law is actually true; for any standard formal-property bearer $\langle\mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$ that satisfies the formal properties $P_{2\leq}, P_{3\leq}, \ldots$, the domain set $\mathcal{D}$ is an infinite set, and any infinite set satisfies the property described by P_C. The argument does hold because of this particular property of particular sets (i.e., infinite sets).

The second example is an argument whose conclusion indirectly expresses Cantor's theorem by the following sentence φ_C:

$$\neg\exists Z[\forall X\exists x\forall y(Zxy \leftrightarrow Xy)].$$ [3]

φ_C is true in all standard structures, and therefore the argument $\langle\emptyset, \varphi_C\rangle$ deriving φ_C from no premise is valid. The standard formal law that validates the argument is the one that is expressed by φ_C, which holds due to Cantor's theorem itself. The argument holds by virtue of the standard formal law involving the particular property of particular binary relations described by the open formula

[3] Assume, as opposed to Cantor's theorem, that there was a one-to-one correspondence between a set $\mathcal{X}$ and its power set $\wp(\mathcal{X})$, by which $a \in \mathcal{X}$ corresponds to $X_a \in \wp(\mathcal{X})$. Then, there would have been a binary relation R such that the range of R with respect to a, i.e., the set $\{y : \langle a, y\rangle \in R\}$, is identical to X_a. The sentence above negates the existence of such relation. (Shapiro[103], p. 103).

$\forall X \exists x \forall y (Zxy \leftrightarrow Xy)$. For any domain, there does not exist such a binary relation on it. Hence, the argument is valid.

The validity of these arguments thus depend on standard formal laws governing particular properties of particular sets, and the dependence is problematic from the minimalist's point of view. As we have argued in Chapter 1 (pp. 17–19), the logical validity of an argument has been supposed to be independent of any properties of particular individual objects such as their biological properties. Based on the same reason for the independence, we can suppose that the logical validity should also be independent of any properties of particular sets. If a standard formal law is about particular sets, then the argument validated by the law can be interpreted as indirectly stating a fact about the sets. Such valid arguments are "set-theoretically valid" arguments and should be distinguished from "logically valid" arguments whose validity does not depend on such particular properties of sets.

I do not intend to claim that logical validity has to be completely independent of any properties of any sets. That is impossible, as long as we take the model-theoretic approach to logic. Rather, my point is that formal properties can be classified into two kinds: (i) formal properties such that standard formal laws governing them are allowed to validate arguments under the minimal notion; and (ii) formal properties such that standard formal laws governing them are not allowed to validate arguments under the minimal notion. If the validity of an argument is grounded in some standard formal law that holds due to formal properties of the former kind, then it can be regarded as logically valid under the minimal notion. If, on the other hand, the validity of an argument is due to some standard formal law that involves formal properties of the latter kind, then it should be labeled as "set-theoretically valid." The characterization of the prime logical validity of classical structures can be reduced to the characterization of formal properties of the kind (i).

Logical Property

We will define the special type of formal properties using logical operators that we have identified in Chapter 3. The idea is simple: a formal property belongs to the kind (i) if and only if it's characteristic function is a logical operator. Remember that a logical operator of sets is a function that assigns the same true value to logically similar sets. If a property has a logical operator as its characteristic function, then it can be thought of as describing a "logical aspect" of sets; for any logically similar sets X and Y, X satisfies the property if and only if Y satisfies the property. For example, the emptiness of sets is one of such properties. Whether a set contains some object or no object is an aspect that is shared by logically similar sets. The emptiness property and other properties whose characteristic functions are not logical can be distinguished from a logical point of view. And more generally, the logical distinction among operators can be transformed into the logical distinction among formal properties. Prime logic validity is the validity that takes into account only such selected formal properties, and a prime formal law is a standard formal laws governing them. An argument is logically valid under the minimal notion if, and only if, its validity can be justified by some prime formal law.

In Chapter 3, we have shown that there are eight logical operators of unary relations (p. 76). Consequently, there are eight properties of unary relations that can be distinguished from others:

(i) P_1 such that $f^{P_1}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D})$;

(ii) P_2 such that $f^{P_2}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\mathcal{D}\}$;

(iii) P_3 such that $f^{P_3}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\mathcal{D}, \emptyset\}$;

(iv) P_4 such that $f^{P_4}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\emptyset\}$;

(v) P_5 such that $f^{P_5}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\mathcal{D}\}$;

(vi) P_6 such that $f^{P_6}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \wp(\mathcal{D}) \setminus \{\mathcal{D}, \emptyset\}$;

(vii) P_7 such that $f^{P_7}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \{\emptyset\}$;

(viii) P_8 such that $f^{P_8}_{\mathcal{D}}(X) = \mathsf{T}$ if and only if $X \in \emptyset$.

Here, $f^{P_i}_{\mathcal{D}}$ denotes the characteristic function of the property P_i acting on $\mathcal{D}$. Note that P_5, P_6, and P_7 are bases of P_1, P_2, P_3, and P_4 in that they can be described by disjunctions of P_5, P_6, and P_7. For example, X satisfies P_4 if and only if X satisfies P_5 or P_6. The property P_8 is the formal property that no unary relation can satisfy. We call these 2^3 logically distinguishable properties of unary relations *logical properties*.

According to the logical similarity relation among n-ary relations (pp. 79–81), binary relations can be regarded as logically similar if they are composed of objects with logically similar images. For a binary relation R on $\mathcal{D}$ and for $a \in \mathcal{D}$, the image $R_1(a)$ of a under R is the set $\{b \in \mathcal{D} : \langle b, a\rangle \in R\}$, and the image $R_2(a)$ is the set $\{b \in \mathcal{D} : \langle a, b\rangle \in R\}$. A binary relation R on $\mathcal{D}$ and a binary relation R' on $\mathcal{D}'$ are logically similar if they satisfy the following conditions for $i = 1, 2$:

(S-1)^i For any $a \in \mathcal{D}$, there exists $a' \in \mathcal{D}'$ such that $\langle \mathcal{D}, R_i(a)\rangle \equiv_C \langle \mathcal{D}', R'_i(a')\rangle$;

(S-2)^i For any $a' \in \mathcal{D}'$, there exists $a \in \mathcal{D}$ such that $\langle \mathcal{D}, R_i(a)\rangle \equiv_C \langle \mathcal{D}', R'_i(a')\rangle$.

Since the images are sets in $\wp(\mathcal{D})$ (i.e., first-order objects), they are classified into three kinds based on the logical similarity relation among sets in $\wp(\mathcal{D})$: $\mathcal{D}$, $\emptyset$, and others. For $a \in \mathcal{D}$, the image $R_1(a)$ might be $\mathcal{D}$, $\emptyset$, or a non-empty subset of $\mathcal{D}$. Thus, with respect to the first-component image $R_1(a)$, there are seven kinds of logically similar binary relations.

(i) R such that $R_1(a)$ is $\mathcal{D}$ for any $a \in \mathcal{D}$;

(ii) R such that $R_1(a)$ is $\emptyset$ for any $a \in \mathcal{D}$;

(iii) R such that $R_1(a)$ is a non-empty subset of $\mathcal{D}$ for any $a \in \mathcal{D}$;

(iv) R such that $R_1(a)$ is $\mathcal{D}$ for some $a \in \mathcal{D}$ and $R_1(b)$ is $\emptyset$ for any other $b \in \mathcal{D}$;

(v) R such that $R_1(a)$ is $\mathcal{D}$ for some $a \in \mathcal{D}$ and $R_1(b)$ is a non-empty subset of $\mathcal{D}$ for any other $b \in \mathcal{D}$;

(vi) R such that $R_1(a)$ is $\emptyset$ for some $a \in \mathcal{D}$ and $R_1(b)$ is a non-empty subset of $\mathcal{D}$ for any other $b \in \mathcal{D}$.

(vii) R such that $R_1(a)$ is $\mathcal{D}$ for some $a \in \mathcal{D}$, $R_1(b)$ is $\emptyset$ for some other $b \in \mathcal{D}$, and $R_1(c)$ is a non-empty subset of $\mathcal{D}$ for any other $c \in \mathcal{D}$.

Similarly, with respect to the second-component image $R_2(a)$, there are seven kinds of logically similar binary relations. Although, as a result, 49 combinations of logically similar binary relations are conceivable, some are impossible. For example, if there is $a \in \mathcal{D}$ such that $R_1(a) = \mathcal{D}$, then for any $b \in \mathcal{D}$, $R_2(b) \neq \emptyset$, because $\langle b, a \rangle \in R$.

Among the 49 combinations, only 13 combinations are possible. We express the 13 combinations using the following notation: for a binary relation $R \in \wp(\mathcal{D} \times \mathcal{D})$, "$\exists\langle R_i(x), X\rangle$" means that there exists $x \in \mathcal{D}$ such that $R_i(x) = X$. Let X_0 and Y_0 be some sets in $\wp(\mathcal{D})$ such that $\emptyset \subsetneq X_0, Y_0 \subsetneq \mathcal{D}$. Then, any binary relation on any domain is logically similar to one of the following:

(i) R such that $\exists\langle R_1(a), \mathcal{D}\rangle$ and $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_1(c), \emptyset\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(ii) R such that $\exists\langle R_1(a), \mathcal{D}\rangle$ and $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(a'), \mathcal{D}\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(iii) R such that $\exists\langle R_1(a), \mathcal{D}\rangle$ and $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(iv) R such that $\exists\langle R_1(a), \mathcal{D}\rangle$ and $\exists\langle R_1(c), \emptyset\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(v) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_1(c), \emptyset\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$ and $\exists\langle R_2(c'), \emptyset\rangle$;
(vi) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_1(c), \emptyset\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(vii) R such that $\exists\langle R_1(a), \mathcal{D}\rangle$ and $\exists\langle R_2(a'), \mathcal{D}\rangle$;
(viii) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(a'), \mathcal{D}\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$ and $\exists\langle R_2(c'), \emptyset\rangle$;
(ix) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(a'), \mathcal{D}\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(x) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(a'), \mathcal{D}\rangle$ and $\exists\langle R_2(c'), \emptyset\rangle$;
(xi) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$ and $\exists\langle R_2(c'), \emptyset\rangle$;
(xii) R such that $\exists\langle R_1(b), X_0\rangle$ and $\exists\langle R_2(b'), Y_0\rangle$;
(xiii) R such that $\exists\langle R_1(c), \emptyset\rangle$ and $\exists\langle R_2(c'), \emptyset\rangle$.

In case (vii), $R = \mathcal{D} \times \mathcal{D}$, and in case (xiii), $R = \emptyset$.

For each kind of possible binary relations, an operator that assigns T to binary relations that are logically similar to it and F to others is a logical operator. Therefore, there are 13 logical operators of binary relations that correspond to one of the binary relations above. In addition, there are logical operators that can be described by disjunctions of the 13 logical operators. As a result, there are, in total, 2^{13} logical operators of binary relations, and consequently, there are 2^{13} properties of binary relations that can be logically distinguished from others. We call the 2^{13} logically distinguishable properties of binary relations *logical properties*.

Many major properties of binary relations are excluded from the list of the logical properties. Transitivity is an example of "non-logical" properties. Although R_1 and R_2 above are logically similar binary relations, R_1 is transitive, while R_2

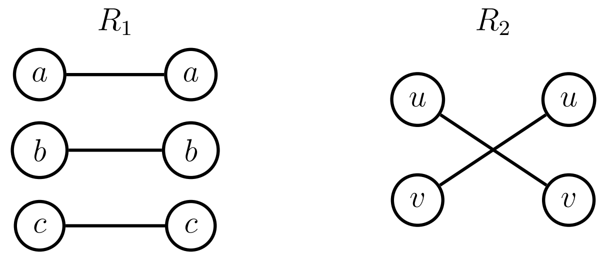

Figure 4.2: Binary relations R_1 and R_2

is not. An operator that assigns T to and only to transitive binary relations is not logical. The logicality of other major properties of binary relations such as reflexivity and symmetricity can be rejected in the same way by the existence of logically similar binary relations such that one satisfies the property and the other does not.

Prime Formal Law

We define a prime formal law as a standard formal law that governs the 2^3 logical properties of unary relations and the 2^{13} logical properties of binary relations above. Every prime formal law, by definition, is a standard formal law, but not every standard formal law is a prime formal law. For example, the standard formal law that validates the argument $\langle \{\varphi_{2\leq}, \varphi_{3\leq}, \ldots\}, \varphi_\infty \rangle$ (pp. 123–124) cannot be taken as a prime formal law, because it involves non-logical properties such as the property of containing infinitely many objects and the property of being transitive. A standard formal law is a formal law that can be satisfied by all standard formal-property bearers. A prime formal law as a standard formal law, therefore, has to be satisfied by all standard formal-property bearers. However, a prime formal law as a special kind of standard formal law is supposed to be satisfied not only by all standard formal-property bearers but also by some Henkin formal-property bearers. In order to determine what standard formal laws are prime formal laws, we need to specify a special type of Henkin formal-property bearers for which prime formal laws have to hold.

Consider, first, Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ to be satisfied by prime formal laws governing the logical property of unary relations whose extension is $\{\mathcal{D}\}$ (the logical property P_5, p.126). The sentence $\exists Z \forall u Z u$ is true in all standard structures, and the standard formal law that justifies the truth of the sentence is:

> For any standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$, there exists a set $X \in \wp(\mathcal{D})$ such that $a \in X$ for all $a \in \mathcal{D}$.

Clearly, the X is the domain set $\mathcal{D}$, and this standard formal law involves the property P_5. Thus, this law is a prime formal law. What Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$ satisfy this prime formal law? The answer is obvious: Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$ such that $\mathcal{D} \in \mathcal{X}(1)$. Other Henkin formal-property bearers do not satisfy this prime formal law.

If a standard formal law is a prime formal law of P_5, then it can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$ such that $\mathcal{D} \in \mathcal{X}(1)$. Conversely, if a standard formal law can be satisfied by all such Henkin formal-property bearers, and if it cannot be satisfied by any other Henkin formal-property bearers, then it can be regarded as related to P_5 but not to any other logical properties. Thus, we can characterize a prime formal law governing P_5 as follows:

> A prime formal law governing P_5 is a standard formal law that can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}\rangle$ such that $\mathsf{Ext}_{\mathcal{D}}(P_5) \cap \mathcal{X}(1) \neq \emptyset$.

Here, by "$\mathsf{Ext}_{\mathcal{D}}(P_5)$", we denote the extension of P_5 on $\mathcal{D}$, which is $\{\mathcal{D}\}$. The condition imposed on Henkin formal-property bearers "$\mathsf{Ext}_{\mathcal{D}}(P_5) \cap D(1) \neq \emptyset$" thus means $\mathcal{D} \in \mathcal{X}(1)$.

The same argument can be made for the logical property whose extension is $\{\emptyset\}$ (the logical property P_7, p.126). The sentence $\exists Z \forall u \neg Z u$ is true in all standard structures and can be grounded in the following standard formal law:

> For any standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D}\times\mathcal{D}), \mathcal{S}\rangle$, there exists a set $X \in \wp(\mathcal{D})$ such that $a \notin X$ for any $a \in \mathcal{D}$.

This standard formal law states the existence of the empty set $\emptyset$, and therefore is a prime formal law involving the logical property P_7. This law can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ such that $\emptyset \in \mathcal{X}(1)$, and all other prime formal laws governing P_7 can also be characterized by such Henkin formal-property bearers:

> A prime formal law governing P_7 is a standard formal law that can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ such that $\mathsf{Ext}_{\mathcal{D}}(P_7) \cap \mathcal{X}(1) \neq \emptyset$.

Since $\mathsf{Ext}_{\mathcal{D}}(P_7) = \{\emptyset\}$, "$\mathsf{Ext}_{\mathcal{D}}(P_7) \cap \mathcal{X}(1) \neq \emptyset$" means $\emptyset \in \mathcal{X}(1)$.

The argument above shows that prime formal laws governing P_5 can be characterized by Henkin formal-property bearers satisfying the condition $\mathsf{Ext}_{\mathcal{D}}(P_5) \cap \mathcal{X}(1) \neq \emptyset$ and prime formal laws governing P_7 by Henkin formal-property bearers satisfying the condition $\mathsf{Ext}_{\mathcal{D}}(P_7) \cap \mathcal{X}(1) \neq \emptyset$. Thus, some might expect that for other six logical properties (P_1, P_2, P_3, P_4, P_6, and P_8) as well, the condition "$\mathsf{Ext}_{\mathcal{D}}(P_i) \cap \mathcal{X}(1) \neq \emptyset$" is a correct condition to specify the Henkin formal-property bearers that characterize prime formal laws governing them. However, that is not true. Consider the logical property P_8 that no unary relation can satisfy. By the definition, we have $\mathsf{Ext}_{\mathcal{D}}(P_8) = \emptyset$ and therefore $\mathsf{Ext}_{\mathcal{D}}(P_8) \cap \mathcal{X}(1) = \emptyset$ for any Henkin formal-property bearer $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$. However, there is a prime formal law governing P_8:

> For any standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle$, there does not exist a set $X \in \wp(\mathcal{D})$ such that $a \in X$ and $a \notin X$ for any $a \in \mathcal{D}$.

This standard formal law states the nonexistence of sets satisfying P_8 and can be satisfied not only by all standard formal-property bearers but also by all Henkin formal-property bearers. On the one hand, there is a prime formal law governing P_8, and therefore there have to be Henkin formal-property bearers that characterize it. On the other hand, however, there is no Henkin formal-property bearer that meets the condition $\mathsf{Ext}_{\mathcal{D}}(P_8) \neq \emptyset$. Hence, it can be concluded that the condition $\mathsf{Ext}_{\mathcal{D}}(P_8) \neq \emptyset$ is not a right condition to be imposed on Henkin formal-property bearers that characterize prime formal laws governing P_8.

Generally, for a logical property P_i, and for a Henkin formal-property bearer $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$, there are three possible cases with respect to the relationship between $\mathsf{Ext}_{\mathcal{D}}(P_i)$ and $\mathcal{X}(1)$:

(i) $\mathsf{Ext}_{\mathcal{D}}(P_i) \neq \emptyset$ and $\mathsf{Ext}_{\mathcal{D}}(P_i) \cap \mathcal{X}(1) \neq \emptyset$;

(ii) $\mathsf{Ext}_{\mathcal{D}}(P_i) \neq \emptyset$ and $\mathsf{Ext}_{\mathcal{D}}(P_i) \cap \mathcal{X}(1) = \emptyset$;

(iii) $\mathsf{Ext}_{\mathcal{D}}(P_i) = \emptyset$ (and therefore $\mathsf{Ext}_{\mathcal{D}}(P_i) \cap \mathcal{X}(1) = \emptyset$).

The examples of the logical properties P_5 and P_7 above show that for a standard formal law to be regarded as a prime formal law governing P_i, it must be satisfied by Henkin formal-property bearers of the case (i) and must not be satisfied by Henkin formal-property bearers of the case (ii).

How about the case (iii)? For a standard formal law to be regarded as a prime formal law governing P_i, does it have to be satisfied by Henkin formal-property bearers such that $\mathsf{Ext}_{\mathcal{D}}(P_i) = \emptyset$? Note that a prime formal law governing P_i, by definition, is a standard formal law, and therefore holds for any standard formal-property bearer $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle$ even if $\mathsf{Ext}_{\mathcal{D}}(P_i) = \emptyset$. In such a standard formal-property bearer, a unary relation satisfying P_i does not exists in $\wp(\mathcal{D})$ and does not appear in $\mathcal{S}$. Whether or not the prime formal law governing P_i holds does not depend on the existence of unary relations satisfying P_i. It then follows that the prime formal law also holds for any Henkin formal-property bearer $\langle \mathcal{D}', \mathcal{X}(1), \mathcal{X}(2), \mathcal{S}' \rangle$ such that $\mathsf{Ext}_{\mathcal{D}}(P_i) = \emptyset$, because there is no difference between the standard formal-property bearer and the Henkin formal-property bearer with respect to the nonexistence of unary relations satisfying P_i. In fact, the prime formal law governing P_8 above holds for any Henkin formal-property bearer such that $\mathsf{Ext}_{\mathcal{D}}(P_8) = \emptyset$ (namely, for any Henkin formal-property bearer).

Based on the consideration above, we characterize prime formal laws governing a logical property of unary relations as follows:

> A prime formal law governing a logical property P of unary relations is a standard formal law that can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ such that either $\mathsf{Ext}_{\mathcal{D}}(P) = \emptyset$ or $\mathsf{Ext}_{\mathcal{D}}(P) \cap \mathcal{X}(1) \neq \emptyset$.

Note that the characterization of prime formal laws governing P_5 (or P_7) obtained from this characterization by replacing P with P_5 (or P_7) is the same as the one given above, because $\mathsf{Ext}_{\mathcal{D}}(P_5) \neq \emptyset$ and $\mathsf{Ext}_{\mathcal{D}}(P_7) \neq \emptyset$.

The same idea can be applied to the characterization of prime formal laws governing the 2^{13} logical properties of binary relations:

> A prime formal law governing a logical property R of binary relations is a standard formal law that can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ such that either $\mathsf{Ext}_{\mathcal{D}}(R) = \emptyset$ or $\mathsf{Ext}_{\mathcal{D}}(R) \cap \mathcal{X}(2) \neq \emptyset$.

How can a prime formal law that governs multiple logical properties be characterized? Consider the sentence $\exists Z \forall x Zx \wedge \exists Z' \forall y \neg Z' y$. This sentence is true in all standard structures, and the standard formal law that validates it is:

> For any standard formal-property bearers $\langle \mathcal{D}, \wp(\mathcal{D}), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle$, there exists a set $X \in \wp(\mathcal{D})$ such that $a \in X$ for any $a \in \mathcal{D}$ and there exists a set $Y \in \wp(\mathcal{D})$ such that $b \notin Y$ for any $b \in \mathcal{D}$.

This standard formal law is a prime formal law that involves the properties P_5 and P_7, which can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ such that $\mathcal{D}, \emptyset \in \mathcal{X}(1)$. Generally, a prime formal law can be thought of as governing logical properties P and P' if it can be satisfied by all and only Henkin formal-property bearers that both of them are supposed to be satisfied by. If a Henkin formal-property bearer $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ is supposed to not satisfy at least one of them, it must fail to satisfy the prime formal law.

If a prime formal law governs all the $(2^3 + 2^{13})$ logical properties, then it can be satisfied by all and only Henkin formal-property bearers that all of them are supposed to be satisfied by. That is to say:

> A prime formal law that governs the logical properties $P_1, \ldots, P_8$ of unary relations and the logical properties $R_1, \ldots, R_{2^{13}}$ of binary relations is a standard formal law that can be satisfied by all and only Henkin formal-property bearers $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ satisfying the following two conditions:
>
> (i) For any P_i, either $\mathsf{Ext}_{\mathcal{D}}(P_i) = \emptyset$ or $\mathsf{Ext}_{\mathcal{D}}(P_i) \cap \mathcal{X}(1) \neq \emptyset$;
> (ii) For any R_j, either $\mathsf{Ext}_{\mathcal{D}}(R_j) = \emptyset$ or $\mathsf{Ext}_{\mathcal{D}}(R_j) \cap \mathcal{X}(2) \neq \emptyset$.

Prime Logic

Each Henkin formal-property bearer $\langle \mathcal{D}, \mathcal{X}(1), \mathcal{X}(2), \mathcal{S} \rangle$ corresponds to a Henkin structure $\langle \mathcal{D}, D(1), D(2), I \rangle$ such that $\mathcal{X}(1) = D(1)$ and $\mathcal{X}(2) = D(2)$

and such that $\mathcal{S}$ is composed of objects that are assigned to constants symbols, unary relation symbols, and binary relation symbols by the interpretation function I (see, pp. 118–119). We call a Henkin structure a *prime Henkin structure* if its corresponding Henkin formal-property bearer satisfies the two conditions above. Suppose that an argument holds in all and only prime Henkin structures. Since every standard structure is a prime Henkin structure, the argument is valid in standard semantics and there is a standard formal law that validates it. The standard formal law holds true in all and only prime Henkin structures. Thus, the law can be satisfied by all and only Henkin formal-property bearers satisfying the two conditions. It then follows that the standard formal law is a prime formal law governing all the $(2^3 + 2^{13})$ logical properties.

More generally, we can show that if an argument holds all, but not necessarily only, prime Henkin structures, then there is a prime formal law governing some of the $(2^3 + 2^{13})$ logical properties that justifies its validity. Suppose that an argument holds all prime Henkin structures. Since the argument is valid in standard semantics, there is a standard formal law that validates it, which can be satisfied by any Henkin formal-property bearer associated with a prime Henkin structure. Assume that the standard formal law governs a formal property P of unary relations that is not a logical property. For example, we can take the property of cardinality κ as P. Then, for some domain $\mathcal{D}$ that contains more than κ-many objects, consider a Henkin formal-property bearer $\mathfrak{H} = \langle \mathcal{D}, \mathcal{X}(1), \wp(\mathcal{D} \times \mathcal{D}), \mathcal{S} \rangle$ such that $\mathcal{X}(1) = \{X \in \wp(\mathcal{D}) : |X| \neq \kappa\}$. Since $\mathcal{X}(1)$ does not contain any unary relation satisfying P, the standard formal law cannot be satisfied by $\mathfrak{H}$. However, $\mathfrak{H}$ meets the two conditions (i) and (ii) above, and therefore the standard formal law has to be satisfied by $\mathfrak{H}$. This is a contradiction. Therefore, P is a logical property. For any formal property that is governed by the standard formal law, we can show that it is a logical property in a similar way (by a Henkin formal-property bearer whose $\mathcal{X}(i)$ does not contain any set satisfying the property). Hence, the standard formal law is a prime formal law.

Under the minimal notion, an argument can be regarded as logically valid if it holds by virtue of some prime formal law. Above, we have characterized prime formal laws and a special kind of Henkin structures in which they are supposed to hold (i.e., prime Henkin structures). We can now define the prime logical validity of classical structures. We call a prime Henkin structure a *prime Henkin model* of a set Γ of sentences if all sentences in Γ are true in it.

> An argument $\langle \Gamma, \varphi \rangle$ is said to be *logically valid* if every prime Henkin model of Γ is also a prime Henkin model of $\{\varphi\}$.

We call the logical system with this new semantics *prime logic*.

With respect to the range of validity, prime logic is located between second-order logic with standard semantics and second-order logic with Henkin semantics. All valid arguments in Henkin semantics are valid in prime logic, and all valid arguments in prime logic are valid in standard semantics. However, the opposite inclusion relation does not hold. There are some arguments that are valid in standard semantics but invalid in prime logic, and there are some arguments that are valid in prime logic but invalid in Henkin semantics. The two arguments $\langle \{\varphi_{2\leq}, \varphi_{3\leq}, \ldots\}, \varphi_\infty \rangle$ and $\langle \emptyset, \varphi_C \rangle$ that we have examined (pp. 123–125) are examples of the former kind.[4] An example of the latter kind is the argument $\langle \emptyset, \exists Z \forall x \neg Zx \rangle$ deriving the existence of the empty unary relation from no premise.

[4]An example of prime Henkin structures that invalidates the argument $\langle \{\varphi_{2\leq}, \varphi_{3\leq}, \ldots\}, \varphi_\infty \rangle$ is $\langle \mathbb{N}, \wp(\mathbb{N}), D(2), I \rangle$ where $D(2)$ contains the following binary relations:

- (i) $R^{(1)}$ such that $\langle n, 0 \rangle \in R^{(1)}$ for any $n \in \mathbb{N}$, such that $\langle m, m \rangle \in R^{(1)}$ for any odd number m, and such that $\langle n, m' \rangle \notin R^{(1)}$ for any $n \in \mathbb{N}$ and for any positive even number m';
- (ii) $R^{(2)}$ such that $\langle n, 0 \rangle \in R^{(2)}$ and $\langle 0, n \rangle \in R^{(2)}$ for any $n \in \mathbb{N}$ and such that $\langle n, n \rangle \in R^{(2)}$ for any positive natural number n;
- (iii) $R^{(3)}$ such that $\langle n, 0 \rangle \in R^{(3)}$ and $\langle n, n \rangle \in R^{(3)}$ for any $n \in \mathbb{N}$;
- (iv) $R^{(4)}$ such that $\langle n, 0 \rangle \in R^{(4)}$ for any $n \in \mathbb{N}$;
- (v) $R^{(5)}$ such that $\langle n, n \rangle \in R^{(5)}$ for any positive natural number $n \in \mathbb{N}$;
- (vi) $R^{(6)}$ such that $\langle 0, 0 \rangle \in R^{(6)}$ and such that $\langle n, m \rangle \in R^{(6)}$ for any positive natural number n and for any odd number m;
- (vii) $R^{(7)}$ is $\mathcal{D} \times \mathcal{D}$
- (viii) $R^{(8)}$ such that $\langle 0, n \rangle \in R^{(8)}$ for any $n \in \mathbb{N}$, such that $\langle m, m \rangle \in R^{(8)}$ for any odd number m, and such that $\langle m', n \rangle \notin R^{(8)}$ for any $n \in \mathbb{N}$ and for any positive even number m';
- (ix) $R^{(9)}$ such that $\langle 0, n \rangle \in R^{(9)}$ and $\langle n, n \rangle \in R^{(9)}$ for any $n \in \mathbb{N}$;
- (x) $R^{(10)}$ such that $\langle 0, n \rangle \in R^{(10)}$ for any $n \in \mathbb{N}$;
- (xi) $R^{(11)}$ such that $\langle 0, 0 \rangle \in R^{(11)}$ and such that $\langle m, n \rangle \in R^{(11)}$ for any positive natural number n and for any odd number m;
- (xii) $R^{(12)}$ such that $\langle n, n \rangle \in R^{(12)}$ for any $n \in \mathbb{N}$;
- (xiii) $R^{(13)}$ is $\emptyset$.

Each $R^{(i)}$ above satisfies the condition of logically distinguishable binary relations of the same number given at p. 128. However, none of them is transitive, irreflexive, and serial (in particular, $(1), \ldots, (12)$ are not irreflexive, and (13) is not serial). Therefore, the conclusion φ_∞ is false in the prime Henkin structure, although all premises $\varphi_{2\leq}, \varphi_{3\leq}, \ldots$ are true in it.

The difference among these logical systems is the difference among the formal properties that they take into account. In standard semantics, any standard formal law governing any formal properties is entitle to validate arguments. In prime logic, prime formal laws governing logical properties play the special role. Henkin semantics is a semantic system in which the validity can be justified by Henkin formal laws.[5]

From the minimalist's point of view, standard semantics overgenerates valid arguments; some arguments hold by virtue of formal laws about non-logical properties. Henkin semantics undergenerates valid arguments; some formal laws, although they govern logical properties, are not available for validating arguments. The bounds of logic should be located between those created by the two major logical systems, which can be drawn by prime logic.

4.2 Prime Logical Validity of Non-Classical structures

Characterizing prime logical validity means characterizing both valid arguments in classical structures and valid arguments in non-classical structures. The characterization of prime logical validity of classical structures has been achieved above. The characterization of prime logical validity of non-classical structures will be investigated below. The important difference between classical structures and non-classical structures is that domains of classical structures are independent of one another, while those of non-classical structures can be connected by accessibility relations. For the characterization, we will first consider what non-classical

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An counterexample of the argument $\langle \emptyset, \varphi_C \rangle$ is a prime Henkin structure $\langle \mathcal{D}', \mathcal{D}'(1), \mathcal{D}'(2), I' \rangle$, where $\mathcal{D}' = \{0, 1, 2\}$, $\mathcal{D}'(1) = \{\emptyset, \{0\}, \mathcal{D}'\}$, and $\mathcal{D}'(2) = \wp(\mathcal{D}' \times \mathcal{D}')$. The binary relation $\{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$ in $\mathcal{D}'(2)$ satisfies the property expressed by the open formula $\forall X \exists u \forall v (Zuv \leftrightarrow Xv)$. Thus, this is not a prime Henkin model of $\{\varphi_C\}$.

[5] A Henkin formal law is a formal law that can be satisfied by all Henkin formal-property bearers. There are four kinds of formal properties that Henkin formal laws can govern: (i) formal properties that any unary relation satisfies; (ii) formal properties that no unary relation satisfies; (iii) formal properties that any binary relation satisfies; (ii) formal properties that no binary relation satisfies. A formal law governing some formal properties other than these fails to be satisfied by some Henkin formal-property bearers.

structures can be regarded as logically similar under the minimal notion.

Logical Similarity among Non-Classical structures

Non-classical logics with a Kripke semantics are diverse. The diversity is mainly due to the diversity of accessibility relations. A *non-classical structure* of a non-classical logical system $\mathcal{L}$ is a triple $\langle \mathfrak{D}, R, I \rangle$, where $\mathfrak{D}$ is a set of domains and R is an accessibility relation associated with $\mathcal{L}$. I is an interpretation function for extra logical terms of a formal language of $\mathcal{L}$. An argument $\langle \Gamma, \varphi \rangle$ is said to be valid in $\mathcal{L}$ if it holds in all domains of all non-classical structures of $\mathcal{L}$. An accessibility relation R is supposed to be a binary relation on $\mathfrak{D}$ in many standard non-classical logics, but can be a ternary relation and more generally an n-ary relation (e.g., the accessibility relation in terms of which the truth condition of the conditional in relevant logic can be defined). A binary accessibility relation on $\mathfrak{D}$ is a set composed of pairs $\langle \mathcal{D}, \mathcal{D}' \rangle$ of domains $\mathcal{D}, \mathcal{D}' \in \mathfrak{D}$, which can be characterized by conditions imposed on it. Reflexivity, symmetricity, and transitivity are examples of conditions to be used for defining well-known non-classical logical systems such as modal logics T, B, and $S4$. Different accessibility relations produce different non-classical structures and different non-classical logical systems. There are various possible accessibility relations, and therefore there are various non-classical logics.

The prime logical validity of non-classical structures is the validity that can be grounded in prime formal laws. An argument can be regarded as logically valid under the minimal notion if, and only if, there exists some prime formal law that validates it. Thus, even if an argument is valid in a non-classical logical system $\mathcal{L}$ based on some justifiable notion of logical validity, it cannot be taken as valid under the minimal notion unless there is some prime formal law that justifies its validity.

What is a prime formal law of non-classical structures? How can they be characterized? Remember that a standard formal law of classical structures holds in standard structures and a Henkin formal law of classical structures holds in Henkin structures. Also, a prime formal law of non-classical structures holds in prime Henkin structures. Each kind of formal laws has a range of application, and they can be distinguished by their ranges. Despite the difference between classical-structures and non-classical structures, prime formal laws of non-classical structures can also be supposed to have a range: all prime formal laws hold in non-classical structures within that range, and some of them fail to hold in some non-classical structures outside of it. In order to define prime formal laws of non-classical structures, their range needs to be determined.

Remember that binary relations on a set can be classified into thirteen kinds by the logical similarity relation (pp. 127–128). By applying logical similarity, binary accessibility relations R of non-classical structures $\langle \mathfrak{D}, R, I \rangle$ can be divided into thirteen kinds. For example, "universal" binary accessibility relations $R = \mathfrak{D} \times \mathfrak{D}$ form one kind, and "empty" binary accessibility relations $R = \emptyset$ form another kind. Reflexivity, symmetricity, and transitivity define collections of binary relations respectively, but none of them can characterize one of the thirteen kinds. Some reflexive binary relation and some irreflexive binary relation belong to the same kind.[6] Some transitive binary relation belongs to one kind and another transitive binary relation belongs to another kind.[7]

If binary accessibility relations R and R' are logically similar, then non-classical structures $\langle \mathfrak{D}, R, I \rangle$ and $\langle \mathfrak{D}', R', I' \rangle$ can also be regarded as logically similar under the minimal notion. Although they might have different collections of domains ($\mathfrak{D}$ and $\mathfrak{D}'$) or different interpretation functions (I and I'), logic does not have to take into account the difference made by them. Non-classical structures $\langle \mathfrak{D}, R, I \rangle$ and $\langle \mathfrak{D}', R', I' \rangle$ are logically similar if $R = \mathfrak{D} \times \mathfrak{D}$ and $R' = \mathfrak{D}' \times \mathfrak{D}'$; these "universal" binary accessibility relations belong to the same kind. For the same reason, $\langle \mathfrak{D}, R, I \rangle$ and $\langle \mathfrak{D}', R', I' \rangle$ are logically similar if $R = R' = \emptyset$

Generally, for a statement to be called a "law," it has to hold not only in particular selected situations but also in any situations that can be thought of as similar to them in a given context. Standard formal laws, Henkin formal laws, and

[6] Let $\mathcal{D}$ be the set $\{a, b, c\}$. And let $R_1 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle\}$ and $R_1' = \{\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle\}$. R_1 is reflexive and R_1' is irreflexive. However, they are logically similar (see p. 81).

[7] For the $\mathcal{D}$ above, let $R_1'' = \mathcal{D} \times \mathcal{D}$. R_1'' and R_1 above are both transitive. However, they are logically dissimilar; R_1'' belongs to the type (xii) which is introduced at p. 128 and R_1 belongs to the type (vii).

prime formal laws hold in all structures associated with them respectively. If a statement holds in some situation and does not hold in another similar situation, it is not a law. Based on this general view of what a law is, we can suppose that prime formal laws satisfies the following condition:

> If a prime formal law is supposed to hold in a non-classical structure, then it holds in all non-classical structures that are logically similar to it.

Note that this condition does not require a prime formal law to hold in any non-classical structure. Under the minimal notion, non-classical structures $\langle \mathfrak{D}, R, I \rangle$ and $\langle \mathfrak{D}', R', I' \rangle$ can be distinguished from each other if their accessibility relations R and R' are logically dissimilar. Collections of non-classical structures of different kinds can be taken as specifying different ranges in which prime formal laws are supposed to hold. Thus, it is allowed that a prime formal law holds in all non-classical structures of one kind but does not hold in some non-classical structures of another kind. Different kinds of binary accessibility relation determine different ranges of prime formal laws. By the classification of ranges, prime formal laws can be divided into different kinds.

Note also that the condition above does not imply that the range of a prime formal law is composed of logically similar non-classical structures of one kind: a union of multiple ranges can be the range of some prime formal laws. It is possible that a prime formal law holds in non-classical structures of one kind and also in non-classical structures of another kind. In fact, if a statement holds in every non-classical structure, then it should be regarded as a prime formal law that has the widest range. There are thirteen kinds of binary accessibility relations, and as a result there are $(2^{13} - 1)$ ranges.[8] Prime formal laws are characterized for each range.

The logical similarity relation among non-classical structures with an n-ary accessibility relation can be defined in the same way: non-classical structures $\langle \mathfrak{D}, R, I \rangle$ and $\langle \mathfrak{D}', R', I' \rangle$ are logically similar if R and R' are logically similar as n-ary relations (see p. 81). Also, for defining the logical similarity relation among non-classical structures $\langle \mathfrak{D}, R_1, \ldots, R_n, I \rangle$ with multiple accessibility relations, the

[8] The meaning of "-1" is that a prime formal law has to hold in at least one range.

same method can be applied: $\langle \mathfrak{D}, R_1, \ldots, R_n, I\rangle$ and $\langle \mathfrak{D}', R'_1, \ldots, R'_n, I'\rangle$ are logically similar if R_i and R'_i are logically similar for all i. Prime formal laws of non-classical structures of any form can be classified based on these logical similarity relations.

Prime Formal Law of Non-Classical Structures

A prime formal law of non-classical structures is a formal law governing formal properties that can be applied to set-theoretical constructs of a certain form. We call a bearer of formal properties a *non-classical formal-property bearer*. To define it, we fix a first-order formal language L^{NC} of a non-classical logical system. For the sake of simplicity, we suppose that L^{NC} contains only unary and and binary relation symbols. We do not deal with n-ary relations for $n \geq 3$. We also suppose that function symbols are not in the vocabulary of L^{NC}. Modal operator symbols such as $\Box$ and $\Diamond$ may or may not be contained in L^{NC}. For any logical constant in L^{NC}, we suppose that its truth condition can be defined in terms of only one binary accessibility relation between domains. Thus, in what follows, we will consider only non-classical structures $\langle \mathfrak{D}, R, I\rangle$ with a binary accessibility relation R. With respect to domains $\mathcal{D} \in \mathfrak{D}$, there are two options: constant domains and variable domains. For simplicity, we suppose that our non-classical logical system is a constant-domain logical system and that all domains of a given non-classical structure contain the same objects.

A *non-classical formal-property bearer* is a quadruple $\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S}\rangle$, where $\mathcal{D}_0 \in \mathfrak{D}$ and $\mathcal{S}$ is a set of quadruples $\langle \mathcal{D}, \langle a_1, a_2, \ldots\rangle, \langle X_1, X_2, \ldots\rangle, \langle Y_1, Y_2, \ldots\rangle\rangle$. Here, $\mathcal{D} \in \mathfrak{D}$, $a_i \in \mathcal{D}$, $X_j \in \wp(\mathcal{D})$, and $Y_k \in \wp(\mathcal{D} \times \mathcal{D})$. For each $\mathcal{D} \in \mathfrak{D}$, we suppose that $\mathcal{S}$ contains only one quadruple $\langle \mathcal{D}, \langle a_1, a_2, \ldots\rangle, \langle X_1, X_2, \ldots\rangle, \langle Y_1, Y_2, \ldots\rangle\rangle$ whose first component is $\mathcal{D}$. Each non-classical structure corresponds to a collection of non-classical formal-property bearers. Let $c_1, c_2, \ldots$ be constant symbols of L^{NC}. Also, let $Q_1, Q_2, \ldots$ be unary relation symbols, and let $R_1, R_2, \ldots$ be binary relation symbols. Then, for a non-classical structure $\langle \mathfrak{D}, R, I\rangle$, the corresponding collection is:

$$\{\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S}\rangle : I(\langle \mathcal{D}, c_i\rangle) = a_i,\ I(\langle \mathcal{D}, Q_j\rangle) = X_i, \text{ and } I(\langle \mathcal{D}, R_k\rangle) = Y_k\}.$$

Let $\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S} \rangle$ and $\langle \mathfrak{D}', \mathcal{D}'_0, R', \mathcal{S}' \rangle$ be non-classical formal-property bearers. They are said to be *isomorphic* if there exist a bijection $\eta : \mathfrak{D} \to \mathfrak{D}'$ and bijections $\pi_{\mathcal{D}} : \mathcal{D} \to \eta(\mathcal{D})$ for any $\mathcal{D} \in \mathfrak{D}$ such that:

(i) $\eta(\mathcal{D}_0) = \mathcal{D}'_0$

(ii) For any $\mathcal{D}_1, \mathcal{D}_2 \in \mathfrak{D}$, it holds that $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle \in R$ if and only if $\langle \eta(\mathcal{D}_1), \eta(\mathcal{D}_2) \rangle \in R'$;

(iii) For any quadruple $\langle \mathcal{D}, \langle a_1, a_2, \ldots \rangle, \langle X_1, X_2, \ldots \rangle, \langle Y_1, Y_2, \ldots \rangle \rangle \in \mathcal{S}$, it holds that $\langle \eta(\mathcal{D}), \langle \pi_{\mathcal{D}}(a_1), \pi_{\mathcal{D}}(a_2), \ldots \rangle, \langle \pi_{\mathcal{D}}(X_1), \pi_{\mathcal{D}}(X_2), \ldots \rangle, \langle \pi_{\mathcal{D}}(Y_1), \pi_{\mathcal{D}}(Y_2), \ldots \rangle \rangle \in \mathcal{S}'$.

A *formal property* is a property such that for any two isomorphic non-classical formal-property bearers, one satisfies the property if and only if the other satisfies the property.

The truth condition of a sentence in L^{NC} describes a formal property. For example, the sentence $\forall x \Box Q_1 x$ is true in a domain $\mathcal{D}_0 \in \mathfrak{D}$ of a non-classical structure $\langle \mathfrak{D}, R, I \rangle$, if the corresponding non-classical formal-property bearer $\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S} \rangle$ satisfies the following formal property:

> For any object $a \in \mathcal{D}_0$, and for any $\mathcal{D}$ such that $\langle \mathcal{D}_0, \mathcal{D} \rangle \in R$, it holds that $a \in X_1$ for $\langle \mathcal{D}, \langle a_1, a_2, \ldots \rangle, \langle X_1, X_2, \ldots \rangle, \langle Y_1, Y_2, \ldots \rangle \rangle \in \mathcal{S}$.

Whether or not a non-classical formal-property bearer satisfies this formal property is determined based on whether or not $\mathcal{D}_0 \subseteq X_1$. Other components in $\mathcal{S}$ (i.e., $a_1, a_2, \ldots, X_2, \ldots Y_1, Y_2, \ldots$) have nothing to do with it. This is because the sentence $\forall x \Box Q_1 x$ contains only one unary relation symbol Q_1, which corresponds to the component X_1 in $\mathcal{S}$.

Let $P_1, P_2, \ldots, P_C$ be formal properties. Consider a statement that can be expressed in the following form:

> If a non-classical formal-property bearer $\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S} \rangle$ satisfies the formal properties $P_1, P_2, \ldots$, then it also satisfies the formal property P_C.

We define a *prime formal law* as a statement of this form that satisfies the following condition:

> There exists some non-classical formal-property bearer $\langle \mathfrak{D}, \mathcal{D}_0, R, \mathcal{S} \rangle$ such that the statement holds true for any non-classical formal-property bearer

$\langle \mathfrak{D}', \mathcal{D}'_0, R', \mathcal{S}' \rangle$ whose accessibility relation R' is logically similar to R.

According to this definition, for some non-classical structure $\langle \mathfrak{D}, R, I \rangle$, a prime formal law is satisfied by any non-classical formal-property bearers associated with any non-classical structures that are logically similar to $\langle \mathfrak{D}, R, I \rangle$. The application range of the prime formal law is composed of the logically similar non-classical structures.

Let P'_1 be the formal property described by the sentence $\forall x \Box Q_1 x$ above. Also, let P'_C be the formal property defined as follows:

(P'_C) For any domain $\mathcal{D}$ such that $\langle \mathcal{D}_0, \mathcal{D} \rangle \in R$, it holds that $a \in X_1$ for any object $a \in \mathcal{D}$.

This formal property P'_C expresses the truth condition of the sentence $\Box \forall x Q_1 x$ in the domain $\mathcal{D}_0$. The statement that any non-classical formal-property bearer satisfying P'_1 satisfies P'_C is a prime formal law. This prime formal law holds true for any non-classical formal-property bearer regardless of what its accessibility relation is.

As we have argued above, there are $(2^{13} - 1)$ kinds of prime formal laws. Each kind of prime formal laws has its application range, which can be determined by the kind of binary accessibility relations associated with it. There are prime formal laws that can be satisfied by any non-classical formal-property bearer with any accessibility relation. The prime formal law above is an example of such formal laws that hold universally. On the other hand, there are prime formal laws that can be satisfied by non-classical formal-property bearers with an accessibility relation of a particular kind. For example, some prime formal laws hold only for non-classical formal-property bearers with a serial accessibility relation.[9] An example of such prime formal raws is the one governing the following formal properties:

(P''_1) For any domain $\mathcal{D}$ such that $\langle \mathcal{D}_0, \mathcal{D} \rangle \in R$, it holds that $a \in X_1$ for any object $a \in \mathcal{D}$;

(P''_C) For some domain $\mathcal{D}$ such that $\langle \mathcal{D}_0, \mathcal{D} \rangle \in R$, it holds that $a \in X_1$ for any object $a \in \mathcal{D}$;

[9]Binary accessibility relations having seriality define a range of prime formal laws, which is the union of the ranges characterized by the logically similar binary relations of the kinds (i), (ii), (iii), (iv), (vi), (vii), (ix), (xii) given p. 128. These binary relations are all serial, while others are not.

(P''_1) is identical to (P'_C) above. (P''_C) is the formal property that can be described by the sentence $\Diamond\forall x Q_1 x$.

For sentences $\varphi_1, \varphi_2, \ldots, \varphi_C$, let $P_1, P_2, \ldots, P_C$ be formal properties that describe their truth conditions. Also, let l be a statement that if a non-classical formal-property bearer satisfies $P_1, P_2, \ldots$, then it also satisfies P_C. If a sentence φ_i is true in all domains in a non-classical structure, then the formal property P_i can be satisfied by all non-classical formal-property bearers associated with the non-classical structure. More generally, if the argument $\langle\{\varphi_1, \varphi_2, \ldots,\}, \varphi_C\rangle$ holds in all domains in a non-classical structure, then l can be satisfied by all non-classical formal-property bearers associated with it.

Let $\mathfrak{S}$ be a collection of non-classical structures satisfying the following condition:

> For any non-classical structures $\langle\mathfrak{D}, R, I\rangle$ and $\langle\mathfrak{D}', R', I'\rangle$ such that R is logically similar to R', it holds that $\langle\mathfrak{D}, R, I\rangle \in \mathfrak{S}$ if and only if $\langle\mathfrak{D}', R', I'\rangle \in \mathfrak{S}$.

If a non-classical structure is in $\mathfrak{S}$, any non-classical structure that is logically similar to it is also in $\mathfrak{S}$. Suppose that an argument $\langle\Gamma, \varphi\rangle$ holds in any domain in any non-classical structure in $\mathfrak{S}$. Then, the corresponding statement l can be satisfied by any non-classical formal-property bearer associated with any non-classical structure in $\mathfrak{S}$. The statement l is a prime formal law.

Under the minimal notion, an argument can be regarded as logically valid if it holds by virtue of some prime formal law. For a non-classical logical system $\mathcal{L}$, suppose that the collection of non-classical structures of $\mathcal{L}$ meets the condition above. If an argument is valid in $\mathcal{L}$, then there is a prime formal law that justifies its validity. The argument is logically valid under the minimal notion. We call such logical system $\mathcal{L}$ a *prime logic*. Since there are $(2^{13} - 1)$ kinds of prime formal laws, there are $(2^{13} - 1)$ prime logics, in each of which all and only logically valid arguments of some kind can be validated.

The modal logical system K, in which any accessibility relation is allowed, is an example of a prime logic. Various arguments can be validated in K, all of which are logically valid under the minimal notion. The modal logical system D, in which binary accessibility relations are supposed to satisfy the seriality condition is another example of a prime logic. Consider the following argument:

$$\begin{array}{ll} & \Box\forall x Q_1 x. \\ \hline \therefore & \Diamond\forall x Q_1 x. \end{array}$$

This argument is valid in D but invalid in K, yet can be regarded as logically valid under the minimal notion, because its validity is based on a prime formal law (the prime formal law governing P''_1 and P''_C above). The logical validity of an argument is relative to a kind of prime formal laws

The modal logical system $S4$ is not a prime logic. In $S4$, a binary accessibility relation is supposed to be reflexive and transitive. Consider the following binary accessibility relations:

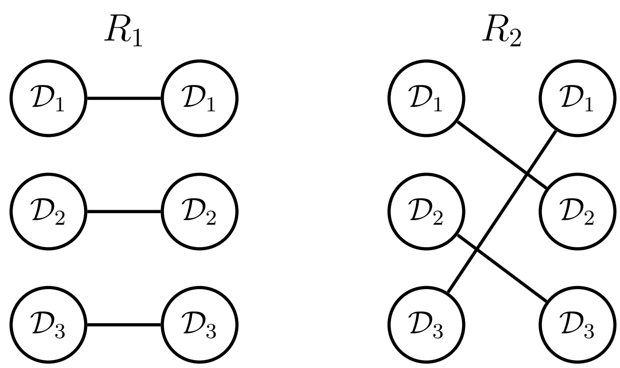

Figure 4.3: Binary accessibility relations R_1 and R_2

R_1 is reflexive and transitive, while R_2 is not. For $\mathfrak{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$, therefore, a non-classical structure $\langle \mathfrak{D}, R_1, I_1 \rangle$ is a legitimate non-classical structure of the modal logical system $S4$, while a non-classical structure $\langle \mathfrak{D}, R_2, I_2 \rangle$ is not. However, R_1 and R_2 are logically similar binary relations. There is no prime formal law that holds in $\langle \mathfrak{D}, R_1, I_1 \rangle$ but does not hold in $\langle \mathfrak{D}, R_2, I_2 \rangle$. Hence, an argument that holds in all and only $S4$-non-classical structures is not logically valid under the minimal notion.

A prime logic can be defined not only for non-classical structures with a binary accessibility relation but also for non-classical structures with an n-ary accessibility relation and for non-classical structures with multiple accessibility relations. The idea of the definition is the same as above. First, classify accessibility relations based on their logical similarity. Second, characterize prime formal laws for each logically similar accessibility relations. Finally, define a prime logic as a non-classical logical system in which arguments can be validated by all and only prime formal laws of a kind.

Relativity of Prime Logic

There are $(2^{13} - 1)$ prime logics. The prime logical validity of non-classical structures can be characterized only in some of these systems, and any "valid" arguments sanctioned by other systems cannot be regarded as logically valid under the minimal notion. Note that our rejection of logical systems is based on the minimal notion of logical validity. We can suppose that any well-established logical system is genuinely logical under some other notion. Our arguments for the $(2^{13}-1)$ logical systems depend on the classification of accessibility relations by the logical similarity relation among them. This aspect is essential for the minimal notion, but might be just one related aspect among several others under another notion.

When we have multiple options for a subject matter, we ask the question of which one is the best or correct. Here, we can ask the question: Among the $(2^{13}-1)$ non-classical logical system, which one is more appropriate for the minimal notion of logical validity?

My position is relativistic but not pluralistic. A relativistic position regarding a philosophical issue says that what option is appropriate is relative to, for example, one's purpose, view, context, and other factors. Based on the items for selection, the appropriate candidate will be determined. A pluralistic position, on the other hand, claims that multiple options are equally correct. There is more than one candidate that satisfies sufficient conditions to be regarded as a correct option.

My claim is the following. All $(2^{13} - 1)$ logical systems reflect the minimal notion of logical validity. However, we cannot further narrow them down by the minimal notion alone. From the minimalist's point of view, they are equally good, and no comparison of their accessibility relations is possible. In order to claim that some system is correct or better than others, the purpose for which we use the non-classical logical system or the view of what the accessibility relation is supposed to represent has to be set out. The correct system(s) can then be determined in relation to these factors.

Consider, for example, a major purpose for the use of non-classical systems: a characterization of modal notions. Philosophers have developed theories concerning the notions of necessity and possibility using possible world semantics. One important property that has been attributed to necessity is that if a sentence is necessarily true, then it is (actually) true. In formal notation, the sentence $\Box\varphi \rightarrow \varphi$ has to be a logical truth. It is well known that for the sentence to be a logical truth in a non-classical logical system, a non-classical structure of the logical system has to have a reflexive accessibility relation. That is, in every legitimate non-classical structure, any domain is supposed to be accessible to itself. Among the $(2^{13} - 1)$ legitimate logical systems, only one system meets this condition, which is the logical system in which the accessibility relation R of $\langle \mathfrak{D}, R, \mathcal{S} \rangle$ is supposed to be $\mathfrak{D} \times \mathfrak{D}$.[10] It thus can be concluded that under the minimal notion, this system is the correct system for the characterization of modal notions.

Formal aspects of deontic notions such as obligation and permission have been investigated in a non-classical logical framework. In deontic logic, the symbol O is used to mean "it is obligatory that" and P is used for "it is permitted that." As opposed to the modal logic with the universal accessibility relation $\mathfrak{D} \times \mathfrak{D}$ above, in deontic logic, $O\varphi \rightarrow \varphi$ cannot be thought of as a logical truth; what is obligatory is not always what is actually the case. Thus, the accessibility relation of deontic logic is not reflexive in some non-classical structures. Instead, $O\varphi \rightarrow P\varphi$ is normally taken as a logical truth; if something is obligatory, then it is permitted. It can be shown that for this to be true in any domain of any non-classical structures, the accessibility relation has to be serial. There is a prime logic that satisfies these conditions (see p. 142). The study of deontic notions is possible under the minimal

[10]Let $\mathfrak{S}$ be the collection of non-classical logical structures of a logical system in which $\Box\varphi \rightarrow \varphi$ is a logical truth, and let $\langle \mathfrak{D}_0, R_0, I_0 \rangle$ be a non-classical structure in $\mathfrak{S}$. Assume that $R_0 \neq \mathfrak{D}_0 \times \mathfrak{D}_0$. Then, there exists a non-classical logical structure $\langle \mathfrak{D}'_0, R'_0, I'_0 \rangle$ such that the accessibility relation R'_0 is logically similar to R_0 and such that R'_0 is not reflexive. Since $\langle \mathfrak{D}'_0, R'_0, I'_0 \rangle$ is logically similar to $\langle \mathfrak{D}_0, R_0, I_0 \rangle$, we have that $\langle \mathfrak{D}'_0, R'_0, I'_0 \rangle \in \mathfrak{S}$ and therefore that $\Box\varphi \rightarrow \varphi$ is not a logical truth. This is a contradiction. Hence, $R_0 = \mathfrak{D}_0 \times \mathfrak{D}_0$.

notion.

Generally, for given conditions of binary accessibility relations, the prime logic that satisfies all of them can be uniquely specified, if it exists.[11] The identification of the prime logic can be done by checking if each kind of binary accessibility relations meets the given conditions. Although all $(2^{13} - 1)$ logical systems reflect the minimal notion of logical validity, they are just candidates for the logical system to be used for a philosophical investigation. To find the right logical system for the investigation, an analysis of what is to be represented by accessibility relations is necessary. Based on the analysis, the conditions to be imposed on accessibility relations are determined, and the appropriate prime logic is specified.

Chapter 4, in part, has been submitted for publication of the material as it may appear in Journal of Philosophical Logic, Springer, 2016. The dissertation author was the sole investigator and author of this paper.

[11]Whether or not there exists such a prime logic depends on conditions to be imposed on binary accessibility relations. For some (actually many) collections of conditions, there is no kind of logically similar binary accessibility relations such that any binary accessibility relation of that kind satisfies all the conditions. In intuitionism, the accessibility relation among possible worlds (as stages of constructive proofs) is supposed to be reflexive and transitive. However, there is no kind of logically similar binary relations that contains only reflexive and transitive ones. This is because some reflexive-transitive binary relation is logically similar to some irreflexive binary relation (an example is R_1 and R_2 above, p. 144). The nonexistence of appropriate prime logics shows that intuitionism is incompatible with the minimal notion. A logical analysis of intuitionistic notions is impossible under the minimal notion. For the investigation of intuitionism, another notion of logical validity has to be employed.

5 Conclusion

Summary of Results

In this dissertation, I have proposed a characterization of logical validity based on the minimal notion. The main result is prime logic. An argument is logically valid under the minimal notion if it holds by virtue of some prime formal law. A prime logic is a logical system in which only such arguments can be validated. The characteristic feature of prime formal laws is that they govern formal properties shared by logically similar set-theoretical constructs. Because of their governing formal properties, prime logical validity can be regarded as absolutely certain. And due to the governed properties involving logical aspects of objects, prime logical validity can be distinguished from other kinds of validity such as set-theoretically validity. Arguments we can make are restricted by prime formal laws in the sense that an argument can make sense only if it is consistent with prime formal laws and no argument can violate them. Our inferential activity is possible only within the limits set by prime logics.

For the characterization of prime logical validity, I have addressed related problems in each chapter. What have been achieved by considering them are as follows: (i) Argument to the effect that the model-theoretic method is appropriate for the characterization (Chapter 2); (ii) Formulation of logical similarity relations (Chapter 3); (iii) Identification of logical operators (Chapter 3); (iv) Distinction between logical properties and set-theoretical properties (Chapter 4); (v) Classification of accessibility relations (Chapter 4). Prime logics have been developed based on these results.

Two Problems of the Future Research

The characterization of prime logical validity is a necessary component of a comprehensive theory of logic under the minimal notion. A theory of logic in general is supposed to answer, first of all, the question of what arguments are logically valid. In addition, the way of how logically valid arguments can be identified needs to be explained by the theory. These can be done by using the characterization proposed in this dissertation: according to our theory, an argument is logically valid if its validity is due to some prime formal law; and logically valid arguments can be identified in a prime logic.

Among other important problems to be solved by the comprehensive theory is the justification problem, which has been introduced in Chapter 2 (pp. 49–52). After characterizing logically valid arguments, what is expected to be done next is to identify the source of their certainty. Under the minimal notion, the justification of a logically valid argument can be reduced to the justification of the prime formal law that validates it. Considering that a prime formal law is a law governing formal properties, its certainty itself can be supposed. However, without clarifying why it certainly holds, the nature of logic will never be revealed. An answer to the justification problem is another necessary component of the comprehensive theory of logic, which I hope to develop in the future research.

Another subject of the future research is to investigate mathematical properties of prime logics. In particular, whether or not they have the completeness property is of great concern. Completeness is not just one desirable property among many. Rather, it is *the* property that has to be examined first when a new logical system was established. From an epistemological point of view, what is meant by completeness is crucial: if the new logical system is complete, then the logical validity of an argument can be known in finite steps. Besides, the importance of completeness can also be understood in the light of the necessity of prime logical validity. In Chapter 2 (p. 39), we have argued that the necessity of the model-theoretic validity can be straightforwardly shown provided that the associated model-theoretic system has a complete proof-theoretic system. Hence, if the completeness theorem of a prime logic can be proven, it can be confirmed without difficulty that the consequence relation of a logically valid argument is necessary.

Closing Remarks

Philosophy graduate students learn a variety of things through performing research for their dissertation. One thing that I think most of them come to understand is the importance of developing a solid notion associated with one's research subject. To any philosophical problem (if it is actually a "problem"), multiple plausible answers can be given. One form of philosophical research is to advocate one answer and reject others. This dissertation contains several discussions of this form. A justification for an answer to a philosophical problem forms a layer. An argument justifies the answer, and the justifying argument itself can be justified by another more fundamental argument: the truth of argument-n depends on the truth of argument-$(n+1)$. What are located at the bottom layer are simple claims and beliefs, without which arguments at a higher-level layer cannot be adequately justified. One's notion of the subject matter is composed of them. A solid notion of a subject matter X is necessary for answering philosophical problems concerning X. This is a lesson that philosophy graduate students learn at the end of their doctorate study. The X is "existence" for some students and "justice" for others. For me, the X is "logic."

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