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Big bang nucleosynthesis and active-sterile neutrino mixing: Evidence for maximal $\nu_\mu \leftrightarrow \nu_\tau$ mixing in Super Kamiokande?

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We discuss big bang nucleosynthesis constraints on maximal $\nu_\mu \leftrightarrow \nu_s$ mixing. Vacuum $\nu_\mu \leftrightarrow \nu_s$ oscillation has been proposed as one possible explanation of the Super Kamiokande atmospheric neutrino data. Based on the most recent primordial abundance measurements, we find that the effective number of neutrino species for big bang nucleosynthesis (BBN) is $N_\nu \lesssim 3.3$. Assuming that all three active neutrinos are light (with masses $\ll 1$ MeV), we examine BBN constraints on $\nu_\mu \leftrightarrow \nu_s$ mixing in two scenarios: (1) a negligible lepton asymmetry (the standard picture) and (2) the presence of a large lepton asymmetry which has resulted from an amplification by $\nu_\tau \leftrightarrow \nu_{s'}$ mixing ($\nu_{s'}$ being ν_s or another sterile neutrino species). The latter scenario has been proposed recently to reconcile the BBN constraints and large-angle $\nu_\mu \leftrightarrow \nu_s$ mixing. We find that the large-angle $\nu_\mu \leftrightarrow \nu_s$ mixing in the first scenario, which would yield $N_\nu \approx 4$, is ruled out as an explanation of the Super Kamiokande data. It is conceivably possible for the $\nu_\mu \leftrightarrow \nu_s$ solution to evade BBN bounds in the second scenario, but only if $200 \text{ eV}^2 \lesssim m_{\nu_\tau}^2 - m_{\nu_{s'}}^2 \lesssim 10^4 \text{ eV}^2$ is satisfied, and if ν_τ decays non-radiatively with a lifetime $\lesssim 10^3$ yr. This mass-squared difference implies $15 \text{ eV} \lesssim m_{\nu_\tau} \lesssim 100 \text{ eV}$ if $\nu_{s'}$ is much lighter than ν_τ . We conclude that maximal (or near maximal) $\nu_\mu \leftrightarrow \nu_\tau$ mixing is a more likely explanation of the Super Kamiokande data. [S0556-2821(99)00504-4]

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I. INTRODUCTION

The recent Super Kamiokande (SuperK) data on atmospheric neutrinos show strong evidence for neutrino oscillation in a channel which involves muon neutrinos and another neutrino species, and with a vacuum mass-squared difference $3 \times 10^{-4} \text{ eV}^2 < \delta m^2 < 7 \times 10^{-3} \text{ eV}^2$ and vacuum mixing angle satisfying $0.8 < \sin^2 2\theta \leq 1$ [1,2]. Measurements of the muon-like events and limits on the electron-like events preclude mixing between ν_μ and ν_e as a significant channel for ν_μ reduction [1]. A $\nu_\mu \leftrightarrow \nu_e$ mixing with these parameters would also yield a uniform suppression of the solar neutrino flux by roughly a factor of ~ 2 across the entire solar neutrino energy spectrum. However, such an energy independent suppression contradicts what is implied from the combined solar neutrino data [3]. This leaves two maximal or near maximal vacuum neutrino oscillation channels as possible avenues for solution: $\nu_\mu \leftrightarrow \nu_\tau$ mixing or $\nu_\mu \leftrightarrow \nu_s$ mixing, where ν_s is a sterile neutrino [e.g., a singlet under the $SU(3) \times SU(2) \times U(1)$ gauge symmetry of the standard model].

However, it has been pointed out by many groups that $\nu_\mu \leftrightarrow \nu_s$ mixing with the aforementioned parameters would violate the constraints from big bang nucleosynthesis (BBN) if all three active neutrinos are light compared to 1 MeV. This violation results from $\nu_\mu \leftrightarrow \nu_s$ mixing, essentially bringing an extra ν_s into chemical equilibrium with the three known species at the BBN epoch. In turn, these extra degrees of freedom imply a larger expansion rate and so would yield a primordial ${}^4\text{He}$ abundance that is too high to accommodate observationally determined abundance bounds [4–7].

There has been some confusion on this result lately due to a larger systematic uncertainty in the measured primordial ${}^4\text{He}$ abundance as argued by some groups and due to recent measurements of the primordial deuterium abundance. Fur-

thermore, the BBN constraint has been obtained by assuming a negligible lepton number asymmetry ($\leq 10^{-7}$), which has been shown to be an invalid assumption under circumstances where the lepton asymmetries in the neutrino sector can be amplified by active-sterile neutrino transformations [8,9]. It has been argued that a ν_τ mixing with a lighter sterile $\nu_{s'}$ may amplify an initially negligible lepton asymmetry to a large level that either is sufficient to suppress a subsequent $\nu_\mu \leftrightarrow \nu_s$ mixing [8] or can be partly converted to a positive chemical potential in the $\nu_e \bar{\nu}_e$ sector to lower the effective N_ν [10]. Both may conceivably lead to an evasion of the BBN bound on the $\nu_\mu \leftrightarrow \nu_s$ mixing parameters.

It is traditional in BBN studies to parametrize both the ${}^4\text{He}$ yield and the expansion rate of the universe at the BBN epoch in terms of an effective number of relativistic neutrino flavors, N_ν . In the context of neutrino oscillation discussions this convention is potentially confusing and misleading, since an efficient neutrino-matter-enhanced transformation could result in both a larger neutrino energy density and a lower ${}^4\text{He}$ yield (by way of a $\nu_e \bar{\nu}_e$ asymmetry) than in the standard picture. Here we will also employ N_ν in the traditional manner to facilitate comparison with the results of previous work. However, we will attempt to point out what the true underlying picture is in every case.

In this paper we intend to clarify the bound from BBN in light of the latest development in the primordial ${}^4\text{He}$ and D abundance measurements, and critique the possibility that the $\nu_\mu \leftrightarrow \nu_s$ mixing parameters required to explain the SuperK data might survive the BBN bound because of an amplified lepton asymmetry. We implicitly assume that all three active neutrinos are light, being relativistic at the BBN epoch.

II. FORMALISM

Here we briefly outline the formalism for active-sterile neutrino transformation used in this paper. A detailed treat-

ment of active-sterile neutrino mixings in the BBN epoch can be found in Ref. [5] or [7].

We adopt units in which $\hbar=c=k=1$. We denote by n_i the number density of a particle i relative to its equilibrium value at a temperature T . These equilibrium values are $2\zeta(3)T^3/\pi^2$ for photons, $3\zeta(3)T^3/2\pi^2$ for electrons and positrons, and $3\zeta(3)T^3/4\pi^2$ for each neutrino species. We describe the $\nu_\alpha \leftrightarrow \nu_s$ ($\alpha=e, \mu$ or τ) transformation channel at the BBN epoch with the following two differential equations:

$$\frac{dP_0}{dt} = \sum_{i=e, \nu_\beta; \beta \neq \alpha} \langle \Gamma(\nu_\alpha \bar{\nu}_\alpha \rightarrow i \bar{i}) \rangle (n_i n_{\bar{i}} - n_{\nu_\alpha} n_{\bar{\nu}_\alpha}), \quad (1)$$

$$\frac{d\mathbf{P}}{dt} = \mathbf{V} \times \mathbf{P} + \frac{dP_0}{dt} \hat{\mathbf{z}} - D\mathbf{P}_\perp. \quad (2)$$

In the equations, P_0 denotes the total number density of the mixture of active and sterile neutrino species, $P_0 = n_{\nu_\alpha} + n_{\nu_s}$, and \mathbf{P} denotes the composition of the mixture. In particular, $P_z = n_{\nu_\alpha} - n_{\nu_s}$. The other component of \mathbf{P} , $\mathbf{P}_\perp = P_x \hat{\mathbf{x}} + P_y \hat{\mathbf{y}}$, is an indication of the phase coherence of the oscillation channel. Quantities $\langle \Gamma \rangle$ are thermally averaged reaction rates. The vector \mathbf{V} represents the frequency and the axis of the oscillation in \mathbf{P} -space, and the D -term represents the damping of \mathbf{P}_\perp due to neutrino interactions. This latter term constantly acts to reduce a mixed neutrino state into flavor eigenstates of ν_α and ν_s .

At the epoch of BBN,

$$V_x = \frac{\delta m^2}{2E} \sin 2\theta, \quad V_y = 0, \quad V_z = -\frac{\delta m^2}{2E} \cos 2\theta + V_\alpha^L + V_\alpha^T, \quad (3)$$

where $\delta m^2 = m_{\nu_s}^2 - m_{\nu_\alpha}^2$ and θ are the usual vacuum mixing parameters of the mass-squared difference and effective vacuum 2×2 mixing angle ($\delta m^2 > 0$ if ν_s is heavier than ν_α), and E is the energy of the neutrinos. V_α^L is the contribution of the Mikheyev-Smirnov-Wolfenstein (MSW) matter-enhanced effect driven by asymmetries in the background plasma [11], and is approximately

$$V_\alpha^L \approx 0.13 G_F T^3 \left[8L_0/3 + 2(n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}) + \sum_{\nu_\beta \neq \nu_\alpha} (n_{\nu_\beta} - n_{\bar{\nu}_\beta}) \right] \quad (4)$$

where G_F is the Fermi constant, T is the plasma temperature, and L_0 represents the contributions from the baryonic asymmetry and electron-positron asymmetry, $\sim 10^{-9}$. Here V_α^T is the contribution due to finite temperature neutrino mass renormalization effects [11] and, for $\alpha = \mu$ or τ , is

$$V_\alpha^T \approx -11(n_{\nu_\alpha} + n_{\bar{\nu}_\alpha}) G_F^2 E T^4. \quad (5)$$

Meanwhile, for $\alpha = \mu$ or τ , we also have

$$D \sim 0.5 G_F^2 T^4 E. \quad (6)$$

It has been shown that adopting the average neutrino energy $E \approx 3.151T$ in Eqs. (3)–(6) and thermally averaging interaction rates give a fairly good description of active-sterile neutrino transformations in BBN [12].

The same equations (1)–(5) also apply to the $\bar{\nu}_\alpha \leftrightarrow \bar{\nu}_s$ mixing if we switch n_{ν_α} (n_{ν_β}) and $n_{\bar{\nu}_\alpha}$ ($n_{\bar{\nu}_\beta}$) in the expression of V_α^L . Here we add an overbar on top of all variables associated with the $\nu_\alpha \leftrightarrow \nu_s$ system to denote the corresponding variables in the $\bar{\nu}_\alpha \leftrightarrow \bar{\nu}_s$ system.

The initial condition of the universe is taken to be a pure $\nu_\alpha \bar{\nu}_\alpha$ system (i.e., $P_0 = \bar{P}_0 = P_z = \bar{P}_z = 1$ and $P_x = \bar{P}_x = P_y = \bar{P}_y = 0$) at an initial temperature

$$T_{\text{init}} \gtrsim 15 \left| \frac{\delta m^2 \cos 2\theta}{eV^2} \right|^{1/6}. \quad (7)$$

This temperature is taken to be high enough that V_α^T dominates over $\delta m^2/2E$, guaranteeing that any neutrino transformation is severely suppressed. (Note, however, that T_{init} has to be below the temperature of the quark-hadron phase transition, ~ 150 MeV [13].)

Barring the existence of a large initial lepton asymmetry generated by unknown processes at $T \gtrsim 150$ MeV, we will assume a negligible initial lepton asymmetry L (e.g., at the same level as L_0). Here a significant L can be generated only by an active-sterile neutrino transformation during the BBN epoch.

III. BBN BOUND WHEN LEPTON ASYMMETRY IS NEGLIGIBLE

The primordial ${}^4\text{He}$ yield Y in standard BBN is principally a function of the expansion rate of the universe, as this determines the temperature of weak freeze-out and, hence, the overall neutron-to-proton ratio. Secondly, the ${}^4\text{He}$ abundance also depends on the baryon density N_B (usually expressed as $\eta \equiv N_B/N_\gamma$), albeit only weakly. Therefore, given an η inferred from the primordial deuterium abundance D/H (which depends sensitively on η), a bound on N_ν in the standard BBN picture can be obtained based on the measured primordial ${}^4\text{He}$ abundance [14]. The same bound can be used to constrain active-sterile neutrino mixings, because such mixings can produce extra neutrino energy density. However, one should use caution here because these scenarios could also result in a larger number of ν_e 's over $\bar{\nu}_e$'s (a positive $\nu_e \bar{\nu}_e$ asymmetry), which could lead to a reduced neutron-to-proton ratio and, hence, a lower Y .

Previous bounds on these mixing channels have been calculated assuming a small η ($\approx 3 \times 10^{-10}$) because an accurate measurement of the primordial D/H was lacking. It is now, however, established that the primordial D/H lies at the low end of the range previously thought: $\text{D}/\text{H} \approx 3.4 \pm 0.3 \times 10^{-5}$ [15]. This low value infers a higher $\eta \approx 5.2 \pm 0.3 \times 10^{-10}$, which predicts a slightly higher primordial ${}^4\text{He}$ abundance Y for the standard picture $N_\nu = 3$. The upper limit on N_ν based on the measured Y should then become tighter [16].

However, complicating this issue is the debate over the

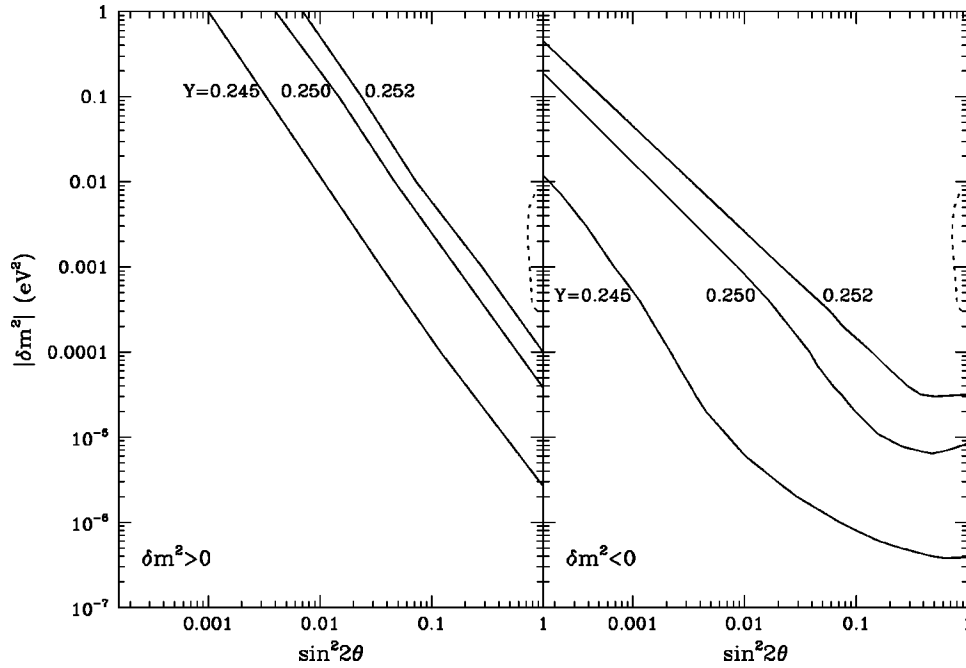


FIG. 1. ${}^4\text{He}$ yields of the ν_μ (or ν_τ) $\leftrightarrow \nu_s$ mixing (for $\eta = 4.7 \times 10^{-10}$). Mixings that yield $Y \geq 0.25$, including the region inside the dashed line that is compatible with the Super Kamiokande data, are ruled out by BBN.

systematic uncertainties in the observationally inferred Y . Based on 62 low metallicity HII regions, Olive *et al.* [17] estimated $Y = 0.234 \pm 0.002(\text{stat.}) \pm 0.005(\text{syst.})$; or $Y = 0.237 \pm 0.003(\text{stat.}) \pm 0.005(\text{syst.})$, when the controversial object I Zw 18 (also the most metal-deficient object of the sample) is ignored. Based on another sample which partially overlaps with that of Olive *et al.*, Izotov and Thuan [18] have claimed $Y = 0.244 \pm 0.002$, partly because they deduced a higher Y for I Zw 18. Disagreement aside, the two values do not seriously contradict each other given the systematic uncertainty ~ 0.005 of the estimates. With the systematic uncertainty in mind, we can still set a conservative upper limit $Y < 0.25$ for the purpose of constraining the neutrino oscillation physics at the BBN epoch. A higher Y would not only be inconsistent with both estimates, it could also possibly be inconsistent with the morphology of the horizontal branch stars in globular clusters [19].

Thus, given $D/H \approx 3.4 \pm 0.3 \times 10^{-5}$ and $Y < 0.25$, we can set a rather conservative bound on N_ν at the 95% C.L.,

$$N_\nu \leq 3.3. \quad (8)$$

This bound can be used to constrain the active-sterile neutrino transformations in the BBN epoch, but it is more precise and convenient to calculate directly the helium yield of neutrino oscillations, because it is ultimately the helium mass fraction that is compared to observations. We took $\eta = 4.6 \times 10^{-10}$ and recalculated the ${}^4\text{He}$ yield in the presence of active-sterile neutrino mixing. (Our calculations are similar to those in Shi, Schramm and Fields [7].) Figure 1 shows the result. A parametrization of the bound on the $\nu_\mu \leftrightarrow \nu_s$ mixing can be obtained:

$$|\delta m^2| \sin^4 2\theta \leq 10^{-5} (10^{-7}) \text{eV}^2 \quad \text{for } \delta m^2 > (<) 0. \quad (9)$$

The $\nu_\mu \leftrightarrow \nu_s$ mixing solution to the SuperK atmospheric neutrino data is clearly ruled out by BBN if lepton asymmetry is negligible ($L < 10^{-7}$ in this case). To put this result in perspective, our results suggest that a $\nu_\mu \leftrightarrow \nu_s$ mixing solution to the SuperK data in this scenario would imply $N_\nu = 3.9$.

IV. BBN BOUND IN THE PRESENCE OF AN AMPLIFIED LEPTON ASYMMETRY

A resonant $\nu_\alpha \leftrightarrow \nu_s$ transformation can amplify the lepton asymmetry in the ν_α sector to a level

$$L_{\nu_\alpha} \equiv \frac{3(n_{\nu_\alpha} - n_{\bar{\nu}_\alpha})}{8} \sim \pm \frac{|\delta m^2 / \text{eV}|}{10T_6^4}, \quad (10)$$

where $T_6 = T/\text{MeV}$ [9]. Generation of this lepton number does not necessarily force ν_s into chemical equilibrium if the mixing is small enough to satisfy the constraints in Fig. 1 [9].

To calculate the amplification of the lepton asymmetry by $\nu_\alpha \leftrightarrow \nu_s$ mixing, we take the initial lepton number to be negligibly small. Note that in this limit $L_{\nu_\alpha} = 3(P_z - \bar{P}_z)/16$. From Eq. (2) and its anti-neutrino counterpart, L_{ν_α} satisfies the approximate equation

$$\begin{aligned} \frac{dL_{\nu_\alpha}}{dt} &\approx \frac{DV_x^2}{V_x^2 + [V_0 - \beta(2L_{\nu_\alpha} + L_0)]^2} \\ &\times \left\{ \frac{3V_0\beta(2L_{\nu_\alpha} + L_0)P_z}{4\{V_x^2 + [V_0 + \beta(2L_{\nu_\alpha} + L_0)]^2\}L_{\nu_\alpha}} - 1 \right\} L_{\nu_\alpha}, \end{aligned} \quad (11)$$

which is valid in the adiabatic limit, $|\mathbf{V}| \gg |d\mathbf{V}/dt|/|\mathbf{V}|$. In Eq. (11) we use $V_0 = (V_z + \bar{V}_z)/2$, and $\beta = (V_z - \bar{V}_z)/2(2L_{\nu_\alpha} + L_0) \approx 0.35G_F T^3$. We have also assumed $P_z \sim 1$, because the $\nu_\alpha \leftrightarrow \nu_s$ mixing would violate the BBN bound in Fig. 1 if $P_z \ll 1$.

Equation (11) is a damping equation for $2L_{\nu_\alpha} + L_0$, unless $V_0 > 0$. Here $V_0 > 0$ if $\delta m^2 < 0$ (i.e., a heavier ν_α) and if the temperature T drops below the resonant temperature for $\nu_\alpha \leftrightarrow \nu_s$ (see the Appendix),

$$T_{\text{res}} \approx 22 \left| \frac{\delta m^2}{1 \text{ eV}^2} \right|^{1/6} \text{ MeV} \quad \text{for } \alpha = \mu, \tau. \quad (12)$$

For $T > T_{\text{res}}$, $2L_{\nu_\alpha} + L_0$ will be driven toward zero. In this case $V_{\nu_\alpha}^L$ approaches zero, unless the initial asymmetry L_0 (whose definition may be expanded here to also include the neutrino asymmetry generated by other active-sterile neutrino mixings) is too large for the damping to be efficient. Efficient damping occurs when the generated lepton asymmetry in other neutrino species satisfies

$$|L| \lesssim 10^{-4} |\delta m^2 \sin 2\theta|^{1/2}, \quad (13)$$

so that $(dL_{\nu_\alpha}/dt)/L_{\nu_\alpha} < H$, where H is the Hubble expansion rate.

Equation (2) and its anti-neutrino counterpart are non-linear equations. It has two attractors which give the asymptotic values of Eq. (10). The asymmetry L_{ν_α} calculated through numerical integrations of these equations exhibits chaotic behavior for T just below T_{res} [9]. In this phase L_{ν_α} oscillates chaotically between the two attractors. For sufficiently low T , L_{ν_α} approaches asymptotically one of the two attractors. It is in this asymptotic phase when the adiabatic condition is satisfied and Eq. (11) applies.

It is conceivable that a $\nu_{\tau \leftrightarrow \nu_s}$ mixing satisfying

$$\left| \frac{\delta m'^2}{\text{eV}^2} \right|^{1/6} \sin^2 2\theta \gtrsim 10^{-11} \quad (14)$$

and the the BBN bound Eq. (9) can generate $|L_{\nu_\tau}| \gtrsim 10^{-7}$ [9,20], which in turn suppresses the bad BBN effects of $\nu_\mu \leftrightarrow \nu_s$ transformation with $\delta m^2 \sim 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta \sim 1$, and so relaxes the BBN bound on the $\nu_\mu \leftrightarrow \nu_s$ mixing. (Here $\nu_{s'}$ and ν_s could be in principle the same species but are distinguished here for clarity.) But as we will demonstrate below, while this is not impossible, there are severe constraints on the required $\nu_{\tau \leftrightarrow \nu_{s'}}$ mixing parameters for this loophole to be realized.

To distinguish between the $\nu_{\tau \leftrightarrow \nu_{s'}}$ transformation and the $\nu_\mu \leftrightarrow \nu_s$ transformation, all quantities applying to the former process are marked with a prime. To suppress the transformation due to a near maximal $\nu_\mu \leftrightarrow \nu_s$ mixing, the lepton asymmetry has to be amplified early by the $\nu_{\tau \leftrightarrow \nu_{s'}}$ transformation channel before any significant $\nu_\mu \leftrightarrow \nu_s$ transformation can occur. This requires the ν_s production rate at the $\nu_{\tau \leftrightarrow \nu_{s'}}$ resonance temperature T'_{res} to satisfy

$$D \left(\frac{\delta m^2 \sin 2\theta}{V_\mu^T} \right)^2 < H, \quad (15)$$

with $H = 5.5T^2/m_{\text{pl}}$. Here $m_{\text{pl}} \approx 1.22 \times 10^{28} \text{ eV}$ is the Planck mass. Employing T'_{res} and V_μ^T we obtain, from Eq. (15),

$$|\delta m'^2| > 10 |\delta m^2 \sin 2\theta|^{4/3} \sim 10^{-2} \text{ eV}^2. \quad (16)$$

Once the lepton asymmetry is amplified, the $\nu_\mu \leftrightarrow \nu_s$ mixing will have

$$V_x = -\frac{\delta m^2}{2E} \sin 2\theta,$$

$$V_z = -\frac{\delta m^2}{2E} \cos 2\theta + V_\mu^T \pm V_\mu^L \approx V_\mu^T \pm \beta(2L_{\nu_\mu} + L_{\nu_\tau}), \quad (17)$$

where “+” (“−”) applies to the $\nu_\mu \leftrightarrow \nu_s$ ($\bar{\nu}_\mu \leftrightarrow \bar{\nu}_s$) transformation channel. If L_{ν_τ} is positive (negative), the $\nu_\mu \leftrightarrow \nu_s$ ($\bar{\nu}_\mu \leftrightarrow \bar{\nu}_s$) channel will experience a resonance, regardless of the sign of δm^2 at a neutrino energy

$$E_{\text{res}} \approx \frac{\beta |L_{\nu_\tau}|}{V_\mu^T/E} \approx \frac{|L_{\nu_\tau}|}{63G_FT} \quad (18)$$

(assuming initially $L_{\nu_\mu} = 0$). The resonance moves from lower energies to higher energies in the mu neutrino spectrum as $|L_{\nu_\tau}|$ grows. At the same time, the $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_s$ ($\nu_\mu \leftrightarrow \nu_s$) transformation channel is suppressed by L_{ν_τ} as a consequence of having an even larger $|V_z|$. The net result is a newly generated L_{ν_μ} with a sign opposite to that of L_{ν_τ} . If this L_{ν_μ} becomes as large as $\sim -L_{\nu_\tau}/2$, the $\nu_\mu \leftrightarrow \nu_s$ mixing will be unhindered and will eventually bring ν_s into equilibrium. Based on this consideration, Foot and Volkas [20,21] argued for a requirement on the $\nu_{\tau \leftrightarrow \nu_s}$ mixing

$$|\delta m'^2| \gtrsim 20 \text{ eV}^2. \quad (19)$$

Here we show that this requirement is too weak. We find that a more stringent requirement can be obtained based on the same consideration at the beginning stage of the L_{ν_τ} growth, when both L_{ν_τ} and the mu neutrino resonance energy E_{res} are very small. For definitiveness, we assume that $L_{\nu_\tau} > 0$ so that both the $\bar{\nu}_\tau \leftrightarrow \bar{\nu}_{s'}$ transformation and the $\nu_\mu \leftrightarrow \nu_s$ transformation encounter resonances.

Each of these two resonances occurs in a finite neutrino energy bin (resonance width). Outside the resonance energy bins, the effective mixing between active neutrinos and sterile neutrinos quickly diminishes and becomes negligible. This feature enables a simple semi-analytical approach to accurately calculate the growth of lepton number asymmetries by tracking only neutrinos in resonance regions. The details of this semi-analytical approach are presented in the Appendix.

The constraint on $\delta m'^2$ results from the requirement that $|L_{\nu_\tau}|$ be much larger than $|L_{\nu_\mu}|$ at any temperature as the $\nu_\mu \leftrightarrow \nu_s$ resonance sweeps through the entire neutrino energy spectrum. In other words, we must have

$$f(\epsilon_{\text{res}}) \delta\epsilon_{\text{res}} R \left| \frac{\delta\epsilon_{\text{res}}}{d\epsilon_{\text{res}}/dt} \right| < \frac{4}{3} (L_{\nu_\tau} + 2L_{\nu_\mu}) \quad (20)$$

for any ϵ_{res} . In the equation, $f(\epsilon) = [2/3\zeta(3)][\epsilon^2/(e^\epsilon + 1)]$ is the neutrino distribution function. $\delta\epsilon_{\text{res}}$ is the energy width of the resonance. $f(\epsilon_{\text{res}}) \delta\epsilon_{\text{res}}$ is therefore the fraction of mu neutrinos undergoing resonance. R is the resonant transition rate. $\delta\epsilon_{\text{res}}/|d\epsilon_{\text{res}}/dt|$ is the duration of the resonance at ϵ_{res} . The energy width of the resonance depends on whether the resonant transition is collision-dominated ($D > V_x$), or oscillation-dominated ($D < V_x$):

$$\delta\epsilon_{\text{res}} \sim \begin{cases} 2|D\partial\epsilon_{\text{res}}/\partial V_z| & \text{if } D > V_x \\ 2|V_x\partial\epsilon_{\text{res}}/\partial V_z| & \text{if } D < V_x \end{cases} \quad (21)$$

Likewise, the resonant transition rate depends on these parameters:

$$R \approx \begin{cases} V_x^2/D & \text{if } D > V_x \\ V_x & \text{if } D < V_x \end{cases} \quad (22)$$

We only consider the case where $\epsilon_{\text{res}} \ll 1$, because this is when L_{ν_τ} is in its initial stage of growth ($L_{\nu_\tau} \ll 10^{-7}$) and is most easily matched by a competing L_{ν_μ} . In this case, $T \approx T'_{\text{res}}$ (the temperature at which L_{ν_τ} growth starts), and $f(\epsilon_{\text{res}}) \approx \epsilon_{\text{res}}^2/1.8$. We can further rewrite $|\dot{\epsilon}_{\text{res}}| \equiv H\epsilon_{\text{res}}|d \ln \epsilon_{\text{res}}/d \ln T|$. Then with Eq. (12) and the expression for H , Eq. (20) becomes

$$|\delta m'^2/1 \text{ eV}^2|^{11/6} > 2 \times 10^4 \epsilon_{\text{res}}^{-1} |d \ln \epsilon_{\text{res}}/d \ln T|^{-1} |\delta m^2/10^{-3} \text{ eV}^2|^2 \quad \text{if } D > V_x; \quad (23a)$$

$$|\delta m'^2/1 \text{ eV}^2|^{17/6} > 10^3 \epsilon_{\text{res}}^{-3} |d \ln \epsilon_{\text{res}}/d \ln T|^{-1} |\delta m^2/10^{-3} \text{ eV}^2|^3 \quad \text{if } D < V_x. \quad (23b)$$

In the initial stage of rapid L_{ν_τ} growth, $|d \ln \epsilon_{\text{res}}/d \ln T|$ can be related to the growth rate of L_{ν_τ} by $|d \ln \epsilon_{\text{res}}/d \ln T| \approx |d \ln L_{\nu_\tau}/d \ln T - 2| \approx |d \ln L_{\nu_\tau}/d \ln T|$ (with L_{ν_μ} safely ignored). Our semi-analytical calculations show that in the initial exponential stage of L_{ν_τ} growth when L_{ν_τ} is $\ll 10^{-7}$, $|d \ln L_{\nu_\tau}/d \ln T|$ is well approximated by

$$\left| \frac{d \ln L_{\nu_\tau}}{d \ln T} \right| \approx 6 \times 10^6 \sin 2\theta', \quad (24)$$

independent of $\delta m'^2$. Of course, $\sin 2\theta'$ here must satisfy the BBN bound Eq. (9).

Equations (23) and (24) show that the most stringent requirement on $m_{\nu_\tau}^2 - m_{\nu_s}^2$, does indeed come not from $\nu_\mu \leftrightarrow \nu_s$ resonances at $\epsilon_{\text{res}} \sim 3$, but from resonances centered at the smallest possible ϵ_{res} as long as the ν_μ or $\bar{\nu}_\mu$ transition probability in that resonance energy bin is $\ll 1$. This condition, expressed as

$$R \left| \frac{\delta\epsilon_{\text{res}}}{d\epsilon_{\text{res}}/dt} \right| \leq 0.1, \quad (25)$$

can be rewritten as

$$m_{\nu_\tau}^2 - m_{\nu_s}^2 \geq \begin{cases} 200(|m_{\nu_\mu}^2 - m_{\nu_s}^2|/10^{-3} \text{ eV}^2)^{3/4} \text{ eV}^2 & \text{for } \sin^2 2\theta' \geq 10^{-8} \\ (\sin 2\theta')^{-1/2} (|m_{\nu_\mu}^2 - m_{\nu_s}^2|/10^{-3} \text{ eV}^2) \text{ eV}^2 & \text{for } \sin^2 2\theta' \leq 10^{-8} \end{cases} \quad (28)$$

Equation (28) implies that we must have

$$m_{\nu_\tau} \geq 15 \text{ eV} \quad (29)$$

to successfully suppress the $\nu_\mu \leftrightarrow \nu_s$ transformation in BBN.

$$\epsilon_{\text{res}} \geq \left| \frac{\delta m'}{1 \text{ eV}^2} \right|^{-1/2} \left| \frac{d \ln \epsilon_{\text{res}}}{d \ln T} \right|^{-1/3} \left| \frac{\delta m^2}{10^{-3} \text{ eV}^2} \right|^{2/3}, \quad (26)$$

regardless of the value of D/V_x . It can be shown that ϵ_{res} is in the oscillation-dominated regime if

$$\epsilon_{\text{res}} \lesssim 0.25 \left| \frac{\delta m^2}{10^{-3} \text{ eV}^2} \right|^{1/2} \left| \frac{\delta m'^2}{1 \text{ eV}^2} \right|^{-1/2}. \quad (27)$$

Therefore, the most stringent requirement on $\delta m'^2$ comes from the oscillation-dominated regime for $\sin^2 2\theta' \geq 10^{-8}$ (while $|d \ln \epsilon_{\text{res}}/d \ln T| \geq 10^3$), and from collision-dominated regime for $\sin^2 2\theta' \leq 10^{-8}$ (while $|d \ln \epsilon_{\text{res}}/d \ln T| \leq 10^3$).

Combining Eqs. (23), Eq. (24), and Eq. (26) yields a requirement on the mass-squared-difference necessary to effect suppression of $\nu_\mu \leftrightarrow \nu_s$ transformation at the Super Kamio-kande level:

Tau neutrinos this massive would contribute a fraction of the critical density $\Omega_\nu \approx 0.5h_{50}^{-2}$ (where h_{50} is the Hubble constant in the units of 50 km/sec/Mpc) in the form of hot dark matter (HDM) [22]. When normalized to yield the observed structure today, a matter-dominated flat ($\Omega_m=1$) model universe with $\Omega_\nu \geq 0.3$ (the remainder being mostly cold dark matter) yields too few damped Lyman- α systems (protogalaxies) at a redshift $z \geq 3$ to accommodate observations [23,24]. Because of the free-streaming of neutrinos, the hot dark matter component reduces the density fluctuation amplitudes at the galaxy mass scale and causes galaxy-sized structures to form too late ($z \lesssim 1$). Even an $\Omega_\nu = 0.2$ flat, matter-dominated model may still be in disagreement with observations [25]. Therefore, any light stable and weakly interacting neutrino more massive than $m_\nu \gtrsim 5h_{50}^2$ eV will have trouble with structure formation considerations. Models with these neutrinos cannot be rescued by pushing h_{50} much higher, because both the age of the universe and the observed structure today require $h_{50} \approx 1$ [26]. If, as implied by the cosmic deceleration measurements [27], the total matter density in our universe is less than the critical density, the incompatibility of structure formation at high z and $\Omega_\nu \geq 0.2$ will become even worse—HDM would then comprise a larger fraction of the matter density and so reduce the density fluctuations at small scales even more effectively.

To simultaneously have $m_{\nu_\tau} \gtrsim 15$ eV and successful structure formation at high redshifts, the massive tau neutrinos would have to decay before making an imprint on the cosmic density fluctuation spectrum. Since sub-horizon-sized density fluctuations (i.e., those affected by the free streaming of neutrinos) were frozen before the epoch of matter-radiation equality and grew only after that, the density fluctuations residing in other dark matter components would not be affected by the density fluctuations in the tau neutrino spatial distribution as long as these ν_τ 's decayed before the matter-radiation equality epoch. The lifetime of ν_τ must then be

$$\tau_{\nu_\tau} \lesssim 10^3 \text{ yr.} \quad (30)$$

The decay cannot be radiative, as it would violate the bound on neutrino electromagnetic dipole moments [28].

For those $\nu_\tau \leftrightarrow \nu_s$ mixings with $10^{-2} \text{ eV}^2 \leq |\delta m'^2| \leq 10^{-1} \text{ eV}^2$, the subsequent $\nu_\mu \leftrightarrow \nu_s$ resonance occurs during the chaotic phase of the $\nu_\tau \leftrightarrow \nu_{s'}$ evolution. Our numerical calculations show that in this case the two mixing systems are coupled and continue to be chaotic. The $\nu_\mu \leftrightarrow \nu_s$ transformation will in fact soon dominate the chaotic oscillation process because this transformation channel has a larger $|V_x|$ than the BBN-constrained $\nu_\tau \leftrightarrow \nu_{s'}$ transformation channel. Therefore, for $10^{-2} \text{ eV}^2 \leq |\delta m'^2| \leq 10^{-1} \text{ eV}^2$, the $\nu_\mu \leftrightarrow \nu_s$ transformation cannot be suppressed by the presence of the $\nu_\tau \leftrightarrow \nu_{s'}$ transformation channel.

The $\nu_\tau \leftrightarrow \nu_{s'}$ system is also bounded by BBN at large $\delta m'^2$. If a large ($L_{\nu_\tau} \gtrsim 0.1$) asymmetry is generated by the $\nu_\tau \leftrightarrow \nu_{s'}$ transformation before $T \sim 5$ MeV, the produced $\nu_{s'}$

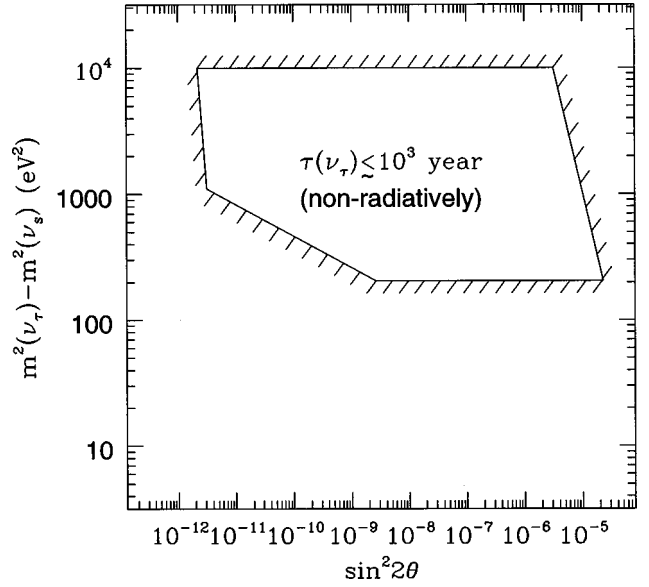


FIG. 2. The parameter space of $\nu_\tau \leftrightarrow \nu_{s'}$ mixing and the additional requirements on ν_τ that may alleviate the BBN constraint on the $\nu_\mu \leftrightarrow \nu_s$ mixing parameters which can accommodate the Super Kamiokande data.

or $\bar{\nu}_{s'}$, and the re-thermalized $\nu_\tau \bar{\nu}_\tau$ will increase N_ν by 0.5, violating the BBN bound. This imposes an upper bound on $|\delta m'^2|$ [10,29]:

$$|\delta m'^2| \lesssim 10^4 \text{ eV}^2, \quad (31)$$

implying $m_{\nu_\tau} \lesssim 100$ eV if $\nu_{s'}$ is much lighter.

In Fig. 2 we plot the conditions that ν_τ and the $\nu_\tau \leftrightarrow \nu_{s'}$ mixing must satisfy in order to alleviate the BBN constraint on the maximal vacuum $\nu_\mu \leftrightarrow \nu_s$ mixing that may be implied by the SuperK data.

We finally very briefly comment on another scenario in which a positive lepton asymmetry generated by a $\nu_\tau \leftrightarrow \nu_{s'}$ mixing may be partially converted into the $\nu_e \bar{\nu}_e$ sector by a matter-enhanced $\nu_\tau \leftrightarrow \nu_e$ transformation. This positive ν_e chemical potential can reduce the effective N_ν and may make some room for extra neutrino energy density. A previous paper has argued that the reduction in N_ν can be as large as $\Delta N_\nu \approx 0.5$ [10]. This will not accommodate for an extra $\Delta N_\nu = 0.9$ produced by the $\nu_\mu \leftrightarrow \nu_s$ mixing in question, given the ^4He -derived bound $N_\nu \lesssim 3.3$. We also note that even the 0.5 reduction in N_ν may be overestimated [29].

V. SUMMARY

We have shown that based on the observationally inferred primordial abundance of ^4He ($Y < 0.25$) and deuterium ($\text{D}/\text{H} \approx 3.4 \pm 0.3 \times 10^{-5}$), big bang nucleosynthesis yields a stringent bound on the effective number of neutrino species (energy densities) during the BBN epoch, $N_\nu \lesssim 3.3$. This bound can be employed to constrain active-sterile neutrino mixings, as plotted in Fig. 1. With the preassumption that all

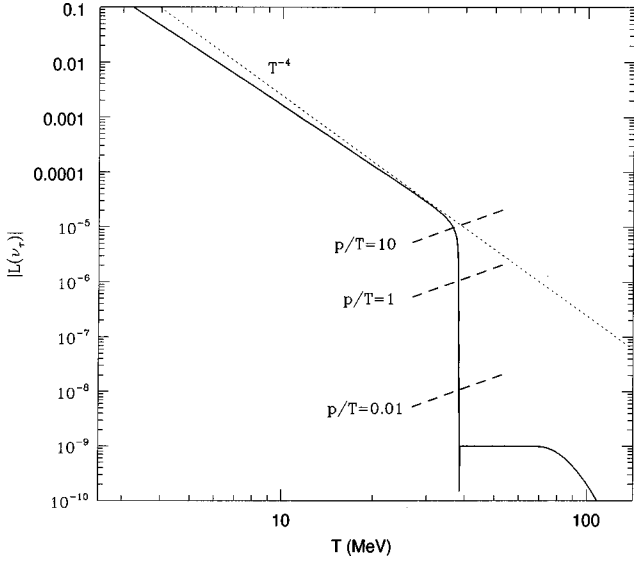


FIG. 3. The growth of the tau neutrino asymmetry as a result of the tau neutrino-sterile neutrino mixing, assuming $m_{\nu_\tau}^2 - m_{\nu_{s'}}^2 = 50 \text{ eV}^2$ and $\sin^2 2\theta' = 10^{-4}$. The intersections between the growth curve for L_{ν_τ} and the dashed lines indicate when resonances occur for ν_μ (if $L_{\nu_\tau} > 0$) or $\bar{\nu}_\mu$ (if $L_{\nu_\tau} < 0$) neutrinos with momentum p .

three active neutrinos are light compared to 1 MeV, it rules out the $\nu_\mu \leftrightarrow \nu_s$ mixing explanation of the Super Kamiokande atmospheric neutrino data. The only way to circumvent this bound is if there is a simultaneous mixing between ν_τ and a lighter sterile neutrino $\nu_{s'}$ with the following properties:

- (1) A mixing mass-squared difference $200 \text{ eV}^2 \leq m_{\nu_\tau}^2 - m_{\nu_{s'}}^2 \leq 10^4 \text{ eV}^2$ and a non-radiative ν_τ decay lifetime $\tau_{\nu_\tau} \leq 10^3 \text{ yr}$. The mass-squared difference implies a ν_τ mass between $\sim 15 \text{ eV}$ and $\sim 100 \text{ eV}$ if $\nu_{s'}$ is much lighter than ν_τ .
- (2) $[(m_{\nu_\tau}^2 - m_{\nu_{s'}}^2)/\text{eV}^2]^{1/6} \sin^2 2\theta \gtrsim 10^{-11}$ and $[(m_{\nu_\tau}^2 - m_{\nu_{s'}}^2)/\text{eV}^2] \sin^4 2\theta \leq 10^{-7}$, where θ is the vacuum mixing angle between ν_τ and $\nu_{s'}$.

We conclude that the $\nu_\mu \leftrightarrow \nu_\tau$ vacuum oscillation channel with maximal or near maximal mixing angle provides a more natural explanation for the Super Kamiokande data.

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APPENDIX

Our semi-analytical calculations track the movement (in energy space) of resonances in the neutrino energy spectrum and the resonant conversion rate of neutrinos in the resonance regions. For $\nu_\alpha \leftrightarrow \nu_s$ mixing with vacuum mixing parameters $\delta m^2 = m_{\nu_s}^2 - m_{\nu_\alpha}^2 < 0$ and $\sin^2 2\theta$, both the conver-

sion of ν_α and $\bar{\nu}_\alpha$ are calculated. If we denote $\epsilon_{\text{res}}^{(\pm)}$ as the resonant neutrino energy (divided by the temperature) for ν_α (“+”) and $\bar{\nu}_\alpha$ (“-”), the rate of resonant conversion, when normalized by the total ν_α or $\bar{\nu}_\alpha$ number density, is

$$\mathcal{R}^{(\pm)} \approx \pi f[\epsilon_{\text{res}}^{(\pm)}] \left| \frac{\partial \epsilon_{\text{res}}^{(\pm)}}{\partial V_z^{(\pm)}} \right| V_x^2 = \pi f[\epsilon_{\text{res}}^{(\pm)}] \frac{|\delta m^2| \sin^2 2\theta}{4T}. \quad (\text{A1})$$

This equation is essentially the left-hand side of Eq. (20) except for a factor π which comes from a detailed integration of transition probability over the entire resonance region. The rate of change of L_{ν_α} is therefore

$$\frac{dL_{\nu_\alpha}}{dt} = \frac{3}{8} (\mathcal{R}^{(+)} - \mathcal{R}^{(-)}). \quad (\text{A2})$$

When L_{ν_α} is small (i.e., $|V_\alpha^L| \ll |V_\alpha^T|$), the effect of matter-antimatter asymmetry on the $\nu_\alpha \leftrightarrow \nu_s$ and $\bar{\nu}_\alpha \leftrightarrow \bar{\nu}_s$ oscillation is small. Both oscillation systems will encounter resonances, at an energy

$$\epsilon_{\text{res}}^{(\pm)} \approx \left(\frac{|\delta m^2|}{44 G_F^2 T^6} \right)^{1/2} \left(1 \pm \frac{0.35 G_F T^3 (2L_{\nu_\alpha} + L_0)}{|\delta m^2|/2T} \right) \quad (\text{A3})$$

[see Eqs. (3) to (5)]. Equation (A2) is approximately

$$\frac{dL_{\nu_\alpha}}{dt} \approx \frac{\pi}{4} \frac{df(\epsilon)}{d\epsilon} G_F T^3 \sin^2 2\theta (L_{\nu_\alpha} + L_0/2), \quad (\text{A4})$$

with $df(\epsilon)/d\epsilon$ evaluated at $\epsilon = |\delta m^2/44 G_F^2 T^6|^{1/2}$. This is apparently a damping equation for $(L_{\nu_\alpha} + L_0/2)$ if $df(\epsilon)/d\epsilon < 0$ (when $\epsilon < 2.217$), and a growth equation for $(L_{\nu_\alpha} + L_0/2)$ if $df(\epsilon)/d\epsilon > 0$ (when $\epsilon > 2.217$). Therefore $(L_{\nu_\alpha} + L_0/2)$ is always damped to 0 at temperatures $T > T'_{\text{res}}$, when $\epsilon_{\text{res}}^{(\pm)}$ is small; and $(L_{\nu_\alpha} + L_0/2)$ grows for $T < T'_{\text{res}}$ when $\epsilon_{\text{res}}^{(\pm)}$ is large enough.

In Fig. 3 we plot the result of our numerical calculation based on Eqs. (A1) and (A2) for $\nu_\tau \leftrightarrow \nu_{s'}$ mixing, assuming $m_{\nu_\tau}^2 - m_{\nu_{s'}}^2 = 50 \text{ eV}^2$ and $\sin^2 2\theta' = 10^{-4}$. It is for the most part similar to the thick solid line in Fig. 1 of Foot [21], which employs the same parameters. There are minor differences that are readily identifiable: (1) our result tracks T^{-4} more closely in the “power-law growth” epoch; (2) L_{ν_τ} in our results does not switch sign at the initial point of growth. The sign difference is not surprising because of the chaotic character of the growth, which introduces a sign ambiguity to the problem [9]. Similar calculations for other choices of $m_{\nu_\tau}^2 - m_{\nu_{s'}}^2$ and $\sin^2 2\theta'$ show that the temperature at which the L_{ν_τ} growth starts is approximately given by Eq. (12).

As an illustration, also plotted in Fig. 3 are the $|L_{\nu_\tau}|$ required for the $\nu_\mu \leftrightarrow \nu_s$ or $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_s$ resonance to occur at ν_μ or $\bar{\nu}_\mu$ energies $\epsilon_{\text{res}} \equiv p_{\text{res}}^{(\mu)}/T = 0.01, 1, 10$. (The parameters for the $\nu_\mu \leftrightarrow \nu_s$ mixing are $\delta m^2 = 10^{-3} \text{ eV}^2$ and $\sin 2\theta = 1$.) It can

be seen from the figure that the lower energy component of the ν_μ or $\bar{\nu}_\mu$ neutrinos encounters the resonances first when $|L_{\nu_\tau}|$ is very small, and the resonance region moves through the ν_μ or $\bar{\nu}_\mu$ spectrum to higher neutrino energies as $|L_{\nu_\tau}|$ is very small, and the resonance region moves through the ν_μ or $\bar{\nu}_\mu$ spectrum to higher neutrino energies as $|L_{\nu_\tau}|$ becomes larger. Essentially all ν_μ or $\bar{\nu}_\mu$ encounter resonances at $T \approx T'_{\text{res}}$.

In Fig. 4 we show $|d \ln L_{\nu_\tau} / d \ln T|$ as a function of the $\nu_\mu \leftrightarrow \nu_s$ vacuum mixing parameters, $m_{\nu_\tau}^2 - m_{\nu_s}^2$ and $\sin^2 2\theta'$, in the initial exponential stage of L_{ν_τ} growth when L_{ν_τ} is $\ll 10^{-7}$. This figure shows that $|d \ln L_{\nu_\tau} / d \ln T|$ is approximately a linear function of $\sin 2\theta'$, and is insensitive to $m_{\nu_\tau}^2 - m_{\nu_s}^2$. An analytical fit to this numerical result yields Eq. (24).

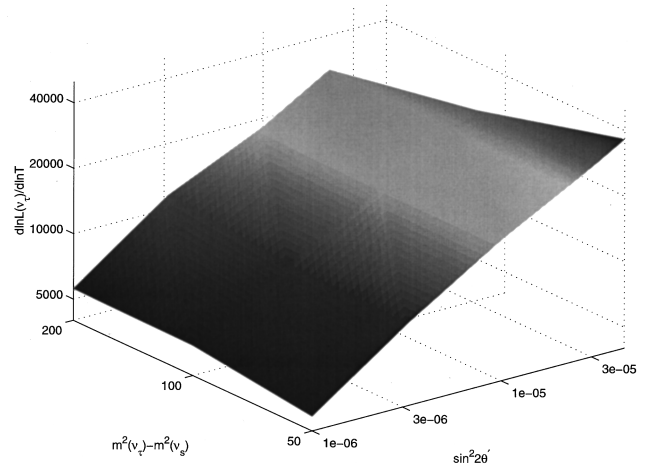


FIG. 4. The initial rate for L_{ν_τ} growth, $d \ln L / d \ln T$, as a function of the vacuum tau neutrino-sterile neutrino mixing parameters.

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