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RANDOM EVENT GENERATION WITH PREFERRED FREQUENCY DISTRIBUTIONS[1]

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ABSTRACT

A method of random event generation is described that allows wide choices for the frequency distribution of the generated events in Lorentz-invariant phase space.

I. Introduction

The convenience and generality of Monte Carlo methods for predicting the results of experiments from theoretical models, and for simulating experiments for the measurement of efficiencies and biases, has led to their extensive use in particle physics. For these applications the Monte Carlo method is used to evaluate phase-space[2] integrals of the form $R = \int_v r(\rho) d^n\rho$, where ρ is a point in the 3n-dimensional phase space, $d^n\rho$ is a volume element, and v is the phase-space volume over which the function is to be integrated. This volume could be the total phase space accessible to the n-particle final state or a subvolume delimited by equipment configuration or restrictions placed on experimental measurables.

More specifically, $\rho = (p_1 \cdots p_n)$ and

$$d^n\rho = \prod_{i=2}^{n} d^4p_i \, \delta(p_i^2 - m_i^2) \, \theta(p_i^0) \, \delta(P_0 - \sum_{j=1}^{n} p_j), \tag{1}$$

where p_i is the four-momentum of the ith final-state particle and P_0 is the four-momentum of the initial state. Then $r(\rho)d^n\rho$ is the transition rate from the initial state to the volume element $d^n\rho$ centered at the point ρ, and R is the total rate into the volume v.

To evaluate R by a Monte Carlo method one could generate N random points with constant frequency in the volume v. Then,

$$R = \langle r \rangle_v V = \frac{V}{N} \sum_{i=1}^{N} r(\rho_i),$$

where $r(\rho_i)$ is the value of $r(\rho)$ for the ith random point and $V = \int_v d^n\rho$. The principal limitation of this Monte Carlo method is that the number of points needed for the required statistical accuracy may be prohibitive.

The fractional statistical uncertainty in the above evaluation of R is $(\delta R/R)_v = F_v/N^{1/2}$, where F_v is the fractional root-mean-square deviation of $r(\rho)$ defined as $F_v = [\langle r^2 \rangle_v - \langle r \rangle_v^2]^{1/2}/\langle r \rangle$. For a given statistical accuracy, $(\delta R/R)_v$, the smaller the F_v [i.e., the smoother the function $r(\rho)$], the smaller the number of random points required. However, functions that vary rapidly or have appreciable value only for small regions of the volume of integration are characterized by large F_v, and many random points may be required to give reasonable statistical accuracy.

Sometimes a priori knowledge of the behavior of the function $r(\rho)$ can be used to reduce the number of points required. Let $g(\rho)$ be another function defined over the volume v that approximates the behavior of $r(\rho)$. Consider a transformation of variables from those of ρ to a new set η, such that $d^n\eta = g(\rho)d^n\rho$. Let $G = \int_u d^n\eta = \int_v g(\rho)d^n\rho$, where u is the same volume as v, but expressed in the variables of η. One could than express the integral for R as $R = \int_u [r(\rho)/g(\rho)]d^n\eta$ and evaluate it by the Monte Carlo method. One generates N random points with constant frequency in the variables of η in the volume u. This is the same as generating the points with frequency $g(\rho)$ in the variables of ρ in the volume v. Then

$$R = \left\langle \frac{r}{g} \right\rangle_u \quad G = \frac{G}{N} \sum_{i=1}^{N} \frac{r(\rho_i)}{g(\rho_i)}.$$

The fractional statistical uncertainty in this evaluation of R is

$$(\delta R/R)_u = F_u/N^{1/2}, \text{ where } F_u = [\langle (r/g)^2 \rangle_u - \langle r/g \rangle_u^2]^{1/2}/\langle r/g \rangle_u .$$

If $F_u << F_v$, which is the case if $g(\rho)$ is a good approximation to $r(\rho)$,

the number of Monte Carlo points needed for a given statistical accuracy is greatly reduced.

Thus, in order to use Monte Carlo techniques efficiently, one needs a generator that randomly generates points in Lorentz-invariant phase space[3] with a variety of choices for the frequency distribution of the points. In particle physics most transition rates, $r(\rho)$, have sharp dependencies on the invariant masses and four-momentum transfers squared of various final-state particle combinations. This report describes a method of random-event generation which allows the general frequency distribution

$$g(\rho) = \prod_{i=2}^{n-1} BW(\mu_i) \prod_{j=2}^{n} e^{a_j t_j} , \qquad (2)$$

where n is the number of final-state particles, the μ_i are the invariant masses of various particle combinations, and the t_j are the four-momentum transfer squared to various particle combinations from either the beam or target particle. BW can be either a constant or a primitive Breit-Wigner function

$$BW(\mu_i) = 1/\left\{ [(\mu_i - E_i) 2/\Gamma_i]^2 + 1 \right\} .$$

The choices of the particle combinations for the μ_i and t_j are very general; the restrictions are described in Section III. The set of parameters E_i, Γ_i, and a_j may have arbitrary values, thereby allowing for wide choices in the phase-space frequency distribution of the generated events.

II. Phase Space

The starting point for this random-event generation method is the

reexpression of $d^n\rho$ from the four vectors p_i of Eq. (1) to invariant masses and angles of subsystems of particles. This is accomplished by application of the factorization property of Lorentz-invariant phase space [1]. One way of expressing this property is

$$d^n\rho(E_0,\ m_1 \cdots m_n) = \frac{q_k}{4E_0}\, d\Omega_k\, d\mu_k^2\, d\mu_{n-k}^2\, \theta(q_k^2)\, d^k\rho\,(\mu_k,\ m_1 \cdots m_k) \times d^{n-k}\rho\,(\mu_{n-k},\ m_{k+1} \cdots m_n). \tag{3}$$

Here Eq. (3) divides the n-particle final state into two systems of k and n-k particles respectively, $m_1 \cdots m_n$ are the rest masses of the final-state particles, E_0 is the total invariant mass of the final state, μ_k and μ_{n-k} are the invariant masses of the two final-state systems, and q_k and Ω_k are the magnitude and direction of the total three-momentum of the k system in the overall center of mass. Explicitly one has

$$q_k = \left[\left(\frac{E_0^2 + \mu_k^2 - \mu_{n-k}^2}{2E_0}\right)^2 - \mu_k^2\right]^{1/2}.$$

The factor $d^k\rho(\mu_k,\ m_1 \cdots m_k)$ is k-particle phase space for the system of particles $m_1 \cdots m_k$ with invariant mass μ_k; $d^{n-k}\rho(\mu_{n-k},\ m_{k+1} \cdots m_n)$ is the analog for the recoiling system of n-k particles; $d\Omega_k q_k/4E_0$ is phase space for two bodies of total invariant mass E_0, having invariant masses μ_k and μ_{n-k} respectively; and $\theta(q_k^2)$ expresses[4] energy conservation by allowing only physical values for μ_k and μ_{n-k}. Equation (3) explicitly expresses the dependence of $d^n\rho$ on $d\mu_k^2$, $d\mu_{n-k}^2$, and $d\Omega_k$.

Next $d^k\rho(\mu_k,\ m_1 \cdots m_k)$ and $d^{n-k}\rho(\mu_{n-k},\ m_{k+1} \cdots m_n)$ are themselves each factored into two subsystems in the same way. By repeated application of this factorization, the n-particle phase space can be represented as a product of n-1 two-particle phase spaces. Thus,

$$d^n\rho(E_0,\ m_1 \cdots m_n) = \delta(\mu_n^2 - E_0^2) \prod_{i=2}^{n} \frac{q_i}{4\mu_i}\, d\mu_i^2 d\Omega_i \theta(q_i^2)\, \theta(\text{connected}), \qquad (4a)$$

or, since $d\mu_i^2 = 2\mu_i\, d\mu_i$,

$$d^n\rho(E_0,\ m_1 \cdots m_n) = \delta(\mu_n^2 - E_0^2) \prod_{i=2}^{n} \frac{q_i}{2} d\mu_i\, d\Omega_i\, \theta(q_i^2)\, \theta(\text{connected}). \qquad (4b)$$

Each term in the product can be thought of as the decay of a parent system into two daughters: μ_i is the invariant mass of a parent system, q_i and Ω_i are the magnitude and direction, respectively, of the momentum of one of the daughters in the parent's rest frame (q_i and $-\Omega_i$ represent the momentum of the other daughter). The delta function requires that the invariant mass of the whole final state be the center-of-mass energy E_0. The first step function, $\theta(q_i^2)$, requires that energy be conserved at each step in the decay chain, and the second step function, θ(connected), requires that the invariant masses and solid angles appearing in Eq. (4) correspond to subsystems that connect together to form such decay chains. Thus, although there is a wide choice for the n-2 invariant masses and n-1 decay angles appearing in Eq. (4), they are not completely arbitrary. Figure 1 diagrams the three possible ways to connect a five-particle final state.

III. Event Generation

The procedure described in the preceding section for factoring n-body phase space into n-1 connected two-body decay phase spaces is the basis for this random-event generation method. First the final state is factored into an arbitrary set of n-1 two-body "decay" vertices (like those diagrammed in Fig. 1 for a five-body final state). Then each

vertex is generated one by one. Thus, in order to generate an entire final state, one need only be able to generate the two-body "decay" vertex for the reaction $p \rightarrow p_1 + p_2$ and connect these vertices to form the entire final state. One must start with the highest-mass vertex and work down to the next highest and so on, since energy conservation imposes constraints on the generation of the lower vertices.

Consider the generation of the two-body decay $p \rightarrow p_1 + p_2$. Let p and μ be the four-momentum and invariant mass of the parent in an arbitrary Lorentz frame and $p_1 \mu_1$ and $p_2 \mu_2$ be those of the daughters. Here p is input to the problem and p_1 and p_2 are the desired answer; p_1 or p_2 (or both) may represent single particles or particle combinations.

First consider the most general case, in which both p_1 and p_2 represent multiparticle combinations. The invariant mass μ_1 is first generated in the in interval $(S_1, \mu - S_2)$, where S_1 and S_2 are the sums of the rest masses of the particles combining to form p_1 and p_2. Next μ_2 is generated in the interval $(S_2, \mu - \mu_1)$. From these masses one can evaluate

$$| \underset{\sim}{p}_1 | = | \underset{\sim}{p}_2 | = q = \left[\left(\frac{\mu^2 + \mu_1^2 - \mu_2^2}{2\mu} \right)^2 - \mu_1^2 \right]^{1/2} .$$

Next, the direction for p_1 ($= \hat{k}$) is generated over the full range of solid angle, and $\underset{\sim}{p}_1 = q \hat{k}$, $\underset{\sim}{p}_2 = - q\hat{k}$, $p_1^0 = (q^2 + \mu_1^2)^{1/2}$, and $p_2^0 = (q^2 + \mu_2^2)^{1/2}$.

These four vectors, p_1 and p_2, so evaluated, are expressed in the Lorentz frame $p = (\underset{\sim}{0}, \mu)$. They may then be transformed to the arbitrary Lorentz frame $p = (\underset{\sim}{p}, p^0)$.

If one of the daughter systems (say p_2) represents a single particle

with mass m_2, then only μ_1 and $\hat{k}$ need to be generated, since $\mu_2 = S_2 = m_2$. If both daughters are single particles then only $\hat{k}$ needs to be generated, since $\mu_1 = S_1 = m_1$ as well as $\mu_2 = S_2 = m_2$.

The manner in which μ_1, μ_2 and $\hat{k}$ are generated is dictated by the dependence of the frequency distribution function $g(\rho)$, of Eq. (2), on these variables.

If the n-2 μ_i's in Eq. (2) correspond to the invariant masses of the subsystems into which the final state has been factored, and if the n-1 t_j's correspond to the four-momentum transfers squared to these subsystems, then Eq. (2) may be rewritten as

$$g(\rho) = \prod_{i=2}^{n} g_i(\rho) = \prod_{i=2}^{n} g_i\left[\mu_1^{(i)}, \mu_2^{(i)}, t^{(i)}\right], \tag{5}$$

where each $g_i(\rho)$ corresponds to a particular two-body decay vertex. This constitutes the restrictions (referred to in the introduction) on the choices for the μ_i's and t_j's (of Eq. 2) imposed by this method.

If a Breit-Wigner distribution for an invariant mass is <u>not</u> desired, then the mass is distributed randomly according to a constant frequency distribution within its limits. If a Breit-Wigner distribution for μ_1 or μ_2 or both is contained in $g_i(\rho)$, then one defines a new variable

$$\eta = \int_S^x BW(x')dx' = \frac{\Gamma}{2}\left[\tan^{-1}\left(\frac{x-E}{\Gamma/2}\right) - \tan^{-1}\left(\frac{S-E}{\Gamma/2}\right)\right], \tag{6a}$$

where $x = \mu_1$ or μ_2 and $S = S_1$ or S_2. One then distributes η randomly with constant frequency in the interval $[0, \eta^{(+)}]$, where

$$\eta^{(+)} = \frac{\Gamma}{2}\left[\tan^{-1}\left(\frac{x_{max}-E}{\Gamma/2}\right) - \tan^{-1}\left(\frac{S-E}{\Gamma/2}\right)\right]. \tag{6b}$$

Once η is generated, inverting Eq. (6a) gives the corresponding value for x.

If $g_i(\rho)$ contains no functional dependence on $\hat{k}$, then $\hat{k}$ is generated randomly in solid angle with a constant frequency distribution. That is, the polar angle θ is generated so that $\cos\theta$ is uniformly distributed in the interval (-1, 1) and the azimuthal angle ϕ is uniformly generated in the interval $(0, 2\pi)$. Then $\hat{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

If $\hat{k}$ is to be given an e^{at} frequency distribution with respect to some momentum vector represented by $A = (\underset{\sim}{A}, A^0)$ in the Lorentz frame $p = (\underset{\sim}{0}, \mu)$, then a more involved procedure must be employed. Proceeding as in the case of invariant masses, one defines a new variable

$$\delta = \int_{t^-}^{t} e^{at'}\, dt' \tag{7a}$$

and randomly generates δ with a constant frequency in the interval $[0, \delta^{(+)}]$, where

$$\delta^{(+)} = \frac{1}{a}\left(e^{at^+} - e^{at^-}\right), \tag{7b}$$

and

$$t^{\pm} = m_a^2 + x^2 - 2A^0(q^2 + x^2)^{1/2} \pm 2\,|\,\underset{\sim}{A}\,|\,q \tag{7c}$$

and

$$m_a^2 = (A^0) - |\,\underset{\sim}{A}\,|^2 .$$

Here, $x = \mu_1$ or μ_2 depending upon whether p_1 or p_2 is to have the e^{at} distribution. Equation (7a) is inverted to solve for t once δ is generated. Solving further for $\cos\theta$, one has

$$\cos\theta = 2\,\frac{t - t^-}{t^+ - t^-} - 1. \tag{7d}$$

A more complicated angular frequency distribution may be given to $\hat{k}$. Let $I(\cos\theta)$ be some decay angular distribution desired for $\hat{k}$ with respect to the direction defined by $\underset{\sim}{A}$. Define a new variable

$$\gamma = \int_{-1}^{\cos\theta} I(\cos\theta')\, d(\cos\theta') \tag{8}$$

and randomly distribute γ with uniform frequency in the interval $[0, \gamma^{(+)}]$, where

$$\gamma^{(+)} = \int_{-1}^{1} I(\cos\theta')\, d(\cos\theta')\,.$$

Equation (8) is then inverted (if such inversion is possible in closed form) to obtain $\cos\theta$ in terms of the generated γ.[5]

Once $\cos\theta$ is known, a unit vector $\hat{\ell}$ is generated such that $\ell_z = \cos\theta$ and $\hat{\ell}$ has a random azimuth angle with respect to $\underset{\sim}{A}$; an azimuthal angle ϕ is generated in the interval $(0, 2\pi)$. Then $\hat{\ell} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ is the representation of $\hat{k}$ in the coordinate system, where $\underset{\sim}{A}/|\underset{\sim}{A}| = (0, 0, 1)$. Then $\hat{\ell}$ is rotated to the coordinate system, where $\underset{\sim}{A}/|\underset{\sim}{A}| = (A_x, A_y, A_z)/|\underset{\sim}{A}|$ to obtain $\hat{k}$, i.e., $k = R(\underset{\sim}{A}/|\underset{\sim}{A}|)\hat{\ell}$, where R is the rotation operator.

The vector A may be any momentum in the problem. For example, it could be the beam or target momentum, expressed in the Lorentz frame $p = (\underset{\sim}{0}, \mu)$, or the helicity direction $\underset{\sim}{p}$. Also, the e^{at} or $I(\cos\theta)$ frequency distribution can be given to either p_1 or p_2.

If p_1 or p_2 (or both) represents multiparticle systems then they become parents and are in turn factored into daughter subsystems, and the

four-vectors of these daughters are generated in the same manner as above. This factoring of multiparticle systems into subsystems is continued until the four-vectors of all of the final-state particles are generated.

IV. Phase-Space Weighting

The random-event generation method described in the preceding section uses invariant masses and decay solid angles of particle subsystems as the generation variables. These particular variables were chosen because the limits imposed upon them by momentum and energy conservation are easily obtained. Events distributed uniformly in Lorentz-invariant phase space are distributed uniformly in these solid angles but not, however, in these invariant masses. Conversely if events are generated uniformly in these invariant masses [which is the prescription of this generation method for $g(\rho) = 1$] they will <u>not</u> be uniformly distributed in Lorentz-invariant phase space. This is apparent from Eq. (4b), which may be rewritten as

$$d^n\rho(E_0, m_1 \cdots m_n) = \left(\frac{1}{2^n E_0} \prod_{i=2}^{n} q_i\right) d\Omega_n \prod_{i=2}^{n-1} d\mu_i d\Omega_i \theta(q_i^2)\theta(\text{connected}). \tag{4c}$$

Inspection of Eq. (4c) shows that events generated uniformly in the n-2 μ_i's will have the frequency distribution $\left(\prod_{i=2}^{n} q_i/2^n E_0\right)^{-1}$ in Lorentz-invariant phase space. Thus, in order to achieve distributions that correspond to Lorentz-invariant phase space, each event generated by this method must be weighted by the quantity

$$w_{ps} = \frac{1}{2^n E_0} \prod_{i=2}^{n} q_i . \tag{9}$$

Equation (4c) may be written as

$$d^n\rho = J(\rho;\mu,\Omega)d\Omega n \prod_{i=2}^{n-1} d\Omega_i d\mu_i, \text{ where } J(\rho;\mu,\Omega) \qquad (10)$$

represents the Jacobian of the transformation from the mass and angle variables to those of the phase space. Thus, the phase-space weight is just this Jacobian. For the general case consider, as was done in the introduction, the phase-space integral $R = \int_v r(\rho)d^n\rho = \int_w r(\rho)J(\rho;\alpha)d^n\alpha$, where $J(\rho;\alpha)$ is the Jacobian of the transformation from the set of variables, α, to those of phase space and w is the volume, v, expressed in the α variables. Evaluating R by a Monte Carlo method, one has $R = (W/N)\sum_{i=1}^{N} r(\rho_i)J(\rho_i;\alpha)$, where the N Monte Carlo events are distributed randomly with constant frequency in the variables of α. Here $W = \int_w d^n\alpha = \int_v d^n\rho/J(\rho;\alpha)$. In the introduction the special case of $J(\rho;\alpha) = 1/g(\rho)$ was considered. The equation for the evaluation of R may be rewritten as $R = (1/N)\sum_{i=1}^{N} r(\rho_i)w(\rho_i)$, where $w(\rho_i) = W\,J(\rho_i;\alpha)$ is referred to as the "phase-space weight" for the ith event. Thus, whenever the basic generation variables are not those which are uniformly distributed in the phase space, a phase-space weight must be applied.

For the event-generation method discussed in this report we choose as basic generation variables those which are uniformly distributed in a multidimensional cube of unit length on each side. That is, each of these basic generation variables is a dimensionless random number, generated uniformly in the interval (0, 1). Then, $W = 1$ and $J(\rho)$ will be the Jacobian of the transformation from this unit cube to the phase space. This transformation is accomplished in two steps. First the phase space is expressed in terms of the masses and decay solid angles appearing in

Eq. (4c), and then these masses and solid angles are expressed in terms of the random numbers. The first transformation was accomplished by employing the factorization property of Lorentz-invariant phase space as described in Section II, and its Jacobian, $J(\rho;\Omega,\mu)$, is given by the right-hand side of Eq. (9). The second transformation and its Jacobian, $J(\mu,\Omega)$, are detailed below. The Jacobian of the complete transformation is the phase-space weight, $w(\rho)$, for the Monte Carlo events, and is given by $w(\rho) = J(\rho;\Omega,\mu)J(\Omega,\mu)$.

Consider the generation of the two-body decay $p \rightarrow p_1 + p_2$ discussed in Section III (the notation used in the following discussion is the same as in Section III). The invariant mass μ_1 is generated uniformly or according to a Breit-Wigner distribution in the interval $(S_1, \mu\text{-}S_2)$ imposed by energy conservation. For the case of uniform generation, this can be accomplished by setting $\mu_1 = (\mu - S_2 - S_1)r_1 + S_1$, where r_1 is a random number generated uniformly in the interval (0, 1). The Jacobian of this transformation is simply $J(\mu_1) = (\mu - S_2 - S_1)$. For the case of Breit-Wigner generation of μ_1 the transformation is a two-step process. First η_1 is generated in the interval $[\,0, \eta_1^{(+)}]$, then Eq. (6a) is inverted to solve for μ_1. The generation of η_1 is accomplished by setting $\eta_1 = \eta_1^{(+)}r_1$, thus $J(\eta_1) = \eta_1^{(+)}$. The Jacobian of the transformation involved in solving for μ_1 in terms of η_1 is $1/BW(\mu_1)$ by construction. The Jacobian of the complete transformation for the Breit-Wigner generation of μ_1 is $J(\mu_1) = J(\eta_1)/BW(\mu_1) = \eta_1^{(+)}/BW(\mu_1)$.

The second invariant mass μ_2 is generated in the interval $(S_2, \mu\text{-}\mu_1)$, also imposed by energy conservation, in the same manner as the generation of μ_1 described above. Thus, $J(\mu_2) = \mu - \mu_1 - S_2$ for uniform

generation or $J(\mu_2) = \eta_2^{(+)}/BW(\mu_2)$ for Breit-Wigner generation. Specifically for Breit-Wigner generation of μ_1 or μ_2 (or both),

$$J(\mu_1) = \frac{\Gamma_1}{2}\left[\tan^{-1}\left(\frac{\mu-S_2-E_1}{\Gamma_1/2}\right) - \tan^{-1}\left(\frac{S_1-E_1}{\Gamma_1/2}\right)\right]\left\{\left[(\mu_1-E_1)2/\Gamma_1\right]^2+1\right\}, \quad (11a)$$

$$J(\mu_2) = \frac{\Gamma_2}{2}\left[\tan^{-1}\left(\frac{\mu-\mu_1-E_2}{\Gamma_2/2}\right) - \tan^{-1}\left(\frac{S_2-E_2}{\Gamma_2/2}\right)\right]\left\{\left[(\mu_2-E_2)2/\Gamma_2\right]^2+1\right\}. \quad (11b)$$

The direction $\hat{k} = (\cos\theta, \phi)$ of $\underset{\sim}{p}_1$ (or $-\underset{\sim}{p}_2$) in the rest frame of p is generated in the full range of solid angle. For the case of uniform frequency distribution this is accomplished by setting $\cos\theta = 2r_z - 1$ and $\phi = 2\pi r_\phi$, where r_z and r_ϕ are random numbers generated in the interval $(0, 1)$. Then $d(\cos\theta)d\phi = 4\pi\, dr_z\, dr_\phi$, so that $J(\hat{k}) = 4\pi$.

Consider now the case in which $\hat{k}$ is given an e^{at} frequency distribution. First δ is uniformly generated in the interval $[0, \delta^{(+)}]$ (Eq. 7). That is, $\delta = \delta^{(+)}r_z$, where r_z is a random number generated in the interval $(0, 1)$. Thus $J(\delta) = \delta^{(+)}$. Then Eq. (7a) is inverted to solve for t. The Jacobian of this transformation by construction is e^{-at}, thus $J(t) = \delta^{(+)}e^{-at}$. Then Eq. (7d) is inverted to solve for $\cos\theta$, so that $d(\cos\theta)/dt = 2/(t^+ - t^-)$. When these results are combined, $J(\cos\theta) = 2\delta^{(+)}e^{-at}/(t^+ - t^-)$. Next the azimuth angle, ϕ, is generated in the same manner as above, so that $J(\phi) = 2\pi$. Combining these results and inserting the explicit expression for $\delta^{(+)}$, one has

$$J(\hat{k}) = J(\cos\theta)J(\phi) = \frac{4\pi(e^{at^+} - e^{at^-})}{a(t^+ - t^-)\, e^{at}} \quad (12)$$

for the case of an e^{at} generation for $\hat{k}$.

The final Jacobian for the decay vertex $p \rightarrow p_1+p_2$ is $J(\mu_1, \mu_2, \hat{k})$ $= J(\mu_1)J(\mu_2)J(\hat{k})$. The Jacobian for the entire final state is the product of the Jacobians of each of the decay vertices that combine to form the final state. Therefore,

$$J(\mu,\Omega) = \prod_{i=2}^{n} J(\mu_1^{(i)}) J(\mu_2^{(i)}) J(\hat{k}_i), \tag{13}$$

where the product is over all the decay vertices.

Equation (13) represents the Jacobian of the transformation from the 3n-4 random numbers to the invariant masses and decay solid angles. This must be combined with $J(\rho;\mu,\Omega)$, which is the Jacobian of the transformation from the invariant masses and decay angles to the phase space, in order to form the phase-space weight for events generated by this method. Thus

$$w(\rho) = J(\rho;\mu,\Omega)J(\mu,\Omega) = \frac{1}{2^n E_0} \prod_{i=2}^{n} q_i J(\mu_1^{(i)})J(\mu_2^{(i)})J(\hat{k}_i). \tag{14}$$

Equation (14) represents the Jacobian of the transformation to the n-particle phase space from the (3n-4)-dimensional unit cube.

V. Applications

This random-event generator may now be used to evaluate phase-space integrals and simulate experiments as discussed in the introduction. To evaluate phase-space integrals of the form $R = \int_V r(\rho) d\rho^n$ or simulate an experiment with the matrix element squared, $r(\rho)$, a generation function $g(\rho)$ (Eq. 2) is first chosen that most closely approximates $r(\rho)$. The n-particle final state is factored into the appropriate subsystems whose invariant masses and four-momentum transfers squared enter explicitly

into $g(\rho)$. A sample of N Monte Carlo events is generated so as to correspond to a random distribution in Lorentz-invariant phase space with frequency $g(\rho)$. From the discussion in the introduction it is clear that the more closely $g(\rho)$ approximates $r(\rho)$ in the volume v, the fewer the number of Monte Carlo events needed for a given statistical accuracy, and thus the greater the efficiency of the method.

For each event, $r(\rho)$ as well as $w(\rho)$ (Eq. 14) is evaluated, and then one has

$$R(v) = \int_{v \leq V} r(\rho)\, d^n\rho = (1/N) \sum_{i=1}^{N} r(\rho_i)\, w(\rho_i)\, \theta(v-\rho_i). \quad (15)$$

The sum is over the generated events; V is the total phase-space volume accessible to the system, and $v \leq V$ is the subvolume of phase space into which the rate is to be determined. The step function $\theta(v-\rho)$ is zero for the region of phase space outside v and unity inside. This subvolume could be, for example, the phase-space volume included by the experimental apparatus or that defined by an interval of a kinematic variable.

The phase-space integral, $R(v)$, as evaluated in Eq. (15), may be used directly to determine interaction rates. This is because $w(\rho)$ is the Jacobian of the transformation from a unit volume to the phase space. If the n-particle final state results from the decay of a particle with energy E in some Lorentz frame, then the decay rate (number of decays per particle per time) into the subvolume, v, in that Lorentz frame is

$$\lambda(v) = 1/\bar{t}(v) = R(v)/2E\,(2\pi)^{3n-4}. \quad (16)$$

If the final state results from the collision of two particles, then the total collision cross section into the subvolume, v, is

$$\sigma(v) = R(v)/4m\,|\underset{\sim}{p}|\,(2\pi)^{3n-4}\,, \tag{17}$$

where m is the mass of one of the incident particles and $|\underset{\sim}{p}|$ is the momentum of the other particle in the rest frame of the first.

Differential cross sections are evaluated by employing Eqs. (15) and (17) with appropriate definitions of the subvolume, v. Let $d\sigma/d\alpha$ be the differential cross section with respect to some measurable quantity, α. Then $d\sigma = (d\sigma/d\alpha)\,d\alpha$ or, approximately, $\Delta\sigma = \sigma(\Delta v_\alpha) = (d\sigma/d\alpha)\,\Delta\alpha$. Here $\Delta\alpha$ is a small interval in the variable, α, centered at α_0, and Δv_α is the phase-space volume subtended by the interval $\Delta\alpha$. Combining this with Eqs. (15) and (17), one has

$$\frac{d\sigma}{d\alpha}(\alpha_0) \approx \sum_{i=1}^{N} \frac{r(\rho_i)\,w(\rho_i)\,\theta\,(\alpha_i-\alpha_0+\Delta\alpha/2)\,\theta\,(\alpha_0+\Delta\alpha/2-\alpha_i)}{(\Delta\alpha)N\,4m\,|\underset{\sim}{p}|\,(2\pi)^{3n-4}}\,. \tag{18}$$

Here α_i is the value of α for each Monte Carlo event. The approximation becomes an equality in the limit that $\Delta\alpha \to 0$ and $N \to \infty$. The product of the step functions effectively accepts only those events which lie in a bin of width $\Delta\alpha$ centered at α_0. To obtain $d\sigma/d\alpha$ as a function of α one may simply vary α_0 in steps of $\Delta\alpha$. According to Eq. (18) this is equivalent to generating a sample of N Monte Carlo events and histogramming the variable α in bins of width $\Delta\alpha$ and with weight $r(\rho_i)\,w(\rho_i)/(\Delta\alpha)\,N\,4m\,|\underset{\sim}{p}|\,(2\pi)^{3n-4}$ applied to each entry.[6]

From the above discussion it is easily seen that all Monte Carlo efficiency calculations essentially reduce to comparing the relative cross

sections (or decay rates) into various phase-space subvolumes for the same model, $r(\rho)$. These subvolumes need not be explicitly known. One generates Monte Carlo events in the whole phase-space volume and, according to Eq. (15), discards those which are outside the subvolume in question. The decision to discard an event can be made on any basis whatsoever (for example, detector geometry). This ability to define phase space subvolumes and also the integrand, $r(\rho)$, in any set of variables, rather than the integration variables of some phase-space integral, as well as the ability to integrate any function, $r(\rho)$, makes the Monte Carlo method a useful tool.

The Monte Carlo event generator described in this report allows the generation of events with frequency distributions $g(\rho)$, whose form is given by Eq. (2). However, this generator may be used to generate events with more general frequency distributions. Let $g_i(\rho)$ represent a particular frequency distribution of the form in Eq. (2). Then Monte Carlo events with the frequency distribution

$$g(\rho) = \sum_{i=1}^{M} N_i g_i(\rho)/G_i \tag{19}$$

may be generated with this event generator. Equation (19) represents a frequency distribution that is a linear combination of frequency distributions of the form in Eq. (2). N_i is the number of events generated with frequency $g_i(\rho)$, and M is the number of such frequency distributions. Here $G_i = \int g_i(\rho) d\rho^n = \langle g_i(\rho) w(\rho) \rangle$, where the average is taken for the events generated with frequency $g_i(\rho)$.

First, N_1 events are generated with frequency $g_1(\rho)$ as prescribed

in Section III. Then N_2 events are generated with frequency $g_2(\rho)$, and so on until N_M events are generated with frequency $g_M(\rho)$. The total number of events generated is $N = \sum_{i=1}^{M} N_i$. The phase-space weight applied to each event for this generalized frequency generation is

$$W_i(\rho) = \left[N \frac{g_i(\rho)}{G_i} \Big/ \sum_{j=1}^{M} N_j \frac{g_j(\rho)}{G_j} \right] w(\rho),$$

where $W_i(\rho)$ is the generalization of $w(\rho)$ (Eq. 14) for the case of more than one simple frequency distribution, $g_i(\rho)$. Note that the form of $W_i(\rho)$ for an event depends upon the simple frequency distribution, $g_i(\rho)$, from which it came, and for the special case of $M = 1$, $W_1(\rho) = w(\rho)$. Substituting this generalized weight, $W_i(\rho)$, for $w(\rho)$ in any of the above equations makes the equation valid for the generalized frequency distribution, $g(\rho)$ of Eq. (19).

This Monte Carlo method allows the simulation of experiments with transition rate $r(\rho)$ by weighting each event generated by $r(\rho)w(\rho)$, or $r(\rho)W_i(\rho)$. For some applications, however, unweighted events are required whose density in Lorentz-invariant phase space is given by $r(\rho)$. This can be accomplished by generating along with each event a random number, x, in the interval $(0, x_{max})$, where x_{max} is an upper bound for the event weights. Events are discarded whose weight is less than x, and those not so discarded are each given unit weight. These events are distributed randomly with a frequency $r(\rho)$ in Lorentz-invariant phase space. The upper bound, x_{max}, need not be the least upper bound for the event weights, but the closer it is to the least upper bound the more efficient the method becomes.[7] However, this method is always less

efficient than using all the events with their corresponding weights.

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Appendix A

More Efficient Invariant-Mass Generation

This section discusses in more detail the generation of invariant masses of particle subsystems and the resultant phase-space weights introduced by the generation method. As discussed in Sections III and IV, these invariant masses are distributed randomly with a frequency that is either a constant or a Breit-Wigner distribution. Associated with this generation method is a weight to be applied to each event which is the Jacobian of the transformation from the random numbers to the corresponding invariant masses. If this Jacobian is not a constant it can introduce additional statistical uncertainty to calculations using these events.

Consider first the case in which no invariant masses are to have Breit-Wigner generation. Then each invariant mass μ_i is generated with constant frequency within its limits. That is, $\mu_i = R_i r_i + S_i$, where R_i is the range allowed by energy conservation for μ_i, r_i is a random number generated in the interval (0, 1), and S_i is the sum of the rest masses of the particles combining to form the subsystem. In general R_i depends upon other invariant masses in the final state, specifically those that are generated before μ_i. The Jacobian of the transformation from r_i to μ_i is simply $J(\mu_i, r_i) = \partial\mu_i/\partial r_i = R_i$. The total Jacobian for the transformation from all the random numbers to all the invariant masses is then

$J(\mu, r) = \prod_{i=2}^{n-1} R_i(\mu_{i+1} \cdots \mu_{n-1})$. Here n is the number of final-state particles. For large values of n this Jacobian as a function of the μ_i's has a very rapid variation. Thus, when it is applied as part of the phase-space weight for the generated Monte Carlo events a great loss of efficiency results. A measure of the efficiency is the value of the quantity $\epsilon = \langle J \rangle^2 / \langle J^2 \rangle$. This quantity has value unity for $J(\mu, r)$ equal to a constant, and becomes smaller as $J(\mu, r)$ becomes more rapidly varying.

If the final state is factored into cascading subsystems all of which recoil from a single particle, then it is straightforward to show

$$\epsilon(n) = \frac{[2(n-2)]!}{2^{n-2}[(n-2)!]^3} \approx_{n>6} \frac{2^{n-2}}{\sqrt{(n-2)\pi}\,(n-2)!}, \qquad \text{(A1)}$$

where the approximation employs Sterling's formula for $(n-2)!$. Thus, this method of invariant-mass generation becomes very inefficient for large multiplicity in the final state. Factoring the final state in a different way increases the efficiency, but it still has the same general dependence on n as Eq. (A1) for large multiplicity.

It is possible to increase the efficiency by generating the invariant masses with a nonconstant frequency distribution such that the Jacobian of the transformation from the random numbers to the masses has a less rapid variation than for the case of constant-frequency generation. Let $\mu_i = R_i \rho_i + S_i$, where ρ is a dimensionless number generated in the interval $(0, 1)$ but with frequency $h_i(\rho_i)$. Then $J_i(\mu_i, r_i) = R_i(\partial\rho_i/\partial r_i) = R_i/h_i(\rho_i)$, so that the total Jacobian is

$$J(\mu, r) = \left(\prod_{i=2}^{n-1} R_i\right) \Big/ \left[\prod_{i=2}^{n-1} h_i(\rho_i)\right]. \qquad \text{(A2)}$$

If a set of functions $h_i(\rho_i)$ can be found so that $J(\mu, r)$ is a slowly varying function then the efficiency can be increased.

Consider the generation of a two-body decay vertex $p \rightarrow p_1 + p_2$, as discussed in Section III. (The notation used here is the same as in Section III.) Let $\mu_1 = R_1 \rho_1 + S_1$ and $\mu_2 = R_2 \rho_2 + S_2$, and let there be M_1 and M_2 particles in the p_1 and p_2 systems respectively. If

$$h_1(\rho_1) = \frac{(M_1 + M_2 - 2)!}{(M_1 - 2)!\,(M_2 - 1)!}\, \rho_1^{M_1 - 2}\, (1 - \rho_1)^{M_2 - 1} \qquad \text{(A3)}$$

and

$$h_2(\rho_2) = (M_2 - 1)\, \rho_2^{M_2 - 2} \qquad \text{(A4)}$$

for every decay vertex in the final state, then it is shown in Appendix B that the total final-state Jacobian has the constant value

$$J(\mu, r) = \frac{(E_0 - S_n)^{n-2}}{(n - 2)!} \,. \qquad \text{(A5)}$$

Here E_0 is the center-of-mass energy for the final state and S_n is the sum of rest masses in the final state. Thus, the efficiency due to the invariant-mass generation is unity, independent of the final-state multiplicity.[8]

Equation (A4) implies that $\rho_2 = r_2^{(M_2 - 1)^{-1}}$, where r_2 is a random number generated with constant frequency in the interval (0, 1). Equation (A3) can be inverted only for the cases in which $M_2 = 1$ or $M_1 = 2$. For these cases $\rho_1 = r_1^{(M_1 - 1)^{-1}}$ or $\rho_1 = 1 - r_1^{M_2^{-1}}$. For all other cases Eq. (A3) cannot be solved explicitly for ρ_1 as a function of r_1. However, it is easy to show that if $M_1 + M_2 - 2$ random numbers are generated with constant frequency in the interval (0, 1), and then ordered from smallest to largest, the $(M_1 - 1)$th ordered random number will have the frequency distribution given by Eq. (A3).

For Breit-Wigner generation of an invariant mass the Jacobian of the transformation from the random number to the invariant mass is given by Eq. (11). In most applications, the motivation for employing Breit-Wigner invariant-mass generation is to concentrate Monte Carlo events in a region where the matrix element squared $r(\rho)$ has rapid variation approximated by a Breit-Wigner shape. Thus, the major inefficiency in $r(\rho)\, J(\mu)$ will be caused by the variation with invariant mass of the first term contained within the brackets of Eq. (11). This term will have rapid variation only when the upper limit of the invariant mass is less than or near the central value of the Breit-Wigner. Therefore, it is always most efficient to generate invariant masses with Breit-Wigner frequencies as early as possible in the generation of the event. The later in the event an invariant mass is generated the more restricted is its average range and the closer the upper limit will be to the central value of a Breit-Wigner. This will cause more rapid variation of $r(\rho)\, J(\mu)$ and a lowering of the efficiency. This effect is also present for e^{at} momentum-transfer-squared distributions, so that subsystems which are to be generated in this manner should also be generated as early as possible within the event for maximum efficiency.

The result of the discussion in this appendix is, then, an alternative prescription to that given in Section III for the generation of invariant masses that leads to greater statistical efficiency. Consider once again the generation of a two-body decay vertex $p \rightarrow p_1 + p_2$ (notation is the same as in Section III). If both invariant masses μ_1 and μ_2 have Breit-Wigner frequencies the prescription of Section III is used. If only one of them is to have a Breit-Wigner frequency then it is generated as in Section III and the other is generated with the frequency given by Eq. (A4). If neither

invariant mass is to have a Breit-Wigner frequency distribution, the first one is given the frequency distribution of Eq. (A3) and the second that of Eq. (A4).

This prescription will always result in greater statistical efficiency than the prescription of Section III. The increase in efficiency depends upon the number of non-Breit-Wigner invariant-mass generations. In the limit that all of the final-state invariant masses are non-Breit-Wigner, the increase in efficiency resulting from this prescription is closely approximated by the inverse of Eq. (A1).

Appendix B

This section will prove the assertion of the previous section that generating the final-state invariant masses with the frequency distributions of Eqs. (A3) and (A4) leads to the constant Jacobian given by Eq. (A5).

Consider first the special case in which the final state is factored into a set of cascading subsystems, each one recoiling from a single particle. (Notation used here is the same as in Appendix A.) For this case one has the recursion relation $R_i = R_{i+1}\,\rho_{i+1}$ and $R_{n-1} = E_0 - S_n$ so that $R_i = (E_0 - S_n)\prod_{k=i+1}^{n-1}\rho_k$ and $\prod_{i=2}^{n-1} R_i = (E_0 - S_n)^{n-2}\prod_{i=2}^{n-1}\rho_i^{i-2}$. Also for this case Eq. (A3) reduces to $h_i(\rho_i) = (i-1)\rho_i^{i-2}$, so that $\prod_{i=2}^{n-1}(i-1)\rho_i^{i-2} = (n-2)!\prod_{i=2}^{n-1}\rho_i^{i-2}$, and therefore from Eq. (A2) one has the result expressed by Eq. (A5).

Next consider a slightly more general case in which the final state is factored first into two subsystems of M_1 and M_2 particles respectively, and then each of these subsystems is factored into cascading subsystems as above. The contribution to the total final-state Jacobian from the factoring of each of the cascading subsystems is

$$J_{M_1}(\mu, r) = \frac{(\mu_1 - S_{M_1})^{M_1-2}}{(M_1 - 2)!} = \frac{(E_0 - S_n)^{M_1-2}}{(M_1 - 2)!} \rho_1^{M_1-2} \tag{B1}$$

and

$$J_{M_2}(\mu, r) = \frac{(\mu_2 - S_{M_2})^{M_2-2}}{(M_2 - 2)!} = \frac{(E_0 - S_n)^{M_2-2}}{(M_2 - 2)!} (1-\rho_1)^{M_2-2} \rho_2^{M_2-2}, \tag{B2}$$

where μ_1 and μ_2 are the invariant masses of the two recoiling subsystems.

From Eqs. (A2) and (A3) one has

$$\frac{\partial \mu_1}{\partial r_1} = \frac{R_1}{h_1(\rho_1)} = \frac{(M_1 - 2)!\ (M_2 - 1)!\ (E_0 - S_n)}{(n-2)!\ \rho_1^{M_1-2}\ (1-\rho_1)^{M_2-1}}, \tag{B3}$$

and from Eq. (A4),

$$\frac{\partial \mu_2}{\partial r_2} = \frac{R_2}{h_2(\rho_2)} = \frac{(E_0 - S_n)(1-\rho_1)}{(M_2 - 1)\rho_2^{M_2-2}}. \tag{B4}$$

Combining Eqs. (B1), (B2), (B3), and (B4), one has, for the total final-state Jacobian,

$$J(\mu, r) = J_{M_1}(\mu, r) J_{M_2}(\mu, r) \frac{\partial \mu_1}{\partial r_1} \frac{\partial \mu_2}{\partial r_2} = \frac{(E_0 - S_n)^{n-2}}{(n-2)!}. \tag{B5}$$

Equation (B5) shows that for the purpose of calculating the Jacobian $J(\mu, r)$, each pair of recoiling particle subsystems in the final state may be replaced by a single set of cascading subsystems. This replacement can be repeated for all the multiparticle recoiling subsystems in the final

state. Then the Jacobian is just given by Eq. (A5).

Footnotes

[1] Work done under auspices of the U. S. Atomic Energy Commission.

[2] Phase space, as discussed here and defined by Eq. (1), is the momentum-space factor of phase space as usually defined in statistical mechanics. The spatial integrations of phase space are ignored in this report, since the results of scattering experiments are stated in terms of the momenta of the interacting particles. If one normalizes in a box of volume H, then the phase-space integral contains the factor H^n for an n-particle final state. However, since no physical quantity measurable in a scattering experiment depends upon this spatial volume factor, it is ignored. For a more complete discussion see reference [1], p. 80.

[3] A point in Lorentz-invariant phase space will be referred to as an "event" in this report.

[4] $\theta(x)$ is the Heavyside step function, $\theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$.

[5] Equation (8) can always be inverted by simple numerical methods.

[6] The fractional statistical uncertainty in the value of $d\sigma/d\alpha$ for each bin of the histogram is $\sqrt{\langle w^2 \rangle / (N \langle w \rangle^2)}$, where N is the number of Monte Carlo events in the bin, $\langle w \rangle$ is the mean weight, and $\langle w^2 \rangle$ the mean square weight for the events in the bin.

[7] A method of obtaining the least upper bound is to perform a search in the (3n-4)-dimensional cube for the maximum of the function $r(\rho)w(\rho) = f(x_1 \cdots x_i \cdots x_{3n-4})$, where n is the number of final-state particles and the x_i are the random numbers in the intervals (0, 1). The

function $f(x_1 \cdots x_i \cdots x_{3n-4})$ is obtained by generating an event corresponding to a specific set of x_i^s and evaluating $r(\rho)$ and $w(\rho)$ for the event. The search can be performed by using any one of many computer codes that optimize a function of several variables. (See, for example: S. Derenzo, MINF68--A General Minimizing Routine, Lawrence Radiation Laboratory Group A Programming Note No. P-190 July 1969.)

[8]Note that this applies only to the Jacobian of the transformation from the random numbers to the invariant masses (Eq. A2). The total phase-space weight for the event (Eq. 14) still contains the Jacobian of the transformation from the invariant masses to the phase space $J(\rho, \mu\Omega) = \left(\prod_{i=2}^{n} q_i \right) / 2^n E_0$, which also has a dependence on the invariant masses. The effect of this variation on the event efficiency is much less dramatic than that given by Eq. (A1), but is still not negligible. For $n \lesssim 15$ the efficiency is approximated by $\epsilon(n) \simeq e^{-(n-2)/\bar{n}}$, where the value of $\bar{n}$ depends greatly upon the center-of-mass energy E_0 and the rest masses of the final-state particles. The least efficient case results when all the final-state particles have zero rest mass. In this case $\bar{n} \approx 5.5$ independent of E_0. However, when $E_0 = 10$ GeV and the final state is composed of a proton and n - 1 pions, $\bar{n} \approx 9.0$. For $n > 15$, $\epsilon(n)$ falls more slowly with increasing n. For n = 25 the efficiency is 0.03 for the massless case and 0.13 for the later example.

Reference

1. R. Hagedorn, "Relativistic Kinematics," W.A. Benjamin, Inc., New York (1964), p. 94.

Figure Legend

Fig. 1. The three ways to connect two-body "decay" vertices to form a five-particle final state.

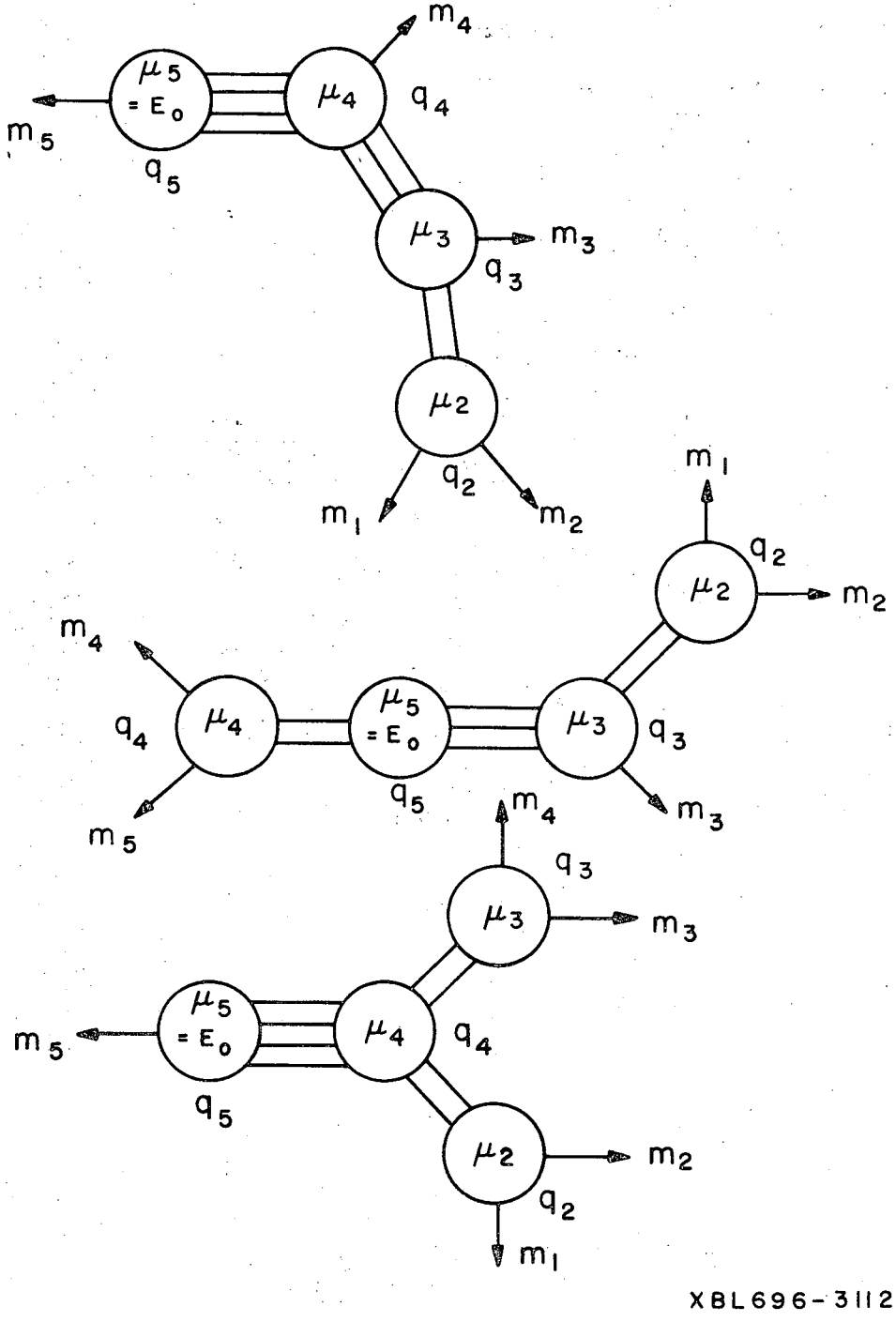

Fig. 1

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