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Cox Rings and Partial Amplitude

by

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Abstract

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In algebraic geometry, we often study algebraic varieties by looking at their codimension one subvarieties, or divisors. In this thesis we explore the relationship between the global geometry of a variety X over $\mathbb{C}$ and the algebraic, geometric, and cohomological properties of divisors on X. Chapter 1 provides background for the results proved later in this thesis. There we give an introduction to divisors and their role in modern birational geometry, culminating in a brief overview of the minimal model program.

In chapter 2 we explore criteria for Totaro's notion of q-amplitude. A line bundle L on X is q-ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer m_0 such that $m \geq m_0$ implies $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL)) = 0$ for $i > q$. We show that a line bundle L on a complex projective scheme X is q-ample if and only if the restriction of L to its augmented base locus is q-ample. In particular, when X is a variety and L is big but fails to be q-ample, then there exists a codimension 1 subscheme D of X such that the restriction of L to D is not q-ample.

In chapter 3 we study the singularities of Cox rings. Let (X, Δ) be a log Fano pair, with Cox ring R. It is a theorem of Birkar, Cascini, Hacon and McKernan that R is finitely generated as a $\mathbb{C}$ algebra. We show that Spec R has log terminal singularities.

To my parents.

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Chapter 1

Introduction

The material in this chapter is intended as background for the results of chapters 2 and 3. The main prerequisite is a knowledge of algebraic geometry at the level of Hartshorne's book[17], but some knowledge of toric varieties, as in Fulton [13], is helpful for some of the examples. Section 1.1 covers the basics of the theory of divisors, culminating in the definition of the canonical divisor for singular varieties. In the first half of section 1.2 we go over various notions of positivity for divisors on algebraic varieties as in Lazarsfeld [24]. The second half introduces Totaro's definition of partial amplitude for divisors [38] as well as some results. Section 1.3 introduces Cox rings and Mori Dream Spaces as in Hu and Keel's paper [19]. The final section of this chapter is a short introduction to the minimal model program and birational geometry in higher dimensions. The main reference is the book of Kollár and Mori [22], though Matsuki's book [26] is a gentler introduction.

1.1 Divisors

We work over the complex numbers $\mathbb{C}$. Classically, a variety was thought of as the vanishing set of a number of homogeneous polynomials in a projective space $\mathbb{P}^n$. A more modern approach is to define varieties abstractly as a topological space with some data describing the local polynomial functions. To reconcile the two approaches, it is necessary to find a way of describing a map $f : X \to \mathbb{P}^n$ via information intrinsic to X. Assuming that X is irreducible and the image of f lies in no hyperplane, then for any hyperplane H, the pullback $f^{-1}(H \cap f(X))$ will be a codimension 1 subset of X. We can actually recover the map f up to coordinate change on $\mathbb{P}^n$ from the set of all the codimension 1 subsets arising from pullbacks, and this observation motivates the theory of divisors.

Let X be a normal variety. A Weil divisor is a formal $\mathbb{Z}$-linear combination of finitely many irreducible codimension 1 subvarieties of X. One way to construct a divisor is via the zeros and poles of a rational function: For every codimension 1 subvariety $Z \subset X$, there is a discrete valuation v_Z on the field of rational functions $K(X)$. To the rational function f we associate the divisor $\sum v_{Z_i}(f)Z_i$. We say that two divisors D and E are linearly equivalent if $D - E$ is the divisor of a rational function f. Weil divisors naturally form an abelian group under addition; after quotienting by linear equivalence we get the class group $\mathrm{Cl}(X)$. A divisor is called effective if all coefficients are nonnegative. A vector space of linearly equivalent effective divisors is called a linear system. The first example of a linear system of divisors is the set of hyperplane sections of an embedded projective variety.

Example 1.1. Consider the point $P = (0 : 1)$ on $\mathbb{P}^1 = \mathrm{Proj} k[s,t]$, and the divisor $D = 3P$. The rational functions on $\mathbb{P}^1$ are the rational functions $k(s/t)$ in the single variable s/t. Given any 3 points A, B, C of $\mathbb{P}^1$, there is a homogeneous cubic $f(s,t)$ vanishing on those points. The divisor of the rational function f/s^3 is $A+B+C-3P$ so the divisors $A+B+C$ and $3P$ are linearly equivalent.

Let $Q = (1 : 0)$. We can think of $3P$, $2P + Q$, $P + 2Q$, and $3Q$ as the vanishing of the homogeneous cubics s^3, s^2t, st^2, and t^3 respectively. These form a basis for the homogeneous cubic polynomials, and hence for the linear system of effective divisors of degree 3 on $\mathbb{P}^1$. If we want to recover a map $\mathbb{P}^1 \to \mathbb{P}^n$ such that these divisors are pullbacks of the hyperplane, we should create a map from a polynomial ring to $k[s,t]$ such that these cubic monomials are the images of the generators:

$$k[x_0, x_1, x_2, x_3] \to k[s,t];\ x_i \to s^{3-i}t^i$$

This induces a map $\mathbb{P}^1 \to \mathbb{P}^3$, recovering the twisted cubic from its hyperplane sections.

Weil and Cartier Divisors

In example 1.1 we took a linear system of divisors on a variety X and constructed a map $X \to \mathbb{P}^N$. In our construction we implicitly used the fact that these divisors were locally given by the vanishing of a single function. Such a divisor is called a Cartier divisor. The Cartier divisors form a subgroup of the Weil divisors, and on a projective variety, the group of Cartier divisors modulo linear equivalence is isomorphic to the Picard group of line bundles under the tensor product[28], so we will treat the two concepts interchangeably.

Another way to distinguish Cartier divisors is via their associated sheaves. Given a Weil divisor $\sum a_i D_i$, we can associate the sheaf $\mathcal{O}(\sum a_i D_i)$ which is locally given by the rational functions which have at worst poles of order a_i along the D_i. We say that a sheaf on X is invertible if it is locally isomorphic to the structure sheaf $\mathcal{O}_X$. A divisor D is Cartier if and only if the sheaf $\mathcal{O}(D)$ is invertible, just as invertible sheaves correspond to line bundles. Conversely we can get a divisor from an invertible sheaf by taking a rational section.

On a nonsingular variety, every Weil divisor is Cartier. In birational geometry, it often becomes necessary to introduce singular varieties, so we cannot count on every Weil divisor being Cartier. However, if we are attempting to study divisors via constructing a map to projective space it is enough for the Weil divisor to have a multiple which is Cartier. Such a divisor is called $\mathbb{Q}$-Cartier. A variety on which every Weil divisor is $\mathbb{Q}$-Cartier is said to have $\mathbb{Q}$-factorial singularities, or to be $\mathbb{Q}$-factorial.

Example 1.2. We will construct a variety along with a Weil divisor which is not $\mathbb{Q}$-Cartier. Let Z be the affine quadric cone cut out by the equation $xy - zw$ in $\mathbb{A}^4$. The variety Z is the cone over a smooth quadric surface Y in $\mathbb{P}^3$, and so Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The class group of Y is generated by the two rulings. Choose a ruling, and let l_1 and l_2 be two lines in that ruling. Let D_1 and D_2 be the Weil divisors on Z given by the cones over l_1 and l_2.

Because l_1 and l_2 do not intersect, D_1 and D_2 intersect only in the cone point P of X. Now suppose nD_1 were a Cartier divisor. Then in some neighborhood of P, a single equation would cut out nD_1. Thus in some neighborhood of P inside D_2, this equation must cut out a scheme supported on P. But P has codimension 2 inside D_2, so it cannot be set theoretically cut out by a single equation by Krull's Principle Ideal Theorem [11][Thm 10.1]. Hence D_1 is not $\mathbb{Q}$-Cartier.

Another way to see that the Z given above is not $\mathbb{Q}$-factorial is by using the language of toric varieties: A toric variety is $\mathbb{Q}$-factorial iff the associated fan is simplicial, and the fan of Z is 3 dimensional with one cone which has 4 rays, so it is not simplicial.

The Canonical Divisor

Given an arbitrary variety X, it can be difficult to find nontrivial divisors on X. If X is smooth, X comes automatically equipped with a locally free sheaf Ω_X, which is the sheaf of differentials on X or equivalently the cotangent bundle. The rank of Ω_X is $n = \dim X$, so $\omega_X = \bigwedge^n \Omega_X$ is an invertible sheaf. We write K_X for the associated divisor, which we call the canonical divisor.

The canonical divisor plays an important role in duality theory, but for our purposes its importance lies in the fact that the canonical divisor arises automatically from the data of a smooth variety. Under certain circumstances, we may use this some multiple of this divisor to get a canonical birational model of our variety embedded in projective space. In addition, the canonical divisor behaves well in families, so this canonical model can be used to construct moduli spaces. For example, the moduli space $\mathcal{M}_g$ of genus g curves is constructed by taking a quotient of the family of genus g curves embedded by $3K_C$[9].

We will also need a way of defining K_X for singular varieties. Assume X is normal. Then let $j : U \hookrightarrow X$ be the smooth locus of X, and the boundary of U in X has codimension at least 2. Since U is smooth, the canonical sheaf ω_U is invertible on U. We then define $\omega_X = j_*\omega_U$ as our canonical sheaf on X. One can show that when X is Cohen-Macaulay, ω_X enjoys the appropriate duality properties. Also, the sheaf ω_X corresponds to a Weil divisor K_X which is the strict transform of K_U. Thus when two varieties are isomorphic in codimension 2, their canonical divisors are strict transforms of each other. See [32] for details.

Often we would like to compute K_X by relating X to another variety on which we already know the canonical divisor. A formula that compares the canonical divisors on two varieties related by a map is called an adjunction formula. The first adjunction formula we will need is for a resolution of singularities. Assume X is normal and K_X is $\mathbb{Q}$-Cartier. Let $f : \tilde{X} \to X$

be a resolution of singularities. Then the exceptional locus of f on X has codimension at least 2, so away from the exceptional divisors of f the canonical divisors of X and $\tilde{X}$ agree. Thus for some rational numbers a_i we must have the following formula:

$$K_{\tilde{X}} = f^* K_X + \sum a_i E_i$$

Another adjunction formula is used to compute the canonical divisor of a divisor on X. Let X be a smooth variety, and let D be a normal divisor on X. Then $K_D = (K_X + D)|_D$. This formula can also be extended beyond the smooth case, the key requirement is that D must be a Weil divisor which is Cartier in codimension 2, otherwise a correction term is necessary[22, Rmk 5.47].

1.2 Positivity

Ample, Big, and Nef

In what follows, we assume that X is an irreducible projective variety and that D is a $\mathbb{Q}$-Cartier divisor unless otherwise specified. We motivated our definition of divisors by looking at hyperplane sections of embeddings into projective space. However, not every Cartier divisor D can arise this way; one immediate necessary condition on D is that D must be effective.

We say that D is very ample if D is the hyperplane section for an embedding of X into $\mathbb{P}^N$. Often we are concerned with the behavior of mD for large values of m, which lets us work with $\mathbb{Q}$-Cartier divisors, so we say that D is ample if some positive multiple of D is very ample.

More precisely, for all $m > 0$, there is a canonical map of sheaves on X given by $\phi_m : H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \to \mathcal{O}_X(mD)$. Let $U_m \subset X$ be the largest open set of X on which this map is surjective. Then ϕ_m induces a map from U_m to the projective space $\mathbb{P}(H^0(X, \mathcal{O}_X(mD))$. If for some m, ϕ_m induces an embedding of all of X into projective space, we say that D is ample. Likewise, if for some m, ϕ_m induces an embedding on some open set of X, we say that D is big. Thus the ample divisors of X correspond to embeddings, and the big divisors correspond to birational maps. In particular, every ample divisor is big.

On every smooth variety, we have the canonical divisor K_X. If X is a variety with K_X big, we say that X is of general type. If K_X is ample, X is called canonical, where is if $-K_X$ is ample, X is called a Fano variety.

While we defined amplitude by a geometric property, amplitude has important cohomological and numerical characterizations. Serre's criterion says that a divisor D on X is ample if and only if for any coherent sheaf $\mathcal{F}$ there is an integer m_0 (depending on $\mathcal{F}$), such that for $m > m_0$, $H^i(X, \mathcal{F} \otimes \mathcal{O}(mD)) = 0$ for all $i > 0$.

Numerical Criteria

The first numerical criterion for amplitude we will introduce is the Nakai-Moishezon criterion. Suppose we have $X \subset \mathbb{P}^{\Bbbk}$. Let Y be a dimension l subvariety of X, and call the hyperplane class H. Then the intersection number $H^l \cdot Y$ is the degree of Y, which is positive. Thus a necessary condition for D on X to be ample is that for every subvariety Y, $H^{\dim Y} \cdot Y > 0$. The Nakai-Moishezon criterion says that this is also a sufficient condition.

There is another, related criterion for amplitude known as Kleiman's criterion which only relies on knowing the intersection numbers of D with curves on X. Before stating Kleiman's criterion, it is helpful to have the notion of numerical equivalence of divisors. A divisor D on X is said to be numerically trivial if for every curve C on X, $D \cdot C = 0$. Two divisors are said to be numerically equivalent if their difference is numerically trivial. It turns out that whether or not a divisor is ample or big depends only on its numerical equivalence class, so it is convenient to with numerical equivalence classes of divisors.

To that end, let $\mathrm{Pic}(X)$ be the Picard group of X, and let $\mathrm{Num}(X)$ be the subgroup consisting of numerically trivial divisors. Then define the Neron-Severi group $N^1(X) = \mathrm{Pic}(X)/\mathrm{Num}(X)$. The key fact about the Neron-Severi group is that it is always a finitely generated free abelian group. The rank of $N^1(X)$ is called the Picard number, $\rho(X)$. We also define the Neron-Severi space, $N^1_{\mathbb{R}}(X) = N^1(X) \otimes \mathbb{R}$.

Each curve C in X defines a linear function on $N^1_{\mathbb{R}}(X)$. Since X was assumed to be projective, there is some divisor on X which is positive on every curve. Thus there is a nonempty closed convex cone consisting of divisors which have nonnegative intersection with every curve of X. The classes in this cone are called are called the nef classes, and the cone is called the nef cone, $\mathrm{Nef}(X)$. Kleiman's criterion says that the cone of ample divisors is the interior of $\mathrm{Nef}(X)$. Equivalently, the ample divisors are precisely the divisors which are positive on any limit of effective curve classes.

It is important to note that not every limit of effective curve classes is itself an effective curve class. Thus it is not enough in Kleiman's criterion to simply test whether D is positive on every curve. See [24][Example 1.5.2] for details.

One consequence of Kleiman's criterion is that if D' and D are numerically equivalent, D is ample iff D' is ample. The same is true of big divisors; whether or not a divisor is big depends only on its numerical equivalence class. Big divisors form an open, convex cone in $N^1_{\mathbb{R}}(X)$. The closure of the big cone is called the pseudoeffective cone, so called because effective divisor on X lies in the pseudoeffective cone.

Partial Amplitude

In the complex analytic setting, a holomorphic line bundle L on an n-dimensional complex manifold X is ample if and only if it admits hermitian metric whose curvature form has n positive eigenvalues everywhere. Andreotti and Grauert [1] showed that if L has $n-q$ positive eigenvalues everywhere, then for every coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL)) = 0$ for $i > q$ and sufficiently large m.

This suggests a generalization of the notion of an ample bundle, which Totaro [38] explored in a recent paper. For a natural number q, a line bundle is L is called q-ample if for every coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL)) = 0$ for $i > q$ and sufficiently large m. Totaro, building on work of Demailly, Peternell, and Schneider [10], Sommese [36], and others showed that whether or not L is q-ample depends only on its numerical class, and that the cone of such bundles is open in $N^1(X)$. For example, the 0-ample cone coincides with the ample cone, while by Serre duality the $(n-1)$-ample cone is the complement of the negative of the pseudoeffective cone.

Given that q-ample bundles have nice numerical properties, it makes sense to look for analogues of Kleiman's criterion. One might naively hope that since it is possible to check 0-amplitude on curves, we could check q-amplitude on the $(q+1)$-dimensional subvarieties. This is not the case; Totaro [38] has given an example of a smooth toric 3-fold with a line bundle L such that L is not in the closure of the 1-ample cone, but the restriction of L to every 2-dimensional subvariety is in the closure of the 1-ample cone of that subvariety. Thus a line bundle L may be in the closure of the q-ample cones of every proper subvariety of X, but not in the closure of the q-ample cone of X.

Given a variety X and a line bundle L which is not q-ample, when is there a proper subvariety $Z \subset X$ such that $L|_Z$ is not q-ample? It is helpful to have a notion of the set of points where L fails to be ample, which we call the augmented base locus of L. More concretely, the stable base locus of L is the algebraic set given by the intersection of the base loci of mL as m goes to infinity. The augmented base locus is the stable base locus of $L - \varepsilon H$, where H is any ample line bundle and ε is a suitably small positive real number. It is a theorem of Nakamaye [29] that the augmented base locus is well defined. The main theorem of chapter 2 is that restriction to the augmented base locus gives a criterion for q-amplitude for line bundles:

Theorem 1.3. *[4] Let X be complex projective scheme, and let L be a line bundle on X. Let Y be the scheme given by the augmented base locus of L with the unique scheme structure as a reduced closed subscheme of X. Then L is q-ample on X if and only if the restriction of L to Y is q-ample.*

As a corollary to Theorem 1.3, we obtain a Kleiman-type criterion for $(n-2)$-amplitude of big divisors when X is smooth.

Corollary 1.4. *[4] Let X be a nonsingular complex projective variety. A big line bundle L on X is $(n-2)$-ample iff the restriction of $-L$ to every irreducible codimension 1 subvariety is not pseudoeffective.*

1.3 Cox Rings

The Cox ring, or total coordinate ring, of a normal projective variety is a generalization of the homogeneous coordinate ring of projective space. On a normal projective variety X,

$\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$. Note that $\mathrm{Cox}(X)$ is a graded ring, graded by the abelian group $\mathrm{Cl}(X)$.

When X is projective space, the class group $\mathrm{Cl}(\mathbb{P}^n)$ is $\mathbb{Z}$, generated by the hyperplane section H. The degree m monomials in $x_0, \ldots, x_n$ form a basis for $H^0(\mathbb{P}^n, \mathcal{O}_X(mH))$. Thus the Cox ring of $\mathbb{P}^n$ is the homogeneous coordinate ring $k[x_0, \ldots x_n]$, with the standard grading by total degree.

In example 1.1, we looked at the cubic embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$. The homogeneous coordinate ring for the twisted cubic was given by $k[s^3, s^2t, st^2, t^3]$, which is a subring of $k[s,t]$. More generally, for any embedding of a variety X into $\mathbb{P}^n$, the homogeneous coordinate ring S consists of sections of multiples of some ample divisor on X, hence S is a subring of the Cox ring. In a similar way we can think of $\mathrm{Cox}(X)$ as somehow encoding all of the maps from X to a projective space.

Let us assume now that $\mathrm{Cl}(X)$ is a finitely generated free abelian group of rank r. Let $K(X)$ be the field of rational functions on X, and fix a $\mathbb{Z}$-basis D_i of effective divisors for $\mathrm{Cl}(X)$. Then we can alternatively define the Cox ring as the subring of $K(X)[t_i^{\pm}]$ generated by terms of the form $f \prod t_i^{\alpha_i}$ where $v_{D_i}(f) \geq \alpha_i$. This definition is often easier to work with.

Mori Dream Spaces

Cox [7] showed that the Cox ring of X is a polynomial ring iff X is a toric variety. Projective toric varieties correspond to lattice polytopes, and one can encode all their algebraic data combinatorially. For example, the effective and nef cones of a toric variety are always rational polyhedral cones. Hu and Keel [19] realized that many of these properties still hold when $\mathrm{Cox}(X)$ is finitely generated as a k-algebra. We say that X is a Mori Dream Space if X is a normal projective variety which is $\mathbb{Q}$-factorial, satisfies $\mathrm{Pic}_{\mathbb{Q}}(X) = N^1(X) \otimes \mathbb{Q}$ (i.e. any numerically trivial line bundle is torsion), and has finitely generated Cox ring. This definition differs from the one given by Hu and Keel, but they prove that the two are equivalent.

Mori Dream Spaces are so called because their birational geometry is particularly nice, and so it is easy to establish all the steps of the Mori program. One such aspect is that if X is a Mori Dream Space, then the effective cone of X can be divided into finitely many rational polyhedral chambers corresponding the distinct birational maps from X to a projective space. This is best illustrated by example:

Example 1.5. Let X be the blowup of $\mathbb{P}^2$ in two distinct points. This a toric variety, and hence a Mori Dream Space. The class group of X is a free abelian group of rank 3, generated by L, E_1, and E_2, where L is the pullback of the class of a line, and E_1 and E_2 are the exceptional divisors. The effective cone is generated by the -1 curves E_1, E_2, and the strict transform of the line connecting the two points, which has class $L - E_1 - E_2$.

In figure 1.5 we show a 2-dimensional slice of the effective cone of X with its chamber decomposition. There are 5 chambers, corresponding to different birational maps: The ample cone corresponds to the identity map on X, while the other chambers correspond to the maps that blow down E_1, E_2, both E_1 and E_2, or the line $L - E_1 - E_2$.

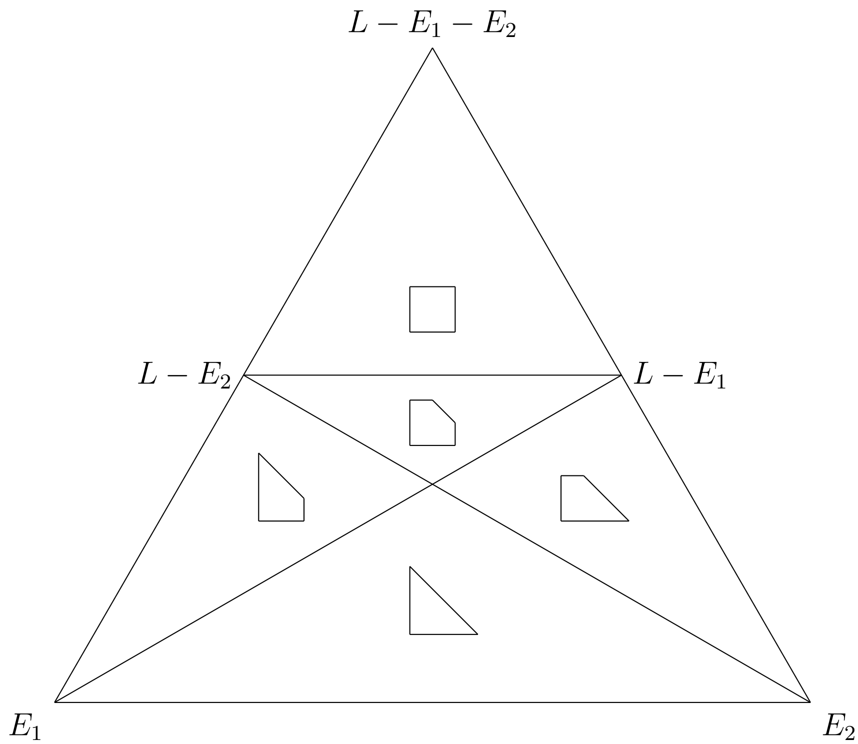


Figure 1.1: A slice of the chamber decomposition of the effective cone of $X = \mathrm{Bl}_2\mathbb{P}^2$. Each chamber corresponds to a birational model of X, and is labeled with the corresponding polytope. The ample cone is in the center.

Given a Cox ring, we would like to be able to recover the original variety X. In the case of projective space, the Cox ring is the $\mathbb{Z}$-graded polynomial ring, and we recover $\mathbb{P}^n$ by taking Proj of the ring $k[x_0, \dots x_n]$. The functor Proj is an example of a GIT quotient; equivalently one takes $\mathbb{A}^{n+1} = \mathrm{Spec} k[x_0, \dots x_n]$, and $\mathbb{P}^n$ is the GIT quotient of $\mathbb{A}^{n+1}$ by the $\mathbb{G}_m$ action induced by the $\mathbb{Z}$ grading.

Let us now assume that $\mathrm{Cl}(X)$ is freely generated by r line bundles. Then $\mathrm{Cox}(X)$ is $\mathbb{Z}^r$ graded, and X can be thought of as a GIT quotient of Spec $\mathrm{Cox}(X)$ by the r dimensional torus $\mathbb{G}_m^r$. In this case, there is more than one possible GIT quotient because there may be many possible linearizations of the group action. These different linearizations correspond to different birational models of X. In this way, Hu and Keel [19] use the theory of variation of GIT (VGIT) to explore the decomposition of the effective cone into chambers.

1.4 Higher Dimensional Birational Geometry

Surfaces

One of the main goals of modern algebraic geometry over the last half century has been the classification of algebraic varieties up to birational equivalence. This was known classically for curves, as every projective curve has exactly one smooth birational model. For surfaces one can always find new birational models by blowing up points. If one blows up a smooth surface at a point, the exceptional divisor is always a rational curve with self intersection -1 (henceforth called a -1 curve. In fact the converse is also true; Castelnuovo's contractibility criterion says that on a smooth surface any -1 curve can be blown down to a smooth point. So if we are looking for a representative X for a particular birational equivalence class, it is natural to require that X be minimal in the sense that X contains no -1 curves.

Let Y be a smooth surface, not necessarily minimal. Every time we blow down a curve, the Picard rank of Y decreases by 1, so given a smooth surface we must eventually reach a birational minimal surface by contracting curves. We say that Y is of general type if the canonical divisor K_Y is big. In the case where Y is of general type there is a unique minimal surface birational to Y. This is not generally true; for example the surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are both minimal and birational to each other.

Zariski showed that for a surface Y of general type, the canonical ring is finitely generated. Thus one can make a canonical model $Y_{\text{can}} = \text{Proj } R(K_Y)$. The canonical ring is a birational invariant, and so Y_{can} is a birational invariant. It is possible that the canonical model has singularities, but in the surface case there is a minimal resolution of singularities which is the minimal model.

The Minimal Model Program

The goal of the minimal model program was to extend the picture for surfaces to varieties of higher dimension. More explicitly, given a variety of general type X, one wants to show that the canonical ring $R(K_X)$ is finitely generated. The general approach involves making a series of birational modifications to X, eventually reaching X_{can}. This approach is due to Kawamata, Kollár, Mori, Reid, Shokurov, and many others, and eventually the finite generation of the canonical ring was shown in work of Birkar, Cascini, Hacon and McKernan[2].

The first step is to formulate the right analogue of the -1 curves in Castelnuovo's criterion. The primary insight is due to Mori. Consider a -1 curve E on a smooth surface X. By adjunction, $K_E = (K_X + E)|_E$. So $-2 = K_X \cdot E - 1$, and $K_X \cdot E = -1$, and every -1 curve is a curve which has negative intersection with the canonical divisor. When we blow down the -1 curves to reach a minimal model, we are eliminating all the curves with negative intersection with the canonical divisor, so on the minimal model, K_X is nef.

In higher dimensions, we would like to proceed by systematically contracting K_X negative curves. This is possible because of Mori's cone theorem, which says that the part of the cone of curves such that $K_X \cdot C > \epsilon$ is generated by finitely many rational curves. So we can always find a rational curve C to contract. At this point two things can happen. In the first case, the exceptional locus of the contraction has codimension 1, and we say we have a divisorial contraction. This is the only possibility in the surface case. The second, more problematic, case occurs when a the exceptional locus has codimension at least 2. This is called a small contraction.

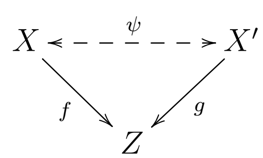
After contracting a curve, we may introduce singularities. However, we do not want to introduce arbitrarily singular varieties. One basic prerequisite for a variety Y appearing in the minimal model program is that we must be able to make sense of the intersection numbers $K_Y \cdot C$ for curves in Y. This means that some multiple of K_Y must be Cartier, i.e. the Weil divisor K_Y is $\mathbb{Q}$-Cartier. Such a Y is called $\mathbb{Q}$-Gorenstein.

Let $f : X \to Z$ be the contraction. When f is a divisorial contraction, Z is $\mathbb{Q}$-Gorenstein, and we will be able to continue running the minimal model program. Consider what happens when f is a small contraction. Suppose mK_Z is $\mathbb{Q}$-Cartier. Then because f is small, $f^*(mK_Z) = mK_X$. Since the extremal curve C was contracted to a point in Z, $f^*(mK_Z) \cdot C = 0$. But C was K_X negative, which is a contradiction. Thus for a small contraction f, K_Z is not $\mathbb{Q}$-Gorenstein, and so has unacceptable singularities.

Flips and Flops

The solution to this difficulty is a birational modification known as a flip. To illustrate, let us consider a slightly different situation: Suppose X is a Mori Dream Space, and D is a big divisor on X. Then the nef cone of X is rational polyhedral. Let C be a curve such that $C \cdot D < 0$, and intersection with C forms one of the walls of the nef cone of X. Now, choose a general divisor nef divisor D' such that $C \cdot D' = 0$. Then we have a map $f : X \to Z = \operatorname{Proj} \bigoplus H^0(X, \mathcal{O}(mD'))$. Assume that f is a small contraction.

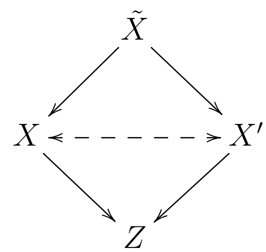
In this case the variety Z is not $\mathbb{Q}$-factorial, so we would like to find a variety X' which fits in the following diagram



where ψ is an isomorphism in codimension 1, and such that any curves contracted by g have positive intersection with the strict transform of D under ψ. We will call the diagram (or the variety X' by abuse of notation) a D-flip. If we think about the chamber decomposition of the effective cone $\mathrm{Eff}(X)$, it is straightforward to find such an X'. The wall of the ample cone $\mathrm{Amp}(X)$ corresponding to C forms the interface between $\mathrm{Amp}(X)$ and a second chamber Q. Choose a divisor F inside Q. Then we can take $X' = \operatorname{Proj} \bigoplus H^0(X, \mathcal{O}(mF))$, so that X' is the model corresponding to the chamber Q. It is a general fact about Mori Dream Spaces that the models for each chamber are $\mathbb{Q}$-factorial [19]. Any curve contracted by g corresponds to the wall that Q shares with $\mathrm{Amp}(X)$, but D lies on Q's side of this wall so any such curve is positive with respect to D. In effect, we have escaped the problem of non $\mathbb{Q}$-factorial singularities by 'jumping' a wall in the chamber decomposition.

In the context of the MMP, we are interested in the sign of $K_X \cdot C$. When $K_X \cdot C < 0$, we have a flip, when it is 0 the small modification is called a flop, and when it is positive we have an anti-flip. If a flip exists, it is unique. While it is easy to establish the existence and termination of flips for a Mori Dream Space, it is one of the difficult technical obstacles in the general theory.

The simplest examples of flips and flops occur in dimension 3. The most famous example is the Atiyah flop. Recall from Example 1.2 that the quadric cone Z cut out by $xy - zw$ in $\mathbb{A}^4$ is not $\mathbb{Q}$-factorial. One way to resolve the singularity at the cone point by blowing up. Let $\tilde{X}$ be this blowup, with exceptional divisor E. Then E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. However, $\tilde{X}$ is not a minimal resolution of Z, and in fact there are two distinct minimal resolutions of Z. For we can contract E by either ruling to get a smooth variety. Call the two resulting varieties X and X' (which are isomorphic). We have the following diagram, where X and X' are related by a flop.



All the morphisms in this picture are toric, so we can represent the flop using combinatorial pictures. See figure 1.4.

Another way to understand the Atiyah flop is as a variation of GIT quotients. The coordinate ring of Z can be thought of as the ring of invariants of $A = k[x_0, x_1, y_0, y_1]$ under the action of $\mathbb{G}_m$ via $t : (x_0, x_1, y_0, y_1) \to (tx_0, tx_1, t^{-1}y_0, t^{-1}y_1)$. This action induces a $\mathbb{Z}$ grading on A. Now Z is one possible quotient of Spec(A), but by varying the linearization we can get other quotients. Let A_+ and A_- be the subrings of A with grading ≥ 0 and ≤ 0 respectively. Then the small resolutions X and X' are given by $\text{Proj}A_+$ and $\text{Proj}A_-$. See Reid's talk on flips [33] for details.

Singularities

For a surface S, we were able to find a smooth model with K_S nef. We cannot do this in higher dimensions. Let Y be the cone over the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ by quadrics. Then the cone point of Y is a singularity, let $\pi : \tilde{Y} \to Y$ be the blowup, with exceptional divisor E. Now $E \cong \mathbb{P}^2$ and $E|_E = \mathcal{O}(2)$, so by adjunction $\mathcal{O}(-3) = (K_{\tilde{Y}} + E)|_E$, so $K_{\tilde{(Y)}} = \pi^*(K_Y) + \frac{1}{2}E$.

Now let L be a line in E. The map π contracts L, so $\pi^*(K_Y) \cdot L = 0$. Since $E|_E = \mathcal{O}(-2)$, we have $E \cdot L = -2$. Hence the preceding calculations show that $K_{\tilde{Y}} \cdot L = -1$. So if X is

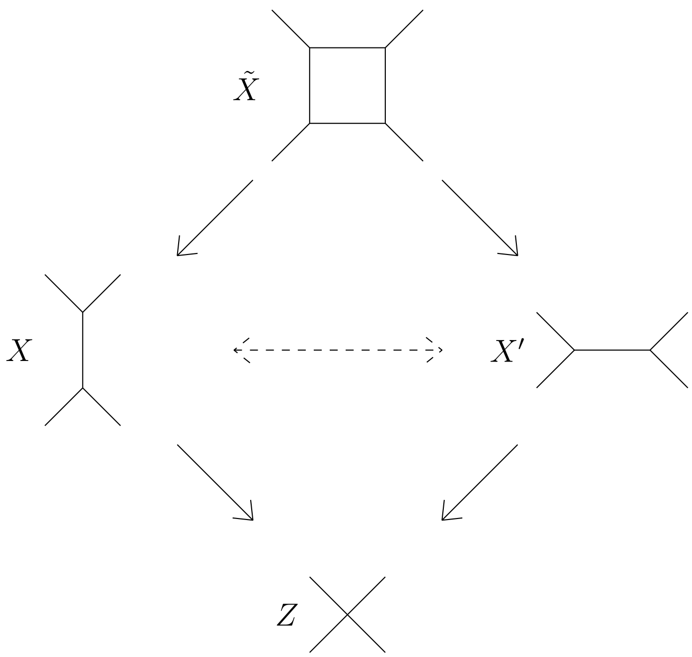


Figure 1.2: Toric picture of the Atiyah flop. The pictures shown are dual to the fans of each toric variety.

any variety that has an analytic neighborhood isomorphic to an analytic neighborhood of E in $\tilde{Y}$, then the image of E will be contracted to a singularity in X when we run the MMP. In particular X has no smooth model with nef canonical divisor.

For an explicit example, let A be an abelian 3-fold. Then A has an involution, corresponding to taking the inverse. Let X be the quotient of A by this involution. The quotient X is smooth away from the images of the 2^6 points of order 2, and each image is a singularity analytically isomorphic to the cone over the Veronese. The canonical divisor K_X is nef, but there is no smooth model of X with nef canonical divisor[39][16.17].

Thus we must be willing to allow some kinds of singularities. We will first require that any variety X appearing in the MMP is normal and $\mathbb{Q}$-Gorenstein, that is that K_X is $\mathbb{Q}$-Cartier. This is so that we can define the intersection number $C \cdot K_X$ for any curve C.

Since X is normal, X is smooth away from a codimension 2 locus. If we take any resolution of singularities $f : \tilde{X} \to X$, we can relate the canonical divisors of X and $\tilde{X}$ using adjunction:

$$K_{\tilde{X}} = f^* K_X + \sum a_i E_i$$

The a_i are rational numbers called the discrepancies for the exceptional divisors E_i. If K_X is Cartier, then the a_i are integers.

We say that X is terminal, canonical, log terminal, or log canonical, if X is normal and $\mathbb{Q}$-Gorenstein and for some resolution of X the discrepancies a_i are > 0, ≥ 0, > -1, or

≥ -1 respectively. It can be shown that these definitions are independent of the resolution chosen. Terminal singularities are so called because they are the singularities that must be included for the MMP to terminate. Canonical singularities are the singularities that appear in canonical models.

Any surface with terminal singularities is smooth, and surface canonical singularities are the Du Val singularities. As we have seen, there are singular 3-folds with terminal singularities. If Y is the cone over the quadratic Veronese embedding of $\mathbb{P}^2$, we have seen that the Y has a resolution $\pi : \tilde{Y} \to Y$ such that $K_{\tilde{Y}} = \pi^*(K_Y) + \frac{1}{2}E$. Thus the cone point of Y gives an example of a terminal 3-fold singularity. Carrying out the MMP may introduce singularities, but if one starts with terminal singularities and carries out the MMP at each step the variety will still have terminal singularities. In particular, discrepancies do not get worse under a flip or a flop.

Pairs

We often expand the MMP to pairs (X, Δ), where X is a normal variety and Δ is a positive $\mathbb{Q}$-linear combination of irreducible divisors. One such reason to do this is that viewing Δ as a boundary lets us study varieties that are not necessarily complete. Given a pair (X, Δ), we define the log canonical divisor $K_X + \Delta$, which plays the role of the canonical divisor in the MMP for pairs. In this context, we now require that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, but impose no such requirement on K_X.

Similarly, one defines singularities for pairs. Let (X, Δ) be a pair. We say that $f : \tilde{X} \to X$ is a log resolution of (X, Δ) if $\tilde{X}$ is smooth and the exceptional divisors and the strict transform of Δ together form a simple normal crossing divisor. Now we can define log discrepancies for a resolution. Let (X, Δ) be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and $f : \tilde{X} \to X$ a log resolution. Set Γ to be the sum of all exceptional divisors of f along with the strict transforms of the components of Δ. By adjunction, we have

$$K_{\tilde{X}} + \Gamma = f^*(K_X + \Delta) + \sum a_i E_i$$

where the E_i range over the components of Γ. The a_i are called the log discrepancies for f. We say that (X, Δ) is Kawamata log terminal, or klt, if all $a_i > 0$, and that (X, Δ) is log canonical, or lc, if the $a_i \geq 0$. These definitions do not depend on the log resolution chosen. Note that if (X, Δ) is klt, then the coefficients of Δ are all less than 1.

With these concepts, we can expand our definition of Fano varieties. We say that (X, Δ) is a log Fano pair if (X, Δ) is klt and $-(K_X + \Delta)$ is ample. If $(X, 0)$ is a log Fano pair, we say that X is a strict Fano variety. One of the nice things about this definition is that while not every toric variety is Fano, it is true that any toric variety is log Fano for some choice of Δ. Also, Birkar, Cascini, Hacon and McKernan [2] have proved that if (X, Δ) is log Fano, then $\mathrm{Cox}(X)$ is finitely generated. Thus log Fano pairs give many examples of Mori Dream Spaces. The primary result of chapter 3 investigates the singularities of the Cox ring of a log Fano variety:

Theorem 1.6. *[5, 15]*

Let (X, Δ) be a $\mathbb{Q}$-factorial log Fano pair over $\mathbb{C}$, and let $D_1, \ldots D_r$ be a basis for the torsion free part of Cl(X).

1. *The ring $\bigoplus_{\mathbb{Z}^r} H^0(X, \mathcal{O}(\sum a_i D_i))$ is normal with log terminal singularities, and in particular is Cohen-Macaulay.*

2. *If X is a smooth complete strict Fano variety, then Cox(X) is Gorenstein with canonical singularities.*

Chapter 2

Big q-Ample Line Bundles

2.1 Background

For ample divisors, there is a clear relationship between geometric, cohomological, numerical properties. For example, the Kleiman criterion tells us that 0-amplitude is determined by the restriction of L to the irreducible curves on X. We would like to develop similar relationships between these properties for q-ample divisors. Indeed, one gets at least some information about the q-ample cone by looking at restrictions to $(q+1)$-dimensional subvarieties.

However, Totaro [38] has given an example of a smooth toric 3-fold with a line bundle L which is not in the closure of the 1-ample cone, but the restriction of L to every 2-dimensional subvariety is in the closure of the 1-ample cone of each subvariety. For completeness, we include this example in section 2.5. The example shows that the most direct generalization of Kleiman's criterion does not hold for even the first open case: the 1-ample cone of a 3-fold. However, we still expect some relationship between the cohomological and geometric properties of line bundles. For example, Küronya [23] has proven an analogue of the Fujita vanishing theorem for line bundles whose augmented base locus has dimension at most q.

In this chapter we will prove Theorem 1.3 and Corollary 1.4. Thus one can in fact test q-amplitude on proper subschemes in the case where L is a big line bundle on a projective variety X. In particular, we show that if L is a big line bundle which is not q-ample, and D is the locus of vanishing of a negative twist of L, then the restriction of L to D is not q-ample either.

S. Matsumura has shown in [27] that a line bundle admits a hermitian metric whose curvature form has all but q eigenvalues positive at every point iff it admits such a metric when restricted to the augmented base locus. A line bundle with such a metric is q-ample, but Ottem [30] has recently found projective varieties with q-ample bundles that are not q-positive.

When X is a 3-fold, a big line bundle L is 1-ample iff its dual is not in the pseudoeffective cone when restricted to any surface contained in X. Since a big line bundle on a 3-fold is always 2-ample, our results give a complete description of the intersection of the q-ample cones with the big cone of a 3-fold in terms of restriction to subvarieties.

In the final section we examine possible geometric criteria for an effective line bundle to be q-ample. The first case is that on an n-dimensional Cohen Macaulay variety, any line bundle which admits a disconnected section must fail to be $(n-2)$-ample. This helps to explain some features of Totaro's example, and may lead to more general criteria for q-amplitude.

2.2 The Restriction Theorem

In this section we prove that a line bundle L which fails to be q-ample is still not q-ample when restricted to any section of $L - H$, where H is any ample line bundle.

Theorem 2.1. *Let X be a reduced projective scheme over $\mathbb{C}$. Suppose L is a line bundle on X which is not q-ample on X, and let L' be a line bundle with a nonzero section such that $\mathcal{O}(\alpha L - \beta L')$ is ample for some positive integers α, β. Let D be the subscheme of X given by the vanishing of some nonzero section of L'. Then $L|_D$ is not q-ample on D.*

Before proving Theorem 2.1, we will need a lemma:

Lemma 2.2. *Let X be a projective scheme over $\mathbb{C}$. Fix an ample line bundle H on X. Suppose L is a q-ample line bundle on X for some $q \geq 0$. Then for every coherent sheaf $\mathcal{F}$ on X there exist integers a_0 and b_0 such that given $a, b \geq 0$, $H^i(X, \mathcal{F} \otimes \mathcal{O}(aL + bH)) = 0$ for $i > q$ whenever $a \geq a_0$ or $b \geq b_0$.*

Proof. Every coherent sheaf has a possibly infinite resolution by bundles of the form $\bigoplus \mathcal{O}(-dH)$. By [24, Appendix B], it thus suffices to check for finitely many sheaves of the form $\mathcal{O}(-dH)$. The proof follows by induction on the dimension of X. In the base case, dimension 0, the lemma follows because for every coherent sheaf the groups H^i vanish for $i > 0$.

Since every ample line bundle has some multiple which is very ample it suffices to prove the lemma when H is very ample. It is also enough to find the constants a_0 and b_0 such that the cohomology vanishes for a fixed $i > q$. Assume H is very ample, and fix an $i > q$. Now, suppose X has dimension n and the lemma is true for projective schemes of dimension $n-1$.

Because L is q-ample, we know there exists a_1 such that $H^i(X, \mathcal{O}(aL - dH)) = 0$ whenever $a \geq a_1$. Let D be a hyperplane section under the embedding given by H. By the inductive hypothesis, there exists a_2 such that $H^i(D, \mathcal{O}(aL + (b-d)H)) = 0$ whenever $a \geq a_2$ and $b \geq 0$. By abuse of notation, we use L to refer to both the line bundle on X and its pullback to D. The projection formula [17, II, Ex 5.1] along with the preservation of cohomology under push forward by a closed immersion shows that this will not change the cohomology. Thus we have an exact sequence in cohomology:

$$\ldots \to H^i(X, \mathcal{O}(aL + (b-d)H)) \to H^i(X, \mathcal{O}(aL + (b+1-d)H)) \to H^i(D, \mathcal{O}(aL + (b+1-d)H)_{|D}) \to \ldots$$

Set $a_0 = \max\{a_1, a_2\}$. Then for $a \geq a_0$, we know that $H^i(D, \mathcal{O}((aL + (b+1-d)H))_{|D}) = 0$ so by induction on b we know that $H^i(X, \mathcal{O}(aL + (b-d)H))$ vanishes for all $b > 0$. To find

b_0, we know that for each $a < a_0$, there exists b' such that the cohomology vanishes for $b > b'$ since H is ample. Take b_0 as the maximum of all the b'. □

Proof of Theorem 2.1. L is q-ample iff αL is, so we may assume $\alpha = 1$. Likewise Totaro [38, Cor 7.2] shows that L is q-ample on a scheme X iff its restriction to the reduced scheme is q-ample, so we may assume $\beta = 1$. At this point we are assuming that $H = L - L'$ is ample.

We recall another result of Totaro [38, Thm 7.1]: Given H ample there exists a global constant C such that L is q-ample iff there exists N such that $H^i(X, \mathcal{O}(NL - jH)) = 0$ for all $i > q$, $1 \leq j \leq C$. Let us assume L is $(q+1)$-ample but not q-ample. Since L is not q-ample, for all N one of the above groups is nonzero. Since L is $(q+1)$-ample that group must have $i = q+1$ for large enough N. Now, H is ample so for sufficiently large e, $H^i(X, \mathcal{O}((e-j)H)) = 0$ for $i > q$, $1 \leq j \leq C$.

Likewise, for all sufficiently large $e \geq 1$, we know that $H^{q+1}(X, \mathcal{O}((e-j)H)) = 0$, and that for some $1 \leq j \leq C$, $H^{q+1}(X, \mathcal{O}(eL - jH)) \neq 0$. Since $\mathcal{O}(L') = \mathcal{O}(L - H)$ there exist j and k such that $1 \leq j \leq C$, and $1 \leq k \leq e$ such that $H^{q+1}(X, \mathcal{O}((e-j)H + (k-1)L')) = 0$ and $H^{q+1}(X, \mathcal{O}((e-j)H + kL')) \neq 0$. To simplify notation we set $l = e - j$.

Consider the exact sequence:

$$0 \to \mathcal{F} \to \mathcal{O}_X(-L') \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

The section defining D may be given by a section which is not regular when X is reducible and so the sheaf $\mathcal{F}$ may be nonzero. Now write $\mathcal{G} = \operatorname{coker}(\mathcal{F} \to \mathcal{O}_X(-L')) = \ker(\mathcal{O}_X \to \mathcal{O}_D)$. After twisting by $\mathcal{O}(lH + kL')$ we have two resulting long exact sequences in cohomology. The first is

$$\ldots \to H^{q+1}(X, \mathcal{O}(lH+(k-1)L')) \to H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH+kL')) \to H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}(lH+kL')) \ldots$$

Since $k \leq l + j$ and $\mathcal{O}(H + L') = \mathcal{O}(L)$, for sufficiently large e, $H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}(lH + kL')) = H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}((l-k)H + kL)) = 0$, by Lemma 2.2 . Thus $H^{q+1}(X, \mathcal{O}(lH + (k-1)L')) = 0$ implies $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) = 0$.

The second long exact sequence is given by

$$\ldots \to H^i(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) \to H^i(X, \mathcal{O}(lH + kL')) \to H^i(D, \mathcal{O}(lH + kL')_{|D}) \to \ldots$$

The group $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) = 0$, and $H^{q+1}(X, \mathcal{O}(lH + kL')) \neq 0$, so we see that $H^{q+1}(D, \mathcal{O}(lH + kL')_{|D}) \neq 0$. $\mathcal{O}(lH + kD) = \mathcal{O}((l-k)H + kL)$, which has the form $\mathcal{O}(aL + (b-d)H)$, where $d = C$, $a, b \geq 0$, and $a + b \geq e$. Since we could choose e arbitrarily large, by Lemma 2.2 L is not q-ample when restricted to D.

□

In the case where X is irreducible, every nonzero section of a line bundle is regular, and we get the following corollary:

Corollary 2.3. *If X is a complex projective variety (irreducible and reduced) and L is a big line bundle which is not q-ample, there exists a codimension 1 subscheme of X on which L is not q-ample.*

Proof. The cone of big line bundles on a projective variety is open, so we may pick L' also big, so some large multiple of L' has a nonzero section whose vanishing is an effective Cartier divisor. □

One subtlety of the Kleiman criterion for ample divisors is that is possible to have a divisor class which is positive on every irreducible curve but is not ample. One such example is due to Mumford and can be found in [24, Example 1.5.2]. In particular this shows that in Corollary 2.3 the hypothesis 'big' cannot be replaced by 'pseudoeffective', already when $q = 0$.

2.3 Augmented Base Loci

Here we review the concepts of stable and augmented base loci. Let L be a Cartier divisor on a variety X. Write $\mathrm{Bs}(|L|)$ for the base locus of the full linear series of L. It is also helpful to have a notion of the base locus for large multiples of L, as well as for small perturbations by the inverse of an ample line bundle.

Definition 2.4. [24, Def 2.1.20] The stable base locus of L is the algebraic set

$$\mathbf{B}(L) = \bigcap_{m \geq 1} \mathrm{Bs}(|mL|).$$

There exists an integer m_0 such that $\mathbf{B}(L) = \mathrm{Bs}(|km_0L|)$ for $k >> 0$ [24, Prop 2.1.20].

Definition 2.5. [25, Def 10.3.2] The augmented base locus of L, denoted by $\mathbf{B}_+(L)$, is the closed algebraic set given by $\mathbf{B}(L - \epsilon\mathcal{H})$, for any ample $\mathcal{H}$, and sufficiently small $\epsilon > 0$.

It is a theorem of Nakamaye [29] that the augmented base locus is well defined. Note that stable and augmented base loci are defined as algebraic sets, not as schemes.

Geometric properties of $\mathbf{B}_+(L)$ reveal information about how much L fails to be ample. For example, $\mathbf{B}_+(L)$ is empty if and only if L is ample. More generally, Küronya has proved in [23] a Fujita-vanishing type result for the cohomology groups H^i where $i > \dim \mathbf{B}_+(L)$.

Theorem 2.6. *[23, Thm C] Let X be a projective scheme, L a Cartier divisor, and $\mathcal{F}$ a coherent sheaf on X. Then there exists m_0 such that $m \geq m_0$ implies $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL + D)) = 0$ for all $i >$dim $\mathbf{B}_+(L)$ and any nef divisor D.*

In particular, Küronya's theorem implies that L is q-ample, for all q at least as big as the dimension of $\mathbf{B}_+(L)$. We show that in fact L is q-ample if and only if the restriction of L to $\mathbf{B}_+(L)$ is q-ample:

Proof of Theorem 1.3. Certainly if L is q-ample on X it must be q-ample on Y. For the converse, we apply 2.1 inductively. Suppose L is not q-ample. We may assume all schemes are reduced by [38, Cor 7.2]. Choose an ample divisor H, and choose a and b such that $L' = aL - bH$ satisfies $\mathrm{Bs}(|L'|) = \mathbf{B}_+(L)$.

Suppose there is a point $x \in X$ which is not contained in Y. Then since Y is the base locus of L', there is a section of L' which does not vanish at x, and let X' be the vanishing of this section. Then by 2.1 L is not q-ample on X'. The process only terminates when $X' = Y$, and it must terminate because X was a noetherian topological space. □

2.4 Towards a Numerical Criterion for q-ample Line Bundles

The cone of ample line bundles in $N^1(X)$ has a nice description in terms of the geometry of curves in X due to a theorem of Kleiman. (See for example [24, 1.4.23].)

Theorem 2.7. *(Kleiman's criterion) Let Nef(X) be the cone of nef divisors. Nef(X) is a closed cone, and the cone of ample divisors is the interior of Nef(X).*

One would like similar criteria to test the q-amplitude of L. A duality argument gives a criterion for the $(n-1)$-ample cone:

Theorem 2.8. *[38, Thm 9.1] On a variety X, the $(n-1)$-ample cone is the negative of the complement of the pseudoeffective cone.*

The Kleiman criterion says that L is in the closure of the ample cone iff $-L$ is not big on any curve. Theorem 2.8 says that L is in the closure of the $(n-1)$-ample cone iff $-L$ is not big on X, which is the only subvariety of X having dimension n. Thus in some sense, both criteria say that to test if a divisor is in the closure of the q-ample cone it suffices to show that its dual is not in the big cone of any subvarieties of dimension $q+1$. While one would hope that such a criterion holds for all q, we will see in 2.5 an example of Totaro which shows this fails for even the case of 3-folds. However, if we also require the divisor to be big, we may combine Corollary 2.3 with a modification of the duality argument to yield Corollary 1.4.

Proof of Corollary 1.4. Certainly if L is $(n-2)$-ample on X it is $(n-2)$-ample on every subvariety. For the other direction, using 2.3 if L fails to be $(n-2)$-ample we have an effective Cartier divisor D on which L is not $(n-2)$-ample. By [38, Cor 7.2] we may assume D is reduced. Since X is nonsingular, D is a still a Cartier divisor, and the dualizing sheaf $\mathcal{K}_D$ is a line bundle given by $\mathcal{K}_D = (\mathcal{K}_X \otimes \mathcal{O}(D))|_D$.

Let D_i be the components of D, and let $f : \coprod D_i \to D$ be the canonical map. Then the map $\mathcal{O}_D \to f_* \bigoplus \mathcal{O}_{D_i}$ is injective, and so yields an injective map $H^0(D, J) \to \bigoplus H^0(D_i, J|_{D_i})$ for any line bundle J on D. Suppose $-L$ is not pseudoeffective on any of the D_i. Then for any line bundle J and sufficiently large m depending on J, $H^0(D_i, \mathcal{O}(J - mL)|_{D_i}) = 0$, so $H^0(D, \mathcal{O}(J - mL)) = 0$.

It follows by duality that $H^{n-1}(D, \mathcal{K}_D \otimes \mathcal{O}(mL - J)) = 0$ for any line bundle J and sufficiently large m. But by [38, Thm 7.1] this means L is $(n-2)$-ample on D, a contradiction. □

2.5 Totaro's Example

In this section we reproduce Totaro's example from [38] of a line bundle L on a smooth toric Fano 3-fold X such that L is not in the closure of the 1-ample cone of X, but L is in the closure of the 1-ample cone of every proper subvariety of X. Our goal is investigate what sort of additional obstacles beyond the numerical criterion must be considered to say when an effective bundle is q-ample.

Definition 2.9. A line bundle L on X is called q-nef if for every dimension $q+1$ subvariety $V \subset X$ the restriction of $-L$ to V is not big.

The q-nef cone is a closed cone in $N^1(X)$. By Theorem 2.8, a q-ample bundle must be q-nef. Also, when $q = 0$ or $q = n-1$, the q-ample cone is the interior of the q-nef cone. Let X be the projectivization of the rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is a smooth toric Fano 3-fold. One can show that the corresponding fan Σ in $\mathbb{Z}^3 \otimes \mathbb{R}$ has rays

$$f_1 = (0,0,-1), f_2 = (0,0,1), f_3 = (1,0,1), f_4 = (0,1,-1), f_5 = (-1,0,0), f_6 = (0,-1,0)$$

The two dimensional cones are given by

$$(13), (14), (15), (16), (23), (24), (25), (26), (34), (36), (45), (46)$$

The maximal cones are

$$(134), (136), (145), (146), (234), (236), (245), (246)$$

Line bundles on X are given by piecewise linear functions on Σ which are integral linear functions on each cone. Let $\langle \Sigma(1) \rangle$ be the $\mathbb{R}$ vector space spanned by the rays of Σ. Since X is simplicial we have an identification

$$\mathrm{Pic} \otimes \mathbb{R} \cong \langle \Sigma(1) \rangle^* / (\mathbb{Z}^3 \otimes \mathbb{R})^*$$

Write F_i for the function which sends f_i to 1 and $f_{j, j \neq i}$ to 0. Then we can identify F_i with the divisor which is the closure of the torus orbit corresponding to the ray f_i. Let $L = 3F_1 + 3F_2 - F_3 - F_4 - F_5 - F_6$. Then L is not in the closure of the 1-ample cone, but L is 1-nef.

To see that L is not in the closure of the 1-ample cone it suffices to show that a positive twist of L is not 1-ample. For example, take $H = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$. Then for any sufficiently small rational $\lambda > 0$, a large integral multiple of $L + \lambda H$ has a nonvanishing

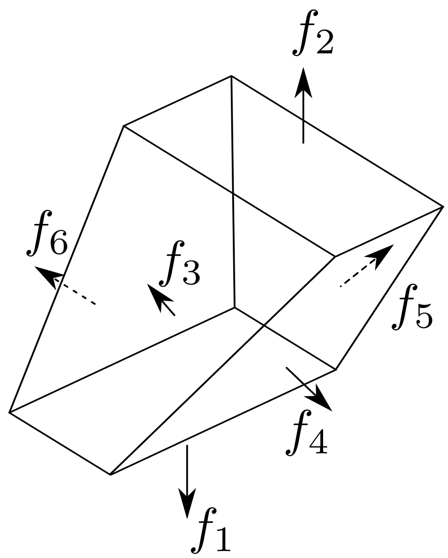


Figure 2.1: The dual polytope to Σ

H^2. This follows from the formula for cohomology of line bundles given in [13, p. 74], along with the fact that the rays with negative coefficients form a nontrivial 1-cycle in $|\Sigma| \setminus \{0\}$.

The 1-nef cone of a toric variety consists of divisors whose restriction to each torus invariant surface is not the negative of a big divisor. It can be shown that L is 1-nef by restricting to each F_i. As an example we explicitly work out the restriction of L to F_1.

The divisor F_1 is a toric variety and its fan is given by $\Sigma_{F_1} = \mathrm{Star}(f_1)/\langle f_1 \rangle$. Denote the image of the ray f_i in Σ_{F_1} by f_i'. This fan is isomorphic to the fan of $\mathbb{P}^1 \times \mathbb{P}^1$. The most straightforward way of restricting L to F_1 is to choose a linearly equivalent representative in $\langle \Sigma(1) \rangle^*$ which vanishes on f_1. Take $L' = 6F_2 - 4F_3 + 2F_4 - F_5 - F_6$. Then the resulting piecewise linear function ψ on Σ_{F_1} has

$$\psi(f_3') = -4, \psi(f_4') = 2, \psi(f_5') = -1, \psi(f_6') = -1$$

This corresponds to the divisor $\mathcal{O}(1, -3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which is not the negative of a big divisor. A similar calculation for the other F_i shows that L is actually 1-nef.

Figure 2.2 shows a slice of $N^1(X)$, where the effective cone is shaded. The numbers in each region are the largest q such that a line bundle in the interior of that region is q-ample.

2.6 Further Questions

Let X be a variety and L a line bundle on X. When L is not big, $\mathbf{B}_+(L)$ is all of X, and so yields no new information about whether L is q-ample. However, when L is effective, we may hope to see other geometric consequences of q-amplitude reflected in the geometry of a

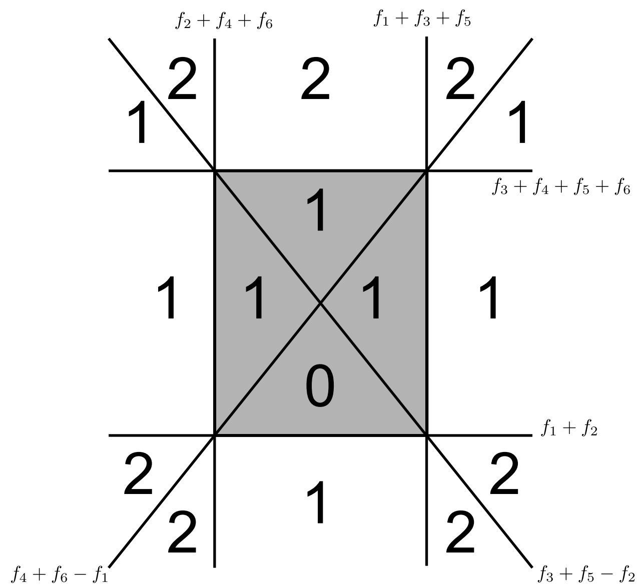


Figure 2.2: Chambers in $N^1(X)$. The effective cone is shaded, and each chamber is marked with the smallest q such that a line bundle in the interior of the chamber is q-ample. The planes are labelled by the corresponding linear dependence among rays in $\Sigma(1)$.

section. In the example in section 2.5, the divisor $F_1 + F_2$ is not 1-ample, and this cannot be seen via any sort of restriction to proper subvarieties of X. However, $F_1 + F_2$ cannot be 1-ample because it admits a section with disconnected zero set.

Proposition 2.10. *Let X be a normal irreducible Cohen-Macaulay variety of dimension n. If L is a line bundle on X which admits a global section with disconnected zero set, then L is not $(n-2)$-ample.*

Proof. Let D be the vanishing of section of L, which is disconnected. Then we can take the infinitesimal thickening mD as the vanishing of a section of $\mathcal{O}(mL)$. Consider the restriction exact sequence:

$$0 \to \mathcal{O}(-mL) \to \mathcal{O}_X \to \mathcal{O}_{mD} \to 0$$

Since X is connected $H^0(X, \mathcal{O}_X)$ is one dimensional, but mD is not connected so $H^0(mD, \mathcal{O}_{mD})$ is at least two dimensional. Thus the associated map $H^0(X, \mathcal{O}_X) \to H^0(mD, \mathcal{O}_{mD})$ is not surjective and so taking the associated long exact sequence we see that $H^1(X, \mathcal{O}(-mL))$ is nonzero. Let $\mathcal{K}_X$ be the dualizing sheaf on X. By Serre duality, $H^{n-1}(X, \mathcal{K}_X \otimes \mathcal{O}(mL))$ is nonvanishing for all m so L is not $(n-2)$-ample. □

Question 2.11. Given a smooth variety X with an effective line bundle L which is $(n-2)$-nef and such that there is a neighborhood U in $N^1(X)$ that no line bundle in U admits a section with disconnected vanishing set, must L be $(n-2)$-ample?

One possible way to interpret Proposition 2.10 is as a sort of Lefschetz hyperplane theorem for $(n-2)$-ample divisors. Bott has proved the following generalization of the Lefschetz hyperplane theorem:

Theorem 2.12. *[3, Thm III] Let X be a smooth variety of dimension n, and L a line bundle which admits a Hermitian metric whose curvature form has at least $n-q$ positive eigenvalues (counted with multiplicity) at every point. Suppose also that Y is the vanishing set of a section of L. Then X is obtained from Y as a topological space by attaching cells of dimension at least $n-q$.*

A line bundle is called q-positive if it admits such a Hermitian metric. If Y has 'too much' homology in dimension $n-q-2$ it cannot be a section of a q-positive line bundle. It is a well known theorem of Andreotti and Grauert [1] that a q-positive line bundle is q-ample. The problem of determining when the converse holds was posed by [10], but little progress had been made until recently. Ottem [30] has given examples of line bundles which are q-ample but not q-positive when $\frac{1}{2}\dim X - 1 < q < \dim X - 2$. These examples are effective, and the analogue of the Lefschetz hyperplane theorem holds over $\mathbb{Q}$ but not $\mathbb{Z}$. S. Matsumura has shown in [27] that if X is a compact n dimensional complex manifold with a Kähler form ω, and L is a line bundle such that the intersection $\omega^{n-1} \cdot L > 0$, then L is 1-positive.

Chapter 3

Singularities of Cox Rings of Fano Varieties

3.1 Introduction

Like the homogeneous coordinate ring of a smooth projective variety, the Cox ring of a smooth projective variety X may have singularities. For a smooth projective variety, the singularities of the coordinate ring are governed by the geometry of the embedding. Likewise the global geometry of X will govern the singularities of $\mathrm{Cox}(X)$. In this section we prove Theorem 1.6 which says that if X is a log Fano variety then the Cox ring has log terminal singularities.

Popov [31] has established in characteristic 0 that Cox rings of smooth del Pezzo surfaces have rational singularities, and are hence Cohen-Macaulay. Castravet and Tevelev [6] extended these results to higher dimensional projective spaces at general points. Since del Pezzo surfaces are Fano, and log terminal singularities are always rational, Theorem 1.6 recovers Popov's results and extends them to higher dimensional log Fano varieties.

It is helpful to have a more general notion of Cox ring.

Definition 3.1. Let $D_1, \ldots D_r$ be Weil divisors which form a $\mathbb{Z}$ basis for the torsion free part of $\mathrm{Cl}(X)$. Then a Cox ring of X is given by

$$\mathrm{Cox}(X; D_1, \ldots, D_r) = \bigoplus_{\langle a_1, \ldots a_r \rangle \in \mathbb{Z}^r} H^0(X, \mathcal{O}_X(a_1 D_1 + \ldots a_r D_r)) \subset K(X)[t_1^{\pm}, \ldots t_r^{\pm}].$$

That is, $\mathrm{Cox}(X; D_1, \ldots, D_r)$ is generated by elements of the form $f t_1^{\alpha_1} \ldots t_r^{\alpha_r}$, where f is a rational functions with at worst poles of order α_i along D_i, and no other poles.

We will generally assume that $\mathrm{Cl}(X)$ has no torsion; in this case the Cox ring is independent of the generators chosen, and the above definition corresponds with the one given in section 1.3.

Elizondo, Kurano, and Watanabe [12] showed in all characteristics that the Cox ring of a normal variety with finitely generated torsion free class group is a unique factorization domain (UFD). Recent work of Hashimoto and Kurano [18] computes the canonical modules of Cox rings. They show that when X is a normal variety whose class group is a finitely generated free abelian group and Cox(X) is Noetherian, the canonical module of Cox(X) is a rank one free module. On a smooth Fano variety, the Picard group and hence the class group is a finitely generated free abelian group [35, Prop 2.1.2]. Also log terminal singularities on a Gorenstein variety are canonical. Hence the second statement of Theorem 1.6 follows from the first.

Much of the material in this chapter is based on the author's work in [5]. The author had earlier conjectured that conjectured that finitely generated Cox rings are generally Cohen-Macaulay. I am very grateful to Yoshinori Gongyo [14] for pointing out a counterexample: A very general algebraic hyper-Kähler 4-fold will have Picard number 1 [8], and hence have finitely generated Cox ring. But if X is hyper-Kähler, then $H^2(X, \mathcal{O}_X)$ is nonzero, so this ring cannot be Cohen-Macaulay.

In the earlier version, 1.6 was stated only for the case $\Delta = 0$. The arguments work equally well for the log Fano case, and this is a more natural setting for many of the constructions. Independently, Gongyo, Okawa, Sannai, and Takagi [15] proved the log Fano case using reduction to positive characteristic. They also proved a converse to 1.6 which relates it to work of Schwede and Smith on global F-regularity [34] and investigated the case of log Calabi-Yau Mori Dream Spaces.

Theorem 3.2. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial log Calabi-Yau pair over $\mathbb{C}$ such that X is a Mori Dream Space, and let $D_1, \ldots D_r$ be a basis for the torsion free part of Cl(X). Then the ring Cox$(X; D_1 \ldots D_r)$ is normal with log canonical singularities.*

Gongyo, Okawa, Sannai, and Takagi [15] have shown Thm 3.2 for case where X is a surface, and more recently, Kawamata and Okawa [20] proved this in arbitrary dimension along with its converse.

We use ideas from the minimal model program as well as those of Hu and Keel [19] relating MMP, Cox rings, and GIT.

Our proof proceeds by induction on the Picard rank ρ of X. The case of Fano varieties with $\rho = 1$ follows from general work of Tomari and Watanabe on normal $\mathbb{Z}$-graded rings [37, Thm 2.6], although for completeness we provide a proof up to cyclic covering using different techniques in section 3.2.

For $\rho > 1$ we may recover X from Cox(X) by means of a GIT quotient by a torus $\mathbb{G}_m^{\rho_X}$. We will filter this quotient into a series of $\mathbb{G}_m$ quotients, and show inductively that the singularities never get too bad. This requires constructing a variety X' from X which is a small compactification of the total space of a nontrivial $\mathbb{G}_m$ bundle on X. We must show that with a suitable choice of bundle, X' is also Fano with $\mathbb{Q}$-factorial and log terminal singularities.

In section 3.3, we review facts about Mori Dream Spaces, and show that if X is a Mori Dream Space and L a line bundle on X, then the projectivized vector bundle $Y =$

$\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$ is also a Mori Dream Space. We also show that the Cox ring of X is a cyclic cover of the Cox ring of X'. The goal of section 3.4 is to make birational modifications to Y until we arrive at a normal $\mathbb{Q}$-factorial log Fano pair (X', Δ) which is a small compactification of the chosen $\mathbb{G}_m$ bundle. Standard techiniques of MMP show that (X', Δ) is klt. In section 3.5 the same is done for the log Calabi-Yau case.

Finally, in section 3.6 we address the cyclic covers introduced in sections 3.2 and 3.3 to complete the argument.

3.2 Fano varieties with $\rho = 1$

When the Picard number of X is just one, the Cox ring is the ring of sections of multiples of a divisor. This makes it easier to study than the multigraded case, and questions of whether singularities of such rings are Cohen-Macaulay, Gorenstein, or rational were studied in detail by Watanabe [40].

Theorem 3.3. *1. Let (X, Δ) be a $\mathbb{Q}$-factorial log Fano pair such that $\rho_X = 1$, and let L be the ample generator of Pic(X). Then the homogeneous coordinate ring $\bigoplus H^0(X, \mathcal{O}(nL))$ has log terminal singularities.*

2. Let (X, Δ) be a projective $\mathbb{Q}$-factorial log Calabi-Yau pair such that $\rho_X = 1$, and let L be the ample generator of Pic(X). Then the homogeneous coordinate ring $\bigoplus H^0(X, \mathcal{O}(nL))$ has log canonical singularities.

When X is strictly Fano, 3.3 is a special case of a result of Tomari and Watanabe [37, Thm 2.6].

Proof. Since X is $\mathbb{Q}$-factorial and Δ is effective the pair $(X, 0)$ does not have worse singularities than (X, Δ). So we assume $\Delta = 0$. Let $Z = \operatorname{Spec} \bigoplus H^0(X, \mathcal{O}(nL))$. We must show that Z is normal, $\mathbb{Q}$-factorial, and then compute discrepancies of a resolution of Z. The variety Z is normal since the ring $\bigoplus H^0(X, \mathcal{O}(nL))$ is integrally closed. We will resolve the singularities of Z in two steps.

Let $\psi : \tilde{X} \to X$ be a resolution of singularities of X. Now, let $S(L)$ be the symmetric algebra on $\mathcal{O}(L)$, and let $\mathbb{A}_X(L) = \mathbf{Spec}_X(S(L))$. This is the total space of the line bundle L, and has a projection map $\pi : \mathbb{A}_X(L) \to X$. We will likewise define $\mathbb{A}_{X'}(\psi^* L) = \mathbf{Spec}_{X'}(S(\psi^* L))$. There is a birational maps $\psi' : \mathbb{A}_{X'}(\psi^* L) \to \mathbb{A}_X(L)$ which is a resolution of $\mathbb{A}_X(L)$. There is a second birational map $f : \mathbb{A}_X(L) \to Z$ which contracts the zero section E to a point. Note that $E|_E = -L$.

Together, these two maps resolve the singularities of Z. Note that Z is $\mathbb{Q}$-factorial since the relative Picard and Weil class groups of f both have rank 1. It remains only to check discrepancies.

The $\mathbb{A}^1$ bundle $\mathbb{A}_X(L)$ is smooth in codimension 2 so by adjunction, $K_{\mathbb{A}_X(L)} = \pi^* K_X - E$. On Z, a multiple of K_Z is trivial at the cone point, so $f^*(K_Z) = \pi^* K_X - mE$, where m is the nonnegative rational number satisfying $mL = -K_X$. In the log Fano case, m is strictly positive. Thus $K_{\mathbb{A}_X(L)} = f^*K_Z + (m-1)E$.

Since none of the centers of blowups of ψ' are contained in E, ψ'^*E is the strict transform of E. Thus in the log Fano case, the discrepancy of E is $m-1 > -1$, and in the log Calabi-Yau case the discrepancy $m-1 \geq -1$. The other discrepancies are the same as those of ψ, which are large enough by hypothesis. □

Example 3.4. Let X be a hypersurface of degree $d > n$ in $\mathbb{P}^n$, where $n \geq 4$. By the Lefschetz hyperplane theorem, $\mathrm{Pic}(X) = \mathbb{Z}$, generated by $\mathcal{O}(1)$. For all $m \in \mathbb{Z}$, $H^1(X, \mathcal{O}(m)) = 0$, so the ring $\mathrm{Cox}(X) = \bigoplus H^0(X, \mathcal{O}(m)) = k[x_0 \dots x_n]/f$, which is finitely generated. Thus X is a Mori Dream Space. Note that our choice $d > n$ guarantees that X is not a Fano variety. However, $H^{n-1}(X, \mathcal{O}_X) \neq 0$, so by [21, Theorem 1] the ring $\mathrm{Cox}(X)$ does not have rational singularities. This ring is however Cohen-Macaulay.

3.3 Mori Dream Spaces

Mori Dream Spaces were first introduced by Hu and Keel[19], and are so called because it is relatively easy to carry out the operations of the Mori Program on such a space. Let X be a projective variety, and R a Cox ring for X. The variety X is called a Mori Dream Space if X is $\mathbb{Q}$-factorial, $\mathrm{Pic}_{\mathbb{Q}}(X) = N^1(X)$, and R is finitely generated as a $\mathbb{C}$ algebra. This is a very special property, and has many nice consequences for the birational geometry of X. In particular, the nef and psuedoeffective cones of X are both rational polyhedral cones, and every nef divisor on X is semiample.

Note that Hu and Keel's definition of the Cox ring of X [19] requires the divisors D_i to be Cartier. When X is $\mathbb{Q}$-factorial their definition differs from ours by only a finite extension. Thus it doesn't matter which ring we consider for questions of finite generation. Also, when X is smooth and $\mathrm{Pic}(X) \cong \mathbb{Z}^r$, the definitions coincide.

Theorem 3.5. *[2] If (X, Δ) is a $\mathbb{Q}$-factorial log Fano pair then X is a Mori Dream Space.*

Thus log Fano varieties give a nice class of Mori Dream Spaces to study, though these are not the only examples of Mori Dream Spaces. Given a Mori Dream Space X, we want an appropriate compactification X' of a $\mathbb{G}_m$ bundle on X. One obvious way to get a compactification is to take a projectivized vector bundle. Given a vector bundle $\mathcal{E}$ on X, let $\mathbb{P}_X(\mathcal{E}) = \mathbf{Proj}_X(\oplus \mathrm{Sym}^n(\mathcal{E}))$. We will be solely concerned with the case where $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X(L)$ where L is a Cartier divisor on X. In this case $\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$ is a compactification of the $\mathbb{G}_m$ bundle on X associated to L. There are two irreducible boundary divisors, corresponding to the 0 section of L and the section at ∞. To better understand the birational geometry of $Y = \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$ we calculate the Cox ring:

Theorem 3.6. *Let X be a Mori Dream Space. Choose $D_1, \dots D_r$ Weil divisors generating the torsion free part of $Cl(X)$), and let L be a nontrivial Cartier divisor which is in the* *subgroup generated by the D_i. Let $Y = \mathbb{P}_X(\mathcal{O}_X(L) \oplus \mathcal{O}_X)$, which is a $\mathbb{P}^1$ bundle over X with projection $\pi : Y \to X$. Then*

1. *The divisors $\pi^* D_i, E_\infty$, where E_∞ is the section of X at infinity form a $\mathbb{Z}$-basis for the torsion free part of Cl(Y).*

2. *Cox($Y; \pi^* D_1, \ldots, \pi^* D_r, E_\infty$) $\cong$ Cox($X; D_1, \ldots, D_r$)$[s, t]$*

3. *Y is also a Mori Dream Space.*

Proof. For the first statement, $Y \setminus E_\infty$ is an $\mathbb{A}^1$ bundle over X, so its class group is isomorphic to that of X, and is generated by the pullbacks of generators for Cl(X). We have an exact sequence [17, II, Prop 6.5]:

$$\mathbb{Z} \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X) \to 0$$

It remains to show that the first map is injective. This will follow since L was nontorsion: The restriction of E_∞ to itself is it's normal bundle, which is $-L$. This is nontorsion, so E_∞ is not torsion in Y either.

For the second statement, set $R = \mathrm{Cox}(X; D_1, \ldots, D_r)$, $S = \mathrm{Cox}(Y; \pi^* D_1, \ldots, \pi^* D_r, E_\infty)$. Now, since $\mathbb{P}(\mathcal{O}(L) \oplus \mathcal{O}) \cong \mathbb{P}(\mathcal{O}(-L) \oplus \mathcal{O})$ with the only difference being that the tautological invertible sheaf $\mathcal{O}(1)$ is twisted by $\pi^*(-L)$, we have that the zero section E_0 and the infinity section E_∞ are related by $E_0 \sim L + E_\infty$. Let y be the rational function with a zero of order one at E_0 and poles along L and E_∞.

Choose an affine open set U in X such that L is trivial on U, and y has no poles in U. Then $\mathbb{A}^1 \times U$ is an affine open set of Y, with coordinate ring $\Gamma(U)[y]$. Thus the function field $K(Y) = K(X)(y)$.

We are ready to define the new variables s and t in $K(Y)[t_1^\pm, \ldots t_{r+1}^\pm]$. The variable t is defined to be t_{r+1}. This is an element of S since the rational function 1 has no zeros or poles.

Next we define the element s. Let α_i be the coefficient of D_i in L for $1 \leq i \leq r$, and let $\alpha_{r+1} = 1$. Then set $s = y \prod t_i^{\alpha_i}$, which is in S since y has poles only along L and E_∞. y and t_{r+1} are algebraically independent over R, so $R[s, t] \subset S$. It remains to be shown that every element of S is in $R[s, t]$. It suffices to check homogeneous elements under the $\mathbb{Z}^{r+1}$ grading, so let $\lambda = \frac{f(U,y)}{g(U,y)}$ be a rational function such that $\lambda \prod t_i^{\beta_i}$ is an element of S, where f and g are functions in the coordinate ring of $\mathbb{A}^1 \times U$.

The rational function λ has no poles along E_0, so if $g(U, y)$ is divisible by y^n so is $f(x, y)$. Assume therefore that $g(U, y)$ is not divisible by y. Likewise, λ cannot have any poles along any horizontal divisor, so we may assume g is actually the pullback of a function in $\Gamma(U)$. Now if $f(U, y)$ is divisible by y^m, we may divide $\lambda \prod t_i^{\beta_i}$ by s^m and still have an element of S. Thus we assume the lowest y-degree term of λ is constant in y. Note that $\frac{1}{s^m} \lambda \prod t_i^{\beta_i}$ has nonnegative degree in t_{r+1}, since the rational function has no zeros at E_∞. Now, the constant term of $\frac{\lambda}{y^m}$ is the restriction to this function to $E_0 \cong X$, and this rational function has zeros and poles of the prescribed orders along the D_i, so the constant term belongs to $R[u, v]$. If the only term was the constant term, we are finished, otherwise we may proceed by induction on the number of terms.

For the third statement we know that Y has a finitely generated Cox ring. Since Y has $\mathbb{Q}$-factorial singularities if X does, and every numerically trivial divisor on Y is torsion, we conclude that Y is also a Mori Dream Space.

□

Warning! For an arbitrary vector bundle $\mathcal{E}$ on a Mori Dream Space X, $\mathbb{P}_X(\mathcal{E})$ might not be a Mori Dream Space! In fact, this may fail even if X is a toric variety[16].

Our goal is, given X, build a variety X' with a similar Cox ring and a lower Picard number. If X is log Fano, then X' should be also, likewise if X is log Calabi-Yau X' should be too. Thus we may inductively reduce to the $\rho = 1$ case. The variety $Y = \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$ constructed is not appropriate for our purposes since the Picard number has gone up. However for the right choice of L, a suitable birational modification of Y will produce such an X'.

Let $Y_0 \subset Y$ be the open set consisting of the complement of the boundary divisors E_0 and E_∞. Then Y_0 is a $\mathbb{G}_m$ bundle over X. We will choose X' to be a particular small compactification of Y_0; that is the boundary will have codimension at least 2 in X'. Thus we can canonically identify $\mathrm{Cl}(X') \cong \mathrm{Cl}(Y_0)$.

Theorem 3.7. *Take X, Y, Y_0 as above. Let $D_1 \dots D_r$ be a basis for the torsion free part of Cl(X), and assume $L = mD_r$ for some m. Fix μ_m a primitive mth root of unity. Let $\pi_0 : Y_0 \to X$ be the projection morphism. Let X' be a projective small compactification of Y_0.*

Then $\pi_0^* D_1 \dots \pi_0^* D_{r-1}$ generate the torsion free part of Cl(X'), and Cox($X'; \pi_0^* D_1, \dots \pi_0^* D_{r-1}$) is isomorphic to the $\mathbb{Z}/m$ invariant part of Cox($X; D_1, \dots D_r$) under the action induced by $t_r^i \to \mu_m^i t_r^i$.

Proof. By assumption, L is nontorsion. $\mathrm{Cl}(Y_0) \cong \mathrm{Cl}(X')$ is given by the quotient of $\mathrm{Cl}(Y)$ by the subgroup generated by E_0 and E_∞, and the only classes in $\pi_0^* \mathrm{Cl}(X)$ of this form are multiples of L. This proves the first statement.

For the second statement we must write down a map

$$\alpha : R = \mathrm{Cox}(X'; \pi_0^* D_1, \dots \pi_0^* D_{r-1}) \to \mathrm{Cox}(X; D_1, \dots D_r)$$

Note that because of the grading we need only define α on homogeneous elements of R. As before, $K(X', y) = K(X)$, where y has a zero of order m along $\pi_0^* D_r$ and no other poles or zeros. Choose $U \subset X$ so that rational functions are the ratios of regular functions on U.

A homogeneous element of R has the form $\frac{f(U,y)}{g(U,y)} \prod_{i<r} t_i^{a_i}$. Since $\frac{f(U,y)}{g(U,y)}$ has no poles along horizontal divisors in Y_0, it is actually a Laurent polynomial in y. So such an element of R really has the form $\sum \lambda_i y^i \prod_{i<r} t_i^{a_i}$ where $\lambda_i \in K(X)$ are rational functions with poles of appropriate orders along $\pi_0^* D_1, \dots \pi_0^* D_{r-1}$ and a pole of order at most mi along $\pi_0^* D_r$. Thus we get a well defined map α by sending $\sum \lambda_i y^i \prod_{i<r} t_i^{a_i}$ to $\sum \lambda_i s_r^{mi} \prod_{i<r} s_i^{a_i}$

This map α certainly injective since t_r is algebraically independent of the other variables. Also, given a homogeneous element $\lambda \prod_{i \leq r} s_i^{a_i}$ of $\mathrm{Cox}(X; D_1, \dots D_r)$ which is $\mathbb{Z}/m$ invariant,

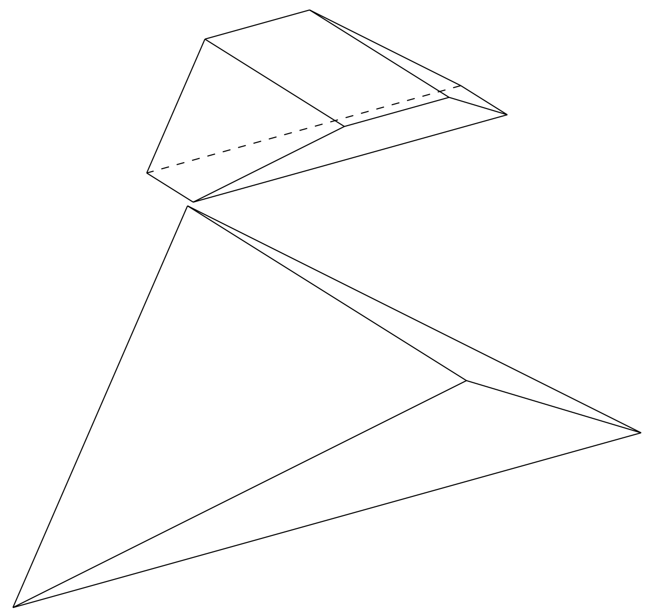


Figure 3.1: The construction for $X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = (-1, 1)$. The $\mathbb{P}^1$ bundle $\mathbb{P}_X(\mathcal{O}_X(1,-1) \oplus \mathcal{O}_X)$ is contracted to $\mathbb{P}^3$.

it is the image of the element $\lambda y^{a_i/m} \prod_{i<r} s_i^{a_i}$ of r. So the image of the map is the $\mathbb{Z}/m$ invariant elements of $\mathrm{Cox}(X; D_1, \ldots D_r)$. $\square$

Before moving on it is instructive to consider a simple example.

Example 3.8. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and take L to be the line bundle $(1, -1)$. Then X is a toric Fano variety, and $\mathrm{Cox}(X) = k[a_0, a_1, b_0, b_1]$, where the a_i have grading $(1, 0)$ and the b_i have grading $(0, 1)$. In our construction the variety $Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(1, -1))$.

Y is also toric, and both boundary divisors are isomorphic to $X = \mathbb{P}^1 \times \mathbb{P}^1$. There is a contracting morphism which contracts each boundary divisor along a ruling down to a $\mathbb{P}^1$, and the image of this morphism is $\mathbb{P}^3$. This is another toric Fano variety, whose Cox ring is again the polynomial ring in 4 variables, this time graded by total degree. In this case there is no cyclic cover, since $(1, -1)$ can be extended to a $\mathbb{Z}$ basis for the class group of X and so $m = 1$.

The variety $X' = \mathbb{P}^3$ is our expected small compactification of the $\mathbb{G}_m$ bundle on X corresponding to L. In fact the map from the open locus can be seen geometrically since projection from each line in $\mathbb{P}^3$ gives a map from the complement of that line to a $\mathbb{P}^1$. Since the example is toric, we exhibit this contraction in terms of polytopes in Figure 3.8.

While this example gives some flavor of the general construction, we cannot in general expect the variety X' to be smooth, even when X is.

3.4 Singularities

Theorem 3.9. *Let (X, Δ) be a log Fano pair with $\rho_X > 1$ which is $\mathbb{Q}$-factorial. Then there exists a line bundle L on X and a small compactification X' of the associated $\mathbb{G}_m$ bundle to L with a $\mathbb{Q}$-divisor Δ' such that the pair (X', Δ') is also a $\mathbb{Q}$-factorial log Fano variety.*

Let X be a Mori Dream Space. The effective cone of X has a decomposition into finitely many Mori chambers [19]. Therefore we may choose a Cartier divisor L such that

1. Both L and $-L$ are not effective.

2. The intersections of the walls of the Mori chamber decomposition of the effective cone with the line segment connecting $-K_X$ and L are transverse, and this segment only intersects one wall at a time, likewise for $-K_X$ and $-L$.

The first condition will ensure that we will construct a small compactification; the second ensures that this compactification will be $\mathbb{Q}$-factorial.

Let Y be the projectivized vector bundle $\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$, and Y_0 the complement of $E_0 \cup E_\infty$ in Y. Then by 3.6 Y is also a Mori Dream Space. Hence Y has finitely many small $\mathbb{Q}$-factorial modifications and these correspond to chambers in the effective cone of Y. To understand these chambers, we need to understand divisors on Y, and in particular we need an ample divisor on Y.

There is the map $\pi : Y \to X$ which exhibits Y as a $\mathbb{P}^1$ bundle over X. Let A be an ample line bundle on X. The pullback π^*A has positive top intersection with every subvariety of Y which isn't the pullback of a variety on X. The divisors E_0 and E_∞ intersect the fibers positively hence by the Nakai-Moishezon criterion [24, 1.2.19] $\pi^*A + \varepsilon E_0 + \varepsilon E_\infty$ is ample for sufficiently small ε. In the log Fano case we will be taking $A = -(K_X + \Delta)$.

The variety X is normal, so Y is smooth in codimension 2. Hence divisors on Y are Cartier in codimension 2, and we can use the adjuction formula[22, Rmk 5.47]. By adjunction, $K_Y = \pi^*K_X - E_0 - E_\infty$. In the case where (X, Δ) is log Fano, we will see that there is a divisor Γ on Y such that (Y, Γ) is log Fano, but even if X is strictly Fano, Y may not be.

Next we will construct X'. By hypothesis, neither L nor $-L$ are effective. Since A is ample, there are positive rational numbers a_+ and a_- such that $A + a_+L$ and $A + a_-L$ lie on the boundary of the effective cone. Choose b greater than both a_+ and a_-. Set $H = A + b(E_0 + E_\infty)$. Let $X' = \operatorname{Proj} \bigoplus H^0(Y, \mathcal{O}(nH))$. There is an induced rational map $f : Y \to X'$ which is regular away from the base locus of H. Since $-pi^*A + \varepsilon E_0$ is ample, the map f is regular and an isomorphism away from the divisors E_0 and E_∞. Thus X' is a compactification of Y_0, which is the geometric realization of the $\mathbb{G}_m$ bundle L. We define the divisor Δ' as the strict transform of $\pi^*\Delta$.

The divisors E_0 and E_∞ are the exceptional loci for the rational map f. Since b was larger than both a_+ and a_-, both divisors are in the stable base locus, and when the base components are removed, the linear series H is not big on either divisor. Thus the images of E_0 and E_∞ are of strictly smaller dimension, so X' is a small compactification of Y_0.

Proposition 3.10. *X' is normal and $\mathbb{Q}$-factorial.*

Proof. It is equivalent to show that the divisor H lies in the interior of a Mori chamber of Y[19, Proposition 1.11].

To show this, we will construct a path in $N^1(Y)$ which connects H with the ample divisor $-\pi^*(K_X + \Delta) + \varepsilon E_0$ and intersects the walls of the Mori decomposition in a finite set. The path will consist of two line segments:

First, connect $H = \pi^*A + bE_0 + bE_\infty$ to $\pi^*A + bE_0 + \varepsilon E_\infty$ with a line segment. Then connect $\pi^*A + bE_0$ to $\pi^*A + \varepsilon E_0 + \varepsilon E_\infty$ with another line segment.

We will need to analyze what it means geometrically to cross a wall between chambers. Say we start in the chamber corresponding to the variety Z. Each wall of this chamber corresponds to a curve in the nef cone of Z. By [19, Proposition 1.11], there are two things that may happen. If the curve comes in a family which covers a divisor, that divisor will be contracted, and the result is the variety of the other chamber.

If the curve does not cover a divisor, then contracting the curve yields a variety which is not $\mathbb{Q}$-factorial. The solution to this difficulty is an operation called a D flip, which is a small modification which creates a new $\mathbb{Q}$-factorial variety, which will be the variety on the other side of the wall. Specifically, let $g : Z \to W$ be the contracting morphism, and D a $\mathbb{Q}$-Cartier divisor such that $-D$ is g-ample. Then a D-flip is a map $g : Z' \to W$ where Z' is a small modification of Z, and the strict transform of D in Z' is $\mathbb{Q}$-Cartier and g'-ample. This is always unique when it exists, and will exist when Z is a Mori Dream Space. See [19],[22] for details.

For our purposes, what is important is that when our path hits a wall this corresponds to the divisor becoming trivial on some curve class. Consider without loss of generality the part of the path where we are adding E_∞. Then up until E_∞ is contracted, the curve in question must be contained in the strict transform of E_∞. Once E_∞ is contracted, E_∞ is a component in the base locus so increasing the coefficient of E_∞ will not change the rational map, and so we will not cross anymore walls.

Now the strict transform of E_∞ is the image of E_∞ under the induced rational map, and this is the birational model of X given by $\operatorname{Proj} \bigoplus_n H^0(X, \mathcal{O}(nA - naL))$. Since L was chosen so that as a ranges from 0 to a_- only one curve becomes negative with respect to $A - aL$ at a time, the same is true along the path in $N^1(Y)$.

Since the path began in the interior of a chamber and intersects chamber walls in only finitely many places, it must end in the interior of a chamber. □

For the rest of this section we will consider only the log Fano case:

Proposition 3.11. *When (X, Δ) is log Fano, the divisor $-(K_{X'} + \Delta')$ is ample on X'.*

Proof. Since E_0 and E_∞ are contracted by f, the strict transform of $-\pi^*K_X - \pi^*\Delta$ is equivalent to that of $-\pi^*K_X - \pi^*\Delta + a_+E_0 + a_-E_\infty$ on X' and hence ample. By adjunction, $\pi_0^*K_X$ is the canonical divisor on Y_0 (where π_0 is the restriction of π to Y_0). But X' is a small compactification of Y_0, so the closure of $\pi_0^*K_X$ in X' is $K_{X'}$, and this is the strict transform of $\pi^* K_X$. Likewise by definition Δ' is the strict transform of $\pi^*\Delta$. Hence $-(K_{X'} + \Delta')$ is ample. $\square$

We will soon show that the pair (X', Δ') has log terminal singularities. We know that X' is related to the variety Y by a sequence of D-flips and divisorial contractions. The contracted curves in every case are actually $(K_Y + \Gamma)$ negative, once we know that (Y, Γ) is klt, the resulting log flips and contractions will also be klt.

Lemma 3.12. *For $0 < \varepsilon < 1$ the pair $(Y, \pi^*\Delta + (1-\varepsilon)E_0 + (1-\varepsilon)E_\infty)$ is klt.*

Proof. Set $\Gamma = \pi^*\Delta + (1-\varepsilon)E_0 + (1-\varepsilon)E_\infty$. Recall that $Y = \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$. Let $\mu : Z \to X$ be a log resolution of the pair (X, Δ). Then we can construct a resolution of singularities of Y by taking $W = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(\mu^* L))$. We claim the map $\nu : W \to Y$ is a log resolution of the pair (Y, Γ). Let F_i be the exceptional divisors of μ. The exceptional divisors of W are the pullbacks of those in Z by the projection ϕ. Thus, they along with the strict transforms of the boundary divisors $\tilde{E}_0$ and $\tilde{E}_\infty$ intersect transversely. The strict transforms $\tilde{E}_0$ and $\tilde{E}_\infty$ are smooth since they are each isomorphic to Z.

$$\begin{array}{ccc} W & \xrightarrow{\nu} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{\mu} & X \end{array}$$

Now we must compare discrepancies. The pair (X, Δ) is klt, so the log discrepancies a_i are postive:

$$K_Z + \sum F_i = \mu^*(K_X + \Delta) + a_i F_i$$

$$K_W = \phi^* K_Z - \tilde{E}_0 - \tilde{E}_\infty$$

$$K_W + \sum \phi^* F_i + \tilde{E}_0 + \tilde{E}_\infty = \phi^*\mu^*(K_X + \Delta) + a_i \phi^* F_i = \nu^*\pi^*(K_X + \Delta) + a_i \phi^* F_i$$

But $\pi^*(K_X + \Delta) = K_Y + \Gamma + \varepsilon(E_0 + E_\infty)$, so

$$K_W + \sum \phi^* F_i + \tilde{E}_0 + \tilde{E}_\infty = \nu^*(K_Y + \Gamma) + \varepsilon\nu^* E_0 + \varepsilon\nu^* E_\infty + a_i \phi^* F_i$$

The pullbacks $\nu^* E_0$ and $\nu^* E_\infty$ are effective, so all the log discrepancies are positive, and (Y, Γ) is klt. $\square$

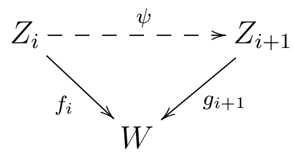
Thus while Y may not be strictly Fano, for sufficiently small positive ε the pair (Y, Γ) is log Fano, where $\Gamma = \pi^*\Delta + (1-\varepsilon)E_0 + (1-\varepsilon)E_\infty$. Now we consider the singularities of X'.

Proposition 3.13. *When (X, Δ) is log Fano the pair (X', Δ') is klt.*

Proof. We have seen already that X' is normal and $\mathbb{Q}$-factorial, hence $K_{X'}+\Delta'$ is $\mathbb{Q}$-Cartier. By lemma 3.12 the pair (Y,Γ) is klt.

Since Y is a Mori Dream Space, the rational map $Y \dashrightarrow X'$ factors into a series of D-flips and divisorial contractions $Y = Z_0 \dashrightarrow Z_1 \ldots \dashrightarrow Z_n = X'$ by moving along a path in $N^1(Y)$. Since X' is $\mathbb{Q}$-factorial, we can choose that path to be a straight line connecting $-(K_Y+\Gamma)$ to $-(K_Y+\Gamma)+F$, where F is effective and satisfies $X' = \operatorname{Proj} \bigoplus H^0(n(-(K_Y+\Gamma)+F))$.

Let Γ_i be the divisor given by the closure of the strict transform of Γ in Z_i. Then $\Gamma = \Gamma_0$, and $\Gamma_n = \Delta'$. Moreover, in the following diagram, we have that $f_{i*}\Gamma_i = g_{i+1*}\Gamma_{i+1}$.



Assume (Z_i,Γ_i) is klt. There is a single curve class C contracted by f_i, and this curve must have negative intersection with the strict transform of F. Thus C is positive on $-(K_{Z_i}+\Gamma_i)$, as this is the strict transform of $-(K_Y+\Gamma)$, and therefore ψ is either a $K_{Z_i}+\Gamma_i$ flip or divisorial contraction of an extremal curve. In either case, the pair (Z_{i+1},Γ_i) is klt [22, Cor 3.42, Cor 3.43].

Thus (X',Δ') is klt.

□

Proof of Theorem 3.9. Construct (X',Δ') as above. Then by 3.10 X' is normal and $\mathbb{Q}$-factorial. By 3.11 $-(K_{X'}+\Delta')$ is ample, and by 3.13 the pair (X',Δ') is klt, hence (X',Δ') is a log Fano pair. □

Example 3.14. Let X be the blowup of $\mathbb{P}^2$ at one point. This is the Hirzebruch surface $\mathbb{F}_1$, a toric Fano variety. The Picard group of X is a free abelian group generated by H and E where H is the pullback of the class of a hyperplane in $\mathbb{P}^2$, and E is the exceptional divisor of the blowup. In this case $K_X = -3H + E_1$, and one can check that $-K_X$ is ample. We will take $2E - H$ as our divisor L. In Figure 3.2 we show the birational transformations from $Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(L))$ to X'. Note that the first of these is a small modification, a D-flip of a curve in one of the exceptional divisors.

3.5 The log Calabi-Yau case

The theorems and proofs of this section are much the same as for section 3.4, but for the case where the pair (X,Δ) is log Calabi-Yau.

Theorem 3.15. *Let (X,Δ) be a projective log Calabi-Yau pair with $\rho_X > 1$ which is $\mathbb{Q}$-factorial. Then there exists a line bundle L on X and a small compactification X' of the associated $\mathbb{G}_m$ bundle to L with a $\mathbb{Q}$-divisor Δ' such that the pair (X',Δ') is also a projective $\mathbb{Q}$-factorial log Calabi-Yau pair.*

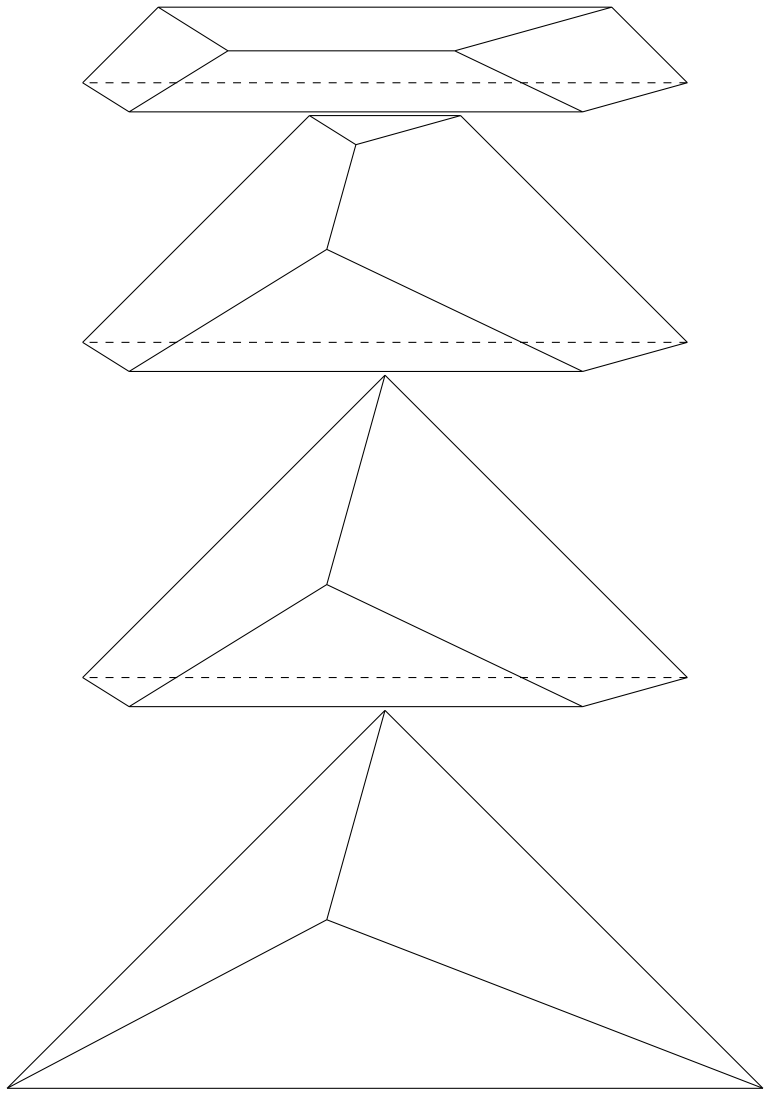


Figure 3.2: A toric example of the construction of X' from X. Here X is the Hirzebruch surface $\mathbb{F}_1$. We start with a compactified $\mathbb{G}_m$ bundle over X, and gradually make birational modifications until both exceptional divisors are contracted.

The construction of (X', Δ') proceeds as in section 3.4. As above take L generic such that neither L nor $-L$ is effective, and set $Y = \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$, with $\Gamma = \pi^*\Delta' + E_0 + E_\infty$. The variety X' is obtained by contracting the boundaries E_0 and E_∞, and Δ' is the strict transform of $\pi^*\Delta$.

Lemma 3.16. *X' is projective.*

Proof. By construction, X' is Proj of a ring of sections of a line bundle on Y, hence X' is projective. □

Lemma 3.17. *The pair (Y, Γ) is log Calabi-Yau.*

Proof. By adjunction, $K_Y = \pi^*K_X - E_0 - E_\infty$, so $K_Y + \Gamma$ is trivial. Thus it remains to show that the pair (Y, Γ) is lc. Let $\mu : Z \to X$ be a log resolution of the pair (X, Δ). Then we can construct a resolution of singularities of Y by taking $W = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(\mu^*L))$. We claim the map $\nu : W \to Y$ is a log resolution of the pair (Y, Γ). Let F_i be the exceptional divisors of μ. The exceptional divisors of W are the pullbacks of those in Z by the projection ϕ. Thus, they along with the strict transforms of the boundary divisors $\tilde{E}_0$ and $\tilde{E}_\infty$ intersect transversely. The strict transforms $\tilde{E}_0$ and $\tilde{E}_\infty$ are smooth since they are each isomorphic to Z.

$$\begin{array}{ccc} W & \xrightarrow{\nu} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{\mu} & X \end{array}$$

Now we must compare discrepancies. The pair (X, Δ) is lc, so the log discrepancies a_i are nonnegative:

$$K_Z + \sum F_i = \mu^*(K_X + \Delta) + a_i F_i$$

$$K_W = \phi^* K_Z - \tilde{E}_0 - \tilde{E}_\infty$$

$$K_W + \sum \phi^* F_i + \tilde{E}_0 + \tilde{E}_\infty = \phi^*\mu^*(K_X + \Delta) + a_i \phi^* F_i = \nu^*\pi^*(K_X + \Delta) + a_i \phi^* F_i$$

But $\pi^*(K_X + \Delta) = K_Y + \Gamma$, so

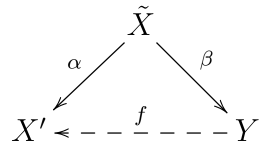
$$K_W + \sum \phi^* F_i + \tilde{E}_0 + \tilde{E}_\infty = \nu^*(K_Y + \Gamma) + a_i \phi^* F_i$$

All the log discrepancies are nonnegative, and so (Y, Γ) is klt. □

Proposition 3.18. *The pair (X', Δ') is log Calabi-Yau.*

Proof. The boundary of Y_0 has codimension 2 in X', and $K_{X'} + \Delta'$ is trivial on the open locus, so it must be trivial on all of X'.

Let $\tilde{X}$ be a common log resolution of (X', Δ') and (Y, Γ), with exceptional divisors F_i.



Both $K_Y + \Gamma$ and $K_{X'} + \Delta'$ are trivial, so their pullbacks are also trivial.

$$K_{\tilde{X}} + \sum F_i = \alpha^*(K_Y + \Gamma) + \sum a_i F_i = \sum a_i F_i = \beta^*(K_{X'} + \Delta') + \sum a_i F_i$$

The discrepancies are the same, and (Y, Γ) is lc, so (X', Δ') is also lc. □

Proof of Thm 3.15. Follows from 3.10, 3.16, and 3.18 □

3.6 Abelian Covers

So far we have been able to construct new Fano varieties with a smaller Picard number at the cost of replacing the Cox ring with a cyclic quotient. To recover the original ring we will have to repeatedly take certain cyclic covers, and show that the singularities do not get any worse. Fortunately, by [22, Prop 5.20 (iv)], if $X' \to X$ is a morphism of normal varieties which is étale in codimension 2, then X' is log terminal iff X is.

In the case where the class group of X is a free abelian group, Cox(X) is a UFD and hence integrally closed [11, Prop 4.10]. In general there may be torsion in the class group however.

Proposition 3.19. *Let X be a normal scheme, and let $D_1 \dots D_r$ generate the torsion free part of the class group of X. Then $R =$ Cox$(X; D_1, \dots, D_r)$ is normal.*

Proof. Since R does not depend on the choice of generators, merely the subgroup of $\mathrm{Cl}(X)$ we may assume for simplicity the D_i are irreducible and effective. The Cox ring is a subring of $K(X)[t_i^{\pm}]$, which is integrally closed. Let S be the integral closure of R. Then $S \subset K(X)[t_i^{\pm}]$. Now, let z be an element of S. We have that $z^n = a_{n-1}z^{n-1} + \dots a_0$ where the $a_i \in R$. We wish to show that $z \in R$. Expand z as a Laurent polynomial $\sum(\lambda \prod t_i^{\alpha_i})$. We must show for each term that λ has at worst poles of the prescribed orders along the D_i.

Consider an irreducible effective Weil divisor D on X. Then D induces a valuation v on $K(X)$. This valuation may be extended to v' on $K(X)(t_i)$ as follows. On a Laurent monomial $v'(f(X)t_i^{\alpha_i} = v(f(X)) + \alpha_i$ if D is one of the D_i, otherwise $v'(f(X)t_i^{\alpha_i} = v(f(X))$. On a Laurent polynomial take the minimum of the valuations of each term.

Now we need to confirm that the new valuation satisfies $v'(\lambda\mu) = v'(\lambda) + v'(\mu)$ for Laurent polynomials λ, μ. Let λ' and μ' consist of the terms of λ and μ which attain the lowest value of v' respectively. Then since $\lambda'\mu' \neq 0$, The lowest term of $\lambda\mu$ has valuation $v'(\lambda) + v'(\mu)$

On a rational function, take the difference of the valuation on the numerator and denominator. Now, by definition of R, if $v'(z) \geq 0$ for every D on X, $z \in R$. Apply v' to both sides of $z^n = a_{n-1}z^{n-1} + \ldots a_0$. Assuming z is nonzero, we see that

$$nv'(z) \geq \min_{0 \leq i < n}(v'(a_i) + iv'(z)) \geq min_{0 \leq i < n}(iv'(z))$$

Thus $v'(z) \geq 0$, so $z \in R$.

□

The next step is to show that the covers described by 3.7 are étale in codimension 2. This is essentially a version of Reid's cyclic covering trick [32, 3.6].

Lemma 3.20. *Assume X is a Mori Dream Space. Let $L_1 \ldots L_\rho$ be Cartier divisors which form a vector space basis for $Pic_{\mathbb{Q}}(X)$. Take $R = \bigoplus H^0(X, \mathcal{O}(\sum a_1 L_1 \ldots a_\rho L_\rho))$. Let D_i be a finite set of Weil divisors such that the subgroup of $Cl(X)$ generated by the L_i is contained in the subgroup generated by the D_i, and such that this subgroup is torsion free. Set $S = \bigoplus H^0(X, \mathcal{O}(\sum a_i D_i))$. Then S is a finite extension of R, and this extension is étale in codimension 2.*

Proof. The fact that the extension is finite follows since R is Noetherian and consists of the invariants of S under the action of an abelian group.

We will find an open set $U \subset \mathrm{Spec} R$ which is étale. Now, we can recover X from $\mathrm{Spec} R$ via a GIT quotient, as in [19]. By [19, Lemma 2.7], the unstable locus has codimension 2, and the remaining open set is a $\mathbb{G}_m^\rho$ bundle over X. Since X is normal, the singular locus of X has codimension at least 2, so we will take U to be the preimage of the smooth locus of X in the $\mathbb{G}_m^\rho$ bundle. We must show that for any $u \in U$, the covering is étale in a neighborhood of u.

Now, let G be the group generated by the D_i modulo the group generated by the L_i. Since X is $\mathbb{Q}$-factorial, G is a finite abelian group, and so $G \cong \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_k$. Set $g_1 \ldots g_k$ as a set of generators, and choose $F_1 \ldots F_k$ Weil divisors which represent these.

Let b_{gi} be a set of generators for S as an R module, indexed so that b_{gi} has the class g in G as a divisor. We can think of each b_{gi} as a section of $L_{gi} + \sum g_j F_j$, where $L_{gi} \in \langle L_1 \ldots L_\rho \rangle$. Now, choose ample divisors $A_1 \ldots A_k$ on X such that $-L_{gi} + \sum g_j A_j$ is a base point free divisor on the nonsingular locus of X for each generator and so that $A_j + F_j$ is base point free.

Now, choose a_{gj} sections of $-L_{gi} + \sum g_j A_j$ which do not pass through u, and x_j sections of $A_j + F_j$ which don't pass through u. In a neighborhood of u, we can assume the a_{gj} and the $x_j^{n_j}$ are all units. The element $b_{gj} a_{gj} \prod x_j^{n_j - g_j}$ is in $H^0(X, \sum(n_i A_i + n_i F_i))$, which is in R. Thus b_{gj} is in the module generated by the x_i. So locally the extension is given by adjoining the x_i, and for each x_i, $x_i^{n_i}$ is a unit. Thus the extension is étale at u. □

Proof of Thm 1.6, 3.2. The proof is by induction on the Picard rank ρ_X. In the case $\rho = 1$, by Theorem 3.3, the ring $\bigoplus H^0(X, \mathcal{O}(nL))$ has log terminal singularities. Let D be a generator of the class group of X. The ring $\operatorname{Cox}(X, D) = \bigoplus H^0(X, \mathcal{O}(nD))$ is normal and so by Lemma 3.20 also has log terminal singularities.

Now, assume that Theorem 1.6 is true for varieties with $\rho = n$. Given (X, Δ) log Fano $\mathbb{Q}$-factorial and log terminal with $\rho_X = n+1$, by Theorem 3.9, there is a $\mathbb{G}_m$ bundle $Y_0 \subset \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$on X with a compactification X' which has $\rho_{X'} = n$, such that X' is Fano and $\mathbb{Q}$-factorial and log terminal. Choose $D_1 \ldots D_r$ generators of the torsion free part of $\operatorname{Cl}(X)$, where D_r is a multiple of L. We abuse notation to write $\pi_0^* D_i$ for the image of $\pi_0^* D_i$ under the isomorphism $\operatorname{Cl}(X') \cong \operatorname{Cl}(Y)$. By the inductive hypothesis, $\operatorname{Cox}(X'; \pi_0^* D_1 \ldots \pi_0^* D_{r-1})$ has log terminal singularities. By 3.7, $\operatorname{Cox}(X'; \pi_0^* D_1 \ldots \pi_0^* D_{r-1})$ consists of the invariants of $\operatorname{Cox}(X; D_1 \ldots D_r)$ under a cyclic group, so $\operatorname{Cox}(X; D_1 \ldots D_r)$ is a cyclic cover of $\operatorname{Cox}(X'; \pi_0^* D_1 \ldots \pi_0^* D_{r-1})$. There is a choice of $\mathbb{Q}$-basis $L_1 \ldots L_{n+1}$ for the Picard group of X such that both rings are abelian covers of $H^0(X, \sum a_i L_i)$, and so by 3.20 these covers are étale in codimension 2. Hence the map from $\operatorname{Cox}(X'; \pi_0^* D_1 \ldots \pi_0^* D_{r-1})$ to $\operatorname{Cox}(X; D_1 \ldots D_r)$ is étale in codimension 2. Since $\operatorname{Cox}(X; D_1 \ldots D_r)$ is normal, the singularities of $\operatorname{Cox}(X)$ are log terminal.

The log Calabi-Yau case is the same but uses 3.15 instead of 3.9. □

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