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A note on uncertainty and perception concerning measurable utility[☆]



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ABSTRACT

A linkage to reconcile measurable utility derived from intensity comparisons or from probability mixtures is provided in this note. This brief note is in honor of Lloyd Shapley whose relatively unknown seminal paper on measurable utility from axioms involving the fineness of perception offered a different view on utility measurement.

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1. Introduction

There are two fundamentally different ways in which one can derive a measurable utility scale from a set of outcomes, over which there are completely ordered preferences. They can be approached from the viewpoint of risk or alternatively fineness of perception. The one that is most commonly discussed involves the introduction of lottery tickets or probability mixtures over the set of certain outcomes. The original axioms for the existence of a measurable utility function defined up to a linear transformation were given by von Neumann and Morgenstern (1944). Herstein and Milnor (1953) provided a somewhat different but simpler axiomatization.

Shapley (1975) considered the possibility for deriving a cardinal utility function from the ranking of outcomes in order of desirability. He showed that if a domain of outcomes can be preference-ordered by a numerical utility function with convex range, and if an intensity ordering also exists, satisfying certain axioms, that compares the relative desirability of different changes from one outcome to another, then there is an essentially unique numerical utility function that simultaneously describes both the preferences

among the outcomes and their intensities. His work enables one to pass from one utility scale not necessarily cardinal, which determines the domain of outcomes for intensity comparison, to a cardinal utility scale.

The purpose of this note is to reconcile the two systems of axioms. We show that if the domain of outcomes can be preference-ranked by a numerical utility functions representing a preference relation over a mixture set and satisfying the Herstein–Milnor axioms, then the Shapley axioms imply an extension of the preference relation to the space of pairs of outcomes embodying intensity comparison.

2. Measurable utility

A set \mathcal{S} is a *mixture set* if for any two elements $a, b \in \mathcal{S}$ and a number $\mu \in [0, 1]$, there is another element in \mathcal{S} , denoted by $\mu a + (1 - \mu)b$, such that (i) $1a + (1 - 1)b = a$, (ii) $\mu a + (1 - \mu)b = (1 - \mu)b + \mu a$, (iii) $\lambda[\mu a + (1 - \mu)b] + (1 - \lambda)b = \lambda\mu a + (1 - \lambda\mu)b$ for all $\lambda \in [0, 1]$. Any convex set in a real vector space with $\mu a + (1 - \mu)b$ as the usual convex combination of a and b is a mixture space. Following the literature, a function $u : \mathcal{S} \rightarrow \mathfrak{R}$ is a measurable utility for a preference relation \succeq on \mathcal{S} if $u(a) \geq u(b) \Leftrightarrow a \succeq b$ and

$$u(\alpha a + (1 - \alpha)b) = \alpha u(a) + (1 - \alpha)u(b), \quad (1)$$

for all $a, b \in \mathcal{S}$ and $\alpha \in [0, 1]$.

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2.1. Herstein and Milnor axioms and theorem

The following axioms were considered in Herstein and Milnor (1953):

Axiom 1. \mathcal{S} is completely ordered by \succeq .

Axiom 2. For any $x, y, z \in \mathcal{S}$, $\{\alpha : \alpha x + (1 - \alpha)y \succeq z\}$ and $\{\alpha : z \succeq \alpha x + (1 - \alpha)y\}$ are closed.

Axiom 3. If $x, z \in \mathcal{S}$ is such that $x \sim z$, then for any $y \in \mathcal{S}$, $\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}z + \frac{1}{2}y$.

Herstein and Milnor proved the following theorem:

Theorem 1. If \mathcal{S} is a mixture set and \succeq satisfies Axioms 1–3 on \mathcal{S} , then there is a measurable utility for \succeq . Moreover, this function is unique up to an order-preserving linear transformation.

Herstein and Milnor's (1953) work provides a simplification of von Neumann and Morgenstern's utility theory.

2.2. Shapley axioms and theorem

Let \succeq be a preference relation on \mathcal{S} . Assume that \succeq is representable by u such that the range $\mathcal{D} = \{u(s) | s \in \mathcal{S}\}$ is convex. Denote a preference relation on $\mathcal{D} \times \mathcal{D}$ by \succeq' . Shapley (1975) considered the following axioms:

Axiom 4. $(a, c) \succeq'(b, c) \Leftrightarrow a \geq b, a, b, c \in \mathcal{D}$.

Axiom 5. $(a, b) \sim'(c, d) \Leftrightarrow (a, c) \sim'(b, d), a, b, c, d \in \mathcal{D}$.

Axiom 6. $\{(a, d, c, d) : (a, b) \succeq'(c, d)\}$ is closed in $\mathcal{D} \times \mathcal{D} \times \mathcal{D} \times \mathcal{D}$.

Shapley established the following theorem:

Theorem 2. Assume \succeq' satisfies Axioms 4–6. Then, there exists a function $v : \mathcal{D} \rightarrow \mathfrak{R}$ such that

$$a \geq b \Leftrightarrow v(a) \geq v(b), \quad a, b \in \mathcal{D} \quad (2)$$

and

$$(a, b) \succeq'(c, d) \Leftrightarrow v(a) - v(b) \geq v(c) - v(d), \quad a, b, c, d \in \mathcal{D}. \quad (3)$$

Moreover, this function is unique up to an order-preserving affine transformation.

Property (2) implies that function v is another utility representation for \succeq . Notice that v and u are not necessarily the same or affine transformations of each other. In comparison, property (3) implies that the preference ordering of the pairs of outcomes is equivalent to the ranking of relative desirability.

3. Lotteries versus intensity comparisons

The collection of Axioms 1–3 on preference relation \succeq resulting in measurable utility and the collection of Axioms 4–6 on preference relation \succeq' giving rise to intensity comparison operate over different domains. There is no reason for them to lead to similar measures, unless the domain of Axioms 4–6 is determined by a utility function characterized by Axioms 1–3.

Theorem 3. Let \mathcal{S} be a mixture set and \succeq satisfies Axioms 1–3 on \mathcal{S} with utility representation u . Let \succeq' on $u(\mathcal{S}) \times u(\mathcal{S})$ satisfying Axioms 4–6. Then, \succeq' and utility function u satisfy (3).

Proof. By Theorem 2, \succeq' can be represented by a utility function v satisfying (2) and (3). Thus, for any $x, y, z \in \mathcal{S}$, by (2),

$$v(u(x)) \geq v(u(y)) \Leftrightarrow u(x) \geq u(y). \quad (4)$$

Since \succeq is representable by u , (4) implies that \succeq is also representable by the composite $v \circ u$ of u and v . Since u is unique for \succeq up to order-preserving affine transformations, we have

$$v(u(x)) = k_1 u(x) + k_2 \quad (5)$$

for some numbers k_1 and k_2 with $k_1 > 0$. For any $x, y, z, w \in \mathcal{S}$, (3) and (5) imply

$$(u(x), u(y)) \succeq'(u(z), u(w)) \Leftrightarrow k_1[u(x) - u(y)] \geq k_1[u(y) - u(z)].$$

Since $k_1 > 0$, the preceding equivalence in turn implies

$$(u(x), u(y)) \succeq'(u(z), u(w)) \Leftrightarrow [u(x) - u(y)] \geq [u(y) - u(z)]. \quad (6)$$

Together, (4) and (6) imply that \succeq' is represented by the utility function u . ■

Remark 1. The domain \mathcal{D} is assumed to be the range of an ordinal utility function in Shapley (1975). It follows that utility function u representing \succeq on \mathcal{S} is not necessarily substitutable for utility function v representing \succeq' on $u(\mathcal{S}) \times u(\mathcal{S})$. Nevertheless, it follows from property (2) that v is another utility function for \succeq on \mathcal{S} .

Remark 2. By (1), utility function u for \succeq on \mathcal{S} satisfies

$$\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}z + \frac{1}{2}w \Leftrightarrow u(x) - u(w) = u(z) - u(y).$$

Thus, if we begin with a measurable utility function u for \succeq on \mathcal{S} , (1) implies

$$\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}z + \frac{1}{2}w \Leftrightarrow (x, w) \sim'(z, y). \quad (7)$$

However, Axioms 4–6 are not completely implied by (7) alone. In other words, if we begin with a measurable utility function, Axioms 4–6 imply a natural extension from unilateral comparison to pairwise comparison, making the difference $u(x) - u(y)$ between elements $x, y \in \mathcal{S}$ meaningful.

4. Conclusion

Both sets of axioms are amenable to experimental testing. The inter linkage connecting risk behavior and perception of intensity differences among pairs of alternatives appears to be normatively attractive. We suspect that empirically it will not hold.

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