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A SIMPLE METHOD FOR DETERMINING WAIST-TO-WAIST TRANSFER PROPERTIES OF QUADRIPOLE DOUBLETS AND TRIPLETS

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Abstract

A new method is reported which allows the exact determination of the waist-to-waist transfer properties of quadrupole doublets and triplets. For any given geometry and finite emittance beam the entire set of available solutions can be found and the quadrupole strengths evaluated. Two examples are given for an asymmetric triplet, for the purpose of illustrating the application of the method.

* On leave of absence from University of Milan, Milano, Italy.

1. Introduction

The theory of beam transport by means of quadrupole doublets or triplets has been reviewed in detail by many authors¹⁾, and their basic optical properties have been determined using either the thin or thick lens approach. Sets of graphs useful in simple cases have been developed by Enge^{2,3)}.

Numerical computations, using the exact transfer matrices are, however, necessary in most practical cases, in order to determine the strengths of the lenses, namely when: a) The thin lens approach gives too rough an approximation. b) The beams to be transmitted have such a finite emittance and size that point-optics concepts cannot be applied. Moreover, many problems are encountered even in carrying out exact calculations, unless one has close enough guesses for the strengths of the lenses and a good knowledge of the possible solutions to a given beam-optics problem. In these circumstances graphical methods may be very useful and a few of them have been reported⁴⁾. The method reported here, based on a simple combination of transfer-matrix tabulation and graphical plotting, has proved quite successful in the determination of the quadrupole gradients for any given beam transfer. Some typical examples are reported for a triplet, the application of the method to a doublet being straightforward.

2. Mathematical summary

We recall some definitions which are of customary use in beam-transport theory. A supposed monochromatic beam travelling along the z-axis is represented in a general way by referring to the phase spaces associated with

the (x,z) radial and (y,z) vertical planes. The (x,x') and (y,y') variables, i.e., position and divergence, describe the beam behavior in phase space, the units being usually mm and mrad. As is well known, most beams delivered by accelerators can be represented in each phase space by an ellipse, whose area, constant by the Liouville theorem, is defined as the beam emittance. The beam has a waist, at some position along the z axis, when it is represented there by upright ellipses in both phase spaces, i.e., the two axes of both ellipses coincide with the (x,x') and (y,y') axes.

Along a drift length the waist position is thus the place where the beam has its minimum size. For beams of finite emittance, the concept of waist is thus similar to that of focus in ordinary optics, and in the following we shall refer to waist-to-waist (WW) transfer, instead of source to image.

For the purpose of beam optics the ellipses are most simply represented, at any waist, through their characteristic length X, defined as $X_r = x_0/x'_0$ and $X_v = y_0/y'_0$ for the radial and vertical planes respectively, the subscripts (0) referring to the semiaxes of the ellipses. When WW transfer is obtained, the shape of the image ellipse will generally be different from that of the source and characterized by a different X'_r or X'_v . The matching ratio ρ is then defined by $X'_{r,v} = \rho_{r,v} X_{r,v}$ and it can easily be shown that ρ is related to the usual linear magnification by $\rho_{r,v} = M_{r,v}^2$.

The transfer matrix for the radial or vertical phase space associated with a transfer line, can be written as a 2×2 matrix: in order to have WW transfer with matching ratio ρ , for a beam with a given X_{in} , it must be possible to reduce this matrix to the form¹⁾

$$\begin{vmatrix} \rho^{1/2} \cos\phi & X_{in} \rho^{1/2} \sin\phi \\ -(X_{in} \rho^{1/2})^{-1} \sin\phi & \rho^{-1/2} \cos\phi \end{vmatrix} \quad (1)$$

where ϕ is an arbitrary phase angle used in describing the image ellipse. The expression (1) is the starting point of the method described here.

3. Outline of the method

Referring to Fig. 1, we consider the following problem of beam transport: given the upstream and downstream distances U and D (which in the most general case could be different for the radial and vertical planes) and given $X_{r,in}$ and $X_{v,in}$, the characteristic lengths of the initial ellipses, find the strengths of the quadrupoles which allow WW transfer between U and D , together with the radial and vertical magnifications.

If we deal with a triplet in which the outer quadrupoles are equally excited, as in Fig. 1, or a doublet, then the problem is one with two free parameters, i.e., the gradients of the quadrupoles. In either case the problem can be treated conveniently by the method suggested here.

Let us write symbolically the transfer matrices of Fig. 1 as

$$T_r = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \quad T_v = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}$$

with self-explanatory notation.

The elements α_{ij} and β_{ij} are, for a fixed geometry, functions only of the lengths L and strengths K_1 and K_2 of the outer and central elements respectively, defined in the magnetic case as usual by $K^2 = B_{\text{tip}}^2 / (Br) \cdot d$ where Br is the magnetic rigidity of the particles, B_{tip} the magnetic field at the pole tip and d the aperture of the quadrupoles. By convention the radial plane refers, in the triplet, to the sequence DFD, i.e., the first quadrupole is defocusing.

Recalling now the matrix (1) we recognize at once that the following equations should be satisfied in order to have WW transfer in both planes:

$$\frac{\alpha_{12}}{\alpha_{21}} = -X_{r,\text{in}}^2 \rho_r \quad (2a)$$

$$\frac{\alpha_{11}}{\alpha_{22}} = \rho_r \quad (2b)$$

$$\frac{\beta_{12}}{\beta_{21}} = -X_{v,\text{in}}^2 \rho_v \quad (3a)$$

$$\frac{\beta_{11}}{\beta_{22}} = \rho_v \quad (3b)$$

where for generality we allow different ρ 's and X 's in the two planes.

By dividing eq. (2a) by (2b) and (3a) by (3b) we obtain:

$$\frac{\alpha_{12}}{\alpha_{21}} \frac{\alpha_{22}}{\alpha_{11}} = -X_{r,\text{in}}^2 \quad (4)$$

$$\frac{\beta_{12}}{\beta_{21}} \frac{\beta_{22}}{\beta_{11}} = -X_{v,\text{in}}^2 \quad (5)$$

The solutions (K_1, K_2) of eqs. (4) and (5) correspond to WW transfer in both planes for the given beam. Once they are found, the ρ_r and ρ_v and hence the magnifications are determined through eqs. (2) and (3).

The best way to solve this system is to tabulate and plot the ratios (4) and (5) of the matrix elements as functions of K_1 , for different values of K_2 .

In the (K_1, X^2) plane any beam of characteristic length X_{in} is then simply represented by a line parallel to the K_1 axis and with ordinate $-X_{in}^2$. The intersections of this line with the plotted curves mentioned above provide pairs of values (K_1, K_2) and thus in a K_1, K_2 plane generate a curve which corresponds to radial (vertical) WW transfer for the given beam. The intersections in the K_1, K_2 diagram of these curves for the radial and vertical matrices then give, finally, the required quadrupole gradients for the simultaneous WW transfer and by a straightforward calculation, the magnifications.

Two examples will be given for the triplet of Fig. 1 which illustrate the usefulness of the method and directly depict results not easily obtainable by other means.

4. Examples

4.1 EQUAL UPSTREAM AND DOWNSTREAM DISTANCES

The triplet considered here consists of three quadrupoles of identical length $L=18$ cm separated by a distance $S=3$ cm. For $U=D=50$ cm, Figs. 2 and 3 present the ratios (4) and (5) tabulated as functions of $(K_1 L)^2$, for different values of $(K_2 L)^2$. The units are so chosen that the beam size and divergence are expressed in mm and mrad and the quoted X^2 are consistent with this choice.

By using $(KL)^2$ instead of KL , the abscissa scale becomes linear with respect to the gradients of the quadrupole lenses.

Only the negative halfplane of each plot, corresponding to negative (X_{in}^2) is of physical significance and is shown here. Points on the $X^2 = 0.0$ axis correspond to WW transfer for a point source. In the "radial" plot, Fig. 2, for every K_2L there is only one curve which lies in the negative halfplane. In the "vertical" plot instead, Fig. 3, the curves for each K_2L split into two branches, each of which allows WW transfer in the sense of eq. (5). The branches corresponding to the larger value of K_1L are almost coincident for different K_2L , and for clarity only two curves are plotted. This last solution corresponds to an axis crossing inside the triplet for rays leaving the source with large divergence.

Thus, for any $(-X_{in}^2)$ value we have, in the (K_1L, K_2L) diagram, one line for the radial matrix and two for the vertical, like those shown in Fig. 4 for $X_{in}^2 = 1$. The two points of intersection represent the only solutions to the desired WW transfer. Point O gives a magnification of unity in both planes, an obvious case for this geometry. The point O' yields a magnification of unity in the radial plane and less than unity, $M_v = 0.43$, in the vertical.

The same pattern of one radial magnification and two vertical values is observed for any other symmetric upstream-downstream triplet geometry for which WW transfer is possible.

It is to be noted that once graphs like Figs. 2 and 3 are constructed they immediately give the solutions for any X_{in} . In terms of the quadrupole gradients, these solutions of course are different for different values of X_{in} . This is illustrated in Fig. 5, where the solutions with unit magnification in both planes are presented for values of X_{in}^2 between 0 and 1.

4.2 UNEQUAL UPSTREAM AND DOWNSTREAM DISTANCES

The behavior is more complicated in the case of $U \neq D$. For $U = 80$ cm and $D = 40$ cm, the plots of eqs. (4) and (5) are given in Figs. 6 and 7. This pattern is typical of the asymmetric case, as it has been checked in many other geometries.

For each K_2L , one has now 2 different behaviors as a function of K_1L . (For convenience the different branches have been labeled according to their behavior.)

For any beam X_{in}^2 we will have in the (K_1L, K_2L) diagram two lines, A and B, corresponding to radial WW transfer, and four for the vertical, C, D, E, F, as is apparent from Fig. 8, drawn for $X_{in}^2 = 1.0$. (The lines in Fig. 8 are labeled in accordance with the plots from which they are derived.) There are thus 8 different solutions to the desired WW transfer which give different magnifications. Interesting features have been established:

a) Along any one of the lines A, B, C, D, E, F, in the (K_1L, K_2L) diagram the radial or vertical magnification is constant within a few percent for a constant X_{in}^2 . So for practical purposes there are just 2 different radial magnifications and 4 vertical values from which to choose.

b) In the radial plane a magnification $M_r^A < 1$ is associated with the line A, as one would expect for $U > D$, while line B has $M_1^B > 1$.

c) In the vertical plane lines D and F give two different magnifications, which for any beam satisfy: $M_v^D < M_r^A < M_v^F < 1.0$. In contrast, the magnifications corresponding to the lines C and E are in the range: $1 < M_v^C < M_r^B < M_v^E$.

d) In solutions E and F, the large divergence rays cross the axis inside the triplet. Only the magnifications less than unity would be of interest in practical cases, if the upstream distance is, as here, less than the downstream one.

Nevertheless, it is worth noticing, from Fig. 8, that solutions with very different magnifications are only a few percent different in terms of quadrupole gradients. The set of magnifications less than unity, available for the geometry considered here, is plotted in Fig. 9 as a function of X_{in} . The results obtained would not generally be expected from the simple approximation usually quoted for triplets.

5. Final remarks

Although computer programs like the well known code "TRANSPORT"⁵⁾ have been developed, in recent years, which should determine the strengths of the lenses along any beam transport line, their use may be somewhat limited by the multiplicity of solutions demonstrated here, and by the fact that solutions with very different magnifications lie very close to one another. It is obvious that without a previous knowledge of the possible magnifications, it is very difficult to determine the quadrupole gradients on the basis of WW requirement only. Conversely a knowledge of the magnifications almost implies prior knowledge of the entire solution.

The method described here enables the designer to visualize the possible solutions with a minimum computational effort and no trial-and-error procedure at all, doing the work only once for all the possible beams he needs to transfer and match. It is clear that the same method can also be applied if the upstream and downstream distances are not equal in the two planes, a case very common for beams delivered from accelerators and not easily solvable in other ways. The same tabulating and plotting procedure is used in this case.

It is, in principle, possible to construct tables of strengths and magnifications for general use, for either the doublet or the triplet system. In fact, it can be shown that all geometrical quantities involved, including the beam size, can be expressed in a dimensionless form through their ratio to the length L of the quadrupoles. In this way all the calculations have general validity. It is felt, however, that due to the extremely large variety of cases occurring in practice, both for the geometries and beams involved, these tables would provide more a general guideline than a complete coverage of the problem, so that the amount of work needed for their preparation is probably not justified.

A modification of the above mentioned "TRANSPORT" program, which could be quite useful is however being developed at L.R.L.⁶⁾ This version would incorporate to some extent the present considerations and evaluate automatically the entire set of solutions to a given WW transfer problem, thus eliminating the plotting work altogether.

Acknowledgments

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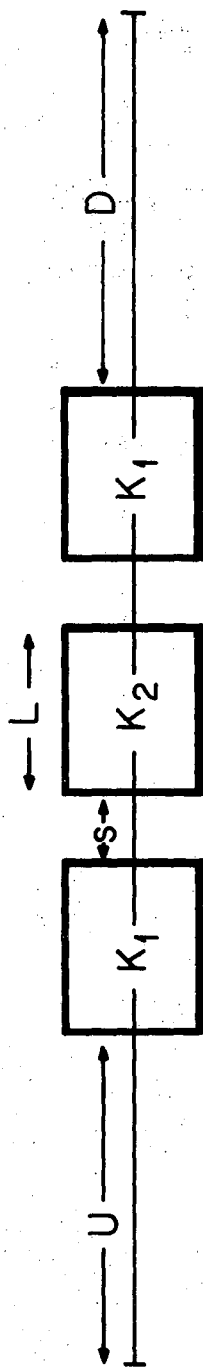
This work was performed under the auspices of the United States Atomic Energy Commission.

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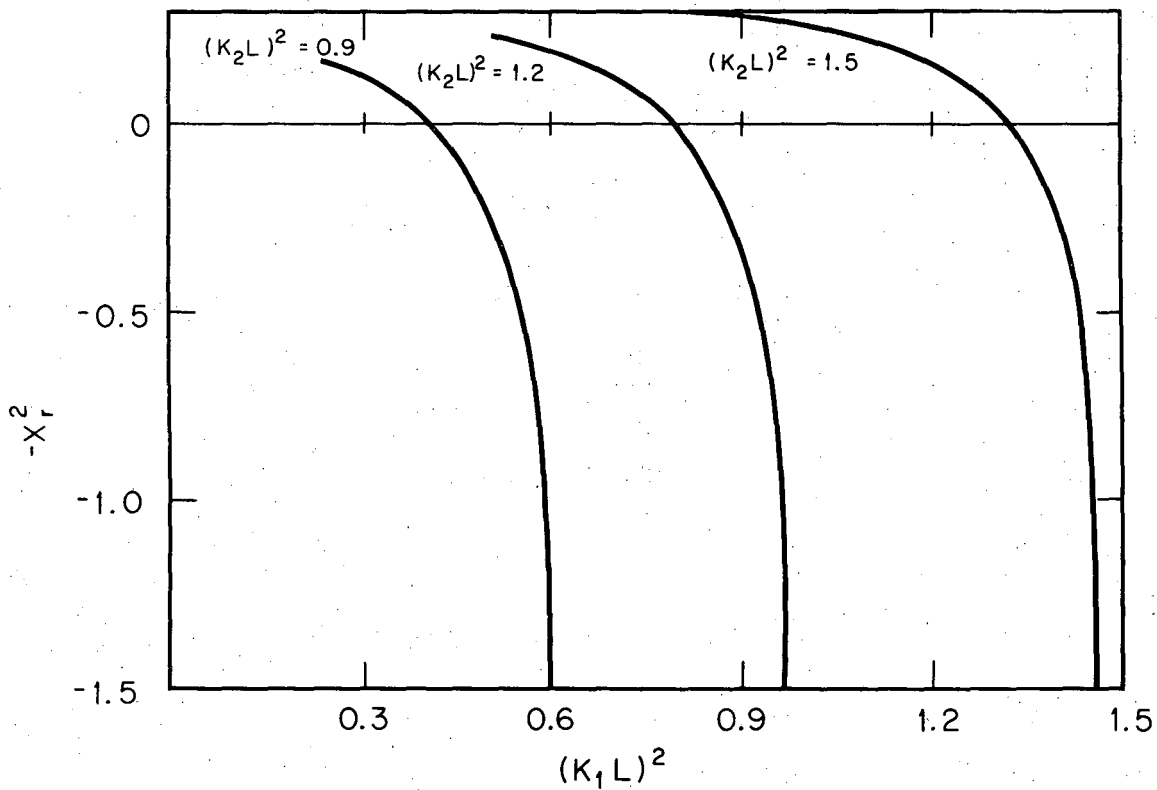
Figure Captions

- Fig. 1 Schematic triplet geometry.
- Fig. 2 Plots of the ratio (4) for the radial transfer matrix, in the case $U = D = 50$ cm.
- Fig. 3 Plots of the ratio (5) for the vertical transfer matrix, in the case $U = D = 50$ cm.
- Fig. 4 Solutions for waist-to-waist transfer in the case $U = D = 50$ cm, for a beam characterized by $X_{in}^2 = 1$.
- Fig. 5 Solutions for waist-to-waist transfer in the case $U = D = 50$ cm, with magnification unity in both planes. Three cases are shown with $X_{in}^2 = 0.1, 0.5, 1$.
- Fig. 6 Plots of the ratio (4) for the radial transfer matrix, in the case $U = 80$ cm, $D = 40$ cm.
- Fig. 7 Plots of the ratio (5) for the vertical transfer matrix, in the case $U = 80$ cm, $D = 40$ cm.
- Fig. 8 Solutions for waist-to-waist transfer in the case $U = 80$ cm, $D = 40$ cm, for a beam characterized by $X_{in}^2 = 1$. The various magnifications obtainable are indicated.
- Fig. 9 Less than unity radial and vertical magnifications, available in the case $U = 80$ cm, $D = 40$ cm, as a function of the beam X_{in} .



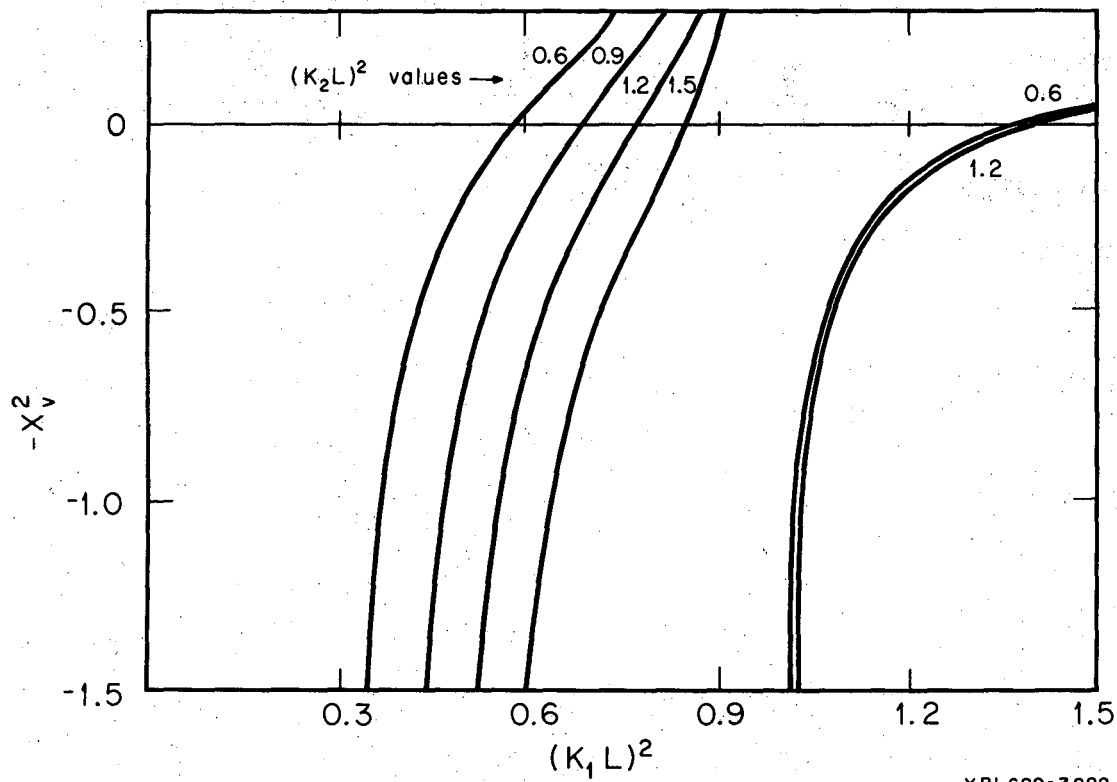
XBL689-3880

Figure 1



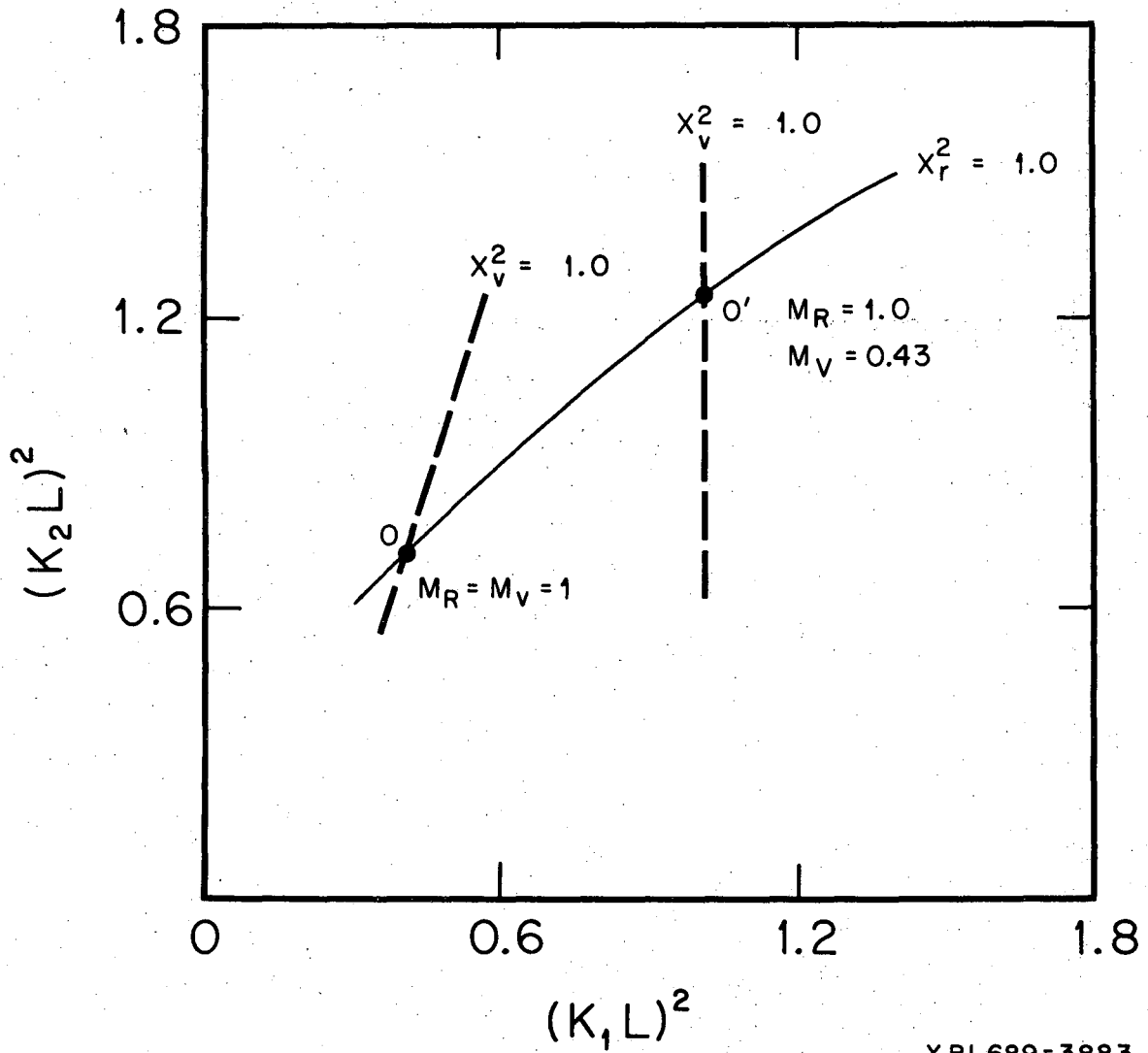
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Figure 2



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Figure 3



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Figure 4

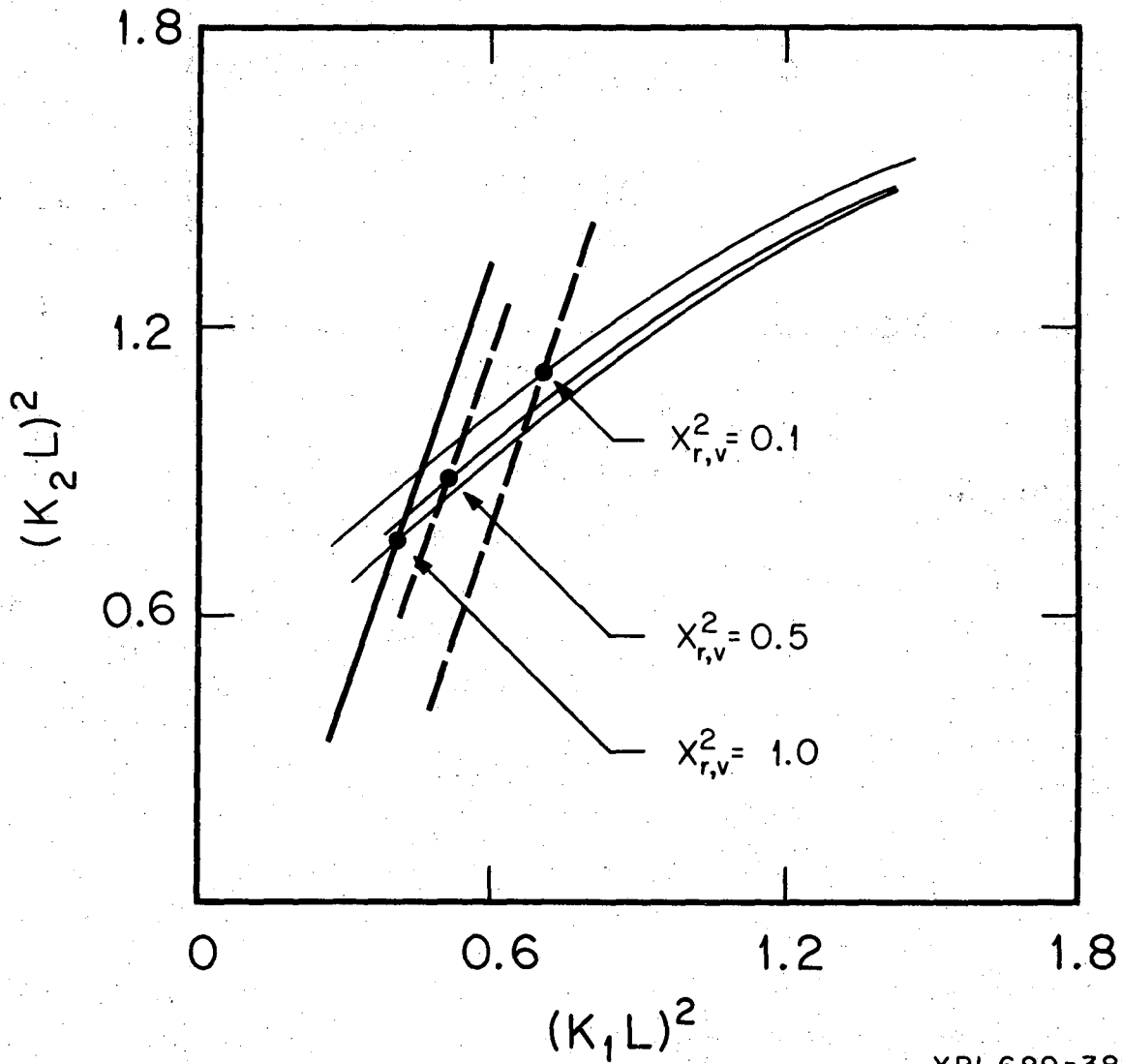
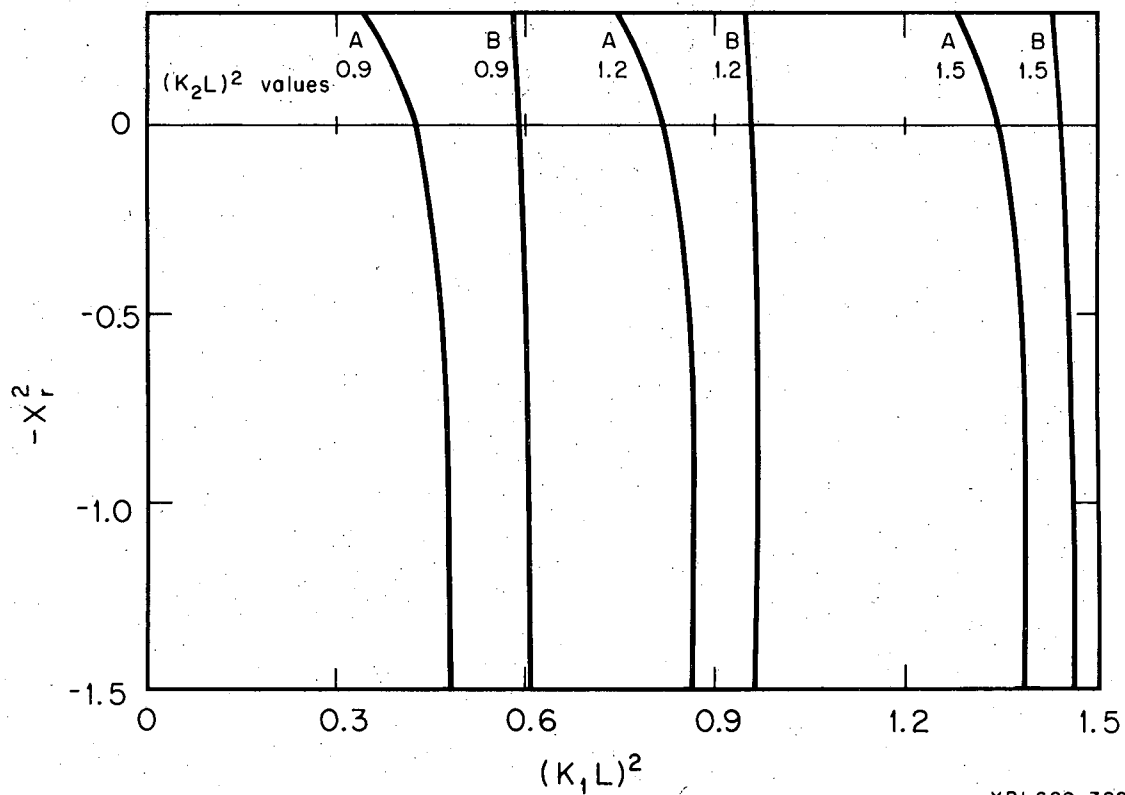
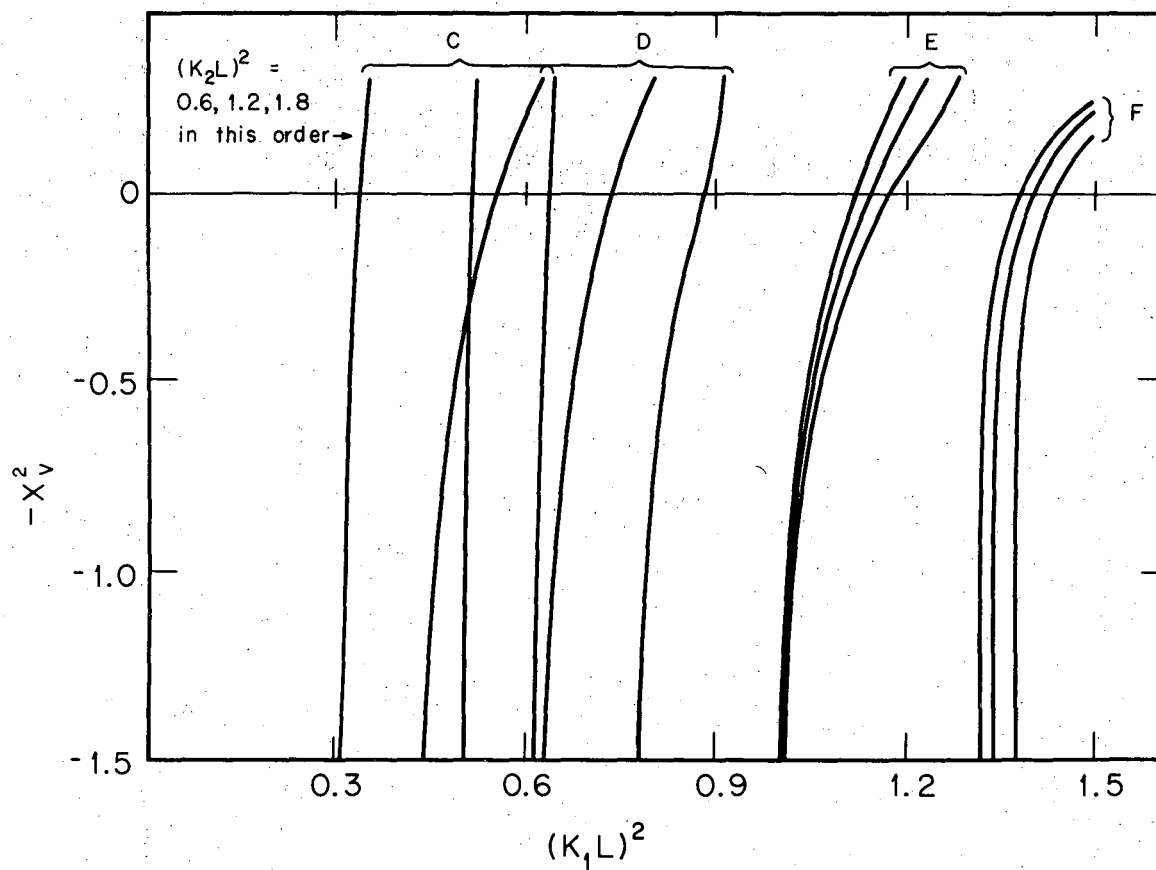


Figure 5



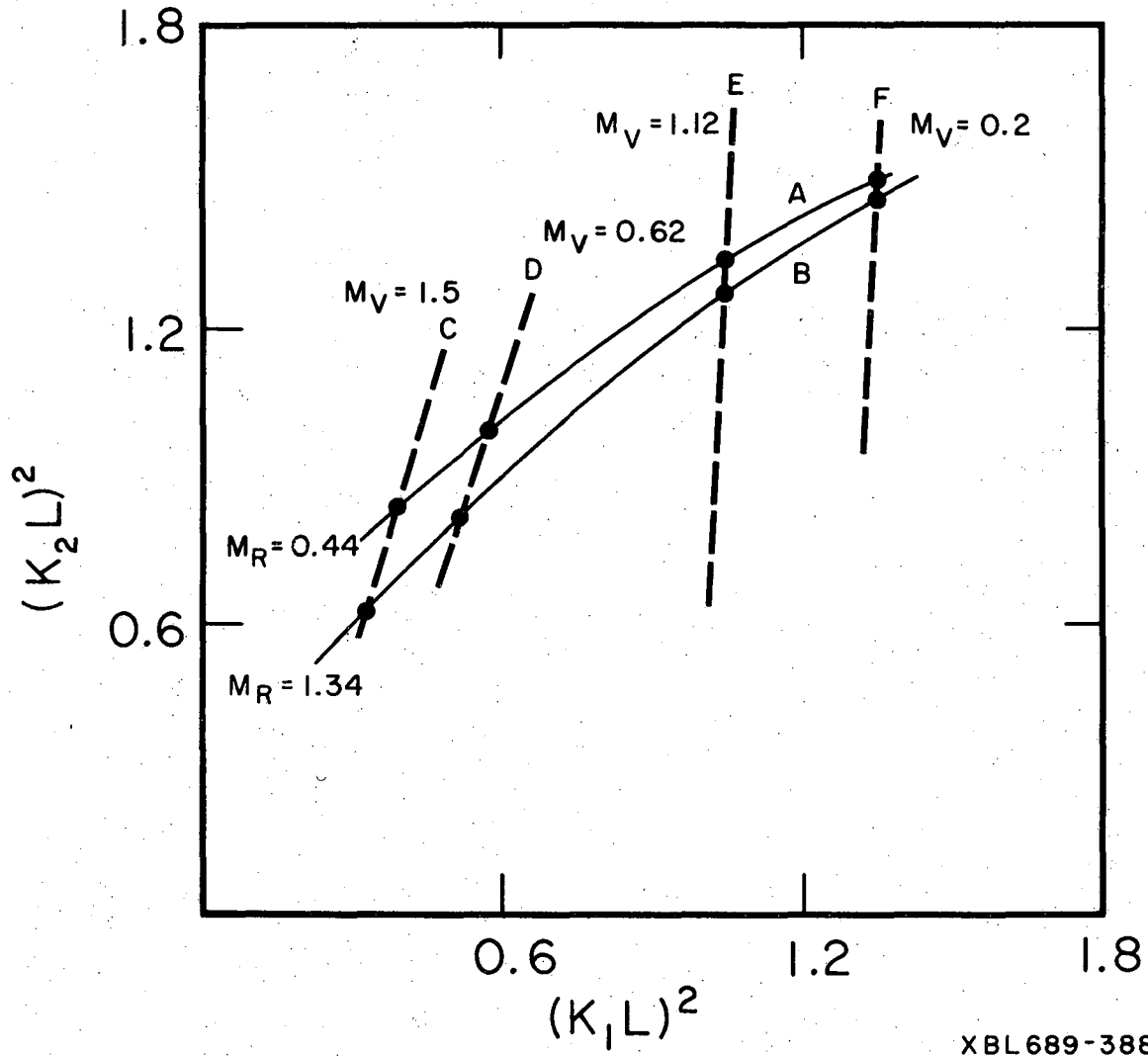
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Figure 6



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Figure 7



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Figure 8

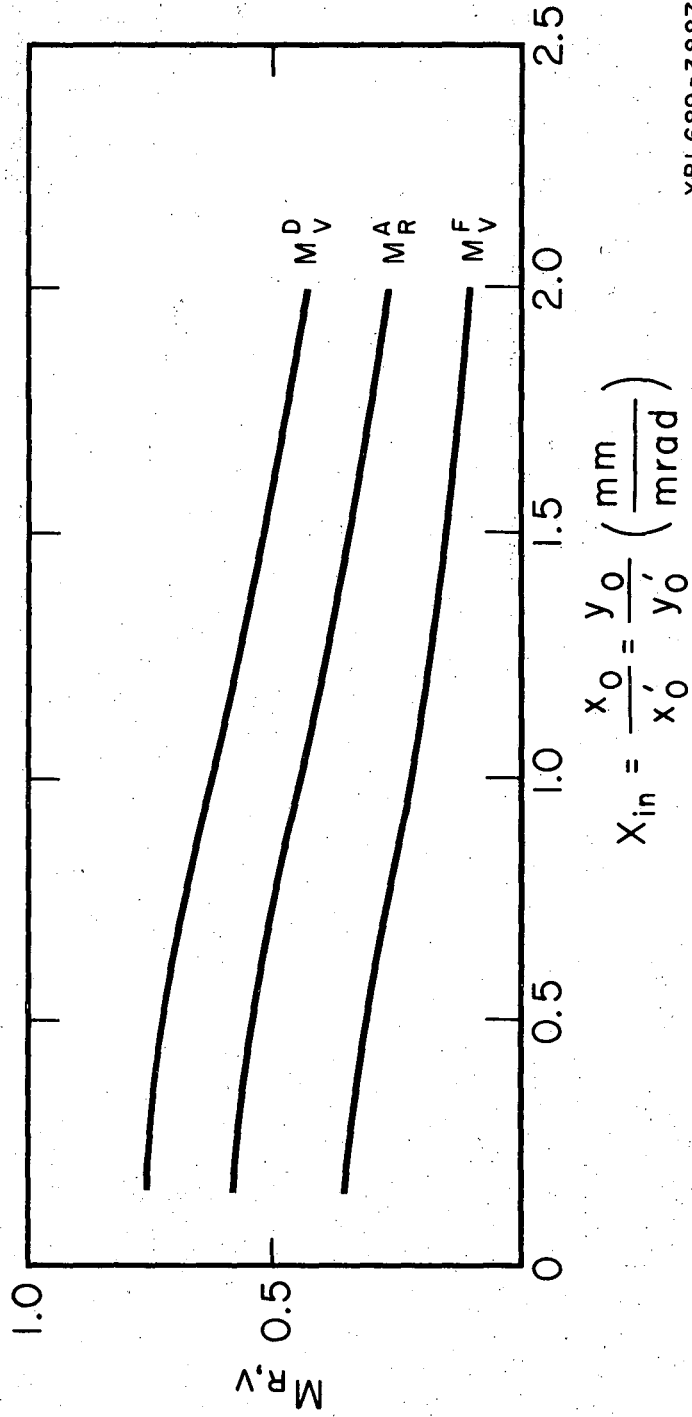


Figure 9

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