EQUIVARIANT SCHRÖDINGER MAPS IN TWO SPATIAL DIMENSIONS: THE $\mathbb{H}^2$ TARGET

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Abstract. We consider equivariant solutions for the Schrödinger map problem from $\mathbb{R}^{2+1}$ to $\mathbb{H}^2$ with finite energy and show that they are global in time and scatter.

1. Introduction

The Schrödinger map equation in $\mathbb{R}^{2+1}$ with values into $\mathbb{S}_\mu \subset \mathbb{R}^3$ is given by

$$u_t = u \times_\mu \Delta u, \qquad u(0) = u_0 \tag{1.1}$$

where $\mu = \pm 1$, the connected Riemannian manifolds S_μ,

$$\begin{aligned} &S_1 = \mathbb{S}^2 = \{y = (y_0, y_1, y_2) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\};\\ &S_{-1} = \mathbb{H}^2 = \{y = (y_0, y_1, y_2) \in \mathbb{R}^3 : -y_1^2 - y_2^2 + y_3^2 = 1,\ y_3 > 0\}, \end{aligned} \tag{1.2}$$

with the Riemannian structures induced by the Euclidean metric $\mathbf{g}_1 = dy_0^2 + dy_1^2 + dy_2^2$ on S_1, respectively the Minkowski metric $\mathbf{g}_{-1} = -dy_0^2 + dy_1^2 + dy_2^2$ on S_{-1}. Thus S_1 is the 2-dimensional sphere $\mathbb{S}^2$, while S_{-1} is the 2-dimensional hyperbolic space $\mathbb{H}^2$. With $\eta_\mu = \mathrm{diag}(1, 1, \mu)$, the cross product $\times_\mu$ is defined by $v \times_\mu w := \eta_\mu \cdot (v \times w)$.

This equation admits a conserved energy,

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|_\mu^2 dx$$

and is invariant with respect to the dimensionless scaling

$$u(t, x) \to u(\lambda^2 t, \lambda x).$$

The energy is invariant with respect to the above scaling, therefore the Schrödinger map equation in $\mathbb{R}^{2+1}$ is *energy critical*.

The local theory for classical data was established in [25] and [21]. We recall

Theorem 1.1 (McGahagan)**.** *If $u_0 \in \dot{H}^1 \cap \dot{H}^3$ then there exists a time $T > 0$, such that (1.1) has a unique solution in $L_t^\infty([0, T] : \dot{H}^1 \cap \dot{H}^3)$.*

The local and global in time of the Schrödinger map problem with small data has been intensely studied for the case $\mu = 1$ corresponding to $\mathbb{S}^2$ as target, see [3], [4], [5], [6], [9], [15], [16]. The state of the art result for the problem with small data was established by the authors in [6] where they proved that classical solutions (and in fact rough solutions too) with small energy are global in time. These results are expected to extend to the case $\mu = -1$, corresponding to $\mathbb{H}^2$ as a target.

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To gain some intuition about the large data problem, one needs to describe the solitons for (1.1). The solitons for this problem are the harmonic maps, which are solutions to $u \times \Delta u = 0$. Since $\mathbb{H}^2$ is negatively curved there are no finite energy nontrivial harmonic maps. In the case of $\mathbb{S}^2$ there are finite energy harmonic maps, but they cannot have arbitrary energy. The trivial solitons are points, i.e. $u = Q$ for some $Q \in \mathbb{S}^2$ and their energy is 0. The next energy level admissible for solitons is 4π; the corresponding soliton is, up to symmetries, the stereographic projection. Based on this, it is natural to make the following

Conjecture 1.2. *a) Global well-posedness and scattering for Schrödinger maps from $\mathbb{R}^2 \times \mathbb{R}$ into $\mathbb{H}^2$ holds for all finite energy data.*

b) Global well-posedness and scattering for Schrödinger maps from $\mathbb{R}^2 \times \mathbb{R}$ into $\mathbb{S}^2$ holds for all data with energy below 4π.

In full generality this remains an open problem. Recently, some progress was made for the problem with large data in the case of $\mathbb{S}^2$. Smith established in [24] a conditional result for global existence of smooth Schrödinger maps with energy $< 4\pi$.

In this article we confine ourselves to a class of *equivariant* Schrödinger maps. These are indexed by an integer m called the equivariance class, and consist of maps of the form

$$u(r, \theta) = e^{m\theta R} \bar{u}(r) \tag{1.3}$$

Here R is the generator of horizontal rotations, which can be interpreted as a matrix or, equivalently, as the operator below

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Ru = \overrightarrow{k} \times_\mu u.$$

Here and thereafter we denote by $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ the standard orthonormal basis in $\mathbb{R}^3$, i.e. the vectors with coordinate representation $(1,0,0), (0,1,0)$ respectively $(0,0,1)$. The case $m = 0$ corresponds to radial symmetry.

The energy for equivariant maps takes the following form:

$$E(u) = \pi \int_0^\infty \left(|\partial_r \bar{u}(r)|_\mu^2 + \frac{m^2}{r^2} (\bar{u}_1^2(r) + \bar{u}_2^2(r)) \right) r dr \tag{1.4}$$

If $m \neq 0$, then $E(u) < \infty$ implies better information about the behavior of u versus the radial case $m = 0$, in particular it implies that u_1 and u_2 have limit zero as $r \to 0$ and $r \to \infty$.

The global regularity question in the case $m = 0$ and target $\mathbb{S}^2$, corresponding to radial symmetry, has been considered recently by Gustafson and Koo, see [14]. The global regularity in the case $m = 1$ and target $\mathbb{S}^2$ was considered by the authors in [8] where they have shown that the 1-equivariant solutions of (1.1) with energy $< 4\pi$ are globally well-posed.

In this paper we consider the case when the target manifold is $\mathbb{H}^2$ and prove the following

Theorem 1.3. *i) Let $\mu = -1$, $m \neq 0$ and $u_0 \in \dot{H}^1 \cap \dot{H}^3$ be an m-equivariant function. Then (1.1) has a unique global in time solution $u \in L^\infty(\mathbb{R} : \dot{H}^1 \cap \dot{H}^3)$. In addition ∇u, in a particular frame, scatters to the free solution of a particular linear Schrödinger equation.*

ii) The above solution is Lipschitz continuous with respect to the initial data in $\dot{H}^1$. In particular if $u_0 \in \dot{H}^1$ is a m-equivariant function, $m \neq 0$ then (1.1) has a global solution

$u(t) \in L^\infty \dot{H}^1$ defined as the unique limit of smooth solutions in $\dot{H}^1 \cap \dot{H}^3$. Scattering also holds for this solution in a suitable frame.

The statement of the scattering cannot be made precise at this time. We need to introduce a moving frame on $\mathbb{H}^2$, write the equation of the coordinates of ∇u in that frame and identify there the linear part of the Schrödinger equation. This will be carried out in Section 2.

The result in Theorem 1.3 is natural since the failure of the well-posedness of (1.1) is expected to be closely related to the existence of finite energy harmonic maps. In the case of $\mathbb{H}^2$ there are no harmonic maps, so no obstacles are present. In the case of $\mathbb{S}^2$ ($\mu = 1$) the lowest energy nontrivial is 4π and it was shown in [23] that blow-up can occur for maps with energy $4\pi+$.

1.1. **Definitions and notations.** While at fixed time our maps into the sphere or the hyperbolic space are functions defined on $\mathbb{R}^2$, the equivariance condition allows us to reduce our analysis to functions of a single variable $|x| = r \in [0, \infty)$. One such instance is exhibited in (1.3) where to each equivariant map u we naturally associate its radial component $\bar{u}$. Some other functions will turn out to be radial by definition, see, for instance, all the gauge elements in Section 2. We agree to identify such radial functions with the corresponding one dimensional functions of r. Some of these functions are complex valued, and this convention allows us to use the bar notation with the standard meaning, i.e. the complex conjugate.

Even though we work mainly with functions of a single spatial variable r, they originate in two dimensions. Therefore, it is natural to make the convention that for the one dimensional functions all the Lebesgue integrals and spaces are with respect to the rdr measure, unless otherwise specified.

Since equivariant functions are easily reduced to their one-dimensional companions via (1.3), we introduce the one dimensional equivariant version of $\dot{H}^1$,

$$\|f\|_{\dot{H}^1_e}^2 = \|\partial_r f\|_{L^2(rdr)}^2 + m^2 \|r^{-1} f\|_{L^2(rdr)}^2. \tag{1.5}$$

This is natural since for functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $u(r, \theta) = e^{m\theta R} \bar{u}(r)$ (here $Ru = \overrightarrow{k} \times u$ or, as a matrix, it is the upper left 2×2 block of the original matrix R) we have

$$\|u\|_{\dot{H}^1} = (2\pi)^{\frac{1}{2}} \|\bar{u}\|_{\dot{H}^1_e}.$$

It is important to note that functions in $\dot{H}^1_e$ enjoy the following properties: they are continuous and have limit 0 both at $r = 0$ and $r = \infty$, see [11] for a proof.

We introduce $\dot{H}^{-1}_e$ as the dual space to $\dot{H}^1_e$ with respect to the L^2 pairing, i.e.

$$\|f\|_{\dot{H}^{-1}_e} = \sup_{\|\phi\|_{\dot{H}^1_e} = 1} \langle f, \phi \rangle$$

The elements from $\dot{H}^{-1}_e$ can be represented in the form $f = \partial_r f_1 + r^{-1} f_2$ with $f_1, f_2 \in L^2$.

Three operators which are often used on radial functions are $[\partial_r]^{-1}, [r^{-m} \bar{\partial}_r]^{-1}$ and $[r \partial_r]^{-1}$ defined as

$$[\partial_r]^{-1} f(r) = -\int_r^\infty f(s) ds, \quad [r^{-m} \bar{\partial}_r]^{-1} f(r) = \int_0^r f(s) s^m ds$$

$$[r \partial_r]^{-1} f(r) = -\int_r^\infty \frac{1}{s} f(s) ds$$

A direct argument shows that

$$
\begin{aligned}
&\|[r\partial_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \qquad 1 \le p < \infty, \\
&\|r^{-m-1}[r^{-m}\bar\partial_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \qquad 1 < p \le \infty, \\
&\|[\partial_r]^{-1} f\|_{L^2} \lesssim \|f\|_{L^1}.
\end{aligned} \tag{1.6}
$$

The equivariance properties of the functions involved in this paper requires that the two-dimensional Fourier calculus is replaced by the Hankel calculus for one-dimensional functions which we recall below.

For $k \ge 0$ integer, let J_k be the Bessel function of the first kind,

$$J_k(r) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - r\sin\tau)d\tau$$

If $H_k = \partial_r^2 + \frac{1}{r}\partial_r - \frac{k^2}{r^2}$, then J_k solves $H_k J_k = -J_k$.

We recall some formulas involving Bessel functions

$$\partial_r J_k = \frac{1}{2}(J_{k-1} - J_{k+1}), \quad (r^{-1}\partial_r)^m \left(\frac{J_k}{r^k}\right) = (-1)^m \frac{J_{k+m}}{r^{k+m}}, \tag{1.7}$$

where $J_{-k} = (-1)^k J_k$.

For each $k \ge 0$ integer one defines the Hankel transform $\mathcal{F}_k$ by

$$\mathcal{F}_k f(\xi) = \int_0^\infty J_k(r\xi) f(r) r dr$$

The inversion formula holds true

$$f(r) = \int_0^\infty J_k(r\xi)\mathcal{F}_k f(\xi)\xi d\xi$$

The Plancherel formula holds true, hence in particular, the Hankel transform is an isometry.

For a radial function f and for an integer k we define its two-dimensional extension

$$R_k(r,\theta) = e^{ik\theta} f(r) \tag{1.8}$$

If $f \in L^2$ then $R_k f \in L^2$; if $R_k f$ has additional regularity, this is easily read in terms of $\mathcal{F}_k f$. Indeed for any $s \ge 0$ integer the following holds true

$$R_k f \in \dot H^s \Leftrightarrow \xi^s \mathcal{F}_k f \in L^2 \tag{1.9}$$

For even values of s this is a consequence of $\Delta R_k f = R_k H_k f$, while for odd values of s it follows by interpolation.

By direct computation, we also have that for $k \neq 0$,

$$R_k f \in \dot H^1 \Leftrightarrow f \in \dot H^1_e, \qquad R_0 f \in \dot H^1 \Leftrightarrow \partial_r f \in L^2. \tag{1.10}$$

We will use the following result

Lemma 1.4. *i) If $f \in L^2$ is such that $H_k f \in L^2$, with $k \neq 1$, then the following holds true*

$$\|\partial_r^2 f\|_{L^2} + \|\frac{\partial_r f}{r}\|_{L^2} + k\|\frac{f}{r^2}\|_{L^2} \lesssim \|H_k f\|_{L^2}$$

ii) If $f \in L^2$ is such that $H_1 f \in L^2$, then the following holds true

$$\|\partial_r^2 f\|_{L^2} + \|\frac{\partial_r f}{r} - \frac{f}{r^2}\|_{L^2} \lesssim \|H_1 f\|_{L^2}$$

iii) If $f \in L^2$ is such that $\partial_r H_0 f \in L^2$, then the following holds true

$$\|\partial_r^3 f\|_{L^2} + \|\frac{\partial_r^2 f}{r} - \frac{\partial_r f}{r^2}\|_{L^2} \lesssim \|\partial_r H_0 f\|_{L^2}$$

iv) If $f \in L^2$ is such that $H_1 f \in \dot{H}^1_e$, then the following holds true

$$\|\partial_r^3 f\|_{L^2} + \|\frac{\partial_r^2 f}{r}\|_{L^2} + \|\frac{\partial_r f}{r^2} - \frac{f}{r^3}\|_{L^2} \lesssim \|H_1 f\|_{\dot{H}^1_e}$$

v) If $f \in L^2$ is such that $H_2 f \in \dot{H}^1_e$, then the following holds true

$$\|\partial_r^3 f\|_{L^2} + \|\frac{\partial_r^2 f}{r} - \frac{\partial_r f}{r^2}\|_{L^2} + \|\frac{\partial_r f}{r^2} - \frac{2f}{r^3}\|_{L^2} \lesssim \|H_2 f\|_{\dot{H}^1_e}$$

vi) If $f \in L^2$ is such that $H_k f \in \dot{H}^1_e$, with $k \geq 3$, then the following holds true

$$\|\partial_r^3 f\|_{L^2} + \|\frac{\partial_r^2 f}{r}\|_{L^2} + \|\frac{\partial_r f}{r^2}\|_{L^2} + \|\frac{f}{r^3}\|_{L^2} \lesssim \|H_k f\|_{\dot{H}^1_e}$$

vii) If $f, \partial_r f \in L^2$, then for any $2 \leq p < +\infty$ the following holds true

$$\|f\|_{L^p} \lesssim_p \|\partial_r f\|_{L^2} + \|f\|_{L^2}$$

viii) If $f, H_k f \in L^2$, with $k \geq 0$, then for any $2 \leq p < +\infty$ the following holds true

$$\|\partial_r f\|_{L^p} \lesssim_p \|H_k f\|_{L^2} + \|f\|_{L^2}$$

Proof. Part i) for $k \in \{0, 2\}$ are established in Lemma 1.3 in [8], and the general result for all $k \geq 3$ follows along the same lines.

For part ii) we use the inversion formula for f and (1.7) to compute

$$\partial_r^2 f = \int (J_3 - 3J_1)(r\xi)\xi^2 \mathcal{F}_1 f(\xi)\xi d\xi$$

and the first part of the estimate follows. The estimate for the second term follows from the form of $H_1 f$.

For part iii) we proceed as above, i.e. use the inversion formula for f and (1.7) to write

$$\partial_r^3 f = \int (J_3 - 3J_1)(r\xi)\xi^3 \mathcal{F}_0 f(\xi)\xi d\xi$$

and conclude with the estimate for $\|\partial_r^3 f\|_{L^2}$, while the estimate for the second term follows from the expression of $\partial_r H_0$.

Parts iv)-vi) follow in a similar manner by using the Hankel transform and (1.7) to derive the estimates. The details are left to the reader.

vii) and viii) are consequences of the standard Sobolev embeddings.

□

1.2. **A few calculus rules.** We recall that given $\mu = \pm 1$ and two vectors $v = {}^t(v_1, v_2, v_3)$ and $w = {}^t(w_1, w_2, w_3)$ in $\mathbb{R}^3$, their inner product is defined as

$$v \cdot_\mu w = \mathbf{g}_{-1}(v, w) = {}^t v \cdot \eta_\mu \cdot w = v_1 w_1 + v_2 w_2 + \pm v_3 w_3, \tag{1.11}$$

where $\eta_\mu = \mathrm{diag}(1, 1, \mu)$. We define also the cross product

$$v \times_\mu w := \eta_\mu \cdot (v \times w), \tag{1.12}$$

where $v \times w$ denotes the usual vector product of vectors in $\mathbb{R}^3$. Simple computations show that, for $\mu = \pm 1$ and $v, w \in \mathbb{R}^3$

$$
\begin{aligned}
& v \cdot_\mu (v \times_\mu w) = w \cdot_\mu (v \times_\mu w) = 0, \\
& (v \times_\mu w) \cdot_\mu (v \times_\mu w) = \mu (v \cdot_\mu v)(w \cdot_\mu w) - \mu (v \cdot_\mu w)^2 \\
& (a \times_\mu b) \cdot_\mu c = a \cdot_\mu (b \times_\mu c)
\end{aligned} \tag{1.13}
$$

1.3. **Energy estimates.** In this section we derive properties of u from the finiteness of its energy $E(u)$ in (1.4) in the case $\mu = -1$ (in the case $\mu = 1$ the corresponding estimates are trivial as all terms come with + sign). We recall that

$$
E(u) = \pi \int_0^\infty \left(|\partial_r \bar{u}_1(r)|^2 + |\partial_r \bar{u}_2(r)|^2 - |\partial_r \bar{u}_3(r)|^2 + \frac{m^2}{r^2} (\bar{u}_1^2(r) + \bar{u}_2^2(r)) \right) r dr
$$

Since $u_1 \partial_r \bar{u}_1 + u_2 \partial_r \bar{u}_2 = u_3 \partial_r \bar{u}_3$ and $\bar{u}_3^2 = 1 + \bar{u}_1^2 + \bar{u}_2^2$ it follows that

$$
E(u) = \pi \int_0^\infty \left(\frac{|\partial_r \bar{u}_1(r)|^2 + |\partial_r \bar{u}_2(r)|^2}{\bar{u}_3^2} + (\frac{\bar{u}_2 \partial_r \bar{u}_1 - \bar{u}_2 \partial_r \bar{u}_1}{\bar{u}_3})^2 + \frac{m^2}{r^2} (\bar{u}_1^2(r) + \bar{u}_2^2(r)) \right) r dr
$$

We also have that

$$
\begin{aligned}
\bar{u}_1^2(r) + \bar{u}_2^2(r) - \bar{u}_1^2(1) - \bar{u}_2^2(1) &= \int_1^r \partial_r (\bar{u}_1^2 + \bar{u}_2^2) ds \\
&\lesssim \int_1^r (|\bar{u}_1| + |\bar{u}_2|)(|\partial_r \bar{u}_1| + |\partial_r \bar{u}_2|) ds \\
&\lesssim \int_1^r (|\bar{u}_1| + |\bar{u}_2|) \frac{|\bar{u}_1| + |\bar{u}_2|}{s} \frac{|\partial_r \bar{u}_1| + |\partial_r \bar{u}_2|}{\bar{u}_3} s ds \\
&\lesssim \sup_{s \in [1,r]} (|\bar{u}_1(s)| + |\bar{u}_2(s)|) E(u)
\end{aligned}
$$

from which we conclude that $\sup_{r \in (0,\infty)} |u_1(r)| + |u_2(r)| \lesssim \bar{u}_1(1) + \bar{u}_2(1) + E(u)$. Therefore $\sup_{r \in (0,\infty)} |\bar{u}_3(r)| \lesssim m + \bar{u}_1(1) + \bar{u}_2(1) + E(u)$, hence from the last expression of $E(u)$ we obtain that $\bar{u}_1, \bar{u}_2 \in \dot{H}^1_e$. In particular it follows that $\bar{u}_1(0) = \bar{u}_2(0) = 0$ (in the sense that the limits exists and equal 0), hence rewriting the above argument on $(0, r]$ instead gives $|\bar{u}_1|_{L^\infty} + |\bar{u}_2|_{L^\infty} \lesssim E(u)$, $|\bar{u}_3|_{L^\infty} \lesssim m + E(u)$. Recalling the last expression of $E(u)$ we obtain

$$
\|\bar{u}_1\|_{\dot{H}^1_e} + \|\bar{u}_2\|_{\dot{H}^1_e} \lesssim E(u)^{\frac{1}{2}} (m + E(u))
$$

In addition we obtain $\bar{u}_3 - 1 \in \dot{H}^1_e$ with

$$
\|\bar{u}_3 - 1\|_{\dot{H}^1_e} \lesssim E(u)^{\frac{1}{2}} (m + E(u))
$$

2. The Coulomb gauge representation of the equation

In this section we rewrite the Schrödinger map equation for equivariant solutions in a gauge form. This approach originates in the work of Chang, Shatah, Uhlenbeck [9]. However, our analysis is closer to the one in [5] and [7]. The computations in subsections 2.1 and 2.2 follow exactly the same lines as the one used in [8]. Then we fix $\mu = -1$ as the analysis becomes more specific to this case.

2.1. **The Coulomb gauge.** The computations below are at the formal level as we are not yet concerned with the regularity of the terms involved in writing various identities and equations. Implicitly we use only the information $u \in \dot{H}^1$. In subsection 2.4 we prove that if $u \in \dot{H}^3$ then all the gauge elements, their compatibility relations and the equations they obey are meaningful in the sense that they involve terms which are at least at the level of L^2.

We let the differentiation operators $\partial_0, \partial_1, \partial_2$ stand for $\partial_t, \partial_r, \partial_\theta$ respectively. Our strategy will be to replace the equation for the Schrödinger map u with equations for its derivatives $\partial_1 u$, $\partial_2 u$ expressed in an orthonormal frame $v, w \in T_u\mathbb{S}_\mu$. We choose $v \in T_u\mathbb{S}_\mu$ such that $v \cdot_\mu v = 1$ and define $w = u \times_\mu v \in T_u\mathbb{S}_\mu$; to summarize

$$v \cdot_\mu v = 1, \qquad v \cdot_\mu u = 0, \qquad w = u \times_\mu v \tag{2.1}$$

From this, we obtain

$$w \cdot_\mu v = 0, \qquad w \cdot_\mu w = 1, \qquad v \times_\mu w = \mu u, \qquad w \times_\mu u = v \tag{2.2}$$

Since u is m-equivariant it is natural to work with m-equivariant frames, i.e.

$$v = e^{m\theta R}\bar{v}(r), \qquad w = e^{m\theta R}\bar{w}(r).$$

where $\bar{v}, \bar{w}$ (as well as $\bar{u}$ from (1.3)) are unit vectors in $\mathbb{R}^3$.

Given such a frame we introduce the differentiated fields ψ_k and the connection coefficients A_k by

$$\psi_k = \partial_k u \cdot_\mu v + i\partial_k u \cdot_\mu w, \qquad A_k = \partial_k v \cdot w. \tag{2.3}$$

Due to the equivariance of (u, v, w) it follows that both ψ_k and A_k are spherically symmetric (therefore subject to the conventions made in Section 1.1). Conversely, given ψ_k and A_k we can return to the frame (u, v, w) via the ODE system:

$$\begin{cases} \partial_k u = (\Re\psi_k)v + (\Im\psi_k)w \\ \partial_k v = -\mu(\Re\psi_k)u + A_k w \\ \partial_k w = -\mu(\Im\psi_k)u - A_k v \end{cases} \tag{2.4}$$

If we introduce the covariant differentiation

$$D_k = \partial_k + iA_k, \quad k \in \{0, 1, 2\}$$

it is a straightforward computation to check the compatibility conditions:

$$D_l\psi_k = D_k\psi_l, \quad l, k = 0, 1, 2. \tag{2.5}$$

The curvature of this connection is given by

$$D_l D_k - D_k D_l = i(\partial_l A_k - \partial_k A_l) = i\mu\Im(\psi_l\bar{\psi}_k), \quad l, k = 0, 1, 2. \tag{2.6}$$

An important geometric feature is that ψ_2, A_2 are closely related to the original map. Precisely, for A_2 we have:

$$A_2 = m(\overrightarrow{k} \times_\mu v) \cdot_\mu w = m\overrightarrow{k} \cdot_\mu (v \times_\mu w) = m\overrightarrow{k} \cdot_\mu (\mu u) = mu_3 \tag{2.7}$$

and, in a similar manner,

$$\psi_2 = m\mu(w_3 - iv_3) \tag{2.8}$$

Since the (u, v, w) frame is orthonormal, it follows that $|\psi_2|^2 = m^2(u_1^2 + u_2^2)$ and the following important conservation law

$$|\psi_2|^2 + \mu A_2^2 = \mu m^2 \tag{2.9}$$

Now we turn our attention to the choice of the $(\bar{v}, \bar{w})$ frame at $\theta = 0$. Here we have the freedom of an arbitrary rotation depending on t and r. In this article we will use the Coulomb gauge, which for general maps u has the form div $A = 0$. In polar coordinates this is written as $\partial_1 A_1 + r^{-2}\partial_2 A_2 = 0$. However, in the equivariant case A_2 is radial, so we are left with a simpler formulation $A_1 = 0$, or equivalently

$$\partial_r \bar{v} \cdot_\mu \bar{w} = 0 \tag{2.10}$$

which can be rearranged into a convenient ODE as follows

$$\partial_r \bar{v} = \mu(\bar{v} \cdot_\mu \bar{u})\partial_r \bar{u} - \mu(\bar{v} \cdot_\mu \partial_r \bar{u})\bar{u} \tag{2.11}$$

The first term on the right vanishes and could be omitted, but it is convenient to add it so that the above linear ODE is solved not only by $\bar{v}$ and $\bar{w}$, but also by $\bar{u}$. Then we can write an equation for the matrix $\mathcal{O} = (\bar{v}, \bar{w}, \bar{u})$:

$$\partial_r \mathcal{O} = M\eta_\mu \mathcal{O}, \qquad M = \partial_r \bar{u} \wedge \bar{u} := \partial_r \bar{u} \otimes \bar{u} - \bar{u} \otimes \partial_r \bar{u} \tag{2.12}$$

with an antisymmetric matrix M.

An advantage of using the Coulomb gauge is that it makes the derivative terms in the nonlinearity disappear. Unfortunately, this only happens in the equivariant case, which is why in [6] we had to use a different gauge, namely the caloric gauge.

The ODE (2.11) needs to be initialized at some point. A change in the initialization leads to a multiplication of all of the ψ_k by a unit sized complex number. This is irrelevant at fixed time, but as the time varies we need to be careful and choose this initialization uniformly with respect to t, in order to avoid introducing a constant time dependent potential into the equations via A_0. Since in our results we start with data which converges asymptotically to $\vec{k}$ as $r \to \infty$, and the solutions continue to have this property, it is natural to fix the choice of $\bar{v}$ and $\bar{w}$ at infinity,

$$\lim_{r\to\infty} \bar{v}(r, t) = \vec{i}, \qquad \lim_{r\to\infty} \bar{w}(r, t) = -\mu\vec{j} \tag{2.13}$$

The existence of a unique solution $\bar{v} \in C((0, +\infty) : \mathbb{R}^3)$ of (2.11) satisfying (2.13) is standard, we skip the details. Moreover the solution is continuous with respect to u in the following sense

$$\|\bar{v} - \bar{\tilde{v}}\|_{L^\infty} \lesssim \|u - \tilde{u}\|_{\dot{H}^1} \tag{2.14}$$

2.2. **Schrödinger maps in the Coulomb gauge.** We are now prepared to write the evolution equations for the differentiated fields ψ_1 and ψ_2 in (2.3) computed with respect to the Coulomb gauge.

Writing the Laplacian in polar coordinates, a direct computation using the formulas (2.3) shows that we can rewrite the Schrödinger Map equation (1.1) in the form

$$\psi_0 = i\left(D_1\psi_1 + \frac{1}{r}\psi_1 + \frac{1}{r^2}D_2\psi_2\right) \tag{2.15}$$

Applying the operators D_1 and D_2 to both sides of this equation and using the relation (2.6) for $l, k = 1, 2$ we obtain

$$\begin{aligned} D_1\psi_0 &= i\left(D_1(D_1+\frac{1}{r})\psi_1 + \frac{1}{r^2}D_2D_1\psi_2\right) - \frac{\mu}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ D_2\psi_0 &= i\left((D_1+\frac{1}{r})D_2\psi_1 + \frac{1}{r^2}D_2D_2\psi_2\right) - \mu\Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned} \tag{2.16}$$

Using now (2.5) for $(k, l) = (0, 1)$ respectively $(k, l) = (0, 2)$ on the left and for $(k, l) = (1, 2)$ on the right we can derive the evolution equations for ψ_m, $m = 1, 2$:

$$\begin{aligned} D_0\psi_1 &= i\left(D_1(D_1+\frac{1}{r}) + \frac{1}{r^2}D_2D_2\right)\psi_1 - \frac{\mu}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ D_0\psi_2 &= i\left((D_1+\frac{1}{r})D_1 + \frac{1}{r^2}D_2D_2\right)\psi_2 - \mu\Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned} \tag{2.17}$$

In our set-up all functions are radial and we are using the the Coulomb gauge $A_1 = 0$. Then these equations take the simpler form

$$\begin{aligned} \partial_t\psi_1 + iA_0\psi_1 =& i\Delta\psi_1 - i\frac{1}{r^2}A_2^2\psi_1 - i\frac{1}{r^2}\psi_1 + \frac{2}{r^3}A_2\psi_2 - \frac{\mu}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ \partial_t\psi_2 + iA_0\psi_2 =& i\Delta\psi_2 - i\frac{1}{r^2}A_2^2\psi_2 - \mu\Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned}$$

The two variables ψ_1 and ψ_2 are not independent. Indeed, the relations (2.5) and (2.6) for $(k, l) = (1, 2)$ give

$$\partial_r A_2 = \mu\Im(\psi_1\bar{\psi}_2), \qquad \partial_r\psi_2 = iA_2\psi_1 \tag{2.18}$$

which at the same time describe the relation between ψ_1 and ψ_2 and determine A_2.

From the compatibility relations involving A_0, we obtain

$$\partial_r A_0 = -\frac{\mu}{2r^2}\partial_r(r^2|\psi_1|^2 - |\psi_2|^2) \tag{2.19}$$

from which we derive

$$A_0 = -\frac{\mu}{2}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) - \mu[r\partial_r]^{-1}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) \tag{2.20}$$

This is where the initialization of the Coulomb gauge at infinity is important. It guarantees that $A_0 \in L^p$, provided that $|\psi_1|^2 - r^{-2}|\psi_2|^2 \in L^p$ for $1 \leq p < \infty$. In particular, without any additional regularity assumptions, we know that $A_0 \in L^1$. A direct computation using integration by parts gives that

$$\int A_0(r)rdr = 0. \tag{2.21}$$

The system satisfied by ψ_1 and $\frac{\psi_2}{r}$ (this being in fact the correct variable instead of ψ_2) is given by:

$$\begin{aligned} (i\partial_t + \Delta - \frac{m^2+1}{r^2})\psi_1 + \frac{2\mu mi}{r^2}\frac{\psi_2}{r} =& A_0\psi_1 + \frac{A_2^2 - m^2}{r^2}\psi_1 + 2i\frac{A_2+\mu m}{r^3}\psi_2 - i\mu\Im(\psi_1\frac{\bar{\psi}_2}{r})\frac{\psi_2}{r} \\ (i\partial_t + \Delta - \frac{m^2+1}{r^2})\frac{\psi_2}{r} - \frac{2\mu mi}{r^2}\psi_1 =& A_0\frac{\psi_2}{r} + \frac{A_2^2 - m^2}{r^2}\frac{\psi_2}{r} - 2i\frac{A_2+\mu m}{r^2}\psi_1 - i\mu\Im(\frac{\psi_2}{r}\bar{\psi}_1)\psi_1 \end{aligned}$$

The problem with this system is that its linear part is not decoupled. This can be remedied by a change of variables. Indeed consider

$$\psi^- = \psi_1 - i\frac{\psi_2}{r}, \qquad \psi^+ = \psi_1 + i\frac{\psi_2}{r}$$

It turns out that $\psi^\pm$ satisfy a similar system (described below) whose linear part is decoupled. The relevance of the variables $\psi^\pm$ comes also from the following reinterpretation. If $\mathcal{W}^\pm$ is defined as the vector

$$\mathcal{W}^\pm = \partial_r u \pm \frac{1}{r} u \times \partial_\theta u \in T_u(\mathbb{S}_\mu)$$

then $\psi^\pm$ is the representation of $W^\pm$ with respect to the frame (v, w). On the other hand, a direct computation leads to

$$\begin{aligned} E(u) &= \pi \int_0^\infty \left(|\partial_r \bar{u}|^2 + \frac{m^2}{r^2} |\bar{u} \times R\bar{u}|^2 \right) r dr \\ &= \pi \|\bar{\mathcal{W}}^\pm\|_{L^2}^2 \mp 2\pi m(\bar{u}_3(\infty) - \bar{u}_3(0)) \end{aligned}$$

where we recall that $u(r,\theta) = e^{m\theta R}\bar{u}(r)$ and $\bar{u}_3(\infty) = \lim_{r\to\infty} \bar{u}_3(r), \bar{u}_3(0) = \lim_{r\to 0} \bar{u}_3(r)$ are well-defined since $\bar{u}_1, \bar{u}_2 \in \dot{H}^1_e$ and if $f \in \dot{H}^1_e$ then $\lim_{r\to 0} f(r) = \lim_{r\to 0} f(r) = 0$, see [11] or [7]. From Section 1.3 it follows that, in the case $\mu = -1$, $\bar{u}_3(\infty) = \bar{u}_3(0) = 1$. In the case $\mu = 1$ one needs the energy restriction $E(u) < 4\pi$ to obtain that $\bar{u}_3(\infty) = \bar{u}_3(0) = 1$, see [8]. In both cases we obtain the following identity

$$\|\psi^\pm\|_{L^2}^2 = \|\bar{\mathcal{W}}^\pm\|_{L^2}^2 = \frac{E(u)}{\pi}. \tag{2.22}$$

From (2.14) it follows that the following continuity property holds true

$$\|\psi^\pm - \tilde{\psi}^\pm\|_{L^2} \lesssim \|u - \tilde{u}\|_{\dot{H}^1} \tag{2.23}$$

A direct computation yields the following system for $\psi^\pm$:

$$\begin{aligned} (i\partial_t + H_m^-)\psi^- &= \left(A_0 - 2\frac{A_2 + \mu m}{r^2} + \frac{A_2^2 - m^2}{r^2} - \frac{\mu}{r}\Im(\psi_2 \bar{\psi}_1) \right) \psi^- \\ (i\partial_t + H_m^+)\psi^+ &= \left(A_0 + 2\frac{A_2 + \mu m}{r^2} + \frac{A_2^2 - m^2}{r^2} + \frac{\mu}{r}\Im(\psi_2 \bar{\psi}_1) \right) \psi^+ \end{aligned}$$

where

$$H_m^- = \Delta - \frac{(m+\mu)^2}{r^2}, \qquad H_m^+ = \Delta - \frac{(m-\mu)^2}{r^2}.$$

Here and whenever Δ acts on radial functions, it is known that $\Delta = \partial_r^2 + \frac{1}{r}\partial_r$. By replacing $\psi_1 = \psi^\pm \mp i r^{-1}\psi_2$ and using $\mu A_2^2 + |\psi_2|^2 = \mu m^2$, we obtain the key evolution system we work with in this paper,

$$\begin{cases} (i\partial_t + H_m^-)\psi^- = (A_0 - 2\frac{A_2+\mu m}{r^2} - \frac{\mu}{r}\Im(\psi_2\bar{\psi}^-))\psi^- \\ (i\partial_t + H_m^+)\psi^+ = (A_0 + 2\frac{A_2+\mu m}{r^2} + \frac{\mu}{r}\Im(\psi_2\bar{\psi}^+))\psi^+ \end{cases} \tag{2.24}$$

We will use this system in order to obtain estimates for $\psi^\pm$. The old variables ψ_1 and $\frac{\psi_2}{r}$ are recovered from

$$\psi_1 = \frac{\psi^+ + \psi^-}{2}, \qquad \frac{\psi_2}{r} = \frac{\psi^+ - \psi^-}{2i} \tag{2.25}$$

From the compatibility conditions (2.18) we derive the formula for A_2

$$A_2(r) + \mu m = -\mu \int_0^r \frac{|\psi^+|^2 - |\psi^-|^2}{4} s ds \tag{2.26}$$

From (2.20) A_0 is given by

$$A_0 = -\frac{\mu}{2} \Re(\overline{\psi}^+ \psi^-) + \mu [r\partial_r]^{-1} \Re(\overline{\psi}^+ \psi^-) \tag{2.27}$$

The compatibility condition (2.18) reduces then to

$$\partial_r [r(\psi^+ - \psi^-)] = -A_2(\psi^+ + \psi^-) \tag{2.28}$$

Next assume that $\psi^\pm \in L^2$ are given such that they satisfy the compatibility conditions (2.28). We reconstruct A_2, ψ_2, ψ_1 using the (2.25) and (2.26). From (2.26) and (2.28) it follows that (2.18) hold true. From (2.26) it follows that $A_2 \in L^\infty$ and it is continuous and has limits both at 0 and ∞. From the definition of ψ_2 we have $\frac{\psi_2}{r} \in L^2$ and from (2.28) we derive $\partial_r \psi_2 \in L^2$, hence $\psi_2 \in \dot{H}^1_e$. From this and (2.26) it follows that $\partial_r A_2 \in L^2$, while by invoking (1.6) we obtain $\frac{A_2 + \mu m}{r} \in L^2$, therefore $A_2 + \mu m \in \dot{H}^1_e$. In particular $A_2(\infty) = \lim_{r\to\infty} A_2(r) = -\mu m$ which implies that $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2}$.

In fact one can keep track of a single variable, ψ^- or ψ^+ since it contains all the information about the map, provided that the choice of gauge (2.13) was made. To be more precise, (2.18) gives the following

$$\partial_r A_2 = \mu \Im(\psi^- \bar{\psi}_2) + \frac{\mu}{r} |\psi_2|^2, \qquad \partial_r \psi_2 = i A_2 \psi^- - \frac{1}{r} A_2 \psi_2 \tag{2.29}$$

We will show that given $\psi^- \in L^2$, this system has a unique solution $A_2 + \mu m, \psi_2 \in \dot{H}^1_e$. From this we can reconstruct ψ_1, ψ^+, A_0. Finally, given ψ^-, A_2 and ψ_2, we can return to the Schrödinger map u via the system (2.4) with the boundary condition at infinity given by (2.13). Eventually we show that if ψ^- satisfies its corresponding equation from (2.24), then the u obtained is a Schrödinger map. A similar procedure can completely reconstruct u from ψ^+.

The reason to keep both variables $\psi^\pm$ (instead of just one) has to do with the nonlinear analysis of the system (2.24). The reason we want to understand how to recover all information from only one variable, say ψ^-, has to do with the elliptic part of the profile decomposition in Proposition 4.3.

2.3. **Fix $\mu = -1$.** The theory with $\mu = 1$ was developed in [8]. From this point on we fix $\mu = -1$ as the theory becomes more specific to this case. When comparing the results obtained here and those in [8] the reader may notice a few differences. First, one sees that $\psi^\pm$ come with operators $H_{m\mp\mu}$ and all the consequences associated, see for instance the regularity below. This is a consequence of the way we chose the limits $\lim_{r\to 0} \bar{u}_3 = \lim_{r\to\infty} \bar{u}_3 = -\mu$. Second, the analytic theory of the system (2.29) with $\mu = -1$, see Proposition 2.3, is somehow different then its counterpart for $\mu = 1$. The Cauchy theory in Section 3 and the Concentration compactness argument in Section 4 are very similar. Finally, the arguments in Section 5 are again specific to the case $\mu = -1$, as in particular no restriction on the size of the energy/mass is needed to rule out the possibility of blow-up.

2.4. **Regularity of the gauge elements.** In this section we clarify the regularity of the gauge elements. Our main claim is the following

Proposition 2.1. *If $u \in \dot{H}^3$ then $R_{m\pm 1}\psi^{\pm} \in H^2$ and*

$$\|u\|_{\dot{H}^1\cap\dot{H}^3} \approx \|R_{m+1}\psi^+\|_{H^2} + \|R_{m-1}\psi^-\|_{H^2} \tag{2.30}$$

The proof of this result will be provided in the Appendix.

Therefore, in the context of $u \in \dot{H}^1 \cap \dot{H}^3$, we have that $R_{m\pm 1}\psi^{\pm} \in H^2 \subset L^\infty$. The H^2 regularity cannot be extended to (two-dimensional extensions of) ψ_1 and $\frac{\psi_2}{r}$ since the ψ^+ and ψ^- require different phases for regularity. However, all the Sobolev embeddings are inherited by ψ_1 and $\frac{\psi_2}{r}$, in particular $\psi_1, \frac{\psi_2}{r} \subset L^\infty$. Since $A_2 = u_3$ it follows that $A_2 \in \dot{H}^1 \cap \dot{H}^3$ and $\partial_t A_2 \in H^1$. Finally by differentiating with respect to t the system (2.11), one can show that $\partial_t \bar{v} \in H^1$, hence $A_0 \in H^1$ which in turn gives $\partial_r A_0 \in L^2$. With these in mind, all the compatibility conditions in the previous two subsections are at least at the level of L^2.

2.5. **Recovering the map from ψ^-.** In this section we address the issue of re-constructing the Schrödinger map u together with its gauge elements from only one of its reduced variable, say ψ^-. Reconstructing ψ_2, A_2 such that $\psi_2, A_2 - m \in \dot{H}^1_e$ is a unique process; however, the reconstruction of the actual map with its frame, i.e. of (u, v, w) is unique provided one prescribes conditions at ∞. The map u satisfies $u(\infty) = \vec{k}$, while the gauge is subjected to the choice (2.13).

The main result of this section is the following

Proposition 2.2. *Given $\psi^- \in L^2$, there is a unique map $u : \mathbb{R}^2 \to \mathbb{S}^2$ with the property that ψ^- is the representation of $\mathcal{W}^-$ relative to a Coulomb gauge satisfying* (2.13)*. This also satisfies $E(u) = \pi\|\psi^-\|^2_{L^2}$.*

If $\tilde{\psi}^- \in L^2$ and $\tilde{u}$ is the corresponding map as above, then the following holds true

$$E(u - \tilde{u}) \lesssim \|\psi^- - \tilde{\psi}^-\|^2_{L^2} \tag{2.31}$$

Here ψ^+ can be reconstructed from ψ^-. Moreover the equations (2.29) which we use for reconstruction force the compatibility condition (2.28) between $\psi^{\pm}$. The result remains true if we start from ψ^+ just that we would start the reconstruction (described below) from the analogue of the (2.29) written in terms of ψ^+. The two problems are in effect equivalent via an inversion. The uniqueness of the reconstruction guarantees that starting from either ψ^+ or ψ^- (which are assumed to be compatible) gives the same u.

The proof consists of several steps. The first one deals with recovering the two gauge elements ψ_2, A_2 from ψ^- by using the system (2.29).

Lemma 2.3. *Given $\psi^- \in L^2$, the system* (2.29) *has a unique solution (A_2, ψ_2) satisfying $\psi_2, A_2 - m \in \dot{H}^1_e$. This solution satisfies*

$$\|\psi_2\|_{\dot{H}^1_e} + \|A_2 - m\|_{\dot{H}^1_e} + \|\frac{A_2 - m}{r}\|_{L^1(dr)} \lesssim \|\psi^-\|_{L^2}(m + \|\psi^-\|^2_{L^2}) \tag{2.32}$$

In addition we have the following properties:

i) given $\epsilon > 0$, and R such that $\|\psi^-\|_{L^2(\mathbb{R}\setminus[R^{-1},R])} \leq \epsilon$, then the following holds true

$$\|\psi_2\|_{\dot{H}^1_e(\mathbb{R}\setminus[R^{-1},\epsilon^{-1}R])} + \|A_2 - m\|_{\dot{H}^1_e(\mathbb{R}\setminus[R^{-1},\epsilon^{-1}R])} \lesssim \epsilon\|\psi^-\|_{L^2} \tag{2.33}$$

ii) if $(\tilde{A}_2, \tilde{\psi}_2)$ is another solution (as above) to (2.29) with $\tilde{\psi}^-$, then

$$\|\psi_2 - \tilde{\psi}_2\|_{\dot{H}^1_e} + \|A_2 - \tilde{A}_2\|_{\dot{H}^1_e} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2} \tag{2.34}$$

iii) if $(\tilde{A}_2, \tilde{\psi}_2)$ satisfy $\tilde{\psi}_2, \tilde{A}_2 - m \in \dot{H}^1_e$ and solve

$$\begin{aligned} \partial_r \tilde{\psi}_2 &= i\tilde{A}_2\tilde{\psi}^- - \frac{1}{r}\tilde{A}_2\tilde{\psi}_2 + E_1 \\ \partial_r \tilde{A}_2 &= -\ \Im(\tilde{\psi}^-\bar{\tilde{\psi}}_2) - \frac{1}{r}(\tilde{A}_2^2 - m^2) + E_2 \end{aligned} \tag{2.35}$$

where $\||E_1| + |E_2|\|_{L^1(dr)+L^2} \lesssim \epsilon$ then

$$\|\psi_2 - \tilde{\psi}_2\|_{\dot{H}^1_e} + \|A_2 - \tilde{A}_2\|_{\dot{H}^1_e} \lesssim C(\|\psi^-\|_{L^2}, \|\tilde{\psi}^-\|_{L^2})(\|\psi^- - \tilde{\psi}^-\|_{L^2} + \epsilon) \tag{2.36}$$

iv) if $\psi^- \in L^p$ with $1 \le p < \infty$ then $\psi^+, \frac{\psi_2}{r}, \frac{A_2 - m}{r} \in L^p$ and

$$\|\psi^+\|_{L^p} + \|\frac{\psi_2}{r}\|_{L^p} + \|\frac{A_2 - m}{r}\|_{L^p} \lesssim C(\|\psi^-\|_{L^2})\|\psi^-\|_{L^p} \tag{2.37}$$

v) if $R_{m-1}\psi^- \in H^s$ then $R_{m+1}\psi^+ \in H^s$ for any $s \in \{1, 2, 3\}$, and

$$\|R_{m-1}\psi^-\|_{H^s} \approx \|R_{m+1}\psi^+\|_{H^s} \tag{2.38}$$

with implicit constants depending on $\|\psi^-\|_{L^2}$.

The reason for having the second type of statement in (2.36) is of technical nature and will be apparent in Section 4. The equation for $\tilde{A}_2$ in (2.35) is more convenient in that form when taking differences. For the original system (2.29) it does not matter how one writes the equation for A_2 thanks to the conservation law $A_2^2 - |\psi_2|^2 = m^2$; however in the case of (2.35) this conservation law does not hold true, hence we write the system in the more convenient form (2.35).

Proof. Our strategy is to solve the ode system (2.29) from zero. Since $\psi_2, A_2 - m \in \dot{H}^1_e$, it follows that $\lim_{r\to 0} \psi_2 = 0, \lim_{r\to 0} A_2 = m$. These two conditions play the role of boundary conditions at zero. Since $\partial_r(A_2^2 - |\psi_2|^2) = 0$, it follows from the conditions at ∞ that $A_2^2 - |\psi_2|^2 = m^2$ holds on all of $\mathbb{R}_+$.

To prove existence, we begin by solving the system in a neighborhood $(0, R^{-1})$ of the origin. By choosing R large enough we can assume without any restriction in generality that

$$\|\psi^-\|_{L^2(0,R^{-1})} \le \epsilon \tag{2.39}$$

and seek (ψ_2, A_2) with the property that

$$\|\psi_2\|_{\dot{H}^1_e(0,R^{-1})} \lesssim \epsilon \tag{2.40}$$

Since $\lim_{r\to\infty} A_2 = m$, $A_2^2 = m^2 + |\psi_2|^2 > 0$ and A_2 is continuous, it follows that $A_2 = \sqrt{m^2 + |\psi_2|^2}$. We substitute this in the ψ_2 equation and discard the dependent A_2 equation. We rewrite the ψ_2 equation as

$$(\partial_r + \frac{m}{r})\psi_2 = im\psi^- + i(A_2 - m)\psi^- - \frac{(A_2 - m)\psi_2}{r}$$

or equivalently

$$r^{-m}\partial_r r^m \psi_2 = im\psi^- + i(A_2 - m)\psi^- - \frac{(A_2 - m)\psi_2}{r}$$

and further

$$\psi_2 = imr^{-m}[r^{-m}\partial_r]^{-1}\psi^- + r^{-m}[r^{-m}\partial_r]^{-1}(i(A_2-m)\psi^- - \frac{(A_2-m)\psi_2}{r})$$

We know from (1.6) that $r^{-m-1}[r^{-m}\bar{\partial}_r]^{-1}$ maps L^2 to L^2, which easily implies that

$$r^{-m}[r^{-m}\bar{\partial}_r]^{-1} : L^2 \to \dot{H}^1_e$$

Hence in order to obtain ψ_2 via the contraction principle it suffices to show that for ψ as in (2.39) and ψ_2 as in (2.40) the map

$$\psi_2 \to i(A_2-m)\psi^- - \frac{(A_2-m)\psi_2}{r}$$

is Lipschitz from $\dot{H}^1_e \to L^2$ with a small ($O(\epsilon)$ in this case) Lipschitz constant. But this is straightforward due to the embedding $\dot{H}^1_e \subset L^\infty$. Thus the existence of ψ_2 in $(0, R^{-1}]$ follows, and the corresponding A_2 is recovered via $A_2(r) = \sqrt{m^2 + |\psi_2(r)|^2}$. The same argument also gives Lipschitz dependence of ψ_2 on ψ^- in $(0, R^{-1}]$.

The solution obtained above on $(0, R^{-1}]$ can be extended locally via standard arguments since $L^2(rdr) \subset L^1_{loc}(dr)$. This extension is global provided we have an a-priori estimate which guarantees that A_2 and ψ_2 stay in a bounded set. Indeed, integrating the equation of A_2 gives

$$A_2(r) - m \leq \|\psi^-\|_{L^2(0,r]}\|\frac{\psi_2}{r}\|_{L^2(0,r]} - \|\frac{\psi_2}{r}\|^2_{L^2(0,r]} = \|\frac{\psi_2}{r}\|_{L^2(0,r)}(\|\psi^-\|_{L^2(0,r]} - \|\frac{\psi_2}{r}\|_{L^2(0,r]})$$

and since $A_2(r) \geq m$ it follows that $\|\frac{\psi_2}{r}\|_{L^2(0,r)} \leq \|\psi^-\|_{L^2(0,r]}$ for any $r \geq 0$, in particular we obtain $\|\frac{\psi_2}{r}\|_{L^2} \leq \|\psi^-\|_{L^2}$. From above estimate we also obtain

$$\|A_2\|_{L^\infty} \leq m + \|\psi^-\|^2_{L^2}. \tag{2.41}$$

This in turn guarantees that the solution (A_2, ψ_2) extends globally up to $r = \infty$. Also, using these estimates in (2.29) gives the (2.32).

For proving (2.33) we use an energy type argument. Denoting

$$F = \frac{\psi_2}{A_2 + m}$$

its derivative satisfies

$$\left|\frac{d}{dr}|F|^2 + \frac{2m}{r}|F|^2\right| \lesssim |\psi^-||F|$$

This further leads to

$$\left|\frac{d}{dr}(r^{2m}|F|)\right| \lesssim r^{2m}|\psi^-|$$

Integrating from infinity we obtain

$$|F| \lesssim r^{-2m}[r^{-2m}\bar{\partial}_r]^{-1}|\psi^-|$$

Returning to ψ_2 we get the pointwise bound

$$|\frac{\psi_2}{A_2+m}| \lesssim r^{-2m}[r^{-2m}\bar{\partial}_r]^{-1}|\psi^-| \tag{2.42}$$

Note that if $|\frac{\psi_2}{A_2+m}| \leq \frac{1}{8m}$ then $|\frac{\psi_2}{A_2+m}| \approx |\psi_2|$. The construction of the solution on $(0, R^{-1})$ gives the corresponding part of (2.33) since (2.42) holds on any such interval. Getting the

$(0, \epsilon^{-1}1R]$ part of (2.33) is slightly more delicate. It suffices to get the L^2 bound for $\frac{\psi_2}{r}$. From (2.42) we have

$$|\psi_2| \lesssim r^{-2m}[r^{-2m}\bar\partial_r]^{-1}(1_{(0,R]}|\psi^-|) + r^{-2m}[r^{-2m}\partial_r]^{-1}(1_{[R,\infty)}|\psi^-|)$$

For the second term we use the smallness of ψ_2 in the hypothesis. For the first one we instead produce a pointwise bound using Cauchy-Schwarz:

$$r^{-2m}[r^{-2m}\bar\partial_r]^{-1}(1_{(0,R]}|\psi^-|) \lesssim r^{-2m}\int_0^R s^{2m}|\psi^-(s)|ds \lesssim (r^{-1}R)^{2m}\|\psi^-\|_{L^2}, \qquad r > R$$

This implies the desired L^2 bound.

Next we turn our attention to (2.34) and (2.36). In fact, in the case of (2.34), in light of the conservation law $\tilde A_2^2 - |\tilde\psi_2|^2 = m^2$, (2.34) follows from (2.36) with $E_1 = E_2 = 0$. Hence we focus our attention on (2.36). We denote

$$\delta\psi = \tilde\psi - \psi, \qquad \delta A_2 = \tilde A_2 - A_2, \qquad \delta\psi_2 = \tilde\psi_2 - \psi_2$$

Without any restriction in generality we can make the assumption $\|\delta\psi\|_{L^2} \ll 1$ and the bootstrap assumption

$$\|\delta\psi_2\|_{L^\infty} + \|\delta A_2\|_{L^\infty} + \|\frac{\delta\psi_2}{r}\|_{L^\infty} + \|\frac{\delta A_2}{r}\|_{L^\infty} \lesssim \epsilon^{\frac12} + \|\delta\psi\|_{L^2}^{\frac12} \tag{2.43}$$

Then we derive the equations for them modulo error terms. We have

$$\begin{aligned}
\partial_r\delta\psi_2 &= i\delta A_2\tilde\psi^- + iA_2\delta\psi - \frac1r A_2\delta\psi_2 - \frac1r\delta A_2\tilde\psi_2 + E_1\\
\partial_r\delta A_2 &= -\ \Im(\psi^-\overline{\delta\psi_2}) - \Im(\delta\psi\overline{\tilde\psi_2}) - \frac2r A_2\delta A_2 - \frac1r(\delta A_2)^2 + E_2
\end{aligned}$$

The following terms $iA_2\delta\psi, \Im(\delta\psi\overline{\tilde\psi_2})$ can be directly included into the error terms E_1, E_2, while the quadratic term $\frac1r(\delta A_2)^2$ can be included in the error term E_2 based on (2.43). We obtain the following linear system for $(\delta\psi_2, \delta A_2)$:

$$\begin{aligned}
\partial_r\delta\psi_2 &= -\frac{m}{r}\delta\psi_2 + i\tilde\psi^-\delta A_2 - \frac1r(A_2 - m)\delta\psi_2 - \frac1r\delta A_2\tilde\psi_2 + E_1\\
\partial_r\delta A_2 &= -\frac{2m}{r}\delta A_2 + \Im(\psi^-\overline{\delta\psi_2}) - \frac2r(A_2 - m)\delta A_2 + E_2
\end{aligned}$$

By considering the $\Re\delta\psi_2, \Im\delta\psi_2$ separately, this is a system of the form

$$\partial_r X = -\frac{m}{r}LX + BX + F, \qquad L = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

where the matrices B, F satisfy $B \in L^2$ and $F \in L^2 + L^1(dr)$. This system needs to be solved with zero Cauchy data at infinity. For this system we need to establish the bound

$$\|X\|_{L^\infty} + \|\frac{X}{r}\|_{L^2} \lesssim \|F\|_{L^2+L^1(dr)} \tag{2.44}$$

If $B = 0$ then

$$X = \begin{pmatrix} r^{-m}[r^{-m}\bar\partial_r]^{-1} & 0 & 0\\ 0 & r^{-m}[r^{-m}\bar\partial_r]^{-1} & 0\\ 0 & 0 & r^{-2m}[r^{-2m}\partial_r]^{-1}\end{pmatrix} F$$

and the conclusion easily follows from argument of type (1.6). If B is small in either $L^2(rdr)$ or in $r^{-1}L^\infty$ then we can treat the BX term perturbatively. If B is large then some more work is needed. We decompose $B = B_1 + B_2$ where $B_1 \in L^1(dr)$ and $|B_2| \ll \frac{1}{r}$. We can construct the bounded matrix $e^{\int B_1}$ as a solution of $\partial_r e^{\int B_1} = e^{\int B_1} B_1$ which also has a bounded inverse. Then we can eliminate B_1 by conjugating with respect to $e^{\int B_1}$, and then treat the part with B_2 perturbatively.

iv) From (2.42), (2.41) and (1.6) we obtain

$$\|\frac{\psi_2}{r}\|_{L^p} \lesssim \|A_2 + m\|_{L^\infty}\|\psi^-\|_{L^p} \lesssim \|\psi^-\|_{L^p}(m + \|\psi^-\|_{L^2}^2)$$

from which (2.37) follows since $\psi^+ = 2i\frac{\psi_2}{r} + \psi^-$ and $A_2 - m = \frac{|\psi_2|^2}{A_2+m}$.

v) Throughout this argument, the use of Sobolev embedding refers to the two-dimensional standard Sobolev embeddings which apply to $R_{m\pm1}\psi^\pm$, which then can be read in terms of $\psi^\pm$.

If $s = 1$ then we use (2.28) to obtain

$$(r\partial_r + (m+1))\psi^+ = (r\partial_r - (m-1))\psi^- - (A_2 - m)(\psi^+ + \psi^-) \tag{2.45}$$

from which

$$\psi^+ = r^{-m-1}[r^m\bar\partial_r]^{-1}\left((r\partial_r - (m-1))\psi^- - (A_2 - m)(\psi^+ + \psi^-)\right)$$

From the Sobolev embedding and (2.37) we obtain

$$\|\frac{(A_2 - m)(\psi^+ + \psi^-)}{r}\|_{L^2} \lesssim \|\frac{A_2 - m}{r}\|_{L^4}\|\psi^+ + \psi^-\|_{L^4} \lesssim \|R_{m-1}\psi^-\|_{H^1}^2$$

which combined with (1.6) gives $\|\frac{\psi^+}{r}\|_{L^2} \lesssim \|R_{m-1}\psi^-\|_{H^1}$. Plugging this back in (2.45) gives $\|\partial_r\psi^+\|_{L^2} \lesssim \|R_{m-1}\psi^-\|_{H^1}$ from which the statement follows for $s = 1$.

If $s = 2$ we differentiate (2.45) to obtain

$$\begin{aligned} H_{m+1}\psi^+ &= (\frac{1}{r}\partial_r - \frac{m+1}{r^2})(r\partial_r + (m+1))\psi^+ \\ &= (\frac{1}{r}\partial_r - \frac{m+1}{r^2})\left[(r\partial_r - (m-1))\psi^- - (A_2 - m)(\psi^+ + \psi^-)\right] \\ &= H_{m-1}\psi^- + (-\frac{2m}{r}\partial_r + \frac{2m^2 - 2m}{r^2})\psi^- - (\frac{1}{r}\partial_r - \frac{m+1}{r^2})\left[(A_2 - m)(\psi^+ + \psi^-)\right] \end{aligned}$$

From Lemma 1.4 it follows that $\|(-\frac{2m}{r}\partial_r + \frac{2m^2-2m}{r^2})\psi^-\|_{L^2} \lesssim \|H_{m-1}\psi^-\|_{L^2}$. From part vii) of Lemma 1.4 we have that $\|\psi^-\|_{L^6} \lesssim \|R_{m-1}\psi^-\|_{H^1}$ and by (2.37) $\|\frac{\psi_2}{r}\|_{L^6} + \|\psi^+\|_{L^6} \lesssim \|R_{m-1}\psi^-\|_{H^1}$, hence we estimate

$$\|\frac{1}{r}\partial_r A_2(\psi^+ + \psi^-)\|_{L^2} \lesssim (\|\psi^-\|_{L^6} + \|\frac{\psi_2}{r}\|_{L^6})\|\frac{\psi_2}{r}\|_{L^6}(\|\psi^-\|_{L^6} + \|\psi^+\|_{L^6}) \lesssim \|R_{m-1}\psi^-\|_{H^1}^3$$

$$\|\frac{A_2 - m}{r^2}(\psi^+ + \psi^-)\|_{L^2} \lesssim \|\frac{\psi_2}{r}\|_{L^6}^2(\|\psi^-\|_{L^6} + \|\psi^+\|_{L^6}) \lesssim \|R_{m-1}\psi^-\|_{H^1}^3$$

Using Lemma 1.4 we estimate

$$\|\frac{A_2 - m}{r}\partial_r\psi^-\|_{L^2} \lesssim \|\frac{A_2 - m}{r}\|_{L^4}\|\partial_r\psi^-\|_{L^4} \lesssim \|R_{m-1}\psi^-\|_{H^2}^3$$

If $s = 3$ then from the above expression for $H_{m+1}\psi^+$ we obtain

$$\frac{1}{r}H_{m+1}\psi^+ = \frac{1}{r}H_{m-1}\psi^- + \frac{1}{r}(-\frac{2m}{r}\partial_r + \frac{2m^2-2m}{r^2})\psi^- - (\frac{1}{r^2}\partial_r - \frac{m+1}{r^3})\left[(A_2 - m)(\psi^+ + \psi^-)\right]$$

If $m = 1$ then the linear part becomes $(\frac{\partial_r^2}{r} - \frac{\partial_r}{r^2})\psi^- \in L^2$ by Lemma 1.4. If $m = 2$ then we have $\frac{1}{r}H_1\psi^- \in L^2$, and from Lemma 1.4, it follows that $4(-\frac{\partial_r}{r^2} + \frac{1}{r^3})\psi^- \in L^2$. If $m \geq 3$, then all the linear terms belong to L^2 in light of Lemma 1.4. As for the nonlinear terms, we have

$$(\frac{1}{r^2}\partial_r - \frac{2}{r^3})\left[(A_2 - 1)(\psi^+ + \psi^-)\right] = \partial_r\left[\frac{A_2 - 1}{r^2}(\psi^+ + \psi^-)\right] = \frac{1}{4}\partial_r\left[\frac{1}{A_2 + 1}|\psi^+ + \psi^-|^2(\psi^+ + \psi^-)\right]$$

which can be easily shown to belong to L^2 by using vii) and viii) of Lemma 1.4.

Finally we apply ∂_r to the expression giving $H_{m+1}\psi^+$ and show that $\partial_r H_{m+1}\psi^+ \in L^2$ in a similar manner. The details are left to the reader.

□

Proof of Proposition 2.2. With ψ_2, A_2 constructed above, we can reconstruct $\psi_1 = \psi^- + i\frac{\psi_2}{r}$. Then we solve the system (2.4) at the level of $(\bar{u}, \bar{v}, \bar{w})$. We would like to solve this system with condition at ∞, $\bar{u} = \overrightarrow{k}, \bar{v} = \overrightarrow{i}, \bar{w} = \overrightarrow{j}$. But this cannot be done apriori. Indeed, consider the coefficient matrix in (2.4)

$$M = \begin{pmatrix} 0 & \Re\psi_1 & \Im\psi_1 \\ \Re\psi_1 & 0 & 0 \\ \Im\psi_1 & 0 & 0 \end{pmatrix}$$

Since $M \notin L^1(dr)$, it is not meaningful to initialize the problem (2.4) at ∞. However M has another structure which is a consequence of (2.18) rewritten as $\psi_1 = (A_2 + 1)\psi_1 + i\partial_r\psi_2$. Therefore $M = N + \partial_r K$ and, by (2.32), N, K satisfy

$$\|N\|_{L^1(dr)} + \|K\|_{\dot{H}^1_e} \lesssim \|\psi^-\|_{L^2}$$

This inequality localizes on intervals $[r, \infty)$ due to (2.33). This allows us to construct solutions with data at $r = \infty$ by using the iteration scheme

$$X = \sum_i X_i, \qquad X_0 = X(\infty), \qquad X_i(r) = \int_r^\infty M(s)X_{i-1}ds$$

We run the iteration scheme in the space $C([r, \infty])$ of continuous functions on (r, ∞) which have limits at ∞. Under the assumption that $X_{i-1} \in C([r, \infty])$ we obtain

$$\begin{aligned} X_i(r) &= \int_r^\infty (N(s) + \partial_s K(s))X_{i-1}ds \\ &= \int_r^\infty N(s)X_{i-1}ds - K(r)X_{i-1}(r) - \int_r^\infty K(s)\partial_s X_{i-1}(s)ds \end{aligned}$$

and further that

$$\|\partial_r X_i\|_{L^2([r,\infty))} + \|X_i\|_{C([r,\infty])} \lesssim \|\psi^-\|_{L^2([r,\infty))}(\|X_{i-1}\|_{L^\infty([r,\infty])} + \|\partial_r X_{i-1}\|_{L^2([r,\infty))})$$

Therefore, inductively, we obtain

$$\|\partial_r X_i\|_{L^2([r,\infty))} + \|X_i\|_{C([r,\infty])} \lesssim \|\psi^-\|^i_{L^2([r,\infty))}$$

By choosing R large such that $\|\psi\|_{L^2([R,\infty))}$ is small, we can rely on an iteration scheme to construct the solution X on $[R,\infty)$.

The uniqueness of this solution is guaranteed by the conservation law $|\bar{u}|, |\bar{v}|, |\bar{w}| = constant$ which follows from the particular form of M.

This also guarantees that the orthonormality conditions imposed at ∞ are preserved (recall that ∞, $\bar{u} = \overrightarrow{k}$, $\bar{v} = \overrightarrow{i}$, $\bar{w} = \overrightarrow{j}$). The solution constructed above can be extended to $(0,\infty)$ by running a similar argument on intervals where $\|\psi^-\|_{L^2(I)}$ is small, where the last interval is of the form $(0,r]$.

The above argument leads to an estimate of the form

$$\|X - X_0\|_{C([0,\infty])} + \|\partial_r X\|_{L^2} \lesssim \|\psi^-\|_{L^2}$$

where by $C([0,\infty])$ we mean continuous functions on $(0,\infty)$ which have limits at 0 and ∞.

Additional information on $\bar{u}, \bar{v}, \bar{w}$ will be obtained in a different manner. Notice that $\bar{u}_3$ and $\zeta = \bar{w}_3 - i\bar{v}_3$ solve the system

$$\partial_r \bar{u}_3 = -\Im(\psi_1 \bar{\zeta}), \qquad \partial_r \zeta = i\bar{u}_3\psi_1$$

which is the same as the one satisfied by A_2, ψ_2. Since the conditions at ∞ are proportional with a constant m, we conclude that $m\bar{u}_3 = A_2, -m\zeta = \psi_2$. From this and the fact that $A_2^2 - |\psi_2|^2 = m^2$ it follows also that $m^2(|\bar{u}_1|^2 + |\bar{u}_2|^2) = |\psi_2|^2$.

Next, we extend the system of vectors to u, v, w using the equivariant setup, i.e. by multiplying them with $e^{m\theta R}$. Using the identification just described above and the orthonormality conditions, it follows that (2.4) is satisfied for $k = 2$. Therefore we have just established the existence of an equivariant map u whose vector field $\mathcal{W}^-$ in the gauge (v, w) is ψ^- and whose gauge elements are ψ_1, ψ_2, A_2. Moreover, we have that

$$E(u) = \pi\|\psi^-\|_{L^2}$$

Given two fields $\psi^-, \tilde{\psi}^-$ we reconstruct X and $\tilde{X}$ as above. Since the construction is iterative it also follows that

$$\|X - \tilde{X}\|_{C[0,\infty]} + \|\partial_r(X - \tilde{X})\|_{L^2} \lesssim \|\psi - \tilde{\psi}\|_{L^2}$$

from which the derivative part in $E(u - \tilde{u})$ follows. Since $u_1 = v_2w_3 - v_3w_2$, $\tilde{u}_1 = \tilde{v}_2\tilde{w}_3 - \tilde{v}_3\tilde{w}_2$, $\psi_2 = -m(\bar{w}_3 - i\bar{v}_3)$ and $\tilde{\psi}_2 = -m(\bar{\tilde{w}}_3 - i\bar{\tilde{v}}_3)$ it follows that

$$\|\frac{u_1 - \tilde{u}_1}{r}\|_{L^2} \lesssim \|\frac{\psi_2 - \tilde{\psi}_2}{r}\|_{L^2}\|X\|_{L^\infty} + \|X - \tilde{X}\|_{L^\infty}\|\frac{\tilde{\psi}_2}{r}\|_{L^2} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2}$$

A similar argument shows that $\|\frac{u_1 - \tilde{u}_1}{r}\|_{L^2} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2}$ which completes the proof of (2.31).

□

3. The Cauchy problem

In this section we are concerned with the nonlinear system of equations (2.24) which we recall here

$$\begin{cases} (i\partial_t + H_{m-1})\psi^- = (A_0 - 2\frac{A_2 - m}{r^2} + \frac{1}{r}\Im(\psi_2\bar{\psi}^-))\psi^- \\ (i\partial_t + H_{m+1})\psi^+ = (A_0 + 2\frac{A_2 - m}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}^+))\psi^+ \end{cases}$$

where ψ_2, A_2, A_0 are given by (2.25), (2.26), respectively (2.27). The problem comes with an initial data $\psi^{\pm}(t_0) = \psi_0^{\pm}$ and we would like to understand its well-posedness on intervals $I \subset \mathbb{R}$ with $t_0 \in I$.

We will be mainly interested in solutions of this system which come from Schrödinger maps, i.e. they satisfy the compatibility conditions (2.28).

For simplicity we denote the nonlinearities by

$$N_m^{\pm}(\psi^{\pm}) = (A_0 \pm 2\frac{A_2 - m}{r^2} \mp \frac{1}{r}\Im(\psi_2\bar{\psi}^{\pm}))\psi^{\pm} \tag{3.1}$$

We define the mass of a function f by $M(f) := \|f\|_{L^2}^2$. The system (2.24) formally conserves the mass, i.e. $M(\psi^-(t)) = M(\psi^-(0))$ and $M(\psi^+(t)) = M(\psi^+(0))$ for all t in the interval of existence. Moreover, as discussed in subsection 2.2, a compatible pair also satisfies $\|\psi^+(0)\|_{L^2} = \|\psi^-(0)\|_{L^2}$.

3.1. **Strichartz estimates.** We begin our analysis by understanding the linear equation

$$(i\partial_t + H_k)u = f, \qquad u(0) = u_0 \tag{3.2}$$

where we recall $H_k = \partial_r^2 + \frac{1}{r}\partial_r - \frac{k^2}{r^2}$.

Our first claim is that, for each k, u satisfies the standard Strichartz estimates

$$\||\nabla|^s R_k u\|_{L_t^p L_r^q} \lesssim \||\nabla|^s R_k u_0\| + \||\nabla|^s R_k f\|_{L_t^{\tilde{p}'} L_r^{\tilde{q}'}} \tag{3.3}$$

where $|\nabla|^s = (-\Delta)^{\frac{s}{2}}$ (defined in the usual manner), $(p,q), (\tilde{p},\tilde{q})$ are admissible pairs in two dimensions ($\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, 2 < p \leq \infty$) and $(\tilde{p}', \tilde{q}')$ is the dual pair of $(\tilde{p},\tilde{q})$. Indeed, $R_k u$ satisfies the following equation

$$(i\partial_t + \Delta)R_k u = R_k f, \qquad R_k u(0) = R_k u_0$$

Then the Strichartz estimates follow from the standard Strichartz in two dimensions. We need to read the Strichartz estimates at the level of the radial functions. For even powers of s we use the identity $\Delta R_k v = R_k H_k v$, hence

$$\|H_k v\|_{L_t^p L_r^q} = \|\Delta R_k v\|_{L_t^p L_x^q} \tag{3.4}$$

and this can be extended to higher regularity but we will not need it.

For odd values of s we use that $|\nabla|^s = |\nabla|(-\Delta)^{\frac{s-1}{2}}$ and that for $k \neq 0$

$$\|\partial_r v\|_{L_t^p L_r^q} + \|\frac{v}{r}\|_{L_t^p L_r^q} \lesssim \||\nabla| R_k v\|_{L_t^p L_x^q} \tag{3.5}$$

while for $k = 0$

$$\|\partial_r v\|_{L_t^p L_r^q} \lesssim \||\nabla| R_k v\|_{L_t^p L_x^q} \tag{3.6}$$

In the context of additional regularity, we need to make improved versions of the Strichartz estimates. We recall the following result from [8].

Lemma 3.1. *Assume that u satisfy (3.2) with initial data u_0 and forcing f.*

i) If $u_0 \in L^2$ is such that $H_k u_0 \in L^2$, for $k \geq 2$, then the following holds true

$$\||\partial_r^2 u| + |\frac{\partial_r u}{r}| + |\frac{u}{r^2}|\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} \lesssim \|H_k u_0\|_{L^2} + \|H_k f\|_{L^1 L^2}$$

ii) If $u_0 \in L^2$ is such that $H_1 u_0 \in L^2$ then the following holds true

$$\|\partial_r^2 u\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} + \|\frac{1}{r}(\partial_r - \frac{1}{r})u\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} \lesssim \|H_1 u_0\|_{L^2} + \|H_1 f\|_{L^1 L^2}$$

These are improved versions of Strichartz estimates from the following point of view. In i) the inequality for $(\partial_r^2 + \frac{1}{r}\partial_r - \frac{k^2}{r^2})u = H_2 u$ is the Strichartz estimate for $H_2 u$ which follows from (3.3) and (3.4); our statement is stronger in saying that each term satisfies the Strichartz estimate. A similar remark is in place for part i). Note the consistency with Lemma 1.4.

3.2. **Setup and Cauchy theory.** In order to make estimates shorter, we make the following notation convention $\|f^\pm\| = \|f^+\| + \|f^-\|$ for various f's and $\|\cdot\|$ involved in the rest of the paper.

Since our non-linear analysis relies mostly on the $L^4_{t,r}$ norm, we define the Strichartz norm of $f : I \times \mathbb{R}^2 \to \mathbb{C}$ by $S_I(f) := \|f\|^4_{L^4(I\times\mathbb{R})}$. If $t_0 \in I$ then we define $S_{I,\leq t_0} f = \|1_{I\cap(-\infty,t_0]} f\|^4_{L^4}$ and $S_{I,\geq t_0} f = \|1_{I\cap[t_0,\infty)} f\|^4_{L^4}$.

We say that a solution $\psi^\pm : I \times \mathbb{R} \to \mathbb{C}$ blows up forward in time if $S_{I,\geq t}\psi^\pm = +\infty, \forall t \in I$. Similarly $\psi^\pm$ blows up backward in time if $S_{I,\leq t}\psi^\pm = +\infty, \forall t \in I$.

A possibility that may occur is that for some interval I, $S_{I,\geq t_0}\psi^+ = +\infty$ while $S_{I,\geq t_0}\psi^- < \infty$, or any other combination. However from (3.9) it follows that solutions satisfying the compatibility condition (2.28) we have that $S_J(\psi^+) \approx S_J(\psi^-)$ on any time interval J. Therefore for such solutions (which we will be mainly interested in) the above scenario is ruled out.

Let $\psi_+^\pm \in L^2$. We say that the solution $\psi^\pm : I \times \mathbb{R} \to \mathbb{C}$ scatters forward in time to $\psi_+^\pm$ iff $\sup I = +\infty$ and $\lim_{t\to\infty} M(\psi^\pm(t) - e^{itH_{m\pm1}}\psi_+^\pm) = 0$. We say that the solution $\psi^\pm : I\times\mathbb{R} \to \mathbb{C}$ scatters backward in time to $\psi_-^\pm$ iff $\inf I = -\infty$ and $\lim_{t\to-\infty} M(\psi^\pm(t) - e^{itH_{m\pm1}}\psi_-^\pm) = 0$.

Our first theorem provides the general Cauchy theory for (2.24).

Theorem 3.2. *Consider the problem* (2.24) *(with ψ_2, A_2, A_0 given by* (2.25), (2.26), (2.27)*) with $\psi_0^\pm \in L^2$. Then there exists a unique maximal-lifespan solution pair $(\psi^+, \psi^-) : I \times \mathbb{R}^2$ with $t_0 \in I$ and $\psi^\pm(t_0) = \psi_0^\pm$ with the additional properties:*

i) I is open.

ii) (Forward scattering) If $\psi^\pm$ do not blow up forward in time, then $I_+ = [0, \infty)$ and $\psi^\pm$ scatters forward in time to $e^{itH_{m\pm1}}\psi_+^\pm$ for some $\psi_+^\pm \in L^2$.

Conversely, if $\psi_+^\pm \in L^2$, then there exists a unique maximal-lifespan solution $\psi^\pm$ which scatters forward in time to $e^{itH_{m\pm1}}\psi_+^\pm$.

iii) (Backward scattering) A similar statement to ii) holds true for the backward in time problem.

iv) (Small data scattering) There exist $\epsilon > 0$ such that if $M(\psi_0^\pm) \leq \epsilon$ then $S_{\mathbb{R}}(\psi^\pm) \lesssim M(\psi_0^\pm)^2$. In particular, the solution does not blow up and we have global existence and scattering in both directions.

v) (Uniformly continuous dependence) For every $A > 0$ and $\epsilon > 0$ there is $\delta > 0$ such that if $\psi^\pm$ is a solution satisfying $S_J(\psi^\pm) \leq A$ and $t_0 \in J$, and such that $M(\psi_0^\pm - \tilde{\psi}_0^\pm) \leq \delta$, then there exists a solution such that $S(\psi^\pm - \tilde{\psi}^\pm) \leq \epsilon$ and $M(\psi(t) - \tilde{\psi}(t)) \leq \epsilon, \forall t \in J$.

vi) (Stability result) For every $A > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $S_J(\psi^\pm) \leq A$, $\psi^\pm$, approximate (2.24), *in the sense*

$$\|(i\partial_t + H_{m\pm1})\psi^\pm - N^\pm(\psi^\pm)\|_{L^{\frac{4}{3}}(J\times\mathbb{R})} \leq \delta,$$

$t_0 \in J, \tilde{\psi}_0^{\pm} \in L^2$ and $S_J(e^{i(t-t_0)H^{\pm}}(\psi^{\pm}(t_0) - \tilde{\psi}_0^{\pm})) \leq \delta$, then there exists a solution $\tilde{\psi}^{\pm}$ on I to (2.24) with $\tilde{\psi}^{\pm}(t_0) = \tilde{\psi}_0^{\pm}$ and $S_J(\psi^{\pm} - \tilde{\psi}^{\pm}) \leq \epsilon$.

vii) (Additional regularity) Assume that, in addition, $R_{m\pm 1}\psi_0^{\pm} \in H^s$ for $s \in \{1, 2, 3\}$. If J is an interval such that $S_J(\psi^{\pm}) \leq A < +\infty$, then the solution $\psi^{\pm}$ satisfies

$$\|R_{m\pm 1}\psi^{\pm}(t)\|_{H^s} \lesssim_A \|R_{m\pm 1}\psi_0^{\pm}\|_{H^s}, \qquad \forall t \in J \tag{3.7}$$

and it also has Lipschitz dependence with respect to the initial data.

The above results are concerned with general solutions of (2.24). However, our interest lies in solutions which correspond to geometric maps. The next result completes the Cauchy theory for solutions of (2.24) which satisfy the compatibility condition (2.28). The system (2.24) does not directly involve the variable ψ_0 which is defined in this context by (2.15).

Theorem 3.3. *i) If $\psi_0^{\pm} \in L^2$ satisfying the compatibility condition (2.28), then $\psi^{\pm}(t)$ satisfies the compatibility condition (2.28) for each $t \in I$. If, in addition, $R_{m\pm 1}\psi_0^{\pm} \in H^3$ then (2.5) and (2.6) are satisfied.*

ii) If the solution satisfies the compatibility condition (2.28) and it does not blow up in time then the two scattering states (described in ii)) are related by

$$\partial_r r(\psi_+^+ - \psi_+^-) = -m(\psi_+^+ + \psi_+^-) \tag{3.8}$$

Conversely, if $\psi_+^{\pm} \in L^2$ satisfy (3.8), then the unique maximal-lifespan solution $\psi^{\pm}$ which scatters to $e^{itH^{\pm}}\psi_+^{\pm}$ (constructed in part ii)) satisfy the compatibility condition (2.28). A similar statement holds true for the backward in time scattering.

iii) If $\psi^{\pm}$ satisfy the compatibility conditions, then for every interval $J \subset I$ (I being the maximal-lifespan interval) the following holds true

$$\|\psi^+\|_{L^4(J)} \approx \|\psi^-\|_{L^4(J)} \tag{3.9}$$

where the constants involved in the use $\approx$ are independent of the interval J.

As a consequence of these theorems we are able to prove the following result

Proposition 3.4. *If $\psi_0^{\pm} \in L^2$ satisfies the compatibility conditions (2.28), $R_{m\pm 1}\psi_0^{\pm} \in H^2$ and $\psi^{\pm}(t)$ is the solution of (2.24) on I then the map $u(t)$ constructed in Proposition 2.2 (for each t) is a Schrödinger map.*

Proof of Theorem 3.2. Parts i)-vi) are standard. Our particular setup is very similar to the one in the Theorem 3.2 in [8], and the proof there can be easily adapted to our problem.

As discussed in [8], part vii) is usually standard, with the exception of one term in it. We rewrite the nonlinear terms as follows

$$A_0 \pm 2\frac{A_2 - m}{r^2} \mp \frac{1}{r}\Im(\psi_2\bar{\psi}^-) = \frac{|\psi^-|^2}{2} - [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-) \pm \frac{1}{2r^2}\int_0^r (|\psi^+|^2 - |\psi^-|^2)sds$$

Without the term $[r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)$, the analysis would be standard, see [8] for more commentaries. We will provide a full analysis of the term

$$N_1^{\pm} = [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\psi^{\pm}$$

This analysis can be extended to the other two terms in $N^{\pm}(\psi^{\pm})$.

The analysis in the case $m = 1$ is similar to the one in [8]. We now proceed with the cases $m \geq 2$. Since $S_I(\psi^{\pm}) \leq A$, the standard theory gives also that

$$\|\psi^{\pm}\|_{L^3L^6(I\times\mathbb{R})} \lesssim_A 1$$

Therefore it makes sense to define

$$\begin{aligned}
B &= \|\partial_r\psi^{\pm}\|_{L^3L^6} + \|\frac{\psi^{\pm}}{r}\|_{L^3L^6} \\
C &= \|\partial_r^2\psi^{\pm}\|_{L^3L^6} + \|\frac{1}{r}\partial_r\psi^{+}\|_{L^3L^6} + \|\frac{\psi^{+}}{r^2}\|_{L^3L^6} + \|\frac{1}{r}(\partial_r - \frac{1}{r})\psi^{-}\|_{L^3L^6} \\
D &= \|\partial_r H^{\pm}\psi^{\pm}\|_{L^3L^6} + \|\frac{1}{r}H^{\pm}\psi^{\pm}\|_{L^3L^6}
\end{aligned}$$

We will prove the following estimates

$$\begin{aligned}
&\|\partial_r N_1^{\pm}\|_{L^1L^2} + \|\frac{1}{r}N_1^{\pm}\|_{L^1L^2} \lesssim_A B \\
&\|H_{m\pm1}N_1^{\pm}\|_{L^1L^2} \lesssim_A C + B^2 \\
&\|\partial_r H_{m\pm1}N_1^{\pm}\|_{L^1L^2} + \|\frac{1}{r}H_{m\pm1}N_1^{-}\|_{L^1L^2} \lesssim_A D + BC
\end{aligned} \tag{3.10}$$

Similar estimates hold true for the other two terms in $N^{\pm}(\psi^{\pm})$. Based on these estimates, the Strichartz estimates 3.3 and the result of Lemma 3.1, a standard argument establishes the conclusion in (3.7).

We now turn to the proof of (3.10). We compute

$$\partial_r N_1^{\pm} = \partial_r\left([r\partial_r]^{-1}\Re(\bar\psi^{+}\psi^{-})\right)\psi^{\pm} + [r\partial_r]^{-1}\Re(\bar\psi^{+}\psi^{-})\partial_r\psi^{\pm}$$

and estimate

$$\|\partial_r N_1^{\pm}\|_{L^1L^2} \lesssim \|\psi^{+}\|_{L^3L^6}\|\frac{\psi^{-}}{r}\|_{L^3L^6}\|\psi^{\pm}\|_{L^3L^6} + \|\psi^{\pm}\|_{L^3L^6}^2\|\partial_r\psi^{\pm}\|_{L^3L^6}$$

from which half of the first estimate in (3.10) follows; the second half follows in a similar manner.

We continue with

$$\begin{aligned}
H_{m\pm1}N_1^{\pm} =& \Delta\left([r\partial_r]^{-1}\Re(\bar\psi^{+}\psi^{-})\right)\psi^{\pm} + 2\partial_r\left([r\partial_r]^{-1}\Re(\bar\psi^{+}\psi^{-})\right)\partial_r\psi^{\pm} \\
&+ \left([r\partial_r]^{-1}\Re(\bar\psi^{+}\psi^{-})\right)H_{m\pm1}\psi^{\pm}
\end{aligned}$$

The last term is estimated by $\lesssim_A C$, the second one is estimated by $\lesssim_A B^2$, while the first one equals

$$(\partial_r + \frac{1}{r})\frac{\Re(\bar\psi^{+}\psi^{-})}{r}\cdot\psi^{\pm} = \frac{\Re(\partial_r\bar\psi^{+}\cdot\psi^{-}) + \Re(\bar\psi^{+}\cdot\partial_r\psi^{-})}{r}\cdot\psi^{\pm}$$

and its L^1L^2 norm is estimated by

$$\lesssim (\|\partial_r\psi^{+}\|_{L^3L^6}\|\frac{\psi^{-}}{r}\|_{L^3L^6} + \|\frac{\psi^{+}}{r}\|_{L^3L^6}\|\partial_r\psi^{-}\|_{L^3L^6})\|\psi^{\pm}\|_{L^3L^6}$$

from which the second estimate in (3.10) follows.

For the third estimate we start with

$$\begin{aligned}\partial_r H_{m\pm1} N_1^{\pm} &= \partial_r \Delta \left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)\psi^{\pm} + \Delta\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)\partial_r\psi^{\pm}\\ &+ 2\partial_r^2\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)\partial_r\psi^{\pm} + 2\partial_r\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)\partial_r^2\psi^{\pm}\\ &+ \partial_r\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)H_{m\pm1}\psi^{\pm} + \left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right)\partial_r H_{m\pm1}\psi^{\pm}\end{aligned}$$

The L^1L^2 norm of the sixth terms above is bounded by $\lesssim_A D$. Using the previous arguments, the L^1L^2 norm of the second, fourth and fifth term is bounded by $\lesssim_A BC$. Since

$$\partial_r^2\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right) = \frac{\Re(\partial_r\bar\psi^+\cdot\psi^-)}{r} + \Re(\frac{\bar\psi^+}{r}(\partial_r - \frac{1}{r})\psi^-)$$

it follows that the L^1L^2 norm of the third term above is bounded by $\lesssim_A BC$.

The first terms is further expanded

$$\begin{aligned}\partial_r\Delta\left([r\partial_r]^{-1}\Re(\bar\psi^+\psi^-)\right) &= \partial_r\left(\frac{\Re(\partial_r\bar\psi^+\cdot\psi^-) + \Re(\bar\psi^+\cdot\partial_r\psi^-)}{r}\right)\\ &= \frac{\Re(\partial_r^2\bar\psi^+\cdot\psi^-) + 2\Re(\partial_r\bar\psi^+\cdot\partial_r\psi^-)}{r} + \Re(\frac{\bar\psi^+}{r}(\partial_r - \frac{1}{r})\partial_r\psi^-)\end{aligned}$$

and estimated by BC. The estimate for $\frac{1}{r}H_{m\pm1}N_1^{\pm}$ is obtained along the same lines, though the argument is much easier. The details are left to the reader. This finishes the argument for (3.10). □

Proof of Theorem 3.3. i) The proof follows exactly the same steps as in [8], with the only adjustments coming from the value of $\mu = -1$ and that we work with a general m.

It is useful to rephrase this in terms of ψ_1, ψ_2, which are recovered linearly from $\psi^{\pm}$. Reverting the algebraic computation from Sections 2.1 and 2.2, ψ_1, ψ_2 solve the system (2.17). Then we seek to show that the relation $D_1\psi_2 = D_2\psi_1$ is preserved along the flow. For this we will derive an equation for the quantity

$$F = D_2\psi_1 - D_1\psi_2$$

Following the lines of the argument in [8] we derive the following equation for F:

$$iD_0F = (\frac{A_2^2}{r^2} - \partial_1(\partial_1 + \frac{1}{r}))F + \Re(F\bar\psi_1)\psi_1 - \frac{1}{r^2}\Re(F\bar\psi_2)\psi_2$$

It is more convenient to recast this as an equation for

$$\frac{F}{r} = -(\partial_r + \frac{1}{r})\frac{\psi_2}{r} + \frac{iA_2}{r}\psi_1$$

which is exactly the quantity in (2.28). We obtain

$$(i\partial_t + H_m)\frac{F}{r} = (A_0 + \frac{A_2^2 - m^2}{r^2})F + \Re(\frac{F}{r}\bar\psi_1)\psi_1 - \frac{1}{r^2}\Re(\frac{F}{r}\bar\psi_2)\psi_2 \tag{3.11}$$

In view of the L^4 Strichartz bounds for ψ_1 and ψ_2 and the derived L^2 bounds for A_0 and $\frac{A_2^2-m^2}{r^2}$, standard arguments show that this linear equation is well-posed in L^2. Hence the conclusion follows provided that $\frac{F}{r}$ has sufficient regularity. Indeed, we have

$$\frac{F}{r} = \frac{i}{2}\left(\partial_r\psi^+ + \frac{1+A_2}{r}\psi^+ - \partial_r\psi^- - \frac{1-A_2}{r}\psi^-\right)$$

It is obvious that if $R_{m\pm1}\psi^{\pm} \in H^1$ then $\frac{F}{r} \in L^2$.

If $R_{m\pm1}\psi^{\pm} \in H^2$ then by using the results in Lemma 1.4 and Sobolev embeddings one easily shows that $\frac{F}{r} \in \dot{H}^1_e$.

We will show in detail that if $R_{m\pm1}\psi^{\pm} \in H^3$, then $H_m\frac{F}{r} \in L^2$. Indeed,

$$\begin{aligned}-2iH_m\frac{F}{r} &= (\partial_r + \frac{1+A_2}{r})H_{m+1}\psi^+ + 2(m+1)\frac{A_2-m}{r^3}\psi^+ + 2\frac{m-A_2}{r^2}\partial_r\psi^+\\ &+ \frac{(\partial_r - \frac{1}{r})\partial_r A_2}{r}\psi^+ + 2\frac{\partial_r A_2}{r}\partial_r\psi^+ - (\partial_r + \frac{1-A_2}{r})H_{m-1}\psi^-\\ &+ 2\frac{m-A_2}{r^2}\partial_r\psi^- + 2(m-1)\frac{m-A_2}{r^3}\psi^- + \frac{(\partial_r - \frac{1}{r})\partial_r A_2}{r}\psi^- + 2\frac{\partial_r A_2}{r}\partial_r\psi^-\end{aligned}$$

The above expression is easily shown to belong to L^2 based on that $R_{m\pm1}\psi^{\pm} \in H^3$, by using that $R_{m\pm1}H_{m\pm1}\psi^{\pm} \in H^1$, the Sobolev embeddings $\psi^{\pm}, \partial_r\psi^{\pm} \in L^6$ and (2.26).

Hence we can conclude that $H_m\frac{F}{r} \in L^2$. This allows us to run a standard energy argument by pairing the equation, with $\bar{F}$, to conclude that

$$\partial_t\|\frac{F}{r}\|^2_{L^2} \lesssim (\|\psi_1\|^2_{L^\infty} + \|\frac{\psi_2}{r}\|^2_{L^\infty})\|\frac{F}{r}\|^2_{L^2}$$

which by using the Gronwall inequality and the fact that $F(0) = 0$ leads to $F(t) = 0$ for all $t \in I$.

In order to run the energy argument it suffices to have $\frac{F}{r} \in \dot{H}^1_e$ and use the pairing of $\dot{H}^{-1}_e$ and $\dot{H}^1_e$. This is useful in the proof of Proposition 3.4 where we assume only $R_{m\pm1}\psi^{\pm} \in H^2$.

In the general case when $\psi^{\pm}_0 \in L^2$ only we regularize them as follows. We produce $R_{m-1}\psi^-_{n,0} \in H^3$ so that $\|\psi^-_0 - \psi^-_{n,0}\|_{L^2} \leq \frac{1}{n}$. By using Lemma (2.3), and particularly part v), we obtain that the compatible pair $R_{m+1}\psi^+_{n,0} \in H^3$ and $\|\psi^+_0 - \psi^+_{n,0}\|_{L^2} \lesssim \frac{1}{n}$. We also recast the compatibility condition to

$$\psi^+ - \psi^- = -[r\partial_r]^{-1}\left(\psi^+ - \psi^- + A_2(\psi^+ + \psi^-)\right)$$

so that all terms involved belong to L^2. Using the conservation of the compatibility condition for $\psi^{\pm}_n(t)$ under the flow (2.24) and part v) of the Theorem, we obtain the desired result.

ii) The key observation is that the equation for ψ_2 in (2.29) becomes linear in the following sense:

$$\lim_{t\to\infty}\|\partial_r\psi_2 - im\psi^- + m\frac{\psi_2}{r}\|_{L^2} = 0 \tag{3.12}$$

under the hypothesis that $\lim_{t\to\infty}\|\psi^-(t) - e^{itH_{m-1}}\psi^-_+\|_{L^2} = 0$. This is easily shown to follow from the following estimate

$$\lim_{t\to\infty}\sup_{r\in(0,\infty)}|r^{-m}\int_0^r e^{itH_{m-1}}f(s)s^m ds| = 0 \tag{3.13}$$

which holds true for $f \in L^2$. The proof of (3.13) is similar to the corresponding statement in the Appendix of [8]. Based on this, it follows that $\lim_{t\to\infty}\|\psi_2(t)\|_{L^\infty} = 0$, and that

$$\lim_{t\to\infty}\|i(A_2 - m)\psi^- - \frac{1}{r}(A_2 - m)\psi_2\|_{L^2} = 0$$

which justifies (3.12).

With the notation (essentially the linearized version of $\frac{F}{r}$ above)

$$f(t) = \partial_r(e^{itH_{m+1}}\psi_+^+ - e^{itH_{m-1}}\psi_+^-) - 2\frac{e^{itH_{m-1}}\psi_+^-}{r},$$

the scattering relation (3.8) can be rewritten as $\lim_{t\to\infty}\|f(t)\|_{\dot{H}_e^{-1}} = 0$. A direct computation gives that f obeys the equation

$$(i\partial_t + H_m)f = 0$$

Since $\lim_{t\to\infty}\|f(t)\|_{\dot{H}_e^{-1}} = 0$ it follows from the conservation of the $\dot{H}_e^{-1}$ norm that $f(0) = 0$ which is (3.8). Alternatively, one could carry out this argument as we did in viii).

Assume now that given $\psi_+^\pm$ satisfying (3.8) we construct (as in ii)) solutions $\psi^\pm(t)$ to (2.24) on some $[T, +\infty)$ which scatter forward to $e^{itH_{m\pm1}}\psi_+^\pm$. Following the argument in part i) we construct F which satisfies (3.11). Assuming additional regularity on the states $\psi_+^\pm$, $R_{m\pm1}\psi_+^\pm \in H^3$, we have by part vii) of Theorem 3.2 that $R_{m\pm1}\psi^\pm(t) \in H^3$, hence by the argument in i) $H_m\frac{F}{r} \in L^2$ and the right-hand side of (3.11) belongs to L^2. Then the Duhamel formula applies to (3.11) and in turn the Strichartz estimate

$$\|\frac{F}{r}\|_{L^4([T,\infty)\times\mathbb{R})} \lesssim \|\psi^\pm\|^2_{L^4([T,\infty)\times\mathbb{R})}\|\frac{F}{r}\|_{L^4([T,\infty)\times\mathbb{R})}$$

where we have used that $\lim_{t\to\infty}\|\frac{F(t)}{r}\|_{L^2} = 0$ (this follows as above because of (3.13)). Next, by taking T large enough, we obtain that $F(t) \equiv 0$ for $t \geq T$ and the conclusion follows by invoking part i).

For general states $\psi_+^\pm \in L^2$ satisfying (3.8) we proceed as above. We approximate them by sequences $\psi_{n,+}^\pm$ with $R_{m\pm1}\psi_{n,+}^\pm \in H^3$; this can be done by regularizing $R_{m-1}\psi_+^-$ first and then showing that the corresponding $R_{m+1}\psi_+^+$ has the same regularity as we did in Lemma 2.3 part v) - in fact this argument involves only the linear part of the argument there. Then we write (3.8) at the level of L^2

$$\psi_+^+ - \psi_+^- = -[r\partial_r]^{-1}((m+1)\psi_+^+ + (m-1)\psi_+^-),$$

use the above argument and a limiting argument.

iii) One side of (3.9) follows from the fixed time bound (2.37). The other side is similar, and it consists and replicating the result of Lemma 2.3 starting from ψ^+ instead.

□

Proof of Proposition 3.4. With the given $\psi_0^\pm$ we reconstruct $u_0 \in \dot{H}^1 \cap \dot{H}^3$ as in Proposition 2.2. The additional regularity $R_{m\pm1}\psi_0^\pm \in H^2$ implies, by (2.30), that $u_0 \in \dot{H}^1 \cap \dot{H}^3$. For the classical Schrödinger Map $u(t)$ with data u_0 we construct its Coulomb gauge, its field components and write the system (2.24) whose initial data is $\psi_0^\pm$. Invoking the uniqueness part of Theorem 3.2, it follows that $\psi^\pm(t)$ are the gauge representation of $\mathcal{W}^\pm(t)$, hence the reconstruction in Proposition 2.2 gives the Schrödinger Map $u(t)$ for each t.

□

We can now identify the critical threshold for global well-posedness and scattering. For any $m \geq 0$, we define $A(m)$ by

$A(m) := sup\{S_{I_{max}}(\psi^-) : M(\psi^-) \leq m$ where $\psi^\pm$ is a solution to (2.24) satisfying(2.28)$\}$

where $\psi^\pm$ is assumed to be a solution of (2.24), satisfying the compatibility condition (2.28) and I_{max} is its maximal interval of existence.

Obviously A is a monotone increasing functions, it is bounded for small m by part iv) and it is left-continuous by part v) of Theorem 3.2. Therefore there exists a critical mass $0 < m_0 \leq +\infty$ such that $A(m)$ is finite for all $m < m_0$ and it is infinite $m \geq m_0$. Also any solution ψ with $M(\psi) < m_0$ is globally defined and scatters.

Note that from (3.9) and the fact that $M(\psi^+) = M(\psi^-)$ (due to the compatibility relation), it follows that we could have used $S_{I_{max}}(\psi^+), M(\psi^+)$ in the definition of $A(m)$ and arrive to the same conclusion as above with the same critical mass m_0.

4. Concentration compactness

The main goal of this section is to prove that if the above critical mass m_0 is finite, then there exists a critical element $\psi^\pm$ with mass m_0 which blows up, see Theorem 4.1. Moreover we can be more precise about the behavior of "scale" of the critical element, see Theorem 4.2. The information provided by the two results aforementioned will be crucial in the next section where we rule out the possibility that m_0 is finite.

We start by exhibiting the symmetries of the system (2.24). The system is invariant under the time reversal transformation $\psi^\pm(r,t) \to \psi^\pm(r,-t)$. This allows us to focus our attention on positive times, i.e. $t \geq 0$. Next, the system is invariant under two other transformations: scaling, $\psi^\lambda = \lambda^{-1}\psi(\lambda^{-1}r, \lambda^{-2}t)$ with $\lambda \in \mathbb{R}$, and phase multiplication, $\psi^\alpha(r,t) = e^{i\alpha}\psi(r,t)$ with $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. The phase multiplication can be ignored as the group generated is compact. This way we generate the first (non-compact) group G of transformations g_λ defined by

$$g_\lambda f(r) = \lambda^{-1} f(\lambda^{-1} r)$$

From (2.25), (2.26) and (2.27), the effect of the action g_λ on $\psi^\pm$ is translated in the action of g^1_λ on ψ_2, A_2 and g^2_λ on A_0 where

$$g^1_\lambda f(r) = f(\lambda^{-1} r), \qquad g^2_\lambda f(r) = \lambda^{-2} f(\lambda^{-1} r)$$

The action of g is extended to space-time functions by

$$T_{g_\lambda} f(r,t) = \lambda^{-1} f(\lambda^{-1} r, \lambda^{-2} t)$$

The equations in (2.24) are also time translation invariant and this suggests enlarging the group G to $G-$ as follows. Given $\lambda > 0$ and $t \in \mathbb{R}$, we define

$$g^-_{\lambda,t} f = \lambda^{-1} [e^{itH_{m-1}} f](\lambda^{-1} r)$$

We denote by G' the group generated by these transformations. Given two sequences $g^n, \tilde{g}^n \in G^-, \forall n \in \mathbb{N}$, we say that they are asymptotically orthogonal iff

$$\frac{\lambda_n}{\tilde{\lambda}_n} + \frac{\tilde{\lambda}_n}{\lambda_n} + |t_n \lambda_n^2 - \tilde{t}_n \tilde{\lambda}_n^2| = \infty \tag{4.1}$$

We are now ready to state the two main results of this section.

Theorem 4.1. *Assume that the critical mass m_0 is finite. Then there exists a critical element, i.e. a maximal-lifespan solution $\psi^\pm$ to (2.24) and satisfying (2.28), with mass m_0 which blows up forward in time. In addition this solution has the following compactness property: there exists a continuous function $\lambda(t) : I_+ = [0, T_+) \to \mathbb{R}_+$ such that the sets*

$$K^\pm := \left\{ \frac{1}{\lambda(t)} \psi^\pm(\frac{r}{\lambda(t)}, t), t \in I_+ \right\}$$

are precompact in L^2.

Remark. *As a consequence of the compactness property it follows that there exists a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that the above critical element satisfies*

$$\int_{r \geq C(\eta)\lambda(t)^{-1}} |\psi^{\pm}(t,r)|^2 r dr \leq \eta, \quad \forall t \in I_+. \tag{4.2}$$

One can construct critical elements whose function $\lambda(t)$ has more explicit behavior.

Theorem 4.2. *Assume that the critical mass m_0 is finite. Then we can construct a critical element as in Theorem 4.1 such that one of the two scenarios holds true:*
i) $T_+ = \infty$ and $\lambda(t) \geq c > 0, \forall t \geq 0$.
ii) $T_+ < \infty$ and $\lim_{t \to T_+} \lambda(t) = \infty$.

The proofs of the Theorems 4.1 and 4.2 follow the same steps as their counterparts in [8], which in turn were inspired by the seminal work of Kenig and Merle, see [17]. We will not reproduce the proofs here due to their lengthy repetitive argument. Instead we state the intermediate Propositions which then lead to the proof of Theorem 4.1.

It is standard, see for instance [17] and [26] that the result in Theorem 4.1 follows from the following

Proposition 4.3. *Assume $m_0 < +\infty$. Let $\psi_n^{\pm} : I_{n+} = [0, T_{n+}) \times \mathbb{R} \to \mathbb{C}, n \in \mathbb{N}$ be a sequence of solutions to (2.24), satisfying (2.28) and such that $\lim_{n\to\infty} M(\psi_n^{\pm}) = m_0$ and $\lim_{n\to\infty} S_{I_{n+}}(\psi_n^{\pm}) = \infty$. Then there are group elements $g_n \in G$ such that the sequence $g_n \psi_n^{\pm}(t_n)$ has a subsequence which converges in L^2.*

One of the main ingredients in the proof of Proposition 4.3 is the classical linear profile decomposition result. These type of results originate in the work of Bahouri and Gerard [1], for the case of nonlinear wave equation and independently, in the work of Merle and Vega [22], for the case of the nonlinear Schrödinger equation. For the case of nonlinear Schrödinger equations see also [2], [18], [26].

Proposition 4.4. *Let $\psi_0^n, n \in \mathbb{N}$ be a bounded sequence in L^2. Then (after passing to a subsequence if necessary) there exists a sequence $\phi^j, j \in \mathbb{N}$ of functions in L^2 and $g^{n,j} \in G^-, n, j \in \mathbb{N}$ such that we have the decomposition*

$$\psi_0^n = \sum_{j=1}^{l} g^{n,j} \phi^j + w^{n,l}, \qquad \forall l \in \mathbb{N} \tag{4.3}$$

where $w^{n,l}$ satisfies

$$\lim_{l\to\infty} \lim_{n\to\infty} S(e^{itH_{m-1}} w^{n,l}) = 0 \tag{4.4}$$

Moreover $g^{n,j}$ and $g^{n,j'}$ are asymptotically orthogonal for any $j \neq j'$ and we have the following orthogonality condition

$$weak \lim_{n\to\infty} (g^{n,j})^{-1} w^{n,l} = 0, \quad \forall 1 \leq j \leq l \tag{4.5}$$

As a consequence the mass decoupling property holds

$$\lim_{n\to\infty} (M(u^n) - \sum_{j=1}^{l} M(\phi^j) - M(w^{n,l})) = 0 \tag{4.6}$$

A similar statement holds true also for the operator H_{m+1}. We explained in [8] how this result follows as an equivariant counterpart of the result in Theorem 7.3 in [26].

Based on Proposition 4.4 and the results in the previous sections, one proves Proposition 4.3 by following the same steps as in [8]. The details are left as an exercise.

5. Momentum and localized momentum.

In this section we rule out the possible scenarios exhibited in Theorem 4.2. With the language used in Section 4, we claim the following

Theorem 5.1. *Critical elements do not exist.*

This will be based on virial type identities. Virial identities for the Schrödinger Map problem originate in the work of Grillakis and Stefanopoulos via a Lagrangian approach, see [10]. In their work the formulation of these identities is at the level of the conformal coordinate, obtained by using the stereographic projection. Our approach is different in the sense that we derive the virial identities at the level of the gauge components. However our results can be derived from [10].

5.1. **Virial type identities.** This section is concerned with identities involving solutions of (2.24) which satisfy the compatibility condition (2.28).

Given $a : \mathbb{R}_+ \to R$ a smooth function, i.e. $|(r\partial_r)^\alpha a| \lesssim_\alpha 1$, and which decays at infinity we claim that

$$\frac{d}{dt}\int a(r)(A_2 - m)rdr = \int r\partial_r a(r)\Re(\psi_1\frac{\bar{\psi}_2}{r})rdr \tag{5.1}$$

By using part i) of Theorem 3.3, the proof of (5.1) goes as follows

$$\begin{aligned}\frac{d}{dt}\int a(r)(A_2 - m)rdr &= \int a(r)\partial_t A_2 rdr = -\int a(r)\Im(\psi_0\bar{\psi}_2)rdr\\ &= -\int a(r)\Im(i(\partial_r\psi_1 + \frac{1}{r}\psi_1 + \frac{iA_2}{r^2}\psi_2)\bar{\psi}_2)rdr\\ &= -\int a(r)\left(\Im(i\partial_r(r\psi_1\bar{\psi}_2)) - \Im(ir\psi_1\partial_r\bar{\psi}_2)\right)dr\\ &= \int \partial_r a(r)\Im(i\psi_1\bar{\psi}_2)rdr = \int r\partial_r a(r)\Re(\psi_1\frac{\bar{\psi}_2}{r})rdr\end{aligned}$$

This computation is valid in a classical sense provided that $R_{m\pm1}\psi^\pm \in H^2$. For general functions $\psi^\pm$ this is done by using a regularization argument as we did in the proof of part i) of Theorem 3.2. Note that the quantities involved on both sides of (5.1) are meaningful in light of the fact that $\psi_0 \in \dot{H}_e^{-1}$ and $a\psi_2 \in \dot{H}_e^1$.

We now introduce the two momenta, the radial and the temporal one, as follows

$$M_1 = \frac{\Re(\psi_1\bar{\psi}_2)}{A_2 + m}, \qquad M_0 = \frac{\Re(\psi_0\bar{\psi}_2)}{A_2 + m}$$

Using the covariant calculus, the time momentum can be further written as follows

$$\begin{aligned}(A_2+m)M_0 &= \Re(\psi_0\bar{\psi}_2)\\ &= \Re\bigg(i(D_1\psi_1+\frac{1}{r}\psi_1+\frac{1}{r^2}D_2\psi_2)\bar{\psi}_2\bigg)\\ &= -\Im(\partial_r\psi_1\bar{\psi}_2)-\frac{1}{r}\Im(\psi_1\bar{\psi}_2)-\frac{A_2}{r^2}|\psi_2|^2\\ &= -\partial_r\Im(\psi_1\bar{\psi}_2)-\Im(\psi_1\partial_r\bar{\psi}_2)+\frac{1}{r}\partial_r A_2-\frac{A_2}{r^2}|\psi_2|^2\\ &= \partial_r^2 A_2+\frac{1}{r}\partial_r A_2-A_2(|\psi_1|^2+\frac{|\psi_2|^2}{r^2})\end{aligned}$$

which leads to

$$M_0=\Delta\ln(A_2+m)+\bigg(\frac{\partial_r A_2}{A_2+m}\bigg)^2-\frac{A_2}{A_2+m}(|\psi_1|^2+\frac{|\psi_2|^2}{r^2}) \tag{5.2}$$

The following identity plays a fundamental role in our analysis

$$\partial_t M_1-\partial_r M_0=-\partial_r A_0 \tag{5.3}$$

This is established by using the covariant rules of calculus,

$$\begin{aligned}\partial_t M_1 &= \frac{\Re(D_0\psi_1\bar{\psi}_2)}{A_2+m}+\frac{\Re(\psi_1\overline{D_0\psi_2})}{A_2+m}-\frac{\Re(\psi_1\bar{\psi}_2)}{(A_2+m)^2}\partial_t A_2\\ &= \frac{\Re(D_1\psi_0\bar{\psi}_2)}{A_2+m}+\frac{\Re(\psi_1\overline{D_2\psi_0})}{A_2+m}+\frac{\Re(\psi_1\bar{\psi}_2)}{(A_2+m)^2}\Im(\psi_0\bar{\psi}_2)\\ &= \partial_r M_0-\frac{\Re(\psi_0\partial_r\bar{\psi}_2)}{A_2+m}+\frac{\Re(\psi_0\bar{\psi}_2)}{(A_2+m)^2}\partial_r A_2+\frac{\Re(\psi_1\overline{D_2\psi_0})}{A_2+m}+\frac{\Re(\psi_1\bar{\psi}_2)}{(A_2+m)^2}\Im(\psi_0\bar{\psi}_2)\\ &= \partial_r M_0-\frac{A_2\Im(\psi_0\bar{\psi}_1)}{A_2+m}-\frac{\Re(\psi_0\bar{\psi}_2)}{(A_2+m)^2}\Im(\psi_1\bar{\psi}_2)+\frac{A_2\Im(\psi_1\overline{\psi_0})}{A_2+m}+\frac{\Re(\psi_1\bar{\psi}_2)}{(A_2+m)^2}\Im(\psi_0\bar{\psi}_2)\\ &= \partial_r M_0-2\frac{A_2\Im(\psi_0\bar{\psi}_1)}{A_2+m}+\frac{|\psi_2|^2\Im(\psi_0\bar{\psi}_1)}{(A_2+m)^2}\\ &= \partial_r M_0-\Im(\psi_0\bar{\psi}_1)\\ &= \partial_r M_0-\partial_r A_0\end{aligned}$$

The above computation is meaningful provided that $R_{m\pm1}\psi^{\pm}\in H^3$.

Next we derive a localized version of (5.3) which has also the advantage that it makes sense for $\psi^{\pm}\in L^2$ only. We take $a:\mathbb{R}_+\to\mathbb{R}$ to be a smooth function which decays at infinity and satisfies also $|\frac{1}{r}\partial_r a|\lesssim 1$ and $|\partial_r^2 a|\lesssim 1$. As a consequence we have that if $f\in\dot{H}^1_e$ then $\frac{1}{r}f\partial_r a\in\dot{H}^1_e$.

We multiply (5.3) by a and integrate by parts as follows

$$\int a(r)M_1(r)dr\,\big|_0^T+\int_0^T\int\partial_r a(r)M_0 dr=\int_0^T\int\partial_r a(r)A_0 dr \tag{5.4}$$

This identity is now meaningful for $\psi^{\pm}\in L^2$. Indeed each term is well-defined for the following reasons:

- the first since a is bounded and $\frac{1}{r}M_1 \in L^2$,
- the second since $\psi_0 \in \dot{H}_e^{-1}$ and $\frac{1}{r}\partial_r a \cdot \psi_2 \in \dot{H}_e^1$,
- the third since $\frac{1}{r}\partial_r a$ is bounded and $A_0 \in L^1$.

The justification of (5.4) for general $\psi^{\pm} \in L^2$ is done by regularizing $\psi^{\pm}$ as above.

It will be useful to rewrite the second term on the left-hand side as follows

$$\int \partial_r a(r) M_0 dr = \int \frac{1}{r}\partial_r a(r) \left(\Delta \ln(A_2+m) + \left(\frac{\partial_r A_2}{A_2+m}\right)^2 - \frac{A_2}{A_2+m}(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2}) \right) r dr$$
$$= -\int \partial_r(\frac{1}{r}\partial_r a(r))\partial_r \ln(A_2+m) r dr - \int \frac{1}{r}\partial_r a(r) G(r) dr$$

where

$$G(r) = -\left(\frac{\partial_r A_2}{A_2+m}\right)^2 + \frac{A_2}{A_2+m}(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2})$$

Using (2.29) one can easily see that G is positive definite,

$$G \geq |\psi_1|^2(\frac{A_2}{A_2+m} - \frac{|\psi_2|^2}{(A_2+m)^2}) + \frac{A_2}{A_2+m}\frac{|\psi_2|^2}{r^2} \geq \frac{m}{m+m_0}|\psi_1|^2 + \frac{1}{2}\frac{|\psi_2|^2}{r^2} \tag{5.5}$$

where $m+m_0$ is an upper bound for A_2, obtained from (2.26).

5.2. **Proof of Theorem 5.1.** The argument is in the spirit of the corresponding one in [17].

Based on a localized version of (5.1) and (5.4) we rule out the possibilities exhibited in parts i) and ii) of Theorem 4.2.

By using (2.25) and (2.33), the concentration property (4.2) implies that all of the differentiated variables ψ_1, ψ_2 and A_2 are concentrated in a compact set,

$$\int_{r \gtrsim C(\eta)c^{-1}\eta^{-1}} (|\psi_1(r)|^2 + \frac{|\psi_2(r)|^2}{r^2} + \frac{(A_2(r)-m)^2}{r^2}) r dr \lesssim \eta, \qquad \forall t \in I_+. \tag{5.6}$$

We start by ruling out the existence of a critical element from part i) of Theorem 4.2, i.e. the global element with $\lambda(t) \geq c > 0, \forall t > 0$. In (5.4), we take $a(r) = r^2\phi(\frac{r}{R})$ where ϕ is smooth and equals 1 for $r \leq 1$ and 0 for $r \leq 2$, and obtain

$$\int a(r) M_1(r) dr \Big|_0^T = \int_0^T \int \partial_r(\frac{1}{r}\partial_r a(r))\partial_r \ln(A_2+m) r dr dt - \int_0^T \int \frac{1}{r}\partial_r a(r) G(r) r dr dt$$
$$+ \int_0^T \int \partial_r a(r) A_0 dr dt \tag{5.7}$$

In this identity there are two main terms which we compare against each other: the one the left-hand side and the second on the right-hand side. All the other terms are controlled by one of the two main terms just mentioned.

We choose $\eta \ll 1$ small enough (the exact choice is derived from the inequalities on the error terms below) and $R = C(\eta)c^{-1}\eta^{-1} \gg c^{-1}$; we estimate the main terms in the above expression by

$$\left| \int a(r) M_1 dr \right| \lesssim \int r^2 |\psi_1| |\frac{\psi_2}{r}| r dr \lesssim R^2 \|\psi_1\|_{L^2} \|\frac{\psi_2}{r}\|_{L^2} \lesssim R^2 m_0$$

which is valid both at $t = 0$ and $t = T$, and, by (5.5) and (5.6)

$$\int_0^T \int \frac{1}{r}\partial_r a(r) G(r) r dr dt \gtrsim T$$

By choosing $T \gg R^2 m_0$ we obtain a contradiction, provided that we establish that all the other terms involved in (5.7) are of error type.

The first term on the left-hand side of (5.7) is bounded as follows

$$\left| \int_0^T \int \partial_r (\frac{1}{r}\partial_r a(r)) \partial_r \ln(A_2 + m) r dr dt \right| \lesssim \int_0^T \int_{r \approx R} |\partial_r A_2| dr dt \lesssim T\eta \ll T$$

For the third term on the right-hand side of (5.7) we use (2.21) and write

$$\left| \int_0^T \int \partial_r a(r) A_0 dr dt \right| = \left| \int_0^T \int (-2 + \frac{1}{r}\partial_r a(r)) A_0 r dr dt \right|$$

which is then bounded by

$$\lesssim \int_0^T \|\psi_1\|_{L^2[R,\infty)} \|\frac{\psi_2}{r}\|_{L^2[R,\infty)} dt \lesssim T\eta \ll T$$

We have just shown that the other two terms in (5.7) are of error type and this finishes the contradiction argument. With this we conclude ruling out the possibility exhibited in part i) of Theorem 4.2.

Next we rule out the critical element of type exhibited in part ii). In this case the assumption is that we have a critical element with $T_+ < \infty, \lim_{t \to T_+} \lambda(t) = +\infty$.

For fixed R we claim that

$$\lim_{t \to T_+} \int \phi(\frac{r}{R})(A_2 - m) r dr = 0 \tag{5.8}$$

Indeed, for given $\epsilon > 0$, pick η such that $\eta^{\frac{1}{2}} R^2 < \epsilon$. Using (2.33) we obtain

$$\begin{aligned}
&\|\phi(\frac{r}{R})(A_2 - m)\|_{L^1} \\
\lesssim &(C(\eta)\lambda(t)^{-1}\eta^{-1})^2 \|\frac{A_2 - m}{r}\|_{L^2[0, C(\eta)\lambda^{-1}(t)\eta^{-1}]} + R^2 \|\frac{A_2 - m}{r}\|_{L^2[C(\eta)\lambda^{-1}(t)\eta^{-1}, R]} \\
\lesssim &(C(\eta)\eta^{-1}\lambda(t)^{-1})^2 m_0^{\frac{1}{2}} + \eta^{\frac{1}{2}} R^2
\end{aligned}$$

By choosing t close enough to T_+, we obtain $(C(\eta)\eta^{-1}\lambda(t)^{-1})^2 m_0^{\frac{1}{2}} < \epsilon$, and this establishes (5.8).

Next we choose $a(r) = \phi(\frac{r}{R})$, fix $\eta > 0$, integrate (5.1) on $[t, T_+)$ and use (5.8) to obtain

$$\int \phi(\frac{x}{R})(A_2(r,t) - m) r dr \lesssim (T_+ - t) \|\psi_1(t)\|_{L^2(|x| \approx R)} \|\frac{\psi_2(t)}{r}\|_{L^2(|x| \approx R)} \lesssim (T_+ - t)\eta$$

provided that $R \gtrsim C(\eta)\eta^{-1}\lambda(t)^{-1}$. By fixing t and taking $\eta \to 0$ (which also forces $R \to \infty$), it follows that

$$\int (A_2(r,t) - m) r dr = 0$$

which implies $A_2(t) \equiv m$ hence, by (2.9) and then by (2.18) it follows that $\psi_2(t) \equiv 0$ and $\psi_1(t) \equiv 0$. Finally this implies by (2.25) that $\psi^{\pm}(t) \equiv 0$ which contradicts the blow-up hypothesis at time T_+ (since the solution is globally in time $\equiv 0$).

6. Proof of the main result

This section is dedicated to the proof of Theorem 1.3. Given an initial data $u_0 \in \dot{H}^1 \cap \dot{H}^3$, by using Theorem 1.1 it follows that it has a unique local solution on $[0,T]$ for some $T > 0$. On this interval we use sections 2.1 and 2.2 to construct the associated compatible fields $\psi^{\pm}$ obeying the system (2.24). By using Theorem 5.1 (and the previous reduction from Section 4) it follows that the solution $\psi^{\pm}$ is globally defined on $[0, +\infty)$ and with $\|\psi^{\pm}\|_{L^4(\mathbb{R}_+ \times \mathbb{R}_+)} < +\infty$. By part vii) of Theorem 3.2 the H^2 regularity of $R_{m\pm 1}\psi_0^{\pm}$ is propagated at all times $t \geq 0$. Invoking Proposition 2.1 this implies that $u(t) \in \dot{H}^1 \cap \dot{H}^3$ with bounds depending on $\|\psi^{\pm}\|_{L^4(\mathbb{R}_+ \times \mathbb{R}_+)}, \|R_{m\pm 1}\psi_0^{\pm}\|_{H^2}$ and t. Using again Theorem 1.1, this means that the solution $u(t)$ can be continued past time T and in fact for all times $t \geq 0$ with $u(t) \in L_t^{\infty}(\mathbb{R}_+ : \dot{H}^1 \cap \dot{H}^3)$. The scattering statement refers to the scattering for $\psi^{\pm}(t)$, which follows from the Cauchy theory for the system (2.24), see Theorem 3.2.

Part ii) of the Theorem 1.3 is standard (see [8] for details) and it follows from (2.23), the Cauchy theory for the system (2.24) and (2.31).

7. Appendix

Proof of Proposition 2.1. We write the arguments below in a qualitative fashion in order to have a concise argument. However one easily sees that the argument below provides quantitative bounds which lead to (2.30).

We first read the information $u \in \dot{H}^2$. Using the equivariance property of u, we obtain

$$H_m u_1, H_m u_2 \in L^2, H_0 u_3 \in L^2. \tag{7.1}$$

Since $u_3^2 = 1 + u_1^2 + u_2^2$ it follows that

$$\frac{u_1 \partial_r u_1 + u_2 \partial_r u_2}{r} = \frac{u_3 \partial_r u_3}{r} \in L^2$$

and by invoking $\frac{1}{r}(\partial_r - \frac{m}{r})(u_1, u_2) \in L^2$, we obtain $\frac{u_1^2+u_2^2}{r^2} \in L^2$.

Since $D_r(v + iw) = 0$ it follows that

$$\partial_r \psi^{\pm} = \partial_r \left(\mathcal{W}^{\pm} \cdot (v + iw) \right) = (\partial_r \mathcal{W}^{\pm}) \cdot (v + iw)$$

where we recall that

$$\mathcal{W}^{\pm} = \partial_r u \pm \frac{1}{r} u \times \partial_\theta u \in T_u(\mathbb{S}^2)$$

From this we compute

$$\frac{1}{r}\mathcal{W}^{\pm} = \frac{1}{r}\left((\partial_r \mp \frac{m}{r})u_1, (\partial_r \mp \frac{m}{r})u_2, \partial_r u_3 \right) \pm m(\frac{(u_3-1)u_1}{r^2}, \frac{(u_3-1)u_2}{r^2}, -\frac{u_1^2+u_2^2}{r^2})$$

From (7.1), Lemma 1.4 and the fact that $\frac{u_1^2+u_2^2}{r^2} \in L^2$, it follows that $\frac{\mathcal{W}^{\pm}}{r} \in L^2$ if $m \geq 2$ and $\frac{\mathcal{W}^{+}}{r} \in L^2$ if $m = 1$. This implies the corresponding result for $\frac{\psi^{\pm}}{r}$.

A direct computation gives

$$\begin{aligned}\partial_r \mathcal{W}^{\pm} &= \partial_r^2 u \pm m\partial_r(\frac{u\times Ru}{r})\\ &= \partial_r^2 u \mp m\frac{\partial_r u_3\cdot u + u_3\cdot\partial_r u}{r} \mp m\frac{\overrightarrow{k}-u_3\cdot u}{r^2}\\ &= (\partial_r^2 \mp \frac{m}{r}\partial_r \pm \frac{m}{r^2})u \mp m\frac{u_3-1}{r}\partial_r u \mp m\frac{\overrightarrow{k}}{r^2} + f^{\pm}u\end{aligned}$$

where

$$f^{\pm} = \mp m\frac{\partial_r u_3}{r} \pm m\frac{u_3-1}{r^2}$$

We then continue with

$$\begin{aligned}\partial_r\psi^{\pm} &= \Big((\partial_r^2 \mp \frac{m}{r}\partial_r \pm \frac{m}{r^2})u_1, (\partial_r^2 \mp \frac{m}{r}\partial_r \pm \frac{m}{r^2})u_2, (\partial_r^2 \mp \frac{m}{r}\partial_r)u_3\Big)\cdot(v+iw)\\ &\mp m\frac{u_3-1}{r}\psi_1 \pm i\frac{u_3-1}{r^2}\psi_2\\ &= F^{\pm} \mp m\frac{u_3-1}{r}\frac{(m-1)\psi^+ + (m+1)\psi^-}{2}\end{aligned}$$

where $F^{\pm}\in L^2$ from (7.1). From the expression of $\mathcal{W}^{\pm}$ and the Sobolev embeddings it follows that $\|\psi^{\pm}\|_{L^4}\lesssim \|u\|_{\dot{H}^1\cap\dot{H}^2}$, hence $\|\frac{\psi_2}{r}\|_{L^4}\lesssim\|u\|_{\dot{H}^1\cap\dot{H}^2}$. Therefore $\frac{u_3-1}{r} = \frac{1}{u_3+1}\frac{u_1^2+u_2^2}{r} = \frac{1}{u_3+1}\frac{|\psi_2|^2}{m^2 r}\in L^4$, which implies that $\frac{u_3-1}{r}\frac{(m-1)\psi^+ + (m+1)\psi^-}{2}\in L^2$ and we conclude with $\partial_r\psi^{\pm}\in L^2$.

Hence we have just established that $R_{m\pm1}\psi^{\pm}\in H^1$. The procedure can be easily reversed, i.e. if $R_{m\pm1}\psi^{\pm}\in H^1$ then $u\in\dot{H}^2$, the details are left to the reader.

Next we transfer third derivatives of u to second derivatives for $\psi^{\pm}$ and vice-versa. From $\Delta u\in\dot{H}^1$, using the equivariance properties of u, it follows

$$H_m u_1, H_m u_2\in\dot{H}^1_e, \partial_r H_0 u_3\in L^2 \tag{7.2}$$

Using the above computation for $\partial_r\psi^+$, we have

$$\begin{aligned}H_{m\pm1}\psi^{\pm} &= (\partial_r + \frac{1}{r})F^{\pm} \mp \frac{u_3-1}{r}\frac{(m-1)\partial_r\psi^+ + (m+1)\partial_r\psi^-}{2} \mp \frac{\partial_r A_2}{r}\frac{(m-1)\psi^+ + (m+1)\psi^-}{2m}\\ &\quad - \frac{(m\pm1)^2}{r^2}\psi^{\pm}\end{aligned}$$

The derivative in $\partial_r F^{\pm}$, can fall on either term in the expression of $F^{\pm}$. From (7.2) and Lemma 1.4 it follows that in all cases $\partial_r(\partial_r^2\mp\frac{m}{r}\partial_r\pm\frac{m}{r^2})u_1, \partial_r(\partial_r^2\mp\frac{m}{r}\partial_r\pm\frac{m}{r^2})u_2, \partial_r(\partial_r^2\mp\frac{m}{r}\partial_r)u_3\in L^2$. Using Lemma 1.4, it follows that if $m\neq1$, then $\partial_r u_1, \partial_r u_2, \partial_r u_3\in\dot{H}^1_e\subset L^\infty$, hence by (2.11) implies that $\partial_r v\in L^\infty$, and similarly $\partial_r w\in L^\infty$. If $m=1$ then by the same Lemma 1.4, $(\partial_r^2\mp\frac{m}{r}\partial_r\pm\frac{m}{r^2})u_1, (\partial_r^2\mp\frac{m}{r}\partial_r\pm\frac{m}{r^2})u_2, (\partial_r^2\mp\frac{m}{r}\partial_r)u_3\in\dot{H}^1_e\subset L^\infty$ and since $\partial_r u\in L^2$, then by (2.11) $\partial_r v, \partial_r w\in L^2$. Hence we have completed the proof of the fact that $\partial_r F^{\pm}\in L^2$.

Next, if $m=1$, then from (7.2) and Lemma 1.4 it follows that $\frac{1}{r}F^{\pm}\in L^2$. The other linear term left is $4\frac{\psi^+}{r^2}$ (in the expression of $H_2\psi^+$), which is estimated from

$$\frac{\psi^+}{r^2} = \frac{1}{r^2}\Big((\partial_r - \frac{m}{r})u_1, (\partial_r-\frac{m}{r})u_2, \partial_r u_3\Big)\cdot(v+iw) - \frac{u_1^2+u_2^2-u_3(u_3-1)}{r^3}(v_3+iw_3)$$

Indeed, from Lemma 1.4 it follows that $(\partial_r - \frac{m}{r})u_1, (\partial_r - \frac{m}{r})u_2, \partial_r u_3 \in L^2$, and from $|\frac{u_1^2+u_2^2-u_3(u_3-1)}{r^3}(v_3+iw_3)| \lesssim \frac{|\psi_2|^3}{r^3}$ and the Sobolev embedding $\frac{\psi_2}{r} \in L^6$ it follows that all the linear terms in $H_{m\pm1}\psi^{\pm} \in L^2$.

If $m = 2$, then $\frac{1}{r}F^+ \in L^2$ on behalf of Lemma 1.4 and $\frac{\psi^+}{r^2} \in L^2$ is shown as above. On the other hand,

$$\frac{1}{r}F^- - \frac{1}{r^2}\psi^- = \frac{1}{r}\left((\partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2})u_1, (\partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2})u_2, (\partial_r^2 - \frac{1}{r}\partial_r)u_3\right) \cdot (v + iw)$$

belongs to L^2 on behalf of Lemma 1.4.

If $m \geq 3$, then it is a simple exercise to show that all the linear terms belong to L^2.

Moving on to the nonlinear terms in the expression of $H_{m\pm1}\psi^{\pm}$, we notice that $\psi^{\pm} \in L^4 \cap L^6$ by using the Sobolev embeddings. Using (2.18), it then follows that $\frac{\partial_r A_2}{r}\frac{(m-1)\psi^+ + (m+1)\psi^-}{2m} \in L^2$ by using the L^6 estimate for all terms involved.

For the last term we claim that $\partial_r\psi^{\pm} \in L^3$, from which $\frac{u_3-1}{r}\frac{(m-1)\partial_r\psi^+ + (m+1)\partial_r\psi^-}{2} \in L^2$ follows by using the L^6 estimate for $\frac{\psi_2}{r}$. The claim follows from the formula above for $\partial_r\psi^{\pm}$, the L^6 estimate for $\psi^{\pm}$ and the Sobolev embedding $(\partial_r^2 \mp \frac{m}{r}\partial_r \pm \frac{m}{r^2})u_1, (\partial_r^2 \mp \frac{m}{r}\partial_r \pm \frac{m}{r^2})u_2, (\partial_r^2 \mp \frac{m}{r}\partial_r)u_3 \in L^3$ (which can be derived using the Hankel calculus along the lines of the arguments in Lemma 1.4).

□

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