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Global Symmetries of Six Dimensional Superconformal Field Theories

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Global Symmetries of Six Dimensional Superconformal Field Theories

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by

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This work is dedicated to my parents, Karen L. Remmer and Gilbert W. Merkx, to whom I am grateful for more than I can express.

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Abstract

Global Symmetries of Six Dimensional Superconformal Field Theories

Peter R. Merkx

In this work we investigate the global symmetries of six-dimensional superconformal field theories (6D SCFTs) via their description in F-theory. We provide computer algebra system routines determining global symmetry maxima for all known 6D SCFTs while tracking the singularity types of the associated elliptic fibrations. We tabulate these bounds for many CFTs including every 0-link based theory. The approach we take provides explicit tracking of geometric information which remained implicit in the classification of 6D SCFTs found in [16]. We derive a variety of new geometric restrictions on collections of singularity collisions in elliptically fibered Calabi-Yau varieties and collect data from local model analyses of these collisions. The resulting restrictions are sufficient to match the known gauge enhancement structure constraints for all 6D SCFTs without appeal to anomaly cancellation and enable our global symmetry computations for F-theory SCFT models to proceed similarly.

Contents

Chapter 1

Global symmetries of 6D SCFTs with 1D Coulumb branch

We begin with a brief review of pertinent aspects of superconformal quantum field theories in six dimensions and their description in F-theory. We then discuss the global symmetries of these theories in cases with discriminant locus consisting of a single compact curve to be contracted in an appropriate limit (i.e. having at most a 1D Coulomb branch) based on work in [6, 24] via field theoretic constraints and separately through geometric constraints on singular elliptic fibrations occurring in F-theory models. We review the resulting geometric constraints for singular elliptic fibrations leading to permissible F-theory SCFT bases (i.e. fibrations having Calabi-Yau threefold resolution) that are crucial to our work in the following chapter.

1.1 Introduction

1.1.1 Context and motivation

Type II string theories formulated on a ten dimensional manifold of the form $M^{3,1} \times X$ with $M^{3,1}$ flat Minkowski spacetime and X a Calabi-Yau threefold enable one to study features of the resulting 4D quantum field theories via geometric properties of X. Alternatively, 4D field theories can be described in terms of "F-theory," beginning with type IIB string theory on $M^{3,1} \times B$ with $\pi : X \to B$ an elliptic fibration of a Calabi-Yau fourfold through the use of a multivalued function $\tau(b)$ on the elliptic curves of the fibration.

The setting of our work here follows a variant of the latter approach which grants access to six-dimensional superconformal field theories via F-theory by lowering the dimension of B to make X a complex elliptically fibered threefold. Widely studied and well-understood properties of the moduli space of these elliptic fibrations (for which there are currently only partial analogues in the fourfold case) are then able to come to bear on the problems one encounters in describing the 6D SCFT landscape. This setup provides certain tools for study of 6D SCFTs which are not as immediately available in a closely related route which begins with M-theory on a manifold of the first form above and proceeds by varying the metric on X, enabling one to draw conclusions in a limit where the elliptic fiber areas approach zero.

The first demonstrations of the existence of superconformal field theories in six dimensions date to the mid-1990s [43, 35, 36], almost twenty years after their existence was proposed in [29]. The F-theory framework was introduced shortly thereafter [40, 26, 27], making new features of these theories accessible in terms of the algebraic geometry of elliptic fibrations. Major advances in characterizing 6D SCFTs have been made in the last several years through F-theory constructions. Descriptions of many previously unknown CFTs have been provided. Recently, a conjecturally complete classification of SCFTs in six dimensions [23, 17, 16] has been detailed. Flavor symmetries (i.e. global symmetries) are among the features left implicit in this classification. Work in [6, 24] has provided constraints on these symmetries for a limited subclass of 6D SCFTs.

Our primary task here is to broaden this class to include all 6D SCFTs. We stress that the approach we take reduces our central task to a mathematically well-defined problem which consists of providing a series of constraints on Weierstrass models for elliptically fibered Calabi-Yau varieties with certain auxiliary data we describe shortly. The constraints we obtain involve a somewhat detailed analysis of local models for these elliptic fibrations and treat sufficiently many cases that it becomes natural to tackle constraining the global Weierstrass models through implementation of exhaustive computer search routines. Note that we do not claim the existence of global models achieving the maximal symmetries we report (though proofs to that end are often possible and even trivial in some cases). Instead, our focus is to derive constraints on elliptic fibrations sufficient to match those coming from quantum field theoretic constraints involving "anomaly cancellation" requirements. In some cases, the constraints we find are strictly tighter (eliminating some of the enhanced SCFTs appearing in [16] as detailed in Appendix B). Since restrictions on the geometry of Weierstrass fibrations are (seemingly necessarily) involved in reaching the precise conclusions of [16], one might hope for the most parsimonious approach to result from a purely geometric characterization of all 6D SCFT constraints.

While our work is often framed in CFT language owing to its motivating context, we are simply deriving constraints on elliptically fibered Calabi-Yau varieties.

We let $\widetilde{\pi} : \widetilde{X} \to B$ be a smooth elliptically fibered Calabi-Yau threefold with a section having all fibers one-dimensional[1]. Let $\pi : X \to B$ be the (possibly) singular fibration obtained by blowing down all fibers of $\widetilde{\pi}$ which do not meet the section. Note that reversing this procedure in the case of interest in which B is an orbifold dates to work of Hirzebruch [18] and Riemenschneider [34]; a modern treatment is available in [19]. An essential ingredient in this process to yield a Calabi-Yau *manifold* from our threefold (when possible) is the celebrated proof of Yau [44]. F-theory compactified (i.e. dimensionally reduced) on $\widetilde{X}$ is then a theory in six dimensions partially specified by appropriate powers of line bundles over B (discussed shortly) along with the assumption that B is compact[2]. We carry out our work directly on the singular fibration π, relating

[1]Considering the case with B two-dimensional and X a (potentially non-smooth) Calabi-Yau variety, our fibration can be desingularized giving an equidimensional elliptically fibered Calabi-Yau variety with at worst $\mathbb{Q}$-factorial terminal singularities [13]; requiring that we have a section ensures equidimensionality. Recent work appearing in [1] offers an interpretation for the role in F-theory of these $\mathbb{Q}$-factorial terminal singularities which prevent a desingularization maintaining the Calabi-Yau condition.

[2]While our work here is confined to 6D theories, dimensional reduction on an elliptic curve enables certain conclusions to be drawn on a corresponding collection of 4D counterparts, as elaborated in [28].

its features to physics of the 6D theories decoupled from gravity (i.e. the SCFTs) by allowing non-compact B and taking a limit in which we contract a collection of curves in B over which X is singular. In section 1.1.4 we discuss constraints on B shown in [17] to follow in this context. (Note that the theories resulting from such constructions are not themselves conformal, but flow to a conformal theory under renormalization.)

Introductions to F-theory and pertinent aspects of six-dimensional F-theory can be found in [9, 17], respectively. In the following section, we review the ingredients of this setup immediately relevant to our work. We then outline the problems treated in each chapter along with several details of the classification scheme which sets the stage for our work.

1.1.2 Outline and objectives

In [6, 24], constraints on global symmetries of 6D SCFTs which can be realized in F-theory constructions (more than) sufficient to match field theoretic anomaly cancellation restrictions were derived for all SCFT constructions in F-theory involving an elliptic fibration with discriminant locus consisting of a single compact curve (i.e. in the cases with zero or one-dimensional Coulomb branch). A general treatment of cases with more than one such curve remains untreated in the literature. In Chapter 1, we review these existing results and many of their derivations.

In Chapter 2 we extend the methods of [6] to all known 6D SCFTs. We introduce a collection of constraints and an algorithm (as well as implementation via computer algebra routines) providing (strictly) tighter geometric restrictions than the field theoretic constraints on gauge enhancements found in [16] and further compute global symmetry algebra maximums consistent with these constraints on F-theory models also via purely geometric restrictions. To make this more precise, we have the following result.

Result 1.1.1. *The gauge algebra constraints given in [16] are (strictly) weaker than the gauge constraints found through restrictions on elliptic fibrations derived in this work extending those of [6]. The eliminated enhancements and methods for comparison are detailed in Appendix B.*

Similarly, systematic comparison of global symmetries in the case with discriminant locus consisting of a single curve was shown in [6, 24] to give the following proposition. Proof of this proposition is reviewed in the present chapter.

Result 1.1.2. *For the field theories corresponding to a discriminant locus with one compact component, the global symmetries permitted by known field theory methods are (strictly) less constrained than those which can arise meeting the constraints on elliptic fibrations via our F-theory global symmetry construction.*

From these results, we are lead to expect the same in general, though we have not systematically confirmed inclusion in all cases. In other terms, we have the following:

Conjecture 1.1.3. *The global symmetry constraints resulting from imposing anomaly cancellation requirements outlined in [16] are (strictly) weaker than the constraints on global symmetry resulting from the algorithm we provide.*

Our approach consists of two basic steps. First, we determine a class of F-theory models compatible with a designated gauge algebra up to certain geometric features dictated by orders of vanishing of polynomials associated to the curves over which our elliptic fibration is singular. We then build from this data of fibrations (locally) consistent with our gauge algebra a collection of constraints on global symmetries which can be constructed in F-theory. This leaves us with a collection of possibilities for the (relatively) maximal global symmetries for the gauge theory. We derive locally consistent models for our constructions (which we might expect to generally be associated to some consistent global model); as noted above, we stop short of explicitly constructing these models globally[3]. The local analysis we provide extends that of [6, 24] to treat all known 6D SCFTs. This approach allows us to provide an algorithm (which we implement) characterizing the global symmetry maxima for any known 6D SCFT realizable in F-theory and reveals some surprising features. We collect the resulting constraints on a certain class of "building-block" theories, thus in effect characterizing the global symmetries of all 6D SCFTs.

We now turn to review geometric features of elliptic fibrations which enter into our work. We phrase our discussion in terms of structures determining SCFTs via a classification of 6D SCFTs (within F-theory) appearing in [16], so we begin with a summary

[3]Though in some cases consistent global constructions follow from our local analysis with little additional effort, we do not attempt in this work to track which of the global symmetry algebra maxima we report are associated to provably consistent constructions. There are many cases in which gluing local models consistently is highly nontrivial. Even fully analyzing all local models adequately to prove local consistency is a major task beyond the scope of this work. Relevant results for pair collisions and multiple point collisions can be found in [13, 32], respectively, but our requirements involving multiple transverse singular curve collisions are somewhat stronger than addressed in the literature.

this result and a few of the essentials for bringing the tools of F-theory to bear on the problem of classifying flavor symmetries of superconformal field theories and their underlying geometric features while minimizing introduction of mathematical machinery not required to derive new constraints.

1.1.3 Weierstrass models and gauge algebras in F-theory

The essential geometric ingredient for an F-theory formulation of an SCFT is a Weierstrass model determining a singular elliptic fibration given by $\pi : X \to B$ with fibers determined by a Weierstrass equation of the form

$$y^2 = x^3 + fx + g \ , \tag{1.1}$$

where f, g are locally defined polynomials on a complex surface (more precisely, f, g are sections of $\mathcal{O}(-4K_B)$, $\mathcal{O}(-6K_B)$ with K_B the canonical bundle over B) with B given by $\mathbb{C}^2/\Gamma$ in the case of 6D F-theory. Here Γ is a discrete subgroup of $U(2)$ determined by the discriminant locus, as discussed further in [28] and briefly in section 2.12, where we derive a simplified expression concerning the generators of these subgroups (for 'A-type endpoints'). The discriminant of this equation,

$$\Delta := 4f^3 + 27g^2 \ , \tag{1.2}$$

is a section of $\mathcal{O}(-12K_B)$ with its "discriminant locus,"

$$\{\Delta = 0\}\ , \tag{1.3}$$

determining where the fibration is singular. The types of singularities that are permitted without being so severe as to prevent a Calabi-Yau resolution are given by the Kodaira classification [22, 21, 30] and summarized in Table 1.1. The last entry in the table is designated "non-minimal" since after the singularities in the fibers are resolved, the resolution contains a curve which can be blown down, i.e., it is not a "minimal model" in the sense of birational geometry. Blowing that curve down is associated to a new Weierstrass model in which the orders of vanishing of (f, g, Δ) along Σ have been reduced by $(4, 6, 12)$. For this reason we need not consider such cases. A two-dimensional Weierstrass model is minimal at P defined by $\{\sigma = 0\}$ provided $\mathrm{ord}_{\sigma=0}(f) < 4$, or $\mathrm{ord}_{\sigma=0}(g) < 6$. In this work, we confine our study to those models lacking non-minimal points. (From the Calabi-Yau condition, we could have blown up such points. Without loss of generality, we take such a model as our starting point.)

The precise gauge algebra which occurs in the cases of ambiguity is determined by inspection of the auxiliary polynomials appearing in Table 1.2 where Σ is a curve along the singular locus $\{z = 0\}$, with larger gauge algebras resulting with more complete factorizations. In 1.3.1 we review a few aspects of this issue in the form presented in [6]. A significant amount of our work will concern analysis determining when these splittings can take place in various contexts.

ord(f)	ord(g)	ord(Δ)	type	singularity	non-abelian algebra
≥ 0	≥ 0	0	I_0	none	none
0	0	1	I_1	none	none
0	0	$n \geq 2$	I_n	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}([n/2])$
≥ 1	1	2	II	none	none
1	≥ 2	3	III	A_1	$\mathfrak{su}(2)$
≥ 2	2	4	IV	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
≥ 2	≥ 3	6	I_0^*	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$
2	3	$n \geq 7$	I_{n-6}^*	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
≥ 3	4	8	IV^*	$\mathfrak{e}_6$	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
3	≥ 5	9	III^*	$\mathfrak{e}_7$	$\mathfrak{e}_7$
≥ 4	5	10	II^*	$\mathfrak{e}_8$	$\mathfrak{e}_8$
≥ 4	≥ 6	≥ 12	non-minimal	-	-

Table 1.1: Singularity types with associated non-abelian algebras.

type	equation of monodromy cover
$\mathrm{I}_n^{s/ns}$, $n \geq 3$	$\psi^2 + (9g/2f)\vert_{z=0}$
$\mathrm{IV}^{s/ns}$	$\psi^2 - (g/z^2)\vert_{z=0}$
$\mathrm{I}_0^{*s/ss/ns}$	$\psi^3 + (f/z^2)\vert_{z=0} \cdot \psi + (g/z^3)\vert_{z=0}$
$\mathrm{I}_{2n-5}^{*s/ns}$, $n \geq 3$	$\psi^2 + \frac{1}{4}(\Delta/z^{2n+1})(2zf/9g)^3\vert_{z=0}$
$\mathrm{I}_{2n-4}^{*s/ns}$, $n \geq 3$	$\psi^2 + (\Delta/z^{2n+2})(2zf/9g)^2\vert_{z=0}$
$\mathrm{IV}^{*s/ns}$	$\psi^2 - (g/z^4)\vert_{z=0}$

Table 1.2: Monodromy cover polynomials determining non-abelian gauge algebras.

The association of non-abelian algebras to Kodaira types indicated in Table 1.1 is carefully elucidated in [3], to which we refer the reader for additional details. The essential idea is that the resolution of singularities, M_{orb}, for a given Kodaira type gives rise to a graph of curves we may naturally associate to a Dynkin diagram determining a Lie algebra $\mathfrak{g}$. This same $\mathfrak{g}$ arises as the gauge algebra of the corresponding physical theories, namely F-theory models compactified on a Calabi-Yau threefold M with metric furnished via [44] from M_{orb}.

1.1.4 Summary of 6D SCFT classification in F-theory

We now turn to an abbreviated summary of the 6D SCFT classification of [16], augmenting our discussion with a reorganization provided by [28]. This classification proceeds by describing all possible choices of base, B, determined by a connected graph of compact curves $\Sigma_j \subset \mathbb{C}^2$ over which the fibration is singular. The codimension one vanishings in B of Δ which give rise to the gauge algebra of our SCFT in the appropriate limit constitute this graph. We return to discuss the ingredients in the limiting process relevant to our discussion in 1.2.3.

The curves Σ_j must have transverse intersections (in at most a single point) meeting certain restrictions and further decorated with an allowed gauge enhancement, both described shortly. We require these curves be contractible in a certain sense (more precisely, we require contractibility at finite distance in the Calabi-Yau moduli space with Weil-Peterson metric dating in this context to [38, 39]). This leads to a pair of conclusions via [15, 41]: i) $\Sigma_j \cong \mathbb{P}^1$ with negative self-intersection (that is, $\Sigma_i \cdot \Sigma_i < 0$), and ii) the graph consisting of the Σ_j must have positive definite adjacency matrix given by

$$A_{ij} = -\Sigma_i \cdot \Sigma_j. \tag{1.4}$$

The latter implies that our graph contains no closed paths (i.e. it is a tree). The statement that distinct curves intersect at most in a single point translates to the condition that for $i \neq j$, we have $\Sigma_i \cdot \Sigma_j \in \{0, 1\}$.

To facilitate our discussion, we typically omit the minus signs giving curve self-intersections with the understanding that all self-intersections are negative (e.g., in place of the chain $-3, -2$, writing instead $3, 2$). Any two digit self-intersections will be given with parentheses where ambiguous. For example, when writing $1, 12, 1, 2$ without commas, we write 1(12)12.

The discriminant locus must also meet the strong restriction that it "blow down consistently," meaning iterated blow-down of all -1 curves must yield a permitted "end-point" given in [23] and restructured in [28]. We discuss certain finer details after reviewing the main result of [16]: with few exceptions we note shortly, every 6D SCFT base in F-theory is of the form

$$S_0 \overset{S_1}{g}_1 L_1 \overset{I^{\oplus r}}{g_2} L_2 g_3 L_3 \cdots g_k L_k g_{k+1} \cdots \overset{I^{\oplus s}}{g_{m-1}} L_{m-1} \overset{I^{\oplus t}}{g_m} S_m,$$

where $g_i \in \{4, 6, 7, 8, 9, (10), (11), (12)\}$ are dubbed "DE-type" nodes (referring to the gauge algebras supported on these curves), $I^{\oplus l}$ is a subgraph of the form 122....2 consisting of l curves (called an "instanton link"), and S_i, L_i are "side links" and "interior links", respectively, where a link is graph of curves allowing attachment to DE-type nodes along its exterior -1 curves, e.g. 131. Briefly, allowed bases are linear chains of curves with highly restricted branching allowed only near the ends. Truncations of the above form are also valid base forms. We refer the reader to [16] (5.101-5.107) for the full details of two exceptions to the above structure and several simplifying restrictions. The first exception allows a limited class of bases with a single 4-valent curve that are linear away from this curve (including the three possible 4-valent links). The only other exception is at precisely five nodes, where there can be up to four instanton branches rather than three as above.

To classify all 6D SCFTs obtainable in this scheme, the remaining ingredient is to determine permitted gauge enhancements for each valid graph. These too were characterized in [16]; some further enhancement elimination is possible as we outline in B. [4] The classification 6D SCFTs built in F-theory is hence simplified to determining the set of allowed links, to which nodes these may attach, and the compatible gauge enhancements for a given graph through prescription of link enhancements gluing coherently. The complete listing of links appears in [16]. We display this result in Tables 1.3,1.4 to

[4] As a step in the present study, these enhancement constraints are confirmed for all links via geometric methods, meaning via features of fibrations having Calabi-Yau resolution without invoking representation theoretic restrictions (gauge anomaly cancellation and mixed anomaly cancellation, which we discuss briefly in section 1.2.2). This gives a strong check on the enhancement structure restrictions of [16].

clarify our primary task: computation of global symmetries for all theories given by an enhancement of a link (since these are the building blocks of all bases). [5,6] Note that not all outer -1 curves permit attachment of nodes. For example, the 4-valent links are all 'noble molecules' permitting no attachments of any kind (since the resulting bases do not blow-down consistently).

A key restriction employed in this classification dates from [23] (and in restructured form from [28]), where it is shown that every valid base blows down one of an A-type, D-type, or E-type "endpoint", with A-type endpoints of the form $\alpha A_n \beta$ where

$$\alpha, \bar{\beta} \in \{\emptyset, 3, 32, 322, 3222, 4, 42, 5, 6, 7\}$$

(with $\bar{\beta}$ denoting the reverse of the string β and A_n denotes a linear chain of n pairwise intersecting -2 curves), D-type endpoints given by $D_n\gamma$, with $\gamma \in \{32, 24\}$, where D_n is D-type Dynkin diagram consisting of n pairwise intersecting -2 curves, e.g. D_5 is given by $2\overset{2}{2}22$, and E-type endpoints consisting of only -2 curves similarly associated to the E-series Dynkin diagrams. [7] The remaining restrictions yielding the classification largely follow from the allowed Kodaira singularity types (i.e. barring non-minimality at any

[5] We refrain from including single curves and instanton links. Note that all 4-valent links are 'noble', i.e. they cannot attach to any nodes. Since we can decompose these few cases into trivalent types with an additional -1 having fairly 'small' global symmetry contribution, we treat these cases by hand rather than provide computer algorithmic treatment for such bases.

[6] Curves Σ with $1 \leq m := -\Sigma \cdot \Sigma \leq 12$ are allowed bases. When $9 \leq m \leq 11$, we must perform $12 - m$ blowups. Thus, one may alternatively consider the valid single curve bases to require $1 \leq m \leq 8$ or $m = 12$. We follow the latter approach.

[7] One exception must be made for the base $2\overset{2}{3}2$, which does not fit any of the above categories.

Interior links	End links		0-links	
131	132	13	232	2315123
1231	1322	123	2312	22315123
12231	1232	1223	23132	23151313
12321	13132	1313	21322	223151313
151321	123132	12313	21512	215131513
151231	131322	122313	223132	2315131513
1513221	131512	131513	215132	215
1512321	1223132	1231513	2315132	2315
1315131	1315132	1315123	2231322	2215
12315131	1231322	1321513	2151322	22315
122315131	1231512	12231513	2215132	23215
123151321	12315132	12321513	2151232	22215
151315131	12231322	12315123	22315132	231315
1223151321	13151322	122315123	23215132	2151315
1513151321	13151232	1315131513	22151322	23151315
12231513221	13215132	12315131513	223151322	21513215
15131513221	12231512	122315131513	232151322	231513215
	122315132	1315	23151315132	2315132215
	123215132	12315	21513151322	2231513215
	123151322	13215	2231513151322	22315132215
	123151232	122315	23	313
	132151322	123215	223	3213
	1223151322	132215	213	31513
	1223151232	13151315	2313	315123
	1232151322	123151315	22313	3131513
	13151315132	131513215	23213	315131513
	123151315132	1223151315	21513	315
	131513151322	1231513215	231513	3215
	1223151315132	12231513215	221513	32215
	1231513151322		215123	31315
	12231513151322		2231513	313215
			2321513	3151315

Table 1.3: All non-instanton linear links with length at least two.

3-Valent links			
0-links	1-links	2-links	3-links
$2\overset{2}{3}13$	$2\overset{2}{3}1$	$2\overset{1}{3}1$	$1\overset{1}{5}1$
$2\overset{2}{3}132$	$3\overset{2}{2}1$	$22\overset{1}{3}1$	$1\overset{1}{5}131$
$2\overset{2}{3}1322$	$51\overset{1}{3}2$	$12\overset{2}{3}1$	$1\overset{1}{5}1321$
$31\overset{3}{\overset{1}{5}}13$	$31\overset{1}{5}12$	$1\overset{1}{5}12$	$1\overset{1}{5}13221$
$231\overset{3}{\overset{1}{5}}13$	$31\overset{1}{5}13$	$31\overset{1}{5}1$	
$231\overset{3}{\overset{1}{5}}132$	$512\overset{1}{3}2$	$2\overset{1}{3}21$	
$2231\overset{3}{\overset{1}{5}}13$	$51\overset{1}{3}22$	$151\overset{1}{3}2$	
$2231\overset{3}{\overset{1}{5}}132$	$231\overset{1}{5}12$	$231\overset{1}{5}1$	
	$231\overset{1}{5}13$	$2231\overset{1}{5}1$	
	$3151\overset{1}{3}2$	$31\overset{1}{5}1321$	
	$2231\overset{1}{5}12$	$231\overset{1}{5}131$	
	$2231\overset{1}{5}13$	$31\overset{1}{5}13221$	
	$23151\overset{1}{3}2$	$231\overset{1}{5}1321$	
	$231\overset{1}{5}132$	$2231\overset{1}{5}131$	
	$223151\overset{1}{3}2$	$231\overset{1}{5}13221$	
	$2231\overset{1}{5}132$	$2231\overset{1}{5}1321$	
	$2231\overset{1}{5}1322$	$2231\overset{1}{5}13221$	
4-Valent links			
$1\overset{1}{\underset{1}{5}}132$	$1\overset{1}{\underset{1}{5}}13$	$1\overset{1}{\underset{1}{5}}1$	

Table 1.4: All branching links

codimension 2 point of B) and the number of required vanishings of f, g, Δ determined by curve self-intersection numbers. [8]

Description of the blow-down process is available in the references above, but we describe it here for convenience. A single step in the process involves blowing down a -1 curve which inflicts a reduction on the negative self-intersection of its neighboring curves. The entire procedure consists of blowing down all -1 curves via iteration. A simple example is the following blow-down process yielding an allowed ADE graph (given by 22) after blow-down:

$$31512 \to 3141 \to 313 \to 22.$$

1.1.5 Further comments

Since our main goal is the computation of global symmetries for every 6D SCFT base with specified enhancement, it should now be plausible that it suffices to proceed by solving this problem for links and then to treat longer quivers containing DE-nodes by sewing our results together on overlaps (as DE-nodes can only appear in a base by themselves or via attachment to a -1 curve of a link). We separate our discussion of bases consisting only of bare instanton links, bare A_n graphs consisting of only -2 curves, and those form $(2)(1)4\ldots1\ldots4(1)$ consisting of alternating -1 and -4 curves optionally decorated by a short instanton link. These bases allow infinitely many non-isomorphic

[8] We wonder if these are in fact sufficient conditions. They are (strictly) tighter constraints on the permitted enhancement structure for all bases appearing [16].

gauge enhancements, thus making their algorithmic treatment and subsequent discussion slightly different. We refer to these as “rogue” bases.

The global symmetry algebras we present throughout this work are constraints on global symmetries realizable in F-theory with the exception of the single curve bases where we explicitly compare the correspondence with field theory. As noted above, while many of the global symmetry maxima we find can likely be realized via our general method outlined in Section 1.2 and generalized in the second chapter, we do not in all cases explicitly construct f, g, and Δ determining the fibration globally. Rather, we provide constraints via analysis of local models for curve intersections without determining whether a consistent global description of the base is possible while satisfying all local intersection requirements for prescription of a given SCFT with associated global symmetry arising in the appropriate limit.

To clarify Results 1.1.1,1.1.2 and Conjecture 1.1.3, we have explicitly confirmed that constraints on global symmetry for single curve theories are (often strictly) tighter in F-theory constructions than predictions arising from field theory. While we have confirmed gauge enhancement constraints from F-theory are (strictly) tighter than their field theoretic counterparts from [16], we have not carried out the analogue of the one compact curve discriminant locus comparisons of global symmetry in the more general base context (though, at least in principle, doing so is fairly straightforward).

In many (but not all) cases, the constraints of [32] (often referred to as “Persson’s List”) have been confirmed via our methods. Configurations remaining after imposition of these constraints lead to consistent global models in many cases even when the non-abelian algebra bearing curves are required to intersect the discriminant locus transversely. However, we do not pursue the complete generalization of [32] relevant to our setting in F-theory where we require collections of transversely intersecting curve stubs of specified singularity types which are permissible as point singularities via [32]. While further constraints on F-theory fibrations might be obtained with such an approach, the close matching between field theory predictions and F-theory constraints for global symmetries of theories with 1D Coulomb branch may indicate that the variety of cases we describe which do not permit a valid global construction is limited. It may be worth noting that the gaugeless and 1D Coulomb branch cases we review in this chapter can be viewed as a particular extension where we allow arbitrary Kodaira type (rather than only type I_0) along the single curve being met by families of transverse curves with the caveat that we only record fiber configurations which carry the (relatively) maximal product algebra(s).

1.2 Global symmetries for SCFTs with 1D Coulumb branch

We summarize here the results of [6, 24] determining (relatively) maximal global symmetry algebras for 6D SCFTs having a zero or one dimensional Coulomb branch as they arise via two separate computational approaches. The first employs field theoretic methods relying on anomaly cancellation restrictions. The second method is based upon constraints on the geometry of singular fibrations giving valid F-theory bases. We discuss the few cases where we find disagreement between these two approaches. New restrictions for intersections of curves over which the elliptic fibration is singular are described. Many of these find a role in the more general situation treated in the following chapter where the Coulomb branch is allowed dimension larger than one. We begin with general overview of the physical context in which these calculation arise before embarking on the summary of results and some of their derivations.

1.2.1 Physical overview

The essential idea in the F-theory constructions of SCFTs for which we investigate global symmetries is that consideration of a contractible collection of curves $\Sigma_i \subset B$ allows for a scaling limit in which gravity decouples and a conformal fixed point emerges. The above SCFT classification result begins with an F-theory model in which the singularities of high-multiplicity have been blown up. Our discussion takes place on this 'Coulomb branch' of the SCFT in which nonzero vacuum expectation values arise for the tensor multiplets of the theory. We are interested in studying the Weierstrass models of these SCFTs directly, but we note that our conclusions only follow after the compact components of the discriminant locus are contracted.

In the present work, we follow the approach used in [6, 24] to derive our constraints on global symmetry. The underlying idea originates from [26, 42, 27] and has also been employed more recently in [17, 8, 16]. This involves determining which collections of non-compact curves carrying non-abelian gauge algebras are permitted to intersect the compact curves in the discriminant locus (these dictating the gauge algebra of our SCFT). In the limit where we contract the compact curves in the discriminant locus, the sum of gauge algebras supported on the non-compact curves arises as a global symmetry algebra of our SCFT. Since the global symmetries of the SCFT act as global symmetries of the Coulomb branch, constraints on those of the Coulumb branch apply to those of the SCFT as well.

We note that the key features of an SCFT are captured by the leading terms in the Weierstrass equation obtained by truncating f, g appropriately as

$$y^2 = x^3 + \tilde{f}(z, w)x + \tilde{g}(z, w) \ ,$$

where $\tilde{f}$ and $\tilde{g}$ contain only the low-order terms of f and g (with the permitted truncations being SCFT dependent). The additional freedoms in an F-theory model for an SCFT captured by the higher order terms and compatible choices of lower order term coefficients may be viewed as specifying irrelevant deformations of the SCFT corresponding to points in the configuration space of field theories which flow to the same SCFT under renormalization. Further relevant comments addressing the physical context in which our work takes place may be found in [6].

As discussed in [6], there are several interesting features of our results in the 1D Coulomb branch case. We do not always find agreement with field theory, and perhaps more interestingly, we do not always find a maximal global symmetry algebra arising via our F-theory constructions. In some cases, we find more than one relatively maximal subalgebra of the field theoretic prediction for global symmetry. In other words, several algebras are realizable in this way, with all others properly contained in one of them. We return to discuss this further in the following chapter.

Finally, we pause to mention that the tools we present in following chapter may be helpful when we consider coupling to gravity within F-theory. It is generally accepted that full theories of supergravity cannot have any global symmetries (see for example arguments in [2]). The local model analyses we present exclude a minority of the compact-compact pairs one might consider in this alternative setting where we can relax our positive-definite adjacency matrix condition and then scan for SCFTs having all global symmetry summands promoted to gauge symmetries upon requiring compactness of the corresponding components of the discriminant locus.

1.2.2 Global symmetries from field theory

We now briefly the review results of the field theoretic predictions for global symmetries of theories with 1D Coulomb branch. These were first reported in [6], to which we refer the reader for additional details of the calculations. Briefly, we arrive at global symmetry constraints from field theory as collected in Table 1.5 by considering which representations of non-abelian gauge algebras (leading to global symmetries) are compatible with those of the gauge algebra along a curve while meeting gauge anomaly cancellation and mixed anomaly cancellation restrictions. Gauge anomaly cancellation determines from a curve of specified self-intersection restrictions on which gauge algebras and representations of those gauge algebras are permitted in addition to the number of multiplets present for each allowed representation. Mixed anomaly cancellation confines allowed pairings of intersecting curves via restrictions on multiplet pairings for chosen representations compatible with gauge anomaly cancellation. A concise summary of restrictions is provided in [16].

We consider a curve Σ with gauge algebra $\mathfrak{g}$. The possible gauge algebras and matter representations were determined from anomaly cancellation in [7, 4]; a convenient place to find these results in F-theory language is Table 10 of [14].[9] Field theory predicts (see, for example, [12]) that N hypermultiplets in a complex representation of the gauge group support an $\mathrm{SU}(N)$ global symmetry group, which is enhanced to $\mathrm{SO}(2N)$ for quaternionic (i.e. pseudoreal) representations, and to $\mathrm{Sp}(N)$ for real representations.[10] In the quaternionic case, the underlying complex representation is a representation by half-hypermultiplets, so that N is allowed to be a half-integer. (We will emphasize this by declaring that the representation consists of M copies of $\frac{1}{2}V$, where M is allowed to be an arbitrary integer, leading to global symmetry $\mathrm{SO}(M)$.)

[9] Our notation for representations follows [14] and is fairly standard: for $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$, F denotes the fundamental representation and Λ^k denotes its exterior powers. Note that for $\mathfrak{g} = sp(n)$, the representation Λ^2 contains a one-dimensional summand; the remainder is denoted by Λ^2_{irr}. For $\mathfrak{so}(n)$, V denotes the vector representation, S_+ and S_- denote half-spin representations, and S_* (according to [14]) denotes a spin or half-spin representation, depending on the parity of n. For the exceptional algebras, representations are denoted by their dimension (in bold face type).

[10] Our convention is that $\mathrm{Sp}(N)$ is a subgroup of $\mathrm{SU}(2N)$.

Our conventions here differ slightly with those of [14]: to compare the two, one may substitute $-K_B$ for L in [14]. Moreover, we interpret $(\dots)\big|_\Sigma$ as an intersection number $(\dots)\cdot\Sigma$, and do not worry about the algebraic cycle class discussed in [14]. Also, we note that the adjoint representation never occurs in our models since Σ is rational, and thus we will simply substitute

$$L\big|_\Sigma = -K_B \cdot \Sigma = 2 + \Sigma^2 \ , \tag{1.5}$$

into the formulas of [14]. Note that [14] would have included $(1+\Sigma^2)\Lambda^2_{\text{irr}}$ as part of the representation content of $\mathfrak{sp}(n),\ n \geq 2$, but since $\Sigma^2 < 0$ we conclude that $\Sigma^2 = -1$ and this representation does not occur.

We have used the following reality conditions. For $\mathfrak{su}(n)$, the fundamental representation is quaternionic for $n = 2$ and is complex for $n > 2$; the anti-symmetric representation is real for $n = 4$ and is complex for $n > 4$. For $\mathfrak{sp}(n)$, the situation is more straightforward, as the fundamental representation is always quaternionic and the irreducible part of the anti-symmetric representation (which does not occur) is always real. For the spinor representation(s) of $\mathfrak{so}(n)$:

- If $n = 0 \mod 8$, the two spinor representations are real.
- If $n = 1, 7 \mod 8$, the unique spinor representation is real.
- If $n = 2, 6 \mod 8$, the two spinor representations are complex.
- If $n = 3, 5 \mod 8$, the unique spinor representation is quaternionic.

$\mathfrak{g}$	representation	global symmetry
$\mathfrak{su}(2)$	$(32+12\Sigma^2)\frac{1}{2}F$	$\mathfrak{so}(32+12\Sigma^2)$
$\mathfrak{su}(3)$	$(18+6\Sigma^2)F$	$\mathfrak{su}(18+6\Sigma^2)$
$\mathfrak{su}(4)$	$(16+4\Sigma^2)F+(2+\Sigma^2)\Lambda^2$	$\mathfrak{su}(16+4\Sigma^2)\oplus\mathfrak{sp}(2+\Sigma^2)$
$\mathfrak{su}(5)$	$(16+3\Sigma^2)F+(2+\Sigma^2)\Lambda^2$	$\mathfrak{su}(16+3\Sigma^2)\oplus\mathfrak{su}(2+\Sigma^2)$
$\mathfrak{su}(6)$	$(16+2\Sigma^2)F+(2+\Sigma^2)\Lambda^2$	$\mathfrak{su}(16+2\Sigma^2)\oplus\mathfrak{su}(2+\Sigma^2)$
$\mathfrak{su}(6)^*$	$(16+\Sigma^2)F+\frac{1}{2}(2+\Sigma^2)\Lambda^3$	$\mathfrak{su}(16+\Sigma^2)\oplus\mathfrak{so}(2+\Sigma^2)$
$\mathfrak{su}(n),\ n\geq 7$	$(16+(8-n)\Sigma^2)F+(2+\Sigma^2)\Lambda^2$	$\mathfrak{su}(16+(8-n)\Sigma^2)\oplus\mathfrak{su}(2+\Sigma^2)$
$\mathfrak{sp}(n),\ n\geq 2$	$(16+4n)\frac{1}{2}F$	$\mathfrak{so}(16+4n)$
$\mathfrak{so}(7)$	$(3+\Sigma^2)V+2(4+\Sigma^2)S_*$	$\mathfrak{sp}(3+\Sigma^2)\oplus\mathfrak{sp}(8+2\Sigma^2)$
$\mathfrak{so}(8)$	$(4+\Sigma^2)V+(4+\Sigma^2)(S_++S_-)$	$\mathfrak{sp}(4+\Sigma^2)\oplus\mathfrak{sp}(4+\Sigma^2)\oplus\mathfrak{sp}(4+\Sigma^2)$
$\mathfrak{so}(9)$	$(5+\Sigma^2)V+(4+\Sigma^2)S_*$	$\mathfrak{sp}(5+\Sigma^2)\oplus\mathfrak{sp}(4+\Sigma^2)$
$\mathfrak{so}(10)$	$(6+\Sigma^2)V+(4+\Sigma^2)S_*$	$\mathfrak{sp}(6+\Sigma^2)\oplus\mathfrak{su}(4+\Sigma^2)$
$\mathfrak{so}(11)$	$(7+\Sigma^2)V+(4+\Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(7+\Sigma^2)\oplus\mathfrak{so}(4+\Sigma^2)$
$\mathfrak{so}(12)$	$(8+\Sigma^2)V+(4+\Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(8+\Sigma^2)\oplus\mathfrak{so}(4+\Sigma^2)$
$\mathfrak{so}(13)$	$(9+\Sigma^2)V+(2+\frac{1}{2}\Sigma^2)\frac{1}{2}S_*$	$\mathfrak{sp}(9+\Sigma^2)\oplus\mathfrak{so}(2+\frac{1}{2}\Sigma^2)$
$\mathfrak{so}(n),\ n\geq 14$	$(n-8)V$	$\mathfrak{sp}(n-8)$
$\mathfrak{e}_6$	$(6+\Sigma^2)\mathbf{27}$	$\mathfrak{su}(6+\Sigma^2)$
$\mathfrak{e}_7$	$(8+\Sigma^2)\frac{1}{2}\mathbf{56}$	$\mathfrak{so}(8+\Sigma^2)$
$\mathfrak{e}_8$	none	none
$\mathfrak{f}_4$	$(5+\Sigma^2)\mathbf{26}$	$\mathfrak{sp}(5+\Sigma^2)$
$\mathfrak{g}_2$	$(10+3\Sigma^2)\mathbf{7}$	$\mathfrak{sp}(10+3\Sigma^2)$

Note: $\Sigma^2=-1$ for $\mathfrak{su}(6)^*$; $\Sigma^2=-1$ for $\mathfrak{sp}(n),\ n\geq 2$;
$\Sigma^2=-4$ for $\mathfrak{so}(n),\ n\geq 14$; and $\Sigma^2=-12$ for $\mathfrak{e}_8$.

Table 1.5: Global symmetries as predicted from field theory.

- If $n=4 \mod 8$, the two spinor representations are quaternionic.

Note that Table 1.5 respects the exceptional isomorphisms of Lie algebras. If we extend the $\mathfrak{so}(n)$ formula to the representation of $\mathfrak{so}(6)$, we find $(2+\Sigma^2)V+4(4+\Sigma^2)S_*$, which agrees with the entry for $\mathfrak{su}(4)$ once we identify (V,S_*) with (Λ,F). Similarly, in the case of $\mathfrak{so}(5)$, we find $(1+\Sigma^2)V+8(4+\Sigma^2)\frac{1}{2}S_*$, which implies that $\Sigma^2=-1$ and V does not occur, thus agreeing with the entry for $\mathfrak{sp}(2)$ once we identify S_* of $\mathfrak{so}(5)$ with F of $\mathfrak{sp}(2)$.

We emphasize that the global symmetry only depends on the gauge algebra supported on Σ, and not on the Kodaira type that realizes it. For example, we can realize $\mathfrak{su}(2)$ with Kodaira types I_2, I_3, III and IV (with appropriate monodromies), and in all cases the global symmetry is $\mathfrak{so}(32+12\Sigma^2)$. Hence, Table 1.5 lists only the algebras and not the Kodaira types.

1.2.3 Global symmetries from F-theory geometry

The global symmetries of 6D SCFTs which may be realized in F-theory models were studied in [6, 24] for bases with discriminant locus consisting of a single curve. We reproduce the main results and some of the pertinent discussion in this section, as these are essential to our computations. In deriving our constraints on global symmetries, two ingredients are essential. We frequently refer to these results dating from [6], reproduced in Appendix A. The first are expansions of f, g, and Δ giving general forms for local models of I_n^* and I_n curves (largely from [20]). Our second major tool is a collection of tables giving forbidden curve intersections and intersection contributions via [6] obtained from these local models. While we need to generalize some of these results for our approach in the present work, much of the relevant intersection data and local model analysis plays a crucial role.

To begin, we review aspects of the setup, notation and terminology introduced in [6] that simplify our presentation.

The global symmetries are visible in an F-theory realization in the form of a component of the discriminant locus $\{4f^3 + 27g^2 = 0\}$ which passes through the point P. After rescaling, this component is necessarily non-compact, and the coupling of the associated gauge group has gone to zero, leaving a global symmetry. If we have an F-theory model defined on a neighborhood of a contractible collection of curves, there may be additional non-compact curves meeting the contractible collection over which the elliptic curve degenerates further. If there were a larger open set containing a compactification of one of those curves, then the Kodaria–Tate classification would have determined a gauge group for that curve. In the limit where the curve becomes non-compact, the gauge coupling goes to zero and we see a global symmetry rather than a gauge symmetry in our local model. These are the F-theory global symmetries we wish to compute. In all cases with discriminant locus consisting of a single compact curve, the following basic proposition of [6] holds.

Basic proposition: each relatively maximal algebra occurring in an F-theory construction with discriminant locus consisting of a single curve carrying non-abelian gauge algebra is a subalgebra of the field-theoretic algebras collected in Table 1.5. In most cases, F-theory can realize the field-theoretic symmetry algebra.

Note that in many cases we cannot determine from the local analysis whether the Kodaira fibers over the non-compact curves have monodromy or not. In these cases, we will prescribe the maximal global symmetry algebra compatible with the data (i.e. the one without monodromy). In some cases, constraints can be found on the non-compact curves to have monodromy, thereby reducing the global symmetry algebra rank.

This is a good moment to point out a small assumption we have been forced to make in our analysis. The discussion of Tate's algorithm in [20] and [25] is incomplete for Kodaira types I_n, $7 \leq n \leq 9$, and it is therefore conceivable that our analysis misses some cases associated with those Kodaira types.

Let us consider an irreducible effective divisor $\Sigma = \{z = 0\} \subseteq \{\Delta = 0\}$ with self-intersection number $\Sigma \cdot \Sigma = -m$. We introduce the condensed notation $(a, b, d)_\Sigma$ to indicate the orders of vanishing of f, g and Δ along Σ, respectively. Σ must be topologically a $\mathbb{P}^1$ and its genus is $g = 0$. Thus we have (as in (1.5))

$$K_B \cdot \Sigma = -2 + m \ . \tag{1.6}$$

The quantities

$$\widetilde{f} \equiv \frac{f}{z^a} \ , \qquad \widetilde{g} \equiv \frac{g}{z^b} \ , \qquad \widetilde{\Delta} \equiv \frac{\Delta}{z^d} \ , \tag{1.7}$$

are sections of the following line bundles

$$\widetilde{f} \leftrightarrow \mathcal{O}(-4K_B - a\Sigma) \ , \quad \widetilde{g} \leftrightarrow \mathcal{O}(-6K_B - b\Sigma) \ , \quad \widetilde{\Delta} \leftrightarrow \mathcal{O}(-12K_B - d\Sigma) \ . \tag{1.8}$$

We define the *residual vanishings on* Σ as

$$
\begin{aligned}
\widetilde{a}_\Sigma &= (-4K_B - a\Sigma)\cdot\Sigma = -4(m-2) + ma \ , \\
\widetilde{b}_\Sigma &= (-6K_B - b\Sigma)\cdot\Sigma = -6(m-2) + mb \ , \\
\widetilde{d}_\Sigma &= (-12K_B - d\Sigma)\cdot\Sigma = -12(m-2) + md \ .
\end{aligned} \tag{1.9}
$$

These values, which are required to be non-negative since $\widetilde{f}$, $\widetilde{g}$, $\widetilde{\Delta}$ do not contain Σ as a component, count the number of zeros (with multiplicity) of $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{\Delta}$, respectively, when restricted to Σ. We will refer to $\widetilde{\Delta}$ as the *residual discriminant.* Extending the notation above, we will indicate the triple (1.9) as $(\widetilde{a},\widetilde{b},\widetilde{d})_\Sigma$. Our analysis aims to determine which collections of local configurations are globally compatible with (1.9).

The first step towards our goal is to tackle the intersection between two curves. Let $\Sigma = \{z=0\}$ and $\Sigma' = \{\sigma = 0\}$ be two such curves intersecting at a point $P \equiv \Sigma\cap\Sigma'$. We describe this situation locally by a Weierstrass model, i.e. by specifying the quantities in (1.1). Following the discussion above we demand that the multiplicities of f and g at P do not exceed 4 and 6 respectively.[11] Already this basic requirement yields a first set of constraints. In fact, we can exclude all pairwise intersections that satisfy

$$
a_\Sigma + a_{\Sigma'} \geq 4 \ , \qquad\qquad b_\Sigma + b_{\Sigma'} \geq 6 \ . \tag{1.10}
$$

[11]If that is not the case (i.e. if the multiplicity of f at P is at least 4 and the multiplicity of g at P is at least 6), it follows that the multiplicity of Δ at P is at least 12.

As an example of this, we see from Table 1.1 that a curve carrying singularity type II* is in principle only allowed to have intersections with curves carrying singularity type I_n. These remaining cases are fully treated in appendix A.1. In general, the situation is a bit more complicated as

$$\mathrm{ord}_P f \geq a_\Sigma + a_{\Sigma'} \ , \qquad \mathrm{ord}_P g \geq b_\Sigma + b_{\Sigma'} \ , \qquad \mathrm{ord}_P \Delta \geq d_\Sigma + d_{\Sigma'} \ . \tag{1.11}$$

It is worth noting that there are cases where the strict inequalities hold; that is, we find additional contributions to the multiplicity at P beyond those coming from the degrees of vanishing along the two curves. Geometrically, this means that our local model describes a situation in which other components of the quantities f, g and Δ intersect at P. Hence, our criterion to discard pairwise intersections now reads

$$\mathrm{ord}_P f \geq 4 \ , \qquad \mathrm{ord}_P g \geq 6 \ . \tag{1.12}$$

For each intersection it is then natural to define

$$\widetilde{a}_P \equiv \mathrm{ord}_P \widetilde{f}\big|_{z=0} \ , \qquad \widetilde{b}_P \equiv \mathrm{ord}_P \widetilde{g}\big|_{z=0} \ , \qquad \widetilde{d}_P \equiv \mathrm{ord}_P \widetilde{\Delta}\big|_{z=0} \ , \tag{1.13}$$

and we will use the notation $(\widetilde{a}_P, \widetilde{b}_P, \widetilde{d}_P)_\Sigma$.

The second step in our analysis is to determine how to glue these local models for pairwise intersections into globally well-defined configurations. Let $\Sigma = \{z = 0\}$ be a curve as above, and let Σ_k, $k = 1, \ldots, N$, be a collection of curves, each transversely intersecting pairwise with Σ at the points P_k. We implement the global constraints on the assembly of the local configurations; these read

$$\widetilde{a}_\Sigma \geq \sum_k \widetilde{a}_{P_k} \ , \qquad \widetilde{b}_\Sigma \geq \sum_k \widetilde{b}_{P_k} \ , \qquad \widetilde{d}_\Sigma \geq \sum_k \widetilde{d}_{P_k} \ . \tag{1.14}$$

Before we proceed to a more detailed case by case investigation, we wish to make a couple of general remarks. We say that $\Sigma = \{z = 0\}$ carries *odd* type whenever the discriminant has the form

$$\frac{\Delta}{z^d} = \left(4\widetilde{f}^3 + 27 z^p \widetilde{g}^2\right) \ , \tag{1.15}$$

for some $p > 0$ and $z \nmid \widetilde{f}$. We indicate this as $(a, b + B, d)_\Sigma$, where $B = 0, 1, \ldots$. Setting $z = 0$, the second term in the right hand side vanishes identically and we find that $\widetilde{d}_P = 3\widetilde{a}_P$. Similarly, we say Σ carries *even* type when $(a + A, b, d)_\Sigma$, $A = 0, 1, \ldots$, and the residual discriminant has the form

$$\frac{\Delta}{z^d} = \left(4 z^p \widetilde{f}^3 + 27\widetilde{g}^2\right) \ , \tag{1.16}$$

for some $p > 0$ and $z \nmid \widetilde{g}$. In this case we have instead $\widetilde{d}_P = 2\widetilde{b}_P$. We describe the remaining cases, namely I_n and I_n^*, as *hybrid* types since both $\widetilde{f}\big|_{z=0}$ and $\widetilde{g}\big|_{z=0}$ in principle contribute to the residual discriminant.

In the even and odd cases, we are able to treat the general cases at once, i.e. those for any value of A and B. This is done by relaxing the first condition in (1.14) for the even case and the second in the odd case. For hybrid curves, this is however not possible, though we will argue in the relevant case (type I_0^*) that the maximal group is obtained for $A = B = 0$.

Nontrivially gauged theories

We first present the global symmetries realizable in F-theory for those cases where the lone curve in the base carries non-abelian gauge algebra and then review the arguments leading to these results dating to [6] in section 1.3 up to tightenings beyond the maxima presented in [6] which are indicated with a '†' symbol. Summaries of these results appear in Tables 1.6 and 1.7. In these cases with non-abelian gauge algebra, the global symmetry predictions from field theory are remarkably close to the constraints we find from F-theory geometry, the latter being more restrictive.[12] We discuss details of the few cases where we find disagreement following a review of the field theoretic Coulomb branch constraints on SCFT global symmetries.

Gaugeless theories

The global symmetries which may arise in F-theory models lacking non-abelian gauge algebra (i.e. where discriminant locus consists of a single trivially gauged compact curve

[12] Around the time [6] appeared, new results released in [31] constraining field theory predictions further in one particular case where find mismatch was reported, bringing agreement with predictions from F-theory. Whether yet unknown further constraints on field theory might allow precise matching in all cases remains an intriguing question.

type along Σ	algebra on Σ	$-\Sigma^2$	max. global symmetry algebra(s)
I_2	$\mathfrak{su}(2)$	2	$\mathfrak{su}(4)$
		1	$\mathfrak{so}(20)$
$I_{n\geq 3}$, n odd	$\mathfrak{sp}([n/2])$	1	$\mathfrak{so}(13+2n)$ $\mathfrak{so}(7+2p)\oplus\mathfrak{so}(7+2n-2p)$, $0\leq p\leq \frac{n+1}{2}$
	$\mathfrak{su}(n)$	2	$\mathfrak{su}(2n)$
		1	$\mathfrak{su}(8+n)$
$I_{n\geq 4}$, n even	$\mathfrak{sp}(n/2)$	1	$\mathfrak{so}(16+2n)$
	$\mathfrak{su}(n)$	2	$\mathfrak{su}(2n)$
		1	$\mathfrak{su}(8+n)$
I_6	$\mathfrak{su}(6)^*$	1	$\mathfrak{su}(15)$
III	$\mathfrak{su}(2)$	2	$\mathfrak{so}(7)$
		1	$\mathfrak{so}(7)\oplus\mathfrak{so}(7)\oplus\mathfrak{su}(2)$ $\mathfrak{so}(7)\oplus\mathfrak{sp}(3)$ (†) $\mathfrak{sp}(5)$ (†)
IV	$\mathfrak{su}(2)$	2	$\mathfrak{g}_2$ (†)
		1	$\mathfrak{g}_2\oplus\mathfrak{g}_2\oplus\mathfrak{su}(3)$ (†) $\mathfrak{g}_2\oplus\mathfrak{sp}(2)$ (†) $\mathfrak{sp}(3)$ (†)
	$\mathfrak{su}(3)$	3	-
		2	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)$ $\mathfrak{sp}(2)$
		1	$\mathfrak{su}(3)^{\oplus 4}$ $\mathfrak{su}(3)^{\oplus 2}\oplus\mathfrak{sp}(2)$ $\mathfrak{su}(3)\oplus\mathfrak{sp}(3)$ $\mathfrak{sp}(4)$
I_0^*	$\mathfrak{g}_2$	3	$\mathfrak{sp}(1)$
		2	$\mathfrak{sp}(4)$
		1	$\mathfrak{sp}(7)$
	$\mathfrak{so}(7)$	3	$\mathfrak{sp}(2)$
		2	$\mathfrak{sp}(4)\oplus\mathfrak{sp}(1)$
		1	$\mathfrak{sp}(6)\oplus\mathfrak{sp}(2)$
	$\mathfrak{so}(8)$	4	-
		3	$\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$
		2	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)^{\oplus 2}$
		1	$\mathfrak{sp}(3)\oplus\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)^{\oplus 3}$

Table 1.6: Global symmetries of gauged F-theory models.

type along Σ	algebra on Σ	$-\Sigma^2$	max. global symmetry algebra
I_1^*	$\mathfrak{so}(9)$	4	$\mathfrak{sp}(1)$
		3	$\mathfrak{sp}(2)$
		2	$\mathfrak{sp}(3)$
		1	$\mathfrak{sp}(4)$
	$\mathfrak{so}(10)$	4	$\mathfrak{sp}(2)$
		3	$\mathfrak{sp}(3)$
		2	$\mathfrak{sp}(4)$
		1	$\mathfrak{sp}(5)$
I_2^*	$\mathfrak{so}(11)$	4	$\mathfrak{sp}(3)$
		3	$\mathfrak{sp}(4)$
		2	$\mathfrak{sp}(5)$
		1	$\mathfrak{sp}(6)$
	$\mathfrak{so}(12)$	4	$\mathfrak{sp}(4)$
		3	$\mathfrak{sp}(5)$
		2	$\mathfrak{sp}(6)$
		1	$\mathfrak{sp}(7)$
I_3^*	$\mathfrak{so}(13)$	2,4	$\mathfrak{sp}(5)$
	$\mathfrak{so}(14)$	4	$\mathfrak{sp}(6)$
$\mathrm{I}_{n\geq 4}^*$	$\mathfrak{so}(2n+7)$	4	$\mathfrak{sp}(2n-1)$
	$\mathfrak{so}(2n+8)$	4	$\mathfrak{sp}(2n)$
IV^*	$\mathfrak{f}_4$	1–5	-
	$\mathfrak{e}_6$	1–6	-
III^*	$\mathfrak{e}_7$	1–8	-
II^*	$\mathfrak{e}_8$	12	-

Table 1.7: Global symmetries of gauged F-theory models.

that is contracted in the appropriate limit), first appeared in [24], thus completing a characterization global symmetries for single curve theories. Essential to the present work are the maximal global symmetry inducing configurations and tabulations of intersection contributions in these cases. Generalizations of this data to cases with $A, B > 0$, are treated in the following chapter. We omit the arguments yielding these results; they are similar in flavor to those we shall discuss at length in cases where the single compact curve in the discriminant locus carries a non-abelian gauge algebra. (Details can be found in [24].) The aforementioned Chapter 2 extensions are required for our approach to allow our treatment of arbitrary bases containing gaugeless components of the discriminant locus since it is often necessary in bases containing more than a single curve that certain Kodaira types in the compact components of the discriminant locus can only appear with $A, B > 0$.

type along Σ	algebra on Σ	$-\Sigma^2$	max. global symmetry algebra/ (max. transverse curve gauge algs.)
I_0	-	2	-
		1	$\mathfrak{e}_8$
			$(\mathfrak{e}_7 \oplus \mathfrak{su}(2))$
			$(\mathfrak{e}_6 \oplus \mathfrak{su}(3))$
			$(\mathfrak{su}(9))$
			$(\mathfrak{so}(16))$
			$(\mathfrak{so}(8) \oplus \mathfrak{so}(8))$
			$(\mathfrak{so}(12) \oplus \mathfrak{su}(2))$
			$(\mathfrak{so}(10) \oplus \mathfrak{su}(4))$
I_1	-	2	$\mathfrak{su}(2)$
		1	$\mathfrak{f}_4 \oplus \mathfrak{su}(3)$
			$\mathfrak{so}(15)$
			$\mathfrak{so}(7) \oplus \mathfrak{so}(9)$
			$\mathfrak{su}(7)$
			$\mathfrak{so}(2N+7) \oplus \mathfrak{su}(M),\ M + N \leq 4$
II	-	2	$\mathfrak{su}(2)$
		1	$\mathfrak{f}_4 \oplus \mathfrak{g}_2$
			$\mathfrak{so}(9) \oplus \mathfrak{g}_2$
			$\mathfrak{g}_2 \oplus \mathfrak{sp}(N),\ N \leq 4$
			$\mathfrak{sp}(3) \oplus \mathfrak{su}(3)$

Table 1.8: Global symmetries of gaugeless F-theory models.

A few comments in the case of a single type I_0 curve are merited. In the analysis leading to the above bounds on global symmetry, the maximal algebras listed are those permitted as collections of singular points along an I_0 curve via constraints derived in [32], but we have not checked the somewhat stronger condition that these can coexist along curves which simultaneously transversely intersect a type I_0 curve. Note that here, as with many of the other configurations of curves we study in this work, the maximal gauged curve collections which can arise may place distinct restrictions on the singularity type of the compact curves they intersect. As an example, among the maximal gauged curve configurations above for a type I_0 are [III*,III] and [IV^{*s},IV^s], from which the algebras $\mathfrak{e}_7 \oplus \mathfrak{su}(2)$ and $\mathfrak{e}_6 \oplus \mathfrak{su}(3)$ arise. However, the former requires $B > 0$, and the latter $A > 0$, where the orders of vanishing along the I_0 curve are given by $(A, B, 0)$. Since A, B cannot simultaneously be non-zero for type I_0, a given fixed compact component of the discriminant locus consisting only allows at most one of these transverse curve collections. In this sense, they can only arise in distinct geometries. This phenomenon is one motivation for tracking the geometric data captured by the orders of f, g, Δ in our work. In our example, we have a larger algebra in which these are both subalgebras, namely $\mathfrak{e}_8$. We might hastily conclude the global symmetry algebra of all models with a lone I_0 curve is then always $\mathfrak{e}_8$. However, when $B > 0$, intersection with any e_8 bearing curve is non-minimal. Our approach intends to enable a broader determination of whether the distinct global symmetries we find may arise from *distinct SCFTs* with their data specified by the geometry of the fibration at a level of precision beyond specification of the gauge algebra, instead tracking the orders of f, g, Δ along each of the compact and non-compact components of the discriminant locus.

1.2.4 Comparison with field theory

Gauged theories

Tables 1.6 and 1.7 list the (relatively) maximal symmetry algebras of F-theory constructions for the class of 6D SCFTs with discriminant locus consisting of a single non-compact curve with non-abelian gauge algebra meeting the additional hypotheses of [6]. Comparing with Table 1.5, we conclude that in some cases the global symmetry from field theory can be realized; in other cases, this cannot happen and we determine the (relatively) maximal global symmetry algebras.

In fact, the agreement between the F-theory constructions and the Coulomb branch predictions is quite good, and we can easily list the cases where there is a mismatch. Note that in some cases, there is more than one way to produce the gauge algebra in F-theory; it is enough for our purposes if we can find agreement between the field theory prediction and at least one of the F-theory realizations. With that being said, the cases where there is a mismatch are:

1. In the case of $\mathfrak{su}(2)$ with $m = 2$, we have obtained only $\mathfrak{so}(7)$ rather than the Coulomb branch expectation of $\mathfrak{so}(8)$. This is the case which has already been explained in [31], which found a field-theoretic reason that the SCFT should have a smaller global symmetry than the one observed on the Coulomb branch.

2. In the case of $\mathfrak{su}(3)$ with $m = 1$, we see the most severe mismatch: the predicted global symmetry is $\mathfrak{su}(12)$, but we find only a variety of different subalgebras of this (different ones for different realizations of the gauge algebra).

3. For $\mathfrak{so}(8)$ with $m = 1$ or $m = 2$, we only realized $\mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m) \oplus \mathfrak{sp}(1)^{\oplus(4-m)}$ rather than the predicted $\mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m) \oplus \mathfrak{sp}(4-m)$.

4. For $\mathfrak{so}(n)$, $9 \leq n \leq 13$ and $m < 4$, the predicted global symmetry associated to the spinor representation is never realized.

5. For $\mathfrak{so}(13)$ with $m = 2$, only a $\mathfrak{sp}(5)$ subalgebra of the predicted $\mathfrak{sp}(7)$ algebra is realized.

6. Finally, for $\mathfrak{f}_4$, $\mathfrak{e}_6$, and $\mathfrak{e}_7$, none of the predicted global symmetries are realized.

Gaugeless theories

Reading from Table 1.8 which is reproduced from [24], we note that in these cases there is a larger gap between what we expect from field theory models and F-theory constructions, the latter usually generating algebras having rank lower than 8, while field theory offers little in the way of constraints to tighten beyond an $\mathfrak{e}_8$ global symmetry algebra.

We return later to discuss an issue hinted at by the following observation. Even when the discriminant locus consists of an I_0 curve, we find significant constraints on the global symmetries that can be realized in F-theory beyond those we might expect from field theory via the $\mathfrak{e}_8$ gauging condition of [16]. When $B > 0$ along Σ of type I_0, we cannot achieve the $\mathfrak{e}_8$ global symmetry algebra we expect from field theory since the resulting model in such a construction is non-minimal. One might take this to suggest that a certain amount of data beyond which Kodaira types occur in Δ may be important in full description of a geometry–field theory dictionary.

1.3 Review of gauged 1D F-theory global symmetry computatons

In this section, we review the arguments presented in [6] leading to the results discussed above and summarized in Tables 1.6,1.7. We proceed through the various possibilities for the Kodaira type along the single compact curve in the discriminant locus carrying a non-abelian gauge algebra.

1.3.1 Comments on monodromy

This "monodromy" determining the gauge algebra is part of Tate's algorithm [37], originally appearing in the physics literature in [3]. Specification of the gauge algebra is reached via Table 4 of [14], which we summarize here. In each case, there is a covering of Σ that can be described by means of an algebraic equation in an auxiliary variable ψ, taking its values in a line bundle over Σ.

For all cases except I_0^*, the equation of the monodromy cover takes the form "$\psi^2 -$ something", and this cover splits (leading to no monodromy) if and only if the expression "something" is a perfect square. In the remaining case I_0^*, the monodromy cover equation defines a degree 3 cover of Σ, and one must analyze this further to determine if that cover is irreducible ($\mathfrak{g}(\Sigma) = \mathfrak{g}_2$), splits into two components ($\mathfrak{g}(\Sigma) = \mathfrak{so}(7)$), or splits into three components ($\mathfrak{g}(\Sigma) = \mathfrak{so}(8)$). Each of these possibilities is reflected in the behavior of the residual discriminant, as we now describe.

First, suppose that the monodromy cover splits completely, resulting in gauge algebra $\mathfrak{so}(8)$. We have

$$\psi^3 + (f/z^2)|_{z=0} \cdot \psi + (g/z^3)|_{z=0} = (\psi - \alpha)(\psi - \beta)(\psi - \gamma) \ , \tag{1.17}$$

with $\alpha + \beta + \gamma = 0$. One can calculate the residual discriminant $(\Delta/z^6)|_{z=0}$ to obtain

$$\begin{aligned} 4(\alpha\beta + \beta\gamma + \gamma\delta)^3 + 27\alpha^2\beta^2\gamma^2 &= -(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 \\ &= -(\alpha - \beta)^2(\alpha + 2\beta)^2(2\alpha + \beta)^2 \ . \end{aligned} \tag{1.18}$$

This form of the residual discriminant is a necessary consequence of having gauge algebra $\mathfrak{so}(8)$. The analysis of [14] then associates to the vanishing of the factors $\alpha - \beta$, $\beta - \gamma$, and $\gamma - \alpha$ the three different 8-dimensional representations of $\mathfrak{so}(8)$.

When the monodromy cover splits into a linear factor and a quadratic factor (corresponding to gauge algebra $\mathfrak{so}(7)$), we have

$$\psi^3 + (f/z^2)|_{z=0} \cdot \psi + (g/z^3)|_{z=0} = (\psi - \lambda)(\psi^2 + \lambda\psi + \mu) \ . \tag{1.19}$$

The residual discriminant $(\Delta/z^6)|_{z=0}$ then reads

$$4(\mu - \lambda^2)^3 + 27\lambda^2\mu^2 = (\mu + 2\lambda^2)^2(4\mu - \lambda^2) \ . \tag{1.20}$$

The factor $\varphi := 4\mu - \lambda^2$ is the discriminant of $\psi^2 + \lambda\psi + \mu = 0$, thus capturing the ramification points of the double cover. If φ is a square, then the double cover splits and we are in the case with $\mathfrak{so}(8)$ gauge algebra. If not, the analysis in [14] then shows that the intersection points are associated to the 7-dimensional representation of $\mathfrak{so}(7)$, while the other points (the zeros of $\mu + 2\lambda^2$) are associated to the spinor representation.

If the monodromy cover splits completely in (1.19), we may choose one of the roots (say α) as λ, and we have $\mu = \beta\gamma$. Using $\gamma = -\alpha - \beta$, this gives

$$\begin{aligned} \mu + 2\lambda^2 &= \beta\gamma + 2\alpha^2 = (2\alpha + \beta)(\alpha - \beta) \ , \\ 4\mu - \lambda^2 &= 4\beta\gamma - \alpha^2 = -(\alpha + 2\beta)^2, \end{aligned} \tag{1.21}$$

readily allowing comparison between the two forms of the discriminant.

1.3.2 Type I_n

Let $\Sigma = \{z = 0\}$ carry type I_n for $n \geq 2$ with self-intersection number $\Sigma^2 = -m$. The residual vanishings on Σ are given by

$$(\widetilde{a}, \widetilde{b}, \widetilde{d})_\Sigma = (8 - 4m, 12 - 6m, 24 + (n - 12)m) \ , \tag{1.22}$$

where $m = 1, 2$. From the analysis in [20] and [25] (reviewed in Appendix A.1), we have

$$f\big|_{z=0} = -\tfrac{1}{48}\phi_n^2 \ , \qquad\qquad g\big|_{z=0} = \tfrac{1}{864}\phi_n^3 \ , \tag{1.23}$$

for some locally-defined function ϕ_n, which depends on n. Except when $n = 2$, where there is no monodromy issue, we can additionally write[13] $\phi_n = \mu\phi_0^2$ with μ square-free and the monodromy is determined by whether μ vanishes[14] somewhere along Σ.

The case $m = 2$ is very simple and we consider it first. Notice that $\widetilde{a}_\Sigma = \widetilde{b}_\Sigma = 0$, implying that neither quantity in (1.23) can vanish. This means that μ cannot vanish anywhere along Σ. Hence, the case with monodromy cannot be realized. This matches with the fact that $\Sigma^2 = -1$ for $\mathfrak{sp}(n)$ as derived in section 1.2.2 via arguments dating to [6]. We can thus assume without loss of generality that there is no monodromy along Σ and the only allowed configurations are chains of type $\mathcal{C}_{n_1,\ldots,n_N} \equiv [\mathrm{I}_{n_1}, \ldots, \mathrm{I}_{n_N}]$ with associated algebras $\mathfrak{su}(n_1) \oplus \cdots \oplus \mathfrak{su}(n_N)$. In particular, the maximal configuration will be

[13]Of course, μ and ϕ_0 depend on n, but we will not keep track of this index.

[14]Note that our previously stated criterion for monodromy in this case asked whether $-9g/2f$, when restricted to $z = 0$, is a square or not. Since $-9g/2f|_{z=0} = \frac{1}{4}\phi_n = \frac{1}{4}\mu\phi_0^2$, this is clearly equivalent.

$\mathcal{C}_{\widetilde{d}_\Sigma}$, yielding the algebra $\mathfrak{su}(\widetilde{d}_\Sigma)$ where $\widetilde{d}_\Sigma = 2n$. We easily see that the basic proposition is verified here, as the algebra from field theory is $\mathfrak{so}(8)$ for $n = 2$ and $\mathfrak{su}(2n)$ for $n \geq 3$.

In the rest of this section, we will deal with the case $m = 1$, for which $(\widetilde{a}, \widetilde{b})_\Sigma = (4, 6)$. We start by proving some simple but useful results for $n \geq 3$, where from (1.23) we have $\deg \phi_n = 2$ and

$$\deg \mu + 2 \deg \phi_0 = 2 \ . \tag{1.24}$$

If Σ has monodromy, then $\deg \mu > 0$ and by (1.24) we get $\deg \mu = 2$ and $\deg \phi_0 = 0$. Otherwise, Σ has no monodromy and $\deg \mu = 0$, $\deg \phi_0 = 1$.

We wish to study the restrictions that this global constraint imposes on different Kodaira singularity types when present in a configuration over Σ. Let $n = 2i$ or $n = 2i+1$ and let $\Sigma' = \{\sigma = 0\}$ carry type $\mathrm{I}_{n'}$, where $n' = 2j$ or $n' = 2j + 1$. To determine the maximal allowed value for n' we will consider for simplicity the general forms given in (A.25). The divisibility conditions read

$$z, \sigma \nmid u \ , \qquad z^i \sigma^j | v \ , \qquad z^{2i} \sigma^{2j} | w \ . \tag{1.25}$$

The residual discriminant for Σ is

$$\left.\frac{\Delta}{z^n}\right|_{z=0} = u^2|_{z=0} \underbrace{\left(4uw - v^2\right)|_{z=0}}_{\widehat{\Delta}} \ , \tag{1.26}$$

and by (1.25) we have that $\sigma^{n'}|\widehat{\Delta}$; that is the degree of vanishing of $\widehat{\Delta}$ at $\sigma = 0$ is at least n'. In other words, though $u\big|_{z=0}$ could vanish at $\sigma = 0$ as well (therefore contributing to $\widetilde{d}_P$), it does not increase the degree of vanishing of Δ along the transverse type $\mathrm{I}_{n'}$ curve. In the remainder of this section, we assume without loss of generality that $u\big|_{z=0}$ does not vanish at the intersection points between Σ and curves of type $\mathrm{I}_{n'}$.

Now we wish to discuss the implications of a transverse intersection with a curve carrying any of the remaining Kodaira singularity types. If Σ has no monodromy, we observe from Table A.2 that type I_0^* and type $\mathrm{I}_{n'}^*$ are not allowed in any configurations, while type IV is forbidden for $n \geq 4$ and type III for $n \geq 5$. Moreover, here $\deg \phi_0 = 1$, thus there can be at most one curve carrying type other than $\mathrm{I}_{n'}$, which will intersect Σ at $\phi_0|_{z=0} = 0$.

In the monodromy case, $\deg \mu = 2$ and configurations can admit up to two curves carrying type other than $\mathrm{I}_{n'}$ intersecting Σ precisely at the roots of $\mu\big|_{z=0}$. To summarize, the relevant configurations are chains of the type $\mathcal{C}_{n_1,\ldots,n_N}$ with one or two other singularity types attached. However, since $\mathcal{C}_{n_1,\ldots,n_N}$ yields the algebra $\oplus_i \mathfrak{su}(n_i)$, when looking for the maximal global symmetry group, we will only consider $\mathcal{C}_{\sum_i n_i}$ with algebra $\mathfrak{su}(\sum_i n_i)$.

Perhaps a comment is useful here. We stated above that Kodaira types other than $\mathrm{I}_{n'}$ can only intersect Σ at the roots of $\phi_n|_\Sigma$. There is an example that seems to contradict this. Let us consider an intersection with type IV along the locus $\{\sigma = 0\}$, then both f and g vanish to order 2 in σ and it seems that $\phi_n = -9g/2f|_{z=0}$ need not vanish. However, the point is that the quantity g/f is evaluated along Σ, where $g|_{z=0}$ and $f|_{z=0}$

can have higher orders of vanishing. These are determined carefully in appendix A.1, and it turns out that for all intersections, except with type $\mathrm{I}_{n'}$, ϕ_n does indeed vanish.

Having summarized some of the general features of the pertinent local models, let us proceed to a detailed analysis.

$\mathbf{n = 2}$

In this case, there is no monodromy ambiguity and we see from Table A.2 that types III, IV, I_0^* and $\mathrm{I}_{n'}^*$ can be part of a configuration on Σ. Here,

$$\left.\frac{\Delta}{z^2}\right|_{z=0} = \phi^2 \left(\phi\widetilde{g}_2 - f_1^2\right) \ , \tag{1.27}$$

where the quantities on the RHS are restricted to the locus $\{z = 0\}$. In the language of (1.26), we define

$$\widehat{\Delta} \equiv \phi\widetilde{g}_2 - f_1^2 \ . \tag{1.28}$$

From (1.23), it follows $\deg\phi = 2$ and there can be up to two curves carrying type other than $\mathrm{I}_{n'}$ in a configuration. Let $\Sigma' = \{\sigma = 0\}$ carry a given Kodaira type and let $P \equiv \Sigma \cap \Sigma'$. Our strategy is to determine the lowest order of vanishing of $\widehat{\Delta}$ at P, and introducing a non-standard notation, we will denote this value by $\mathrm{ord}_P\widehat{\Delta}$. This is easily done by recalling the general form for type I_2

$$f = -\frac{1}{48}\phi^2 + f_1 z + O(z^2) \ , \quad g = \frac{1}{864}\phi^3 - \frac{1}{12}\phi f_1 z + (\widetilde{g}_2 - \frac{1}{12}\phi f_2) z^2 + O(z^3) \ , \quad (1.29)$$

as well as the data from Table 1.1. In fact, let Σ' carry type III, then $\sigma|f$ and $\sigma^2|g$, making $\sigma|\{\phi, f_1\}$ and $\sigma^2|\widetilde{g}_2$. Plugging these into (1.28), we obtain $\sigma^2|\widehat{\Delta}$. If Σ' carries type IV, then $\sigma^2|f$ and $\sigma^2|g$, and hence $\sigma^2|f_1$ while everything else is unchanged from the case above, in turn yielding $\mathrm{ord}_P\widehat{\Delta} = 3$. Next, we consider the case of Σ' carrying type I_p^* for $p \geq 0$. The argument here is somewhat different, as it suffices to realize that

$$6 + p = \mathrm{ord}_P\Delta = 2\mathrm{ord}_P\phi + \mathrm{ord}_P\widehat{\Delta} \ . \quad (1.30)$$

Now, the solution (1.29) is equivalent to (A.25) with the following identifications

$$u = \frac{1}{4}\phi \ , \qquad v_i = f_i \ , \quad i \geq 1 \ , \qquad w_j = \widetilde{g}_j \ , \quad j \geq 2 \ . \quad (1.31)$$

The Tate form for I_p^* (regardless of monodromy) prescribes $\sigma|u$ but $\sigma^2 \nmid u$, hence $\mathrm{ord}_P\phi = 1$.[15] This determines $\mathrm{ord}_P\widehat{\Delta} = 4 + p$. We summarize all of this in the following table.

	I_0^*	I_p^*	III	IV	$\mathrm{I}_{n'}$
$\mathrm{ord}_P\widehat{\Delta}$	4	$4+p$	2	3	n'

[15]In other words, if $\mathrm{ord}_P\phi > 1$ then we are in a different branch of Kodaira's classification.

This also allows us to reduce the types of configurations we need to consider. From the data above, as far as algebras are concerned

$$[\ldots, \mathrm{III}, \ldots, \mathrm{I}_{n'}, \ldots] \subset [\ldots, \mathrm{I}_{n'+2}, \ldots] \ , \quad [\ldots, \mathrm{IV}, \ldots, \mathrm{I}_{n'}, \ldots] \subset [\ldots, \mathrm{I}_{n'+3}, \ldots] \ . \tag{1.32}$$

Hence, we avoid considering type III and type IV. Finally, recalling that $\deg \widehat{\Delta} = 10$, we obtain the following relevant configurations

configuration(s)	algebra(s)
I_{10}	$\mathfrak{su}(10)$
$\mathrm{I}_p^*, \mathrm{I}_{6-p},\ 0 \leq p \leq 6$	$\mathfrak{so}(8+2p) \oplus \mathfrak{su}(6-p)$
$\mathrm{I}_p^*, \mathrm{I}_q^*, \mathrm{I}_{2-p-q},\ 0 \leq \{p, q, p+q\} \leq 2$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(8+2q) \oplus \mathfrak{su}(2-p-q)$

We conclude that the global symmetry of the maximal configuration $[\mathrm{I}_6^*]$ agrees with the field-theoretic prediction of $\mathfrak{so}(20)$.

$\mathbf{n \geq 3}$ odd

Here $\widetilde{d}_\Sigma = 12 + n$, and from appendix A.1 we have

$$\frac{\Delta}{z^n}\bigg|_{z=0} = u_0^2 \left(4u_0 w_{2i+1} + 4u_1 w_{2i} - 2v_i v_{i+1}\right) \ , \tag{1.33}$$

where $u_0 = \mu\phi_0^2$ and $n = 2i+1$. As before, we assume throughout the rest of this section the generic form (A.25), and we can rewrite the last expression as

$$\frac{\Delta}{z^n}\Big|_{z=0} = \frac{1}{16}\mu^3\phi_0^4\left(\phi_0^2 w_{2i+1} + t_i^2 u_1 - \phi_0 t_i v_{i+1}\right) \ , \tag{1.34}$$

denoting by $\widehat{\Delta}$ the term inside the brackets. Even for the cases $n=3$ and $n=5$, the expressions above are completely general, as there is an isomorphism between the explicit forms (A.11) and (A.20) and the inductive form (A.25), given by (for $n=5$)

$$\begin{aligned} &u_0 = \frac{1}{4}\mu\phi_0^2 \ , && u_1 = \phi_1 \ , && v_2 = \frac{1}{2}\mu\phi_0\psi_2 \ , \\ &v_k = f_k \ , \ k \geq 3 \ , && w_4 = \frac{1}{4}\mu\psi_2^2 \ , && w_l = \widehat{g}_l \ , \ l \geq 5 \ . \end{aligned} \tag{1.35}$$

First, let us suppose that there is monodromy on Σ. From above we have that $\deg\mu = 2$ and $\deg\phi_0 = 0$; we set $\phi_0 \equiv 1$ for concreteness. This implies $\deg\widehat{\Delta} = \widetilde{d}_\Sigma - 6 = 6 + n$, yielding I_{6+n} as the largest type $\mathrm{I}_{n'}$ allowed in any configuration.

To take into account the presence of other singularity types we need to determine, as before, the lowest vanishing order of $\widehat{\Delta}$ along $\Sigma' = \{\sigma = 0\}$. We recall from (A.28) the expressions

$$\begin{aligned} u &= \frac{1}{4}\mu + u_1 z + O(z^2) \ , \\ v &= \frac{1}{2}\mu t_i z^i + v_{i+1}z^{i+1} + O(z^{i+2}) \ , \\ w &= \frac{1}{4}\mu t_i^2 z^{2i} + w_{2i+1}z^{2i+1} + O(z^{2i+2}) \ . \end{aligned} \tag{1.36}$$

When Σ' carries type III, we have that $\sigma|\{u,v\}$ and $\sigma^2|w$. Since σ^2 must divide μt_i^2 and μ is square-free, it follows that $\sigma|t_i$. All of this implies that $\sigma^2|\widehat{\Delta}$ for type III. When Σ' carries type IV without monodromy (if there is monodromy, the gauge algebra is $\mathfrak{su}(2)$ and the discussion follows the one above for type III), we have $\sigma|u$ and $\sigma^2|\{v,w\}$. Hence, the terms in

$$g = \frac{2}{27}u^3 - \frac{1}{3}uv + w \tag{1.37}$$

have degrees of vanishing 3, 3 and 2 along $\{\sigma = 0\}$, respectively. Monodromy forces $g/\sigma^2|_{\sigma=0} = w$ to be a square, and from (1.36), we obtain

$$\frac{w}{\sigma^2}\Big|_{\sigma=0} = \widetilde{w}_{2i+1}\Big|_{\sigma=0} z^{2i+1} + \cdots \ , \tag{1.38}$$

where $w_{2i+1} = \sigma^2 \widetilde{w}_{2i+1}$. Whether (1.38) is a square or not is contingent upon whether $\widetilde{w}_{2i+1}$ vanishes identically or it is further divisible by σ. In either case, it is straightforward to verify that $\mathrm{ord}_P \widehat{\Delta} = 3$.

In the case Σ' supports type I_0^*, there are two cases we need to distinguish. In fact, while $\sigma|u$ and $\sigma^2|v$ always hold, we have that $\sigma^3|w$ for $\mathrm{I}_0^{*\mathrm{ns}}$ (algebra $\mathfrak{g}_2$) and $\sigma^4|w$ for both $\mathrm{I}_0^{*\mathrm{ss}}$ and $\mathrm{I}_0^{*\mathrm{s}}$ (algebras $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$, respectively). For the latter case, we will consider only $\mathrm{I}_0^{*\mathrm{s}}$ as it leads to the largest global symmetry. It follows that $\sigma|t_i$ for $\mathrm{I}_0^{*\mathrm{ns}}$ and $\sigma^2|t_i$ for $\mathrm{I}_0^{*\mathrm{s}}$, and (1.34) determines $\sigma^3|\widehat{\Delta}$ for $\mathrm{I}_0^{*\mathrm{ns}}$ and $\sigma^4|\widehat{\Delta}$ for $\mathrm{I}_0^{*\mathrm{s}}$.

The remaining case we need to analyze is when Σ' carries type I_p^* for $p \geq 1$. First suppose that $p = 2n' + 1$; then $\sigma | u$, $\sigma^{n'+3} | v$ and $\sigma^{2n'+4} | w$, which in turn implies $\sigma^{n'+2} | t_i$. This leads to $\mathrm{ord}_P \widehat{\Delta} = 2n' + 4 = p + 3$, and this holds for $\mathrm{I}_p^{*\mathrm{ns}}$. The condition for monodromy is whether

$$\frac{w}{\sigma^{2n'+4}}\Big|_{\sigma=0} = \widetilde{w}_{2i+1}\big|_{\sigma=0} z^{2i+1} + \cdots \ , \tag{1.39}$$

where $w_{2i+1} = \sigma^{2n'+4}\widetilde{w}$, has a square root. As before, in order to have no monodromy $\widetilde{w}_{2i+1}|_{\sigma=0} = 0$, and this makes $\mathrm{ord}_P \widehat{\Delta} = 2n' + 5 = p + 4$. The even case, $p = 2n'$, is similar, but we repeat the argument for completeness. Here we have $\sigma | u$, $\sigma^{n'+2} | v$ and $\sigma^{2n'+3} | w$, which leads to $\sigma^{n'+1} | t_i$. It is convenient to define $\mu = \sigma\tilde{\mu}$, $u_1 = \sigma\widetilde{u}_1$, $t_i = \sigma^{n'+1}\tilde{t}_i$, $v_{i+1} = \sigma^{n'+2}\widetilde{v}_{i+1}$ and $w_{2i+1} = \sigma^{2n'+3}\widetilde{w}_{2i+1}$. This gives $\mathrm{ord}_P \widehat{\Delta} = 2n' + 3 = p + 3$, valid for $\mathrm{I}_p^{*\mathrm{ns}}$. In order to have no monodromy, the quantity

$$4\left(\frac{u}{\sigma}\right)\Big|_{\sigma=0}\left(\frac{w}{\sigma^{2n'+3}}\right)\Big|_{\sigma=0} - \frac{v}{\sigma^{2n'+2}}\Big|_{\sigma=0} = \tilde{\mu}\left(\tilde{t}_i^2\widetilde{u}_1 + \widetilde{w}_{2i+1} - \tilde{t}_i\widetilde{v}_{i+1}\right)\Big|_{\sigma=0} z^{2i+1} + O(z^{2i+2}) \ , \tag{1.40}$$

must be a square. Therefore, the coefficient of the first term on the RHS of (1.40) must vanish and in particular it coincides, up to an irrelevant factor of $\tilde{\mu}$, with (1.34), giving $\mathrm{ord}_P \widehat{\Delta} = p + 4$. The follows table summarizes these results.

	$\mathrm{I}_0^{*\mathrm{ns}}$	$\mathrm{I}_0^{*\mathrm{s}}$	$\mathrm{I}_p^{*\mathrm{ns}}$	$\mathrm{I}_p^{*\mathrm{s}}$	IV	III	$\mathrm{I}_{n'}$
$\mathrm{ord}_P \widehat{\Delta}$	3	4	$3 + p$	$4 + p$	3	2	n'

The argument we provided in the $n = 2$ case for type III and type IV goes through here as well, since for these, the values of $\text{ord}_P\widehat{\Delta}$ are identical. Thus, we will exclude configurations containing them. The remaining relevant configurations are

configuration(s)	algebra(s)
I_{n+6}	$\mathfrak{su}(n+6)$
$\text{I}_p^{*\text{s}}, \text{I}_{n+2-p}, \quad 0 \le p \le n+2$	$\mathfrak{so}(8+2p) \oplus \mathfrak{su}(n+2-p)$
$\text{I}_p^{*\text{ns}}, \text{I}_{n+3-p}, \quad 0 \le p \le n+3$	$\mathfrak{so}(7+2p) \oplus \mathfrak{su}(n+3-p)$
$\text{I}_p^{*\text{s}}, \text{I}_q^{*\text{s}}, \text{I}_{n-2-p-q}, \ 0 \le \{p, q, p+q\} \le n-2$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(8+2q) \oplus \mathfrak{su}(n-2-p-q)$
$\text{I}_p^{*\text{s}}, \text{I}_q^{*\text{ns}}, \text{I}_{n-1-p-q}, \ 0 \le \{p, q, p+q\} \le n-1$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(7+2q) \oplus \mathfrak{su}(n-1-p-q)$
$\text{I}_p^{*\text{ns}}, \text{I}_q^{*\text{ns}}, \text{I}_{n-p-q}, \ 0 \le \{p, q, p+q\} \le n$	$\mathfrak{so}(7+2p) \oplus \mathfrak{so}(7+2q) \oplus \mathfrak{su}(n-p-q)$

By inspection, we see that within each row of the above table except the first, the algebra is maximal when p or $p+q$ assume the highest allowed value, i.e. when the last curve is I_0. This also tells us that $[\text{I}_p^{*\text{s}}, \text{I}^{*\text{s}}_{n-2-p}] \subset [\text{I}^{*\text{s}}_{n+2}]$ and $[\text{I}_p^{*\text{s}}, \text{I}^{*\text{ns}}_{n-1-p}] \subset [\text{I}^{*\text{ns}}_{n+3}]$ and of course $[\text{I}_{n+6}] \subset [\text{I}^{*\text{s}}_{n+2}] \subset [\text{I}^{*\text{ns}}_{n+3}]$. The locally maximal configurations are then as follows.

configuration(s)	algebra(s)
$\text{I}^{*\text{ns}}_{n+3}$	$\mathfrak{so}(13+2n)$
$\text{I}_p^{*\text{ns}}, \text{I}^{*\text{ns}}_{n-p}, \ 0 \le p \le \frac{n+1}{2}$	$\mathfrak{so}(7+2p) \oplus \mathfrak{so}(7+2n-2p)$

The predicted algebra is $\mathfrak{so}(16+2(n-1))$. It is easy to check that the basic proposition holds for each value of n. This example shows two important characteristics of our findings, as we mentioned in the discussion above. First, we verified explicitly that the algebra predicted through field theory is not always realizable in F-theory. Second, this model exhibits multiple relatively maximal algebras, each subalgebras of the maximal algebra from field theory.

If there is no monodromy on Σ, we set $\mu \equiv 1$ and $\deg \phi_0 = 1$. Then $\deg \widehat{\Delta} = \widetilde{d}_\Sigma - 4 = 8 + n$. We recall that type I_0^* and type I_n^* are forbidden in all configurations. However, as mentioned above, for $n = 3$ type III and type IV are still allowed. We thus obtain the following maximal configurations.

$$\begin{array}{cc} \text{configuration} & \text{algebra} \\ \mathrm{I}_{11} & \mathfrak{su}(11) \\ \mathrm{IV}, \mathrm{I}_8 & \mathfrak{su}(3) \oplus \mathfrak{su}(8) \\ \mathrm{III}, \mathrm{I}_9 & \mathfrak{su}(2) \oplus \mathfrak{su}(9) \end{array} \tag{1.41}$$

These are all subalgebras of $\mathfrak{su}(11)$ (which is in turn a subalgebra of field-theoretic prediction $\mathfrak{su}(12)$). For $n \geq 5$, the only relevant configurations are of the form $\mathcal{C}_{n_1,\ldots,n_N}$, and the maximal algebra among these results from $\mathcal{C}_{\widetilde{d}-4}$. This yields $\mathfrak{su}(8+n)$, which coincides with the prediction from field theory.

$\mathbf{n \geq 4}$ even

In this case, we set $n = 2i$ and the residual discriminant assumes the form

$$\left.\frac{\Delta}{z^n}\right|_{z=0} = u_0^2 \left(4u_0 w_{2i} - v_i^2\right) \ , \tag{1.42}$$

where again $u_0 = \frac{1}{4}\mu\phi_0^2$, and

$$u = \frac{1}{4}\mu\phi_0^2 + u_1 z + \ldots \ , \quad v = v_i z^i + v_{i+1} z^{i+1} + \cdots \ , \quad w = w_{2i} z^{2i} + w_{2i+1} z^{2i+1} + \cdots \ . \tag{1.43}$$

We define $\widehat{\Delta} \equiv 4u_0 w_{2i} - v_i^2$.

Suppose first that Σ has monodromy. In this case, the form (1.42) is generic within the assumptions of this paper. In fact, we have already showed that the inductive form (A.25) reproduces the generic form for type I_5 regardless of monodromy, and in particular for type I_4. For type I_6, this is not true in general, but it holds when we impose that Σ has monodromy. In this case, recall that $\deg \mu = 2$ and $\deg \phi_0 = \deg \alpha + \deg \beta = 0$ and without loss of generality we set $\alpha = \beta \equiv 1$. The identification between the two sets of local functions is

$$\begin{aligned}
&u_0 = \frac{1}{4}\mu \ , && u_1 = \nu \ , && u_2 = \phi_2 \ , \\
&v_3 = \frac{1}{3}\nu\phi_2 - 3\lambda \ , && v_4 = f_4 + \frac{1}{3}\phi_2^2 \ , && v_k = f_k \ , \ k \geq 5 \ , \\
&w_6 = \widehat{g}_6 + \frac{1}{3} f_4 \phi_2 + \frac{1}{27}\phi_2^3 \ , && w_l = \widehat{g}_l \ , \ l \geq 7 \ .
\end{aligned} \tag{1.44}$$

Hence, the only case which is not captured in full generality[16] by (1.42) is $n = 6$ with no monodromy, which we will discuss separately.

[16] As stated above, we are making an assumption for $n = 7, 8, 9$.

Let us proceed and determine the lowest order of vanishing of $\widehat{\Delta}$ for intersections with different Kodaira types. Given the simpler form of the solution (1.43) with respect to (1.36), the analysis is less involved. Let $\Sigma' = \{\sigma = 0\}$ carry type III, then $\sigma|\{u, v\}$, $\sigma^2|w$ and $\sigma^2|\widehat{\Delta}$. When Σ' carries type IV, the only modification is that now $\sigma^2|v$, yielding $\sigma^3|\widehat{\Delta}$. Now let Σ' carry type I_p^*, where $p \geq 0$. In what follows, we do not need to distinguish between the different monodromy cases, so we only consider the largest induced algebra. If $p = 2n'+1$ we have $\sigma|u$, $\sigma^{n'+3}|v$, $\sigma^{2n'+4}|w$ and $\sigma^{2n'+5}|\widehat{\Delta}$; if $p = 2n'$ we have $\sigma|u$, $\sigma^{n'+2}|v$, $\sigma^{2n'+3}|w$ and $\sigma^{2n'+4}|\widehat{\Delta}$. We summarize these results in the following table.

$$\begin{array}{c|ccccc} & \mathrm{I}_0^* & \mathrm{I}_p^* & \mathrm{IV} & \mathrm{III} & \mathrm{I}_{n'} \\ \mathrm{ord}_P \widehat{\Delta} & 4 & 4+p & 3 & 2 & n' \end{array} \tag{1.45}$$

As before, we can proceed without consideration of configurations involving type III and type IV. Since $\deg \widehat{\Delta} = \widetilde{d}_\Sigma - 4 = 8 + n$ we obtain the following configurations.

configuration(s)	algebra(s)
I_{8+n}	$\mathfrak{su}(8+n)$
$\mathrm{I}_p^*, \mathrm{I}_{4+n-p}, \quad 0 \leq p \leq n+4$	$\mathfrak{so}(8+2p) \oplus \mathfrak{su}(4+n-p)$
$\mathrm{I}_p^*, \mathrm{I}_q^*, \mathrm{I}_{n-p-q}, \quad 0 \leq \{p, q, p+q\} \leq n$	$\mathfrak{so}(8+2p) \oplus \mathfrak{so}(8+2q) \oplus \mathfrak{su}(n-p-q)$

We see that there is a maximal algebra, namely $\mathfrak{so}(16+2n)$, given by $[\mathrm{I}_{n+4}^*]$. This coincides with the prediction for each n.

If there is no monodromy on Σ (and $n \neq 6$), the only other singularity type other that $\mathrm{I}_{n'}$ is type III for $n = 4$, for which $\widetilde{d}_P \geq 4$ as can be read from (1.45) and (1.42).

The relevant configurations in this case are therefore

configuration	algebra
I_{12}	$\mathfrak{su}(12)$
$\mathrm{III}, \mathrm{I}_{10}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(10)$

The prediction from field theory is $\mathfrak{su}(12)$. Therefore, we have again verified our basic proposition. If $n > 4$, the analysis is similar to the odd case: the only relevant configurations are $\mathcal{C}_{n_1,\ldots,n_N}$ and the maximal is $\mathcal{C}_{\widetilde{d}-4}$. This yields $\mathfrak{su}(8+n)$, also matching the field theory result.

The remaining case to be analyzed is $n = 6$ without monodromy. From (A.17) we have

$$\widetilde{f}\big|_{z=0} = -\tfrac{1}{48}\alpha^4\beta^4 \ , \qquad \widetilde{g}\big|_{z=0} = \tfrac{1}{864}\alpha^6\beta^6 \ , \tag{1.46}$$

as well as

$$\widetilde{\Delta}\big|_{z=0} = \tfrac{1}{432}\alpha^4\beta^3 \underbrace{\left(27\alpha^2\beta^3\widehat{g}_6 + 9\alpha^2\beta^2\phi_2 f_4 + \alpha^2\phi_2^3 - 243\lambda^2\beta^3 + 54\phi_2\nu\lambda\beta^2 - 3\beta\nu^2\phi_2^2\right)}_{\widehat{\Delta}} . \tag{1.47}$$

Now, since $\Sigma^2 = -1$ we have that $(\widetilde{a}, \widetilde{b})_\Sigma = (4, 6)$. Together with (1.46), this implies

$$\deg\alpha + \deg\beta = 1 \ . \tag{1.48}$$

Also, we have that $\widetilde{d}_\Sigma = 18$ and

$$\widetilde{d}_\Sigma = 4 \deg \alpha + 3 \deg \beta + \deg \widehat{\Delta} \ . \tag{1.49}$$

We recall that the relevant configuration $\mathcal{C}_{n_1,\dots,n_N}$ only has support on $\widehat{\Delta}$, that is $\sigma_i \nmid \{\alpha, \beta\}$, where $\{\sigma_i = 0\}$ supports the i-th curve in the configuration. We can then rewrite (1.49) as

$$18 - \sum_i n_i \geq 4 \deg \alpha + 3 \deg \beta \ . \tag{1.50}$$

There are two possibilities:

1. $\deg \alpha = 1, \ \deg \beta = 0$. Then $\sum_i n_i \leq 14$ and the maximal configuration is $\mathcal{C}_{\widetilde{d}_\Sigma - 4}$ with algebra $\mathfrak{su}(14)$. This is the algebra associated to the $\mathfrak{su}(6)$ row of Table 1.5.

2. $\deg \alpha = 0, \ \deg \beta = 1$. Here instead $\sum_i n_i \leq 15$ giving maximal configuration is $\mathcal{C}_{\widetilde{d}_\Sigma - 3}$ with algebra $\mathfrak{su}(15)$. This corresponds to the alternative possibility $\mathfrak{su}(6)^*$.

1.3.3 Type III

Let $\Sigma = \{z = 0\}$ carry singularity type III and $\Sigma \cdot \Sigma = -m$. Plugging $(a, b, d)_\Sigma = (1, 2 + B, 3)_\Sigma$ in (1.9), we compute $(\widetilde{a}, \widetilde{b}, \widetilde{d})_\Sigma = (8 - 3m, 12 - 4m + Bm, 24 - 9m)$. We now proceed to separately analyze the two cases $m = 1, 2$.

Let us start with $m = 2$. First of all, we need to determine which curves locally admit allowed intersections. From (1.10) and appendix A.2 we see that type IV^*, III^* and II^* are forbidden, as well as type I_n^* for $n \geq 1$. From Table A.2, we see that an intersection with I_n is possible for each n, which together with type III, IV and I_0^* constitute the blocks for building configurations. As we remarked above, we do not need to worry about $\widetilde{b}_P$ for odd types. Thus, we list the above singularities types with their contributions to $\widetilde{a}_P$, borrowing the appropriate entries from Table 1.1 and Table A.2.

	III	IV	I_0^*	I_n
$\widetilde{a}_P$	1	2	≥ 2	$\lceil n/2 \rceil$

Now, since $\widetilde{a}_\Sigma = 2$, we must exclude $\mathrm{I}_{n \geq 5}$, and the various possibilities compatible with (1.14) are as follows.

configuration	algebra
III, III	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathrm{I}_2,\ \mathrm{III}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathrm{I}_2,\ \mathrm{I}_2$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
I_3	$\mathfrak{su}(3)$
I_4	$\mathfrak{su}(4)$
IV	$\mathfrak{su}(3)$
I_0^*	$\mathfrak{so}(7)$ or $\mathfrak{g}_2$

Note that we cannot assume that an I_0^* meeting a type III supports an $\mathfrak{so}(8)$ algebra as that would require all intersection points with the residual discriminant (even the ones not near the current intersection) to have even multiplicity. And indeed, the analysis of

[31] shows that we can have at most $\mathfrak{so}(7)$. To make this explicitly, let $\Sigma' = \{\sigma = 0\}$ carry type I_0^*, then

$$f = z\sigma^2 f_0 \ , \qquad\qquad g = z^2\sigma^3 g_0(\sigma) \ , \tag{1.51}$$

where f_0 is a non-zero constant and $\deg g_0(\sigma) = 2B+1$ such that $g_0(0) \neq 0$. The residual discriminant for Σ' reads

$$\left.\frac{\Delta}{\sigma^6}\right|_{\sigma=0} = z^3\left(4f_0^3 + 27zg_0^2(0)\right) \ , \tag{1.52}$$

which is not a square in z and the algebra supported on Σ' cannot be $\mathfrak{so}(8)$. We conclude that there is an algebra, namely $\mathfrak{so}(7)$, that admits as a subalgebra every other entry in the list and is the desired maximal global symmetry algebra in this case.

We now turn to the case $m = 1$. The local models for the intersections are the same as above, here constrained instead by $\widetilde{a}_\Sigma = 5$. This leads to a larger set of possibilities for the intersecting curves, which we do not reproduce here. The result is that there are three different models that exhibit maximal global symmetry. Note than transverse I_n curves must have monodromy to prevent a non-minimal point at the intersection. This fact leads to smaller global symmetry maximums than were reported in [6], differing only in that the last two configurations here have reduced symmetry from the type I_n curves.

configuration	algebra
$\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{I}_2$	$\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$
$\mathrm{I}_0^*,\ \mathrm{I}_6$	$\mathfrak{so}(7) \oplus \mathfrak{sp}(3)$
I_{10}	$\mathfrak{sp}(5)$

As before, we have indicated the monodromy choices in each case making the global symmetry algebra relatively maximal and we used the fact that type I_0^* cannot support an $\mathfrak{so}(8)$ algebra. We notice that these algebras are all contained in the algebra $\mathfrak{so}(20)$ predicted from field theory.

1.3.4 Type IV

We now turn to the analysis of a Kodaira type IV curve along $\Sigma = \{z = 0\}$ with $(a, b, d)_\Sigma = (2 + A, 2, 4)$, for which the residual vanishings are given by $(\widetilde{a}, \widetilde{b}, \widetilde{d})_\Sigma = (8-2m+Am, 12-4m, 24-8m)$. The possibilities for the self-intersection are $m = 1, 2, 3$. Moreover, there are two possibilities for the monodromy on a type IV, the condition being whether $\widetilde{g}\big|_{z=0}$ is a square. If $m = 3$, the gauge algebra must be $\mathfrak{su}(3)$, and there is no matter (and hence no global symmetry).

For $m = 2$, we have $\widetilde{b}_\Sigma = 4$. Let us start by listing configurations that satisfy (1.14). These appear in the following table.

configuration	algebra
I_0^*	$\mathfrak{so}(7)$ or $\mathfrak{g}_2$
III, III	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathrm{I}_3,\ \mathrm{I}_3$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
IV, IV	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
IV, I_3	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$
I_4	$\mathfrak{sp}(2)$

Now we determine how monodromy imposes restrictions in each case.

- $\underline{\mathfrak{su}(2).}$ In this case, $\widetilde{g}\big|_{z=0}$ is not allowed to have all double roots. Therefore, we discard the configurations for which each intersection yields an even value for $\widetilde{b}_P$. All configurations in the list above except the first fall into this category. As in the case of type III, the type I_0^* gives $\mathfrak{so}(7)$ or $\mathfrak{g}_2$. In fact, only $\mathfrak{g}_2$ is possible in this case as discussed in Section 1.3.8.

- $\underline{\mathfrak{su}(3).}$ In this case, $\widetilde{g}\big|_{z=0}$ is a square and hence all its roots are double roots. Moreover, Σ is not allowed to intersect $\Sigma' = \{\sigma = 0\}$ carrying type I_0^* since $\widetilde{b}_P \geq 3$, where $P \equiv \Sigma \cap \Sigma'$, is increased at least to $\widetilde{b}_P = 4$. Thus, at P we are in the case (1.10) and the intersection is non-minimal. The remaining configurations are in principle allowed and yield $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ and $\mathfrak{sp}(2)$ as maximal global symmetries, which indeed are both subalgebras of the algebra $\mathfrak{su}(6)$ predicted from field theory.

If $m = 1$, then $\widetilde{b}_\Sigma = 8$ and we list possible configurations giving a priori relatively maximal algebras before any eliminations.

configuration	algebra
$\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{IV}$	$\mathfrak{so}(7)^{\oplus 2} \oplus \mathfrak{su}(3)$
$\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{I}_3$	$\mathfrak{so}(7)^{\oplus 2} \oplus \mathfrak{su}(3)$
$\mathrm{I}_0^*,\ \mathrm{IV}, \mathrm{IV}$	$\mathfrak{so}(7) \oplus \mathfrak{su}(3)^{\oplus 2}$
$\mathrm{I}_0^*,\ \mathrm{I}_5$	$\mathfrak{so}(7) \oplus \mathfrak{sp}(2)$
IV, IV, IV, IV	$\mathfrak{su}(3)^{\oplus 4}$
$\mathrm{IV},\ \mathrm{I}_6$	$\mathfrak{su}(3) \oplus \mathfrak{sp}(3)$
I_8	$\mathfrak{sp}(4)$
$\mathrm{I}_4,\ \mathrm{IV}, \mathrm{IV}$	$\mathfrak{sp}(2) \oplus \mathfrak{su}(3)^{\oplus 2}$

Again, we discuss the two monodromy cases separately.

- $\underline{\mathfrak{su}(2).}$ We start by showing that the configuration $[\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{IV}]$ leads to a global symmetry that is not larger than $\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$. In fact, let $\{\sigma_1 = 0\}$, $\{\sigma_2 = 0\}$ and $\{\sigma_3 = 0\}$ be the curves carrying Kodaira type IV and the two with type I_0^*, respectively. Then we have the following divisibility conditions

$$f = z^2\sigma_1^2\sigma_2^2\sigma_3^2 f_0(\sigma)\ , \qquad g = z^2\sigma_1^2\sigma_2^3\sigma_3^3 g_0\ , \tag{1.53}$$

where g_0 is a non-zero constant and $\deg f_0(\sigma) = \alpha$ and is such that $f_0(\sigma_i = 0) \neq 0$ for $i = 1, 2, 3$. Looking at $g/\sigma_1^2|_{\sigma_1=0}$, we see it is a perfect square in z, thus the gauge algebra along the intersecting IV is indeed $\mathfrak{su}(3)$ (this holds since the set-up already fills out $\widetilde{b}_\Sigma$). Now, to determine the monodromy on the I_0^* curves, we look at the residual discriminant restricted to one of these curves, say $\{\sigma_2 = 0\}$, which reads

$$\frac{\Delta}{\sigma_2^6}\Big|_{\sigma_2=0} = z^4\left(4z^2\sigma_1^6\sigma_3^6 f_0^3 + 27\sigma_1^4\sigma_3^6 g_0^2\right)\big|_{\sigma_2=0}\ , \tag{1.54}$$

which is not a perfect square in z. Thus, the algebra cannot be $\mathfrak{so}(8)$ for either of the two curves carrying type I_0^*. We are therefore restricted to at most $\mathfrak{so}(7) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(3)$. This argument applies for any of our configurations that contains type I_0^*. As before, we discard configurations in which we must have $g/z^2|_{z=0}$ a square, e.g. a configuration with four type IV curves. The list of configurations to be considered becomes

configuration	algebra
$\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{IV}$	$\mathfrak{g}_2^{\oplus 2} \oplus \mathfrak{su}(3)$
$\mathrm{I}_0^*,\ \mathrm{I}_0^*,\ \mathrm{I}_3$	$\mathfrak{g}_2^{\oplus 2} \oplus \mathfrak{su}(3)$
$\mathrm{I}_0^*,\ \mathrm{I}_5$	$\mathfrak{g}_2 \oplus \mathfrak{sp}(2)$
I_6	$\mathfrak{sp}(3)$,

where we have used that the algebra along each I_0^* is reduced to $\mathfrak{g}_2$ via an argument from [16], as discussed in Section 1.3.8.

The remaining configurations produce several relatively maximal global symmetry algebras, namely

$$\mathfrak{g}_2 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(3)$$
$$\mathfrak{g}_2 \oplus \mathfrak{sp}(2)$$
$$\mathfrak{sp}(3) \ .$$

Note that these configurations differ from those reported in [6]. All are subalgebras of the field theory global symmetry maximum given by $\mathfrak{so}(20)$.

- $\underline{\mathfrak{su}(3).}$ Here we recall that a configuration containing type I_0^* is not allowed. Therefore, we are left with the last four configurations of the list above, and all of these lead to subalgebras of the field-theoretic prediction $\mathfrak{su}(12)$.

1.3.5 Type $\mathbf{I}_0^*$

Let us turn our attention to $\Sigma = \{z = 0\}$ carrying type I_0^* with $\Sigma^2 = -m$ and $(2 + A, 3 + B, 6)_\Sigma$, where A and B cannot be both nonzero. The residual vanishings on Σ are $(\widetilde{a}, \widetilde{b}, \widetilde{d})_\Sigma = (8 - 2m + Am, 12 - 3m + Bm, 24 - 6m)$. The local analysis derived in appendix A.1 still applies, but it must be supplemented by a more careful treatment of the global constraints to be imposed on the Weierstrass model. We are going to prove some simple results about the *global* models involving intersections between type I_0^* and various curves carrying type I_n, for some n. In particular, we look for upper bounds on

n such that type I_n is allowed to intersect type I_0^*, although *locally* the intersection is consistent for any n as can be seen from Table A.2.

There are three possibilities for the gauge algebra on Kodaira type I_0^*, determined by the behavior of the monodromy cover

$$\psi^3 + \frac{f}{z^2}\Big|_{z=0}\psi + \frac{g}{z^3}\Big|_{z=0} . \tag{1.55}$$

If the monodromy cover is irreducible, the gauge algebra is $\mathfrak{g}_2$; if it has two irreducible components, we have $\mathfrak{so}(7)$ and if it has three components, the gauge algebra is $\mathfrak{so}(8)$. Since the prediction from field theory as well as the results we find here differ for these cases, we will treat them separately.

We are going to argue that one needs only to consider configurations of the form

$$\mathcal{C}_{2n_1,\ldots,2n_N} \equiv [\mathrm{I}_{2n_1}, \ldots, \mathrm{I}_{2n_N}]. \tag{1.56}$$

Concerning type III and type IV, we recall from the previous section that type IV intersecting type I_0^* always has monodromy, i.e. it carries algebra $\mathfrak{su}(2)$, so at least as far as global symmetry is concerned, we can replace $[\ldots, \mathrm{IV}, \ldots, \mathrm{III}, \ldots] \to [\ldots, \mathrm{I}_2, \ldots, \mathrm{I}_2, \ldots]$. Moreover, both type III and type IV yield higher values for $\widetilde{d}_P$ with respect to type I_2, resulting in stronger constraints on the remainder of the configuration. Finally, type I_n intersecting type I_0^* always has monodromy as summarized in Table A.2. Thus, type I_{2n} and type I_{2n+1} contribute the same algebra summand, although the former yields a smaller value for $\widetilde{d}_P$. Hence, in the remaining of this section we will restrict our attention to configurations of the form $\mathcal{C}_{2n_1,\ldots,2n_N}$ as defined above.

This has a nice additional consequence; as can be seen from the type I_0^* entry in Table A.1, the local models are highly constrained if either A or B are non-zero. Thus, we conclude that the maximal global symmetry will be realized for configurations $\mathcal{C}_{2n_1,\ldots,2n_N}$ for $A = B = 0$, as anticipated above.

Gauge algebra $\mathfrak{so}(8)$

Let us start by supposing that there is no monodromy on Σ, that is, type I_0^* on Σ carries $\mathfrak{so}(8)$ algebra. In this case the monodromy cover takes the form

$$(\psi - \alpha)(\psi - \beta)(\psi - \gamma) \; , \tag{1.57}$$

where α, β and γ are sections of $-2K_B - \Sigma$ and $\alpha + \beta + \gamma = 0$. Let σ be a transverse coordinate to Σ in some open set, then we can represent the quantities α, β and γ as polynomials of an appropriate degree in σ. The degree is determined by how many times the sections vanish (with multiplicity) on Σ. This means that they vanish exactly

$$(-2K_B - \Sigma) \cdot \Sigma = 4 - m \tag{1.58}$$

times on Σ. In particular, $\deg\alpha = \deg\beta = \deg\gamma = 4-m$. Now, the residual discriminant takes the form

$$\frac{\Delta}{z^6}\bigg|_{z=0} = -(\alpha-\beta)^2(2\alpha+\beta)^2(\alpha+2\beta)^2 \ . \tag{1.59}$$

By construction, $\alpha-\beta, 2\alpha+\beta, \alpha+2\beta$ are also sections of $-2K_B-\Sigma$ and in particular they cannot coincide pairwise (otherwise the third vanishes identically, $\alpha=0$ or $\beta=0$). The maximal algebra will be achieved if we can tune the above sections such that each of the factors of (1.59) has a (different) unique zero of multiplicity $4-m$. If that were the case, the configuration would therefore be $\mathcal{C}_{8-2m,8-2m,8-2m}$, yielding the global symmetry algebra $\mathfrak{sp}(4-m)\oplus\mathfrak{sp}(4-m)\oplus\mathfrak{sp}(4-m)$, which would match the predictions from field theory.

This tuning is only possible when $4-m=1$ and cannot be carried out in the other cases. To see this, consider two homogeneous polynomials of degree $4-m$ on $\mathbb{CP}^1$, each with a zero of multiplicity $4-m$ (representing two of the three factors appearing in (1.59). Introducing an appropriate local coordinate t, we may assume that the zeros of one polynomial are at $t=0$ and those of the other polynomial are at $t=1$. For the third factor, then, we seek constants c_0 and c_1 such that

$$F(t) := c_0 t^{4-m} + c_1 (t-1)^{4-m} \tag{1.60}$$

has a zero of multiplicity $4-m$ at some value of t other than 0 or 1. In particular, if $4-m>1$, then (1.60) must have a multiple root so that $F(t)$ and $F'(t)$ must share a zero (other than $t=0$ or $t=1$). But since $F(t)-\frac{t}{4-m}F'(t)=-c_1(t-1)^{3-m}$, the only possible location of a common zero is $t=1$, contradicting our assumption that it could be located at a third point. This shows that $F(t)$ cannot have roots with multiplicity higher than one; the $4-m$ roots of $F(t)$ of multiplicity one could in principle support $4-m$ curves of type I_2.

The conclusion is that we can achieve $\mathfrak{sp}(4-m)\oplus\mathfrak{sp}(4-m)\oplus\mathfrak{sp}(1)^{\oplus(4-m)}$ symmetry.

Gauge algebra $\mathfrak{so}(7)$

In this case, the monodromy cover (1.55) takes the form

$$(\psi-\lambda)(\psi^2+\lambda\psi+\mu)\ , \tag{1.61}$$

where λ is a section of $-2K_B-\Sigma$ and μ is a section of $-4K_B-2\Sigma$. As before, we think of them as polynomials of the appropriate degree in some transverse coordinate σ in some open set of Σ, and their degrees are given by

$$\deg\lambda=(-2K_B-\Sigma)\cdot\Sigma=4-m\ ,\qquad \deg\mu=(-4K_B-2\Sigma)\cdot\Sigma=8-2m\ . \tag{1.62}$$

The residual discriminant takes the form

$$\widetilde{\Delta} \equiv \frac{\Delta}{z^6}\Big|_{z=0} = -\left(\lambda^2 - 4\mu\right)\left(2\lambda^2 - \mu\right)^2 \ , \tag{1.63}$$

where $\varphi \equiv \lambda^2 - 4\mu$ is not a square (otherwise the algebra is $\mathfrak{so}(8)$). We notice that by construction φ and $2\lambda^2 - \mu$ cannot have common zeros. It is possible that φ has a factor that is a square, and we write $\varphi = \delta\gamma^2$, where δ is not a square. In particular, this means that $\deg\delta$ cannot vanish. Note that for $m = 3$, necessarily $\gamma \equiv 1$. The degree read

$$\deg(2\lambda^2 - \mu) = 8 - 2m \ , \qquad 8 - 2m = 2\deg\gamma + \deg\delta \geq 2\deg\gamma + 2 \ . \tag{1.64}$$

This means that $(2\lambda^2 - \mu)^2$ can support at most type I_{16-4m}, while γ^2 can support at most type I_{6-2m}. The maximal algebra will be achieved if we can tune the above sections as described, yielding the configurations $\mathcal{C}_{16-4m,6-2m}$, with algebras $\mathfrak{sp}(8-2m) \oplus \mathfrak{sp}(3-m)$. These coincide with the predictions from field theory.

Gauge algebra $\mathfrak{g}_2$

When Σ carries gauge algebra $\mathfrak{g}_2$, the Tate form prescribes the divisibility conditions

$$\widetilde{u} \equiv u/z \ , \qquad \widetilde{v} \equiv v/z^2 \ , \qquad \widetilde{w} \equiv w/z^3 \ , \tag{1.65}$$

where (see appendix A.1)

$$f = -\frac{1}{3}u^2 + v \ , \qquad\qquad g = \frac{2}{27}u^3 - \frac{1}{3}uv + w \ . \tag{1.66}$$

In terms of (1.65) the residual discriminant reads

$$\widetilde{\Delta}\Big|_{z=0} = \frac{\Delta}{z^6}\Big|_{z=0} = 4\widetilde{u}^3\widetilde{w} - \widetilde{u}^2\widetilde{v}^2 - 18\widetilde{u}\widetilde{v}\widetilde{w} + 4\widetilde{v}^3 + 27\widetilde{w}^2 \ . \tag{1.67}$$

Let $\mathcal{C}_{2n_1,\ldots,2n_N}$ be a configuration of the type defined above, where $\Sigma_i = \{\sigma_i = 0\}$ carries type I_{2n_i}. If $2\sum_i n_i = \widetilde{d}_\Sigma$, then the residual discriminant must assume the form $\widetilde{\Delta} = c\sigma_1^{2n_1}\cdots\sigma_N^{2n_N}$, where c is a constant. That is, $\widetilde{\Delta}$ is a square and the algebra along Σ cannot be $\mathfrak{g}_2$. This yields a first constraint

$$\sum_i n_i \leq \frac{1}{2}\widetilde{d}_\Sigma - 1 = 11 - 3m \ . \tag{1.68}$$

For each curve in $\mathcal{C}_{2n_1,\ldots,2n_N}$ (following our discussion in appendix A.1), we define the quantities

$$U_{(i)} \equiv \widetilde{u}|_{z=0} \ , \qquad V_{(i)} \equiv \frac{\widetilde{v}}{\sigma^{n_i}}\Big|_{z=0} \ , \qquad W_{(i)} \equiv \frac{\widetilde{w}}{\sigma^{2n_i}}\Big|_{z=0} \ . \tag{1.69}$$

We now show the following. Given $\Sigma = \{z = 0\}$ carrying type I_0^* with gauge algebra $\mathfrak{g}_2$, any configuration $\mathcal{C}_{2n_1,\ldots,2n_N}$ is such that $n_i \leq 10 + 3\Sigma^2$. This follows from the global constraints on $\widetilde{a}_\Sigma$ and $\widetilde{b}_\Sigma$ and on the degrees of U, V, W. Let type I_{2n} be supported along $\Sigma' = \{\sigma = 0\}$ and be part of a configuration on type I_0^*. First, we show that for $n > 10 - 3m$ the following hold

$$\deg V + n = 2 \deg U \ , \qquad \deg W + 2n = 3 \deg U \ . \tag{1.70}$$

While this is trivial in the $m = 1$ and $m = 2$ cases, we need to be careful when $m = 3$. In fact, let $m = 3$ and $n = 2$. Imposing

$$8 - 2m = \deg \widetilde{f}\big|_{z=0} = \deg \left(-\frac{1}{3} U^2 + V \sigma^n \right) \ , \tag{1.71}$$

it follows that either $\deg U \leq 1$ and $\deg V = 0$ or $\deg V = 2 \deg U - 2$. The second constraint

$$12 - 3m = \deg \widetilde{g}\big|_{z=0} = \deg \left(\frac{2}{27} U^3 - \frac{1}{3} U V \sigma^n + W \sigma^{2n} \right) \ , \tag{1.72}$$

instead forces

$$\deg W + 4 = 3 \deg U \qquad \text{or} \qquad \deg W + 4 = 2 + \deg U + \deg V \ . \tag{1.73}$$

The first possibility trivially leads to $\deg U > 1$. The second implies $\deg U + \deg V \geq 2$, which again forces $\deg U > 1$ or $\deg V > 0$. When $n = 3$ (the only other possibility for $m = 3$), the claim is again trivial. Therefore, we verified that (1.70) hold in all cases of interest for this lemma.

Let $m = 3$. We need to show that type I_4 is forbidden. This follows easily from the above since for this curve, we have $\deg U \leq 1$, but (1.70) forces $\deg U \geq 2$.

Let $m = 2$. To show that type I_{10} is not allowed, we explicitly check that (1.71) and (1.72) cannot be both satisfied. Here, we have $\deg U \leq 4$ but (1.70) (for $n = 5$) fixes $\deg U = 4$, $\deg V = 3$ and $\deg W = 2$. We expand these quantities accordingly as

$$
\begin{aligned}
U &= u_0 + u_1\sigma + u_2\sigma^2 + u_3\sigma^3 + u_4\sigma^4 \ , \\
V &= v_0 + v_1\sigma + v_2\sigma^2 + v_3\sigma^3 \ , \\
W &= w_0 + w_1\sigma + w_2\sigma^2 \ ,
\end{aligned}
\tag{1.74}
$$

where $u_4, v_3, w_2 \neq 0$. By imposing (1.71), we obtain

$$
v_2 = \frac{2}{3}u_3u_4 \ , \qquad v_3 = \frac{1}{3}u_4^2 \ , \qquad v_1 = \frac{1}{3}u_3^2 + \frac{2}{3}u_2u_4 \ , \qquad v_0 = \frac{2}{3}u_2u_3 - \frac{2}{3}u_1u_4 \ , \tag{1.75}
$$

and in order for (1.72) to hold, we must satisfy

$$
u_3(u_1u_3 - 2u_4) = 0 \ , \qquad u_2u_3^2 + u_4^2 = 0 \ , \qquad u_3^3 + 6u_2u_3u_4 + 3u_1u_4^2 = 0 \ . \tag{1.76}
$$

The only solution to the above system of equations is $u_3 = u_4 = 0$, which is unacceptable. Thus, $\deg g|_{z=0} \geq 7$ contradicting (1.72).

If $m = 1$, we need to show that I_{16} is forbidden. It is possible to treat this case as well as in the above, that is, by expanding U, V and W to the appropriate degrees and showing that $\deg f|_{z=0} = 6$ and $\deg f|_{z=0} = 9$ cannot both be satisfied. Since the procedure is identical but the algebra is more involved, we do not report these calculations here. This concludes the proof of the lemma.

We can generalize the lemma above in the following way. We want to show that a configuration such that $\sum_i n_i = 11 - 3m$ cannot be supported on type I_0^* with $\mathfrak{g}_2$ gauge algebra. Our argument goes as follows. Suppose $\sum_i n_i = 11 - 3m$, then the residual discriminant takes the following form

$$\widetilde{\Delta} = (\sigma_1^{n_1} \cdots \sigma_N^{n_N})^2 F_{[2]} \ , \tag{1.77}$$

where $F_{[2]}$ can be represented as a polynomial of degree 2 in the local variable σ. If (1.77) can be rewritten as (1.63), this means that the underlying gauge algebra must be enhanced to at least $\mathfrak{so}(7)$. This amounts to investigating whether we can find appropriate sections α and β as in (1.62). As it happens, we can simply count parameters. In fact, for a configuration $\mathcal{C}_{2n_1,\ldots,2n_N}$ the number of parameters that we need to completely specify the form of (1.77) is $N + 3$, where the first summand corresponds to the locations of the N transverse curves and the additional 3 parameters characterize the polynomial $F_{[2]}$. We already see that the most constraining configurations will be the "longest", that is $n_i = 1$ for each $i = 1, \ldots, N = 11 - 3m$. In particular, this means that in every case

$$\#(\widetilde{\Delta}) \leq 11 - 3m + 3 = 14 - 3m \ . \tag{1.78}$$

On the other hand, (1.62) implies that the numbers of parameters that specify the local sections α and β are

$$\#(\alpha) = 4 - m + 1 \ , \qquad \#(\beta) = 8 - 2m + 1 \ . \tag{1.79}$$

Therefore, $\#(\alpha) + \#(\beta) \geq \#(\widetilde{\Delta})$ always holds. This gives us our final bound,

$$\sum_i n_i \leq 10 - 3m \ , \tag{1.80}$$

on allowed configurations. We emphasize the fact that we did not make any statements regarding the realizability of these configurations; we only argued that if the configuration can be consistently constructed, the algebra must be at least $\mathfrak{so}(7)$ (this happens for example in the case $m = 3$ with $\mathcal{C}_{2,2}$).

With these results at our disposal, we conclude that the maximal configurations will be $\mathcal{C}_{20-6m}$, yielding the algebras $\mathfrak{sp}(10-3m)$. These agree with the predictions from field theory.

1.3.6 Type I_n^*

In this section, we analyze configurations on a curve $\Sigma = \{z = 0\}$ carrying type I_n^*, $n \geq 1$, and $(\widetilde{a}, \widetilde{b}, \widetilde{d})_\Sigma = (8 - 2m, 12 - 3m, 24 + (n-6)m)$, where $\Sigma^2 = -m$. (Recall that when $n \geq 4$, we have $m = 4$ and for I_3^* with monodromy, we have $m = 2$ or $m = 4$.) From Table A.3, we see that configurations involving curves carrying other than type II and type $\mathrm{I}_{n'}$ are forbidden, and we discard the former since it does not contribute to the global symmetry. The degrees of vanishing of f and g along Σ are 2 and 3 respectively. Thus the results in appendix A.1 for type I_0^* (for $A = B = 0$) naturally extend here. Hence, each curve $\mathrm{I}_{n'}$ in the configuration has monodromy and the relevant configurations are again $\mathcal{C}_{2n_1,\dots,2n_N}$ with the associated algebras $\mathfrak{sp}(n_1) \oplus \cdots \oplus \mathfrak{sp}(n_N)$.

The following result will uniquely determine the maximal length of the configuration:

Lemma 1.3.1. *Let $\Sigma' = \{\sigma = 0\}$ be a curve carrying any Kodaira type and $P \equiv \Sigma \cap \Sigma'$. Then $(\widetilde{a}_P, \widetilde{b}_P)_\Sigma = (2k, 3k)$ for $k = 0, 1, 2, \dots$.*

In fact, from appendix A.2 we recall

$$\frac{f}{z^2}\bigg|_{z=0} = -\frac{1}{3}u_1^2\ , \qquad\qquad \frac{g}{z^3}\bigg|_{z=0} = \frac{2}{27}u_1^3\ , \tag{1.81}$$

where u_1 is a locally defined function in a neighborhood of Σ. Let k be the degree σ dividing u_1. The result now follows from (1.13). In fact, we can say more; the roots of u_1 cannot contribute to the $2n$ vanishings $\widehat{\Delta}$ required from a transverse type I_{2n} intersection while maintaining non-minimality. The argument here is more involved. The case of an I_2 intersection with I_1^* can be treated without much machinery and makes plain the underlying behavior: non-minimality is induced when requiring roots of u_1 to carry the vanishings of $\widehat{\Delta}$.

We proceed now with separate treatment for the cases with different values of n.

n < 4 odd

Here the residual determinant reads

$$\frac{\Delta}{z^{2i+5}}\bigg|_{z=0} = 4u_1^3 w_{2i+2} \ , \tag{1.82}$$

where $n = 2i - 1$. From Appendix A.1 and result 1.3.1, we have $\sigma_1^{2n_1} \cdots \sigma_N^{2n_N} | w_{2i+2}$. It is then convenient to define the local function ϕ such that

$$w_{2i+2} \equiv \phi \sigma_1^{2n_1} \cdots \sigma_N^{2n_N} \ . \tag{1.83}$$

In terms of degrees, (1.82) reads

$$\widetilde{d}_\Sigma - 2 \sum_i n_i = 3 \deg u_1 + \deg \phi \ . \tag{1.84}$$

As noted above, $\deg u_1 = \widetilde{a}_\Sigma / 2$, and the monodromy on type I_n^* for n odd is determined by whether w_{2n+2} has a square root. This means that $\deg \phi$ must be even for the case without monodromy (indicated by the superscript "s"), while necessarily $\deg \phi > 0$ for the case with monodromy (indicated by the superscript "ns"). In order to impose the least constraint on $\sum_i n_i$, we set $\deg \phi = 0$ for $\mathrm{I}_n^{*\mathrm{s}}$ and $\deg \phi = 2$ for $\mathrm{I}_n^{*\mathrm{ns}}$. For $n = 1$, we have the following table:

m	$\widetilde{d}_\Sigma$	$\deg u_1$	$(\sum_i n_i)^s$	max. algebra	$(\sum_i n_i)^{ns}$	max. algebra
1	19	3	5	$\mathfrak{sp}(5)$	4	$\mathfrak{sp}(4)$
2	14	2	4	$\mathfrak{sp}(4)$	3	$\mathfrak{sp}(3)$
3	9	1	3	$\mathfrak{sp}(3)$	2	$\mathfrak{sp}(2)$
4	4	0	2	$\mathfrak{sp}(2)$	1	$\mathfrak{sp}(1)$

(1.85)

From Table 1.5, we read off the algebras predicted from field theory: $\mathfrak{sp}(6-m)\oplus\mathfrak{su}(4-m)$ and $\mathfrak{sp}(5-m)\oplus\mathfrak{sp}(4-m)$ for the two monodromy cases. We have just determined that our configurations yield algebras that are contained in the first summands.

For $n=3$, the analysis is even simpler, as $\widetilde{d}_\Sigma - 3\deg u_1 = 12$ for any value of m. Therefore, maximal configurations yield $\mathfrak{sp}(6)$ in the case without monodromy and $\mathfrak{sp}(5)$ in the case with monodromy. The prediction from field theory is $\mathfrak{sp}(6)$ for the former case and $\mathfrak{sp}(9-m)\oplus\mathfrak{so}(2-\frac{1}{2}m)$ for the latter (with $m=2$ or $m=4$).

n = 2

The analysis for this case differs from the previous section as the residual discriminant now reads

$$\frac{\Delta}{z^8}\bigg|_{z=0} = u_1^2(4u_1w_5 - v_3^2)\ . \tag{1.86}$$

Considering again a configuration of the form $\mathcal{C}_{2n_1,\dots,2n_N}$, we define

$$(4u_1 w_5 - v_3^2) \equiv \phi \sigma_1^{2n_1} \cdots \sigma_N^{2n_N} \ , \tag{1.87}$$

and we have the following relation

$$\widetilde{d}_\Sigma - 2\sum_i n_i = 2 \deg u_1 + \deg \phi \ . \tag{1.88}$$

The monodromy condition is determined by whether $(4u_1 w_7 - v_3^2)$ admits a square root, and we rephrase this in terms of a (minimal) constraint on the degree of ϕ: $\deg\phi = 0$ for $\mathrm{I}_2^{*\mathrm{s}}$ and $\deg\phi = 2$ for $\mathrm{I}_2^{*\mathrm{ns}}$. The collect these results in the following table.

m	$\widetilde{d}_\Sigma$	$\deg u_1$	$(\sum_i n_i)^s$	max. algebra	$(\sum_i n_i)^{ns}$	max. algebra
1	20	3	7	$\mathfrak{sp}(7)$	6	$\mathfrak{sp}(6)$
2	16	2	6	$\mathfrak{sp}(6)$	5	$\mathfrak{sp}(5)$
3	12	1	5	$\mathfrak{sp}(5)$	4	$\mathfrak{sp}(4)$
4	8	0	4	$\mathfrak{sp}(4)$	3	$\mathfrak{sp}(3)$

(1.89)

Again, these are subalgebras of the first summands of the field theoretical algebras from Table 1.5.

$\mathbf{n \geq 4}$

As before, there are two monodromy cases here, yielding gauge algebras $\mathfrak{so}(2n+7)$ for $\mathrm{I}_n^{*\mathrm{ns}}$ and $\mathfrak{so}(2n+8)$ for $\mathrm{I}_n^{*\mathrm{s}}$. The corresponding predictions from Table 1.5 are $\mathfrak{sp}(2n-1)$ and $\mathfrak{sp}(2n)$, respectively.

Since $m = 4$, we have $\deg u_1 = 0$ and

$$2\sum_i n_i = \widetilde{d}_\Sigma - \deg\phi = 4n - \deg\phi\ , \tag{1.90}$$

where ϕ is defined as in (1.83) for the odd case or as in (1.87) in the even case. With monodromy, we have also $\deg\phi > 0$ and the maximal configuration is given by $\mathcal{C}_{d_\Sigma-2}$, yielding global symmetry algebra $\mathfrak{sp}(2n-1)$; in the case without monodromy, we again set $\deg\phi = 0$ as above and obtain $\mathcal{C}_{d_\Sigma}$ as the maximal configuration with $\mathfrak{sp}(2n)$ as global symmetry. These both agree with the field theory predictions.

1.3.7 Type IV*, III* and II*

Here the situation is quite simple. These Kodaira types are not allowed to intersect any of the singularity types yielding a non-vanishing gauge algebra. Therefore, there is no global symmetry coming from F-theory. In particular, this result matches with the field theory prediction for type II*, for which no matter content is available.

1.3.8 Further constraints

The global symmetry constraints for a few cases treated in [6] can be tightened with a pair of observations used above which we now single out for clairty. The resulting flavor symmetry maxima reductions from those in [6] appear in Table 1.6 indicated with a '†' symbol.

$\mathfrak{sp}(3)$ and $\mathfrak{so}(7)$ summands for $\mathbf{IV}^{ns}$

No $\mathfrak{so}(7)$ flavor symmetry summands are allowed for a type IV^{ns} curve, as originally shown in Appendix E.3.1 of [16]. The argument given there forbids type IV curves from meeting semi-split I_0^* curves, and all details go through for arbitrary values of A, B along the I_0^*. Hence $\mathfrak{so}(7)$ global symmetry summands for type IV curves are reduced to $\mathfrak{g}_2$ summands. This affects results for both the $m = 1, 2$ cases for IV. Furthermore, an $\mathfrak{sp}(3)$ summand does not lie in $\mathfrak{g}_2$ (nor $\mathfrak{so}(7)$) and hence such a global symmetry summand appears (alone) as one of the maximal configurations.

$\mathbf{I}_n$ for $n \geq 5$ meeting III must have monodromy

When $n \geq 5$, I_n curves with a type III intersection must be non-split, as indicated in Table A.2. This implies that the global symmetries for type III curves are reduced from the summaries appearing in [6].

Chapter 2

Classifying global symmetries of all 6D SCFTs

Here we turn our attention to the general problem: classifying the global symmetries of an arbitrary 6D SCFT. We derive a series of constraints on elliptic fibrations corresponding to SCFTs and employ these to provide an algorithm with implementation in a computer algebra system to carry out the calculation of global symmetry algebras for any 6D SCFT. The results of global symmetry calculations are carried out for the majority of links (including all 0-links and all trivalent 1-links) and presented in Appendix D, thus identifying the flavor symmetries of 6D SCFTs more generally via the SCFT classification scheme reviewed in the previous chapter. A surprising side-effect of machinery we develop results: the geometric constraints we uncover more than suffice to match the gauge enhancement structure for all links and hence all 6D SCFTs appearing in [16], the latter relying heavily on anomaly cancellation machinery. The global symmetries we find reveal a distinguished subclass of complex threefolds in correspondence with field theory and shed further light on the geometry to field theory "dictionary." We also present a novel simplifying identity for continued fraction values determining the discrete $U(2)$ subgroup action for SCFT endpoints.

2.1 Overview

We present a series of results enabling our main calculation to proceed via computer algebra system. These fall into three categories. First, we determine which pairs of curves with specified Kodaira types are permitted to intersect without introducing such severe singularities that Calabi-Yau resolution of our fibration would be prevented. This entails generalizing the analysis from [6] which appeared in the previous chapter. A second category consists of computations to determine the minimal intersection contributions from permitted intersections. Finally, we derive a handful of results concerning trios of curves which enable us to reach our conclusions without invoking anomaly cancellation techniques, instead finding all of our restrictions via geometry of the elliptic fibrations in correspondence with 6D SCFTs.

While the restrictions and contribution data found in the previous chapter play a key role here, our route is complicated by a pair of issues we summarize briefly. There, the discriminant locus consists of a single compact curve C. Determining the global symmetry algebras which can be realized in F-theory there is treated via consideration of non-compact curve collections, $\{C_i\}$, each transverse to C and carrying non-abelian gauge algebra which lead to the (relatively) maximal global symmetry algebras as we contract C. The relevant local analysis for transverse curve pair intersections involve only those intersections with one curve compact while the other is non-compact and supports a non-abelian gauge algebra. Any configurations containing transverse curves which result in 'small' global symmetry algebras previously could safely be deemed irrelevant.

The method we pursue in this chapter to generalize our earlier approach now requires an understanding of local models including intersection contribution data for *nearly all permissible pair intersections involving a compact curve*, both for non-compact *and compact* transverse curves, the exceptions being cases of gaugeless non-compact transverse curves and compact curve pair intersections with self-intersections failing to meet the positive-definite intersection matrix condition. To classify global symmetries for all 6D SCFTs realizable in F-theory, we might proceed via analysis of all possible transverse curve collections containing up to three compact curves, with the others non-compact and carrying potentially maximal gauge algebra sums for the given choice of compact curves. For the analysis of linear quiver theories (i.e. those with the compact curves forming a linear chain), we have at most two transverse compact curves in each collection (corresponding to the neighboring curves in the quiver). One might hope that this would suffice to fully constrain the global symmetries available on a quiver, but we find a few subtleties are missed with such an approach.

Consider a quiver $\{\Sigma_j|\ j \in J\}$ with collections of non-compact curves $\{\Sigma_{j,i}|\ i \in I\}$, transverse to Σ_j. Choosing collections $\{\Sigma_{j,i}\}$ which are permitted to intersect Σ_j even in the presence of neighboring curves $\Sigma_{j-1}, \Sigma_{j+1}$ does not guarantee the resulting configuration is permitted (since for example the minimal contributions from Σ_{j+1} to Σ_j in certain cases depend on the curves meeting Σ_{j+1}). In effect, constraints are instead global in the quiver as they involve all curves transverse to the neighboring curves $\Sigma_{j\pm1}$ (thus making possible the propagation of constraints along a quiver). Somewhat surprisingly, however, the global symmetry maxima we find for each Kodaira type specification on each quiver we have surveyed (though having potentially varying values of A, B in our symmetry tables) are *not* more constrained than the set of relatively maximal direct sums of the relatively maximal global symmetry summand algebras arising from each curve of the quiver.

Note that the geometric constraints on a quiver required to permit a specified enhancement are themselves nonlocal in quiver position since the minimum orders of vanishing of f, g to allow a type assignment on a quiver are in some cases necessarily higher than the minimums needed to specify a Kodaira type for a bare curve. This leads to only a very limited constraint propagation and does not appear to be reflected in the global symmetries themselves in the sense noted above.[1,2]

Provided we instead consider all possible Kodaira type assignments for a given quiver resulting in a specified gauge algebra, we instead find a wide variety of cases forbidding curve-by-curve maxima to simultaneously arise as total global symmetry summands.[3]

For instance, every Kodaira type assignment permitting an $\mathfrak{su}(2) \oplus \mathfrak{e}_6$ gauge enhancement of the base 2215 is shown in Table 2.1. While we can realize a $\mathfrak{g}_2$ global symmetry summand from the left-most -2 curve and an $\mathfrak{su}(3)$ global symmetry contribution from the -1 curve, these options are mutually exclusive, i.e. all global symmetry algebras for an $\mathfrak{su}(2) \oplus \mathfrak{e}_6$ gauge theory realizable in F-theory via this quiver are strictly smaller than $\mathfrak{su}(3) \oplus \mathfrak{g}_2$.

As with 1D Coulomb branch, we often find multiple relatively maximal global symmetry algebras even for a fixed Kodaira type assignment along a quiver. A key question here is whether F-theory models with distinct global symmetry maxima sharing isomor-

[1]Note that in our tables, distinct fibrations having differing orders of f, g, Δ are in some cases being compared provided they have the same Kodaira types on the codimension-one singularities in the base. While we track the data to determine minimums for singularity types of the codimension-two singularities and before final comparisons are made, our algorithm distinguishes between theories differing in the A, B assignments affecting codimension-two singularities, these distinctions are discarded to simplify presentation of our results in table format. In other words, we make comparisons for all fibrations with common Kodaira type assignments at codimension-one.

[2]One potential extension of the present work would entail a check of the codimension two singularities in relation to global symmetries which may arise. We hope that our approach in tracking the relevant geometries at the level of precision of orders of f, g, Δ along each curve may help with such codimension-two singularity related investigations.

[3]Another related line of inquiry one might pursue involves consideration of all bases with a specified gauge algebra and fixed $U(2)$ subgroup. Though we comment on such a setup shortly, we do not carry out a systematic investigation the analog of this summand exclusivity phenomenon there.

2	2	1	5	GS Total:
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_1	0	A_1	0	A_1^2
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
g_2	0	A_1	0	$A_1 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_2	0	A_2	0	A_2^2
($\mathfrak{su}(2)$)	(n_0)	(n_0)	($\mathfrak{e}_6$)	
A_2	0	A_2	0	A_2^2
g_2	0	A_1	0	$A_1 \oplus g_2$
(II,n_0)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
0	A_1	0	0	A_1
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
0	g_2	0	0	g_2
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
0	A_2	0	0	A_2
(n_0)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{e}_6$)	
0	g_2	0	0	g_2

Table 2.1: All global symmetry maxima for 2215 with gauge algebra $\mathfrak{su}(2) \oplus \mathfrak{e}_6$ along with each possible Kodaira type assignment to the quiver realizing this gauge algebra and the corresponding F-theory global symmetry maxima for the given Kodaira type assignment.

phic gauge algebras correspond to distinguished SCFTs, i.e. are there terms under which the global symmetries of F-theory models provide SCFT invariants?

Ignoring the question of which differences in global symmetries disappear under renormalization, we can discuss whether it suffices to specify gauge and global symmetries to determine an SCFT uniquely, and if not, precisely which geometric data are also required. First observe that $U(2)$ subgroup data is not captured by this data since for example the $\mathfrak{f}_4$ theories on both -3 and -4 curve have trivial F-theory global symmetry algebra but distinct $U(2)$ subgroup. We will not provide a complete answer; we instead hope to clarify the issues at hand by providing a few comments concerning the reverse direction of the field theory/geometry correspondence motivated by an example illustrating that much (but not all) of the geometric information of our fibration can be reconstructed from CFT symmetry data (even *sans* accompanying representation theoretic information).

Consider an $\mathfrak{e}_6$ gauge theory. There are several bases permitting total gauge algebra $\mathfrak{g}_{\text{gauge}} \cong \mathfrak{e}_6$ including the bases 1,2,21,3,31,131,4,41,141,5,51,151,512,1512. We cannot determine even the number of curves in the base from $\mathfrak{g}_{\text{gauge}}$. When also given the global symmetry algebra, $\mathfrak{g}_{\text{global}}$, of our SCFT, we can say more. Consider the case that $\mathfrak{g}_{\text{global}} \cong \mathfrak{su}(3)$. We can infer that the base contains at least two curves since every single component discriminant locus theory with $\mathfrak{e}_6$ gauge algebra has trivial F-theory global symmetry as we can confirm via Table 1.7. Furthermore, we can easily eliminate the bases $131, 141, 151, 1512$ since their global symmetries are too large. It then becomes necessary to specify $U(2)$ subgroup data to distinguish between the bases 51,512, and others. Consider the relevant geometries corresponding to enhancements of the quivers $51, 512$. There are very few $\mathfrak{e}_6$ gauge compatible Kodaira type assignments for these quivers, all of which appear in Tables 2.2 and 2.3.

5	1	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	
0	A_2	A_2
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	
0	A_1	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	
0	A_2	A_2
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	
0	C_3	C_3
0	g_2	g_2
(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	
0	A_2	A_2
($\mathfrak{f}_4$)	(n$_0$)	
0	C_3	C_3
0	g_2	g_2

Table 2.2: All gauge and global symmetry options for 51 with each possible Kodaira type specification realizing a given gauge theory.

The $\mathfrak{e}_6$ compatible geometries supported on $51, 512$ are determined up to Kodaira type from the global symmetry data provided we fix the quiver. Note that the $U(2)$ subgroup data, a property of the field theory, distinguishes between these bases and $\mathfrak{g}_{\text{global}}$ further specifies the geometry in each case uniquely from field theory data.

This suggests provision of a field theory to geometry "dictionary" in terms of allowable fibrations compatible with field theory data encapsulated in the triple, $(\Gamma, \mathfrak{g}_{\text{gauge}}, \mathfrak{g}_{\text{global}})$, where Γ is the discrete $U(2)$ subgroup associated to our SCFT. As discussed in [16], Γ determines uniquely a quiver which is the minimal blowup of the associated endpoint

5	1	2	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(I_0,n_0)	
0	A_2	0	A_2
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(II,n_0)	
0	A_1	A_1	A_1^2
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(I_1,n_0)	
0	A_2	A_1	$A_1 \oplus A_2$
($\mathfrak{e}_6$)	(n_0)	(n_0)	
0	A_2	A_1	$A_1 \oplus A_2$

Table 2.3: All globlal symmetry maxima for 512 with gauge algebra $\mathfrak{e}_6$ and each Kodaira type choice realizing this gauge algebra shown.

permitting a designated enhancement determined by $\mathfrak{g}_{\text{gauge}}$. The data $\mathfrak{g}_{\text{global}}$ in our example necessitates further blowups to reach a compatible base, for example pushing us from a base with a single -4 curve to 51 or, with a different Γ, from -3 to 512. This triple of field theory data thus potentially gives a means to characterize from field theory the broader collection of geometries in correspondence with CFTs.

Let us pursue this correspondence in our example further by fixing Γ which pairs with the endpoint -3. Since $\mathfrak{g}_{\text{gauge}}$ has only one summand, there can be only one curve in any compatible base with self-intersection -3 or below since these minimally support a non-abelian gauge summand. All curves must have $1 \leq m \leq 6$ since -7 curves and below minimally support an $\mathfrak{e}_7$ or $\mathfrak{e}_8$ gauge algebra. For the base -3, we have $\mathfrak{g}_{\text{global}} = 0$, and this is the only such base. The remaining bases with shared endpoint which can match $\mathfrak{g}_{\text{gauge}}$ have self-intersection string α with

$$\alpha \in \{41, 151, 512, 1612, \overset{1}{161}\}.$$

For 41, we have a unique $\mathfrak{g}_{\text{global}} \cong \mathfrak{su}(3)$. From the data above for 51, we can deduce that when α given by 151, the unique $\mathfrak{g}_{\text{global}} \cong \mathfrak{su}(3)^{\oplus 2}$. For the only trivalent option, this becomes $\mathfrak{su}(3)^{\oplus 3}$ (noting Table 2.4). For 1612, the global symmetry options match those from 512 with an additional $\mathfrak{su}(3)$ summand coming from the left-most -1 curve. To simplify the correspondence, let us consider only the geometries leading to the maximal $\mathfrak{g}_{\text{global}}$ on each quiver. This gives for 512, $\mathfrak{g}_{\text{global}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(3)$ and for 1612, $\mathfrak{g}_{\text{global}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(3)^{\oplus 2}$. To summarize, in each case we have uniquely identified a geometry given the data $(\mathfrak{g}_{\text{gauge}}, \mathfrak{g}_{\text{global}}, \Gamma)$.

As a side-effect of this approach, we single out certain "special" geometries leading to the relatively maximal global symmetries we can associate to a CFT. Calling this collection of varieties $\mathcal{M}_{\mathfrak{g}_{\text{global}}}$, we have shown that in our example we have a one-to-one map from $\mathfrak{g}_{\text{global}}$ to $\mathcal{M}_{\mathfrak{g}_{\text{global}}}$ given by inverting the map

$$\mathcal{M}_{\mathfrak{g}_{\text{global}}}|_{(\mathfrak{g}_{\text{gauge}}=\mathfrak{e}_6, \Gamma_{-3})} \mapsto (\mathfrak{e}_6, \mathfrak{g}_{\text{global}}, \Gamma_{-3})$$

from bases with enhancements specified up to Kodaira type to their relatively maximal global symmetries. The generalization of this statement to all 6D SCFTs is nontrivial. Among the main subtleties are the presence of multiple relative maxima for $\mathfrak{g}_{\text{global}}$. Note that there a corresponding analog of this distinguished class of threefolds in the moduli space of compact Calabi-Yau threefolds upon consideration of a compact base giving an F-theory model coupled to gravity wherein the global symmetries we describe are promoted to gauge symmetries when possible that the non-compact fibers giving global symmetry summands can be assigned negative self-intersection values without leading to inconsistencies.

These distinguished threefolds in contact with F-theory models are remarkably sparse in Calabi-Yau moduli space. A canonical example concerns a -1 curve with type I_0. Its $\mathfrak{e}_8$ global symmetry arising from a single transverse type II^* curve corresponds to the a distinguished geometry among the many others appearing as "Persson's list" entries. Enhancing a -1 curve to reach Kodaira type I_n^{ns} for n odd makes the number of transverse configurations grow exponentially with n while only $\frac{n+3}{2}$ geometries are in correspondence with the global symmetry maxima given in Table 1.6. Moreover, this sparsity is not limited to bases containing only a single compact curve. For example, there are infinitely many minimally enhanced bases with outer links permitting an $e_8 \oplus e_8$ global symmetry. For each, we have a variety with that flavor symmetry arising in the singular limit which is distinguished from the associated varieties leading to any of the 6757 isomorphism classes of proper $e_8 \oplus e_8$ subalgebras.

6	1	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	
0	A_2	A_2
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	
0	A_1	A_1

Table 2.4: All gauge and global symmetry options for 61 with each possible Kodaira type specification realizing a given gauge theory.

In the remainder of this chapter, we focus our attention on geometric details determining the relevant restrictions enabling algorithmic deduction of CFT global symmetry (relative) maxima. Note that tracking the additional vanishings allowed for a given Kodaira type (i.e. A, B) becomes crucial to our study in contrast with the cases treated in Chapter 1. There the absence of neighboring compact curves allowed these to be disregarded during the relevant computations of [6] as discussed in Appendix A.

2.2 Preliminaries

In the following sections, we let Σ be a curve at $\{z = 0\}$ with $-\Sigma \cdot \Sigma = m$, having transverse intersections with curves Σ'_j located at $\{\sigma_j = 0\}$. Let the orders of vanishing along Σ'_j be given by (a'_j, b'_j, c'_j).

2.3 Restriction involving type II curves

We introduce further constraints on collisions involving a Kodaira type II curve in 2.7 for cases involving an I_n^* curve. Here we focus on the remaining non-trivial cases, those involving an I_n curve. Such collisions were studied at length in [24] when the I_n curve is non-compact and $A = 0$ along the type II curve. The tight restrictions we find in sections 2.4,2.5 for type III and IV curves have only weaker analogues here. Underlying this distinction is that we are farther from non-minimality in II,I_n collisions and that as a consequence we are required to consider intersections where the most general form for the relevant local models of type I_n curves is unknown as these include the cases $7 \leq n \leq 9$.

We now proceed to make mild generalizations of the intersection contribution data first appearing in [24] that are required in the present work. Part of our work is dispatched by reading from Table A.1 taken from [6] which states that $A \geq 3$ is not possible for intersections with I_n curves having $n \geq 4$, a bound we revise here since this only holds for non-compact I_n curves. The $A = 0$ results can be read directly from the contribution tables of [24]; in one case we find a small correction. This leaves us to determine the relevant contributions from compact I_n curves for all n, and non-compact curves only for $n \leq 4$. Since type II curves do not carry a non-abelian gauge algebra, we can safely ignore collisions of compact I_n curves with non-compact type II curves.

We collect intersection contributions for the remaining cases in Table 2.5. The '!' symbol there indicates disagreement with [24]. Entries marked with '†' are not permitted via the inductive form in (A.25). The '$*$' symbol indicates the contributions are only valid for a non-compact I_n curve, 'X.' that the intersection is valid only for non-compact I_n curve, and 'X..' that the intersection exceeds numbers of vanishings available for a type II curve even with $m = 1$. Entries to the right of those indicated with an 'X', 'X.', 'X..' are similarly forbidden.

2.4 Restrictions involving type III curves

Let Σ be a curve with Kodaira type III and orders of vanishing $(a, b, 3) = (1, 2 + B, 3)$. Suppose that Σ' has orders of vanishing (a', b', d'). Note Σ has odd type, making $\widetilde{d}$ contributions thrice those of $\widetilde{a}$ contributions. The only technical cases for such intersections

$A=a-1$ \ $n_{ns/s}$	0	1	2	$3\cdots$	$\cdots$	$a-1$
2	$(1,2,4)$		$\cdots(a \mod 2,2,4)(X.\text{ if } A\geq 4)\cdots$			
3_{ns}	$(1,2,4)$	$(0,3,6)$	$(1,3,6)^*X.$	$\cdots(a \mod 2,3,6)^*X.\cdots$		
3_s	$(2,3,6)$	$(2,3,6)$	$(2,3,6)$	$\cdots(2(1-\delta_{4,a}),3,6)(X.\text{ if } A\geq 4)^*\cdots$		
4_{ns}	$(2,4,8)$	$(0,4,8)$	$(2,4,8)^*$X.	$\cdots(2(a \mod 2),4,8)^*X.\cdots$		
4_s	$(2,4,8)^!$	$(2,4,8)$	$\cdots(2(1-\delta_{a,4}),4,8)(X.\text{ if } A\geq 4)^*\cdots$			
5_{ns}	$(2,4,8)$	$(0,5,10),(2,4,8)^*$	$(1,4,8)^*$	$\cdots(2,4,8)^*$X.$\cdots$		
$(\geq 5)_s$	X					
6_{ns}	$(2,4,8)$	$(0,6,12),(2,4,8)^*$		$\cdots(2,4,8)^*$X.$\cdots$		
7_{ns}	$(\leq 3,\leq 6,\leq 12)$	X..†				
8_{ns}	$(\leq 4,\leq 8,\leq 16)^\dagger$X..†					
9_{ns}	$(\leq 4,\leq 8,\leq 16)^\dagger$X..†					
$(n\geq 10)_{ns}$	$(\lfloor\frac{n}{2}\rfloor,2\lfloor\frac{n}{2}\rfloor,4\lfloor\frac{n}{2}\rfloor)$X..					

Table 2.5: Intersection contributions to type II curve from a (non-compact*) I_n curve.

concern transverse curves with type I_n or I_n^*. The latter are restricted for $n\geq 1$ due to non-minimality. We treat intersections with I_0^* curves in section 2.6, focusing here on treating Σ' with type I_n in each case relevant to global symmetry computation, namely those involving at least one compact curve.

2.4.1 III, I_n intersections

Type III with $B\geq 0$ intersection with a non-compact type I_n curve

We now extend the contribution data for cases with $B=0$ analyzed in [6] to those with $B>0$. Reading from Tables A.1,A.2 of [6], we find values for the local intersection contributions to a III with $B=0$ and that the restriction that $B\geq 2$ requires

$n \leq 3$ for non-minimality. We make a correction to this bound; the revisions appear in the appendices as Tables A.1,A.2. First, we collect the results of contributions for intersections analyzed [6] and the remaining $B > 0$ cases of contributions to a type III with $m = 1, 2$ from a non-compact transverse I_n curve in Table 2.6. Note that in the cases with $n \geq 7$, we compute contributions working from the inductive Tate forms for I_n curves from (A.25)-(A.28) that are potentially not the most general when $7 \leq n \leq 9$. Here Σ has residuals given by $(5, 8 + B, 15)$ for $m = 1$ and $(2, 4 + 2B, 6)$ when $m = 2$.

In Table 2.6, entries indicated with '$*$' are only valid for non-compact I_n curves and those with '†' are permitted only for I_n non-compact. Those entries with '!' correct Table A.2 of [6]. The symbol 'X' indicates a non-minimal intersection and entries to the right of an 'X' are also non-minimal. For $m = 2$, those entries with any contribution exceeding those allowed are forbidden; these are indicated with a subscript '1'.

B / $n_{ns/s}$	0	1	2	3	...	$b-2$
2_{ns}	$(1,1,3)$	$(1,0,3)$	$(1,1,3)$	...	...	$(1, b \mod 3, 3)$
3_{ns}	$(2,2,6)$	$(3,0,9)_1, (2,2,6)^*$	$(2,2,6)^*$	$(2,2,6)^\dagger$	...	$(2,2,6)^\dagger$
3_s	$(2,2,6)$	$(2,2,6)$	$(2,2,6)^\dagger$	...	...	$(2,2,6)^\dagger$
4_{ns}	$(2,2,6)^!$	$(3,0,9)_1, (2,2,6)^*$	$(2,2,6)^\dagger$	...	...	$(2,2,6)^\dagger$
4_s	$(2,3,6)$	$(2,3,6)$	$(2,3,6)^\dagger$	...	...	$(2,3,6)^\dagger$
5_{ns}	$(3,3,9)_1$	$(3,0,9)_1$	X			
$(\geq 5)_s$	X					
6_{ns}	$(3,3,9)_1$	$(5,0,15)_1$	X			
$(10 \geq n \geq 7)_{ns}$	$(\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil, 3\lceil \frac{n}{2} \rceil)_1$	X	X			
$(\geq 11)_{ns}$	X					

Table 2.6: Intersection contributions to III from a (non-compact) I_n curve.

Type III with $B \geq 0$ and $m = 2$ intersection with a compact I_n curve with $m = 1$.

The only case of interest for III,I_n intersections with I_n compact arising in F-theory quivers involve a type III curve with $m = 2$ since two -1 curves do not intersect in any valid base and $m' = 2$ on Σ' with type I_n has $(\widetilde{a}, \widetilde{b}) = (0, 0)$, thus preventing intersection with any type III curve. We collect the contributions for valid intersections in Table 2.7. For large n, it will be helpful in making our table succinct to define

$$q(n) = \begin{cases} 4 \text{ if } n \text{ is even} \\ 5 \text{ if } n \text{ is odd.} \end{cases} \tag{2.1}$$

The table entries indicate the minimal intersection contributions to/from a Σ, a (compact unless marked with 'X') type III curve having orders $(1, b, 3) = (1, 2 + B, 3)$ with $m = 2$ transversely intersecting Σ', a compact I_n curve with $m' = 1$. Note the residuals on a (compact) type III curve with $m = 2$ are $(2, 4 + 2B, 6)$. Here, 'X'/'X.' indicate an intersection forbidden by the number of allowed vanishings along Σ/Σ', respectively, and '$-$' a non-minimal intersection. Entries to the right of an '$-$', 'X', or 'X.' are similarly forbidden. Intersections corresponding to entries with an 'X' are permitted only for non-compact Σ.

$n_{ns/s}$ \ B	0	1	2	3	4	≥ 5
1	$(1,1,3)/(2,3,5)$	$(1,0,3)/(2,3,5)$	$\ldots\ (1, b \mod 3, 3)/\left(2\lceil\frac{b}{3}\rceil, 3\lceil\frac{b}{3}\rceil, 4\lceil\frac{b}{3}\rceil+1\right)\ldots$			X.
2	$(1,1,3)/(2,3,4)$	$(1,0,3)/(4,6,6)$, $(2,0,6)/(2,3,5)$	$(2,1,6)/(4,6,8)$	$(2,1,6)/(4,6,9)$	$(2,0,6)/(4,6,9)$	X.
3_{ns}	$(2,2,6)/(2,3,5)$	X$/(2,3,5)$	X/X.			
3_s	$(2,2,6)/(4,6,6)$	$(2,2,6)/(4,6,7)$	–			
4_{ns}	$(2,2,6)/(2,3,4)$	X$/(2,3,5)$	X/X.			
4_s	$(2,3,6)/(4,6,6)$	$(2,2,6)/(4,6,6)$	–			
5_{ns}	X$/(2,3,5)$	X$/(2,3,5)$	X/X.			
$(\geq 5)_s$	–					
6_{ns}	X$/(2,3,4)$	X$/(2,3,5)$	X/X.			
$(\geq 7)_{ns}$	X$/(2,3,q(n))$	X$/(2,3,5)$	X/X.			

Table 2.7: Intersection contributions for (compact $m = 2$) type III, compact type I_n intersections.

Intersection contribution to a compact I_n curve with $m = 1$ from a non-compact III with $B \geq 0$.

The remaining intersections of interest between type III and type I_n curves concern contributions to an I_n curve from a type III curve giving a global symmetry summand. The relevant contributions to the type I_n curves here are also given in Table 2.7, marked with an 'X' when we require the non-compactness (or $m = 1$ if we are not concerned with constructing an F-theory SCFT base but rather an F-theory base with gravity coupled) along the type III condition to hold.

2.5 Restrictions involving type IV curves

Here we collect information about monodromy rules for type IV curves and intersection contributions involving transverse curves. The only subtle cases involve type I_n and I_0^* curves since I_n^* intersections with $n > 0$ results in non-minimality.

2.5.1 Preliminaries

Let Σ be a curve with type IV. Orders of vanishing are then given by $(a, b, 4) = (2 + A, 2, 4)$. Note that Σ has even type. Reading from 1.2, we see the monodromy along Σ is determined by whether $\frac{g}{z^2}|_\Sigma$ is a square; the larger gauge algebra, $\mathfrak{su}(3)$, occurs if so.

2.5.2 Intersections of IV with I_n curves

IV with A $\geq$ 0 meeting I_n curves for global symmetry or quiver intersections

This case is detailed in [6] when the transverse curves are non-compact. From Table 1.6, we have that transverse I_n curves carry at most $\mathfrak{sp}(4)$ symmetry. Table A.2 gives the local intersection contributions to a IV with $A = 0$. Note that as indicated in Table A.1, $n \leq 3$ is required when $A > 0$ for non-minimality. We collect the results of contributions to (and from when applicable) a type IV curve intersecting a type I_n curve for these remaining cases in Tables 2.8 and 2.9, separating the $A = 0$ case in which $n > 3$ is permitted.

$n_{(ns/s)}$ \ $A_{(ns/s)}$	2_{ns}	3_{ns}	3_s
1_{ns}	$(1,2,4)//(4,6,8)$	-$/(1,3,6)$	$(1,3,6)/(1,2,4)//(4,6,8)$
1_s	$(1,2,4)//(4,6,8)$	-$/(1,3,6)^*$	$(1,3,6)^*/(1,2,4)//(4,6,8)^*$
2_{ns}	$(0,2,4)//(4,6,8)$	-$/(0,3,6),(1,2,4)$	$(0,3,6)/(1,2,4)//(4,6,8)$
2_s	$(0,2,4)//(4,6,8)$	-$/(0,3,6)^*,(1,2,4)$	$(0,3,6)^*/(1,2,4)//(4,6,8)^*$
$A \geq 3_{ns},\ 2+A \in 4\mathbb{Z}$	-$/\ (0,2,4)$	-$/(0,3,6)/(1,2,4)$	-$/(0,3,6),(1,2,4)$
$A \geq 3_{ns},\ 2+A \in 2\mathbb{Z} \setminus 4\mathbb{Z}$	-$/\ (0,2,4)$	-$/(0,3,6),(1,2,4)$	-$/(1,3,6),(1,2,4)$
$A \geq 3_{ns},\ A \notin 2\mathbb{Z}$	-$/\ (1,2,4)$	-$/(1,3,6)$	-$/(1,3,6),(1,2,4)$
$A \geq 3_s,\ 2+A \in 4\mathbb{Z}$	-$/\ (0,2,4)$	-$/(0,3,6)^*/(1,2,4)$	-$/(0,3,6)^*,(1,2,4)$
$A \geq 3_s,\ 2+A \in 2\mathbb{Z} \setminus 4\mathbb{Z}$	-$/\ (0,2,4)$	-$/(0,3,6)^*/(1,2,4)$	-$/(1,3,6)^*,(1,2,4)$
$A \geq 3_s,\ A \notin 2\mathbb{Z}$	-$/\ (1,2,4)$	-$/(1,3,6)^*$	-$/(1,3,6)^*,(1,2,4)$

Table 2.8: Intersection contributions to a (compact) $\text{IV}^{ns/s}$ with $A > 0$ from//to a transverse compact/non-compact I_n curve (with 'to contributions' only when compact). An 'X' indicates the intersection is forbidden by non-minimality considerations and a '-' an intersection forbidden by residuals considerations. Here (*) indicates an intersection allowed only when $m = 1$ on IV via limitations of $\widetilde{b}$ on IV.

$\text{IV}^{ns}/\text{IV}^s$ \ $n_{(ns/s)}$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$
2_{ns}	$(0,2,4)$	$(0,2,4)$
3_{ns}	$(0,3,6),(1,2,4)$	$(0,3,6)^*,(1,2,4)$
3_s	$(1,3,6),(1,2,4)$	$(1,3,6)^*,(1,2,4)$
$\geq 4_s$	X	X
4_{ns}	$(0,4,8)$	$(0,4,8)^*$
$9_{ns} > n \geq 5_{ns}$	$(0,n,2n)^*$	$(0,2\left\lceil \frac{n}{2} \right\rceil,4\left\lceil \frac{n}{2} \right\rceil)^*$
$\geq 9_{ns}$	-	-

Table 2.9: Intersection contributions to $\text{IV}^{ns/s}$ with $A = 0$ from a transverse non-compact I_n curve. An X indicates the intersection is forbidden by minimality considerations. Here (*) indicates a permitted intersection only when IV has $m = 1$, and (-) an intersection $\widetilde{b}$ on IV forbids in all cases.

2.6 Restrictions involving type I_0^* curves

Here we compile information about monodromy rules for I_0^* curves and intersection contributions to residuals counts involving transverse curves from an I_0^* curve.

2.6.1 Preliminaries

Let $\Sigma = \{\sigma = 0\}$ be a curve with type I_0^* and orders of vanishing $(a, b, 6) = (2 + A, 3 + B, 6)$. Suppose that $\Sigma' = \{z = 0\}$ is a transverse curve with orders of vanishing (a', b', d'). Recall that we refer to the cases with $2b' = 3a'$ as 'hybrid type', those with $2b' > 3a'$ as 'odd type' and those with $2b' < 3a'$ as 'even type'. The contributions to residual vanishings of Δ along Σ' from intersection with the transverse curve Σ are given by $3\widetilde{a}$ and $2\widetilde{b}$ in the odd and even type cases, respectively, where $\widetilde{a}$ and $\widetilde{b}$ are the residuals contributions to Σ' from a curve Σ for vanishings of f, g, respectively.

The monodromy along Σ is determined by whether $Q(\psi) = \psi^3 + (f/\sigma^2)|_{\{\sigma=0\}}\psi + (g/\sigma^3)|_{\{\sigma=0\}}$ is fully split (giving algebra $\mathfrak{so}(8)$), partially split (giving algebra $\mathfrak{so}(7)$), or irreducible (giving gauge algebra $\mathfrak{g}_2$).

Gauge algebra $\mathfrak{g}_2$:

Here we can write the monodromy cover as $Q = \psi^3 + p\psi + q$, with Q irreducible.

$\mathbf{A > 0}$:

In this case, Q becomes $\psi^3 + q$ with irreducibility implying that q is not a cube; otherwise, we could factor Q as $(\psi-\alpha)(\psi^2+\alpha\psi+\alpha^2)$ with $q = -\alpha^3$. Since $q = (g/\sigma^3)|_{\sigma=0}$, we conclude that Σ cannot have intersections with curves Σ'_j with types $(-, 3b'_j, -)$ using all available g residuals along Σ. Said differently, $g = \sigma^3(\nu\phi^3 + g_4\sigma^4 + O(\sigma^5))$ with ν cube-free and $\deg(\nu) > 0$.

$\mathbf{B > 0}$:

Here $Q = \psi(\psi^2 + p)$, and since this is not an irreducible cubic, this case is not possible.

Gauge algebra $\mathfrak{so}(7)$:

Since the cover is split in this case, $Q = (\psi - \alpha)(\psi^2 + r\psi + s)$. Since the ψ^2 term vanishes, we have $\alpha = r$ and

$$Q = \psi^3 + (s - \alpha^2)\psi - s\alpha.$$

$\mathbf{A > 0}$:

Here we have $s = \alpha^2$, and hence $Q = \psi^3 - \alpha^3$. This means that Q is fully split since $\psi^2 + r\psi + s = \psi^2 + \alpha\psi + \alpha^2$, which has discriminant $\alpha^2 - 4\alpha^2 = -3\alpha^2$, which indeed has a square root. We conclude this case is prohibited.

$\mathbf{B > 0}$:

In this case, $Q = \psi(\psi^2 + s)$, and that this is not fully split implies $s = (f/\sigma^2)|_{\sigma=0}$ is not a square. Hence, Σ cannot have intersections with curves of types $(2a'_j, -, -)$ using all f residuals along Σ and in fact we have a slightly stronger conclusion that this curve cannot receive purely even f intersection contributions using all f residuals available. Said another way, $f = \sigma^2(\mu\phi^2 + f_3\sigma + O(\sigma^2))$ with μ square-free and $\deg(\mu) > 0$.

When $m = 3$, $\widetilde{a} = 2$, but the form of f clearly bars even contributions to $\widetilde{a}$ since $\deg(\mu) > 0$. Hence, intersection with a type II curve with orders $(2, 1, 2)$ is prohibited as are intersections with an I_0 with orders $(2, 0, 0)$.

Gauge algebra $\mathfrak{so}(8)$:

The monodromy cover is fully split here and appears as $(\psi - \alpha)(\psi - \beta)(\psi - \gamma)$. To have the ψ^2 term vanish we have

$$Q = \psi^3 + (-\beta^2 - \alpha^2 - \alpha\beta)\psi - \alpha\beta(\alpha + \beta).$$

Now we note that for $m = 4$, we have that for either $A, B > 0$ this is the only possible gauge algebra. For $A > 0, m = 4$, we have $\widetilde{b} = 0$ and hence the monodromy cover after appropriate rescaling appears as $\psi^3 + 1$ and hence factors completely. For $B > 0, m = 4$ the cover appears as $\psi(\psi^2 + 1)$ after rescaling since here $\widetilde{a} = 0$, and hence the cover can be fully split.

$\mathbf{A > 0}$:

Here $-\beta^2 - \alpha^2 - \alpha\beta = 0$. Substituting for β^2 using this identity gives $(g/\sigma^3)_{\sigma=0} = -\alpha\beta(\alpha + \beta) = \alpha^3$. Thus $g = \sigma^3(\alpha^3 + g_4\sigma + O(\sigma^5))$. This implies that all contributions to the residuals in g come in multiples of 3. For example, we have larger than expected contributions to g residuals from intersections with curves of type II. As another consequence, we see that since $A > 0$, inspecting the case of an intersection with type III shows that we in fact have a $(4, 6, 12)$ point as the remaining terms in g are of total order at least 6.

$\mathbf{B > 0}$:

Here, $\alpha\beta(\alpha+\beta) = 0$, giving three cases: $\alpha = 0, \beta = 0$, and $\alpha = -\beta$. In each, we have $Q = \psi^3 + (\alpha^2)\psi$ (renaming β as α as needed). Thus, $f = \sigma^2(-\alpha^2 + f_3\sigma + O(\sigma^2))$.

Summary of Restrictions

We collect the restricted monodromy assignments in Table 2.10.

	$A > 0$	$B > 0$
$\mathfrak{g}_2$	✓	X
$\mathfrak{so}(7)$	X	✓
$\mathfrak{so}(8)$	✓	✓

Table 2.10: Forbidden monodromy assignments on $\mathrm{I}_0^* \sim (2 + A, 3 + B, 6)$.

2.6.2 Intersections contributions from $\mathbf{I}_0^*$

Here we study the contributions to the residual vanishings along Σ' from a transverse intersection with Σ in each of the cases above.

Gauge algebra $\mathfrak{so}(8)$:

Using the above preliminaries, we have the following table of intersection contributions from Σ with $\mathfrak{so}(8)$ gauge algebra to Σ'. This depends on the orders of vanishing along Σ', whether the order in g there is a multiple of three, and if Σ' is of even or odd type. Recall that Σ' with orders (a', b', d') is of even type if $3a > 2b$, odd type if $3a < 2b$, and hybrid type otherwise. We do not explore the latter case in Table 2.11. Note that I_0 can appear in any of the three types as orders of vanishing for I_0 are given by $(a', b', 0)$ with one of a' or b' necessarily zero.

$A > 0$	$b' \equiv 0 \mod 3$	$b' \not\equiv 0 \mod 3$
	$(2 + A, 3, -)$	$(a, 4, -)$
Even Type on Σ':	$(a, 3, 6)$	$(a, 4, 8)$
Odd Type on Σ':	$(a, 3, 3a)$	$(a, 4, 3a)$

Table 2.11: Intersection contributions from $\mathrm{I}_0^{*s} \sim (a = 2 + A, b = 3 + B, 6)$.

Gauge algebra $\mathfrak{so}(7)$:

Here we only need to study the case $B > 0$. When Σ' has even type, we have minimal contributions to residuals along Σ' given by $(2, b, 2b)$. In the odd type on Σ' case these are instead $(2, b, 6)$.

Gauge algebra $\mathfrak{g}_2$:

Only $A > 0$ is relevant here. We have contributions given by $(a, 3, 6)$ in the even type case and by $(a, 3, 3a)$ in the odd type case.

2.6.3 Restricted tuples

We now discuss some consequences of the above I_0^* restrictions. Residual vanishing counts along Σ before any intersections are $(8 - 2m + 2A, 12 - 3m + 3B, 24 - 6m)$, where $m = \Sigma \cdot \Sigma$. We list forbidden collections of transverse curves simultaneously meeting Σ for various values of m and a given monodromy assignment in Table 2.12. An X indicates the monodromy assignment for specified values of A, B, m is forbidden. Separate forbidden collections are semicolon-separated. Note that we do not use all available vanishings with some collections. Rather, any collection containing a forbidden collection is also ruled out since the required vanishing conditions cannot be met with any of the indicated transverse subcollections. For example, in the case with data given by $\mathfrak{so}(8),\ A > 0,\ m = 3$, the presence of two transverse curves with types $(.,1,.)$ prevents $\widetilde{g} := (g/\sigma^2)|_{\sigma=0}$ from being a cube since these each require 2 additional vanishings of $\widetilde{g}$ at their intersections with Σ, thus contradicting the bound $\widetilde{b} = 3$.

$A > 0$	$m = 4$	$m = 3$	$m = 2$	$m = 1$
$\mathfrak{g}_2$	X	$(.,3,.)$	$2(.,3,.); (.,6,.),$	$3(.,3,.); (.,6,.)(.,3,.); (.,9,.)$
$\mathfrak{so}(7)$	X	X	X	X
$\mathfrak{so}(8)$		$2(.,1,.); (.,2,.)(.,1,.)$	$3(.,1,.); 2(.,2,.)(.,1,.); 3(.,2,.);$ $(.,3,.)(.,2,.)(.,1,.)$ $(.,3,.)2(.,1,.); (.,4,.)(.,1,.)$	$4(.,1,.); 4(.,2,.); 2(.,3,.)2(.,1,.);$ $(.,4,.)(.,2,.)(.,1,.);$ $4(.,1,.); (.,2,.)3(.,1,.); \ldots$
$B > 0$				
$\mathfrak{g}_2$	X	X	X	X
$\mathfrak{so}(7)$	X	$(2,.,.)$	$2(2,.,.); (4,.,.)$	$3(2,.,.); (4,.,.)(2,.,.); (6,.,.)$
$\mathfrak{so}(8)$		$2(1,.,.)$	$(2,.,.)2(1,.,.); 3(1.,.)$	$4(1,.,.); (2,.,.)2(1,.,.);$ $(2,.,.)3(1,.,.); (3,.,.)2(1,.,.); \cdots$

Table 2.12: Forbidden transverse curve collections meeting I_0^* with orders $(a = 2 + A, b = 3 + B, 6)$.

The form of the relevant restricted polynomials for $\mathfrak{so}(8)$ with $A, B > 0$ rule out a significant number of intersections that satisfy naive residuals tallying. Notable cases include $\mathfrak{so}(8), B > 0$ intersection with a type III curve (with orders $(1, \geq 2, 3)$) since this hence induces non-minimality and $\mathfrak{so}(8), B > 1$ intersection with type II curves. Other nontrivial prohibitions include $\mathfrak{so}(8)$, $A > 0$ intersections with any type IV curves or type III curves.

2.6.4 Intersection contributions to $\mathbf{I}_0^*$

$\mathbf{I}_0^*$ with $\mathbf{A > 0}$ meets multiple $\mathbf{I}_n$ curves:

Let Σ be a curve with type I_0^* at $z = 0$ having orders of vanishing $(2 + A, 3, 6)$. Consider first $A > 0$. Here Σ has even type and the contributions to residual vanishings of $\Delta|_{z=0}$ are induced precisely by those of $g|_{z=0}$.

$\boldsymbol{\Sigma \cdot \Sigma = -3}$:

In the case that $m = -\Sigma \cdot \Sigma = 3$, the residual vanishings along Σ are given by $(-, 3, 6)$. Working from the general local form of an I_2 curve found in (A.8), we can place further restrictions to give the forms for I_n with $n > 2$. For an I_2 curve at $\sigma = 0$ to meet Σ, we have $z|\phi$. Since $A > 0$, Σ has even type so each vanishing of $\widetilde{\Delta}$ along Σ corresponds to a vanishing of g there, and hence for two I_2 curves at σ, σ', (using the general form separately for each) restricting to $\sigma = 0$ and $\sigma' = 0$ and using that there is a $\widetilde{b}$ contribution at each intersection, we have $z^2|\phi$ in each expansion (rather than only the a priori requirement that $z|\phi$ needed for intersection with an arbitrary I_0^* curve). Since g goes at $\phi^3 + O(\sigma)$, we thus have a priori contribution to the residuals of the I_0^* given by $(-, \geq 1, \geq 2)$ at each I_2 intersection and contributions $(\geq 4, \geq 6, \geq 8)$ to each I_2 curve. If compact, the I_2 curves must have self intersections given by -1 and they cannot meet other curves of type other than I_n with $n \leq 4$. [4] Since $z^2|f_1$ and $z^2|\phi$, reading from the form of g from (A.8), we in fact have larger residuals contributions to Σ given by $(-, \geq 2, \geq 4)$ from each intersection with an I_n when $n \geq 2$. Hence these triples are disallowed as they exceed the permitted number of vanishings along Σ. Note that results of this kind are taken care of by tracking intersection contributions that we can read from the general forms of I_n unless $A = B = 0$, a case requiring special treatment.

2.6.5 $\mathbf{I}_0^*$ restricted tuples with $A = B = 0$

$\boldsymbol{\Sigma} \cdot \boldsymbol{\Sigma} = -3$:

Here, the number of roots of $\widetilde{f}, \widetilde{g}, \widetilde{\Delta}$ along Σ are $(2, 3, 6)$, respectively. Restricting to Σ, we may assume that any specified pair of transverse singular curves has intersection with Σ at $\sigma = 0$ and $\sigma' = 0$, respectively, with $\sigma = 1/\sigma'$ on the overlap since $\Sigma \cong \mathbb{P}^1$. From the residuals, we read that the homogeneous degrees of $f/z|_{z=0}, g/z^3|_{z=0}$, and $\Delta/z^6|_{z=0}$ (which we respectively define as $\widehat{f}, \widehat{g}, \widehat{\Delta}$) are $2, 3, 6$ respectively.

Claim: $\mathbf{I}_2$ $\mathbf{I}_0^*$ $\mathbf{I}_2$ gives at least a semi-split center curve Σ, with Σ meeting the above hypotheses:

Proof: The first I_2 curve intersection with Σ requires that the form of $\widehat{f}, \widehat{g}$ are those giving the general form for I_2 modified so that the relevant coefficient functions are instead constant. We can homogenize using our knowledge of the degrees of $\widehat{f}, \widehat{g}$, and $\widehat{\Delta}$.

[4] More generally, for $\Sigma' \sim \mathrm{I}_n$ rather than I_2 with $\Sigma' \cdot \Sigma' = -1$, we cannot have intersection with curves other than I_p with $p \leq (n + 12) - 2n$ when n is even and $(n + 12) - 3n$ when n is odd

Proceeding in this fashion while imposing both I_2 intersections, we obtain partial splitting of the monodromy cover. This implies the algebra assignment on the quiver $1, 3, 1$ given by $\mathfrak{su}(2), \mathfrak{g}_2, \mathfrak{su}(2)$ is not possible. The most general $\widehat{f}, \widehat{g}, \widehat{\Delta}$ are given in the patch with coordinates z, σ by

$$\widehat{f} = -3\phi^2 + f_1\sigma - 3\Phi^2\sigma^2$$

$$\widehat{g} = 2\phi^3 - f_1\phi\sigma - f_1\Phi\sigma^2 + 2\Phi^3\sigma^3$$

where ϕ, Φ, f_1 are unspecified constants. One can then partially factor the monodromy cover as

$$\psi^3 + \widehat{f}\psi + \widehat{g} = ((\phi + \Phi) - \psi)((2\phi^2 - f_1\sigma - 2\phi\Phi\sigma + 2\Phi^2\sigma^2) - (p + P\sigma)\psi - \psi^2)$$

in this coordinate patch. Note that we can now also read off the factorization of the monodromy cover polynomial on the other patch. □

Similar methods forbid the triple I_2 I_0^{*ns} III and show that when the III here has orders $(1, 3, 3)$, we can forbid that I_2 I_0^* III regardless of monodromy. Likewise, the triples I_2 I_0^* IV and I_4 I_0^* I_2 are also forbidden for all of monodromy choices.

Anomaly cancellation from geometry

One of the few conditions for which we require further geometric insight (beyond those available via tracking contribution counts and single curve global symmetry maxima) in order to match known constraints derived via anomaly cancellation machinery is the that of the following claim.

Claim: The quiver 322 cannot have algebras $\mathfrak{so}(7), \mathfrak{su}(2), -$.

We begin by noting the first (anomaly cancellation) condition given in [16], namely that "I_0^* on a -3 cannot meet a nonempty 2 and the algebra can only be $\mathfrak{so}(7)$, not $\mathfrak{g}_2$." We now show this follows from geometry without field theory considerations. Along with the other geometric constraints derived here, in [6], and those in the appendices of [16], non-minimality and intersection contribution tallying more than suffice to match all 6D SCFT enhancements constraints on all links and hence on all bases to those one can reach while also employing anomaly cancellation considerations. Some enhancements are eliminated via the present considerations, as discussed in Appendix B.

Proof: With the residuals contribution tracking, we find that the only enhancements of 322 remaining are the options $\mathfrak{so}(7), \mathfrak{su}(2)$ and $\mathfrak{g}_2, \mathfrak{su}(2)$ on the 32. To see that $\mathfrak{so}(7)$ is forbidden on the 3, we first note that in all the available type assignments on the link 32, the 3 has orders precisely $(2, 3, 6)$ and the 2 orders $(2, 2, 4)$. It then suffices to show that under the following conditions, the assignment $\mathfrak{so}(7)$ to the -3 curve is not possible. Our setup leaves only one possibility for the residuals on the -3 curve: they are given by $(0, 1, 2)$ and hence the form of f, g restricted to the -3 curve are given by $f \sim c_1 w^2$ and $g \sim c_2 z w^2$ where c_1, c_2 are nonzero constants and the IV lies at $w = 0$. Observe that $z \neq w$ (or we would have a codimension two $(4, 6, 12)$ point along the I_0^* curve). The resulting monodromy cover is then irreducible. We have the cover given by

$P = \psi^3 + c_1 w^2 \psi + c_2 z w^2$ that cannot be semi-split, since as we saw above the ψ^2 term vanishing requires that

$$Q = (\psi - \alpha)(\psi^2 + \alpha\psi + s) = \psi^3 + (s - \alpha^2)\psi - s\alpha.$$

With self intersection -3, the residuals on are $(2, 3, 6)$ for type I_0^* with orders $(2, 3, 6)$. Hence $\deg(s - \alpha^2) = 2$ and $\deg(s\alpha) = 3$ with the degrees of s, α being $2, 1$ respectively. We can then identify $s \sim w^2$ and $\alpha \sim z$ where z gives the other vanishing of g along the I_0^*. We have $\widehat{g} = c_2 \alpha s = (cw^2)(c'z)$ and $\widehat{f} = (s - \alpha^2) = (cw^2 - c'^2 z^2)$, where c, c' are nonzero constants. This means f has two distinct roots along the I_0^*, contradicting that we use both available vanishings at once in meeting the type IV curve. $\square$

Alternate proof: Suppose we have the $\mathfrak{so}(7)$ algebra on the -3 curve. We know that $\deg s = 2$ and $\deg \alpha = 1$ in the case with orders $(2, 3, 6)$ along the I_0^*. We expand each and imposing that $s - \alpha^2 = \widehat{f} = f_2 w^2$ and $-\alpha s = \widehat{g} = g_2 w^2 + g_3 w^3$ using the divisibility requirements from meeting a IV, where here $\widehat{f} = (f/\sigma^2)|_{\sigma=0}$ and $\widehat{g} = (g/\sigma^3)|_{\sigma=0}$, we have

$$\alpha = \alpha_0 + \alpha_1 \sigma$$

$$s = s_0 + s_1 \sigma + s_2 \sigma^2$$

$$\psi^3 + (s - \alpha^2)\psi - s\alpha = \psi^3 + \widehat{f}\psi + \widehat{g},$$

$$\implies a_0 = s_0 = s_1 = 0.$$

This gives $s\alpha = O(\sigma^3)$, preventing matching the σ^2 term of $\widehat{g}$ unless $g_2 = 0$. The latter induces non-minimality. $\square$

Claim: When Σ is a curve with $m = 3$ and type I_0^{*ss} having orders $(a, b, d) = (2, 3, 6)$, a type III curve Σ' with orders $(1, 3, 3)$ at $\sigma = 0$ imparts beyond minimal intersection contributions (instead contributing at least $(2, 3, 6)$ to the allowed vanishings along Σ).

Proof: We proceed as above, here imposing the requirements that

$$
\begin{aligned}
\alpha &= \alpha_0 + \alpha_1\sigma \\
s &= s_0 + s_1\sigma + s_2\sigma^2, \\
\psi^3 + (s - \alpha^2)\psi - s\alpha &= \psi^3 + \widehat{f}\psi + \widehat{g} \\
\widehat{f} &= f_1\sigma + f_2\sigma^2 \\
\widehat{g} &= g_3\sigma^3.
\end{aligned}
$$

Proceeding to match the terms of $\widehat{f}$ and $s - \alpha^2$ order by order and then those of g, we find before completing the matching that

$$
\begin{aligned}
& a_0 = 0, \quad s_0 = 0, \\
& s_1 = f_1, \quad s_2 = f_2 + \alpha_1^2, \\
\implies & s - \alpha^2 = \widehat{f}, \\
& -s\alpha = -\alpha_1 f_1\sigma^2 - \alpha_1(\alpha_1^2 + f_2)^2\sigma^3.
\end{aligned}
$$

From the latter we see that one of α_1, f_1 must be zero since $\widetilde{g}$ is zero at order σ^2. We rule out the first case as it induces infinite intersection contribution, leaving $f_1 = 0$. Intersection contributions to the I_0^{*ss} are then given by $(2,3,6)$. (Note this forbids any additional intersections along Σ, even with an I_0 curve having $A \geq 1$.) □

The above seemingly mild restriction plays an important role in determining which enhancement configurations are permitted and the degrees of freedom which remain to become global symmetry summands.

2.6.6 Contributions to $\mathbf{I}_0^*$ with $A, B > 0$ from $\mathbf{I}_n$

We now investigate the details of intersection contributions in the few permitted values for n in for I_0^* intersections. When $A, B > 0$, the situation is even more restrictive. The maximal allowed n for I_n meeting I_0^* in each case of $A, B > 0$ is given in Table A.2. So that we may refer to the general forms for I_n type curves, let's suppose our I_0^* here lies at $z = 0$.

A > 0, $\mathfrak{so}(8)$:

We will use the observation that since we have algebra $\mathfrak{so}(8)$, intersection contributions to $\widetilde{b}$ are multiples of 3. Those to $\widetilde{d}$ are then the doubles of those $\widetilde{b}$ contributions as $A > 0$ is an even type I_0^*. From Table A.1, we see that the maximal allowed intersection with I_n in this case with $A > 0$ is for $n = 3$.

$\mathbf{I}_n$ compact:

We will treat the three possible values of n separately when Σ' here is a Kodaira type I_n curve that is compact and has self-intersection -1. Note that we must have $A \leq 2$, since otherwise we exceed the 4 allowed f residual contributions to Σ'. Since $A > 0$, we must be without monodromy along Σ' in the one relevant case where $n = 3$. When $n = 1, 2$ we note that $z^2|\phi$ with ϕ as in (A.4),(A.8), respectively. Considering this fact together with the restriction that we have contributions in threes yields Table 2.13.

A \ n	1	2	3
1	$(1,3,6)/(4,6,10)$	X	X
2	$(0,3,6)/(4,6,10)$	X	X

Table 2.13: Intersection contributions to I_0^{*s} with $A > 0$ from/to I_n. An 'X' indicates the intersection is forbidden by non-minimality.

Note that in the case when $A = 1$ and $n = 1$, we must have at least $z^4|g_1$ and $z^4|g_2$ in (A.4) via the above consideration that g intersection contributions must come in threes here. This has total order of f, g given by $4, 5$ in both cases, falling barely short of non-minimality. The other cases $n > 1$ are barred by similar considerations.

Non-compact I_n curves for global symmetry:

We carry out a similar study here with the change that ϕ is allowed higher degree here, introducing the possibility of intersections with I_2 or I_3. Since we are concerned with global symmetry, we safely ignore $n = 1$ (as I_1 curves carry the trivial algebra). However, intersections for $n > 1$ are forbidden as a result of non-minimality following from the strong requirement that contributions to $\widetilde{b}$ are divisible by 3. This result is indicated in Table 2.14.

$A \backslash n$	2	≥ 3
≥ 1	X	X

Table 2.14: Intersection Contributions to I_0^{*s} with $A > 0$ from an I_n non-compact curve. An 'X' indicates the intersection is forbidden by non-minimality.

$\mathbf{B > 0}$ $\mathfrak{so}(8)$:

This case has similar requirements to those above with the main difference being that the contributions to residuals in f are required to be even here as we saw in the preceding analysis. We collect results for this case in Table 2.15.

$B \backslash n$	1	2
1	$(2,1,6)/(4,6,10)$	X
2	X	X
3	X	X

Table 2.15: Intersection Contributions to I_0^{*s} with $B > 0$ from/to I_n. An 'X' indicates the intersection is forbidden by non-minimality.

$\mathbf{B > 0}$, $\mathfrak{so}(7)$:

Again we consider intersections with a transverse type I_n curve, Σ' using restrictions from Table A.1 which dictate that the maximum value of n here is 2. We will use that we have odd type on the I_0^*; we will not need to use that intersection contributions to the I_0^* in this case are dictated by the lowest order z term in f, say $\mu_1\gamma^2$ with μ_1 square-free and of nonzero degree (since in this case the relevant term cannot be a square, or equivalently, contributions to $\widetilde{a}$ along an I_0^{*ss} cannot have purely even f residual contributions).

I_n compact:

With these constraints, we produce Table 2.16 by reading from the general forms of I_n from (A.4),(A.8),(A.11). Note, we have $B \leq 3$, as the degree of ϕ in (A.4) is 2 in the only relevant compact I_n intersections, those with self-intersection there is -1. For

B \ n	1	2
1	$(1,1,3)/(4,6,10)$	$(1,1,3)/(4,6,8)$
2	$(1,1,3)/(4,6,10)$	X
3	$(1,0,3)/(4,6,10)$	X

Table 2.16: Intersection contributions with I_0^{*ss} with $B > 0$ from/to an I_n curve. An 'X' indicates the intersection is forbidden by minimality considerations.

a non-compact I_n, we can raise the degree of ϕ to allow large values of B for example when meeting I_2. Such intersections are barred for Σ' compact. We have not used the condition that $\widetilde{a}$ contributions cannot be purely even, instead limitations being induced by minimality considerations. We collect our results in Table 2.16.

Non-compact I_n curves for global symmetry:

Our results here only concern $n \geq 2$ since $n = 1$ does not yield global symmetry. From Table A.1, we cannot exceed $n = 2$. We have the identical result when $B = 1$. When $B \geq 2$, we can avoid non-minimality by raising the degree of ϕ. In this case, we only are interested in the intersection contributions to the I_0^*. We collect the relevant result in Table 2.17.

$B \backslash n$	2	≥ 3
1	$(1,1,3)$	X
2	$(1,1,3)$	X
≥ 3	$(1,1,3)$	X

Table 2.17: Intersection Contributions to I_0^{*ss} with $B > 0$ from a transverse non-compactI_n curve. An 'X' indicates the intersection is forbidden by non-minimality considerations.

2.6.7 $A > 0$, $\mathfrak{g}_2$

The a priori restrictions in this case are similar to those for the $\mathfrak{so}(8)$ case with the exception that g contributions are not forced to be multiples of three. On the contrary, we are prevented from having contributions which consist entirely of multiples of three, though this is irrelevant in our analysis here. When $n = 3$, we use that ϕ_0 rather than μ in (A.11) must carry all divisibility (we have $z|\phi_0$ and have thus used all available roots of ϕ_0) since we are considering the case of intersection with a compact I_n curve where the f, g residuals are limited by $4, 6$, respectively. We find there is non-minimal intersection for $n \geq 3$ as this would require $z^2|\psi_1$ with ψ_1 as in (A.11).

$A \backslash n$	1	2	3
1	$(1,1,2)/(4,6,9)$	$(1,2,4)/(4,6,9)$	X
2	$(0,1,2)/(4,6,9)$	$(0,2,4)/(4,6,9)$	X

Table 2.18: Intersection contributions to I_0^{*ns} with $A > 0$ from/to I_n. An 'X' indicates the intersection is forbidden by non-minimality considerations.

2.7 Restrictions involving type I_n^* curves

In this section we collect contributions to residual vanishings from an intersection with an I_n^* curve. The main focus is the case with transverse curve of type I_n.

2.7.1 Single I_n^* intersections with a type I_m curve

In this section, we find the minimal simultaneous contributions to the residual vanishings to each curve from an I_n^*,I_m intersection. The result is that there is a simple general pattern for these contributions which we expect to hold in all cases where intersection should be allowed via the *a priori* residual vanishings allowed on each curve. Some of the general forms of such intersections have been constructed for this analysis, but the most general form in the large n, m case for arbitrary n, m has not. Hence a portion of our results here for $7 \leq m \leq 9$ is conjectural. We expect these contributions to be a generalization of the cases we have compiled in the table below. The general form of the conjecture is then that we have contributions to the residual vanishings along the I_n^* and I_m curves given by $(0, 0, m)$ and $(2, 3, n)$, respectively in the case with or without monodromy along the I_n^* for even m; for m odd instead these contributions are given by

$(0,0,m)$ and $(2,3,n)$ in the case with monodromy and $(0,0,m+1)$, $(2,3,n+1)$ in the case without monodromy, respectively. Note that the I_m is non-split (i.e. has monodromy) since we consider an intersection with I_n^*.

Intersection with type $\mathbf{I}_1^*$

Meeting $\mathbf{I}_1$:

We begin by considering an intersection contributions to an I_1 curve from an I_1^* curve in cases with and without monodromy. This requires working several cases and looking for the minimum. The detailed calculations can be easily carried out with a computer algebra system but are somewhat cumbersome to treat by hand. We find that these minimal contributions to the I_1 are given by $(2,3,7)$ and $(2,3,8)$ from I_1^{*ns} and I_1^{*s}, respectively. Note the agreement with [24].

Meeting $\mathbf{I}_2$:

Here we find that values for the cases with and without monodromy have the same intersection contributions to the I_2, namely $(2,3,7)$ and those to the I_1^* are given by $(0,0,2)$ regardless of monodromy.

Meeting $\mathbf{I}_3$:

Here we find that the monodromy again matters and the contributions to the I_3 are as in the I_1 case given by $(2,3,7)$ and $(2,3,8)$ from I_1^{*ns} and I_1^{*s}, respectively, before we consider the monodromy condition along the I_3. Investigating monodromy involves detailed inspection, but again can be readily treated using a computer algebra system.

We find that for $\mathfrak{su}(3)$ along I_3, the intersection becomes a $(4,6,12)$ point; this same result appears to hold for either monodromy along the I_1^*.

Note that since I_n^* is obtained by imposing further constraints on an I_1^* and we did not use the monodromy information from the I_1^*, this forbids the intersection of I_3^s with higher I_n^* curves as well. Likewise, moving to the cases $\mathrm{I}_{n\geq 3}$ also simply imposes additional constraints on f, g, but the term determining monodromy along $\Sigma = \mathrm{I}_n$ remains the same and its form simply becomes more constrained as we increase n. This method is then an alternative demonstration that I_n^* cannot intersect I_m^s for $(m \geq 3, n \geq 1)$.

Meeting $\mathbf{I}_4$:

In this case, the minimal contributions are $(2,3,7)$ with or without monodromy along the I_1^*.

$\mathbf{I}_n^*$ with $n \geq 4$:

Here we implement the constraints for I_n^* beginning with the inductive form of I_m and vice versa (for large n using the inductive form along the I_n^* and imposing the constraints for I_1, then I_2, etc.) and concluding by imposing monodromy constraints for the I_n^*. This allows one to inspect the resulting form to determine the minimal simultaneous intersection contributions in each case. We find the general pattern for low n appears to continue, but proving this in full generality is beyond our reach. Those cases we have explicitly checked are found in Table 2.19.

2.7.2 Summary

We find the following minimal simultaneous intersection contributions to I_n from I_m^* curves and vice versa, respectively. While it is a priori possible that the minimal contributions could be realized for each with different intersections giving the minimal ones, this does not appear to be the case; it appears possible in all cases studied to simultaneously realize the minimal contributions to both the I_n^* and the I_n curve. Note that only intersections that do not give $(4,6,12)$ points are considered here. Note there is one remaining possible (necessarily simultaneous) singularity of f,g since there are remaining residuals after the I_n^* intersection given by $(2,3,-)$ (which can be located anywhere we choose other than at the original intersection) and there cannot be additional transverse curves over which the fibration is singular not meeting at this point along the I_n curve that are not of type I_p for some p. The transverse curves also must have $(a,b) \leq (2,3)$. In other words, any other intersection with a curve of which the fibration is singular must give either contribution $(2,3,-)$ (there can be at most one of these) or contributions $(0,0,-)$.

2.7.3 Multiple I_n^* curves meeting a type I_m curve

In this case, we must have the self-intersection along the I_m curve, say Σ, with $\Sigma \cdot \Sigma = -1$ since for $-\Sigma \cdot \Sigma > 1$ we have no vanishings of f,g possible. The total residuals are $(4,6,12+m)$ along Σ, so at most two I_n^* curves can meet Σ, and such a pair leaves the no

	I_1^{*ns}	I_1^{*s}	I_2^{*ns}	I_2^{*s}	...	I_4^{*ns}	I_4^{*s}
I_1	$(2,3,7)/(0,0,1)$	$(2,3,8)/(0,0,2)$	$(2,3,8)/(0,0,1)$	$(2,3,9)/(0,0,2)$		$(2,3,10)/(0,0,1)$	$(2,3,11)/(0,0,2)$
I_2	$(2,3,7)/(0,0,2)$	$(2,3,7)/(0,0,2)$	$(2,3,8)/(0,0,2)$	$(2,3,8)/(0,0,2)$		$(2,3,10)/(0,0,2)$	$(2,3,10)/(0,0,2)$
I_3	$(2,3,7)/(0,0,3)$	$(2,3,8)/(0,0,4)$	$(2,3,8)/(0,0,3)$	$(2,3,9)/(0,0,4)$			
I_4	$(2,3,7)/(0,0,4)$	$(2,3,7)/(0,0,4)$					
I_5	$(2,3,7)/(0,0,5)$	$(2,3,8)/(0,0,6)$					
I_6	$(2,3,7)/(0,0,6)$	$(2,3,7)/(0,0,6)$					
⋮							
I_8 (†)	$(2,3,7)/(0,0,8)$	$(2,3,7)/(0,0,8)$	$(2,3,8)/(0,0,8)$	$(2,3,8)/(0,0,8)$	...		
⋮							

Table 2.19: Minimal simultaneous intersection contributions in I_n,I_k^* collisions to I_n and I_k^*, respectively. (The '†' symbol indicates the less general inductive form is used for I_n).

remaining residuals in f, g. Note that intersection with a pair of I_n^* curves requires that the I_m curve has monodromy.

2.7.4 Compact curve intersections for pairs with types II, I_n^*

We first observe that the only relevant case here is $n = 1$, as a type II curve cannot meet an I_n^* curve with $n > 1$; such intersections are non-minimal even when the I_n^* is non-compact. We consider here the case with both curves compact, thus introducing additional constraints not applicable to the situations considered in [6]. Along the I_n^* curve, $(\widetilde{a}, \widetilde{b}) = (2(4-m), 3(4-m))$ and II gives a nontrivial f, g contribution. Hence, $m = 4$ does not need to be considered. We summarize the contributions for all remaining cases of such an intersection in Table 2.20. Since $I_{\geq 1}^{*s}$ and $I_{>2}^{*ns}$ intersections with a type II curve are non-minimal, all relevant local contribution data for II,$I_{n\geq 1}^*$ intersections are captured in our table.

A \ m	3	2	1
0	$(2,3,4)/(3,4,8)$		
1	$(2,3,4)/(2,4,8)$		
2	X..	$(4,6,7)/(3,4,8)$	
3	X..	$(4,6,7)/(2,4,8)$	
4	X..	X..	$(6,9,10)/(3,4,8)$
5	X..	X..	$(6,9,10)/(2,4,8)$
≥ 6	X..	X..	X..

Table 2.20: Intersection contributions to/from a compact I_1^{*ns} with $\Sigma \cdot \Sigma = -m$ intersecting a II. An X indicates the intersection is non-minimal and an (X..)X. that the intersection is banned by (naive) residuals considerations. Entries to the right of an allowed entry have the same values.

Note that Kodaira types beyond II other than I_n family types are banned from I_n^* intersection by non-minimality. Thus we have given here the set of possible intersection contributions for types other than I_0, which we have recorded separately.

2.8 Intersection of a compact I_0 with type I_n^* curve for $n \geq 1$

We now find the intersection contributions to a curve Σ with type I_0 from an I_n^* curve, Σ', which may be non-compact. Here u_1 in the general form of I_n^* given in (A.34)-(A.38) appears to permit any of the *a priori* possible number of roots (at least as far as constraints along the I_n^* are concerned).

Let the vanishings along Σ be given by $(A, B, 0)$, noting that we must have one of A, B being zero. We consider Σ' along $\{\sigma = 0\}$ whose form is given by one of the expansions from (A.34)-(A.38). We compile the total order at the intersection point in 2.21, writing 'X' for non-minimal intersections. By inspecting the form of I_n^*, imposing the required divisibility conditions to have z^A dividing f or z^B dividing g, and then checking the monodromy condition for I_n^*, we arrive at the indicated minimal total orders at the intersection. We do not record the case $A = B = 0$ here since the minimal total order is always $(2, 3, n)$ in such cases and all intersections are allowed except when the Σ' has self intersection -4; for such curves, all $A, B > 0$ intersections are trivially banned by *a priori* residuals tracking. Terminal entries with A or B referenced indicate the general pattern holds for all A or B, applying also to the entries below such an entry. Note that we need only explore $n \leq 4$ since further intersections are forbidden via [32].

	I_1^{*ns}	I_1^{*s}	I_2^{*ns}	I_2^{*s}	I_3^{*ns}	I_3^{*s}	I_4^*
$A = 1$	$(4, 4, 8)$	$(4, 4, 8)$	$(4, 5, 10)$	$(4, 5, 10)$	$(3 + A, 5, 10)$	X	X
$A = 2$	$(4, 4, 8)$	$(4, 4, 8)$	$(4, 5, 10)$	$(5, 5, 10)$		X	X
$A = 3$	$(2 + 2\lceil A/2 \rceil, 4, 8)$	$(2 + 2\lceil A/2 \rceil, 4, 8)$	$(6, 5, 10)$	$(6, 5, 10)$		X	X
$A = 4$			$(3 + A, 5, 10)$	$(3 + A, 5, 10)$		X	X
$B = 1$	$(3, 5, 9)$	$(3, 5, 9)$	$(3, 4 + B, 9)$	$(3, 4 + B, 9)$	X	X	X
$B = 2$	$(3, 6, 9)$	$(3, 6, 9)$			X	X	X
$B = 3$	$(3, 6, 9)$	$(3, 7, 9)$			X	X	X
$B = 4$	$(3, 7, 9)$	$(3, 7, 9)$			X	X	X

Table 2.21: Minimal total order of I_0,I_n^* intersections

2.9 Intersection of a compact I_0 with a compact I_n^* curve for $n \geq 1$

The residual vanishing counts along an I_n^* curve with negative self-intersection m are given by

$$\begin{aligned}
\widetilde{a} &= -4(m-2) + 2m = 8 - 2m \\
\widetilde{b} &= -6(m-2) + 3m = 12 - 3m \\
\widetilde{d} &= -12(m-2) + (n+6)m = 24 + (n-6)m.
\end{aligned}$$

In particular, the constraints on $\widetilde{a}, \widetilde{b}$ and the general forms of I_n^* together imply that u_1 in the general form has the number of roots indicated in Table 2.22. This data allows us

	$m=4$	$m=3$	$m=2$	$m=1$
$\deg(u_1)$	0	1	2	3

Table 2.22: Degree of u_1 for I_n^* with $\Sigma \cdot \Sigma = -m$.

to find the following rules for intersections with type $\mathrm{I}_0 \sim (A, B, 0)$ singularities which are not necessarily compact nor transverse curves but simply singularities along the I_n^* locus. The latter implies that any remaining residuals in purely f or g have such intersection contributions, but we should note that there may in some cases indicated as non-minimal which may in fact be allowed point singularities in f, g to the corresponding order; that is, these tables are intending to track contributions from transverse curves but also give the contributions for point singularities when a transverse curve of the corresponding type is allowed. In other terms, only in cases where the intersections are indicated as allowed rather than non-minimal do we track the data, as our priority is to treat transverse curves rather than simply singular points (but some information for point singularities does result). The contributions to the I_0 are also noted for use in the case that the singularity is an intersection with a it compact transverse I_0 curve. Note that for $m = 4$, we cannot have intersection with a curve of type $\mathrm{I}_0 \sim (A, B, 0)$ with either of $A > 0$ or $B > 0$. This fact and other restrictions of this form are already accounted for via simple residuals tracking without modification from what follows below. For $A = B = 0$, it is easy to see the intersection contributions in both monodromy cases are simply $(2, 3, 6+n)$, the naïve estimate. We collect the relevant results in Table 2.23.

2.9.1 $m = 3$

Here we have at most a single root of u_1. As a result, we find the following intersection contributions and non-minimal intersections. The total residuals along the I_n^* here are given by $(2, 3, 6 + 3n)$. See Table 2.23 below.

2.9.2 $m = 2$

Here the residuals along the I_n^* are given by $(4, 6, 12 + 2n)$. This means that the highest power of z dividing u_1 is two. Hence, A, B are capped for I_0 at $4, 6$. The relevant contribution data is recorded in Table 2.24.

	I_1^{*ns}	I_1^{*s}	I_2^{*ns}	I_2^{*s}	$I_{\geq 3}^{*}$
$A=1$	$(3,4,8)/(2,3,3)$	$(3,4,8)/(2,3,3)$	$(3,5,10)/(2,3,3)$	X.	X.
$A=2$	$(2,4,8)/(2,3,3)$	$(2,4,8)/(2,3,3)$	$(2,5,10)/(2,3,3)$	X.	X.
$A\geq 3$	-	-	-	-	-
$B=1$	$(3,4,9)/(2,3,4)$	$(3,4,9)/(2,3,5)$	$(3,4,9)/(2,3,2)$	$(3,4,9)/(2,3,2)$	X
$B=2$	$(3,4,9)/(2,3,4)$	X.	X.	X.	X
$B=3$	$(3,3,9)/(2,3,4)$	X.	X.	X.	X
$B\geq 4$	-	-	-	-	-

Table 2.23: Intersection Contributions to $I_0 \sim (A,B,0)$ and I_n^* with $m=3$, respectively. 'X' indicates non-minimal intersection and '-' indicates exceeding allowed residuals. 'X.' indicates that the intersection is non-minimal here while in the non-compact case it is was not yet forbidden via the previous analysis in the non-compact case.

	I_1^{*ns}	I_1^{*s}	I_2^{*ns}	I_2^{*s}	I_3^{*ns}	$I_{\geq 3}^{*s}$	$I_{\geq 4}^{*ns}$
$A=1$	$(3,4,8)/(2,3,3)$	$(3,4,8)/(2,3,3)$	$(3,5,10)/(2,3,3)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A=2$	$(2,4,8)/(2,3,3)$	$(2,4,8)/(2,3,3)$	$(2,5,10)/(2,3,3)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A=3$	$(3,4,8)/(4,6,6)$	$(3,4,8)/(4,6,6)$	$(3,5,10)/(4,6,6)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A=4$	$(2,4,8)/(4,6,6)$	$(2,4,8)/(4,6,6)$	$(2,5,10)/(4,6,6)$	$(2,5,10)/(4,6,6)$	X.	X	X
$A\geq 5$	-	-	-	-	-	-	-
$B=1$	$(3,4,9)/(2,3,4)$	$(3,4,9)/(2,3,5)$	$(3,4,9)/(2,3,2)$	$(3,4,9)/(2,3,2)$	X	X	X
$B=2$	$(3,4,9)/(2,3,4)$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,4)$	$(3,4,9)/(4,6,4)$	X	X	X
$B=3$	$(3,3,9)/(2,3,4)$	$(3,4,9)/(4,6,8)$	X.	X.	X	X	X
$B=4$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,8)$	X.	X.	X	X	X
$B=5$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,8)$	X.	X.	X	X	X
$B=6$	$(3,3,9)/(4,6,8)$	$(3,3,9)/(4,6,8)$	X.	X.	X	X	X
$B\geq 7$	-	-	-	-	-	-	-

Table 2.24: Intersection contributions to $I_0 \sim (A,B,0)$ and I_n^* with $m=2$, respectively. 'X' indicates non-minimal intersection or not allowed to have a particular I_n^* on a curve of this self intersection and '-' indicates exceeding allowed residuals. 'X.' indicates that the intersection is non-minimal here while in the non-compact case it is was not yet forbidden via the previous analysis in the non-compact case.

2.9.3 $m = 1$

In this case, the residual counts along the I_n^* are instead given by $(6, 9, 18+n)$. Hence, the highest power of z dividing u_1 is three here, capping A, B for I_0 at $6, 9$. Contribution data for these intersection appears in Table 2.25.

	I_1^{*ns}	I_1^{*s}	I_2^{*ns}	I_2^{*s}	I_3^{*ns}	$\text{I}_{\geq 3}^{*s}$	$\text{I}_{\geq 4}^{*ns}$
$A = 1$	$(3,4,8)/(2,3,3)$	$(3,4,8)/(2,3,3)$	$(3,5,10)/(2,3,3)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A = 2$	$(2,4,8)/(2,3,3)$	$(2,4,8)/(2,3,3)$	$(2,5,10)/(2,3,3)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A = 3$	$(3,4,8)/(4,6,6)$	$(3,4,8)/(4,6,6)$	$(3,5,10)/(4,6,6)$	$(3,5,10)/(4,6,6)$	X.	X	X
$A = 4$	$(2,4,8)/(4,6,6)$	$(2,4,8)/(4,6,6)$	$(2,5,10)/(4,6,6)$	$(2,5,10)/(4,6,6)$	X.	X	X
$A = 5$	$(3,4,8)/(6,9,9)$	$(3,4,8)/(6,9,9)$X	$(3,5,10)/(6,9,9)$	X.	X.	X	X
$A = 6$	$(2,4,8)/(6,9,9)$	$(2,4,8)/(6,9,9)$X	$(2,5,10)/(6,9,9)$	X.	X.	X	X
$A \geq 7$	-	-	-	-	-	-	-
$B = 1$	$(3,4,9)/(2,3,4)$	$(3,4,9)/(2,3,5)$	$(3,4,9)/(2,3,2)$	$(3,4,9)/(2,3,2)$	X	X	X
$B = 2$	$(3,4,9)/(2,3,4)$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,4)$	$(3,4,9)/(4,6,4)$	X	X	X
$B = 3$	$(3,3,9)/(2,3,4)$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(6,9,6)$	$(3,4,9)/(6,9,6)$	X	X	X
$B = 4$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,8)$	X.	X.	X	X	X
$B = 5$	$(3,4,9)/(4,6,8)$	$(3,4,9)/(4,6,8)$	X.	X.	X	X	X
$B = 6$	$(3,3,9)/(4,6,8)$	$(3,3,9)/(4,6,8)$	X.	X.	X	X	X
$B = 7$	$(3,4,9)/(6,9,12)$	X.	X.	X.	X	X	X
$B = 8$	$(3,4,9)/(6,9,12)$	X.	X.	X.	X	X	X
$B = 9$	$(3,3,9)/(6,9,12)$	X.	X.	X.	X	X	X
$B \geq 10$	-	-	-	-	-	-	-

Table 2.25: Intersection contributions to $\text{I}_0\sim (A, B, 0)$ and I_n^* with $m = 1$, respectively. 'X' indicates non-minimal intersection or not allowed to have a particular I_n^* on a curve of this self intersection and '-' indicates exceeding allowed residuals. 'X.' indicates that the intersection is non-minimal here while in the non-compact case it is was not yet forbidden via the previous analysis in the non-compact case.

2.10 Intersections of I_0 and I_0^*

Here we suppose that a curve Σ of type I_0 with vanishings $(A, B, 0)$ at $\{z = 0\}$ meets a curve Σ' of type $\text{I}_0^* = \{\sigma = 0\} := \Sigma'$ and we investigate the restrictions on monodromy on Σ' for various values of A, B, the additional orders of vanishing of f, g along Σ, as above. We let A_0, B_0 give the orders of vanishing along Σ' as $(2 + A_0, 3 + B_0, 6)$. Here we have the monodromy cover given by $\psi^3 + (f/\sigma^2)|_{\{\sigma=0\}} + (g/\sigma^3)|_{\{\sigma=0\}}$ and in the case where this splits as $(\psi - \alpha)(\psi - \beta)(\psi - \gamma)$, it is given by

$$\psi^3 + \psi(-\beta^2 - \alpha^2 - \alpha\beta) - \alpha\beta(\alpha + \beta).$$

In the case that the I_0^* curve Σ' has $B_0 \geq 4$ so that we have orders of vanishing $(2, \geq 6, 6)$, we are near non-minimality. In fact, we find that if the I_0 has orders $(1, 0, 0)$ or beyond in f, non-minimality forbids such an intersection.

To clarify, in this case $\alpha\beta(\alpha + \beta)$ must vanish. Hence one of α, β, or $\alpha + \beta$ must vanish. Suppose $\alpha = -\beta$. Then

$$\alpha^2 + \beta^2 + \alpha\beta = \alpha^2.$$

For $z|f$, we then have $z^2|(f/\sigma^2)|_{\{\sigma=0\}}$, giving a $(4, 6, 12)$ point. To see this, we note that our requirements result in

$$f = \sigma^2(zg_2(z) + zg_3(z)\sigma + zg_4(z)\sigma^2 + \dots)$$

with $(f/\sigma^2)|_{\sigma=0}$ a square, making $zg_2(z)$ is a square. Hence, orders $(3, 6, 9)$ are boosted, with non-minimality resulting after considering monodromy. The other cases are similar. When $B > 0$ along the I_0, there is no analogous result.

In the case that we have $A_0 > 0$, the coefficient on ψ vanishes and hence $(g/\sigma^3)|_{\sigma=0} = (\alpha^2 + \beta^2)(\alpha + \beta)$. The utility of this condition in considering $z^k|g$ to study $B > 0$ is not immediately obvious, so we move on, ignoring this case.

Given $so(7)$ algebra on I_0^*, the monodromy cover splits partially. We can write it as

$$\psi^3 + (\gamma - \alpha^2)\psi - \alpha\gamma.$$

When $A_0 > 0$, this becomes $\psi^3 - \alpha^3$. Since $\alpha^3 = (g/\sigma^3)|_{\sigma=0}$, we see that z divisibility of g along the I_0 becomes z^3 divisibility. In the case that $A_0 \geq 2$, the order in f is already 4 before intersection with the I_0. This results in a $(4, 6, 12)$ point as the $(4, 4, 8)$ point at the intersection is boosted by the monodromy condition. In terms of expanding g, this reads

$$g = \sigma^3(zg_3(z) + zg_4(z)\sigma + zg_5(z)\sigma^2 + \dots)$$

with $zg_3(z)$ a cube. The latter requires that z^3 divides $g|_{\sigma=0}$. Hence, $A_0 \geq 2$ and $B \geq 1$ is forbidden for I_0^{*ss}.

When we have $\mathfrak{g}_2$ algebra, the monodromy cover is irreducible, and appears as $\psi^3 + q\psi + p$. If this had a root, it would appear as $\psi^3 + \psi(\gamma - \alpha^2) - \alpha\gamma$. In the case with

$A_0 > 0$, irreducibility implies that $p \neq \alpha\gamma$ where $\gamma = \alpha^2$, i.e., $(g/\sigma^3)|_{\sigma=0}$ is not a cube. In particular, this prevents $(g/\sigma^3)|_{\sigma=0}$ from being constant, hence barring the case with no residual vanishings along the I_0^* after accounting for any other intersections. Here, $g = \sigma^3(g_3 + \sigma g_4 + \sigma^2 g_5 + \dots)$, and the above is simply to say g_3 is not a cube. This result applies in all configurations involving an I_0^* with $\mathfrak{g}_2$ algebra; there is nothing used about an intersection with I_0. To phrase this differently, if we have an I_0^* with $\mathfrak{g}_2$ algebra and $A_0 > 0$, the other vanishings of g along this curve cannot all appear as cubes. Among other restrictions this imposes, we see that when $\Sigma' \cdot \Sigma' = m$, with m given by $1, 2, 3$, the residual vanishings of g before considering other intersections read $9, 6, 3$, respectively in these cases, implying for example that a g_2 gauged I_0^* with $A_0 > 0$ cannot meet $3, 2, 1$ other type III curves each having types $(1, 3, 3)$ in the respective cases.

Similarly, for $B_0 > 0$ we have monodromy cover given by $\psi^3 + q\psi$. This is not irreducible, thus ruling out the case with $\mathfrak{g}_2$ algebra.

A few of these results are summarized in Table 2.26

	$\mathfrak{g}_2$	$\mathfrak{so}(7)$	$\mathfrak{so}(8)$
	$B_0 > 0$	$A_0 \geq 2$ & $B > 0$	$B_0 \geq 3$ & $A > 0$

Table 2.26: Forbidden intersections and type restrictions

2.11 Global symmetries of 6D SCFTs

In Appendix D, we collect sample tables for the global symmetry maxima for all enhancements of 0-link and branching 1-link quiver theories meeting the constraints derived here and in [6, 24]. We arrive at these results via exhaustive computer algebra routine search methods, discussed in further detail in Appendix C. The global symmetries for distinct enhancements are not compared in our tables[5], as these correspond to the symmetries of potentially distinct physical theories (and certainly so when having distinct gauge algebras[6]). With the exception of a few quiver families permitting infinitely many enhancements, those we have referred to as "rogue bases," our accompanying computer algebra routine can be used to provide the global symmetry maxima for each enhancement of any given quiver provided a machine with sufficient memory [7]. Examples collected in appendices are limited to all 0-links, a few end links, and a sample interior link (the latter being prohibitively lengthy in their full global symmetry prescriptions). This amounts to data for the majority of possible links. The remaining cases of interior link and end link based theories not included in the Appendix D have global symmetry that is loosely characterized as the direct sum of those studied here (via appropriate truncation of outer

[5]Alternative reorganization routines for the output data are provided by methods included in the accompanying computer algebra workbook found in Appendix C. Among these are methods to compare all global symmetry options arising on a given quiver and capabilities to track details of the configurations at the level of orders of vanishings along all curves in each configuration at the other extreme.

[6]We might also ask whether having distinct global symmetry maxima suffices to distinguish SCFTs, and if not, what further data is required.

[7]Rogue bases can also be treated, but as they permit infinitely many gauge enhancements, we require a user to specify a limit on the maximal gauge algebra which should be allowed. Thus a simple listing of all possible global symmetries is not achieved with the same approach. We postpone discussion of these cases to future work, though straightforward.

-1 curves) and the symmetry carried on the outer -1 curves of the link; these latter summands are in most cases the remaining relatively maximal complementary algebras available for the possible enhancements on the outer curves that are compatible with one of the enhancements provided in the tables of Appendix D, where the algebras in which these form relatively maximal compliments are characterized by the possible global symmetries for -1 curves, as detailed originally in [6, 24] and reproduced here in Tables 1.6,1.7,1.8. For complete, direct specification of global symmetry maxima in these cases, the implementation we provide can be run directly on essentially any base [8] directly (even with a typical laptop circa 2016 [9]).

2.12 A comment on $U(2)$ subgroup generators

In this section, we pause briefly to state a new identity capturing the collection of 78 equations for continued fraction values appearing Table 1 of [28]. These correspond to the complete list of A-type endpoint families. Note that these also capture the D-type

[8] Our implementation treats at most 3-valent bases. Hence, for the three 4-valent links (which cannot attach to anything) results may be obtained by inspection of the relevant quiver dropping a -1 curve along with data for the quiver 51. The remaining 4-valent bases are fairly limited and similarly can be readily studied via decomposition to a three valent base and a linear base with little effort.

[9] Computation times depend on the previous computations completed, getting somewhat shorter after some lengthier calculations are stored or imported via the export file from a previous run. Note that such files can be shared between users. The bulk of relevant global symmetry speedups for subsequent non-rogue bases can be obtained from computations for e.g. 131. Many of the implemented gauge algebra calculation speedups can be accessed by running computations determining the gauge algebras available on the full set of linear links; this takes about half an hour on a typical 2016 laptop. The combined runtime for all global symmetry maxima calculations on the set of all 0-links on such a machine is a few dozen hours, mostly occupied by a handful of links for which computations are more involved.

generator number theory via truncation of the D-type branch. [10] The indicated values define the generator of Γ, the discrete $U(2)$ subgroup determining the base orbifold as $B \cong \mathbb{C}^2/\Gamma$, where the action on $\mathbb{C}^2$ is given by a fraction value p/q via

$$(s,t) \mapsto (e^{2\pi i/p}s, e^{2\pi iq/p}t). \tag{2.2}$$

Each endpoint family is indexed in [28] by N, the total number of curves occurring in the string. We will instead write n to count the number of -2 curves inserted between the pair of possibly empty outer strings. We recover the continued fraction values of all collections with a simple identity given in terms of values for the continued fractions of the 12 possible leading or tailing strings, α and β.

2.12.1 A condensed description

To write the following identity, it is convenient to collect values for the continued fractions of the relevant outer strings and their reverses. These appear in Table 2.27.

In our result, we utilize the first column entries of the Hirzebruch-Jung continued fraction matrix representation (as in [33]) for the leading string, α. We list the negatives of these values in Table 2.28 labeling them as $m_{\alpha,1}, m_{\alpha,2}$. Let $\bar{\alpha}$ denote the reverse of the string α. We can now recover the formulas for the values of all continued fractions in

[10] The D-type endpoints have a similar action generator determined by dropping the branch $2\overset{2}{2}$ to obtain the fraction p/q from the remaining string. The other generator has action given by $(z_1, z_2) \mapsto (z_2, -z_1)$, as noted in [17].

$\alpha:$	$\emptyset$	3	4	5	6	7	33	23	32	223	322	2223	3222	22223	32222	24	42
$\frac{p_\alpha}{q_\alpha}:$	0	3	4	5	6	7	$\frac{8}{3}$	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{7}{5}$	$\frac{7}{3}$	$\frac{9}{7}$	$\frac{9}{4}$	$\frac{11}{9}$	$\frac{11}{5}$	$\frac{7}{4}$	$\frac{7}{2}$

Table 2.27: Continued fraction values for outer strings of endpoints.

$\alpha:$	3,4,5,6,7	23	223	2223	22223	33	24
$(m_{\alpha,1}\ m_{\alpha,2}):$	(0 1)	(1 2)	(2 3)	(3 4)	(4 5)	(1 3)	(1 2)

Table 2.28: First column entries of Hirzebruch-Jung continued fraction matrix representation for endpoint strings.

each endpoint family as reported in [28] as

$$C_{\alpha,\beta}(n) = \frac{|(p_{\bar{\alpha}} - q_{\bar{\alpha}})(p_\beta - q_\beta)| \cdot n + |\max\{p_\alpha, 1\}\max\{p_\beta, 1\} - q_\beta m_{\alpha,2}(1 - \delta_{\beta,\emptyset})|}{|(q_\alpha - m_{\alpha,1})(p_\beta - q_\beta)| \cdot n + |q_\alpha \max\{p_\beta, 1\} - q_\beta m_{\alpha,1}(1 - \delta_{\beta,\emptyset})|}.$$

The absolute values, max(−), and Kronecker symbol are included so that we may simultaneously treat the cases with empty strings α, β and may otherwise be ignored. In some terms, the absolute values are never needed, but we leave them in place to emphasize that the resulting function of n consists of nonnegative integer coefficients.

2.13 Conclusions

The geometric constraints we have discussed in the body of this work have allowed an algorithmic approach to computing the gauge and global symmetries for all known 6D SCFTs. We find agreement with results appearing in the literature, for example the handful of cases discussed in [16]. Briefly, those global symmetries for a 6D SCFT which arise appear in an F-theoretic description via contributions from a select few curves in the discriminant locus determining its base while all other curves provide trivial symmetry summands. This agrees with the general structure of allowable bases for SCFTs. Locations permitting global symmetry can be viewed as positions with free hypermultiplets that can often be paired to compact curves (producing more elaborate bases) or to non-compact gauged curves with algebras becoming global symmetry summands. We have provided analysis of many local models for intersections involving pairs of compact curves and completed the remaining details for compact-non-compact pair intersections not elucidated in [6, 24, 16]. We have furnished an algorithm which combines the geometric constraints introduced in these works with the aforementioned analysis to determine the gauge algebra structure of all 6D SCFTs purely via geometric constraints on Calabi-Yau threefold elliptic fibrations (i.e. without hypermultiplet counting restrictions coming from requirements for anomaly cancellation), yielding essential agreement with those prescribed in [16]; the further restrictions on gauge structure we have uncovered are detailed in Appendix B.

A second algorithm and implementation we have included takes this one step further, allowing computation of global symmetries for any 6D SCFT, also via purely geometric constraints. We have collected results of this algorithm applied to all 0-links and branching 1-links (among others) and the resulting global symmetry maxima have been recorded in Appendix D along with results for several miscellaneous quivers. While our constraints are global in the base, we find that this "non-locality" is not often reflected in the global symmetries which can be realized for fixed Kodaira type assignments provided we compare fibrations with potentially differing codimension-two singularities. For a fixed gauge assignment, the summands towards total global symmetry are more often restricted beyond those which may appear along each curve. Thus, the global symmetries which may occur and how they pair with the permissible gauge enhancements of a given base yield a nontrivial structure permitting the possibility that certain features of the geometry beyond a gauge algebra choice and $U(2)$ subgroup determination are necessary to specify a theory uniquely; whether these distinctions disappear after renormalization remains an open question. We have observed that those geometries giving maximal global symmetries for fixed $U(2)$ and gauge data appear to determine a natural class of distinguished threefolds which may be of some mathematical and physical interest. Perhaps further investigation of these structures (including that to appear in [5]) might prove to be a useful tool in future study of the moduli space of Calabi-Yau threefolds and the string landscape.

Appendix A

Local models and intersection data

In this appendix, we review the derivation and results presented in [6] concerning the local models type I_n and type I_n^* curves. We frequently require reference to the general forms for expansions along curves with these types in the present work. As discussed above, the further complications faced in the treatment of local models in the present work stems from our necessity to treat bases containing more than a single compact curve, and hence to treat models where the number of allowed vanishing on each curve in such an intersection are both constrained. In the tables concerning forbidden intersections in this section, we are constraining those configurations where the transverse curve is non-compact. These restrictions then also apply when the transverse curve is compact, this scenario merely imposing further restrictions.

A.1 The Tate algorithm for the I_n case

The expansions in this section provide data necessary to describe a transverse intersection between a Kodaira type I_n curve supported on a divisor $\{\sigma = 0\}$ in the base and curve with another Kodaira type which we may assume without loss of generality to be supported on a divisor $\{z = 0\}$.

The approach in this section generalizes that of [25], which provided careful analysis of Weierstrass models for $\mathfrak{su}(n)$, to treat $\mathfrak{sp}(n/2)$ as well. A similar approach was also employed in [20]. We expand the quantities in (1.1) in powers of σ

$$f = \sum_i f_i \sigma^i \ , \qquad g = \sum_i g_i \sigma^i \ , \qquad \Delta = \sum_i \Delta_i \sigma^i \ , \tag{A.1}$$

where the quantities f_i, g_i and Δ_i can be taken as polynomials in z, at least locally in a neighborhood of $\{\sigma = 0\}$. The requirement for the divisor $\{\sigma = 0\}$ to carry type I_n implies that $\Delta_i = 0$ for all $i < n$. As in [25], we assume that the divisor $\{\sigma = 0\}$ is non-singular, implying that any ring of local functions on a sufficiently small open subset of $\{\sigma = 0\}$ is a unique factorization domain.

In order to obtain a type I_1 singularity we need

$$\Delta_0 = 4f_0^3 + 27g_0^2 = 0 \ , \tag{A.2}$$

and hence a local function ϕ such that [1]

$$f_0 \sim -\tfrac{1}{48}\phi^2 \ , \qquad\qquad g_0 \sim \tfrac{1}{864}\phi^3 \ . \tag{A.3}$$

We set $f_0 = -\frac{1}{48}\phi^2$ and $g_0 = \frac{1}{864}\phi^3$ and we obtain: [2]

Summary for $\mathbf{I}_1$

$$\begin{aligned} f &= -\tfrac{1}{48}\phi^2 + f_1\sigma + O(\sigma^2) \ , \\ g &= \tfrac{1}{864}\phi^3 + g_1\sigma + O(\sigma^2) \ , \\ \Delta &= \tfrac{1}{192}\phi^3 \left(12g_1 + \phi f_1\right)\sigma + O(\sigma^2) \ . \end{aligned} \tag{A.4}$$

The coefficients f_i, g_i above have been redefined to absorb the terms proportional to σ that we might have introduced when substituting for f_0 and g_0. Now, setting $\Delta_1 = 0$ implies

$$g_1 \sim -\tfrac{1}{12}\phi f_1 \ . \tag{A.5}$$

We can now give the most general form for type I_2. In presenting the expansions, introducing the following notation is helpful. For each $i \geq 1$, there are terms in the expansion of Δ appearing as

$$\Delta_i = 4f_0^2 f_i + 27g_0 g_i + \cdots \ . \tag{A.6}$$

[1] Given two local functions ϕ_1 and ϕ_2, the notation $\phi_1 \sim \phi_2$ indicates that the two functions are identical up to powers of σ.

[2] The choice of numerical coefficients aims to match conventions in the existing literature.

This makes it convenient to define

$$\widetilde{g}_i = g_i + \tfrac{1}{12}\phi f_i \ , \qquad i \geq 1 \ . \tag{A.7}$$

Summary for $\mathbf{I}_2$

$$\begin{aligned} f &= -\tfrac{1}{48}\phi^2 + f_1\sigma + f_2\sigma^2 + O(\sigma^3) \ , \\ g &= \tfrac{1}{864}\phi^3 - \tfrac{1}{12}\phi f_1\sigma + (\widetilde{g}_2 - \tfrac{1}{12}\phi f_2)\sigma^2 + O(\sigma^3) \ , \\ \Delta &= \tfrac{1}{16}\left(\phi^3\widetilde{g}_2 - \phi^2 f_1^2\right)\sigma^2 + O(\sigma^3) \ . \end{aligned} \tag{A.8}$$

To ensure that the coefficient of σ^2 in Δ vanishes, we can set

$$\phi \sim \mu\phi_0^2 \ , \qquad\qquad f_1 \sim \tfrac{1}{2}\mu\phi_0\psi_1 \ , \tag{A.9}$$

with μ square-free (for appropriate locally-defined functions). Now, $\Delta_2 = 0$ is solved by

$$\widetilde{g}_2 = \tfrac{1}{4}\mu\psi_1^2 \ . \tag{A.10}$$

Summary for $\mathbf{I}_3$

$$\begin{aligned} f &= -\tfrac{1}{48}\mu^2\phi_0^4 + \tfrac{1}{2}\mu\phi_0\psi_1\sigma + f_2\sigma^2 + f_3\sigma^3 + O(\sigma^4) \ , \\ g &= \tfrac{1}{864}\mu^3\phi_0^6 - \tfrac{1}{24}\mu^2\phi_0^3\psi_1\sigma + \tfrac{1}{4}\left(\mu\psi_1^2 - \tfrac{1}{3}\mu\phi_0^2 f_2\right)\sigma^2 + \left(\widetilde{g}_3 - \tfrac{1}{12}\mu\phi_0^2 f_3\right)\sigma^3 + O(\sigma^4) \ , \\ \Delta &= \tfrac{1}{16}\mu^3\phi_0^3\left(\phi_0^3\widetilde{g}_3 - \psi_1^3 - \phi_0^2\psi_1 f_2\right)\sigma^3 + O(\sigma^4) \ . \end{aligned} \tag{A.11}$$

Now, in order to obtain type I_4 along $\{\sigma = 0\}$, we need to set

$$\Delta_3 = \tfrac{1}{16}\mu^3\phi_0^3\left(\phi_0^3\widetilde{g}_3 - \psi_1^3 - \phi_0^2\psi_1 f_2\right) = 0 \ . \tag{A.12}$$

By writing the terms in the parenthesis as $\psi_1^3 + \phi_0(\dots)$, we have that $\phi_0\big|_{\sigma=0}$ must divide $\psi_1\big|_{\sigma=0}$, i.e.

$$\psi_1 \sim -\tfrac{1}{3}\phi_0\phi_1 \ . \tag{A.13}$$

Now, $\Delta_3 = 0$ is solved by setting

$$\widetilde{g}_3 = -\tfrac{1}{3}\phi_1 f_2 - \tfrac{1}{27}\phi_1^3 \ . \tag{A.14}$$

Again, for each $i \geq 3$ we have terms in the expansion of Δ_i of the form

$$\Delta_i = 24 f_0 f_1 f_{i-1} + 27 g_0 \widetilde{g}_i + \cdots \ , \tag{A.15}$$

so that it is convenient to redefine

$$\widehat{g}_i = \widetilde{g}_i + \tfrac{1}{3}\phi_1 f_{i-1} \ , \qquad \widehat{f}_2 = f_2 + \tfrac{1}{3}\phi_1^2 \ . \tag{A.16}$$

Summary for $\mathbf{I}_4$

$$\begin{aligned}
f &= -\tfrac{1}{48}\mu^2\phi_0^4 - \tfrac{1}{6}\mu\phi_0^2\phi_1\sigma + (\widehat{f}_2 - \tfrac{1}{3}\phi_1^2)\sigma^2 + f_3\sigma^3 + f_4\sigma^4 + O(\sigma^5) \ , \\
g &= \tfrac{1}{864}\mu^3\phi_0^6 + \tfrac{1}{72}\mu^2\phi_0^4\phi_1\sigma + \tfrac{1}{6}\left(\tfrac{1}{3}\mu\phi_0^2\phi_1^2 - \tfrac{1}{2}\mu\phi_0^2\widehat{f}_2\right)\sigma^2 \\
&\quad + \left(-\tfrac{1}{3}\phi_1\widehat{f}_2 + \tfrac{2}{27}\phi_1^3 - \tfrac{1}{12}\mu\phi_0^2 f_3\right)\sigma^3 + \left(\widehat{g}_4 - \tfrac{1}{3}\phi_1 f_3 - \tfrac{1}{12}\mu\phi_0^2 f_4\right)\sigma^4 + O(\sigma^5) \ , \\
\Delta &= \tfrac{1}{16}\mu^2\phi_0^4\left(-\widehat{f}_2^2 + \mu\phi_0^2\widehat{g}_4\right)\sigma^4 + O(\sigma^5) \ .
\end{aligned} \tag{A.17}$$

Next, in order for Δ_4 to vanish along $\{\sigma = 0\}$, we require that $\mu\phi_0\big|_{\sigma=0}$ divides $\widehat{f}_2\big|_{\sigma=0}$, i.e.

$$\widehat{f}_2 \sim \tfrac{1}{2}\mu\phi_0\psi_2 \ , \tag{A.18}$$

for some locally defined function ψ_2. We obtain type I_5 along $\{\sigma = 0\}$ by setting

$$\widehat{g}_4 = \tfrac{1}{4}\mu\psi_2^2 \ . \tag{A.19}$$

Summary for $\mathbf{I}_5$

$$\begin{aligned}
f &= -\tfrac{1}{48}\mu^2\phi_0^4 - \tfrac{1}{6}\mu\phi_0^2\phi_1\sigma + (\tfrac{1}{2}\mu\phi_0\psi_2 - \tfrac{1}{3}\phi_1^2)\sigma^2 + f_3\sigma^3 + f_4\sigma^4 + f_5\sigma^5 + O(\sigma^6) \ , \\
g &= \tfrac{1}{864}\mu^3\phi_0^6 + \tfrac{1}{72}\mu^2\phi_0^4\phi_1\sigma + \tfrac{1}{6}\left(\tfrac{1}{3}\mu\phi_0^2\phi_1^2 - \tfrac{1}{4}\mu^2\phi_0^3\psi_2\right)\sigma^2 \\
&\quad + \left(-\tfrac{1}{6}\mu\phi_0\phi_1\psi_2 + \tfrac{2}{27}\phi_1^3 - \tfrac{1}{12}\mu\phi_0^2 f_3\right)\sigma^3 + \left(\tfrac{1}{4}\mu\psi_2^2 - \tfrac{1}{3}\phi_1 f_3 - \tfrac{1}{12}\mu\phi_0^2 f_4\right)\sigma^4 \\
&\quad + \left(\widehat{g}_5 - \tfrac{1}{3}\phi_1 f_4 - \tfrac{1}{12}\mu\phi_0^2 f_5\right)\sigma^5 + O(\sigma^6) \ , \\
\Delta &= \tfrac{1}{16}\mu^3\phi_0^4\left(\phi_0^2\widehat{g}_5 + \phi_1\psi_2^2 - \phi_0\psi_2 f_3\right)\sigma^5 + O(\sigma^6) \ .
\end{aligned} \tag{A.20}$$

We can go one step further and find a complete general form for type I_6 by setting $\Delta_5 = 0$, which requires

$$\phi_1\psi_2^2 - \phi_0\psi_2 f_3 + \phi_0^2\widehat{g}_5 = 0 \ . \tag{A.21}$$

Now, we factorize the roots of $\phi_0|_{\sigma=0}$ according to which ones divide $\phi_1|_{\sigma=0}$ or $\psi_2|_{\sigma=0}$, as

$$\phi_0 \sim \alpha\beta \ , \qquad \psi_2 \sim -\tfrac{1}{3}\alpha\phi_2 \ , \qquad \phi_1 \sim \beta\nu \ , \tag{A.22}$$

where α, β, ϕ_2 and ν are locally-defined functions, and $\alpha\big|_{\sigma=0}$ the greatest common divisor of $\phi_1\big|_{\sigma=0}$ and $\psi_2\big|_{\sigma=0}$. This implies that $\beta\big|_{\sigma=0}$ and $\phi_2\big|_{\sigma=0}$ do not share any common factor. Now $\Delta_5 = 0$ is satisfied by

$$f_3 \sim -\tfrac{1}{3}\nu\phi_2 - 3\beta\lambda \ , \qquad \widehat{g}_5 \sim \phi_2\lambda \ , \tag{A.23}$$

for some locally-defined function λ. Finally,

Summary for $\mathbf{I}_6$

$$\begin{aligned}
f =& -\tfrac{1}{48}\mu^2\alpha^4\beta^4 - \tfrac{1}{6}\mu\alpha^2\beta^3\nu\sigma + \tfrac{1}{3}(-\tfrac{1}{2}\mu\alpha^2\beta\phi_2 - \beta^2\nu^2)\sigma^2 \\
&+ (-\tfrac{1}{3}\nu\phi_2 - 3\beta\lambda)\sigma^3 + f_4\sigma^4 + f_5\sigma^5 + f_6\sigma^6 + O(\sigma^7) \ , \\
g =& \tfrac{1}{864}\mu^3\alpha^6\beta^6 + \tfrac{1}{72}\mu^2\alpha^4\beta^5\nu\sigma + \tfrac{1}{18}\left(\mu\alpha^2\beta^4\nu^2 + \tfrac{1}{4}\mu^2\alpha^4\beta^3\phi_2\right)\sigma^2 \\
&+ \left(\tfrac{1}{12}\mu\alpha^2\beta^2\nu\phi_2 + \tfrac{2}{27}\beta^3\nu^3 + \tfrac{1}{4}\mu\alpha^2\beta^3\lambda\right)\sigma^3 \\
&+ \left(\tfrac{1}{36}\mu\alpha^2\phi_2^2 + \tfrac{1}{9}\beta\nu^2\phi_2 + \beta^2\nu\lambda - \tfrac{1}{12}\mu\alpha^2\beta^2 f_4\right)\sigma^4 \\
&+ \left(\phi_2\lambda - \tfrac{1}{3}\beta\nu f_4 - \tfrac{1}{12}\mu\alpha^2\beta^2 f_5\right)\sigma^5 + \left(\widehat{g}_6 - \tfrac{1}{3}\beta\nu f_5 - \tfrac{1}{12}\mu\alpha^2\beta^2 f_6\right)\sigma^6 + O(\sigma^7) \ , \\
\Delta =& \tfrac{1}{432}\mu^2\alpha^4\beta^3\left(27\mu\alpha^2\beta^3\widehat{g}_6 + 9\mu\alpha^2\beta^2\phi_2 f_4 + \mu\alpha^2\phi_2^3\right. \\
&\left. -243\lambda^2\beta^3 + 54\phi_2\nu\lambda\beta^2 - 3\beta\nu^2\phi_2^2\right)\sigma^6 + O(\sigma^7) \ .
\end{aligned} \tag{A.24}$$

Solving algebraically $\Delta_6 = 0$ is a daunting challenge to treat in complete generality. In order to deal with type I_n for $n \geq 7$, we instead implement the inductive argument of [20], the results of which we state here.

Summary for $\mathbf{I}_i$, $i \geq 7$

$$\begin{aligned}
f &= -\tfrac{1}{3}u^2 + v \ , \\
g &= \tfrac{2}{27}u^3 - \tfrac{1}{3}uv + w \ , \\
\Delta &= 4u^3w - u^2v^2 - 18uvw + 4v^3 + 27w^2 \ ,
\end{aligned} \tag{A.25}$$

where $u,\ v$ and w are local functions such that

$$\sigma^{[i/2]} | \ v \ , \qquad \sigma^{2[i/2]} | \ w \ , \qquad \sigma \nmid u \ . \tag{A.26}$$

In the even case with $i = 2n$, the expansion of u in powers of σ takes the form

$$u = \tfrac{1}{4}\mu\phi_0^2 + u_1\sigma + \cdots + u_{n-1}\sigma^{n-1} \ , \tag{A.27}$$

where, as before, $\mu\big|_{\{\sigma=0\}}$ is square-free, while v and w are generic. In the odd case with $i = 2n+1$, these have instead the expansions

$$\begin{aligned} v &= \tfrac{1}{2}\mu\phi_0 t_n\sigma^n + v_{n+1}\sigma^{n+1} + \cdots \ , \\ w &= \tfrac{1}{4}\mu t_n^2\sigma^{2n} + w_{2n+1}\sigma^{2n+1} + \cdots \ , \end{aligned} \tag{A.28}$$

where t_n is a locally defined function. This is indeed the most general solution for I_m, $m \geq 10$. It is a solution, but not the most general one, for $m < 10$. We explicitly constructed above the most general solution for $m \leq 6$, so we have three cases, namely $m = 7, 8, 9$, for which the forms above from [6] may not capture the full story. In our present work, as in [6], we treat this cases with forms as described by (A.25)-(A.28).

Local models for intersections

We want to use the above expressions for type I_n in order to construct local models describing intersections of a divisor $\{\sigma = 0\}$ carrying type I_n and a divisor $\{z = 0\}$ carrying any other Kodaira type. In order to do this, let (a, b, d) be the degrees of vanishing of (f, g, Δ) along $\{z = 0\}$. Then we have the divisibility conditions

$$z^a |\ f \ , \qquad\qquad z^b |\ g \ , \qquad\qquad z^d |\ \Delta \ . \tag{A.29}$$

Imposing these conditions on each term of the expansions (A.1) leads in turn to certain divisibility conditions on the various local functions describing the local model for the intersection. In particular, we must check whether the point of intersection $P = \{z = \sigma = 0\}$ has degrees of vanishing equal or higher than $(4, 6, 12)$, in other words, if it describes a non-minimal intersection[3]. If that is that case, we discard this solution.

Monodromy

For type I_n, $n \geq 3$, there is a monodromy condition we need to test in order to fully specify the algebra, i.e. to determine whether the gauge algebra summand is either $\mathfrak{su}(n)$ or $\mathfrak{sp}([n/2])$. The condition for monodromy is determined by the function μ defined in (A.9): we are in the case without monodromy, i.e. the algebra is $\mathfrak{su}(n)$, when $\mu\big|_{\{\sigma=0\}}$

[3]We refer to this condition as the "(4,6,12) condition".

has no zeros, and conversely we are in the monodromy case when μ vanishes somewhere along $\{\sigma = 0\}$. As we will see in the example below, there are cases where the general form of the local model for the intersection forces μ to vanish somewhere. That is, we can prove that in order to have an allowed intersection the monodromy on the type I_n is fixed.

An example

Let us consider the intersection between a type I_4 along the divisor $\{\sigma = 0\}$ and a type I_0^* along the locus $\{z = 0\}$. We take the degrees of vanishing of f, g and Δ along $\{z = 0\}$ to be 2, 3 and 6, respectively. This means that

$$z^2|\ f\ , \quad \text{but } z^3 \nmid f\ , \qquad\qquad z^3|\ g\ , \quad \text{but } z^4 \nmid g\ . \tag{A.30}$$

Imposing these constraints on each term of the solution (A.17) implies that

$$z^2|\ f_i\ , \quad i \geq 3\ , \quad z^3|\ \widehat{g}_i\ , \quad i \geq 4\ , \quad z^2|\ (\widehat{f}_2 - \tfrac{1}{3}\phi_1^2) \quad \Longrightarrow \quad z^2|\ \widehat{f}_2\ ,\ z|\ \phi_1 \tag{A.31}$$

and either $z|\phi_0$ or $z|\mu$. We distinguish between these two cases:

1. $z|\ \phi_0$ and $z \nmid \mu$. Let us rewrite our quantities by factoring out the explicit partial z dependence determined above; for simplicity of notation, we indicate the functions with the same symbol, as

$$\begin{aligned} f &= -\tfrac{1}{48}z^4\mu^2\phi_0^4 - \tfrac{1}{6}z^3\mu\phi_0^2\phi_1\sigma + (z^2\widehat{f}_2 - \tfrac{1}{3}z^2\phi_1^2)\sigma^2 + z^2f_3\sigma^3 + z^2f_4\sigma^4 + z^2O(\sigma^5)\ , \\ g &= \tfrac{1}{864}z^6\mu^3\phi_0^6 + \tfrac{1}{72}z^5\mu^2\phi_0^4\phi_1\sigma + \tfrac{1}{6}z^2\mu\phi_0^2\left(\tfrac{1}{3}z^2\phi_1^2 - \tfrac{1}{2}z^2\widehat{f}_2\right)\sigma^2 \\ &\quad + \left(-\tfrac{1}{3}z^3\phi_1\widehat{f}_2 + \tfrac{2}{27}z^3\phi_1^3 - \tfrac{1}{12}z^4\mu\phi_0^2f_3\right)\sigma^3 + \left(z^3\widehat{g}_4 - \tfrac{1}{3}z^3\phi_1f_3 - \tfrac{1}{12}z^4\mu\phi_0^2f_4\right)\sigma^4 \\ &\quad + z^3O(\sigma^5)\ . \end{aligned} \tag{A.32}$$

   The degrees of vanishing of f and g at P are 4 and 6, respectively. This solution is therefore non-minimal and it is not allowed.

2. $z|\mu$ and $z \nmid \phi_0$. In this case, we have

$$
\begin{aligned}
f &= -\tfrac{1}{48}z^2\mu^2\phi_0^4 - \tfrac{1}{6}z^2\mu\phi_0^2\phi_1\sigma + (z^2\widehat{f}_2 - \tfrac{1}{3}z^2\phi_1^2)\sigma^2 + z^2 f_3\sigma^3 + z^2 f_4\sigma^4 + z^2 O(\sigma^5)\ , \\
g &= \tfrac{1}{864}z^3\mu^3\phi_0^6 + \tfrac{1}{72}z^3\mu^2\phi_0^4\phi_1\sigma + \tfrac{1}{6}z\mu\phi_0^2\left(\tfrac{1}{3}z^2\phi_1^2 - \tfrac{1}{2}z^2\widehat{f}_2\right)\sigma^2 \\
&\quad + \left(-\tfrac{1}{3}z^3\phi_1\widehat{f}_2 + \tfrac{2}{27}z^3\phi_1^3 - \tfrac{1}{12}z^3\mu\phi_0^2 f_3\right)\sigma^3 + \left(z^3\widehat{g}_4 - \tfrac{1}{3}z^3\phi_1 f_3 - \tfrac{1}{12}z^3\mu\phi_0^2 f_4\right)\sigma^4 \\
&\quad + z^3 O(\sigma^5)\ , \\
\Delta &= \tfrac{1}{16}z^2\mu^2\phi_0^4\left(-z^4\widehat{f}_2^2 + z^4\mu\phi_0^2\widehat{g}_4\right)\sigma^4 + z^6 O(\sigma^5)\ . \qquad \text{(A.33)}
\end{aligned}
$$

Now, the degree of vanishing of f and g at P are 2 and 3, respectively, and therefore, this is an allowed intersection. Moreover, the fact that $\mu\big|_{\{\sigma=0\}}$ vanishes along the locus $\{z=0\}$ implies that we have monodromy and the gauge algebra for the type I_4 is $\mathfrak{sp}(2)$.

Summary

Following the procedure described above, we can summarize our results for local models describing intersections of type I_n with curves of even and odd type as well as I_0^* (this case can be extended to $\mathrm{I}_{m\geq 1}^*$). Table A.1 describes the maximum n such that type I_n is allowed by the (4,6,12) condition to intersect the singularity types listed below. The symbol '-' indicates that this possibility is not realizable, for example in all odd types we cannot have $A \neq 0$.

	$A=B=0$	$A=1$	$A\geq 2$	$B=1$	$B\geq 2$
II^*	0	0	0	-	-
III^*	0	-	-	0	0
IV^*	1	1	1	-	-
I_0^*	∞	3	3	2	2
IV	∞	3	3	-	-
III	∞	-	-	∞	$5^!$
II	∞	∞	4	-	-

Table A.1: Allowed intersections for non-compact type I_n curves. Here a '!' superscript denotes a correction to the corresponding data in [6].

The results summarized in this table have some immediate important consequences. First, configurations $\mathcal{C}_{n_1,\cdots,n_N}$ of type I_{n_i} curves do not yield any non-abelian symmetry for type II^*, III^* and IV^*. Second, in the most interesting case of type I_0^*, Table A.1 provides strong constraints if either $A \neq 0$ or $B \neq 0$. In both cases, by using the monodromy result in Table A.2, the only admissible algebra summand is $\mathfrak{sp}(1)$. This is trivially a sub-case of the more general $A = B = 0$ setting, the case of primary interest in [6]. Attention to this case allows one to put all comparisons of global symmetries for given single curve theories on the same footing. Furthermore, those global symmetries which are realizable become maximal among all A, B possibilities in the 1D Coulomb branch case, making the significance of multiple relative global symmetry maxima clear. Treatment of the general A, B case however becomes necessary when there is more than one component of Δ in the base. This is perhaps most clear when we consider that many quivers demand such $A, B > 0$ assignments to achieve a specified gauge enhancement. Tabulation of the global symmetries which may be achieved for general A, B even for single curve theories thus becomes of interest. This data is an immediate outcome of our approach in the present work.

For each allowed model, we can compute the minimal local contributions and in some cases, as we have seen in the example above, we can gather information about the monodromy. Here we restrict to $A = B = 0$, since from the above this is the primary case in the 1D Coulomb branch case; generalizations to $A, B > 0$ are treated in Chapter 2. We restrict our attention to intersections for type I_n, $n \geq 2$, since these are the curves carrying a non-abelian gauge algebra. In Table A.2, we used b.p. to indicate that both monodromies on type I_n are possible.

A.2 The Tate algorithm for the I_n^* case

In this appendix, we collect the general forms for I_n^* for $n \geq 1$, taken from [20]. Let $\{\sigma = 0\}$ carry type I_n^*. We write the quantities f, g and Δ as expansions in the variable σ, where the coefficients are local functions in a sufficiently small open subset of $\{\sigma = 0\}$.

Summary for I_1^*

$$\begin{aligned} f &= -\tfrac{1}{3}u_1^2\sigma^2 + f_3\sigma^3 + O(\sigma^4)\ , \\ g &= \tfrac{2}{27}u_1^3\sigma^3 + (\widetilde{g}_4 - \tfrac{1}{3}u_1 f_3)\sigma^4 + O(\sigma^5)\ , \\ \Delta &= 4u_1^3\widetilde{g}_4\sigma^7 + O(\sigma^8)\ , \end{aligned} \tag{A.34}$$

where we set $\widetilde{g}_j = g_j + \tfrac{1}{3}u_1 f_{j-1}$ for $j \geq 4$.

	n	gauge algebra	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{d}$
I_0^*	≥ 2	$\mathfrak{sp}([n/2])$	0	0	n
IV	2	$\mathfrak{su}(2)$	0	2	4
IV	3	b.p.	1	2	4
IV	3	$\mathfrak{su}(2)$	0	3	6
IV	4	$\mathfrak{sp}(2)$	0	4	8
IV	5	$\mathfrak{sp}(2)$	0	5	10
IV	6	$\mathfrak{sp}(3)$	0	6	12
IV	≥ 7	$\mathfrak{sp}([n/2])$	0	n	$2n$
III	2	$\mathfrak{su}(2)$	1	1	3
III	3	b.p.	2	2	6
III	4	$\mathfrak{sp}(2)$	2	2	6
III	4	$\mathfrak{su}(4)$	2	3	6
III	5	$\mathfrak{sp}(2)$	3	3	9
III	6	$\mathfrak{sp}(3)$	3	3	9
III	≥ 7	$\mathfrak{sp}([n/2])$	$\lceil n/2 \rceil$	$\lceil n/2 \rceil$	$3\lceil n/2 \rceil$

Table A.2: Local contributions from non-compact type I_n curves.

Summary for $\mathbf{I}_2^*$

$$\begin{aligned} f &= -\tfrac{1}{3}u_1^2\sigma^2 + f_3\sigma^3 + f_4\sigma^4 + O(\sigma^5)\ , \\ g &= \tfrac{2}{27}u_1^3\sigma^3 - \tfrac{1}{3}u_1 f_3\sigma^4 + (\widetilde{g}_5 - \tfrac{1}{3}u_1 f_4)\sigma^5 + O(\sigma^6)\ , \\ \Delta &= u_1^2\left(4u_1\widetilde{g}_5 - f_3^2\right)\sigma^8 + O(\sigma^9)\ . \end{aligned} \tag{A.35}$$

Summary for $\mathbf{I}_3^*$

$$\begin{aligned} f &= -\tfrac{1}{48}\mu_1^2 s_0^4\sigma^2 + \tfrac{1}{2}\mu_1 s_0 t_2\sigma^3 + f_4\sigma^4 + f_5\sigma^5 + O(\sigma^6)\ , \\ g &= \tfrac{1}{864}\mu_1^3 s_0^6\sigma^3 - \tfrac{1}{24}\mu_1^2 s_0^3 t_2\sigma^4 + \tfrac{1}{4}\mu_1(t_2^2 - \tfrac{1}{3}s_0^2 f_4)\sigma^5 + (\widetilde{g}_6 - \tfrac{1}{12}\mu_1 s_0^2 f_5)\sigma^6 + O(\sigma^7)\ , \\ \Delta &= \tfrac{1}{16}\mu_1^3 s_0^3\left(\widetilde{g}_6 s_0^3 - f_4 s_0^2 t_2 - t_2^3\right)\sigma^9 + O(\sigma^{10})\ , \end{aligned} \tag{A.36}$$

where μ_1 is square-free.

The case $n \geq 4$ is fully captured by an induction argument similar to the one presented in the previous appendix for type I_n.

Summary for $\mathbf{I}_{n\geq 4}^*$

For the even case, $n = 2m$, we have

$$
\begin{aligned}
u &= u_1\sigma + u_2\sigma^2 + \cdots + u_m\sigma^m \ , \\
v &= v_{m+2}\sigma^{m+2} + v_{m+3}\sigma^{m+3} + \cdots \ , \\
w &= w_{2m+3}\sigma^{2m+3} + w_{2m+4}\sigma^{2m+4} + \cdots \ ,
\end{aligned} \tag{A.37}
$$

while for the odd case, $n = 2m + 1$, we have

$$
\begin{aligned}
u &= u_1\sigma + u_2\sigma^2 + \cdots + u_{m+1}\sigma^{m+1} \ , \\
v &= v_{m+3}\sigma^{m+3} + v_{m+4}\sigma^{m+4} + \cdots \ , \\
w &= w_{2m+4}\sigma^{2m+4} + w_{2m+5}\sigma^{2m+5} + \cdots \ ,
\end{aligned} \tag{A.38}
$$

where f, g and Δ are as in (A.25).

Summary of intersection restrictions

Here we summarize local models of intersections involving a type I_n^* curve and a non-compact transverse curve. Table A.3 gives upper bounds on n for the permitted intersections between I_n^* curve with transverse curves bearing various other Kodaira type singularities. The symbol '-' indicates that this possibility is not permitted for the indicated Kodaira type on the transverse curve, while the symbol 'X' represents a non-minimal intersection for all $n \geq 1$.

	$A = B = 0$	$A = 1$	$A \geq 2$	$B = 1$	$B \geq 2$
II^*	X	X	X	-	-
III^*	X	-	-	X	X
IV^*	X	X	X	-	-
I_0^*	X	X	X	X	X
$\mathrm{I}_{p\geq 1}^*$	X	-	-	-	-
IV	X	X	X	-	-
III	X	-	-	X	X
II	1	1	1	-	-

Table A.3: Intersections permitted for type I_n^*.

Appendix B

New restrictions on 6D SCFT gauge enhancements

In this appendix, we collect the discrepancies between gauge enhancements described in [16] and those found by tracking Kodaira type intersection restrictions. Briefly, the restrictions introduced in this work are (strictly) tighter, differing only in a minority of cases. To make this comparison, we employ the computer algebra routines accompanying [16] given in 'fiber_enhancements.nb' to produce the enhancements permitted by [16]. Our resulting comparisons are properly drawn after editing a few lines of code in 'fiber_enhancements.nb' as detailed in this section. To simplify our comparisons, we replace all occurrences of $\mathfrak{su}(2)$ with $\mathfrak{sp}(1)$.

B.1 Linear link enhancements

B.1.1 Consequences of $\mathfrak{so}(13)$ GS constraints

We first note the restriction that $\mathfrak{so}(13)$ can only occur on a curve of negative self-intersection $m = 2$ or $m = 4$. The F-theory global symmetry from [6] for such a curve, $\mathfrak{sp}(5)$ is independent of m, in contrast to the field theory global symmetry given by $\mathfrak{sp}(9+\Sigma^2)$. When $m = 4$, these agree, but for $m = 2$, we find restrictions beyond the [16] enhancements as a result. For the quivers 21 and $214\dots$, the enhancements allowed by [16] include $\mathfrak{so}(13), \mathfrak{sp}(7), (\dots)$. These are not compatible with the F-theory global symmetry restrictions (though permitted by field theory).

B.1.2 Link enhancement comparisons comments

The base 213

We have constructed a modification of the notebook 'fiber_enhancements.nb' which was supplied with [16]. Selected edits and repackaging allow us to make a direct comparison for link enhancements permitted via our algorithms and those of [16]. In this section, we detail the modifications we make to the core elements of 'fiber_enhancements.nb' and discuss the resulting comparisons for link gauge enhancements.

We first edit a loop index for assigning family of enhancements to the quiver 213 which produces $\mathfrak{sp}(0)$ and $\mathfrak{sp}(-1)$ in the function 'evenmorefor213.'

Among other enhancements permitted on this quiver, the configurations

$$\begin{array}{ccc} 2 & 1 & 3 \\ \mathfrak{su}(2) & \mathfrak{sp}(3) & \mathfrak{so}(11) \\ \mathfrak{su}(2) & \mathfrak{sp}(4) & \mathfrak{so}(11) \end{array} \tag{B.1}$$

also appear in the 'fiber_enhancements.nb' outputs before modification. Since these may not be allowed, we edit the first loop index edit as detailed shortly. Note that by inspection of [16] Figure 3, if we set $P_{\geq 2} = 0$ and $P_1 = 2$, i.e. if we study the configuration 21 with $\mathfrak{su}(2)$ on the -2, we find that on the -1 we have $\mathfrak{sp}(M_0)$ with $M_0 \leq 2$.

More generally, on a -2 with $\mathfrak{su}(n)$, the maximal intersection can be I_{2n}, that is $\mathfrak{sp}(n)$.

For the quiver 213, among the hard-coded triplets in 'fiber_enhancements.nb' is a double appearance of

$$\mathfrak{su}(2), n_0, \mathfrak{so}(11),$$

one occurrence of which we replace with $\mathfrak{su}(2), n_0, \mathfrak{so}(12)$ as the latter appears also obeys the E_8 gauging condition. Other permitted enhancements with bare -1 center curve which obey the $\mathfrak{e}_8$ gauging condition we also add for this reason include

$$\begin{array}{ccc} 2 & 1 & 3 \\ \mathfrak{so}(7) & n_0 & \mathfrak{g}_2 \\ \mathfrak{so}(8) & n_0 & \mathfrak{g}_2 \\ \mathfrak{so}(9) & n_0 & \mathfrak{g}_2 \\ \mathfrak{g}_2 & n_0 & \mathfrak{su}(3) \\ \mathfrak{so}(7) & n_0 & \mathfrak{su}(3) \\ \mathfrak{so}(8) & n_0 & \mathfrak{su}(3) \\ \mathfrak{so}(9) & n_0 & \mathfrak{su}(3) \\ \mathfrak{so}(10) & n_0 & \mathfrak{su}(3)\ . \end{array} \tag{B.2}$$

The aforementioned loop index edit for the 'evenmorefor213' loop revises the indexing to read

```
evenmorefor213 = Flatten[Table[{su[nl], sp[nm], so[nr]}, {nl, 2, 20},
    {nr, 7, 12},
    {nm, Max[Ceiling[(2 nl + nr - 16)/4], 1], Min[nl, nr - 7 + 2 KroneckerDelta[nr, 7]}]],
```

This bounds the loop to prevent the $\mathfrak{sp}(-1)$ and $\mathfrak{sp}(0)$. We revise the final loop index from 'Min[2 nl, nr - 7 + KroneckerDelta[nr, 7]', allowing agreement with p.49 [16] restrictions, the p.57 21(−) convexity condition, and the $\mathfrak{sp}(2), \mathfrak{so}(7)$ permitted assignment for the base 13, the latter taken care of via revision of the loop indices of 'morefor213' to allow $\mathfrak{sp}(2), \mathfrak{so}(7)$ on 13. In hopes of keeping all trivial algebra assignments to a curve consistently written as n_0 terms rather than allowing $\mathfrak{sp}(0)$ terms, we treat all such additions by adding hard-coded triplets. A final change we make to this loop index is an adjustment to make the upper index to be larger, the result involving a replacement of a Kronecker symbol with its double as indicated by the '†' symbol below. The edited loop then reads

```
morefor213 =
    Flatten[Table[{so[nl], sp[nm], so[nr]},
    { nl, 7, 13},
    { nr, 7, 12},
    { nm,
      Max[Ceiling[(nl + nr - 16 + KroneckerDelta[nr, 7])/4], 1],
```

Min[nl - 6 + 3 KroneckerDelta[nl, 7], nr - 7 + $2^{\dagger}$ KroneckerDelta[nr, 7]]}], 2].

The base 13

We edit the hard-coded algebras for the base 13 to include $\mathfrak{sp}(2), \mathfrak{so}(7)$ as an enhancement since this appears to be permitted by all anomaly cancellation and geometric restrictions known to us.

B.1.3 Check of above edits

Running the independent approach of checking Kodaira types with permitted vanishing counts and all intersections obeying global symmetry constraints gives agreement on the full set of linear links *after* making the edits above.

B.2 Enhancements of branching links

B.2.1 Barring a $\mathfrak{g}_2$ trifecta branching from $\mathfrak{f}_4$

There are a number of branching links where enhancements are hard-coded to allow the subconfiguration

$$\begin{array}{ccccc} \underset{\mathfrak{g}_2}{3} & 1 & \underset{\mathfrak{f}_4}{5} & 1 & \underset{\mathfrak{g}_2}{3} \\ & & 1 & & \\ & & \underset{\mathfrak{g}_2}{3} & & \end{array}$$

that we can exclude by residuals counts together with Kodaira type requirements. Note that for a gaugeless curve to support an $\mathfrak{f}_4$,$\mathfrak{g}_2$ neighbor pair, it must be a type II curve. This leaves only

$$\begin{array}{ccccc} 3 & 1 & 5 & 1 & 3 \\ & & 1 & & \\ & & 3 & & \\ (\mathrm{I}_0^{*ns},\mathfrak{g}_2) & (\mathrm{II},\mathrm{n}_0) & (\mathrm{IV}^{*ns},\mathfrak{f}_4) & (\mathrm{II},\mathrm{n}_0) & (\mathrm{I}_0^{*ns},\mathfrak{g}_2) \\ & & (\mathrm{II},\mathrm{n}_0) & & \\ & & (\mathrm{I}_0^{*ns},\mathfrak{g}_2) & & \end{array}$$

for the possibility of Kodaira type assignments. However, even this is not allowed since there are four available vanishings of $\widetilde{\Delta}$ along the -5 curve with $\mathfrak{f}_4$ algebra, though each type II curve intersection contributes two vanishings.

Consequences

This eliminates several enhancements permitted by [16], precisely one on each of the subquivers containing the above enhancement on the relevant subquiver. All such links where these are permitted via [16] can be obtained by truncation of the configuration

$$\begin{array}{cccccccc} 2 & 2 & \underset{\mathfrak{g}_2}{3} & 1 & \underset{\mathfrak{f}_4}{5} & 1 & \underset{\mathfrak{g}_2}{3} & 2 \\ & & & & 1 & & & \\ & & & & \underset{\mathfrak{g}_2}{3} & & & \end{array}$$

B.2.2 Two remaining branching link enhancement mismatches

An additional branching link enhancement

The enhancement given by

$$\begin{array}{ccc} (2,\mathfrak{sp}(1)) & (3,\mathfrak{so}(7)) & (1,-) \\ & (1,-) & \end{array}$$

is not allowed via 'fiber_enhancements.nb' of [16] but appears to be viable.

A restricted branching link enhancement

The enhancement

$$\begin{array}{cccc} (2,\mathfrak{sp}(1)) & (3,\mathfrak{so}(7)) & (1,-) & (3,\mathfrak{so}(9)) \\ & (2,\mathfrak{sp}(1)) & & \end{array}$$

is permitted in [16] 'fiber_enhancements.nb' as it appears to pass the E_8 gauging condition check and anomaly cancellation requirements. However, we can rule out this enhancement through consideration of Kodaira types which realize the relevant algebras as follows.

First, we note that for an intersection with the -2 curves having $\mathfrak{so}(7)$ algebras these curves must have Kodaira type III (IV is not allowed for other reasons but would still prevent the configuration). This means that of the 6 residuals in $\widetilde{\Delta}$ available at the -3 curve, we have used three for each -2 curve intersection, leaving none available for the -1 curve to carry any type other than I_0. However, the -1 curve cannot have type I_0 since it is required to have $6 + 7$ vanishings of its $\widetilde{\Delta}$ to support intersections with its neighboring curves which necessarily have Kodaira types I_0^{*ss} and I_1^{*ns} from their algebra content and hence orders contributions to the -1 given by at least 6 and 7, respectively.

B.2.3 Summary for branching links

On all enhancements other than those mentioned above, we find agreement with those appearing from 'fiber_enhancements.nb' from [16].

B.3 Summary of link comparisons

After compensating for the aforementioned differences, we find agreement with the gauge enhancement structure on all links. In other words, we have explicitly checked that there is a choice of Kodaira assignment and chosen orders of f, g, Δ along each curve of the link which meet all intersection contribution tallying requirements and global symmetry restrictions known to us.

Appendix C

Algorithm summary and implementation

In this section we collect methods written for a computer algebra system. These may allow adaptation for other purposes, though the primary focus in their development has been the computation of 6D SCFT global symmetries. There are three main groupings of methods. The first collection of methods permits computation of the gauge algebras allowed in 6D SCFTs from geometric considerations. The next collection of methods permits handling of semisimple lie algebra inclusion rules in a manner compatible with the goals of this project. The final grouping serves for the study of global symmetries. Several methods in the first grouping also play a critical dual purpose role. The reason for this is that many of the restrictions on which Kodaira types may be paired in curve collisions often hold even when one of the curves is non-compact.

A summary of the algorithm determining gauge enhancements and global symmetries for each enhancement on a quiver follows. In practice, the methods are somewhat more complicated than indicated due to the necessity of computational efficiency boosts. These involve 'sewing' results by combining shorter quivers together to treat longer quivers, storage of partial results during computation, writing data to file for later use and presentation in text, and enabling parallel computation. Since these are non-essential to the underlying algorithm, we will not discuss these aspects here, instead providing the precise work-flow via inclusion of functioning computer algebra code implementing the procedure.

Given a quiver, [1] our algorithm finds all gauge algebras on a quiver while tracking the Kodaira types achieving these algebras. For each of these type assignments, it will then find all possible global symmetry algebras. This process begins by assigning all possible orders of vanishing to each of the curves in the quiver compatible with non-minimality up to some user specified maximum values for additional orders of vanishing in f, g for each Kodaira type. Each type assignment is then endowed with every naively possible monodromy assignment. Each type with monodromy assignment then is sent for tabulation of intersection contributions. If these contributions are less than the maximum number of contributions allowed on each curve in the quiver, this assignment is checked against known restrictions for pairwise intersections of compact curves with gauge algebra assignments. Passing pairs move to the next stage. Each length three subquiver is then checked against known restrictions on length three quiver Kodaira type and gauge assignments. Branching quivers are then checked against restrictions of transverse trios meeting each junction derived in [6, 24]. The resulting list determines the allowed gauge enhancements for the given quiver.

After gauge assignments are completed, the resulting permitted list of type assignments with associated gauge algebra options are checked for allowed global symmetry options. This is achieved by assigning every possible transverse collection of Kodaira types carrying non-trivial gauge algebra at each curve in the quiver. Each collection is checked against known pair restrictions for compact and non-compact pair intersections. Then intersection contributions from collection of curves transverse to a given curve is tabulated after forming all possible transverse curve lists on the entire quiver. If the resulting contributions are less than those allowed at each curve in the quiver, the collection proceeds to further checks and is otherwise discarded. Each pair, triple, and quartet of curves transverse to the curves of the quiver are checked against known global symmetry restrictions for curves with specified enhancement. The resulting list determines all possible global symmetry summands.

Each non-compact collection for a given enhancement is then associated to an algebra. The relatively maximal algebras of those that may occur for the enhancement are determined via semisimple Lie algebra inclusion rules and analyses of maximal semisimple Lie subalgebras due to [11] (available in translation as [10]). These are the global

[1] Rogue bases make the statement here less clean; so, let us separate the statements for these quivers. The core algorithm applies identically for rogue bases. However, since we include an argument giving an upper bound on the algebra rank which shall occur in any enhancement to make the algorithm finite through bypassing the infinity of enhancements permitted on a rogue base, we require somewhat awkward statements extending those we provide in the obvious fashion with this caveat of user maximal algebra rank upper bound entry.

symmetry algebras for the enhancement. Those non-compact collections with strictly smaller associated algebras are discarded and the resulting list returned.

The methods achieving this algorithm appear in the following subsection, written in 'Mathematica' syntax. A repackaged version of the program provided in conjunction with [16] is included along with methods allowing direct comparisons of enhancements described in the literature and those we compute via other methods enabling tracking of geometric data not previously available.[2] We essentially find agreement for the enhancements permitted on all links with those permitted via [16] and, as a consequence, also for all known 6D SCFTs with a handful of exceptions where we find slightly tighter geometric constraints on F-theory bases as detailed in Appendix B.

C.0.1 Computer algebra implementation of gauge and global symmetry computations

We now present a collection of computer algebra routines which enable various computations including the main algorithms discussed above. These have not been optimized for computational efficiency. As noted above, some streamlining has been achieved via 'sewing' of results on shorter quivers, resulting in a reduced algorithmic complexity making computations for links feasible on a typical laptop circa 2016.

[2]Minor revisions have been made in the repackaged version to correct a faulty loop index and a few hard-coded triplets as required to match the literature. These are indicated in the code comments and detailed in Appendix B.

```
(*******************************************************************************************
       SUMMARY.
       The focus here is to compute gauge and global symmetry for all enhancements of F-theory bases.
   *******************************************************************************************)

(* CLEAR VARIABLES AND SET OPTIONS. SET WORKING DIRECTORY FOR READING DATA SAVED TO FILE. *)


ClearAll[ "Global`*"];


Quiet[CloseKernels[];
   ClearAll[ "Global`*"];
  LaunchKernels[];];
Pause[0.5];
ParallelEvaluate[
  $HistoryLength = 0;
 ];
$HistoryLength = 0;

SetDirectory[NotebookDirectory[]];
SetOptions[$FrontEnd, FontSize → 9];
SetOptions[EvaluationNotebook[], FontSize → 9];

outputTexFileName = "gsTexTablesv10.tex";
outputTemplateForTexFileName = "outputTemplateFileGS.txt";

(* OPTION SETTINGS/VARIABLE INITIALIZE *)


Pause[1];
toWriteFileSavingHashTable = False;
toWriteFileWithSavedMonodromyHashTable = False;
toWriteFileWithAlgCache = True;
toReadFileWithSavedHashTable = False;
toReadFileWithSavedMonodromyHashTable = False;
toReadFileWithAlgCache = True;

babyTest = False;
distinguishSu2FromSp1 = False;
recomputeAtomicEnhancementsRatherThanReadFromFile = False;
toSaveRecomputedAtomicEnhancementsToFile = True;
atomicEnhancementsFileNameIn = "atomicEnhancementsRandomSample.mx";
atomicEnhancementsFileNameOut = "atomicEnhancementsRandomSample.mx";

(* IN file names *)
monodromyHashFileNameIn = "hashTablesMonodromy06_03_2017v1.mx";
enhancementHashFileNameIn = "hashTablesEnhancements06_03_2017v1.mx";
algebraCacheFileNameIn = "cacheTablesAlgs06_03_2017v1.mx";
(* OUT file names *)
monodromyHashFileNameOut = "hashTablesMonodromy06_03_2017v1.mx";
enhancementHashFileNameOut = "hashTablesEnhancements06_03_2017v1.mx";
algebraCacheFileNameOut = "cacheTablesAlgs06_03_2017v1.mx";

If[babyTest,
  atomicEnhancementsFileNameIn = "atomicEnhancementsBaby.mx";
  atomicEnhancementsFileNameOut = "atomicEnhancementsBaby.mx";
  monodromyHashFileNameIn = "hashTablesMonodromyBaby.mx";
  enhancementHashFileNameIn = "hashTablesEnhancementsBaby.mx";
  monodromyHashFileNameOut = "hashTablesMonodromyBaby.mx";
  enhancementHashFileNameOut = "hashTablesEnhancementsBaby.mx";
 ];


SetSharedFunction[WriteSharedMonodromyTable, EnhancementToMonodromyFromTable, WriteStoredQuiversTable,
  WriteListToStoredQuiversTable, ReadStoredQuiversTable, cacheOfAlgebrasToAllItsSubAlgebras,
  cacheOfAlgebrasToItsMaximalSubAlgebras, cacheOfRelMaxProperSubAlgsOfSemiSimple,
  indexOfAlgebrasForWhichAllSubsKnown, ReadAlgToIndexTableForWhichAllSubsAreKnown,
  WriteAlgToIndexTableForWhichAllSubsAreKnown, ReadAlgToAllSubalgebrasTable,
  WriteAlgToAllSubalgebrasTable, ReadAlgToIndexTableForWhichAllMaxSubsAreKnown,
  WriteAlgToIndexTableForWhichAllMaxSubsAreKnown, ReadAlgToAllMaxSubalgebrasTable,
  WriteAlgToAllMaxSubalgebrasTable, indexOfAlgebrasForWhichAllMaxSubsKnown
 ];
SetSharedFunction[GiveEnhancementTypes, ReadHashKeyTable, ReadHash, HashIt ];

SetSharedVariable[numberOfQuiversStored, toHash];
SetSharedVariable[nonCompactCurveContributionsErrorMessageHasBeenPrinted, quiversFinished,
  globalCounter, readHashInProgress, writeHashInProgress, readHashKeyInProgress, writeHashKeyInProgress,
  readHashInProgressEnhToMon, writeHashInProgressEnhToMon, readHashInProgressStoredQuivers,
  writeHashInProgressStoredQuivers, readHashInProgressAlgToAllSubsIndex,
```

```
  writeHashInProgressAlgToAllSubsIndex, readHashInProgressAlgToAllSubs, writeHashInProgressAlgToAllSubs,
  readHashInProgressAlgToAllMaxSubsIndex, writeHashInProgressAlgToAllMaxSubsIndex,
  readHashInProgressAlgToAllMaxSubs, writeHashInProgressAlgToAllMaxSubs];


readHashInProgressEnhToMon = False;
writeHashInProgressEnhToMon = False;
readHashInProgressStoredQuivers = False;
writeHashInProgressStoredQuivers = False;

writeHashInProgress = False;
readHashInProgress = False;
writeHashKeyInProgress = False;
readHashKeyInProgress = False;

readHashInProgressAlgToAllSubsIndex = False;
writeHashInProgressAlgToAllSubsIndex = False;
readHashInProgressAlgToAllSubs = False;
writeHashInProgressAlgToAllSubs = False;

readHashInProgressAlgToAllMaxSubsIndex = False;
writeHashInProgressAlgToAllMaxSubsIndex = False;
readHashInProgressAlgToAllMaxSubs = False;
writeHashInProgressAlgToAllMaxSubs = False;

hashKeyTable = {};
globalCounter = 12;
hashTable = {};
storedQuiversTable[] = {};
enhToMonodromyOptionsTable[] = {};
quiversFinished = 0;
numberOfQuiversStored = 0(* This get's incremented as we move along*);
toHash = True (* gets turned off after the max number are hashed *);
nonCompactCurveContributionsErrorMessageHasBeenPrinted = False;
(* This ends the shared variables and functions definitions *)

hashingPauseLength = 0.000000001;
hashKeyReadPauseLength = 0.0000000001(*raise these if performance *);
hashingReadPauseLengthEnhToMon = 0.000000001;
hashingWritePauseLengthEnhToMon = 0.00000000001;
hashingReadPauseLengthStoredQuivers = 0.00000001;
hashingWritePauseLengthStoredQuivers = 0.0000000001;

hashingReadPauseLengthAlgToAllSubs = 0.00000001;
hashingWritePauseLengthAlgToAllSubs = 0.00000001;
hashingReadPauseLengthAlgToAllSubsIndex = 0.00000001;
hashingWritePauseLengthAlgToAllSubsIndex = 0.00000001;

hashingReadPauseLengthAlgToAllMaxSubs = 0.00000001;
hashingWritePauseLengthAlgToAllMaxSubs = 0.00000001;
hashingReadPauseLengthAlgToAllMaxSubsIndex = 0.00000001;
hashingWritePauseLengthAlgToAllMaxSubsIndex = 0.00000001;

maxNumberOfQuiversToHash = 20000;
maxLengthToBeHashed = 250 (* max length of quivers to be hashed *);
minLengthToBeHashed = 1;
minLengthToBeHashedEnhToMon = 2;
maxLengthToBeHashedEnhToMon = 250;
minLengthToSew = 4;
maxAlgRankStoredAllMaxSubs = 27;
maxAlgRankStoredAllMaxProperSubs = 27;
maxAlgRankStoredAllSubAlgs = 27;

lastTime = AbsoluteTime[];
launchTime = AbsoluteTime[];

DistributeDefinitions[
  hashingPauseLength ,
  hashKeyReadPauseLength ,
  hashingReadPauseLengthEnhToMon ,
  hashingWritePauseLengthEnhToMon ,
  hashingReadPauseLengthStoredQuivers ,
  hashingWritePauseLengthStoredQuivers ,
  maxNumberOfQuiversToHash ,
  lastTime ,
  launchTime,
  minLengthToBeHashed ,
  maxLengthToBeHashed ,
  minLengthToSew,
  minLengthToBeHashedEnhToMon,
```

```
  maxLengthToBeHashedEnhToMon,
  maxAlgRankStoredAllMaxSubs,
  maxAlgRankStoredAllMaxProperSubs,
  maxAlgRankStoredAllSubAlgs,
  hashingReadPauseLengthAlgToAllSubs,
  hashingWritePauseLengthAlgToAllSubs ,
  hashingReadPauseLengthAlgToAllSubsIndex ,
  hashingWritePauseLengthAlgToAllSubsIndex
 ];


(* The following global variables should be set by the user to bound the
 highest n for type In and the highest m for Imstar for use in the BMM code. *)

maxNForIn = 19;
maxNForInstar = 7;
globalAdditionalAMax = 5;
globalAdditionalBMax = 6;
maxNForInGS = 19;
maxNForInstarGS = 14;
maxAforIzerostarGS = 2;
maxBforIzerostarGS = 2;
maxBforIIIGS = 1;
maxAforIVGS = 1;
maxAforIVStarGS = 1;


maxAforIzero     = (* 4+ *) globalAdditionalAMax;
maxBforIzero     = (* 6+ *) globalAdditionalBMax;
maxAforII        = globalAdditionalAMax;
maxBforIII       = globalAdditionalBMax;
maxAforIV        = globalAdditionalAMax;
maxAforIzerostar = globalAdditionalAMax;
maxBforIzerostar = globalAdditionalBMax;
maxAforIVstar    = globalAdditionalAMax;
maxBforIIIstar   = globalAdditionalBMax;
maxAforIIstar    = globalAdditionalAMax;

(* Define the list of quivers of interest. *)


(* In babyTest mode the shorter list below is used.  *)



(* We start by defining a global variable to be used elsewhere which is
 the list available type assignments for each specified self intersection.  *)

(* Some example quivers and a loop to sew them together into a longer test list *)
```

```
quiverListFromAtomic[1] = { {1, 3, 1}, {1, 2, 3, 1}, {1, 2, 3, 2, 1}, {1, 2, 2, 3, 1},
    {3, 2, 2, 1}, {2, 3, 1, 3, 2, 2, 1}, {2, 2, 3, 1, 3, 2, 2, 1}, {3, 1, 3, 2, 2, 1}, {3, 2, 1}};
quiverListFromAtomic[2] = { {2, 3, 2, 1}, {2, 3, 1, 3, 2, 1}, {2, 2, 3, 1, 3, 2, 1}, {3, 1, 3, 2, 1},
    {3, 1}, {2, 3, 1, 3, 1}, {2, 2, 3, 1, 3, 1}, {3, 1, 3, 1}, {2, 3, 1}, {2, 2, 3, 1} };
quiverListFromAtomic[3] = { {2, 3, 1, 3, 2}, {2, 2, 3, 1, 3, 2}, {3, 1, 3, 2}, {2, 1, 3, 2}, {3, 1, 2, 3},
    {2, 2, 3, 1, 3, 2, 2}, {3, 1, 3, 2, 2}, {2, 1, 3, 2, 2} , {3, 1, 3}, {2, 3, 2, 1, 3}, {3, 1, 2} };
quiverListFromAtomic[4] = { {1, 3, 1, 5, 1, 3, 1}, {1, 2, 3, 1, 5, 1, 3, 1}, {1, 2, 3, 1, 5, 1, 3, 2, 1},
    {1, 2, 2, 3, 1, 5, 1, 3, 1}, {1, 2, 2, 3, 1, 5, 1, 3, 2, 1}, {1, 2, 2, 3, 1, 5, 1, 3, 2, 2, 1}};

quiverListFromAtomic[5] =
   {{ 1, 5, 1, 3, 2, 2, 1}, {1, 5, 1, 3, 2, 1}, {1, 5, 1, 2, 3, 2, 1}, {5, 1, 3, 2, 2, 1}, {3, 2, 1, 5, 1, 3, 2, 2, 1},
     {2, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 2, 3, 1, 5, 1, 3, 2, 2, 1}, {3, 1, 5, 1, 3, 2, 2, 1},
    {2, 3, 2, 1, 5, 1, 3, 2, 2, 1}, {5, 1, 3, 2, 1}, {5, 1, 2, 3, 2, 1},
    {3, 2, 1, 5, 1, 3, 2, 1}, {2, 3, 1, 5, 1, 3, 2, 1}, {2, 2, 3, 1, 5, 1, 3, 2, 1}, {3, 1, 5, 1, 3, 2, 1},
    {2, 3, 2, 1, 5, 1, 3, 2, 1}, {2, 3, 1, 5, 1, 2, 3, 2, 1}, {2, 2, 3, 1, 5, 1, 2, 3, 2, 1},
    {3, 1, 5, 1, 2, 3, 2, 1}, {5, 1, 3, 1}, {3, 2, 1, 5, 1, 3, 1}, {2, 3, 1, 5, 1, 3, 1},
    {2, 2, 3, 1, 5, 1, 3, 1}, {3, 1, 5, 1, 3, 1}, {2, 3, 2, 1, 5, 1, 3, 1}, {5, 1, 2, 3, 1}, {1, 5, 1, 2, 3, 1},
    {5, 1, 2, 2, 3, 1}, {2, 1, 5, 1, 3, 1}, {2, 1, 5, 1, 3, 2, 1}, {2, 1, 5, 1, 3, 2, 2, 1}};
quiverListFromAtomic[6] = { {1, 3, 2, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 2, 3, 1}, {3, 1, 5, 1, 2, 3, 1}};

quiverListFromAtomic[7] =
   { {3, 2, 1, 5}, {2, 3, 1, 5}, {3, 2, 2, 1, 5}, {2, 2, 3, 1, 5}, {3, 1, 5}, {2, 3, 2, 1, 5}, {5, 1, 2},
    {5, 1, 2, 2}, {5, 1, 2, 2, 2}, {3, 1, 3, 2, 1, 5}, {2, 3, 1, 3, 1, 5}, {3, 1, 3, 1, 5}, {3, 2, 1, 5, 1, 3, 2},
    {2, 3, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 3, 2}, {3, 1, 5, 1, 3, 2}, {2, 3, 2, 1, 5, 1, 3, 2}, {2, 1, 5, 1, 3, 2},
    {2, 2, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 2, 3}, {3, 1, 5, 1, 2, 3}, {2, 1, 5, 1, 2, 3}, {2, 2, 3, 1, 5, 1, 3, 2, 2},
    {3, 1, 5, 1, 3, 2, 2}, {2, 3, 2, 1, 5, 1, 3, 2, 2}, {2, 1, 5, 1, 3, 2, 2}, {2, 2, 1, 5, 1, 3, 2, 2},
    {3, 1, 5, 1, 3}, {2, 3, 2, 1, 5, 1, 3}, {2, 1, 5, 1, 3}, {2, 2, 1, 5, 1, 3}, {2, 1, 5, 1, 2, 3, 2},
    {2, 1, 5, 1, 2}, {3, 1, 3, 1, 5, 1, 3, 2}, {3, 1, 3, 1, 5, 1, 3, 2, 2}, {3, 1, 3, 1, 5, 1, 3}};
quiverListFromAtomic[8] = {{1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {1, 5, 1, 3, 1, 5, 1, 3, 2, 1},
    {1, 5, 1, 3, 1, 5, 1, 3, 1}, {5, 1, 2, 3, 1, 5, 1, 3, 2, 2, 1}, {5, 1, 3, 1, 5, 1, 3, 2, 2, 1},
    {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1},
    {3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {5, 1, 2, 3, 1, 5, 1, 3, 2, 1}, {5, 1, 3, 1, 5, 1, 3, 2, 1},
    {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1},
    {5, 1, 2, 3, 1, 5, 1, 3, 1}, {5, 1, 3, 1, 5, 1, 3, 1}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 1} };
quiverListFromAtomic[9]  = {{2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1},
    {1, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2}, {1, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}};
quiverListFromAtomic[10] = { {5, 1, 3, 2, 1, 5}, {5, 1, 3, 2, 2, 1, 5}, {5, 1, 3, 1, 5}, {5, 1, 2, 3, 2, 1, 5},
    {2, 3, 1, 5, 1, 3, 2, 1, 5}, {2, 2, 3, 1, 5, 1, 3, 2, 1, 5}, {3, 1, 5, 1, 3, 2, 1, 5}, {2, 1, 5, 1, 3, 2, 1, 5},
    {3, 1, 5, 1, 2, 3, 1, 5}, {2, 3, 1, 5, 1, 3, 2, 2, 1, 5}, {2, 2, 3, 1, 5, 1, 3, 2, 2, 1, 5},
    {3, 1, 5, 1, 3, 2, 2, 1, 5}, {3, 2, 1, 5, 1, 3, 1, 5}, {2, 3, 1, 5, 1, 3, 1, 5}, {3, 1, 5, 1, 3, 1, 5},
    {2, 1, 5, 1, 3, 1, 5}, {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 2},
    {2, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}, {2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}, {3, 1, 5, 1, 3, 1, 5, 1, 3},
    {2, 1, 5, 1, 3, 1, 5, 1, 3}, {5, 1, 3, 1, 5, 1, 3, 2, 1, 5}, {5, 1, 3, 1, 5, 1, 3, 1, 5} };
quiverListFromAtomic[11] = {{2, 2, 3}, {2, 3, 2}, {2, 3}};
 quiverListFromAtomic[12] = {{5, 1}, {1, 5, 1}, {1, 5, 1, 3}, {1, 5, 1, 3, 1}, {1, 5, 1, 3, 1, 5},
    {1, 5, 1, 3, 1, 5, 1}, {1, 5, 1, 3, 1, 5, 1, 3}, {1, 5, 1, 3, 1, 5, 1, 3, 2}, {1, 5, 1, 2}};

(* JoinQuiverLists:  *)


JoinQuiverLists [numberOfLists_] := Module[{i, longListOfQuivers},
    longListOfQuivers = {};
    For[i = 1, i ≤ numberOfLists, i ++,
      longListOfQuivers = Join[longListOfQuivers, quiverListFromAtomic[i]];
      ];
    Return[longListOfQuivers];
   ];

(* longListOfQuivers= JoinQuiverLists[11] consolidates the atomic appendix quiver
  groupings above. A few extras are included to make sewing computations quicker. *)
```

```
longListOfQuivers = {{2, 3, 1, 5}, {5, 1, 3, 1, 5}, {2, 2, 3, 1, 5}, {3, 1, 5}, {5, 1, 3, 2, 1, 5}, {2, 2, 3},
   {2, 3, 1, 5, 1, 3, 2}, {5, 1, 3, 2, 2, 1, 5}, {5, 1, 2, 2, 3}, {2, 2, 3, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 3, 2, 2},
   {2, 3, 1, 5, 1, 3, 1, 5}, {2, 3}, {2, 3, 2, 1, 5}, {2, 3, 1, 5, 1, 3, 2, 2, 1, 5}, {5, 1, 3, 2, 2, 1},
   {2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}, {2, 3, 1, 5, 1, 3, 2, 1, 5}, {2, 2, 3, 1, 5, 1, 3, 2, 2, 1, 5},
   {2, 3, 2, 1, 5, 1, 3, 2}, {5, 1, 2, 3, 2, 1, 5}, {5, 1, 2, 3}, {5, 1, 3, 1, 5, 1, 3, 1, 5},
   {2, 2, 3, 1, 5, 1, 3, 2, 1, 5}, {3, 1, 5, 1, 3, 2}, {2, 3, 1, 5, 1, 3, 2, 2, 1}, {3, 1, 5, 1, 3, 2, 2},
   {3, 1, 5, 1, 3, 1, 5}, {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 3, 2, 1, 5, 1, 3, 2, 2},
   {3, 2, 2, 1}, {5, 1, 3, 1, 5, 1, 3, 2, 1, 5}, {2, 3, 2}, {2, 2, 3, 1, 3, 2, 2}, {3, 1, 5, 1, 3},
   {3, 1, 5, 1, 3, 2, 2, 1, 5}, {3, 1, 5, 1, 3, 2, 1, 5}, {5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {3, 1, 5, 1, 2, 3, 1, 5},
   {5, 1, 3, 2, 1}, {2, 3, 2, 1, 5, 1, 3}, {3, 2, 1, 5, 1, 3, 2}, {2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1},
   {2, 2, 3, 1, 3, 2}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 2}, {5, 1, 2, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 2, 3, 1, 5, 1, 2, 3},
   {1, 2, 2, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {3, 1, 5, 1, 3, 2, 2, 1},
   {5, 1, 3, 1}, {5, 1, 2, 3, 2, 1}, {2, 3, 1, 5, 1, 3, 2, 1}, {3, 1, 5, 1, 2, 3}, {2, 3, 1, 5, 1, 2, 3, 2, 1},
   {2, 2, 3, 1, 3, 2, 2, 1}, {2, 3, 2, 1, 5, 1, 3, 2, 2, 1}, {2, 2, 3, 1, 5, 1, 3, 2, 1}, {3, 2, 1, 5, 1, 3, 1, 5},
   {2, 2, 3, 1}, {5, 1, 2, 2, 3, 1}, {3, 2, 1}, {3, 1, 3, 2, 2}, {3, 1, 5, 1, 3, 1, 5, 1, 3}, {2, 3, 1, 3, 1, 5},
   {5, 1, 3, 1, 5, 1, 3, 2, 1}, {2, 2, 3, 1, 5, 1, 2, 3, 2, 1}, {3, 1}, {5, 1, 2}, {5, 1, 3, 1, 3},
   {2, 3, 1, 3, 2, 2, 1}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1}, {2, 3, 2, 1}, {5, 1, 2, 3, 1, 5, 1, 3, 2, 1},
   {2, 2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1}, {3, 2, 1, 5, 1, 3, 2, 2, 1}, {2, 3, 2, 1, 5, 1, 3, 2, 1},
   {3, 1, 3}, {3, 1, 5, 1, 3, 2, 1}, {1, 5, 1, 3, 2, 2, 1}, {2, 3, 1, 3, 2}, {3, 1, 5, 1, 2, 3, 2, 1},
   {2, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1}, {2, 3, 1, 5, 1, 3, 1}, {2, 3, 1}, {2, 3, 1, 3}, {1, 2, 2, 3, 1, 5, 1, 3, 2, 1},
   {2, 2, 3, 1, 5, 1, 3, 1}, {5, 1, 2, 3, 1}, {5, 1, 3, 1, 5, 1, 3, 1}, {3, 2, 1, 3}, {1, 2, 2, 3, 1},
   {3, 1, 3, 2, 2, 1}, {2, 3, 2, 1, 3}, {2, 3, 2, 1, 5, 1, 3, 1}, {2, 2, 3, 1, 3, 2, 1}, {2, 1, 5, 1, 3, 2},
   {5, 1, 2, 3, 1, 5, 1, 3, 1}, {3, 1, 5, 1, 3, 1}, {3, 1, 3, 2, 1, 5}, {3, 2, 1, 5, 1, 3, 2, 1},
   {3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 1}, {3, 1, 3, 1, 5, 1, 3, 2}, {1, 5, 1, 3, 1, 5, 1, 3, 2, 2, 1},
   {2, 2, 1, 5}, {1, 5, 1, 3, 2, 1}, {1, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}, {1, 3, 1, 5, 1, 3, 1, 5, 1, 3, 2},
   {2, 1, 5, 1, 3, 2, 2}, {1, 3, 2, 1, 5, 1, 3, 2}, {3, 1, 3, 1, 5, 1, 3, 2, 2}, {2, 1, 5, 1, 3, 1, 5},
   {2, 1, 5, 1, 3}, {1, 2, 2, 3, 1, 5, 1, 3, 1}, {2, 2, 3, 1, 5, 1, 2, 3, 1}, {1, 2, 3, 1, 5, 1, 3, 2, 1},
   {2, 1, 5, 1, 3, 2, 1, 5}, {1, 2, 3, 2, 1}, {3, 1, 3, 1, 5, 1, 3}, {3, 2, 1, 5, 1, 3, 1}, {2, 1, 3, 2, 2},
   {3, 1, 5, 1, 2, 3, 1}, {3, 1, 5, 1, 3, 1, 5, 1, 3, 1}, {2, 3, 1, 3, 2, 1}, {1, 5, 1, 3, 1, 5, 1, 3, 2, 1},
   {2, 2, 2, 1, 5}, {3, 1, 3, 2, 1}, {1, 2, 3, 1}, {2, 1, 5, 1, 3, 2, 2, 1}, {2, 2, 1, 5, 1, 3, 2}, {2, 1, 3},
   {1, 5, 1, 2, 3, 2, 1}, {2, 1, 5, 1, 3, 1, 5, 1, 3, 2, 2}, {2, 2, 3, 1, 3, 1}, {1, 2, 3, 1, 5, 1, 3, 1},
   {2, 3, 1, 2}, {2, 2, 1, 5, 1, 3, 2, 2}, {2, 2, 1, 5, 1, 3}, {2, 1, 5, 1, 3, 1, 5, 1, 3}, {1, 3, 1},
   {1, 5, 1, 3, 1, 5, 1, 3, 1}, {2, 1, 5, 1, 2, 3}, {2, 1, 5, 1, 2, 3, 2}, {3, 1, 3, 1}, {2, 1, 5, 1, 3, 2, 1},
   {2, 3, 1, 3, 1}, {1, 3, 1, 5, 1, 3, 1}, {1, 5, 1, 2, 3, 1}, {2, 1, 5, 1, 3, 1}, {2, 1, 5, 1, 2}};

(* longListOfQuivers =(* RandomSample[JoinQuiverLists[12]]*)
 SortBy[JoinQuiverLists[12], {Length[#]&}] *)

babyQuiverList =
  Reverse[{{2, 3, 2, 1, 5}, {5, 1, 3, 2, 2, 1}, {5, 1, 3, 2, 2, 1, 5}, {3, 2, 1}, {3, 2, 2, 1}, {2, 2, 3}, {2, 3, 2},
    {2, 3}, {5, 1, 3, 2, 1, 5}, {3, 2, 2, 1, 5}, {3, 2, 1, 5}, {3, 1}, {5, 1, 3, 1, 5}, {2, 3, 1, 5}, {3, 1, 5}}];

If[babyTest, longListOfQuivers = babyQuiverList;];


(* User input notes:  *)


(* Redefine longListOfQuivers below this line
  to have the code run on any list of quivers via running  eg:
                     longListOfQuivers = {{1,3},{2,1,3},{5,1,3,1,5}};
              via cut and paste of this command to the line below this comment.

   NOTE: Avoiding all 2s and 1s quivers eg 222221
    will let the atomic code ignore those left and right gauge algebras
                   specified by the user below  *)

(* TossBadOutputs:   Output cleaning to remove sp[0] and sp[-1]
  from raw atomic code. (Their's below is edited and no such outputs occur.) *)
```

```
TossBadOutputs[list_, inputlink_: Null] := Module[{tempList, newList, i, j, toSkip},
     If[TrueQ[inputlink ≠ Null] && Depth[inputlink] > 2, Return[list];];
     tempList = list;
     newList = {};
     toSkip = False;
      For[i = 1, i ≤ Length[tempList], i ++,
        toSkip = False;
       For[j = 1, j ≤ Length[tempList[[i]]], j ++,
         If[ToString[tempList[[i, j]] ] == ToString[sp[-1]], toSkip = True; Break[]; ];
          If[ToString[tempList[[i, j]] ] == ToString[sp[0]], tempList[[i, j]] = n[0];];
          If[ToString[tempList[[i, j]] ] == ToString[su[2]]
                    && distinguishSu2FromSp1 == False, tempList[[i, j]] = sp[1];];
        ];

        If[toSkip == False, AppendTo[newList, tempList[[i]]]; ,
                                                          Continue[];
          ];
   ];
   Return[newList];
  ];


(* The repackaged and edited atomic code is in the following subsection  *)


(* Set these to get answers for quivers with ONLY all 2s
 or only 2s and 1s with designated left and right curve algebras. *)
leftnodeGauge = su[2];
rightnodeGauge = su[2];

(*The left link--required only if the input link consists only of 1's and 2's.  Not used otherwise.*)
(*The right link--required only if the input link consists only of 1's and 2's.  Not used otherwise.*)

(*Their code is adapted;
   it is not used in our code.  It is used only to give comparisons for gauge algebra outputs). User
 specifies left and right algebra global variables for use iff the quiver
 is purely 1s and 2s and not used otherwise.   *)

(* ATOMIC CODE REPACKAGED. *)


(* Adjacency matrix fcn: *)
```

```
AtomicAdjacencyMatrix[link_] :=
  Module[{trees, poslist, tab, pos, mat, rowtoadd, st},
   trees = (Depth[link] ≥ 3);
   Quiet[If[! trees, Return[  Table[Table[-link[[i]] * KroneckerDelta[i, j]
                                              + KroneckerDelta[i, j - 1] + KroneckerDelta[i, j + 1],
                                  {i, 1, Length[link]}], {j, 1, Length[link]}]
            ]
      ,
      (
       tab = Table[Length[link[[i]]], {i, 1, Length[link]}];
       poslist = Transpose[Position[tab, 2]][[1]];
       mat = Table[Table[If[Count[poslist, i] == 0 && Count[poslist, j] == 0,
           -link[[i]] * KroneckerDelta[i, j] + KroneckerDelta[i, j - 1] + KroneckerDelta[i, j + 1],
           If[Count[poslist, i] == 0 && Count[poslist, j] ≠ 0,
            {-link[[i]] * KroneckerDelta[i, j]
                  + KroneckerDelta[i, j - 1] + KroneckerDelta[i, j + 1], 0},
            If[Count[poslist, i] != 0 && Count[poslist, j] == 0,
             KroneckerDelta[i, j - 1] + KroneckerDelta[i, j + 1],
             If[Count[poslist, i] != 0 && Count[poslist, j] != 0 && i ≠ j,
              {0, 0},
              If[Count[poslist, i] != 0 && Count[poslist, j] != 0 && i == j,
               {-link[[i, 1]], 1},
               0
              ]
             ]
            ]
           ]
          ]
          , {j, 1, Length[link]}], {i, 1, Length[link]}];
       For[st = 1, st ≤ Length[poslist], st ++,
        pos = poslist[[st]];
        rowtoadd = Table[If[i == pos, {1, -link[[i, 2]]},
            If[Count[poslist, i] != 0, {0, 0}, 0]], {i, 1, Length[link]}] // Flatten;
        mat = Table[Flatten[mat[[i]]], {i, 1, Length[mat]}];
        mat = Insert[mat, rowtoadd, pos + st];
       ];
      )
     ];
   ];
   Return[mat];
  ];


(* The quiver check function follows. When there is no output,
the quivers pass. Otherwise it prints an error. *)

CheckQuiver[link_] := Module[{upwardlength},
   Quiet[
     upwardlength = Max[Table[Length[link[[i]]], {i, 1, Length[link]}]];
     If[upwardlength ≤ 3 && Depth[link] ≤ 3
            && ! PositiveDefiniteMatrixQ[-AtomicAdjacencyMatrix[link]],
      Print["Not an acceptable link!", link];];
    ];
  ];

(* The main atomic enhancement listing function follows. The input format is
  AssignEnhancements[quiver, leftGaugeAlg, rightGaugeAlg]
 eg:     AssignEnhancements[{2,1,3}, n[0], n[0]]
   Note: The last two arguments are not used unnless quiver is 1s and 2s only. *)

AssignEnhancements[inputlink_, leftNodeGauge_, rightNodeGauge_] :=
  Module[{leftnode, rightnode, leftopen, rightopen, leftclosed,
    rightclosed, onenumber, leftnumber, rightnumber, infinitetoadd222, infinitetriplet,
     moreforinfinite221, evenmoreforinfinite221, mostforinfinite221, triplet, morefor313,
    evenmorefor313, evenmorefor213, morefor213, quartet, morefor2222, evenmorefor2222,
    doubletlist, morefor22, morefor21, evenmorefor21, mostfor21, doublet, outlist,
    workinglist, startlist, nextlist, currentlist, tpos, lineartable, i, j, k,
    outlistNoDuplicates, finalOutput, cleanedOutlist, sortedFinalOutput, nl, nm, nr},


    CheckQuiver[inputlink];

   outlist = {};

   If[inputlink[[1]] == 1 && ToString[leftNodeGauge] == "su[2]",
    leftnode = sp[1];, leftnode = leftNodeGauge;
   ];
   If[inputlink[[-1]] == 1 && ToString[rightNodeGauge] == "su[2]",
    rightnode = sp[1];, rightnode = rightNodeGauge;
```

```
];

leftopen = StringPosition[ToString[leftnode], "["][[1, 1]];
leftclosed = StringPosition[ToString[leftnode], "]"][[1, 1]];
ToExpression[StringTake[ToString[leftnode], {leftopen + 1, leftclosed - 1}]];

leftopen = StringPosition[ToString[leftnode], "["][[1, 1]];
leftclosed = StringPosition[ToString[leftnode], "]"][[1, 1]];
rightopen = StringPosition[ToString[rightnode], "["][[1, 1]];
rightclosed = StringPosition[ToString[rightnode], "]"][[1, 1]];
onenumber = 0;
If[inputlink[[1]] == 2,
 leftnumber = Max[2 * ToExpression[StringTake[ToString[leftnode], {leftopen + 1, leftclosed - 1}]], 2];,
 (*Otherwise, it's -1.*)
 leftnumber = Max[2 * ToExpression[StringTake[ToString[leftnode], {leftopen + 1, leftclosed - 1}]], 4];
 onenumber = Max[2 * ToExpression[StringTake[ToString[leftnode], {leftopen + 1, leftclosed - 1}]], 4];
];
If[inputlink[[-1]] == 2,
 rightnumber =
   Max[2 * ToExpression[StringTake[ToString[rightnode], {rightopen + 1, rightclosed - 1}]], 2];,
 (*Otherwise, it's -1.*)
 rightnumber =
  Max[2 * ToExpression[StringTake[ToString[rightnode], {rightopen + 1, rightclosed - 1}]], 4];
 onenumber = Max[2 * ToExpression[StringTake[ToString[rightnode], {rightopen + 1, rightclosed - 1}]], 4];
];
infinitetoadd222 =
 Flatten[Table[{su[nl], su[nm], su[nr]}, {nl, 2, Min[leftnumber, rightnumber] * (Length[inputlink] - 1)},
   {nr, 2, Min[leftnumber, rightnumber] * (Length[inputlink] - 1)},
   {nm, Ceiling[(nl + nr) / 2], Min[2 nl, 2 nr]}], 2];
infinitetriplet[2, 2, 2] = Join[{{g[2], su[2], n[0]}, {su[2], so[7], su[2]}(*,
   {su[4],su[2],n[0]}*), {su[3], su[2], n[0]}, {su[2], su[2], n[0]},
   {n[0], su[2], n[0]}, {n[0], n[0], n[0]}, {su[2], g[2], su[2]}}, infinitetoadd222];
infinitetriplet[2, 2, 2] = DeleteDuplicates[Flatten[Table[{infinitetriplet[2, 2, 2][[i]],
     Reverse[infinitetriplet[2, 2, 2][[i]]]}, {i, 1, Length[infinitetriplet[2, 2, 2]]}], 1]];
moreforinfinite221 = Flatten[Table[{su[nl], su[nm], n[0]}, {nm, 2, 9},
   {nl, Max[2, Ceiling[nm / 2]], 2 nm}], 1];
evenmoreforinfinite221 = Flatten[Table[{su[nl], su[nm], su[nr]}, {nr, onenumber / 2, onenumber / 2},
   {nl, 2, 2 (nr + 8 + KroneckerDelta[nr, 3] + KroneckerDelta[nr, 6]) - nr},
   {nm, Ceiling[(nr + nl) / 2], Min[2 nl, (nr + 8 + KroneckerDelta[nr, 3] + KroneckerDelta[nr, 6])]}], 2];
mostforinfinite221 = Flatten[Table[{su[nl], su[nm], sp[nr]}, {nr, onenumber / 2, onenumber / 2},
   {nl, 2, 2 (2 nr + 8) - nr}, {nm, Ceiling[(nr + nl) / 2], Min[2 nl, 2 nr + 8]}], 2];
infinitetriplet[2, 2, 1] = Join[{{g[2], su[2], n[0]}, {so[7], su[2], n[0]}, {su[2], so[7], n[0]},
   {su[2], n[0], n[0]}, {n[0], su[2], n[0]}, {n[0], n[0], n[0]}, {su[2], n[0], n[0]}, {su[2], g[2], n[0]},
   {su[2], g[2], sp[1]}, {su[2], g[2], sp[2]}, {su[2], g[2], sp[3]}, {su[2], so[7], sp[1]},
   {su[2], so[7], sp[2]}, {su[2], so[7], sp[3]}, {su[2], so[7], sp[4]}, {n[0], su[2], g[2]},
   {n[0], su[2], sp[1]}, {n[0], su[2], sp[2]}, {n[0], su[2], su[3]}(*,{n[0],su[2],su[4]}*)},
  moreforinfinite221, evenmoreforinfinite221, mostforinfinite221];
infinitetriplet[1, 2, 2] = Table[Reverse[infinitetriplet[2, 2, 1][[i]]],
  {i, 1, Length[infinitetriplet[2, 2, 1]]}];

(*Note that the following specializes to what actually appears in the list.*)
triplet[2, 2, 2] =
 Join[{{g[2], su[2], n[0]}, {su[2], so[7], su[2]}(*,{su[4],su[2],n[0]}*), {su[3], su[2], n[0]},
   {su[2], su[2], n[0]}, {n[0], su[2], n[0]}, {n[0], n[0], n[0]}, {su[2], g[2], su[2]}},
  Flatten[Table[{su[nl], su[nm], su[2]}, {nm, 2, 4}, {nl, 2, 2 nm - 2}], 1],
  Flatten[Table[{su[nl], su[nm], su[3]}, {nm, 3, 6}, {nl, 3, 2 nm - 3}], 1]];
triplet[2, 2, 2] = DeleteDuplicates[Flatten[Table[{triplet[2, 2, 2][[i]], Reverse[triplet[2, 2, 2][[i]]]},
    {i, 1, Length[triplet[2, 2, 2]]}], 1]];
evenmorefor213 = Flatten[Table[{su[nl], sp[nm], so[nr]}, {nl, 2, 20},
   {nr, 7, 12}, {nm, Max[Ceiling[(2 nl + nr - 16) / 4], 1],
    Min[nl, nr - 7 + 2 KroneckerDelta[nr, 7]          (* This is edited from the atomic
      code. It formerly read as Min[2 nl,nr-7+KroneckerDelta[nr,7]. Now it is capped below
        like the other 213 loops to prevent reading sp0 or sp-1. See first two bullets
        on p.49 Atomic for the rules that should give this correction. Note also they
        allow sp2 so7 on 13 in this configuration elsewhere explicitly. Also note that
        p.57 for 214 gives this convexity condition (there: 2P'>2M_1) *) ]}], 2];
morefor213 = Flatten[Table[{so[nl], sp[nm], so[nr]}, {nl, 7, 13}, {nr, 7, 12},
   {nm, Max[Ceiling[(nl + nr - 16 + KroneckerDelta[nr, 7]) / 4], 1]
    (* Let's handle the n[0] center guys separately by hardcoding *),
    Min[nl - 6 + 3(*This used to read 3 Kronecker....see.p61 of Atomic for the quiver 21.
                      If nl<->so7 on 2, we may only wish to allow sp1,sp2,sp3,
      as on p.61 making this read 2*Kronecker?... *) KroneckerDelta[nl, 7],
      nr - 7 + 2(* This read 1*Kronecker. We may ask if it should read 2*Kronecker?....See
        Atomic for 31. Note so7 [sp1 or sp2] is allowed. We might
        wish to check this obeys the rules for spinor reps on -3,
      -4 for so7 being required as given on p.49 Atomic.  *) KroneckerDelta[nr, 7]]}], 2];
triplet[2, 1, 3] = Join[{{n[0], n[0], su[3]}, {n[0], n[0], g[2]}, {n[0], n[0], so[7]},
   {n[0], n[0], so[8]}, {n[0], n[0], so[9]}, {n[0], n[0], so[10]}, {n[0], n[0], so[11]},
   {n[0], n[0], so[12]}, {n[0], n[0], f[4]}, {n[0], n[0], e[6]}, {n[0], n[0], e[7]}, {su[2], n[0], su[3]},
   {su[3], n[0], su[3]}, {su[4], n[0], su[3]}, {su[5], n[0], su[3]}, {su[6], n[0], su[3]},
```

```
    {su[2], n[0], g[2]}, {su[3], n[0], g[2]}, {su[4], n[0], g[2]}, {su[2], n[0], e[6]},
    {su[3], n[0], e[6]}, {su[2], n[0], e[7]}, {su[2], n[0], so[7]}, {su[3], n[0], so[7]},
    {su[4], n[0], so[7]}, {su[2], n[0], so[8]}, {su[3], n[0], so[8]}, {su[4], n[0], so[8]},
    {su[2], n[0], so[9]}, {su[3], n[0], so[9]}, {su[4], n[0], so[9]}, {su[2], n[0], so[10]},
    {su[3], n[0], so[10]}, {su[4], n[0], so[10]}, {su[2], n[0], so[11]}, {so[7], n[0], so[7]},
    {su[2], n[0], so[12]}(* This has been edited from so[11], already on the list. ,
    {so[7],n[0],g[2]},{so[8],n[0],g[2]},{so[9],n[0],g[2]}We also edit to add these last three. *),
    {so[8], n[0], so[7]}, {so[7], n[0], so[8]}, {so[8], n[0], so[8]}, {so[9], n[0], so[7]}
    (* We now edit to include {g[2],n[0],su[3]},{so[7],n[0],su[3]},{so[8],n[0],su[3]},
    {so[9],n[0],su[3]},{so[10],n[0],su[3]} as all these include in D5-A2\subset E8.  *),
    (* Begin all added atomic omited 213 hardcoded triples. *)
    {so[7], n[0], g[2]}, {so[8], n[0], g[2]}, {so[9], n[0], g[2]}, {g[2], n[0], su[3]},
    {so[7], n[0], su[3]}, {so[8], n[0], su[3]}, {so[9], n[0], su[3]}, {so[10], n[0], su[3]},
    (* End added atomic omited *)
    {so[7], n[0], so[9]}, {n[0], sp[1], g[2]}, {n[0], sp[1], so[7]}, {n[0], sp[1], so[8]},
    {n[0], sp[1], so[9]}, {n[0], sp[1], so[10]}, {n[0], sp[1], so[11]}, {n[0], sp[1], so[12]},
    {e[6], n[0], su[3]}, {su[2], n[0], f[4]}, {su[3], n[0], f[4]}, {g[2], n[0], f[4]}, {g[2], n[0], g[2]},
    {g[2], n[0], g[2]}, {g[2], n[0], so[7]}, {g[2], n[0], so[8]}, {g[2], n[0], so[9]},
    {f[4], n[0], g[2]}, {f[4], n[0], su[3]}, {g[2], sp[1], g[2]}, {g[2], sp[1], so[7]},
    {g[2], sp[1], so[8]}, {g[2], sp[1], so[9]}, {g[2], sp[1], so[10]}, {g[2], sp[1], so[11]},
    {g[2], sp[1], so[12]}, {g[2], sp[2], so[7]}, {g[2], sp[2], so[9]}, {g[2], sp[2], so[10]},
    {g[2], sp[2], so[11]}, {g[2], sp[2], so[12]}, {g[2], sp[3], so[10]}, {g[2], sp[3], so[11]},
    {g[2], sp[3], so[12]}, {g[2], sp[4], so[11]}, {g[2], sp[4], so[12]}, {su[2], sp[1], g[2]},
    {su[3], sp[1], g[2]}, {su[4], sp[1], g[2]}, {su[5], sp[1], g[2]}, {su[6], sp[1], g[2]},
    {so[7], sp[1], g[2]}, {so[8], sp[1], g[2]}, {so[9], sp[1], g[2]}, {so[10], sp[1], g[2]},
    {so[11], sp[1], g[2]}, {so[12], sp[1], g[2]}, {so[13], sp[1], g[2]}}, morefor213, evenmorefor213];
triplet[3, 1, 2] = Table[Reverse[triplet[2, 1, 3][[i]]], {i, 1, Length[triplet[2, 1, 3]]}];
(*Note that the following one specializes to what actually appears in the list.*)
triplet[2, 2, 1] = {{g[2], su[2], n[0]}, {so[7], su[2], n[0]} ,
   {su[4], su[2], n[0]}, {su[3], su[2], n[0]}, {su[2], su[2], n[0]}, {su[2], n[0], n[0]},
   {n[0], su[2], n[0]}, {n[0], n[0], n[0]}, {su[2], su[3], n[0]}, {su[3], su[3], n[0]},
   {su[4], su[3], n[0]}, {su[5], su[3], n[0]}, {su[6], su[3], n[0]}, {su[2], g[2], n[0]}};
triplet[1, 2, 2] = Table[Reverse[triplet[2, 2, 1][[i]]], {i, 1, Length[triplet[2, 2, 1]]}];
triplet[3, 2, 2] = {{g[2], su[2], n[0]}};
triplet[2, 2, 3] = Table[Reverse[triplet[3, 2, 2][[i]]], {i, 1, Length[triplet[3, 2, 2]]}];
triplet[2, 3, 2] = {{su[2], so[7], su[2]}};
triplet[3, 2, 1] = {{g[2], su[2], n[0]}, {so[7], su[2], n[0]}};
triplet[1, 2, 3] = Table[Reverse[triplet[3, 2, 1][[i]]], {i, 1, Length[triplet[3, 2, 1]]}];
triplet[2, 3, 1] = {{su[2], g[2], n[0]}, {su[2], so[7], sp[1]}, {su[2], so[7], n[0]}};
triplet[1, 3, 2] = Table[Reverse[triplet[2, 3, 1][[i]]], {i, 1, Length[triplet[2, 3, 1]]}];
triplet[1, 3, 1] = {{n[0], su[3], n[0]}, {n[0], g[2], n[0]}, {n[0], so[7], n[0]}, {n[0], so[8], n[0]},
   {n[0], so[9], n[0]}, {n[0], so[10], n[0]}, {n[0], so[11], n[0]}, {n[0], so[12], n[0]},
   {n[0], f[4], n[0]}, {n[0], e[6], n[0]}, {n[0], e[7], n[0]}, {sp[1], so[7], n[0]},
   {sp[2], so[7], n[0]}, {sp[1], so[8], n[0]}, {sp[1], so[9], n[0]}, {sp[2], so[9], n[0]},
   {sp[1], so[10], n[0]}, {sp[2], so[10], n[0]}, {sp[3], so[10], n[0]}, {sp[1], so[11], n[0]},
   {sp[2], so[11], n[0]}, {sp[3], so[11], n[0]}, {sp[4], so[11], n[0]}, {sp[1], so[12], n[0]},
   {sp[2], so[12], n[0]}, {sp[3], so[12], n[0]}, {sp[4], so[12], n[0]}, {sp[5], so[12], n[0]},
   {sp[1], g[2], n[0]}, {sp[1], so[7], sp[1]}, {sp[1], so[8], sp[1]}, {sp[1], so[9], sp[1]},
   {sp[1], so[10], sp[1]}, {sp[2], so[10], sp[1]}, {sp[1], so[11], sp[1]}, {sp[2], so[11], sp[1]},
   {sp[3], so[11], sp[1]}, {sp[2], so[11], sp[2]}, {sp[1], so[12], sp[1]}, {sp[2], so[12], sp[1]},
   {sp[3], so[12], sp[1]}, {sp[2], so[12], sp[2]}, {sp[4], so[12], sp[1]}, {sp[3], so[12], sp[2]}};
triplet[1, 3, 1] = DeleteDuplicates[Flatten[Table[{triplet[1, 3, 1][[i]], Reverse[triplet[1, 3, 1][[i]]]},
     {i, 1, Length[triplet[1, 3, 1]]}], 1]];
morefor313 = {{g[2], sp[1], g[2]}, {g[2], sp[1], so[7]}, {g[2], sp[1], so[8]}, {g[2], sp[1], so[9]},
   {g[2], sp[1], so[10]}, {g[2], sp[1], so[11]}, {g[2], sp[1], so[12]}, {so[7], sp[1], so[7]},
   {so[7], sp[2], so[7]}, {so[7], sp[1], so[9]}, {so[7], sp[1], so[10]}, {so[7], sp[1], so[11]},
   {so[7], sp[1], so[12]}, {so[7], sp[2], so[9]}, {so[7], sp[2], so[10]}, {so[7], sp[2], so[11]},
   {so[7], sp[2], so[12]}, {so[8], sp[1], so[7]}, {so[8], sp[1], so[8]}, {so[8], sp[1], so[9]},
   {so[8], sp[1], so[10]}, {so[8], sp[1], so[11]}, {so[8], sp[1], so[12]}};
evenmorefor313 = Flatten[Table[{so[nl], sp[nm], so[nr]}, {nl, 9, 12},
    {nr, 9, 12}, {nm, Ceiling[(nr + nl - 16) / 4], Min[nr - 7, nl - 7]}], 2];
triplet[3, 1, 3] = Join[{{su[3], n[0], su[3]}, {g[2], n[0], su[3]}, {f[4], n[0], su[3]},
    {e[6], n[0], su[3]}, {g[2], n[0], g[2]}, {f[4], n[0], g[2]}, {so[7], n[0], su[3]},
    {so[8], n[0], su[3]}, {so[9], n[0], su[3]}, {so[10], n[0], su[3]}, {so[7], n[0], g[2]},
    {so[8], n[0], g[2]}, {so[9], n[0], g[2]}, {so[7], n[0], so[7]} , {so[8], n[0], so[7]},
    {so[9], n[0], so[7]} , {so[8], n[0], so[8]}}, morefor313, evenmorefor313];
triplet[3, 1, 3] = DeleteDuplicates[Flatten[Table[{triplet[3, 1, 3][[i]], Reverse[triplet[3, 1, 3][[i]]]},
     {i, 1, Length[triplet[3, 1, 3]]}], 1]];
triplet[1, 5, 1] = {{n[0], f[4], n[0]}, {n[0], e[6], n[0]}, {n[0], e[7], n[0]}};
triplet[1, 5, 1] = DeleteDuplicates[Flatten[
    Table[{triplet[1, 5, 1][[i]], Reverse[triplet[1, 5, 1][[i]]]}, {i, 1, Length[triplet[1, 5, 1]]}], 1]];
triplet[3, 1, 5] = {{su[3], n[0], f[4]}, {su[3], n[0], e[6]}, {g[2], n[0], f[4]}};
triplet[5, 1, 3] = Table[Reverse[triplet[3, 1, 5][[i]]], {i, 1, Length[triplet[3, 1, 5]]}];
triplet[2, 1, 5] = {{n[0], n[0], f[4]}, {n[0], n[0], e[6]}, {n[0], n[0], e[7]}, {su[2], n[0], f[4]}, {su[3],
    n[0], f[4]}, {su[2], n[0], e[6]}, {su[3], n[0], e[6]}, {su[2], n[0], e[7]}, {g[2], n[0], f[4]}};
triplet[5, 1, 2] = Table[Reverse[triplet[2, 1, 5][[i]]], {i, 1, Length[triplet[2, 1, 5]]}];
(*For quartets, the ordering convention is left top right middle*)
quartet[2, 2, 1, 3] = {{su[2], su[2], n[0], so[7]}};
quartet[2, 1, 2, 3] = Table[{quartet[2, 2, 1, 3][[i, 1]], quartet[2, 2, 1, 3][[i, 3]],
    quartet[2, 2, 1, 3][[i, 2]], quartet[2, 2, 1, 3][[i, 4]]}, {i, 1, Length[quartet[2, 2, 1, 3]]}];
```

```
quartet[1, 2, 2, 3] = Table[{quartet[2, 2, 1, 3][[i, 3]], quartet[2, 2, 1, 3][[i, 2]],
   quartet[2, 2, 1, 3][[i, 1]], quartet[2, 2, 1, 3][[i, 4]]}, {i, 1, Length[quartet[2, 2, 1, 3]]}];
quartet[2, 1, 1, 3] = {{su[2], n[0], n[0], g[2]}, {su[2], sp[1], n[0], so[7]}};
quartet[1, 1, 2, 3] = Table[{quartet[2, 1, 1, 3][[i, 3]], quartet[2, 1, 1, 3][[i, 2]],
   quartet[2, 1, 1, 3][[i, 1]], quartet[2, 1, 1, 3][[i, 4]]}, {i, 1, Length[quartet[2, 1, 1, 3]]}];
quartet[1, 2, 1, 3] = Table[{quartet[2, 1, 1, 3][[i, 2]], quartet[2, 1, 1, 3][[i, 1]],
   quartet[2, 1, 1, 3][[i, 3]], quartet[2, 1, 1, 3][[i, 4]]}, {i, 1, Length[quartet[2, 1, 1, 3]]}];
quartet[1, 1, 1, 5] = {{n[0], n[0], n[0], f[4]}, {n[0], n[0], n[0], e[6]}, {n[0], n[0], n[0], e[7]}};
quartet[2, 3, 1, 2] = {{n[0], g[2], n[0], su[2]}};
quartet[3, 2, 1, 2] = Table[{quartet[2, 3, 1, 2][[i, 2]], quartet[2, 3, 1, 2][[i, 1]],
   quartet[2, 3, 1, 2][[i, 3]], quartet[2, 3, 1, 2][[i, 4]]}, {i, 1, Length[quartet[2, 3, 1, 2]]}];
quartet[2, 1, 3, 2] = Table[{quartet[2, 3, 1, 2][[i, 1]], quartet[2, 3, 1, 2][[i, 3]],
   quartet[2, 3, 1, 2][[i, 2]], quartet[2, 3, 1, 2][[i, 4]]}, {i, 1, Length[quartet[2, 3, 1, 2]]}];
quartet[1, 3, 2, 2] = Table[{quartet[2, 3, 1, 2][[i, 3]], quartet[2, 3, 1, 2][[i, 2]],
   quartet[2, 3, 1, 2][[i, 1]], quartet[2, 3, 1, 2][[i, 4]]}, {i, 1, Length[quartet[2, 3, 1, 2]]}];
quartet[1, 2, 3, 2] = Table[{quartet[2, 3, 1, 2][[i, 3]], quartet[2, 3, 1, 2][[i, 1]],
   quartet[2, 3, 1, 2][[i, 2]], quartet[2, 3, 1, 2][[i, 4]]}, {i, 1, Length[quartet[2, 3, 1, 2]]}];
quartet[3, 1, 2, 2] = Table[{quartet[2, 3, 1, 2][[i, 2]], quartet[2, 3, 1, 2][[i, 3]],
   quartet[2, 3, 1, 2][[i, 1]], quartet[2, 3, 1, 2][[i, 4]]}, {i, 1, Length[quartet[2, 3, 1, 2]]}];
morefor2222 = Flatten[Table[{su[nl], su[nt], su[nr], su[nm]}, {nl, 2, Min[leftnumber, rightnumber] *
     (Length[inputlink] - 1)}, {nt, 2, Min[leftnumber, rightnumber] * (Length[inputlink] - 1)},
   {nr, 2, Min[leftnumber, rightnumber] * (Length[inputlink] - 1)},
   {nm, Ceiling[(nl + nr + nt) / 2], Min[{2 nl, 2 nr, 2 nt}]}], 3];
quartet[2, 2, 2, 2] = Join[{{su[2], su[2], su[2], so[7]}, {su[2], su[2], su[2], g[2]},
   {n[0], n[0], n[0], su[2]}, {n[0], n[0], su[2], su[2]}(*,{n[0],n[0],su[4],su[2]}*)}, morefor2222];
(*Need to include permutations of first three guys.*)
quartet[2, 2, 2, 2] = DeleteDuplicates[
  Join[quartet[2, 2, 2, 2], Table[{quartet[2, 2, 2, 2][[i, 2]], quartet[2, 2, 2, 2][[i, 1]],
     quartet[2, 2, 2, 2][[i, 3]], quartet[2, 2, 2, 2][[i, 4]]}, {i, 1, Length[quartet[2, 2, 2, 2]]}],
   Table[{quartet[2, 2, 2, 2][[i, 2]], quartet[2, 2, 2, 2][[i, 3]], quartet[2, 2, 2, 2][[i, 1]],
     quartet[2, 2, 2, 2][[i, 4]]}, {i, 1, Length[quartet[2, 2, 2, 2]]}],
   Table[{quartet[2, 2, 2, 2][[i, 1]], quartet[2, 2, 2, 2][[i, 3]], quartet[2, 2, 2, 2][[i, 2]],
     quartet[2, 2, 2, 2][[i, 2]]}, {i, 1, Length[quartet[2, 2, 2, 2]]}],
   Table[{quartet[2, 2, 2, 2][[i, 3]], quartet[2, 2, 2, 2][[i, 1]], quartet[2, 2, 2, 2][[i, 2]],
     quartet[2, 2, 2, 2][[i, 2]]}, {i, 1, Length[quartet[2, 2, 2, 2]]}],
   Table[{quartet[2, 2, 2, 2][[i, 3]], quartet[2, 2, 2, 2][[i, 2]], quartet[2, 2, 2, 2][[i, 1]],
     quartet[2, 2, 2, 2][[i, 2]]}, {i, 1, Length[quartet[2, 2, 2, 2]]}]]];
doubletlist[1, 3] = {(* {sp[2],so[7]},  ATOMIC CORRECTION: ADD THIS sp2 so7. should it
     have been here? should it be here? See screwUps.tex *) {sp[2], so[7]}, {n[0], su[3]},
  {n[0], g[2]}, {n[0], f[4]}, {n[0], e[6]}, {n[0], e[7]}, {n[0], so[7]}, {n[0], so[8]},
  {n[0], so[9]}, {n[0], so[10]}, {n[0], so[11]}, {n[0], so[12]}, {sp[1], g[2]}, {sp[1], so[7]},
  {sp[1], so[8]}, {sp[1], so[9]}, {sp[2], so[9]}, {sp[1], so[10]}, {sp[2], so[10]},
  {sp[3], so[10]}, {sp[1], so[11]}, {sp[2], so[11]}, {sp[3], so[11]}, {sp[4], so[11]},
  {sp[1], so[12]}, {sp[2], so[12]}, {sp[3], so[12]}, {sp[4], so[12]}, {sp[5], so[12]}};
doubletlist[3, 1] = Table[Reverse[doubletlist[1, 3][[i]]], {i, 1, Length[doubletlist[1, 3]]}];
doubletlist[1, 5] = {{n[0], f[4]}, {n[0], e[6]}, {n[0], e[7]}};
doubletlist[5, 1] = Table[Reverse[doubletlist[1, 5][[i]]], {i, 1, Length[doubletlist[1, 5]]}];
doubletlist[2, 3] = {{su[2], g[2]}, {su[2], so[7]}};
doubletlist[3, 2] = Table[Reverse[doubletlist[2, 3][[i]]], {i, 1, Length[doubletlist[2, 3]]}];
morefor22 = Flatten[
  Table[{su[nl], su[nr]}, {nl, 2, Max[leftnumber, rightnumber]}, {nr, Max[2, Ceiling[nl / 2]], 2 nl}], 1];
doublet[2, 2] = Join[{{n[0], n[0]}, {su[2], n[0]}, {so[7], su[2]}, {g[2], su[2]}}, morefor22];
doublet[2, 2] = DeleteDuplicates[
  Join[doublet[2, 2], Table[Reverse[doublet[2, 2][[i]]], {i, 1, Length[doublet[2, 2]]}]]];
morefor21 = Flatten[Table[{so[nl], sp[nr]}, {nl, 9, 13}, {nr, 1, nl - 6}], 1];
evenmorefor21 = Flatten[Table[{su[nl], sp[nr]},
   {nl, 2, Max[leftnumber, rightnumber]}, {nr, Max[1, Ceiling[(nl - 8) / 2]], nl}], 1];
mostfor21 = Flatten[Table[{su[nl], su[nr]}, {nr, 2, Max[leftnumber, rightnumber]},
   {nl, Max[2, Ceiling[nr / 2]], nr + 8 + KroneckerDelta[nr, 3] + KroneckerDelta[nr, 6]}], 1];
doublet[2, 1] = Join[{{n[0], n[0]}, {so[7], sp[1]}, {so[7], sp[2]}, {so[7], sp[3]}, {so[7], sp[4]},
   {so[8], sp[1]}, {so[8], sp[2]}, {g[2], sp[1]}, {g[2], sp[2]}, {g[2], sp[3]}, {g[2], sp[4]},
   {su[2], n[0]}, {su[3], n[0]}, {su[4], n[0]}, {su[5], n[0]}, {su[6], n[0]}, {su[7], n[0]},
   {su[8], n[0]}, {su[9], n[0]}, {so[7], n[0]}, {so[8], n[0]}, {so[9], n[0]}, {so[10], n[0]},
   {so[11], n[0]}, {so[12], n[0]}, {so[13], n[0]}, {f[4], n[0]}, {e[6], n[0]}, {e[7], n[0]},
   {n[0], sp[1]}, {su[2], g[2]}, {su[2], so[7]}}, morefor21, evenmorefor21, mostfor21];
doublet[1, 2] = Table[Reverse[doublet[2, 1][[i]]], {i, 1, Length[doublet[2, 1]]}];

(* now the stitching loop *)
Quiet[
 If[Max[Flatten[inputlink]] == 2,
   (*The list is infinite--we need to use the left and right links.*)
   If[Length[inputlink] == 2,
     outlist = {};
     workinglist = doublet[inputlink[[1]], inputlink[[2]]];
     For[i = 1, i ≤ Length[workinglist], i ++,
      If[ToString[workinglist[[i, 1]]] == ToString[leftnode] &&
         ToString[workinglist[[i, 2]]] == ToString[rightnode],
       outlist = {leftnode, rightnode};
      ];
     ];
```

```
,
If[Length[inputlink] == 3 && Depth[inputlink] == 2,
  startlist = infinitetriplet[inputlink[[1]], inputlink[[2]], inputlink[[3]]];
  nextlist = {};
  For[i = 1, i ≤ Length[startlist], i ++,
   If[ToString[startlist[[i, 1]]] == ToString[leftnode] &&
      ToString[startlist[[i, 3]]] == ToString[rightnode],
     nextlist = Append[nextlist, startlist[[i]]];
    ];
  ];
  outlist = nextlist;
  ,
  (*Now, we know the length is longer than 3,
  or it's t-shaped.  Here we handle the "longer than 3" case.*)
  If[Depth[inputlink] == 2,
    currentlist = infinitetriplet[inputlink[[1]], inputlink[[2]], inputlink[[3]]];
    nextlist = {};
    For[i = 1, i ≤ Length[currentlist], i ++,
     If[ToString[currentlist[[i, 1]]] == ToString[leftnode],
       nextlist = Append[nextlist, currentlist[[i]]];
      ];
    ];
    currentlist = nextlist;
    For[i = 3, i ≤ Length[inputlink] - 1, i ++,
     workinglist = infinitetriplet[inputlink[[i - 1]], inputlink[[i]], inputlink[[i + 1]]];
     nextlist = {};
     For[j = 1, j ≤ Length[currentlist], j ++,
      For[k = 1, k ≤ Length[workinglist], k ++,
        If[currentlist[[j, -2]] == workinglist[[k, 1]] &&
           currentlist[[j, -1]] == workinglist[[k, 2]],
          nextlist = Append[nextlist, Append[currentlist[[j]], workinglist[[k, 3]]]];
         ];
       ];
     ];
     currentlist = nextlist;
    ];
    nextlist = {};
    For[i = 1, i ≤ Length[currentlist], i ++,
     If[ToString[currentlist[[i, -1]]] == ToString[rightnode],
       nextlist = Append[nextlist, currentlist[[i]]];
      ];
    ];
    currentlist = nextlist;
    outlist = currentlist;
    ,
    (*Now, we know the link is T-shaped*)
    tpos = Position[Table[Depth[inputlink[[i]]], {i, 1, Length[inputlink]}], 2][[1, 1]];
    lineartable =
     Table[If[i ≠ tpos, inputlink[[i]], inputlink[[i, 1]]], {i, 1, Length[inputlink]}];
    currentlist = infinitetriplet[lineartable[[1]], lineartable[[2]], lineartable[[3]]];
    nextlist = {};
    For[i = 1, i ≤ Length[currentlist], i ++,
     If[ToString[currentlist[[i, 1]]] == ToString[leftnode],
       nextlist = Append[nextlist, currentlist[[i]]];
      ];
    ];
    currentlist = nextlist;
    For[i = 3, i ≤ Length[lineartable] - 1, i ++,
     workinglist = infinitetriplet[lineartable[[i - 1]], lineartable[[i]], lineartable[[i + 1]]];
     nextlist = {};
     For[j = 1, j ≤ Length[currentlist], j ++,
      For[k = 1, k ≤ Length[workinglist], k ++,
        If[currentlist[[j, -2]] == workinglist[[k, 1]] &&
           currentlist[[j, -1]] == workinglist[[k, 2]],
          nextlist = Append[nextlist, Append[currentlist[[j]], workinglist[[k, 3]]]];
         ];
       ];
     ];
     currentlist = nextlist;
    ];
    nextlist = {};
    For[i = 1, i ≤ Length[currentlist], i ++,
     If[ToString[currentlist[[i, -1]]] == ToString[rightnode],
       nextlist = Append[nextlist, currentlist[[i]]];
      ];
    ];
    currentlist = nextlist;
    (*At this point, we've handled the linear part of the
     link.  Now just need to do one more check, which involves the t-link.*)
    workinglist = quartet[lineartable[[tpos - 1]], inputlink[[tpos, 2]],
```

```
          lineartable[[tpos + 1]], inputlink[[tpos, 1]]];
        nextlist = {};
        For[j = 1, j ≤ Length[currentlist], j ++,
         For[k = 1, k ≤ Length[workinglist], k ++,
           If[currentlist[[j, tpos - 1]] == workinglist[[k, 1]] && currentlist[[j, tpos + 1]] ==
               workinglist[[k, 3]] && currentlist[[j, tpos]] == workinglist[[k, 4]],
             nextlist = Append[nextlist, Join[Take[currentlist[[j]], tpos - 1],
                 {{currentlist[[j, tpos]], workinglist[[k, 2]]}},
                 Take[currentlist[[j]], {tpos + 1, Length[currentlist[[j]]]}]]];
            ];
          ];
        ];
        currentlist = nextlist;
        outlist = currentlist;
       ];
    ];

 ];

,
(*Now, we know that the list in finite.*)
If[Length[inputlink] == 2,
  outlist = doubletlist[inputlink[[1]], inputlink[[2]]];
  ,
  If[Length[inputlink] == 3 && Depth[inputlink] == 2,
    outlist = triplet[inputlink[[1]], inputlink[[2]], inputlink[[3]]];
    ,
    (*Now, we know the length is longer than 3, or it's t-shaped.*)
    If[Depth[inputlink] == 2,
      currentlist = triplet[inputlink[[1]], inputlink[[2]], inputlink[[3]]];
      For[i = 3, i ≤ Length[inputlink] - 1, i ++,
       workinglist = triplet[inputlink[[i - 1]], inputlink[[i]], inputlink[[i + 1]]];
       nextlist = {};
       For[j = 1, j ≤ Length[currentlist], j ++,
        For[k = 1, k ≤ Length[workinglist], k ++,
          If[currentlist[[j, -2]] == workinglist[[k, 1]] &&
              currentlist[[j, -1]] == workinglist[[k, 2]],
            nextlist = Append[nextlist, Append[currentlist[[j]], workinglist[[k, 3]]]];
           ];
         ];
       ];
       currentlist = nextlist;
      ];
      outlist = currentlist;,
      (*Now, we know the link is T-shaped*)
      tpos = Position[Table[Depth[inputlink[[i]]], {i, 1, Length[inputlink]}], 2][[1, 1]];
      lineartable =
       Table[If[i ≠ tpos, inputlink[[i]], inputlink[[i, 1]]], {i, 1, Length[inputlink]}];
      currentlist = triplet[lineartable[[1]], lineartable[[2]], lineartable[[3]]];
      For[i = 3, i ≤ Length[lineartable] - 1, i ++,
       workinglist = triplet[lineartable[[i - 1]], lineartable[[i]], lineartable[[i + 1]]];
       nextlist = {};
       For[j = 1, j ≤ Length[currentlist], j ++,
        For[k = 1, k ≤ Length[workinglist], k ++,
          If[currentlist[[j, -2]] == workinglist[[k, 1]] &&
              currentlist[[j, -1]] == workinglist[[k, 2]],
            nextlist = Append[nextlist, Append[currentlist[[j]], workinglist[[k, 3]]]];
           ];
         ];
       ];
       currentlist = nextlist;
      ];
      (*At this point, we've handled the linear part of the
       link.  Now just need to do one more check, which involves the t-link.*)
      workinglist = quartet[lineartable[[tpos - 1]], inputlink[[tpos, 2]],
        lineartable[[tpos + 1]], inputlink[[tpos, 1]]];
      nextlist = {};
      For[j = 1, j ≤ Length[currentlist], j ++,
       For[k = 1, k ≤ Length[workinglist], k ++,
         If[currentlist[[j, tpos - 1]] == workinglist[[k, 1]] && currentlist[[j, tpos]] ==
             workinglist[[k, 4]] && currentlist[[j, tpos + 1]] == workinglist[[k, 3]],
           nextlist = Append[nextlist, Join[Take[currentlist[[j]], tpos - 1],
               {{currentlist[[j, tpos]], workinglist[[k, 2]]}},
               Take[currentlist[[j]], {tpos + 1, Length[currentlist[[j]]]}]]];
          ];
        ];
      ];

      currentlist = nextlist;
      outlist = currentlist;
```

```
        ];
      ];

    ];
  ];

];

If[inputlink == {1, {1, 5, 1}, 1},
 outlist =
   {{n[0], {n[0], f[4], n[0]}, n[0]}, {n[0], {n[0], e[6], n[0]}, n[0]}, {n[0], {n[0], e[7], n[0]}, n[0]}};
];
If[inputlink == {1, {1, 5, 1}, 1, 3},
 outlist = {{n[0], {n[0], f[4], n[0]}, n[0], su[3]}, {n[0], {n[0], f[4], n[0]}, n[0], g[2]},
    {n[0], {n[0], e[6], n[0]}, n[0], su[3]}};
];
If[inputlink == {3, 1, {1, 5, 1}, 1},
 outlist = {{su[3], n[0], {n[0], f[4], n[0]}, n[0]},
    {g[2], n[0], {n[0], f[4], n[0]}, n[0]},
    {su[3], n[0], {n[0], e[6], n[0]}, n[0]}};
];
If[inputlink == {1, {1, 5, 1}, 1, 3, 2},
 outlist = {{n[0], {n[0], f[4], n[0]}, n[0], g[2], su[2]}};
];
If[inputlink == {2, 3, 1, {1, 5, 1}, 1},
 outlist = {{su[2], g[2], n[0], {n[0], f[4], n[0]}, n[0]}};
];
If[inputlink == {3, 1, {5, {1, 3}}, 1, 3},
 outlist = {{su[3], n[0], {f[4], {n[0], su[3]}}, n[0], su[3]},
    {g[2], n[0], {f[4], {n[0], su[3]}}, n[0], su[3]}, {su[3], n[0], {f[4], {n[0], g[2]}}, n[0], su[3]},
    {su[3], n[0], {f[4], {n[0], su[3]}}, n[0], g[2]},
    {g[2], n[0], {f[4], {n[0], g[2]}}, n[0], su[3]},
    {g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2]},
    {su[3], n[0], {f[4], {n[0], g[2]}}, n[0], g[2]},
    {g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2]},
    {su[3], n[0], {e[6], {n[0], su[3]}}, n[0], su[3]}};
];
If[inputlink == {2, 3, 1, {5, {1, 3}}, 1, 3},
 outlist = {{su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], su[3]},
    {su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], su[3]},
    {su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2]},
    {su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2]}};
];
If[inputlink == {3, 1, {5, {1, 3}}, 1, 3, 2},
 outlist = {{su[3], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2]},
    {su[3], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2]},
    {g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2]},
    {g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2]}};
];
If[inputlink == {2, 2, 3, 1, {5, {1, 3}}, 1, 3},
 outlist = {{n[0], su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], su[3]},
    {n[0], su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], su[3]},
    {n[0], su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2]},
    {n[0], su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2]}};
];
If[inputlink == {3, 1, {5, {1, 3}}, 1, 3, 2, 2},
 outlist = {{su[3], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2], n[0]},
    {su[3], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2], n[0]},
    {g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2], n[0]},
    {g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2], n[0]}};
];
If[inputlink == {2, 3, 1, {5, {1, 3}}, 1, 3, 2},
 outlist = {{su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2]},
    {su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2]}};
];
If[inputlink == {2, 2, 3, 1, {5, {1, 3}}, 1, 3, 2},
 outlist = {{n[0], su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2]},
    {n[0], su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2]}};
];
If[inputlink == {2, 3, 1, {5, {1, 3}}, 1, 3, 2, 2},
 outlist = {{su[2], g[2], n[0], {f[4], {n[0], su[3]}}, n[0], g[2], su[2], n[0]},
    {su[2], g[2], n[0], {f[4], {n[0], g[2]}}, n[0], g[2], su[2], n[0]}};
];

If[Depth[outlist] ≤ 3 || TrueQ[Depth[inputlink] == 3 && Depth[outlist ≤ 4]] ||
  TrueQ[Depth[inputlink] == 4 && Depth[outlist ≤ 5]],
     If[Length[outlist] == 0,
         Print["no atomic enhancement available for", inputlink];
     ];
     If[Depth[outlist] ≤ 3, Return[TossBadOutputs[{outlist}]];];
```

```
        Return[{outlist}];
    ](* If there's only one entry, let's give it braces. Otherwise, delete any duplicates. *);
    outlistNoDuplicates = DeleteDuplicates[Sort[outlist]];
    cleanedOutlist = TossBadOutputs[outlistNoDuplicates, inputlink];
    If[TrueQ[inputlink == Reverse[inputlink]],
      finalOutput = DeleteDuplicates[cleanedOutlist, Reverse[#1] == #2 &];
     ,
      finalOutput = cleanedOutlist;
     ];
    sortedFinalOutput = Sort[finalOutput];

    Return[sortedFinalOutput];
   ];


(* The following method allows treatment of quiver lists *)

AssignEnhancementsToListOfQuivers[quiverList_, leftNodeGauge_, rightNodeGauge_] := Module[{i, quiver,
     enhancementList, quiverEnhancementListPair, listOfQuiversAndAllAllowedEnhancementsOnEach},
     listOfQuiversAndAllAllowedEnhancementsOnEach = {};
    listOfQuiversAndAllAllowedEnhancementsOnEach = Flatten[Reap[
        For[i = 1, i ≤ Length[quiverList], i ++,
            quiver = quiverList[[i]];
            enhancementList = AssignEnhancements[quiver, leftNodeGauge, rightNodeGauge];
            quiverEnhancementListPair = {quiver, enhancementList};
            Sow[quiverEnhancementListPair];
         ];
       ][[2]], 1];
    Return[listOfQuiversAndAllAllowedEnhancementsOnEach];
   ];

(* The following method converts each su[2] appearing to sp[1]. *)


Su2ToSp1InQuiverListEnhancementsList[enhancementsOnQuiverList_] :=
  Module[{i, j, k, quiverListEnhancementComparisonSp, enhancementsOnQuiverListSp1,
     ourEnhancementsOnSomeQuiver,  algToAdd, enhancementOnQuiverToAdd},
    enhancementsOnQuiverListSp1 = {};
    For[i = 1, i ≤ Length[enhancementsOnQuiverList], i ++,
      ourEnhancementsOnSomeQuiver = {};
      For[j = 1, j ≤ Length[enhancementsOnQuiverList[[i, 2]]], j ++,
         enhancementOnQuiverToAdd = {};
       For[k = 1, k ≤ Length[enhancementsOnQuiverList[[i, 2, j]] ], k ++,
        If[ ToString[enhancementsOnQuiverList[[i, 2, j, k]]] == ToString[su[2]],
          algToAdd = sp[1]; ,
          algToAdd =  enhancementsOnQuiverList[[i, 2, j, k]];
         ];
          AppendTo[enhancementOnQuiverToAdd , algToAdd];
        ];
       AppendTo[ourEnhancementsOnSomeQuiver, enhancementOnQuiverToAdd];
     ];
     AppendTo[enhancementsOnQuiverListSp1,
              {enhancementsOnQuiverList[[i, 1]], ourEnhancementsOnSomeQuiver}];
    ];
    Return[enhancementsOnQuiverListSp1];
   ];

(* END ATOMIC CODE *)


(* Import atomic outputs from last run via read from
 file rather than recompution when option is set to 'True'. *)
```

```
Quiet[
  If[recomputeAtomicEnhancementsRatherThanReadFromFile,
        atomicQuiverListEnhancementsOnLongListOfQuivers =
     AssignEnhancementsToListOfQuivers[longListOfQuivers, leftnodeGauge, rightnodeGauge];
        atomicQuiverListEnhancementsOnLongListOfQuiversSp1 =
           Su2ToSp1InQuiverListEnhancementsList[atomicQuiverListEnhancementsOnLongListOfQuivers];
        If[toSaveRecomputedAtomicEnhancementsToFile,
             DumpSave[atomicEnhancementsFileNameOut, {atomicQuiverListEnhancementsOnLongListOfQuivers,
                                        atomicQuiverListEnhancementsOnLongListOfQuiversSp1 }];
            Print["atomic enhancements recomputed and saved"];
         ];
     ,
      (* Otherwise read them from a file *)
       Get[ atomicEnhancementsFileNameIn];
        Print["atomic enhancements read from file"];
   ];
 ];
```

```
atomic enhancements read from file
```

```
(* Convert atomic list with su2 to one with sp1 in
 those places and set this for all cases if we don't distinguish. *)

atomicQuiverListEnhancementsOnLongListOfQuiversSp1 =
       Su2ToSp1InQuiverListEnhancementsList[atomicQuiverListEnhancementsOnLongListOfQuivers];
If[! distinguishSu2FromSp1,

  atomicQuiverListEnhancementsOnLongListOfQuivers = atomicQuiverListEnhancementsOnLongListOfQuiversSp1;
  ];

(*********************************************
     Here begins BM code proper.
   *****************************************)

(* Comparison to atomic outputs from above are near the end. The comparisons are done in Main.nb *)


(* AlgebrasOnQuiverInQuiverList:
 Find the atomic enhancements on a quiver from their stored outputs on the list of quivers. *)

AlgebrasOnQuiverInQuiverList[quiver_, listOfQuiverAndAllAlgebrasOnEach_] :=
  Module[{i, quiverFromListEntry},
   For[i = 1, i ≤ Length[listOfQuiverAndAllAlgebrasOnEach], i++,
     quiverFromListEntry = listOfQuiverAndAllAlgebrasOnEach[[i, 1]];
     If[quiverFromListEntry == quiver, Return[listOfQuiverAndAllAlgebrasOnEach[[i, 2]] ];];
   ];
    Print["no such quiver in that list"];
    Return[-1];
  ];

(* AtomicAlgebrasOnQuiver:
 Find atomic enhancements on a quiver in the longListOfQuivers or babyList in babyTest mode *)

AtomicAlgebrasOnQuiver[quiver_] := Module[{},
   Return[AlgebrasOnQuiverInQuiverList[quiver, atomicQuiverListEnhancementsOnLongListOfQuivers]];
  ];

(* We input the allowed orders of vanishing for the
 various Kodaira types. Then we loop to treat I_n and I_mstar.  *)
```

```
basicKodairaTypesForEachSelfInstersection = { (*for -1 curves: *)
    {{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for -2 curves: *)
    {{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {2, 3, 9}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for -3 curves: *)
   {{2, 2, 4}, {2, 3, 6}, {2, 3, 7}, {2, 3, 8}, {3, 4, 8}, {3, 5, 9}},
   (* for -4 curves *)
    {{2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for -5 curves *)
   {{3, 4, 8}, {3, 5, 9}},
   (* for -6 curves *)
   {{3, 4, 8}, {3, 5, 9}},
   (* for -7 curves *)
   {{3, 5, 9}}  (* Note that {3,6,9}, {3,7,9} are excluded except on a -
     8 since the needed f residual for m<8 will require nonminimality. *) ,
   (* for -8 curves. *)
   {{3, 5, 9}},
   (* for -9 *)
   { }(* We could add II* types for -9,-10,-11. *) ,
   (* for -10 *)
   {},
   (* for -11 *)
   {},
   (* for -12  *)
   {{4, 5, 10}}
            };

basicKodairaTypesForEachSelfInstersection
```

```
{{{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
 {{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {2, 3, 9}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
 {{2, 2, 4}, {2, 3, 6}, {2, 3, 7}, {2, 3, 8}, {3, 4, 8}, {3, 5, 9}}, {{2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
 {{3, 4, 8}, {3, 5, 9}}, {{3, 4, 8}, {3, 5, 9}}, {{3, 5, 9}}, {{3, 5, 9}}, {}, {}, {}, {{4, 5, 10}}}
```

```
ListEnhancementTypesOnSingleCurves[basicKodairaTypes_, maxNForIn_,
   maxNForInstar_, maxAforIzero_, maxBforIzero_ , maxAforII_, maxBforIII_, maxAforIV_,
   maxAforIzerostar_, maxBforIzerostar_, maxAforIVstar_, maxBforIIIstar_, maxAforIIstar_] :=
  Module[ {listToAddForIn, listToAddToForSelfIntersectionFourInstar,
    listToAddForIzeroA, listToAddForIzeroB, listToAddForII,
    listToAddForIII, listToAddForIV, listToAddForIzerostarA,
     listToAddForIzerostarB, listToAddForIVstar, listToAddForIIIstar,
    listToAddForIIstar,  outArray, returnList, i},

   (* First, In, Instar *)
   listToAddForIn = Table[{0, 0, i}, {i, 1, maxNForIn}];
   listToAddToForSelfIntersectionFourInstar = Table[{2, 3, 6 + i}, {i, 1, maxNForInstar}];
   listToAddForIzeroA     = Table[ {i, 0, 0},   {i, 1, maxAforIzero }];
   listToAddForIzeroB      = Table[{0, i, 0},   {i, 1, maxBforIzero }];
   listToAddForII          = Table[ {1 + i, 1, 2}, {i, 1, maxAforII }];
   listToAddForIII         = Table[{1, 2 + i, 3}, {i, 1, maxBforIII}];
   listToAddForIV          = Table[{2 + i, 2, 4}, {i, 1, maxAforIV }];
   listToAddForIzerostarA  = Table[{2 + i, 3, 6}, {i, 1, maxAforIzerostar }];
   listToAddForIzerostarB  = Table[{2, 3 + i, 6}, {i, 1, maxBforIzerostar }];
   listToAddForIVstar      = Table[{3 + i, 4, 8}, {i, 1, maxAforIVstar}];
   listToAddForIIIstar     = Table[{3, 4 + i, 9}, {i, 1, maxBforIIIstar}];
   listToAddForIIstar      = Table[{4 + i, 5, 10}, {i, 1, maxAforIIstar }];

   (* Array[outArray, 12];*)

   outArray[1]  = Join[basicKodairaTypes[[1]], listToAddForIn,
                      listToAddForIzeroA,
     listToAddForIzeroB, listToAddForII, listToAddForIII, listToAddForIV,
                                         listToAddForIzerostarA,
     listToAddForIzerostarB,  listToAddForIVstar];

   outArray[2] = Join[basicKodairaTypes[[2]], listToAddForIn, listToAddForIzeroA, listToAddForIzeroB,
                      listToAddForII, listToAddForIII, listToAddForIV, listToAddForIzerostarA,
                      listToAddForIzerostarB, listToAddForIVstar];
   outArray[3] = Join[basicKodairaTypes[[3]], listToAddForIV,
                           listToAddForIzerostarA, listToAddForIzerostarB,  listToAddForIVstar];

   outArray[4] = Join[basicKodairaTypes[[4]], listToAddToForSelfIntersectionFourInstar,
                             listToAddForIzerostarA, listToAddForIzerostarB,  listToAddForIVstar];

   outArray[5] = Join[basicKodairaTypes[[5]], listToAddForIVstar];
   outArray[6] = Join[basicKodairaTypes[[6]], listToAddForIVstar];
   outArray[7] = basicKodairaTypes[[7]] (* This is the empty list *) ;
   outArray[8] = Join[basicKodairaTypes[[8]], listToAddForIIIstar];
   outArray[9] = basicKodairaTypes[[9]] (* This is the empty list *) ;
   outArray[10] = basicKodairaTypes[[10]] (* This is the empty list *) ;
   outArray[11] = basicKodairaTypes[[11]] (* This is the empty list *) ;
   outArray[12] = Join[basicKodairaTypes[[12]] , listToAddForIIstar];
   returnList = Table[outArray[i], {i, 1, 12}];
   (* Note:
    IIIstar add-ons only go on for -8. Those for IIstar only for -12. Those for Instar only on -4.
                For -7, we add nothing; only (3,5,9) is possible at all.  For -9,
   -10,-11, nothing possible. IIstar is only for -12.  *)
   Return[returnList];
  ];

theKodairaTypesForEachSelfInt  =
  ListEnhancementTypesOnSingleCurves[basicKodairaTypesForEachSelfInstersection,
    maxNForIn, maxNForInstar, maxAforIzero, maxBforIzero , maxAforII, maxBforIII, maxAforIV,
    maxAforIzerostar, maxBforIzerostar,  maxAforIVstar,  maxBforIIIstar, maxAforIIstar];

 theKodairaTypesForEachSelfInt
```

```
{{{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9},
  {0, 0, 1}, {0, 0, 2}, {0, 0, 3}, {0, 0, 4}, {0, 0, 5}, {0, 0, 6}, {0, 0, 7}, {0, 0, 8}, {0, 0, 9},
  {0, 0, 10}, {0, 0, 11}, {0, 0, 12}, {0, 0, 13}, {0, 0, 14}, {0, 0, 15}, {0, 0, 16}, {0, 0, 17},
  {0, 0, 18}, {0, 0, 19}, {1, 0, 0}, {2, 0, 0}, {3, 0, 0}, {4, 0, 0}, {5, 0, 0}, {0, 1, 0}, {0, 2, 0},
  {0, 3, 0}, {0, 4, 0}, {0, 5, 0}, {0, 6, 0}, {2, 1, 2}, {3, 1, 2}, {4, 1, 2}, {5, 1, 2}, {6, 1, 2},
  {1, 3, 3}, {1, 4, 3}, {1, 5, 3}, {1, 6, 3}, {1, 7, 3}, {1, 8, 3}, {3, 2, 4}, {4, 2, 4}, {5, 2, 4},
  {6, 2, 4}, {7, 2, 4}, {3, 3, 6}, {4, 3, 6}, {5, 3, 6}, {6, 3, 6}, {7, 3, 6}, {2, 4, 6}, {2, 5, 6},
  {2, 6, 6}, {2, 7, 6}, {2, 8, 6}, {2, 9, 6}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}},
 {{0, 0, 0}, {1, 1, 2}, {2, 3, 7}, {2, 3, 8}, {2, 3, 9}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8},
  {3, 5, 9}, {0, 0, 1}, {0, 0, 2}, {0, 0, 3}, {0, 0, 4}, {0, 0, 5}, {0, 0, 6}, {0, 0, 7}, {0, 0, 8},
  {0, 0, 9}, {0, 0, 10}, {0, 0, 11}, {0, 0, 12}, {0, 0, 13}, {0, 0, 14}, {0, 0, 15}, {0, 0, 16}, {0, 0, 17},
  {0, 0, 18}, {0, 0, 19}, {1, 0, 0}, {2, 0, 0}, {3, 0, 0}, {4, 0, 0}, {5, 0, 0}, {0, 1, 0}, {0, 2, 0},
  {0, 3, 0}, {0, 4, 0}, {0, 5, 0}, {0, 6, 0}, {2, 1, 2}, {3, 1, 2}, {4, 1, 2}, {5, 1, 2}, {6, 1, 2},
  {1, 3, 3}, {1, 4, 3}, {1, 5, 3}, {1, 6, 3}, {1, 7, 3}, {1, 8, 3}, {3, 2, 4}, {4, 2, 4}, {5, 2, 4},
  {6, 2, 4}, {7, 2, 4}, {3, 3, 6}, {4, 3, 6}, {5, 3, 6}, {6, 3, 6}, {7, 3, 6}, {2, 4, 6}, {2, 5, 6},
  {2, 6, 6}, {2, 7, 6}, {2, 8, 6}, {2, 9, 6}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}},
 {{2, 2, 4}, {2, 3, 6}, {2, 3, 7}, {2, 3, 8}, {3, 4, 8}, {3, 5, 9}, {3, 2, 4}, {4, 2, 4}, {5, 2, 4},
  {6, 2, 4}, {7, 2, 4}, {3, 3, 6}, {4, 3, 6}, {5, 3, 6}, {6, 3, 6}, {7, 3, 6}, {2, 4, 6}, {2, 5, 6},
  {2, 6, 6}, {2, 7, 6}, {2, 8, 6}, {2, 9, 6}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}},
 {{2, 3, 6}, {3, 4, 8}, {3, 5, 9}, {2, 3, 7}, {2, 3, 8}, {2, 3, 9}, {2, 3, 10}, {2, 3, 11}, {2, 3, 12},
  {2, 3, 13}, {3, 3, 6}, {4, 3, 6}, {5, 3, 6}, {6, 3, 6}, {7, 3, 6}, {2, 4, 6}, {2, 5, 6},
  {2, 6, 6}, {2, 7, 6}, {2, 8, 6}, {2, 9, 6}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}},
 {{3, 4, 8}, {3, 5, 9}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}},
 {{3, 4, 8}, {3, 5, 9}, {4, 4, 8}, {5, 4, 8}, {6, 4, 8}, {7, 4, 8}, {8, 4, 8}}, {{3, 5, 9}},
 {{3, 5, 9}, {3, 5, 9}, {3, 6, 9}, {3, 7, 9}, {3, 8, 9}, {3, 9, 9}, {3, 10, 9}}, {},
 {}, {}, {{4, 5, 10}, {5, 5, 10}, {6, 5, 10}, {7, 5, 10}, {8, 5, 10}, {9, 5, 10}}}
```

```
GiveEnhancementTypes[] := Module[{},
       Return[theKodairaTypesForEachSelfInt];
  ];

enhTypes = GiveEnhancementTypes[];


(* Share enhTypes variable to all kernels. *)


DistributeDefinitions[enhTypes];


(* BlowdownNeededOnPair:
 Determines whether we have 4 6 12 points in the intersection of pair of curves. *)

BlowdownNeededOnPair[ type1_, type2_] := Module[{},
    (
     If[ type1[[1]] + type2[[1]] < 4 ||  type1[[2]] + type2[[2]] < 6 ,  Return [False];, Return[True]; ];
    )

  ];

(* ATypeAdjacencyMatrix:
 Finds the adjacency matrix function here for linear quivers only and not for D-types.  *)

ATypeAdjacencyMatrix[quiver_] := Module[{i, j, matrix},
    (
     matrix = ConstantArray[0, {Length[quiver], Length[quiver]}];
     For[i = 1, i ≤ Length[quiver], i ++,
        For [ j = 1, j ≤ Length[quiver], j ++,
            If[i == j, matrix[[i, j ]] = -quiver[[i]]; ,
               If[j == i + 1, matrix[[i, j]] = 1; ,
                  If[j == i - 1, matrix[[i, j]] = 1;,
                     (* if not in one of these three cases it is zero *)
                      matrix[[i, j]] = 0;
                    ];
                  ];
               ];
            ];
        ];
     Return[matrix];
    )
  ]; (* End module *)

(* ComputeResidualVanishings:
 Finds the number of remaining vanishings on each curve a quiver with types chosen. *)
```

```
ComputeResidualVanishingsForLinearQuiver[listWithEnhancements_] :=
  Module[{i, j, at, bt, dt, residualsList, adjacencyMatrix, quiver, length},
   at = 0;
   bt = 0;
   dt = 0;
   length = Length[listWithEnhancements];
   quiver = Table[listWithEnhancements[[i, 1, 1]], {i, 1, length}];
   adjacencyMatrix = ATypeAdjacencyMatrix[quiver];
   (* compute at bt and dt *)
   (* For example in the case of f, that
        a_i=
    (-4k - a1 C_1 - .... an C_n ). C_i  = -4(m_i - 2) - Sum[j ,a_j adjacencyMatrix[[j, i]], {j,1,n}]
      = -4(adjacencyMatrix[[i,i]] - 2) - Sum[j ,a_j adjacencyMatrix[[j, i]], {j,1,n}] *)
   residualsList = {};
   (*the indexes 1,2,3 in adjacencyMatrix below keep track of f,g,Delta *)
   residualsList = Flatten[Reap[
       For[i = 1, i ≤ Length[listWithEnhancements], i++,
         at = -4 * (-adjacencyMatrix[[i, i]] - 2) -
           Sum[listWithEnhancements[[j, 2, 1]] * (adjacencyMatrix[[i, j]]), {j, 1, Length[quiver]}] ;
         bt = -6 * (-1 * (adjacencyMatrix[[i, i]]) - 2) - Sum[listWithEnhancements[[j, 2, 2]] *
             ( adjacencyMatrix[[i, j]]), {j, 1, Length[quiver]}] ;
         dt = -12 * (-adjacencyMatrix[[i, i]] - 2) - Sum[listWithEnhancements[[j, 2, 3]] *
             (adjacencyMatrix[[i, j]]), {j, 1, Length[quiver]}] ;
         Sow[{{quiver[[i]]},   listWithEnhancements[[i, 2]] ,
                    {at, bt, dt}}];
        ];
      ][[2]], 1];
   Return[residualsList];
  ];


TotalAPrioriVanishingsAvailableForLinearQuiverEnhancement[listWithEnhancements_] :=
  Module[{i, j, at, bt, dt, residualsList, adjacencyMatrix, quiver},
   at = 0;
   bt = 0;
   dt = 0;
   quiver = Flatten[Take[listWithEnhancements, All, 1]];
   residualsList =
    Flatten[Reap[
       For[i = 1, i ≤ Length[listWithEnhancements], i++,
         at = -4 * (quiver[[i]] - 2) + listWithEnhancements[[i, 2, 1]] * quiver[[i]] ;
         bt = -6 * (quiver[[i]] - 2) + listWithEnhancements[[i, 2, 2]] * quiver[[i]] ;
         dt = -12 * (quiver[[i]] - 2) + listWithEnhancements[[i, 2, 3]] * quiver[[i]] ;
         Sow[ {at, bt, dt}];
        ];
      ][[2]], 1];
   Return[residualsList ];
  ];

TotalAPrioriVanishingsAvailableDTypes[enh_] := Module[
   {i, length, branchPositions, backboneEnh, totalVanishingsAvailable, backboneVanishingsAvailable},
    length = Length[enh];
    branchPositions = Table[If[Depth[enh[[i]]] > 3, 1, 0], {i, 1, length}];
    backboneEnh = Table[If[branchPositions[[i]] == 0,
                          enh[[i]]
                          ,
                          enh[[i, 1]]
                         ]
                        , {i, 1, length}];
   backboneVanishingsAvailable =
    TotalAPrioriVanishingsAvailableForLinearQuiverEnhancement[backboneEnh];
   totalVanishingsAvailable = Table[ If[branchPositions[[i]] == 0,
                                        backboneVanishingsAvailable[[i]]
                                        ,

      TotalAPrioriVanishingsAvailableForLinearQuiverEnhancement[enh[[i]]]
                                       ]
                                  , {i, 1, length}];
   Return[totalVanishingsAvailable];
  ];
```

```
WrappedTotalAPrioriVanishingsAvailableDTypes[enh_] :=
  Module[{i, unwrappedTotalVanishings, wrappedTotals, length},
    length = Length[enh];
    unwrappedTotalVanishings = TotalAPrioriVanishingsAvailableDTypes[enh];
    wrappedTotals = Table[
     If[Depth[unwrappedTotalVanishings[[i]]] < 3,
      {unwrappedTotalVanishings[[i]]}
      ,
      unwrappedTotalVanishings[[i]]
     ], {i, 1, length}];
   Return[wrappedTotals];
  ];


TotalAPrioriVanishingsAvailable[enh_] := Module[{},
   If[Depth[enh] > 4, Return[TotalAPrioriVanishingsAvailableDTypes[enh]];];
   Return[TotalAPrioriVanishingsAvailableForLinearQuiverEnhancement[enh]];
  ];

ComputeResidualVanishingsDType[enh_] := Module[{i, j, length, branchPositions,
    backboneEnh, enhWithResBeforeSubtractAtBranches, backboneEnhWithRes, enhWithRes},
   length = Length[enh];
   branchPositions = Table[If[Depth[enh[[i]]] > 3, 1, 0], {i, 1, length}];
   backboneEnh = Table[If[branchPositions[[i]] == 0,
                           enh[[i]]
                           ,
                           enh[[i, 1]]
                          ]
                         , {i, 1, length}];
   backboneEnhWithRes = ComputeResidualVanishings[backboneEnh];
   enhWithResBeforeSubtractAtBranches = Table[ If[branchPositions[[i]] == 0,
                                       backboneEnhWithRes[[i]]
                                        ,
                                       ComputeResidualVanishings[enh[[i]]]
                                       ]
                                     , {i, 1, length}];
   enhWithRes = Table[ If[branchPositions[[i]] == 0,
                                       enhWithResBeforeSubtractAtBranches[[i]]
                                        ,
                                       Table[ If[j > 1 ,
                                                  enhWithResBeforeSubtractAtBranches[[i, j]]
                                                 ,
                                                 {backboneEnhWithRes[[i, 1]],
                                                   backboneEnhWithRes[[i, 2]],
                                                   enhWithResBeforeSubtractAtBranches[[i, 1, 3]]
                                                        - If[i > 1, enh[[i - 1, 2]], {0, 0, 0}]
                                                         - If[i < length, enh[[i + 1, 2]], {0, 0, 0}]
                                                   }
                                               ]
                                          , {j, 1, Length[ enhWithResBeforeSubtractAtBranches[[i]]]}]
                        ]
                    , {i, 1, length}];
   Return[enhWithRes];
  ];

ComputeResidualVanishings[enh_] := Module[{},
   If[Depth[enh] > 4,
     Return[ComputeResidualVanishingsDType[enh]];
     ,
     Return[ComputeResidualVanishingsForLinearQuiver[enh]];
    ];
  ];

(* QuiverAndRawEnhancementListToListWithEnhancements:
 Convert a quiver and enhancement to a paired list  *)

QuiverAndRawEnhancementListToListWithEnhancements[quiver_, enhancement_] :=
  Module[{i, length, listWithEnhancements},
   length = Length[quiver];
   If[length ≠ Length[enhancement], Print["error: quiver length does not match enhancement length"];];
   listWithEnhancements = Table[{{quiver[[i]]}, enhancement[[i]] }, {i, 1, length}];
   Return[listWithEnhancements];
  ];

ComputeResidualVanishingsFromQuiverAndEnhancement[quiver_, enhancement_] := Module[{},
   Return[ComputeResidualVanishings[
      QuiverAndRawEnhancementListToListWithEnhancements[quiver, enhancement]] ];
  ];
```

```
(* ResidualsArePositive:    Determines whether the residuals are nonnegative. *)

ResidualsArePositive[residualsList_] := Module[ {isNonneg, i},
   (
    isNonneg = True;
    For[i = 1, i ≤ Length[residualsList], i ++,
         If[ residualsList[[i, 3, 1]] < 0 || residualsList[[i, 3, 2]] < 0 || residualsList[[i, 3, 3]] < 0,
      isNonneg = False;
      Break;
           ]
      ];
    Return[isNonneg];
   )];

(* Hashing for type assignments: *)

(* Now we assign enhancements using the above
 residuals computation to remove types not meeting our creteria. *)

InitializeHashTable[] := Module[{i, j},
   hashKeyTable = Table[{{i}, i}, {i, 1, 12}];
   hashTable = Table[{{{i},
            If[Length[enhTypes[[i]]] > 0, enhTypes[[i, j]], {}]}}, {i, 1, 12},
     {j, 1, Length[enhTypes[[i]]] + KroneckerDelta[ Length[enhTypes[[i]] ] , 0] }];
  ];

InitializeHashTable[];


Quiet[
  If[toReadFileWithSavedHashTable,
      Get[ enhancementHashFileNameIn];

   ];
 ];

SignedHashTableKey[quiver_, keyTable_] := Module[{i, hashTableIndex, maxLegth, toReturn},
        For[i = 1, i ≤ Length[keyTable ], i ++,
            If[quiver == keyTable [[i, 1]],
                  toReturn = keyTable [[i, 2]];
                  Return[toReturn];
              ,
               If[quiver == Reverse[keyTable [[i, 1]]],
                  toReturn = -keyTable [[i, 2]];
                  Return[ toReturn];
              ];
            ];
          ];
   Return[0];
  ];

MaxLengthQuiverInHashTable[keyTable_] := Module[{maxLength, lastLength, i},
   maxLength = 0;
   lastLength = 0;
   For[i = 1, i ≤ Length[keyTable], i ++,
        lastLength = Length[keyTable[[i, 1]]  ];
        If[lastLength > maxLength, maxLength = lastLength; ];
   ];
   Return[maxLength];
  ];

MaxLengthOfHashedQuiverAndItsSignedKey[quiver_, keyTable_] :=
  Module[{i, maxLength, keyToMax, length, key, quiverLength, maxLengthQuiverInHashTable},
   maxLength = 0;
   keyToMax = 0;
   length = 0;
   key = 0;
   quiverLength = Length[quiver];
   maxLengthQuiverInHashTable = MaxLengthQuiverInHashTable[keyTable];

    For[i = 1, i ≤ Min[ quiverLength, maxLengthQuiverInHashTable  ], i ++,
         key = SignedHashTableKey[Take[quiver, i], keyTable];
         If[key ≠ 0,  length = Length[keyTable[[Abs[key], 1]]  ];, length = 0;];
         If[ length > maxLength, maxLength = length; keyToMax = key; ];
     ];
    Return[{maxLength, keyToMax}];
  ];
```

```
ReadHashKeyTable[] := Module[{toReturn},
   While[writeHashKeyInProgress == True,
     Pause[hashKeyReadPauseLength];
   ];
   readHashKeyInProgress = True;
   toReturn = hashKeyTable;
   readHashKeyInProgress = False;
   Return[toReturn];
  ];

HashIt[quiver_, outputList_] := Module[{length, i, alreadyHashed, keyTable},
    alreadyHashed = False;
    length = Length[quiver];
    keyTable = ReadHashKeyTable[];
    While[readHashInProgress,
        Pause[hashingPauseLength];
   ];
    If[minLengthToBeHashed ≤ length ≤ maxLengthToBeHashed,
       If[SignedHashTableKey[quiver, keyTable] ≠ 0,
          alreadyHashed = True;
          Return[];
         ];
     ];
    writeHashInProgress = True;
    writeHashKeyInProgress = True;
       globalCounter = globalCounter + 1;
       AppendTo[hashKeyTable, {quiver, globalCounter}];
       AppendTo[hashTable, outputList];
    writeHashKeyInProgress = False;
    writeHashInProgress = False;
  ];

ReadHash[key_] := Module[{return},
    While[writeHashInProgress, Pause[hashingPauseLength];
   ];
    readHashInProgress = True;
     return = hashTable[[key]];
    Pause[hashingPauseLength];
    readHashInProgress = False;
    Return[return];
  ];

(* ListPossibleTypesOnQuiver:
 Finds Kodaira type assignments for a quiver.  Those shorter than maxLength are stored. *)


ListPossibleTypesOnLinearQuiver[inputChain_] :=
  Module[{i, j, k, r, curveToAdd, partialListWithEnh, partialToAdd, listOfPartialLists,
     nextListOfPartialLists, residualsOnTail, outputList, listIsPossible, lengthOfPartialList,
     partialToAddTail, currentTail, lastCurveType, length, maxGrowLength, hashTableIndex,
     partialFromHashed, startHashedPos, maxStartHashed, maxStartHashedLength, maxStartHashedKey,
     maxToGrowHashed, maxToGrowHashedLength, maxToGrowHashedKey, hashTableEntryLength, tailOfQuiver,
     tailOfQuiverIncludingLast, listOfPartialListsToTryAttaching, headToAdd, mergeSectionEnh,
     residualsOnMerger, addOnLength, curveToAttemptToGlue, listToReverseEntries, keyTable},
   length = Length[inputChain];
   (* Treat the i equals 1 case ie the first curve *)
   If[length == 1, Return[ReadHash[inputChain[[1]] ] ]; ];

   listOfPartialLists = {};
   nextListOfPartialLists = {};
   curveToAdd  = {};
   residualsOnTail = {};
   partialToAddTail = {};
   lastCurveType = {};
   maxStartHashedLength = 0;
   maxStartHashedKey = 0;
   tailOfQuiver = {};
   tailOfQuiverIncludingLast = {};
   maxToGrowHashed = {};
   maxToGrowHashedLength = 0;
   maxToGrowHashedKey = 0;
   headToAdd = {};
   mergeSectionEnh = {};
   residualsOnMerger = {};
   addOnLength = 0;
   curveToAttemptToGlue = {};
   listToReverseEntries = {};

   keyTable = ReadHashKeyTable[];
```

```
listIsPossible = False;
(* We start with the assumption it is not possible until we find a successful assignment *)
maxStartHashed = MaxLengthOfHashedQuiverAndItsSignedKey[inputChain, keyTable];
maxStartHashedLength = maxStartHashed[[1]];
maxStartHashedKey = maxStartHashed[[2]];
  If[maxStartHashedKey > 0,
    listOfPartialLists = ReadHash[maxStartHashedKey];
  ];
  If[maxStartHashedKey < 0,
      listToReverseEntries = ReadHash[-maxStartHashedKey];
      hashTableEntryLength = Length[listToReverseEntries];
      listOfPartialLists = Table[Reverse[listToReverseEntries[[r]] ], {r, 1, hashTableEntryLength}];
   ];
  If[maxStartHashedKey == 0,
     Print["error, incorrect hash table build or type. See  ListPossibleTypesOnQuiver"];
     Return[{{}}];
  ];
 If[maxStartHashedLength == length, Return[listOfPartialLists];];
(* Now add all compatible types on each successive curve recursively checking for
 nonminimality we go and dropping any nonminimal model inducing assignments. *)

For[i = maxStartHashedLength + 1, i ≤ length, i++ ,
   maxToGrowHashedLength = 1;

   If[length - i ≥ 1 (* If we can add more than one, see if we have something hashed to add *),
       tailOfQuiverIncludingLast = Take[inputChain, {i - 1, length}];

  maxToGrowHashed = MaxLengthOfHashedQuiverAndItsSignedKey[ tailOfQuiverIncludingLast, keyTable];
       maxToGrowHashedLength = maxToGrowHashed[[1]];
       maxToGrowHashedKey = maxToGrowHashed[[2]];
       ,
       maxToGrowHashedLength = 1;
     ];

    If[ maxToGrowHashedLength ≥ 3(* if we can add more than 1 at this step,add them *),
       If[maxToGrowHashedKey ≥ 1,
           listOfPartialListsToTryAttaching = ReadHash[maxToGrowHashedKey];
       ];
       If[maxToGrowHashedKey ≤ -1,
         listToReverseEntries = ReadHash[-maxToGrowHashedKey];
         hashTableEntryLength = Length[listToReverseEntries];
         listOfPartialListsToTryAttaching = Table[Reverse[listToReverseEntries[[r]] ],
                                                   {r, 1, hashTableEntryLength}];
       ];
       addOnLength = maxToGrowHashedLength - 1;
        For[k = 1, k ≤ Length[listOfPartialLists], k++,
               residualsOnTail = {};
               partialListWithEnh = listOfPartialLists[[k]];
               lengthOfPartialList = i - 1;
               partialToAdd = {};
               partialToAddTail = {};
               currentTail = Take[partialListWithEnh, -2 + KroneckerDelta[i, 2]];
               lastCurveType = partialListWithEnh[[i - 1, 2]];

               For[ j = 1, j ≤ Length[  listOfPartialListsToTryAttaching ], j++,
                      partialToAdd = {};
                      curveToAdd =  listOfPartialListsToTryAttaching[[j, 2]];
                      curveToAttemptToGlue =  listOfPartialListsToTryAttaching[[j, 1]];
                      If[ curveToAttemptToGlue ≠ partialListWithEnh[[i - 1]], Continue[]; ];
                      mergeSectionEnh = Append[currentTail, curveToAdd];

                          If[(ResidualsArePositive[ComputeResidualVanishings[mergeSectionEnh]]) ,

      If[2 > addOnLength + 1, Print["error see ListPossibleTypesOnQuiver"]; ];
                              headToAdd = Take[ listOfPartialListsToTryAttaching[[j]],
                                                 {2, addOnLength + 1}
                                                ];
                              partialToAdd = Join[partialListWithEnh, headToAdd];
                              AppendTo[nextListOfPartialLists, partialToAdd];
                              If[i == length, listIsPossible = True;
                                 ];
                          ];
               ];
         ](*  We are done growning the tail at this position.
               Now we try to continue growing if possible. *) ;
       listOfPartialLists = nextListOfPartialLists;
       nextListOfPartialLists = {};
       If[ i - 1 + addOnLength == length,
              HashIt[inputChain,  listOfPartialLists];
```

```
              Return[listOfPartialLists];
          ](* This If means we completed growing the full quiver via
                 growing using a hashed chunk of it's enhancements. *);
          i =  i - 1 + addOnLength;
          HashIt[Take[inputChain, i],  listOfPartialLists];
          Continue[];
        ](* We've finished growing with hashed subquivers where possible at this position. *) ;
      (*Otherwise we attach a single curve.*)
      If[ 1 ≤ maxToGrowHashedLength ≤ 2
           (* If not growning from a hashed quiver, we proceed to add any single curves allowed. *) ,
          For[k = 1, k ≤ Length[listOfPartialLists], k ++,
                  residualsOnTail = {};
                  partialListWithEnh = listOfPartialLists[[k]];
                  lengthOfPartialList = i - 1;
                  partialToAdd = {};
                  partialToAddTail = {};
                  currentTail = Take[partialListWithEnh, -2 + KroneckerDelta[i, 2]];
                  lastCurveType = partialListWithEnh[[i - 1, 2]];

              (*If nothing relevant hashed, then add a single curve: *)
                  For[ j = 1, j ≤ Length[ enhTypes[[ inputChain[[i]] ]] ], j ++,
                         partialToAdd = {};
                         curveToAdd = {{inputChain[[i]]}, enhTypes[[ inputChain[[i]] , j ]]};
                         partialToAddTail = Append[currentTail, curveToAdd];
                       If[ ! (BlowdownNeededOnPair[lastCurveType, curveToAdd[[2]] ]) ,
                              residualsOnTail = ComputeResidualVanishings[partialToAddTail];
                              If[ResidualsArePositive[residualsOnTail] ,
                                   partialToAdd = Append[partialListWithEnh, curveToAdd];

           nextListOfPartialLists = Append[nextListOfPartialLists, partialToAdd];
                                   If[i == length, listIsPossible = True;
                                     ];
                              ];
                       ];
                   ];

             ](* End attaching single curve. Now move to next position after updating lists. *) ;
        ](* end if block telling us to grow a single curve. *) ;

        listOfPartialLists = nextListOfPartialLists;
        nextListOfPartialLists = {};
        HashIt[Take[inputChain, i], listOfPartialLists]
    (* Hash any subquiver where we found all assignments.*);
    ](* This ends the i loop.  We've updated i accordingly in each case. We move to the next
    unassigned position in the quiver. We are finished when done assigning on full quiver. *);
    outputList = listOfPartialLists;
    If[listIsPossible,
         Return[outputList];,
         HashIt[inputChain, {{}}];
         Return[{{}}];
    ];
  ];

PassesBranchJuntionNaiveResidualsCheck[dTypeQuiver_, dTypeQuiverEnhancement_] :=
  Module[{i, length, branchPositions, backboneEnhancement, backboneResiduals},
   length = Length[dTypeQuiver];
   (* Careful with
    edits: the call from enhTypes...does not have the enhancement branching as the quiver does. *)
   branchPositions = Table[If[Depth[dTypeQuiverEnhancement[[i]]] > 3, 1, 0], {i, 1, length}];
   backboneEnhancement = Table[If[branchPositions[[i]] == 0,
                                  dTypeQuiverEnhancement[[i]]
                                  ,
                                  dTypeQuiverEnhancement[[i, 1]]
                               ]
                          , {i, 1, length}];
   backboneResiduals = ComputeResidualVanishings[backboneEnhancement];
   For[i = 1, i ≤ length, i ++,
     If[branchPositions[[i]] == 0,
       Continue[];
     ];
     If[(Length[Select[Flatten[backboneResiduals[[i, 3]] - dTypeQuiverEnhancement[[i, 2, 2]]], (# < 0) &]
               ] > 0),
      Return[False];
     ];
   ];
   Return[True];
  ];
```

```
ListPossibleTypesOnBranchingQuiver[inputChain_] :=
  Module[{i, j, k, n, length, branchPositions, branchedQuivers,
    enhsOnBranchedQuivers, backboneQuiver, backboneEnhancements, numEnhancementsOnBranch,
    newEnhancementsIncludingBranchesSoFar, residualsOnBackbone, keyTable, key, readFromHash},

   keyTable = ReadHashKeyTable[];
   key = SignedHashTableKey[inputChain, keyTable];
   If[key > 0,
    readFromHash = ReadHash[key];
    Return[readFromHash];
   ](* If stored, return it. TODO: also allow reverse of quiver in lookup. *) ;
   If[key < 0,
    readFromHash = ReadHash[-key];
    readFromHash = Reverse[readFromHash, 2];
    Return[readFromHash];
   ](* If stored, return it. TODO: also allow reverse of quiver in lookup. *) ;

   length = Length[inputChain];
   branchPositions = Table[If[Depth[inputChain[[i]]] > 1, 1, 0], {i, 1, length}];
   branchedQuivers = Table[If[branchPositions[[i]] == 1, inputChain[[i]], {}], {i, 1, length}];
   enhsOnBranchedQuivers =
    Table[If[branchPositions[[i]] == 1, ListPossibleTypesOnLinearQuiver[branchedQuivers[[i]]], {}],
       {i, 1, length}];
   backboneQuiver =
    Table[If[branchPositions[[i]] == 1, inputChain[[i, 1]], inputChain[[i]]], {i, 1, length}];
   backboneEnhancements = ListPossibleTypesOnLinearQuiver[backboneQuiver];
   (* Now the sewing loop. *)
   For[i = 1, i ≤ length, i ++,
       If[branchPositions[[i]] == 0,
           Continue[];
        ,
          newEnhancementsIncludingBranchesSoFar = Flatten[Reap[
              For[j = 1, j ≤ Length[backboneEnhancements], j ++,
                  numEnhancementsOnBranch = Length[enhsOnBranchedQuivers[[i]]];
                  For[k = 1, k ≤ numEnhancementsOnBranch, k ++,
                      If[! TrueQ[backboneEnhancements[[j, i]] == enhsOnBranchedQuivers[[i, k, 1]]],
                        Continue[];
                        ,
                        Sow[Table[If[branchPositions[[n]] == 0 || n > i,
                                   backboneEnhancements[[j, n]]
                                   ,
                                   enhsOnBranchedQuivers[[i, k]]
                                ]
                             , {n, 1, length}]
                        ];
                      ];
                  ];
              ];
          ][[2]], 1];
          backboneEnhancements = newEnhancementsIncludingBranchesSoFar;
          backboneEnhancements = Select[backboneEnhancements,
                 PassesBranchJuntionNaiveResidualsCheck[inputChain, #] &];
       ];
     ];
   backboneEnhancements = Select[backboneEnhancements,
             PassesBranchJuntionNaiveResidualsCheck[inputChain, #] &];
   HashIt[inputChain, backboneEnhancements];
   Return[backboneEnhancements];
  ];

ListPossibleTypesOnQuiver[quiver_] := Module[{},
   If[Depth[quiver] ≤ 2, Return[ListPossibleTypesOnLinearQuiver[quiver]];];
   Return[ListPossibleTypesOnBranchingQuiver[quiver]];
  ];

(* ListPossibleTypesOnQuiverNoHash:   This is the same as the previous
  method minus the storing of output values or using previous stored values.  *)

(* Now the same function without using hash tables;
This is not used in the main workflow but potentially faster for single quivers
 with no previous calculations having been done and no loaded hash tables.  *)
```

```
ListPossibleTypesOnQuiverNoHash[inputChain_] :=
  Module[{i, j, k, curveWithEnh, partialListWithEnh, partialToAdd, listOfPartialLists,
    nextListOfPartialLists, residualsOnTail, outputList, listIsPossible,
    lengthOfPartialList, partialToAddTail, currentTail, lastCurveType},
   nextListOfPartialLists = {};
   (* Treat the i equals 1 case ie the first curve *)

   listIsPossible = False;
   (* We start with the assumption it is not possible until we find a successful assignment *)
   For[j = 1, j ≤ Length[enhTypes[[inputChain[[1]] ]] ] , j ++,
         If[8 < inputChain[[1]] < 12,
            Print["error: quiver banned by residuals.
      -8,-9,-10,-11 not allowed.  see: ListPossibleTypesOnQuiver"];
            Return[{}];
            ] ;
          nextListOfPartialLists =
                   Append[nextListOfPartialLists,
                        {{{inputChain[[1]]}, enhTypes[[inputChain[[1]], j]] } }];
     ];
     listOfPartialLists = nextListOfPartialLists;
     nextListOfPartialLists = {};
     If[ Length[inputChain] == 1,
         outputList = listOfPartialLists;
         listIsPossible = True;
       ];
   (* now add all compatible types on each successive curve recursively checking
    for 4 6 12 points as we go and dropping any 4 6 12 point inducing assignments *)

    For[i = 2, i ≤ Length[inputChain], i ++ ,

    If[8 < inputChain[[i]] < 12,          Print["error: quiver banned by residuals. -8,-9,-10,-11
        not allowed.  see: ListPossibleTypesOnQuiver"];
              Return[{}];
              ];

        For[k = 1, k ≤ Length[listOfPartialLists], k ++,
              residualsOnTail = {};
              partialListWithEnh = listOfPartialLists[[k]];
              lengthOfPartialList = Length[ partialListWithEnh];
              partialToAddTail = {};
              currentTail = Take[partialListWithEnh, -2 + KroneckerDelta[i, 2]];
              lastCurveType = partialListWithEnh[[i - 1, 2]];

              For[ j = 1, j ≤ Length[ enhTypes[[ inputChain[[i]]  ]] ], j ++,
                    partialToAdd = {};
                    curveWithEnh = {{inputChain[[i]]}, enhTypes[[ inputChain[[i]] , j ]]};
                    partialToAddTail = Append[currentTail, curveWithEnh];
      If[ ! (BlowdownNeededOnPair[lastCurveType, curveWithEnh[[2]] ]) ,  (* In other words if
        nonminimality is possible in the intersection with the proposed curve to add *)
                        residualsOnTail = ComputeResidualVanishings[partialToAddTail];
                         If[ResidualsArePositive[residualsOnTail] ,
                              partialToAdd = Append[partialListWithEnh, curveWithEnh];

        nextListOfPartialLists = Append[nextListOfPartialLists, partialToAdd];
                              If[i == Length[inputChain], listIsPossible = True;
                                ];
                            ];
                        ];
                    ];
            ];
          listOfPartialLists = nextListOfPartialLists;
          nextListOfPartialLists = {};
        ];
       outputList = listOfPartialLists;
       If[inputChain == Reverse[inputChain],
          outputList =  DeleteDuplicates[outputList, Reverse[#1] == #2 &];
       ];
       If[listIsPossible, Return[outputList];, Return[{{}}]];
      ];

(* TypesListToResidualsList:
 Convert a type assingnment list to a list showing the residual vanishings. *)
```

```
TypesListToResidualsList[listOfPossibleTypeAssignments_] := Module[ {i, outputList},
      outputList = {};
     For[i = 1, i ≤ Length[listOfPossibleTypeAssignments], i ++,
        outputList = Append[outputList, ComputeResidualVanishings[listOfPossibleTypeAssignments[[i]]]];
        ];
     Return[outputList];
    ];

(* ResidualsListToEnhancementList:   Tossing of the residuals for list editing. *)


ResidualsListToEnhancementList[listOfPossibleTypeAssignmentsWithResiduals_] :=
  Module[ {i, j, outputList, enhancementOnQuiver},
      outputList = {};
     For[i = 1, i ≤ Length[listOfPossibleTypeAssignmentsWithResiduals], i ++,
       enhancementOnQuiver = {};
     For[j = 1, j ≤ Length[listOfPossibleTypeAssignmentsWithResiduals[[i]]], j ++,
          AppendTo[enhancementOnQuiver, {listOfPossibleTypeAssignmentsWithResiduals[[i, j, 1 ]],
          listOfPossibleTypeAssignmentsWithResiduals[[i, j, 2]]}];
          ];
       AppendTo[outputList, enhancementOnQuiver];
      ];
     Return[outputList];
    ];

(* EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver:
 This drops the m data and keeps only the types. *)

EnhancementWithSelfIntersectionsToTypesOnlyOnLinearQuiver[typeAssignmentWithIntersections_] :=
  Module[{typesOnQuiver, i},
   typesOnQuiver = Flatten[Reap[
        For[i = 1, i ≤ Length[typeAssignmentWithIntersections], i ++,
             Sow[typeAssignmentWithIntersections [[i, 2]] ];
         ];
       ][[2]], 1];
   Return[typesOnQuiver];
  ];

EnhancementWithSelfIntersectionsToTypesOnlyOnDTypeQuiver[typeAssignmentWithIntersections_] :=
  Module[{typesOnQuiver, i, enhTerm},
   typesOnQuiver = Flatten[Reap[
        For[i = 1, i ≤ Length[typeAssignmentWithIntersections], i ++,
            enhTerm = typeAssignmentWithIntersections [[i]] ;
             If[Depth[enhTerm] > 3,
                 Sow[EnhancementWithSelfIntersectionsToTypesOnlyOnLinearQuiver@enhTerm]
                   (*In this case, it's branching enhancement section. *);
               ,
              Sow[enhTerm[[2]] ](* In this case, it's a single curve;
                                 the orders are in the second slot. *);
            ];
         ];
       ][[2]], 1];
   Return[typesOnQuiver];
  ];

(* This is a wrapper method to separate whether
 we need the DType method or the linear quiver method: *)

EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[typeAssignmentWithIntersections_] := Module[{},
   If[Depth[typeAssignmentWithIntersections] ≤ 4,
    Return[EnhancementWithSelfIntersectionsToTypesOnlyOnLinearQuiver[typeAssignmentWithIntersections]];
   ](* Depth of linear enhancement is 4 and DTypes enhancements are 5. *);
    Return[EnhancementWithSelfIntersectionsToTypesOnlyOnDTypeQuiver[typeAssignmentWithIntersections]];
  ];

(* ListAllPossibleEnhancementsWithResidualsOnQuiver:
 Given a quiver this method lists the enhancements and residuals *)

ListAllPossibleEnhancementsWithResidualsOnQuiver[quiver_] := Module[{returnList},
    returnList = TypesListToResidualsList[ListPossibleTypesOnQuiver[quiver]];
   Return[returnList];
  ];

(* MonodromyTypes:    Determine monodromy options on a curve. *)
```

```
(* The following implements the gauge algebra constraints as shown in Table 6.1 of
 BMM paper.  Note these may not be the complete collection of known constraints.  *)
(* First start with monodromy options on a bare curve.  Here
 m is the self intersection on the curve. *)
(* This return value -1 indicates an error in type assignment inconsistent with our old results *)

MonodromyTypes[ curveWithEnhancement_ ] := Module[{m, a, b, d, monodromyTypesOptions},
   m = curveWithEnhancement[[1, 1]];
   a = curveWithEnhancement[[2, 1]];
   b = curveWithEnhancement[[2, 2]];
   d = curveWithEnhancement[[2, 3]];
   monodromyTypesOptions = {-1};
   If[m ≥ 8 && a ≥ 4 && b == 5, Return[{0}];];
   If[m ≤ 8 && a == 3 && b ≥ 5, Return[{0}];];
   If[m == 6 && a ≥ 3 && b == 4, Return[{1}];];
   If[m ≤ 5 && a ≥ 3 && b == 4, Return[{0, 1}];];
   If[m == 4 && a == 2  && b == 3 && d ≥ 10, Return[{0, 1}];];
   If[a == 2 && b == 3 && d == 9,

    If[m == 1 || m == 3 ,      Print["invalid type assignment I3Star with m = 1...see p.35 table GSPaper"];
        Return[{-1}];
         ];
      If[m == 4, Return[{0, 1}];];
      If[m == 2, Return[{0}];];
      ];
   If[m ≤ 4  && a == 2 && b == 3 && ( d == 7 || d == 8), Return[{0, 1}];];
   If[m ≤ 4 && ((a == 2 && b ≥ 3) || (a ≥ 2 && b == 3)) && d == 6 ,
      If[ m == 4 , Return[{2}];]
      (* See table 6.1 in BMM and the Izerostar restrictions section here. *) ;
      If[m ≤ 3 && b ≥ 4, Return[{1, 2}];]
      (* B>0 bars g_2 i.e. requires b = 3.
           See the Izerostar restrictions section. B>0 bars so(8). *) ;
      If[m ≤ 3 && a ≥ 3, Return[{0, 2}];] (* If nonzero residuals,
         allow g_2 or so(8) but a≥3 bans so(7) see the Izerostar restrictions section. *) ;
      Return[ {0, 1, 2}] (* Otherwise allow all options for Izerostar *) ;
   ];
   If[m ≤ 3 && b == 2 && a ≥ 2 && d == 4,
      If[m == 3, Return[{1}];, Return[{0, 1}];];
     ];
   If[m ≤ 2 && a == 1 && b ≥ 2 && d == 3, Return[{0}]; ];
   If[m ≤ 2 && a ≥ 1 && b == 1 && d == 2, Return[{0}];];
   If[m ≤ 2 && a == 0 && b ≥ 1 && d == 0 , Return[{0}]; ];
   If[m ≤ 2 && a ≥ 1 && b == 0 && d == 0 , Return[{0}];];
   If[m ≤ 2 && a == 0 && b == 0 && d ≤ 2, Return[{0}];];
   If[m == 1 && a == 0 && b == 0 && d ≥ 3, Return[{0, 1}];];
   If[m == 2 && a == 0 && b == 0 && d ≥ 3, Return[{1}];];
   Print["bad assignment...see MonodromyTypes
      function and refer to Kodaira table for valid Kodaira types"];
   Print[{m, {a, b, d}}];
   Return[monodromyTypesOptions ];
  ];

(* GaugeAlgebraFromEnhancementAndMonodromy:
 Gauge algebra assigment on curve given monodromy and type *)
```

```
GaugeAlgebraFromEnhancementAndMonodromy[enhancement_, monodromy_] := Module[{a, b, d, algebraToReturn},
   algebraToReturn = -1;
   (* This catches any errors as all actual cases should arise before this is returned *)
   If[monodromy == -1, Return[-1]; ];
   a = enhancement[[1]];
   b = enhancement[[2]];
   d = enhancement[[3]];
   If[distinguishSu2FromSp1 ,

    If[d ≤ 1, Return[ n[0]];];
    If[d == 2 && a ≥ 1 && b == 1, Return[n[0]];];
    If[a == 1 && d == 3, Return[ su[2]]];
    If[a == 0 && b == 0 && d ≥ 2,
        If[monodromy == 1 && d ≠ 2, Return[ su[d]              ]; ];
        If[monodromy == 0 ,         Return[ sp[ Floor[d / 2] ]  ]; ];
       ];
    If[a ≥ 2 && b == 2 && d == 4,
       If[ monodromy == 1, Return[su[3]];, Return[su[2]];];
       ];
    If[ d == 6 && ((a == 2 && b ≥ 3 ) || (a ≥ 2 && b == 3)) ,
       If[monodromy == 2, Return[so[8]];];
       If[monodromy == 1, Return[ so[7]];];
       If[monodromy == 0, Return[ g[2]]; ];
       ];
    If[ a == 2 && b == 3 && d ≥ 7,
        If[monodromy == 1, Return[ so[2 (d - 6) + 8]];];
        If[monodromy == 0, Return[so[2 (d - 6) + 7]];];
      ];
    If[ a ≥ 3 && b == 4 && d == 8,
        If[monodromy == 1, Return[ e[6]];, Return[f[4]]; ];
       ];
    If[ a == 3 && b ≥ 5 && d == 9, Return[e[7]];];
    If[ a ≥ 4 &&  b == 5 && d == 10, Return[e[8]];];
   ];
   If[ ! distinguishSu2FromSp1 ,
    If[d ≤ 1, Return[ n[0]];];
    If[d == 2 && a ≥ 1 && b == 1, Return[n[0]];];
    If[a == 1 && d == 3, Return[ sp[1]]];
    If[a == 0 && b == 0 && d ≥ 2,
        If[monodromy == 1 && d ≠ 2, Return[ su[d]              ]; ];
        If[monodromy == 0 ,         Return[ sp[ Floor[d / 2] ]  ]; ];
       ];
    If[a ≥ 2 && b == 2 && d == 4,
       If[ monodromy == 1, Return[su[3]];, Return[sp[1]];];
       ];
    If[ d == 6 && ((a == 2 && b ≥ 3 ) || (a ≥ 2 && b == 3)) ,
       If[monodromy == 2, Return[so[8]];];
       If[monodromy == 1, Return[ so[7]];];
       If[monodromy == 0, Return[ g[2]]; ];
       ];
    If[ a == 2 && b == 3 && d ≥ 7,
        If[monodromy == 1, Return[ so[2 (d - 6) + 8]];];
        If[monodromy == 0, Return[so[2 (d - 6) + 7]];];
      ];
    If[ a ≥ 3 && b == 4 && d == 8,
        If[monodromy == 1, Return[ e[6]];, Return[f[4]]; ];
       ];
    If[ a == 3 && b ≥ 5 && d == 9, Return[e[7]];];
    If[ a ≥ 4 &&  b == 5 && d == 10, Return[e[8]];];
   ];
   Return[algebraToReturn];
  ];

(* GaugeAlgebraFromEnhancementAndMonodromyForTex:
 Gauge algebra assigment on curve given monodromy and type *)
```

```
GaugeAlgebraFromEnhancementAndMonodromyForTex[enhancement_, monodromy_] :=
  Module[{ordF, ordG, ordDelta, algebraToReturn},
   algebraToReturn = -1; (* This catches any errors as all
    actual cases should arise before this is returned *)
   If[monodromy == -1, Return[-1]; ];
   ordF = enhancement[[1]];
   ordG = enhancement[[2]];
   ordDelta = enhancement[[3]];
   If[ordDelta == 0, Return[ {"\\Izero{}", {{0, 0}, 0}}];];
   If[ordDelta == 1, Return[{"\\Ione{}", {{0, 0}, 0}}];];
   If[ordDelta == 2 && ordF ≥ 1 && ordG == 1, Return[{"\\II{}", {{0, 0}, 0}}];];
   If[ordF == 1 && ordDelta == 3, Return[{"III", {{a, 1}, 1}}]; ];
   If[ordF == 0 && ordG == 0 && ordDelta ≥ 2,
      If[monodromy == 0 && ordDelta == 2, Return[{"\\Itwo{}", {{a, ordDelta - 1}, 1} } ]; ];
      If[monodromy == 1 && ordDelta ≠ 2,
     Return[{"\\Ins<" <> ToString[ordDelta] <> ">{}", {{a, ordDelta - 1}, 1} } ]; ];
      If[monodromy == 0 && ordDelta > 3,
     Return[{"\\Inns<" <> ToString[ordDelta] <> ">{}", {{c, Floor[ordDelta / 2]} , 1} } ]; ];
      If[monodromy == 0 && ordDelta == 3,
     Return[{"\\Inns<" <> ToString[ordDelta] <> ">{}", {{a, 1} , 1} } ]; ];
      ];
   If[ordF ≥ 2 && ordG == 2 && ordDelta == 4,
     If[ monodromy == 1, Return[{"\\IVs{}", {{a, 2}, 1}}];, Return[{"\\IVns{}", {{a, 1}, 1}}];];
     ];
   If[ ordDelta == 6 && ((ordF == 2 && ordG ≥ 3 ) || (ordF ≥ 2 && ordG == 3)) ,
     If[monodromy == 2, Return[{"\\Izerostars{}", {{d, 4}, 1}}];];
     If[monodromy == 1, Return[{"\\Izerostarss{}", {{b, 3}, 1}}];];
     If[monodromy == 0, Return[ {"\\Izerostarns{}", {{g, 2}, 1}}]; ];
     ];
   If[ ordF == 2 && ordG == 3 && ordDelta ≥ 7,

    If[monodromy == 1, Return[{"\\Instars <" <> ToString[ordDelta - 6] <> ">{}", {{d, ordDelta - 2}, 1}}];];
     If[monodromy == 0, Return[{"\\Instarns <" <> ToString[ordDelta - 6] <> ">{}",
        {{b, ordDelta - 3}, 1}}];];
    ];
   If[ ordF ≥ 3 && ordG == 4 && ordDelta == 8,
      If[monodromy == 1, Return[ {"\\IVstars{}", {{e, 6}, 1}}];, Return[{"\\IVstarns{}", {{f, 4}, 1}}]; ];
     ];
   If[ ordF == 3 && ordG ≥ 5 && ordDelta == 9, Return[{"\\IIIstar{}", {{e, 7}, 1}}];];
   If[ ordF ≥ 4 &&  ordG == 5 && ordDelta == 10, Return[{"\\IIstar{}", {{e, 8}, 1}}];];
   Return[-1];
  ];

(* GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy:
 Gauge algebra assigment on quiver curve given monodromy and type *)

GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromy[
   enhancementOnlyOnQuiver_, monodromyOnQuiver_] := Module[
   {i, gaugeAlgebrasOnQuiver, gaugeAlgebraOnCurve, enhancementOnCurve, monodromyOnCurve, length, max},
   length = Min[Length[enhancementOnlyOnQuiver], Length[ monodromyOnQuiver]];
   Catch[
    gaugeAlgebrasOnQuiver = Flatten[Reap[
         For[i = 1, i ≤ length, i ++,
             enhancementOnCurve = enhancementOnlyOnQuiver[[i]];
             monodromyOnCurve = monodromyOnQuiver[[i]];

             gaugeAlgebraOnCurve =
            GaugeAlgebraFromEnhancementAndMonodromy[enhancementOnCurve, monodromyOnCurve];
             If[TrueQ[gaugeAlgebraOnCurve == -1], Throw[Return[{-1}]];];
           (* This is in case of a bad assignment *)
             Sow[gaugeAlgebraOnCurve];
          ];
        ][[2]], 1];
   ];
   Return[gaugeAlgebrasOnQuiver];
  ];

(* GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy:
 Gauge algebra assigment on quiver curve given monodromy and type *)
```

```
GaugeAlgebrasFromEnhancementOnDTypeQuiverAndMonodromy[enhancementOnlyOnQuiver_, monodromyOnQuiver_] :=
  Module[{i, gaugeAlgebrasOnQuiver, gaugeAlgebraOnCurve,
    enhancementOnCurve, monodromyOnCurve, length, max, enhTerm, monTerm},
   length = Min[Length[enhancementOnlyOnQuiver], Length[ monodromyOnQuiver]];
   Catch[
    gaugeAlgebrasOnQuiver = Flatten[Reap[
         For[i = 1, i ≤ length, i ++,
              enhTerm = enhancementOnlyOnQuiver[[i]] ;
              monTerm =  monodromyOnQuiver[[i]];
               If[Depth[enhTerm] ≥ 3,
                   Sow[GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromy[enhTerm, monTerm]];
                ,
                 gaugeAlgebraOnCurve = GaugeAlgebraFromEnhancementAndMonodromy[enhTerm, monTerm ];
                 If[TrueQ[gaugeAlgebraOnCurve == -1], Throw[Return[{-1}]];]
                  (* In this case, we find the algebra for a single curve.
                     The if statement is in case of a bad assignment. *);
                 Sow[gaugeAlgebraOnCurve];

              ];
           ];
        ][[2]], 1];
   ];
   Return[gaugeAlgebrasOnQuiver];
  ];

(* Simple wrapper to treat DTypes separately *)


GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[
   enhancementOnlyOnQuiver_, monodromyOnQuiver_] := Module[{},
   If[Depth[enhancementOnlyOnQuiver] ≤ 3,
    Return[GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromy[
       enhancementOnlyOnQuiver, monodromyOnQuiver]];
   ];
   Return[
    GaugeAlgebrasFromEnhancementOnDTypeQuiverAndMonodromy[enhancementOnlyOnQuiver, monodromyOnQuiver]];
  ];

(* GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex:
 Gauge algebra assigment on quiver curve given monodromy and type in .tex form. *)

GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromyForTex[
   enhancementOnlyOnQuiver_, monodromyOnQuiver_] := Module[
   {i, gaugeAlgebrasOnQuiver, gaugeAlgebraOnCurve, enhancementOnCurve, monodromyOnCurve, length, max},
   length = Min[Length[enhancementOnlyOnQuiver], Length[ monodromyOnQuiver]];
   gaugeAlgebrasOnQuiver = {};

   For[i = 1, i ≤ length, i ++,
      enhancementOnCurve = enhancementOnlyOnQuiver[[i]];
      monodromyOnCurve = monodromyOnQuiver[[i]];

      gaugeAlgebraOnCurve =
     GaugeAlgebraFromEnhancementAndMonodromyForTex[enhancementOnCurve, monodromyOnCurve];
      If[TrueQ[gaugeAlgebraOnCurve == -1], Return[{-1}];];  (* This is in case of a bad assignment *)
      AppendTo[gaugeAlgebrasOnQuiver, gaugeAlgebraOnCurve];
   ];
   Return[gaugeAlgebrasOnQuiver];
  ];
```

```
GaugeAlgebrasFromEnhancementOnDTypeQuiverAndMonodromyForTex[
   enhancementOnlyOnQuiver_, monodromyOnQuiver_] := Module[{i, gaugeAlgebrasOnQuiver,
    gaugeAlgebraOnCurve, enhancementOnCurve, monodromyOnCurve, length, max, enhTerm, monTerm},
   length = Min[Length[enhancementOnlyOnQuiver], Length[ monodromyOnQuiver]];
   Catch[
    gaugeAlgebrasOnQuiver = Flatten[Reap[
         For[i = 1, i ≤ length, i ++,
              enhTerm = enhancementOnlyOnQuiver[[i]] ;
              monTerm =  monodromyOnQuiver[[i]];
               If[Depth[enhTerm] ≥ 3,
                   Sow[GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromyForTex[enhTerm, monTerm]];
                 ,

             gaugeAlgebraOnCurve = {GaugeAlgebraFromEnhancementAndMonodromyForTex[enhTerm, monTerm ]};
                  If[TrueQ[gaugeAlgebraOnCurve == -1], Throw[Return[{-1}]];]
                   (* In this case, we find the algebra for a single curve.
                      The if statement is in case of a bad assignment. *);
                   Sow[gaugeAlgebraOnCurve];
              ];
           ];
        ][[2]], 1];
   ];
   Return[gaugeAlgebrasOnQuiver];
  ];

(* Simple wrapper to treat DTypes separately *)


GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex[
   enhancementOnlyOnQuiver_, monodromyOnQuiver_] := Module[{},
   If[Depth[enhancementOnlyOnQuiver] ≤ 3,
    Return[GaugeAlgebrasFromEnhancementOnLinearQuiverAndMonodromyForTex[
       enhancementOnlyOnQuiver, monodromyOnQuiver]];
   ];
   Return[GaugeAlgebrasFromEnhancementOnDTypeQuiverAndMonodromyForTex[
     enhancementOnlyOnQuiver, monodromyOnQuiver]];
  ];

(* IntersectionContributionsToResiduals:
 Intersection contributions are computed less naively with monodromy. *)


(* Currently residuals are passed to both the curve and the transverse curveand only those on the curve
 are called. The transverse a priori residuals are available if needed in the current call.
  The method can be used for any transverse curve with enhancement and monodromy without residuals. The
 transverse self intersection is not used. Calls to this method can replace this with a null value. *)

IntersectionContributionsToResiduals[curveWithEnhAndResiduals_,
   transverseCurveWithEnhancement_, monodromyOnCurve_, monodromyOnTransverseCurve_,
   enhancementWithResidualsOnQuiver_ , curvePosition_, transverseCurvePosition_,
   numNonTrivialFGNeighborsAtPositionForGS_: 0, fibersCollectionsForGSOnEntireQuiver_: Null,
   nonTrivialFGFiberCountVec_: Null, transverseCurveIsCompact_: Null] :=
  Module[{conts, contributionsToResidualsOfCurve, a , b, d, at, bt, dt, an, bn, dn, mon1,
    mon2, acont, bcont, dcont, max, m, mn,  length, i, j, otherNeighborPosition,
    hasOtherNontrivialFGNeighbors, hasTwoNontrivialFGNeighbors,
    transverseHasOtherNontrivialFGNeighbors, transverseIsCompact, row, col, transRow, transCol},
   a = curveWithEnhAndResiduals[[2, 1]];
   b = curveWithEnhAndResiduals[[2, 2]];
   d = curveWithEnhAndResiduals[[2, 3]];
   at =  curveWithEnhAndResiduals[[3, 1]];
   bt = curveWithEnhAndResiduals[[3, 2]];
   dt = curveWithEnhAndResiduals[[3, 3]];
   an = transverseCurveWithEnhancement[[2, 1]];
   bn = transverseCurveWithEnhancement[[2, 2]];
   dn = transverseCurveWithEnhancement[[2, 3]];
   (* We will return naive contributions in cases we have not treated as special cases.
        Hence we make the following definition. *)
   contributionsToResidualsOfCurve = {an, bn, dn};
   mon1 = monodromyOnCurve;
   mon2 = monodromyOnTransverseCurve;
   acont = an;
   bcont = bn;
   dcont = dn;
   (* Self intersections are m for the curve recieving the intersection.
      The are mn for the transverse curve.*)
   m = curveWithEnhAndResiduals[[1, 1]];
   mn = -1 (* This is a dummy value assignment
     and the assignment of the value is in the j≠-1 block below.  *);
   If[Depth[enhancementWithResidualsOnQuiver] > 4,
    (* In this case, we have a D-type quiver.
```

```
   The call to this method gives the nonTrivialFGFiberCountVec and transverseCurveIsCompact.
    In this case,
the curvePosition argument is of the form {i,j} and the enhancementWithResidualsOnQuiver and
 nonTrivialFGFiberCountVec are wrapped so are ragged arrays of depth 3, i.e. ragged matrices.
 *)
        row = curvePosition[[1]];
        col = curvePosition[[2]];
        If[TrueQ[transverseCurvePosition == Null] || TrueQ[transverseCurvePosition == -1],
          transverseHasOtherNontrivialFGNeighbors = False;
         ,
            transRow = transverseCurvePosition[[1]];
            transCol = transverseCurvePosition[[2]];
            If[nonTrivialFGFiberCountVec[[transRow, transCol]] ≥ 2,
                transverseHasOtherNontrivialFGNeighbors = True;
              ,
              If[nonTrivialFGFiberCountVec[[transRow, transCol]] ≥ 1  && (a == 0 && b == 0),
                  transverseHasOtherNontrivialFGNeighbors = True;
              ];
            ];
            If[nonTrivialFGFiberCountVec[[transRow, transCol]] == 0,
                   transverseHasOtherNontrivialFGNeighbors = False;
            ];
        ];
        If[nonTrivialFGFiberCountVec[[row, col]] ≥  2,
           hasOtherNontrivialFGNeighbors = True;
            If[(an == 0 && bn == 0),
               hasTwoNontrivialFGNeighbors = True;
               ,
               If[nonTrivialFGFiberCountVec[[row, col]] ≥  3,
                   hasTwoNontrivialFGNeighbors = True;
               ];
            ];
            ,
          If[nonTrivialFGFiberCountVec[[row, col]] == 1,
              hasTwoNontrivialFGNeighbors = False;
              If[(an == 0 && bn == 0),
                  hasOtherNontrivialFGNeighbors = True;
              ];
          ];
          If[nonTrivialFGFiberCountVec[[row, col]] == 0,
              hasTwoNontrivialFGNeighbors = False;
              hasOtherNontrivialFGNeighbors = True;
          ];
       ];

,
i = curvePosition;
j = transverseCurvePosition;
If[j ≠ -1,
  transverseIsCompact = True;
  ,
   If[TrueQ[transverseCurveIsCompact],
    transverseIsCompact = True;
    ,
    transverseIsCompact = False;
    ];
];
hasOtherNontrivialFGNeighbors = False;
hasTwoNontrivialFGNeighbors = False;
transverseHasOtherNontrivialFGNeighbors = False;
length = Length[enhancementWithResidualsOnQuiver];
otherNeighborPosition = -1;

If[transverseIsCompact
 (* This case treats the case where the transverse curve is part of the quiver. *),
   If[length == 1,
       transverseHasOtherNontrivialFGNeighbors = False;
       If[TrueQ[numNonTrivialFGNeighborsAtPositionForGS == Null],
         hasOtherNontrivialFGNeighbors = False;
         ,
         If[numNonTrivialFGNeighborsAtPositionForGS ≥ 1,
           hasOtherNontrivialFGNeighbors = True;
           If[numNonTrivialFGNeighborsAtPositionForGS ≥ 2,
              hasTwoNontrivialFGNeighbors = True;
          ];
         ];
      ](* Now we begin the case with more than one compact curve in the quiver. *) ;
     ,
         mn = transverseCurveWithEnhancement[[1, 1]];
         If[i == 1 || i == length,
```

```
                (* We make the following definition in the case that the curve has no
                   other neighbors besides the transverse curve giving contributions. *)
                 hasOtherNontrivialFGNeighbors = False;
                 If[j == 1 || j == length, transverseHasOtherNontrivialFGNeighbors = False;,
                        If[ length > 2,
                                If[i > j, If[ (enhancementWithResidualsOnQuiver[[j - 1, 2, 1]] > 0 ) ||

          (enhancementWithResidualsOnQuiver[[j - 1, 2, 2]] > 0 ),
                                                transverseHasOtherNontrivialFGNeighbors = True;];];

     If[i < j, If[ (enhancementWithResidualsOnQuiver[[j + 1, 2, 1]] > 0 ) ||
                                                 (enhancementWithResidualsOnQuiver[[
             j + 1, 2, 2]] > 0 ),
                                                transverseHasOtherNontrivialFGNeighbors = True;];];
                            ];
                    ];
          ];
    If[ i ≠ 1 && i ≠ length,
          If[j < i,  otherNeighborPosition = i + 1;, otherNeighborPosition = i - 1;
            ];
             If[ enhancementWithResidualsOnQuiver[[otherNeighborPosition, 2, 1]] ≠ 0 ||

    enhancementWithResidualsOnQuiver[[otherNeighborPosition, 2, 2]] ≠ 0 ,
                hasOtherNontrivialFGNeighbors = True;
                   If[ enhancementWithResidualsOnQuiver[[j, 2, 1]] ≠ 0 ||
                                enhancementWithResidualsOnQuiver[[j, 2, 2]] ≠ 0 ,
                            hasTwoNontrivialFGNeighbors = True;
                     ];
              ];
          ];
    If[TrueQ[nonTrivialFGFiberCountVec ≠ Null],
         If[nonTrivialFGFiberCountVec[[j]] ≥  2, transverseHasOtherNontrivialFGNeighbors = True;
         ,
           If[nonTrivialFGFiberCountVec[[j]] == 1 &&  (a == 0 && b == 0),
               transverseHasOtherNontrivialFGNeighbors = True;
             ];
        ];
         If[nonTrivialFGFiberCountVec[[i]] ≥  2,
              hasTwoNontrivialFGNeighbors = True;
              hasOtherNontrivialFGNeighbors = True;
              ,
             If[nonTrivialFGFiberCountVec[[i]] == 1,
                  hasTwoNontrivialFGNeighbors = False;
                  If[(an == 0 && bn == 0),
                      hasOtherNontrivialFGNeighbors = True;
                  ];
              ];
        ];
     ](* This is for the case in the post GS positive residuals check *);
  ](* The loop ends the case with more than one curve loop tail. *) ;
];
(* The following is in place for GS fiber contributions. *)
If[ ! transverseIsCompact,
     transverseHasOtherNontrivialFGNeighbors = False
 (* Provided this transverse GS fiber at most meets the quiver.
     We may find further constraints not imposed here. *);
     If[TrueQ[nonTrivialFGFiberCountVec == Null],

      (* Case1: In this case we are checking validity of a quiver while GS fibers are assigned.
          We start in the case without other transverse fibers and recheck total intersection
          contributions separately in other methods. *)
       If[length == 1,
            hasOtherNontrivialFGNeighbors = False;
            hasTwoNontrivialFGNeighbors = False
        (* There may be more than two neighbors. That is checked in other methods. *);
       ];
       If[i == 1 && length ≠ 1,
                 hasTwoNontrivialFGNeighbors = False;
                 If[ enhancementWithResidualsOnQuiver[[i + 1, 2, 1]] ≠ 0 ||
                              enhancementWithResidualsOnQuiver[[i + 1, 2, 2]] ≠ 0 ,
                    hasOtherNontrivialFGNeighbors = True;
                   ];
        ];
        If[i == length && length ≠ 1,
                 hasTwoNontrivialFGNeighbors = False;
                 If[ enhancementWithResidualsOnQuiver[[i - 1, 2, 1]] ≠ 0 ||
                              enhancementWithResidualsOnQuiver[[i - 1, 2, 2]] ≠ 0 ,
                    hasOtherNontrivialFGNeighbors = True;
                   ];
        ];
```

```
          (* Otherwise it's a curve with two neighbors. *)
          If[i > 1 && i < length,
              If[
                 enhancementWithResidualsOnQuiver[[i - 1, 2, 1]] ≠ 0 ||
                                   enhancementWithResidualsOnQuiver[[i - 1, 2, 2]] ≠ 0 ,
                        hasOtherNontrivialFGNeighbors = True;
               ];
              If[
                 enhancementWithResidualsOnQuiver[[i - 1, 2, 1]] ≠ 0 ||
                                   enhancementWithResidualsOnQuiver[[i - 1, 2, 2]] ≠ 0 ,
                           If[hasOtherNontrivialFGNeighbors,
                                hasTwoNontrivialFGNeighbors =  True;,
                                hasOtherNontrivialFGNeighbors = True;
                                hasTwoNontrivialFGNeighbors = False;
                              ];
               ];
          ];
       If[numNonTrivialFGNeighborsAtPositionForGS ≠ 0,
            If[numNonTrivialFGNeighborsAtPositionForGS == 1,
              hasOtherNontrivialFGNeighbors = True;
              hasTwoNontrivialFGNeighbors = False;
           ];
            If[numNonTrivialFGNeighborsAtPositionForGS ≥ 2,
              hasOtherNontrivialFGNeighbors = True;
              hasTwoNontrivialFGNeighbors = True;
           ];
        ](* In this case, the method calling has already counted
            the number of nontrivial FG neighbors so far including the
            other GS curves assigned thus far. *);

    ,
      (* Case 2: In this case we are checking validity of a quiver with GS fibers assigned. *)
        If[nonTrivialFGFiberCountVec[[i]] ≥  2,
               hasOtherNontrivialFGNeighbors = True;
                If[(an == 0 && bn == 0),
                   hasTwoNontrivialFGNeighbors = True;
                   ,
                   If[nonTrivialFGFiberCountVec[[i]] ≥  3,
                       hasTwoNontrivialFGNeighbors = True;
                   ];
                ];
               ,
              If[nonTrivialFGFiberCountVec[[i]] == 1,
                  hasTwoNontrivialFGNeighbors = False;
                  If[(an == 0 && bn == 0),
                      hasOtherNontrivialFGNeighbors = True;
                  ];
              ];
              If[nonTrivialFGFiberCountVec[[i]] == 1,
                  hasTwoNontrivialFGNeighbors = False;
                  hasOtherNontrivialFGNeighbors = True;
              ];
              If[(! TrueQ[transverseCurvePosition == Null]) || (! TrueQ[transverseCurvePosition == -1]),
                   If[nonTrivialFGFiberCountVec[[transverseCurvePosition]] ≥ 2,
                      transverseHasOtherNontrivialFGNeighbors = True;
                    ,
                    If[nonTrivialFGFiberCountVec[[transverseCurvePosition]] ≥ 1
                         &&  (a == 0 && b == 0),
                        transverseHasOtherNontrivialFGNeighbors = True;
                    ];
                 ];
                   If[nonTrivialFGFiberCountVec[[transverseCurvePosition]] == 0,
                        transverseHasOtherNontrivialFGNeighbors = False;
                   ];
              ];
         ];
     ](* This is used in the post GS positive residuals check *);
 ](* This ends the case for a linear quiver. *);
](* This ends the if block for A-type vs. D-type quivers. *);


(* Case:  Contributions to Izerostar with so(8) with A>0.
   See the Izerostar restrictions section.  We use the conclusions for
   the following calculation of residuals contributions. We have even type in this
   case. Here the g contributions are in multiples of three since \gt is a cube. *)
(* IZEROSTAR In, then the others *)
If[ (a ≥ 3 && b == 3 && d == 6 && mon1 == 2),
    If[(an == 0 && bn == 0 && dn ≥ 1) (* This contribution employs that the In fiber is compact. *),
       If[! transverseIsCompact,
             Return[{-1, -1, -1}](* No GS fibers are allowed in this case since I_1 is the
```

```
                                    only allowed fiber and carries no symmetry.
                                    See the section on Izerstar restrictions. *) ;
          ];
        If[transverseHasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}];]
          (* There are too many f,g residuals needed on In in this case. We have that \mu
              in the general form for In can't hold the divisibility needed since a>2 here. *);
        If[a == 3 && dn == 1, Return[{1, 3, 2*3}];];
        If[a == 4 && dn == 1,  Return[{0, 3, 2*3}];];

   If[a ≥ 5 || dn > 1, (*    Print["banned intersection with so(8) A>2 Izerostar, In>1 COMPACT"];*)
                         Return[{-1, -1, -1}];
        ](* Larger fiber types are not allowed here since both fibers are compact. *) ;
      ](* This finishes contributions from In to so(8) with A>0. *) ;
    acont = an;
    max = Max[Ceiling[bn / 3], Ceiling[dn / 6]];
    bcont = 3 max;
    Return[{acont, bcont, 2 bcont}] (* We recall that contributions to \bt come in multiples
                                     of three and those to \dt are twice those. We take
                                     the max here to achieve the resulting constraint. *);
];
(* Case: Izerostar so(8) with B>0 there.  First we treat such a fiber meeting an In fiber.  *)
If[(a == 2 && b ≥ 4 && d == 6 && mon1 == 2),
    If[(an == 0 && bn == 0 && dn > 0 ),
       If[b == 4 && dn == 1, Return[{2, 1, 6}]; ];
   ](* all other In meeting Izerostar{s} B>
    0 cases are banned in the pairs method.  See IzerostarRestrictions.pdf. *) ;
 bcont = bn;
 max = Max[Ceiling[an / 2], Ceiling[dn / 6]]
 (* We must have even at contributions and we have odd type here. *) ;
 acont = 2 max;
 Return[{acont, bcont, 3 acont}];
];
If[(a == 2 && b ≥  4 && d == 6)  && mon1 == 1,
    If[(an == 0 && bn == 0 && dn ≥  1),
      If[! transverseIsCompact,
          If[dn > 2, Return[{-1, -1, -1}];];
          If[dn == 2, Return[{1, 1, 3}];];
      ](* These are the noncompact cases. See IzerostarIntersections.pdf for details. *);
      If[(b == 4 || b == 5)  && dn == 1, Return[{1, 1, 3*1}];];
      If[b == 6  && d == 1, Return[{1, 0, 3*1}]; ];
      If[b == 4 && d == 2, Return[{1, 1, 3*1}]; ];
   ];
    Return[{an, bn, 3 an}](* Otherwise we employ the naive odd type contributions. This
                            can likely be improved. *) ;
];
If[(a ≥ 3 && b == 3 && d == 6) && mon1 == 0,
    If[(an == 0 && bn == 0 && dn ≥ 1),
      If[a == 3 && d == 1, Return[{1, 1, 2*1}];];
      If[a == 4 && d == 1, Return[{0, 1, 2*1}];];
      If[a == 3 && d == 2, Return[{1, 2, 2*2}];];
      If[a == 4 && d == 2, Return[{0, 2, 2*2}];];
   ](* Those are the only In allowed here. Otherwise we are in the following case. *) ;
   Return[{an, bn, 2 bn}](* semi naive even type contributions for non I_n intersections. *) ;
];
 (* A = B = 0 on the Izerostar getting the contributions *)

(* Special Case: Contributions to Izerostar{ss} from III with
     A=B=0 with m=3 when III has orders of vanishing given by (1,3,3). *)
If[ (a == 2 && b == 3 && d == 6 && m == 3) && mon1 == 1 ,
 If[( an == 1 && bn == 3 && dn == 3), Return[{2, 3, 6} ](* We get slightly higher than expected residuals.
                         See the section for Izerostar restrictions. *) ;
    ];
](* Next hybrid type on the Izerostar *);

If[ (a == 2 && b == 3 && d == 6) && (an == 0 && bn == 0 && dn ≥ 1),
    If[dn == 1,
       Return[{0, 0, 1}];
    ];
    If[dn == 2,
       Return[{0, 0, 2}] (* Improvements may be possible here based on
           the monodromy and self intersection along the Izerostar.*);
    ];
    If[dn ≥  3 && mon2 == 1, Return[{-1, -1, -1}]; ]
    (* This induces nonminimality as we can read from the I_3 general form *);
    If[dn ≥ 1 && mon2 == 0,
      Return[{0, 0, dn}] (* This is the naive value.
           Study of triplets including I_2 Izerostar I_2 may enable improvements. *) ;
    ];
](* Restrictions for the intersections of g_2 and I_1 fibers may also be applicable.
     Extending these to the compact case may enable to further restrictions.
```

```
     This ends contributions to Izerostar from I_n.*);

(* Now we treat contributions to In from Instar with A,B>0.
   The structure here is the same as in the previous section.
   Relevant discussion can be found in the Izerostar restrictions section.  *)
If[ ((an ≥ 3 && bn == 3 && dn == 6) || (an == 2 && bn ≥ 4 && dn == 6))
     && mon2 == 2 && (a == 0 && b == 0 && d == 1),
      Return[{4, 6, 10}] (* This employs compactness of I_n.
        Relevant details include the discussion in the Izerostar restrictions section. *);
  ](* This finishes contributions from In from so(8)A,B>0 as only one case allowed. *) ;

If[(an == 2 && bn ≥ 4 && dn == 6)  && mon2 == 1,
    If[(a == 0 && b == 0 && d ≥ 1),
       If[d == 1, Return[{4, 6, 10}];];
       If[d == 2, Return[{4, 6, 8}]; ](* This also employs compactness of I_n. *) ;
     ];
  ];

If[(an ≥ 3 && bn == 3 && dn == 6) && mon2 == 0,
    If[(an == 0 && bn == 0 && dn ≥ 1),
       Return[{4, 6, 9}](* Only I_1,I_2 are allowed.
            Again we use the compactness of the I_n fiber. *) ;
    ];
];
(* Case: I_n meets Instar with contributions to I_n. *)
If[(an == 2 && bn == 3 && dn ≥ 7),
    If[a == 0 && b == 0 && d ≥ 1,
          If[ EvenQ[d], Return[ {2, 3, dn}]; ,
               If[mon2 == 0, Return[{2, 3, dn}];];
               If[mon2 == 1, Return[{2, 3, dn + 1}];];
               ];
        ];
 ];
(* Case: I_n meets Instar contributions to Instar. *)
If[ (an == 0 && bn == 0 && dn ≥ 1),
      If[(a == 2 && b == 3 && d ≥ 7),
            If[mon1 == 0, Return[{0, 0, dn}]];
            If[mon1 == 1, Return[{0, 0, 2 Ceiling[dn / 2]}];];
         ];
 ];
(* Rechecking these contributions to I_n may allow improvements. *)
If[a == 0 && b == 0 && d ≥ 2,
       (* Here the \dt contributions may allow further tightening. *)
   max = Max[ Ceiling[an / 2], Ceiling[bn / 3]];
        (* We will use that s0 cannot vanish in the general form for I_n when at < 4 or bt < 6
           since this would give to many residual contributions.*)
       If[mon1 == 1 && d ≠ 2,
              If[ max > 0 && hasOtherNontrivialFGNeighbors , Return[{-1, -1, -1} ];]
              (* In this case all residuals are already gone. *);
              If[ max > 0 && ! hasOtherNontrivialFGNeighbors ,
                           max = Max[ Ceiling[an / 4], Ceiling[bn / 6]];
                               If[ d == 3,  dcont  = 3 max;];
                               If[d == 4,   dcont  = 4 max;];
                               If[d == 5,  dcont  = 4 max;];
                               If[d == 6,  dcont  = 3 max;];
                               If[d ≥ 7,  dcont  = 4 max;] (* in both even and odd cases.
                                  To give the more conservative constraints not using
                                   the inductive form of I_n at 7,8,9, this should be
                                    reduced to 3max until d\geq 10.
                                          *);
                          Return[{4 max, 6 max , Max[dcont, dn]}];
              ];
       ];
       If[mon1 == 0,
             If[hasOtherNontrivialFGNeighbors  && max > 0
                      (* Only \mu can (and must) vanish here. *) ,
                         If[max > 1, Return[{-1, -1, -1}]; ]
                                (* We need 4 6 f g contributions but we already used some *) ;
                    If[EvenQ[d],
                         Return[{2 max,  3 max, Max[ 2 max,  dn ]}];
                        ];
                    If[! EvenQ[d],
                        Return[{2 max,  3 max, Max[ 3 max,  dn]}];
                       ];
             ] ;
       ];
];
(* Case: Contributions to I_1, reading from M&R and modifying as needed to track A,B *)
If[(a == 0 && b == 0 && d == 1),
       If[  (an ≥ 3 && bn == 4 && dn == 8) && mon2 == 0,
```

```
        Return[{2 Ceiling[ an / 2], 3 Ceiling[ an / 2], 3 Ceiling[ an / 2] + 4}];
    ];
    If[(an == 2 && bn == 3  &&  dn ≥ 7 ) && mon2 == 1,
       If[ dn ≥ 10,
          Return[{-1, -1, -1}];
       ];
       Return[{2, 3, dn + 1}];
    ];
    If[(an == 2 && bn == 3  &&  dn ≥ 7) && mon2 == 0,
       If[dn ≥ 11,
           Return[{-1, -1, -1}];
       ];
       Return[{2, 3, dn}];
    ];
    If[((an ≥ 2 && bn == 3 && dn == 6) || (an == 2 && bn ≥ 3 && dn == 6 )),
        If[ mon2 == 2,
              If[ an == 2 && bn == 3,
                   Return[{2, 3, 7}];
              ];
              If[an ≥ 3,
                  Return[
                         {2 Ceiling[ an / 2], 3 Ceiling[ an / 2], 3 Ceiling[ an / 2] + 3}
                  ];
              ]
(* This contribution can be read from BMM A.4 after imposing z divisibility. *) ;
              If[bn ≥ 4,
                  Return[
                   {2 Ceiling[bn / 3],
                     3 Ceiling[bn / 3],
                      3 Ceiling[bn / 3] + Min[(Ceiling[bn / 3] + 2), bn]}
                  ];
              ];
         ];
         If[ mon2 == 1,
               If[an == 2 && bn == 3, Return[{2, 3, 6}]; ];
               If[an ≥ 3, Return[ {2 Ceiling[ an / 2], 3 Ceiling[ an / 2], 3 Ceiling[ an / 2] + 3}];
               ](* This contribution can also be read from BMM A.4 after
                    imposing z divisibility. *);
               If[ bn ≥ 4,
                  Return[ { 2 Ceiling[bn / 3],
                            3 Ceiling[bn / 3],
                             3 Ceiling[bn / 3] + Min[(Ceiling[bn / 3] + 2), bn]
                          }
                  ];
               ];
         ];
        If[mon2 == 0,
               If[an ≥ 3,
                    Return[ {2 Ceiling[ an / 2], 3 Ceiling[ an / 2], 3 Ceiling[ an / 2] + 3}];];
               If[ bn ≥ 4,
                     Return[{ 2 Ceiling[bn / 3],
                               3 Ceiling[bn / 3],
                                3 Ceiling[bn / 3] + Min[(Ceiling[bn / 3] + 2), bn]}
                     ];
               ];
               If[an == 2 && bn == 3 && dn == 6,
                     If[hasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}];];
                     Return[{4, 6, 9}];
               ];
          ] (* This concludes the mon2==0 case. One may be able to rule out this case for
                 g_2 I_1 intersections. *) ;
       ];
    If[(an == 0 && bn == 0 && dn > 0 ) ,
        Return[ {0 , 0, dn}](* TODO: RECHECK M&R SINCE THIS READS (0,0,dn+3) there. *) ;
    ];
    If[(an ≥ 2  && bn == 2 && dn == 4) ,
        If[mon2 == 1, Return[{2 Ceiling[an / 2], 3 Ceiling[an / 2], 3 Ceiling[an / 2] + 3}]
             (* again by reading from BMM A.4 imposing z div. in each monodromy case *) ;];
        If[mon2 == 0 , Return[{2 Ceiling[an / 2], 3 Ceiling[an / 2], 3 Ceiling[an / 2] + 2}]; ];
     ];
    If[ (an == 1 && bn ≥ 2 && dn == 3 ),
        Return [{2 Ceiling[bn / 3], 3 Ceiling[bn / 3], 3 Ceiling[bn / 3] + (Ceiling[bn / 3] + 1)}];
    ];
    If[(an ≥ 1 && bn == 1 && dn == 2),
        Return[{2 Ceiling[an / 2], 3 Ceiling[an / 2], 3 Ceiling[an / 2] + 1}];
    ];
    If[(an ≥ 1 && bn == 0 && dn == 0),
        Return[{2 Ceiling[an / 2], 3 Ceiling[an / 2], 3 Ceiling[an / 2] }];
    ];
    If[(an == 0 && bn ≥ 1 && dn == 0),
```

```
            Return[{2 Ceiling[bn / 3], 3 Ceiling[bn / 3], 3 Ceiling[bn / 3] + Ceiling[bn / 3] }];
        ];
](* This completes the I_1 case. *) ;


(* Case: Contributions to Instar. *)
If[a == 2 && b == 3 && d ≥ 7,
      If[(d == 9 &&  ( m == 1 || m == 3)) || (d ≥ 10 && m < 4),
        Print["type assignment error...see IntersectionContributions method,
             Case Contribution to Instar"];
        Return[{-1, -1, -1}];
      ](* This is here to catch errors in type assignments and is nonessential. *);
      If[(an ≥ 1 && bn == 1 && dn == 2) (* Here we treat type II on the transverse fiber*),
          max = Ceiling[an / 2];
          acont = 2 max;
          bcont = 3 max;
          If[d == 7 && mon1 == 0,
            Return[{acont, bcont, 3 max + 1}] (* This term can be read from u1^3 g_4 using z|g_4. *) ;
          ];
          If[d ≥ 8 || (d == 7 && mon1 == 1),
                    Print["pair restriction error. Instar with n>1 or n=1
                            without monodromy meeting II is not allowed. See
                             IntersectionContributions method, Case:
                               Contribution to Instar"];
                    Return[{-1, -1, -1}];
             ](* This is here to catch errors in user type assignments.
                 In the main workflow this would have been caught by the monodromy on pair check
                  via nonminimality restrictions. *);
      ](* This ends contributions to Inster from a II. *) ;
      If[m == 3 && ((an == 0 && bn ≥ 0 && dn == 0) || (an ≥ 0 && bn == 0 && dn == 0 )),
          If[d == 7,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[bn == 1 || bn == 2 || bn == 3, Return[{2, 3, 4}]; ];
             ];
             If[mon1 == 1,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[bn == 1, Return[{2, 3, 5}]; ];
             ];
          ];
         If[d == 8,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[bn == 1 , Return[{2, 3, 2}]; ];
             ];
             If[mon1 == 1,
                 If[bn == 1, Return[{2, 3, 2}     ]; ];
                 If[bn ≥ 2, Return[{-1, -1, -1} ]; ];
             ];
          ];
      ](* See the section on Izerostar restrictions for relevant discussion. *) ;
      If[m == 2 && ((an == 0 && bn ≥ 0 && dn == 0) || (an ≥ 0 && bn == 0 && dn == 0 )),
          If[d == 7,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[bn == 1 || bn == 2 || bn == 3, Return[{2, 3, 4}]; ];
                If[bn == 4 || bn == 5 || bn == 6, Return[{4, 6, 7}]; ];
             ];
             If[mon1 == 1,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[bn == 1, Return[{2, 3, 5}]; ];
                If[bn ≥ 2 && bn ≤ 6, Return[{4, 6, 8}]; ];
             ];
          ];
         If[d == 8,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[bn == 1 ,          Return[{2, 3, 2}]; ];
                If[bn == 2,           Return[{4, 6, 4}     ]; ];
                If[bn ≥ 3,            Return[{-1, -1, -1} ]; ];
             ];
             If[mon1 == 1,
                 If[an ≥ 1 && an ≤ 4, Return[{4, 6, 6}];];
                 If[bn == 1,           Return[{2, 3, 2}     ]; ];
                 If[bn == 2,           Return[{4, 6, 4}     ]; ];
                 If[bn ≥ 3,            Return[{-1, -1, -1} ]; ];
             ];
```

```
        ];
      ](* See the section on Izerostar restrictions for relevant discussion. *) ;
      If[m == 1 && ((an == 0 && bn ≥ 0 && dn == 0) || (an ≥ 0 && bn == 0 && dn == 0 )),
          If[d == 7,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[an == 5 || an == 6, Return[{ 6, 9, 9}];];
                If[bn == 1 || bn == 2 || bn == 3, Return[{2, 3, 4}]; ];
                If[bn == 4 || bn == 5 || bn == 6, Return[{4, 6, 8}]; ];
                If[bn == 7 || bn == 8 || bn == 9, Return[{6, 9, 12}]; ];
             ];
             If[mon1 == 1,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[an == 5 || an == 6, Return[{ 6, 9, 9}];];
                If[bn == 1,          Return[{2, 3, 5}]; ];
                If[bn ≥ 2 && bn ≤ 6, Return[{4, 6, 8}]; ];
                If[bn ≥ 7,           Return[{-1, -1, -1}];];
             ];
          ];
         If[d == 8,
             If[mon1 == 0,
                If[an == 1 || an == 2, Return[{2, 3, 3}];];
                If[an == 3 || an == 4, Return[{4, 6, 6}];];
                If[an == 5 || an == 6, Return[{ 6, 9, 9}];];
                If[bn == 1 ,         Return[{2, 3, 2}]; ];
                 If[bn == 2,          Return[{4, 6, 4}    ]; ];
                 If[bn == 3,          Return[{6, 9, 6}    ];];
                 If[bn ≥ 4,           Return[{-1, -1, -1} ]; ];
             ];
             If[mon1 == 1,
                 If[an ≥ 1 && an ≤ 4, Return[{4, 6, 6}    ];];
                  If[an ≥ 5,          Return[{-1, -1, -1} ];];
                 If[bn == 1,          Return[{2, 3, 2}    ];];
                 If[bn == 2,          Return[{4, 6, 4}    ];];
                 If[bn == 3,          Return[{6, 9, 6}    ];];
                 If[bn ≥ 4,           Return[{-1, -1, -1} ];];
             ];
          ];
    ](* See the section on Izerostar restrictions for relevant discussion. *);
    max = Max[ Ceiling[an / 2], Ceiling[bn / 3]];
    If[d ≠ 9,
        If[! EvenQ[d],
             If[mon1 == 1,
                 Return[{2 max, 3 max, Max[3 max, 2 Ceiling[  (dn - 3 max ) / 2] + 3 max ]}];
                 ,
                 Return[{2 max, 3 max, Max [3 max, dn]}];
              ];
        ];
        If[EvenQ[d],
            If[mon1 == 1,
                 Return[{2 max, 3 max,  Max[2 max, 2 Ceiling[dn / 2]]}];
                 ,
                 Return[ {2 max, 3 max,  Max[2 max, dn]}];
            ];
        ];
     ];
    If[d == 9,
          If[mon1 == 1 (* This requires m=2 or m=4. *) ,
               If[ max ≥ 1 || hasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}];
               ](* This requires exceeding the allowed residuals or inducing
                   nonminimality since t_2 cannot be 1 and mu cannnot carry
                   the z divisibility without causing nonminimality.
                   The only other case here involving a transverse In fiber
                   is treated sepeartely. *);
          ];
          If[mon1 == 0,
                    If[bn > 0, Return[{-1, -1, -1}];] (* otherwise 4 6 12 *);
                    If[an > 0 && bn == 0,
                       Return[ {4 Ceiling[an / 4], 6 Ceiling[an / 4], 3 Ceiling[an / 4]}];
                    ](*  Here we can have s_0 divisibility without inducing
                         nonminimality  via t_2. *);
          ];
    ];
];

(* This completes the contributions to Instar. *)

(* Case I_0: *)
```

```
If[ (a ≥ 0 && b == 0 && d == 0) || (a == 0 && b ≥ 0 && d == 0),
     If[a ≥ 1 (* Here we are in an 'even type' so the
    contributions to \dt are determined at twice the \bt contributions *) ,
         If[(an ≥ 0 && bn == 0 && dn == 0),  Return[{ an, 0 , 0 }];];
         If[(an == 0 && bn ≥ 0 && dn == 0),  Return[{0, bn, 2 bn}]; ];
         If[(an ≥ 4 && bn == 5 && dn == 10), Return[{an, bn, dn}];];
         If[(an == 3 && bn ≥ 5 && dn == 9),  Return[{an, bn, 2 bn}];];
         If[(an ≥ 1 && bn == 1 && dn == 2)
             || (an ≥ 3 && bn == 4 && dn == 8) || (an ≥ 2 && bn == 2 && dn == 4),
               Return[{an, bn, 2 bn}];
         ];
         If[(an == 2 && bn == 3 && dn == 7),
                  If[EvenQ[a],
                      Return[{2, 4, 8}];
                     ,
                      Return[{3, 4, 8}];
                  ] (* Reading from BMM B.1 and imposing z^a|f
                      while looking for lowest \sigma term in f/z^a,
                      g/z^0 Delta/z^0 restricted to z = 0, respectively. *) ;
          ];
          If[(an == 2 && bn == 3 && dn == 8),
            If[a == 2 && mon2 == 1, Return[{3, 5, 10}] (* This is the only exceptional case. *);];
            If[mon2 == 0 || mon2 == 1,
                      If[EvenQ[a],
                       Return[{2, 5, 10}];,
                       Return[{3, 5, 10}];
                      ] (* The other cases can be treated by reading from BMM B.2.
                          We require z^a|f and look for lowest \sigma term in
                          f/z^a, g/z^0,Delta/z^0 restricted to z = 0  respectively.
                          We have even type for I_0 with A>0.
                          This boosts us from 9 to 10 for the \dt contribution.
                          We don't need to distinguish monodromy here.*) ;
              ];
           ];
           If[ ((an == 2 && bn == 3 && dn == 9) ),
                If[mon2 == 0,
                       Return[{3, 5, 10}] (* We always get the larger contribution from f here
                                              since only z|s0 prevents 4 6 12 *) ;
                      ,
                       Return[{-1, -1, -1}] (* The larger algebra on I_3^* is not allowed
                                   when meeting Izero A>0 and induces a 4,6,12 point *) ;
                ];
           ];
  ];
  If[b ≥ 1 (* Here we are in an 'odd type'. Thus the contributions to \dt are thrice
           the \at contributions *) ,
         If[(an ≥ 3 && bn == 4 && dn == 8) || (an ≥ 2 && bn == 2 && dn == 4),
              If[mon2 == 1 &&  ! (EvenQ[b]), Return[{an, bn + 1, 3 an}];, Return[{an , bn, 3 an}];]; ];
         If[(an == 1 && bn ≥ 2 && dn == 3 ) || (an == 3 && bn ≥ 5 && dn == 9 )
               || (an ≥ 1 && bn == 1 && dn == 2) ,
             Return[{an, bn, 3 an}];
         ];
         If[(an == 2 && bn == 3 && dn ≥ 7),

    If[transverseIsCompact (* In this case the transverse curve is compact and we can say more via
                    via consideration of mn, the transverse neg self intersection *),
                If[mn == 3,
                       If[ b == 1 || b == 2, Return[{3, 4, 9}] (* Some intersections are ruled out
                            in the forbidden pairs method.
                            The rest give the contribution to the Izero as (3,4,9) here,
                            where we have Izero with B>0. *);
                       ];
                       If[b == 3, Return[{3, 3, 9}];
                       ];
                ];
                If[mn == 2,
                       If[ Mod[b, 3] ≠ 0, Return[{3, 4, 9}];,
                                             If[(mon2 == 0 && b == 3) || (b == 6) ,    Return[{3, 3, 9}];
                                              ];
                           ];
                ];
                If[mn == 1,
                     If[ Mod[b, 3] ≠ 0,
                        Return[{3, 4, 9}];
                       ,
                        If[(mon2 == 0 && b == 3) || (b == 6) ,
                           Return[{3, 3, 9}];
                        ];
                     ];
```

```
                ];
                If[ Mod[b, 3] == 0,
                    Return[ {2, 3, dn}];
                ] (* We may be able to improve this through accounting for monodromy
                    along the Instar and treating m==1 OR m==2 on the Instar. *);
                If[! Mod[b, 3] == 0, Return[{3, 3, Max[dn, 9]}];
                ];
            ];
        ];
    ] ;
];
(* This ends the cases for I_0. We note the Izerostar contributions to Izero are
   treated via Izerostar below. *)

(* Cases for II: We read from M&R and modify their arguments to track A,B.  *)
     (* Note that in these cases, the \dt contribution is twice the \bt
        contribution since we have even type on a II. *)
 If[ (a ≥ 1 && b == 1 && d == 2),
      If[(an ≥ 3 && bn == 4 && dn == 8 ),
           If[ mon2 == 0 , Return[{an, 4, 8}];
           ];
           If[mon2 == 1 ,
                Return[{-1, -1, -1}];
           ];
      ];
      If[ ( an == 2 && bn == 3 && dn ≥ 7) ,
           If[ mon2 == 0  && dn == 7,
                If[Mod[a, 2] == 0,
                  Return[{2, 4, 8}]
                  (* We note a correction for M&R here. We get lowered contribution here when
                     A=1 for example on the II. Orders (2,1,2) gives a contribution (2,4,8)
                     rather than (3,4,8). This leaves more freedom for other curves. The M&R
                     conclusions might be rechecked for simiar revisions elsewhere.
                *) ;
              ];
              If[Mod[a, 2] ≠ 0,
                 Return[{3, 4, 8}];
              ];
           ];
           If[mon2 == 1  || dn ≥ 8,
             Return[{-1, -1, -1}];
           ];
       ];
       If[(an ≥ 2 && bn ≥ 3 && dn == 6 ),
            If[ mon2 == 0,
               Return[{an, bn, 6}];
            ](* One might be able to improve this intersection in all monodfromy cases
                 via tracking A,B. *) ;
            If[mon2 == 1,
                   If[ an == 2 && bn == 3,
                       Return[{2, 4, 8}];
                   ] (* We note there is an error in M&R for the subcase (2,4,6) since
                        we are on a II. The \dt cont is twice \bt contribution since II
                        has even type. *) ;
                   If[an ≥ 3, Return[{an, 5, 10}];
                   ] (* using here from reading after 3.14 in M&R that
                          g/\sigma^3 |
   _{\sigma = 0 } is a cube and that \sigma^3 z^3 divides g at the intersection
                          so 5 orders beyond the a priori 1 in g along the II *);

  If[ bn ≥ 4, Return[{2, bn, 2 bn}] (* we have no similar argument here as in the an>0 case,
                              but along II, the Deltat contributions are twice those of gt *) ; ];
              ];
             If[mon2 == 2, Return[{-1, -1, -1}]; ];
       ];
       If[(an == 0 && bn == 0 && dn ≥ 10 ),
            If[ mon2 == 0 , Return[{Floor[dn / 2], 2 Floor[dn / 2], 4 Floor[dn / 2]}]; ];
            If[mon2 == 1,
                Return[{-1, -1, -1}];
            ];
         ];
        If[ (an == 0 && bn == 0 && dn ≥ 4 && dn ≤ 9),
          If[a == 1 && dn == 4,
               Return[{2, 4, 8}];
          ];
          If[a == 1 && dn == 5 && mon2 == 0,
               Return[{2, 4, 8}];
          ];
          If[a == 2 && dn == 3 && mon2 == 0,
               Return[{0, 3, 6}];
```

```
          ];
          If[a == 2 && dn == 3 && mon2 == 1,
               Return[{2, 3, 6}];
          ];
          If[a == 2 && dn == 4 && mon2 == 0,
               Return[{0, 4, 8}];
          ];
          If[a == 2 && dn == 4 && mon2 == 1,
               Return[{2, 4, 8}];
          ];
          If[mon2 == 1 && dn ≥ 5,
               Return[{-1, -1, -1}];
          ];
          If[a == 2 && dn == 5 && mon2 == 0 && transverseIsCompact,
               Return[{0, 5, 10}];
          ];
          If[a == 2 && dn == 5 && mon2 == 0 && ! transverseIsCompact,
               If[at ≥ 2, Return[{2, 4, 8}];, Return[{0, 5, 10}];];
          ];
          If[a == 2 && dn == 6 && mon2 == 0 && transverseIsCompact,
               Return[{0, 6, 12}];
          ];
          If[a == 2 && dn == 6 && mon2 == 0 && ! transverseIsCompact,
               If[at ≥ 2, Return[{2, 4, 8}];, Return[{0, 6, 12}];];
          ];
          If[a ≥ 2 && dn ≥ 7 && mon2 == 0,
               Return[{-1, -1, -1}](* Note that this is actually only
                              known banned via the inductive form for I_n
                              and hence the cases n=7...9 may be allowed. *);
          ];
          If[a == 2 && dn == 3 && mon2 == 0,
               Return[{2, 4, 8}];
          ];
          If[mon2 == 0 && dn == 7,
               If[a ≥ 2,
                  Return[{-1, -1, -1}];
                  ,
                  Return[{3, 6, 12}];
                ];
           ];
          If[mon2 == 0 && dn ≥ 8,
               (* Note: the following cases: dn = 7, 8, 9,
are not fully constrained since we do not know the general form for I_n n= 7, 8,9 *)
                Return[{Floor[dn / 2], Max[4, 2 * Floor[dn / 2]], 4 Floor[dn / 2]}];
              ];
          ];
          If[(an == 0 && bn == 0 && dn == 3),
              If[ mon2 == 1,
                     If[Mod[a, 4] == 0,
                        acont = 0;
                        ,
                        acont = 1;
                     ];
                     Return[{acont, 3, 6}];
               ];
              If[ mon2 == 0,
                       If[Mod[a, 2] == 0,
                         acont = 0;
                         ,
                         acont = 1;
                       ];
                       If[a ≥ 3 && transverseIsCompact, Return[{-1, -1, -1}];];
                       If[a == 1,
                         Return[{acont, 2, 4}];
                          ,
                         Return[{acont, 3, 6}];
                        ];

              ];
         ];
          If[(an == 0 && bn == 0 && dn == 4),
              If[ mon2 == 1,
                     If[Mod[a, 4] == 0,
                        acont = 0;
                        ,
                        acont = 2;
                     ];
                     Return[{acont, 4, 8}];
               ];
              If[ mon2 == 0,
```

```
                    If[Mod[a, 2] == 0,
                      acont = 0;
                      ,
                      acont = 2;
                    ];
                     Return[{acont, 4, 8}];
               ];
            ];
           If[(an == 0 && bn == 0 && dn == 2), Return[{1, 2, 4}]; ];
           If[(an == 0 && bn == 0 && dn == 1), Return[{1, 1, 2}]; ];
           If[(an ≥ 2 && bn == 2 && dn == 4),
                 If[mon2 == 1, Return[{an, 3, 6}];];
                 If[mon2 == 0, Return[{an, 2, 4}];];
            ];
            If[(an == 1 && bn ≥ 2 && dn == 3 ), Return[{1, bn, 2 bn }];]
  (* order of Delta along II determined by g rest. *);
            If[(an ≥ 1 && bn == 1 && dn == 2), Return[{an, 1, 2}];];
            If[(an ≥ 1 && bn == 0 && dn == 0 ) , Return[{ an, 0 , 0}]; ];
            If[(an == 0 && bn ≥ 1 && dn == 0), Return[{0, bn, 2 bn}]; ];

];
   (*  This completes the II case *)

(* If even type on C', the curve which is getting the contributions: *)
If[( 3 a > 2 b),
     If[ a ≥ 3 && b == 4 && d == 8 (* IV* s gets even \bt contributions: *) ,
        If[mon1 == 1, Return[{an, 2 Ceiling[bn / 2], 4 Ceiling [bn / 2]}];];
        If[mon1 == 0, Return[{an, bn, 2 bn}];];
     ];
     (* Izerostar so(8) pos A contributions to C' *)
     If[(an ≥ 3 && bn == 3 && dn == 6),
              If[mon2 == 1, Return[{-1, -1, -1}];];
              If[a ≥ 2 && b == 2 && d == 4 && mon1 == 1, Return[{an, 4, 8}];]
   (* If IV s, then g res cont is even.  *) ;
              If[mon2 == 2,
                   If[Mod[b, 3] == 0, Return[{an, 3, 6}]; , Return[{an, 4, 8}];]
     (* Izerostar rules for so(8) with A>0 *) ;
               ];
              If[mon2 == 0,
                   Return[{an, 3, 6}];
               ];
     ];
     (* Izerostar so(8) pos B contributions to C' *)
     If[(an == 2 && bn ≥ 4 && dn == 6),
           If[mon2 == 0, Return[{-1, -1, -1}];];
           If[mon2 == 2,
              If[Mod[a, 2] == 0,
                 If[a ≥ 2 && b == 2 && d == 4 && mon1 == 1,

      Return[ { 2, 2 Ceiling[bn / 2], 4 Ceiling[bn / 2]}](* If IV s, then g res cont is even.  *) ;
                  ];
                 Return[{2, bn, 2 bn}] (*  If not IV s,
     then simply what we expect for even type getting cont from Izerostar *);
                 ];
              If[! (Mod[a, 2] == 0),
                  If[a ≥ 2 && b == 2 && d == 4 && mon1 == 1,

      Return[ { 3, 2 Ceiling[bn / 2], 4 Ceiling[bn / 2]}](* If IV s, then g res cont is even.
                                                                                f
        cont raised when a is odd via Izerostar rules. *) ;
                  ];
                  Return[{3, bn, 2 bn}] (* if not IV s,
     then simply bn g cont and f cont still raised from Izerostar rules. *);
                  ];
            ];
          If[mon2 == 1,
                If[a ≥ 2 && b == 2 && d == 4 && mon1 == 1,

     Return[ { 2, 2 Ceiling[bn / 2], 4 Ceiling[bn / 2]}](* If IV s, then g res cont is even.  *) ;
                  ];
                Return[{2, bn, 2 bn}];
            ];
     ];
      If[((a ≥ 2 && b == 2 && d == 4 ) && (an == 0 && bn == 0 && dn ≥ 1)),
          If[! transverseIsCompact,
              If[a ≥ 3 && dn ≥ 4,
                   Return[{-1, -1, -1}](*I_(n>3) are forbidden for GS fibers on IV as recorded
                                                 in BMM and IVIntersectionContributions.pdf. *);
               ];
              If[dn == 2,
```

```
                    If[Mod[a, 2] == 0, acont = 0;, acont = 1; ];
                    bcont = 2;
                    Return[{acont, bcont, 2 bcont}];
                ];
                If[a ≥ 3 && dn == 3,
                     If[mon2 == 0,
                         If[Mod[a, 2] ≠ 0, Return[{1, 3, 6}];];
                         If[Mod[a, 2] == 0,
                            If[at == 0 && mon1 == 1,
                                If[m == 1, Return[{0, 3, 6}];];
                                If[m ≥ 2, Return[{-1, -1, -1}];];
                            ];
                            If[at ≥ 1,
                               If[m == 1, Return[{1, 2, 4}];];
                               If[m ≥ 2, Return[{1, 2, 4}];];
                            ](* Note that there are actually two choices.
                                (0,3,6) is also an option when m=1.*);
                         ];
                     ];

                     If[mon2 == 1,
                        If[Mod[a, 4] == 0,
                         If[at ≥ 1 && bt ≥ 2 && dt ≥ 4, Return[{1, 2, 4}];];
                         If[m ≥ 2 && mon1 == 1, Return[{1, 2, 4}];];
                         If[at == 0 && mon1 == 0 && m ≥ 2, Return[{0, 3, 6}];];
                         If[at == 0 && mon1 == 1 && m ≥ 2, Return[{-1, -1, -1}];];
                         Return[{1, 2, 4}]
      (* This is here as a fail safe to catch any missed cases. *);
                        ];
                        If[Mod[a, 4] ≠ 0,
                              Return[{1, 2, 4}];
                        ];
                     ];
                ](* End A>0 and dn==3 case. Now the A=0 case. *);
                If[a == 2 && dn ≥ 3,
                   If[dn ≥ 9 && mon1 == 1, Return[{-1, -1, -1}];](* \bt limitations on IV. *);
                   If[dn ≥ 4 && mon2 == 1, Return[{-1, -1, -1}];](* Non minimal intersection *) ;
                   If[dn ≥ 9, Return[{-1, -1, -1}];];
                   If[8 ≥ dn ≥ 4 && mon1 == 0 && mon2 == 0, Return[{0, dn, 2 * dn}];];
                   If[dn == 3 && mon1 == 1 && m ≥ 2, Return[{1, 2, 4}];];
                   If[dn == 3 && mon1 == 1 && mon2 == 0 && at == 0 && m == 1, Return[{0, 3, 6}];];

    If[dn == 3 && mon1 == 1 && mon2 == 1 && at ≥ 1, Return[{1, 2, 4}];](* Make a choice in favor of

     smaller \dt contribution. *) ;
                    Return[{0, dn, 2 dn}](* Catch any remaining cases *) ;
                ];

        ](* The above section for use in GS. Now the compact case for the quiver. *) ;
        If[a ≥ 3, If[transverseHasOtherNontrivialFGNeighbors,
                     Return[{-1, -1, -1}];];] (* Here we have used too many residuals along the In *);
        If[a ≥ 5, Return[{-1, -1, -1}]; ]
                      (* Here we have used too many residuals along the In--
                       this is already tracked by residuals,
but in case of bad user input we add this. *);

        If[dn == 1,
             If[a == 2 , acont = 0; bcont = 1;];
             If[a == 3,  acont = 1; bcont = 1;];
             If[a == 4,  acont = 0; bcont = 1;];
           acont = Max[acont, an];
           bcont = Max[bcont, bn];
             If[mon1 == 1, Return[{acont, 2 Ceiling[bcont / 2], 4 Ceiling [bcont / 2]}];];
             If[mon1 == 0, Return[{acont, bcont, 2 bcont}];];
        ];
       If[dn == 2,
             If[a == 2, acont = 0; bcont = 2;];
             If[a == 3, acont = 1; bcont = 2;];
             If[a == 4, acont = 0; bcont = 2;];
             acont = Max[acont, an];
             bcont = Max[bcont, bn];
             If[mon1 == 1, Return[{acont, 2 Ceiling[bcont / 2], 4 Ceiling [bcont / 2]}];];
             If[mon1 == 0, Return[{acont, bcont, 2 bcont}];];
        ];
       If[dn ≥ 3 && mon2 == 1 && transverseHasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}];]
                             (* In this case we have used too many residuals along the In.   *);
       If[dn ≥ 3 && mon2 == 0 && a > 2, Return[{-1, -1, -1}];]
                             (* In this case we cannot meet the required form along the In
                                                the In since \mu is square free since otherwise
                                                       we use too many residuals along the In. *);
```

```
        If[dn == 3,
             If[mon2 == 1,
              If[a == 2, acont = 1; bcont = 2;];
              If[a == 3,  acont = 1; bcont = 3;];
              If[a == 4, acont = 0; bcont = 3;];
              ];
             If[mon2 == 0,
              If[a == 2, acont = 0; bcont = 3;];
             ];
             If[mon1 == 1, Return[{acont, 2 Ceiling[bcont / 2], 4 Ceiling [bcont / 2]}];];
             If[mon1 == 0, Return[{acont, bcont, 2 bcont}];];
         ];
        If[dn ≥ 4 && mon2 == 1,
              Return[{-1, -1, -1}] (* This is banned since it induces a 4,6,12 point. *);
         ];
        If[mon2 == 0 && a == 2,
             If[dn == 4,  acont = 0; bcont = 4;];
             If[dn == 5, acont = 0; bcont = 5;];
             If[dn == 6,  acont = 0; bcont = 6;] (* These were checked for each value of dn.  *);
             If[dn ≥ 7, acont = 0; bcont = dn;] ;
             If[mon1 == 1,
                bcont = Max [2 Ceiling[bn / 2], Ceiling [dn / 2]];
                bcont = 2 Ceiling[bcont / 2];
                Return[{acont, bcont, 2 bcont}];
             ];
             If[mon1 == 0,
                Return[{acont, bcont, 2 bcont}];
              ];
        ];
   ];
    If[ a ≥ 2 && b == 2 && d == 4 ,
        If[mon1 == 1,
            bcont = Max [2 Ceiling[bn / 2], Ceiling [dn / 2]];
            bcont = 2 Ceiling[bcont / 2];
            Return[{an, bcont, 2 bcont}];
         ];
        If[mon1 == 0,
            bcont = Max [bn, Ceiling [dn / 2]];
            Return[{an, bcont, 2 bcont}];
        ];
    ](* IV s gets even \bt contributions. Intersections with Izerostar are special as \at
                               cont is raised too; treated above. *);
];

(* If odd type on C': *)
If[(3 a < 2 b ),
        (* Contributions to III from In *)
    If[(a == 1 && b ≥ 2 && d == 3) && (an == 0 && bn == 0 && dn ≥ 1) ,
         If[! transverseIsCompact,
              If[dn > 4 && mon2 == 1, Return[{-1, -1, -1}]; ];
              If[dn == 2,
                   If[b == 2 || b ≥ 3, Return[{1, 1, 3}]; ];
                   If[b == 3, Return[{1, 0, 3}]; ];
              ];
              If[dn == 3 && mon2 == 0, Return[{2, 2, 6}]; ];
              If[dn == 3 && mon2 == 1, Return[{2, 2, 6}]; ];
              If[dn == 4 && mon2 == 0, Return[{2, 2, 6}]; ];
              If[dn == 4 && mon2 == 1, Return[{2, 3, 6}]; ];
               acont = Ceiling[dn / 2];
               bcont = acont;
              Return[ {acont, bcont, 3 acont}];
           ](* GS fiber contributions above. *) ;

          (* Compact pair intersection contributions. *)
              If[dn == 1 ,
                    If[ Mod[b, 3] == 0,
                         If[ b == 3 || b == 6,
                             acont = 1;
                             bcont = 0;
                             Return[{acont, bcont, 3 acont}];
                            ];
                     ];
                    If[Mod[b, 3] ≠ 0 ,
                         acont = 1;
                         bcont = 1;
                         Return[{acont, bcont, 3 acont}];
                    ];
                 ];
              If[dn == 2 ,
                    If[b == 2, acont = 1; bcont = 1; Return[{acont, bcont , 3 acont}]; ];
```

```
                    If[b == 3 && transverseHasOtherNontrivialFGNeighbors,
  acont = 2; bcont = 0; Return[{acont, bcont , 3 acont}]; ];
                    If[b == 3 && ! transverseHasOtherNontrivialFGNeighbors,
  acont = 1; bcont = 1; Return[{acont, bcont , 3 acont}]; ];
                    If[b ≥ 4 && transverseHasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}]; ]
                                   (* used too many f and g res on In at \sigma^0 term *) ;
                    If[! transverseHasOtherNontrivialFGNeighbors,

  If[b == 4, acont = 2; bcont = 1; Return[{acont, bcont , 3 acont}]; ] (* total order (3,5,9) *) ;
                       If[b == 5, acont = 2; bcont = 1; Return[{acont, bcont , 3 acont}]; ] ;
                       If[b == 6, acont = 2; bcont = 0; Return[{acont, bcont , 3 acont}]; ]

  (* total order (3,6,9) at intersection. Cancellation needed in g at ord 2*) ;
                      ];
                ];

If[dn ≥ 3 && mon2 == 1 && transverseHasOtherNontrivialFGNeighbors, Return[{-1, -1, -1}]; ];
              If[dn ≥ 3 && mon2 == 0 && b ≥ 4, Return[{-1, -1, -1}]; ];
              If[dn == 3,
                  If[mon2 == 0,
                       If[b == 2 ,  acont = 2;
   bcont = 2;
   Return[{acont, bcont, 3 acont}];];
                       If[b == 3,  acont = 3;
   bcont = 0; Return[{acont, bcont, 3 acont}];];
                    ];
                  If[mon2 == 1,
                       If[b == 2, acont = 2; bcont = 2;
   Return[{acont, bcont, 3 acont}] (* Here z|\phi0, z|f2, z|phi1.  *); ];
                       If[b == 3, acont = 2; bcont = 2;  Return[{acont, bcont, 3 acont}]
                              (* Here z|\phi0, z|f2, z^2|phi1.Total order (3,5,9) *); ];
                       If[b ≥ 4,  Return[{-1, -1, -1}]

    (* This above almost agrees with M&R which says III B≥2 forbidden to meet In s n≥3.
                              It seems n==3 is not allowed.  *); ];
                     ];
                 ];
              If[dn ≥ 4 && b ≥ 4 && mon2 == 1, Return[{-1, -1, -1}];]
(* Since In for n=3 is banned without mon meeting III, this is follows

 as here we only impose futher constraints.  *) ;
              If[dn == 4,
                  If[mon2 == 0,
                          If[b == 2, acont =  2; bcont = 2;  Return[{acont, bcont, 3 acont}];  ];
                          If[b == 3, acont = 4; bcont = 0;  Return[{acont, bcont, 3 acont}];  ];
                    ];
                  If[mon2 == 1 ,
                          If[b == 2, acont = 2; bcont = 3;   Return[{acont, bcont, 3 acont}];  ];
                          If[b == 3, acont = 2; bcont = 2;   Return[{acont, bcont, 3 acont}];  ];
                    ];
                ];
              If[dn ≥ 5 ,
                  If[mon2 == 1, Return[{-1, -1, -1}];] (* again, 4,6,12 points *) ;
                  If[mon2 == 0,

  If[b == 2, acont = Ceiling[dn / 2]; bcont = Ceiling[dn / 2];   Return[{acont, bcont, 3 acont}];];
                          If[b == 3,  acont = dn; bcont = 0;   Return[{acont, bcont, 3 acont}];]
                              (* From residuals on in f on III,
  we have I7 is the maximal fiber that can meet compact III-(1,3,3) ie. B=1 *) ;
                    ];
              ];
    ] (* This ends contributions to III from In *) ;

  (* Izerostar so(8) pos A contributions to C' *)
  If[(an ≥ 3 && bn == 3 && dn == 6),
          If[mon2 == 2,
            If[Mod[b, 3] == 0, Return[{an, 3, 3 an}];, Return[{an, 4, 3 an}];];
         ];
          If[mon2 == 0,
            Return[{an, 3, 3 an}];
          ];
          If[mon2 == 1, Return[{-1, -1, -1}];];
  ];
  (* Izerostar so(8) pos B contributions to C' *)
  If[(an ≥ 2 && bn ≥ 4 && dn == 6),
       If[mon2 == 2,
           If[Mod[a, 2] == 0, Return[{2, bn, 6}];, Return[{3, bn, 9}];];
      ];
      If[mon2 == 1,
```

```
                    Return[ {2, bn, 6}];
              ];
              If[mon2 == 0, Return[{-1, -1, -1}];];
             ];
    ];

    Return[contributionsToResidualsOfCurve];
   ];

(* ComputeAPosterioriResidualsConsideringMonodromy:
 Find the reduced residuals using intersection contributions. *)

ComputeAPosterioriResidualsConsideringMonodromyOnLinearQuiver[
    enhancementWithResiduals_, monodromyOnQuiver_, nonTrivialFGTransverseFiberCountVec_: Null] :=
   Module[{i, j, k, rawResidualsAtSite, residualsAtSite, adjacencyMatrix, quiver, types,
     mon, residualsList, intersectionContributions, enhancementWithResidualsAtSiteAPost,
     enhancementWithResidualsOnQuiverToReturn, naiveIntersectionContributions,
     modifiedIntersectionContributions, inAtBtMinGen, contributions, enhWithResiduals},

    enhancementWithResidualsAtSiteAPost = {};
    enhancementWithResidualsOnQuiverToReturn = {};
    intersectionContributions = {};
    residualsAtSite = {};
    naiveIntersectionContributions = {} ;
    modifiedIntersectionContributions = {};
    contributions = {};
    inAtBtMinGen = 0;
    enhWithResiduals = enhancementWithResiduals;
     If[Length[enhWithResiduals[[1]]] == 2,
       enhWithResiduals = ComputeResidualVanishings[enhWithResiduals];
    ](* In this case of faulty input, we don't have the residuals so we compute them. *);
    types = enhWithResiduals;
    mon = monodromyOnQuiver;
    quiver = {};
    quiver  = Flatten[Take[enhWithResiduals, All, 1]];
    adjacencyMatrix = ATypeAdjacencyMatrix[quiver];
    residualsList = {};
    (*the indexes 1,2,3 in adjacencyMatrix below keep track of f,g,Delta *)
    For[i = 1, i ≤ Length[types], i ++,

     rawResidualsAtSite = { -4 * (quiver[[i]] - 2) + types[[i, 2, 1]] * quiver[[i]], -6 * (quiver[[i]] - 2) +
        types[[i, 2, 2]] * quiver[[i]], -12 * (quiver[[i]] - 2) + types[[i, 2, 3]] * quiver[[i]]}
     (* residuals before subtracting contributions from transverse intersections *) ;
     residualsAtSite = rawResidualsAtSite;
     For[j = 1, j ≤ Length[quiver], j ++,
         If[(i ≠ j) && (adjacencyMatrix[[i, j]] ≠ 0),
        intersectionContributions = IntersectionContributionsToResiduals[types[[i]],
          types[[j]], monodromyOnQuiver[[i]], monodromyOnQuiver[[j]], types, i, j ];
           (* next line is to Catch Errors and toss the bad intersections: *)
           If[(intersectionContributions[[1]] < 0
                    || intersectionContributions[[2]] < 0
                         ||  intersectionContributions[[3]] < 0 ) , Return[{-1}] ;
            ];
            naiveIntersectionContributions = types[[j]][[2]];
            modifiedIntersectionContributions =
                  {Max[naiveIntersectionContributions [[1]], intersectionContributions[[1]] ],
                     Max[naiveIntersectionContributions [[2]], intersectionContributions[[2]] ],
                     Max[naiveIntersectionContributions [[3]], intersectionContributions[[3]] ]};

               If[(types[[i, 2]] == {2, 3, types[[i, 2, 3]]} &&  types[[i, 2, 3]] ≥ 7)
                        || (types[[i, 2]] == {0, 0, types[[i, 2, 3]]} &&  types[[i, 2, 3]] ≥ 1),
                         inAtBtMinGen = Max[Ceiling[modifiedIntersectionContributions[[1]] / 2],
                                            Ceiling[modifiedIntersectionContributions[[2]] / 3]];
                            contributions = {2 inAtBtMinGen, 3 inAtBtMinGen,
                                                   modifiedIntersectionContributions[[3]]}; ,
                            contributions = modifiedIntersectionContributions;
                  ];
               residualsAtSite = residualsAtSite - contributions;
          ];
       ];
     If[residualsAtSite[[1]] < 0 || residualsAtSite[[2]] < 0 || residualsAtSite[[3]] < 0, Return[{-1}];];
     enhancementWithResidualsAtSiteAPost =  {types[[i, 1]], types[[i, 2]], residualsAtSite};
     AppendTo[ enhancementWithResidualsOnQuiverToReturn , enhancementWithResidualsAtSiteAPost];
    ];

    Return[enhancementWithResidualsOnQuiverToReturn];
   ];
```

```
WrapDTypeEnhancement[enh_] := Module[{ branchPositions, length, wrappedEnh },
   length = Length[enh];
    branchPositions = Table[If[Depth[enh[[i]]] > 3, 1, 0], {i, 1, length}];
   wrappedEnh = Table[If[branchPositions[[i]] == 0,
                             {enh[[i]]}
                             ,
                             enh[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have the same depth.*);
   Return[wrappedEnh];
  ];

WrapBareOrders[bareFGDOrders_] := Module[{i, wrappedOrders},
   wrappedOrders = Table[If[Depth[bareFGDOrders[[i]]] ≤ 2,
                                    {bareFGDOrders[[i]]}
                                    ,
                                    bareFGDOrders[[i]]
                             ]
                       , {i, 1, Length[bareFGDOrders]}];
   Return[wrappedOrders];
  ];

ComputeAPosterioriResidualsConsideringMonodromyDType[enh_, monodromyOnQuiver_] :=
  Module[{i, j, enhWithRes, nontrivialFGVec, totalDegrees, length, branchPositions,
    backboneEnh, wrappedMon, wrappedEnh, contributions, contributionsAtSite, leftCont,
    rightCont, topCont, bottomCont, wrappedTotalDegrees, wrappedEnhWithAPostResiduals,
    residuals, enhWithAPostResiduals, wrappedNonTrivialFGVec, contributionsAtCol},
   enhWithRes = ComputeResidualVanishingsDType[enh];
   If[Length[Select[Flatten[enhWithRes], (# < 0) &]] > 0, Return[-1];];
   nontrivialFGVec = TotalNumberOfNontrivialTransverseFGFibersAsVec[enhWithRes];
   totalDegrees = TotalAPrioriVanishingsAvailable[enhWithRes];
   length = Length[enhWithRes];
    branchPositions = Table[If[Depth[enhWithRes[[i]]] > 3, 1, 0], {i, 1, length}];
    backboneEnh = Table[If[branchPositions[[i]] == 0,
                             enhWithRes[[i]]
                             ,
                             enhWithRes[[i, 1]]
                            ]
                          , {i, 1, length}];
   wrappedMon = Table[If[branchPositions[[i]] == 0,
                             {monodromyOnQuiver[[i]]}
                             ,
                             monodromyOnQuiver[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have length 1 rather than 0.*);
   wrappedEnh = Table[If[branchPositions[[i]] == 0,
                             {enhWithRes[[i]]}
                             ,
                             enhWithRes[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have the same depth.*);
   wrappedTotalDegrees = Table[If[branchPositions[[i]] == 0,
                             {totalDegrees[[i]]}
                             ,
                             totalDegrees[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have the same depth.*);
   wrappedNonTrivialFGVec = Table[If[branchPositions[[i]] == 0,
                             {nontrivialFGVec[[i]]}
                             ,
                             nontrivialFGVec[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have the same depth.*);
   Catch[
    contributions = Flatten[Reap[
                        For[i = 1, i ≤ length, i ++,
                         contributionsAtCol = Flatten[Reap[
                             For[j = 1, j ≤ Length[wrappedMon[[i]]], j ++,
                              If[j ≥ 2,
                                        leftCont = {0, 0, 0};
                                        rightCont = {0, 0, 0};
                                        bottomCont =
                      IntersectionContributionsToResiduals[ wrappedEnh[[i, j]], wrappedEnh[[i, j - 1]],
                                                            wrappedMon[[i, j]],
```

```
                    wrappedMon[[i, j - 1]], wrappedEnh, {i, j}, {i, j - 1},
                                                   Null, Null, wrappedNonTrivialFGVec, True];
                                 If[TrueQ[j == Length[wrappedMon[[i]] ] ],
                                    topCont = {0, 0, 0};
                                 ];
                                 If[! TrueQ[j == Length[wrappedMon[[i]] ] ],
                                       topCont =
                     IntersectionContributionsToResiduals[ wrappedEnh[[i, j]], wrappedEnh[[i, j + 1]],
                                                      wrappedMon[[i, j]],
                      wrappedMon[[i, j + 1]], wrappedEnh, {i, j}, {i, j + 1},
                                                      Null, Null, wrappedNonTrivialFGVec, True ];
                                 ];
                           ];
                        If[j == 1,
                                      bottomCont = {0, 0, 0};
                                 If[TrueQ[ branchPositions[[i]] == 0],
                                      topCont = {0, 0, 0};];
                                 If[TrueQ[branchPositions[[i]] == 1],
                                      topCont =
                     IntersectionContributionsToResiduals[ wrappedEnh[[i, j]], wrappedEnh[[i, j + 1]],
                                                   wrappedMon[[i, j]],
                      wrappedMon[[i , j + 1]], wrappedEnh, {i, j}, {i, j + 1},
                                                   Null, Null, wrappedNonTrivialFGVec, True];];
                                 If[i == 1,
                                      leftCont = {0, 0, 0};
                                      rightCont = IntersectionContributionsToResiduals[
                                                        wrappedEnh[[i, j]],
                     wrappedEnh[[i + 1, j]], wrappedMon[[i, j]], wrappedMon[[i + 1, j]],
                                                        wrappedEnh, {i, j},
                     {i + 1, j}, Null, Null, wrappedNonTrivialFGVec, True];];
                                 If[i == length,
                                      rightCont = {0, 0, 0};
                                      leftCont = IntersectionContributionsToResiduals[
                                                         wrappedEnh[[i, j]],
                     wrappedEnh[[i - 1, j]], wrappedMon[[i, j]], wrappedMon[[i - 1, j]],
                                                   wrappedEnh, {i, j}, {i - 1, j},
                     Null, Null, wrappedNonTrivialFGVec, True];];
                                 If[1 < i < length,
                                      leftCont =  IntersectionContributionsToResiduals[
                                                              wrappedEnh[[i, j]],
                     wrappedEnh[[i - 1, j]], wrappedMon[[i, j]], wrappedMon[[i - 1, j]],
                                                       wrappedEnh, {i, j},
                     {i - 1, j}, Null, Null, wrappedNonTrivialFGVec, True];
                                      rightCont = IntersectionContributionsToResiduals[
                                                              wrappedEnh[[i, j]],
                     wrappedEnh[[i + 1, j]], wrappedMon[[i, j]], wrappedMon[[i + 1, j]],
                                                        wrappedEnh, {i, j},
                     {i + 1, j}, Null, Null, wrappedNonTrivialFGVec, True];];
                            ];

                         If[(Length[
                     Select[Flatten[Join[leftCont, rightCont, topCont, bottomCont]], (# < 0) &]] > 0),
                               Throw[Return[-1]];
                               Return[-1];
                               ,
                               contributionsAtSite =  leftCont + rightCont + topCont + bottomCont;
                              ];
                             Sow[contributionsAtSite];
                           ](* This ends the j loop. *);
                          ][[2]], 1];
                       Sow[contributionsAtCol];
                    ](* This ends the i loop. *);
                  ][[2]], 1];
 ];
 residuals = Table[wrappedTotalDegrees[[i, j]] - contributions[[i, j]],
   {i, 1, length}, {j, 1, Length[wrappedEnh[[i]]] }];
 If[(Length[Select[Flatten[residuals], (# < 0) &]] > 0), Return[-1];];
 wrappedEnhWithAPostResiduals = Table[Append[Drop[wrappedEnh[[i, j]], -1], residuals[[i, j]] ],
   {i, 1, length}, {j, 1, Length[wrappedEnh[[i]]]}];
 enhWithAPostResiduals = Table[If[branchPositions[[i]] == 0,
                          Flatten[wrappedEnhWithAPostResiduals[[i]], 1]
                           ,
                           wrappedEnhWithAPostResiduals[[i]]
                         ]
                       , {i, 1, length}]
   (* We wrap the single curve entries so they have the same depth.*);
 Return[enhWithAPostResiduals ];
];
```

```
ComputeAPosterioriResidualsConsideringMonodromy[enhancementWithResiduals_,
   monodromyOnQuiver_, nonTrivialFGTransverseFiberCountVec_: Null] := Module[{},
   If[Depth[enhancementWithResiduals] > 4,
    Return[ComputeAPosterioriResidualsConsideringMonodromyDType[
       enhancementWithResiduals, monodromyOnQuiver]];
   ];
   Return[ComputeAPosterioriResidualsConsideringMonodromyOnLinearQuiver[
     enhancementWithResiduals, monodromyOnQuiver, nonTrivialFGTransverseFiberCountVec]];
  ];

(* IsValidMonodromyAssignmentToPair:
 Test for valid monodromy assignment on a pair of transversely intersecting curves. *)

(* The following method return true or false given a length two quiver, enhancement,
and a monodromy assignment based on whether this assignment is possible. Currently all
 assignments are assumed valid, so this is where we can implement additional constraints.  *)

IsValidMonodromyAssignmentToPair[enhancementWithResiduals_,
   monodromyAssignmentToQuiver_, transverseIsCompact_: True] :=
  Module[{booleanToReturn, length, m, a, b, d, an, bn, dn, at, bt, dt, monodromy,
    monodromyNext, enhancementWithoutResiduals, aPosterioriResiduals},
   booleanToReturn = True;
   length = Length[enhancementWithResiduals];
   If[length ≠ 2 , Print["error: length of pair is not two"]; Return[False]; ];
   If[length ≠  Length[monodromyAssignmentToQuiver],
    Print["error: argument lengths do not match for pair"];
   ];

   m = enhancementWithResiduals[[1, 1, 1]];
   (* orders of vanishing *)
   a = enhancementWithResiduals[[1, 2, 1]];
   b = enhancementWithResiduals[[1, 2, 2]];
   d = enhancementWithResiduals[[1, 2, 3]];
   an =  enhancementWithResiduals[[2, 2, 1]];
   bn = enhancementWithResiduals[[2, 2, 2]];
   dn = enhancementWithResiduals[[2, 2, 3]];

   (* monodromy assignments *)
   monodromy = monodromyAssignmentToQuiver[[1]];
   monodromyNext = monodromyAssignmentToQuiver[[2]];


   If[(a == 0 && b == 0  && d > 0 ),
      If[d > 3 && monodromy == 0 && (an > 2 || bn > 3), Return[False]; ]
     (* The roots of \mu are distinct in this case.
             Compact In only. *) ;
   ];


   (* Case: II *)
   (* rule out a II meeting IV^* with e_6 based on
    4 6 12 considerations using the polynomial information about monodromy
    being a square on IV^* if g restriced appropriately is a square there *)
   If[ (a ≥ 1 && b == 1 && d == 2)   (* if II on left *) ,
          If[ (an ≥ 3 && bn == 4 && dn == 8)   (* and IV^* on right *) ,
            If[ monodromyNext == 1,
                 Return[False](* IV^*s prevented by 4,6,12 point from IV* expansion *);
              ];
            ];

    (* All forbidden cases meeting II with m==
     2 are taken care of by tracking residuals except for the following: *)
          If[m == 2 ,
           (* if m==2 II on left *)
            If[((an ≥ 2 && bn == 2 && dn == 4 ) && monodromyNext == 1 ), Return[False]; ]
     (*  IVs on right banned *) ;
            If[ (an == 0 && bn == 0  && dn == 4), Return[False];] (* I_4 banned *) ;
            If[ (an == 0 && bn == 0 && dn == 3) && monodromyNext == 1, Return[False];] (* I_3s banned *) ;
           ];

          If[m == 1 ,
           (* if m==1 II on left *)
            If[((an ≥ 2 && bn == 2 && dn == 4 ) && monodromyNext == 1 ), Return[False]; ]
     (*  IVs on right banned *) ;
            If[ (an == 0 && bn == 0  && dn ≥  4) && monodromyNext == 1, Return[False];]
     (* I_4s and beyond banned *) ;
            If[ (an == 0 && bn == 0 && dn ≥ 9) , Return[False];] (* I9 and beyond banned;
     this code above forces I5-I9 to be ns *) ;
            If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
```

```
   If[monodromyNext == 2, Return[False];] (* Izerostar meeting II cannot have so(8) (M&R). *);
            ];

          If[(an == 2 && bn == 3 && dn ≥ 8), Return[False]; ] (* Beyond Ionestar is banned *) ;
          If[(an == 2 && bn == 3 && dn == 7), (* Ionestar s is banned *)
                  If[monodromyNext == 1, Return[False];];
            ];
         ];

];
(* Case: I_0: We might benefit from reviewing Instar cases Persson's list banned pairs. *)
If[(a ≥ 0 && b ≥ 0 && d == 0),
   (* Persson's list banned pairs.   *)
    If[ (an == 2 && bn == 3 && dn > 11)
          || (an == 0 && bn == 0 && dn > 9)
         ,
         Return[False](* I4star is the max Instar *) (* I9 is the max In *);
      ](* end of Persson's list banned pairs.   *);


   If[b ≥ 3,
    If[(an ≥ 4 && bn == 2 && dn == 4) && monodromyNext == 1,
      Return[False] (* Ban large A IVs meets I_0 b≥3 since 4 6 12 *) ;
      ];
    ];
   If[a ≥ 2 ,
    If[ (an == 0  && bn == 0 && dn ≥ 8) && monodromyNext == 1,
        Return[False](* In s with n at lesat 8 can't meet Izero A≥ 2--using large In form*) ;
        ];
     ];
 If[b ≥ 2 ,
    If[ (an == 0  && bn == 0 && dn ≥ 8) && monodromyNext == 1,
        Return[False](* In s with n at least can't meet Izero A≥ 2--using large In form *) ;
        ];
    ];
 If[a ≥ 1,
     If[ (an == 2 && bn ≥ 6 && dn == 6) && monodromyNext == 2,
          Return[False] (* Izerostar so(8) (2,≥6,6) cannot meet Izero (≥1,0,0) since 4,6,12 pt. *);
      ];
     If[(an == 2 && bn == 3 && dn == 9) && monodromyNext == 1,
          Return[False] (* I_3^* meeting Izero A>0 must be ns.    *) ;
      ];
     If[(an == 2 && bn == 3 && dn ≥ 10),
          Return[False] (* I_≥4^* meeting Izero A>0 is forbidden *);
      ];
   ];
 If[b ≥ 1,
    If[ (an ≥ 4 &&  bn ≥ 4 && dn ≥ 8) && monodromyNext == 1,
   Return[False](* Ban large A IV^*s meets I_0 b≥1 since 4 6 12 *) ;
      ];
    If[(an ≥ 4 && bn == 3 && dn == 6) && monodromyNext == 1,
          Return[False] (* Izerostar so(7) (≥ 4,3,6) cannot meet Izero (0,≥1,0) since 4,6,

    12 pt. See the IzeroInstar file section Izero Izerostar for details. *);
      ];
    If[(an == 2 && bn == 3 && dn ≥  9),
         Return[False] (* I_≥3^* meeting Izero B>0 not allowed; see IzeroInstar.pdf *);
     ];
   ];
];

(* Case: Izero Instar intersections *)

If[(a == 2 && b == 3 && d ≥ 7) && ((an == 0 && bn ≥ 0 && dn == 0) || (an ≥ 0 && bn == 0 && dn == 0 )),
   If[m == 3,
      If[d ≥ 9 && (an > 0 || bn > 0), Return[False]; ];
      If[d == 8 && monodromy == 1 && an ≥ 1, Return[False];];
      If[d == 8 && (monodromy == 0 || monodromy == 1) && bn ≥ 2, Return[False];];
      If[d == 7 && monodromy == 1 && bn ≥ 2, Return[False];];
    ] (* See IzeroInstar.pdf for details *) ;
   If[m == 2,
      If[d ≥ 9 && (an > 0 || bn > 0), Return[False]; ];
      If[d == 8 && (monodromy == 0 || monodromy == 1) && bn ≥ 3, Return[False];];
    ] (* See IzeroInstar.pdf for details *) ;
   If[m == 1,
      If[d ≥ 9 && (an > 0 || bn > 0), Return[False]; ];
      If[d == 8 && (monodromy == 0 || monodromy == 1) && bn ≥ 4, Return[False];];
      If[d == 8 && (monodromy == 1)                  && an ≥ 5, Return[False];];
      If[d == 7 && (monodromy == 1)                  && bn ≥ 7, Return[False];];
```

```
    ] (* See IzeroInstar.pdf for details *) ;

];

(* Case: I1 *)
If[a == 0 && b == 0 && d == 1 ,
       (* if m==2 I1 on left,
 the residuals are (0,0,2), so this case is completely determined by residuals *)
     If[m == 1 ,
        (* if m==1 I1 on left, the residuals are (4,6,13) *)
           If[(an == 0 && bn == 0 && dn ≥ 8 ) , Return[False];] (*  In for n≥8 banned *) ;
           If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned (M&R)*) ;

   If[ (an ≥ 3 && bn == 4  && dn == 8) && monodromyNext == 1, Return[False];] (* IV*s is banned (M&R)*) ;
           If[(an == 2 && bn == 3 && dn == 10 ) && monodromyNext == 1, Return[False]; ]
   (*  In* for n≥5 banned *) ;
           If[(an == 2 && bn == 3 && dn ≥ 11 ) , Return[False]; ] (*  In* for n≥5 banned *) ;
      ];
];
(* This should conclude the case I1 *)

(* Case: I2 *)
If[a == 0 && b == 0 && d == 2 ,
       (* if m==2 I2 on left,
 the residuals are (0,0,4)  so this case is completely determined by residuals *)
     If[m == 1 ,
        (* if m==1 I2 on left then the residuals are (4,6,14) *)
           If[(an == 0 && bn == 0 && dn ≥ 11 ) , Return[False]; ] (*  In for n≥11 banned *) ;
           If[ (an ≥ 3 && bn ≥ 4  && dn ≥ 9), Return[False];] (* II* and III* and IV* is banned *) ;
           If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
           If[(an == 2 && bn == 3 && dn ≥ 13 ) , Return[False];]  (*  In* for n≥7 banned *) ;
          ];
];
(* This should conclude the case I2 *)

If[a == 0 && b == 0 && d ≥ 3 && monodromy == 1,
   If[ an == 2 && bn == 3 && dn ≥ 7, Return[False]; ]
  (* In s n>2 cannot meet Im star m>0...4,6,12 point. *);
];
(* The following symmetric pairs of conditions
  are imposed with one of the fibers being I_n from BMM Table A.1.
      That they are symmetric follows from the restriction coming from non-
 minimality requirements not using
         compactness of either fiber. *)
If[(an == 0 && bn == 0 && dn ≥ 3) && (a == 2 && b ≥ 4 && d == 6), Return[False];];
If[(a == 0 && b == 0 && d ≥ 3) && (an == 2 && bn ≥ 4 && dn == 6), Return[False];]
      (* See BMM Table A.1. I_0* with B>0 cannot meet I_m with m≥3 *);

If[(an == 0 && bn == 0 && dn ≥ 4) && (a ≥ 3 && b == 3 && d == 6), Return[False];];
If[(a == 0 && b == 0 && d ≥ 4) && (an ≥ 3 && bn == 3 && dn == 6), Return[False];]
      (* See BMM Table A.1. I_0* with A>0 cannot meet I_m with m≥4 *);

If[(an == 0 && bn == 0 && dn ≥ 4) && (a ≥ 3 && b == 2 && d == 4), Return[False];];
If[(a == 0 && b == 0 && d ≥ 4) && (an ≥ 3 && bn == 2 && dn == 4), Return[False];]
      (* See BMM Table A.1. IV with A>0 cannot meet I_m with m≥4 *);

If[(an == 0 && bn == 0 && dn ≥ 4) && (a == 1 && b == 4 && d == 3), Return[False];];
If[(a == 0 && b == 0 && d ≥ 4) && (an == 1 && bn == 4 && dn == 3), Return[False];]
      (* See BMM Table A.1. III with B≥2 cannot meet I_m with m≥4 *);

(* Case: I3 *)
If[a == 0 && b == 0 && d == 3 ,
       (* if m==2 I3 on left,
 the residuals are (0,0,5), so this case is completely determined by residuals *)
     If[m == 1 ,
        (* if m==1 I3 on left, the residuals are (4,6,15), we need to distinguish monodromy *)
           If[monodromy == 0,
                 If[(an == 0 && bn == 0 && dn ≥ 10 ) , Return[False];] (*  In for n≥10 banned *) ;
                 If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                 If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV*s is banned *) ;
                 If[(an == 2 && bn == 3 && dn ≥ 12 ) && monodromyNext == 1 , Return[False];]
    (*  In*s for n≥6 banned *) ;
                 If[(an == 2 && bn == 3 && dn ≥ 13 ) && monodromyNext == 0 , Return[False];]
    (*  In*ns for n≥7 banned *) ;
           ];
           If[monodromy == 1,
                If[(an == 0 && bn == 0 && dn ≥ 12 ) , Return[False];] (*  In for n≥12 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV*s is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False];](*  In* for n≥0 banned *) ;
```

```
            ];
        ];
  ];
(* This should conclude the case I3 *)
(* Case: Id for d≥5 odd *)
     (* if m==2 In on left, the residuals are (0,0,d),
so this case is completely determined by residuals *)
If[a == 0 && b == 0 && d ≥ 3 && Mod[d, 2] ≠ 0,
     If[m == 1 ,
        (* if m==1 Id on left, the residuals are (4,6,12+d), we need to distinguish monodromy *)
           If[monodromy == 0,
                If[(an == 0 && bn == 0 && dn ≥  d + 7) , Return[False];] (*  In for n≥d+7 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[(an == 2 && bn == 3 && dn ≥ d + 9 ) && monodromyNext == 1 , Return[False]; ]
     (*  In*s for n≥d+3 banned *) ;
                If[(an == 2 && bn == 3 && dn ≥ d + 10) && monodromyNext == 0 ,
                      Return[False];
                  ] ;
           ];
           If[monodromy == 1,
                If[(an == 0 && bn == 0 && dn ≥ d + 9 ) , Return[False];] (*  In for n≥9+d banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False];] (*  In* for n≥0 banned *) ;
                If[an ≥ 2 && bn == 2 && dn == 4 , Return[False];] (*  IV is banned *) ;
                If[an == 1 && bn ≥ 2 && dn == 3 , Return[False];] (*  III is banned *) ;
                If[an ≥ 1 && bn == 1 && dn == 2 , Return[False];] (*  II is banned *) ;
            ];
        ];
  ];
(* This should conclude the case In odd *)

(* Case: Id for d≥4 even, except d=6 without monodromy, to be treated separately *)
If[a == 0 && b == 0  && d ≥ 4 && Mod[d, 2] == 0,
       (* if m==2 In on left,
 the residuals are (0,0,d), so this case is completely determined by residuals *)
     If[m == 1 ,
        (* if m==1 Id on left, the residuals are (4,6,12+d), we need to distinguish monodromy *)
           If[monodromy == 0,
                If[(an == 0 && bn == 0 && dn ≥  d + 9) , Return[False];] (*  In for n≥d+9 banned *) ;
                If[(an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ d + 11 , Return[False];] (*  In* for n≥d+5 banned *) ;
           ];
           If[monodromy == 1 && d ≠ 6,
                If[(an == 0 && bn == 0 && dn ≥ d + 9 ) , Return[False];] (*  In for n≥9+d banned *) ;
                If[(an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False];] (*  In* for n≥0 banned *) ;
                If[an ≥ 2 && bn == 2 && dn == 4 , Return[False];] (*  IV is banned *) ;
                If[an == 1 && bn ≥ 2 && dn == 3 , Return[False];] (*  III is banned ???? *) ;
                If[an ≥ 1 && bn == 1 && dn == 2 , Return[False];] (*  II is banned ???? *) ;
            ];
           If[monodromy == 1 && d == 6,
     (* How do we distinguish the two cases???? Here I only imposed the higher one su(6)star *)
                If[(an == 0 && bn == 0 && dn ≥ 16 ) , Return[False]; ] (*  In for n≥16 banned *) ;
                If[(an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV*s is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False];] (*  In* for n≥0 banned *) ;
                If[an ≥ 2 && bn == 2 && dn == 4 , Return[False];] (*  IV is banned *) ;
                If[an == 1 && bn ≥ 2 && dn == 3 , Return[False];] (*  III is banned ???? *) ;
                If[an ≥ 1 && bn == 1 && dn == 2 , Return[False];] (*  II is banned ???? *) ;
            ];
        ];
  ];
(* This should conclude the case In even, modulo refining d=6 *)

(* Case: III *)
If[a == 1 && b ≥ 2 && d == 3 ,
     If[m == 2 ,
       (* if m==2 III on left, the residuals are (2,4,5) *)

  If[(an == 0 && bn == 0 && dn ≥ 5 ) , Return[False];]  (* In for n≥5 banned  CHECK THIS!!! *) ;
              If[ (an ≥ 3 && bn ≥ 5  && dn ≥9), Return[False];] (* II* and III* is banned *) ;
              If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
              If[an == 2 && bn == 3 && dn ≥ 7 , Return[False];]    (* In* for n≥1 banned *) ;
              If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
                   If[monodromyNext > 1, Return[False];]
    (* Izerostar must have g_2 or so(7)  See atomic p.115 for pf. *);
```

```
                ];
        ];
       If[m == 1 ,
         (* if m==1 III on left, the residuals are (5,8,15) *)
                If[(an == 0 && bn == 0 && dn ≥ 11) , Return[False]; ] (*  In for n≥11 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In*s for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
                     If[monodromyNext == 2, Return[False];] ;
                 ](* Izerostar meeting III must have g_2 or so(7). See atomic p.115 for pf.*);
         ];
];
(* This should conclude the case III *)

(* Case: IV *)
If[a ≥ 2 && b == 2 && d == 4 ,
  (* if m==3 IV on left, the residuals are (-,0,0) so this is taken care by residuals *)
     If[(an == 2 && bn == 3 && dn ≥ 7), Return[False]; ] ;

      If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,

  If[monodromyNext ≥ 1, Return[False];] (* Izerostar meeting IV must have g_2 atomic p.114.
                        Note this really does apply for A,B>0 along \Izerostar cases.
                              Hence BMM GS is reduced from \so(7) for IVns to g_2 or A_2+A_1 *);
      ];
      If[m == 2 ,
       (* if m==2 IV on left, the residuals are (4+2A,4,8), we need to distinghuish monodromy *)
           If[monodromy == 0,
                 If[(an == 0 && bn == 0 && dn ≥  4) , Return[False]; ] (*  In for n≥4 banned
                           note this
     follows since otherwise there are purely even contributions to \bt,
                           which is not allowed for IVns.*) ;
                 If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                 If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                 If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                 If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
                      If[monodromyNext ≥ 1, Return[False];] ;
                 ](* Izerostar meeting IV must have g_2 atomic p.114 *);
            ];
           If[monodromy == 1,
                 If[(an == 0 && bn == 0 && dn ≥ 5) , Return[False]; ]
   (*  In for n≥5 banned. Also taken care of int.cont. *) ;
                 If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                 If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                 If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                 If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
    Return[False](* Izerostar not allowed *);
                   ];
            ];
      ];
     If[m == 1 ,
       (* if m==1 IV on left, the residuals are (6+A,8,16) *)
           If[monodromy == 0,
                 If[(an == 0 && bn == 0 && dn ≥ 8) , Return[False]; ]  (*  In for n≥8 banned
                           note this
     follows since otherwise there are purely even contributions to \bt,
                           which is not allowed for IVns.*) ;
                 If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                 If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                 If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                 If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
                      If[monodromyNext ≥ 1, Return[False];] ;
                   ] (* Izerostar must have g_2 by atomic appendix E.3.1 *);
            ];
           If[monodromy == 1,
                 If[(an == 0 && bn == 0 && dn ≥ 9) , Return[False]; ] (*  In for n≥8 banned.
                           Ruled out by intersection contributions to the IV.*) ;
                 If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                 If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                 If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                 If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) ,
                     Return[False];
                 ](* Izerostar not allowed since monodromy from IV gives 4,6,12 point. *);
            ];
      ];
];
(* This should conclude the case IV *)

(* Case: I0star *)
```

```
If[( (a ≥ 2 && b ≥ 3 && d == 6 ) ) ,
    If[transverseIsCompact,
        If[(a > 2 || b > 3) && (an == 0 && bn == 0 && dn ≥ 3 && monodromyNext == 0), Return[False];]
  (* COMPACT ONLY. *) ;
        If[(a > 2 || b > 3) && (an == 0 && bn == 0 && dn > 0),
            If[monodromy == 2 && a ≥ 3 && dn ≥ 2, Return[False];]
   (* COMPACT ONLY, IzerostarRestrictions.*);
            If[monodromy == 2 && b ≥ 4 && dn ≥ 2, Return[False];]
   (* COMPACT ONLY, IzerostarRestrictions.*);
            If[monodromy == 2 && b ≥ 5 && dn == 1, Return[False];]
   (* COMPACT ONLY, IzerostarRestrictions.*);
            If[monodromy == 2 && a ≥ 3 && dn ≥ 2, Return[False];]
   (* COMPACT ONLY, IzerostarRestrictions.*);
            If[monodromy == 1 && b ≥ 5 && dn == 2, Return[False];](* COMPACT, ditto. *);
            If[monodromy == 1 && b ≥ 4 && dn ≥ 3, Return[False];](* COMPACT, ditto. *);
            If[monodromy == 0 && a ≥ 3 && dn ≥ 3,  Return[False];](* COMPACT, ditto. *);
        ];
    ];
    If[(a > 2 || b > 3) && monodromy == 2,
          If[ (an == 1 && bn ≥ 2 && dn == 3) || (an ≥ 2 && bn == 2 && dn == 4), Return[False];]
  (* (4,6,12) IzerostarRestrictions *) ;
          If[b ≥ 5 && (an ≥ 1 && bn == 1 && dn == 1), Return[False]; ]
  (* 4,6,12; IzerostarRestrictions *) ;
    ];
      If[(m == 3) && (an == 0 && bn == 0 && dn ≥ 5), Return[False];]
 (* infinite intersection contribution to the Izerostar *);
      If[ (m == 3) && (an == 0 && bn == 0 && dn ≥ 4) && monodromy == 0, Return[False]; ]
           (* general form for I_4 restricted to the curve explicitly splits.
                                   I4 sp2
  Izerostar so7 on 1 3.  This extends to I_5, but there it's is impossible
                                         to meet Izerostar on a -3 for any monodromy. We
                                            already knew this is forbidden from global symmetry.
                        Note the g_2 case
   cannot meet I_3 or beyond without monodromy on I_3.  This argument
                                 is also in the workbook I5 meets
   IzerostarOnMinus3 not possible.nb *) ;
      If[ (m == 3 && a == 2 && b == 3 && d == 6) && (an == 2 && bn == 2 && dn == 4),
              If[monodromy ≥  1, Return[False];] (* so(7) not allowed since this gives

    resdiduals after meeting the IV preventing splitting the
                                                monodromy cover even partially.
                                                     See
    IzerostarRestrictions.pdf for details. *);
       ];
      If[ (a == 2 && b ≥ 4 && d == 6) && monodromy == 1,
              If[m == 3 && an ≥ 2, Return[False];];
              If[m == 2 && an ≥ 4, Return[False];];
              If[m == 1 && an ≥ 6, Return[False];];
      ](* so(7) not allowed since this gives resdiduals
   after meeting the IV or II with order (2,1,2) prevents splitting the
          monodromy cover even partially for example.  See IzerostarRestrictions.pdf section on B>
  0 so(7) for details. *);
      If[ (a ≥ 3 &&  b == 3 && d == 6) && monodromy == 0,
              If[ m == 3 &&  bn == 3 , Return[False];];
              If[m == 2 && bn == 6, Return[False];];
              If[m == 1 && bn == 9, Return[False];];
       ]
  (* g2 Izerostar (≥3,3,6) cannot have g restricted to the Izerostar curve being a cube since
              then the cover is not irreducible. *);
      If[  (a == 2 &&  b ≥ 4 && d == 6) && monodromy == 0,
           Return[False]; ] (* g_2 Izerostar not possible with B>0 *) ;
 (* if m==4 I0star on left,
 the gauge algebra is so(8) and only intersection with (0,0,0) is allowed *)
     If[monodromy == 2 && (an == 1 && bn ≥ 2 && dn == 3), Return[False];]
 (* so(8) cannot meet III. See atomic p. 115. *) ;
     If[monodromy ≥ 1 && (an ≥ 2 && bn ==  2 && dn == 4), Return[False];]
 (* so(\geq 7) cannot meet IV. See atomic p. 114. *) ;
     If[monodromy == 2 && a ≥  3 && (an == 1 && bn ≥ 2 && dn == 3), Return[False];]
        (* This can be removed via the above restriction. *) ;
      If[m == 4 && monodromy ≠ 2, Return[False];];
      If[m == 3 ,
       (* if m==3 I0star on left,
  the residuals are (2+3A,3+3B,6), we need to distinghuish monodromy *)
         If[monodromy == 0,

   If[(an == 0 && bn == 0 && dn ≥ 4 ) , Return[False]; ] (*  In for n≥4 banned by GS max. *) ;
              If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ]
   (*  In_s for n≥3 banned *) ;
              If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
              If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
```

```
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False]; ] (*  In* for n≥0 banned *) ;
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];] ;
            ];
          If[monodromy == 1,
                If[(an == 0 && bn == 0 && dn ≥ 6 ) , Return[False]; ] (*  In ns for n≥6 banned *) ;

  If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ] (*  Ins for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False]; ]
  (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
  (* IV can only have monodromy*);
            ];
          If[monodromy == 2,
                If[(an == 0 && bn == 0 && dn ≥ 4 ) , Return[False]; ] (*  In  ns for n≥4 banned *) ;
                If[(an == 0 && bn == 0 && dn ≥ 3 && monodromyNext == 1 ) , Return[False]; ]
  (*  In s for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 10), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False];]
  (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
  (* IV can only have monodromy*);
            ];
     ];
          If[m == 2,
      (* if m==2 I0star on left,
 the residuals are (4+2A,6+2B,12), we need to distinghuish monodromy *)
          If[monodromy == 0,
                If[(an == 0 && bn == 0 && dn ≥ 10 ) , Return[False];] (*  In for n≥10 banned *) ;

  If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ] (*  In s for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False];] (*  In* for n≥0 banned *) ;

  If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];] (* IV can only have monodromy*);
            ];
          If[monodromy == 1,
                If[(an == 0 && bn == 0 && dn ≥ 10 )  && monodromyNext == 0, Return[False]; ]
  (*  In ns for n≥10 banned *) ;
                If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ]
  (*  Ins for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8 , Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False]; ]
  (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
  (* IV can only have monodromy*);
            ];
          If[monodromy == 2,
                If[(an == 0 && bn == 0 && dn ≥ 6 )  && monodromyNext == 0, Return[False]; ]
  (*  In ns for n≥5 banned *) ;
                If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ]
  (*  Ins for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8 , Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False]; ]
  (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
  (* IV can only have monodromy*);
            ];
     ];
             If[m == 1,
      (* if m==1 I0star on left,
 the residuals are (6+A,9+B,18), we need to distinghuish monodromy *)
          If[monodromy == 0,
                If[(an == 0 && bn == 0 && dn ≥ 16 ) , Return[False]; ] (*  In for n≥16 banned *) ;

  If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ] (*  In s for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8 , Return[False];]   (* IV* is banned *) ;
                If[an ≥ 2 && bn ≥ 3 && dn ≥ 6 , Return[False]; ] (*  In* for n≥0 banned *) ;

  If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];] (* IV can only have monodromy*);
```

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            ];
            If[monodromy == 1,
                If[(an == 0 && bn == 0 && dn ≥ 14 ) , Return[False]; ] (*  In ns for n≥14 banned *) ;

    If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ] (*  In s for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In* for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False]; ]
    (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
    (* IV can only have monodromy*);
            ];
            If[monodromy == 2,
                If[(an == 0 && bn == 0 && dn ≥ 8 ) && monodromyNext == 0, Return[False]; ]
    (*  In ns for n≥7 banned *) ;
                If[(an == 0 && bn == 0 && dn ≥ 3 ) && monodromyNext == 1, Return[False]; ]
    (*  In s for n≥3 banned *) ;
                If[ (an ≥ 3 && bn ≥ 5  && dn ≥ 9), Return[False];] (* II* and III* is banned *) ;
                If[ an ≥ 3 && bn == 4  && dn == 8 &, Return[False];]   (* IV* is banned *) ;
                If[an == 2 && bn == 3 && dn ≥ 7 , Return[False]; ] (*  In*s for n≥1 banned *) ;
                If[( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)) , Return[False]; ]
    (* Izerostar not allowed *);
                If[(an ≥ 2 && bn == 2 && dn == 4) && monodromyNext == 1, Return[False];]
    (* IV can only have monodromy*);
            ];
        ];
];
(* This should conclude the case I0star *)

(* IV* and beyond general restriction. *)
If[ a ≥ 3 && b ≥ 4 && d ≥ 8,
    If[dn ≥ 2 && (! (an ≥ 1 && bn == 1 && dn == 2)), Return[False]; ]; ]
(* IV* and beyond can't meet anything with symmetry *) ;
If[ a ≥ 4 && b == 4 && d == 8 && monodromy == 1,
    If[(bn == 1), Return[False]; ]
  (* IV*s with A≥ 1 cannot meet Izero B==1 since this creates 4,6,12 point *);
];
If[ a ≥ 3 && b == 2 && d == 4 && monodromy == 1,
    If[(an == 1 && bn == 3 && dn == 3), Return[False]; ]
  (* IVs with A≥ 1 cannot meet III B==1 since this creates 4,6,12 point *);
];

(* Case Instar GS restrictions *)
 If[(a == 2 && b == 3 && d ≥ 7) && (an == 0 && bn == 0 && dn ≥ 3 && monodromyNext == 1) , Return[False];
](* In s cannot for n>2 meet Imstar. *) ;

(* The following symmetric pairs of
  conditions appear from BMM Table B.1 from non-minimality contraints. *)
If[(an == 2 && bn == 3 && dn ≥ 7) && (a ≥ 1 && b ≥ 1 && d ≥ 3) , Return[False];];
If[(a == 2 && b == 3 && d ≥ 7) && (an ≥ 1 && bn ≥ 1 && dn ≥ 3) , Return[False];];
If[(an == 2 && bn == 3 && dn ≥ 8) && (a ≥ 1 && b ≥ 1 && d ≥ 2) , Return[False];];
If[(a == 2 && b == 3 && d ≥ 8) && (an ≥ 2 && bn ≥ 1 && dn ≥ 2) , Return[False];
](* See BMM Table B.1 for details noting the
  only fiber with symmetry that can met Instar with n≥1 have type I_m.
       This restriction is imposed to prevent a non-minimal model. Similarly,
I_n* with n≥2 cannot meet II. *) ;

If[(a == 2 && b == 3 && d == 7),
    If[monodromy == 0,
        If[m == 4 && (an == 0 && bn == 0 && (dn > 3 ) && monodromyNext == 0 ), Return[False];];
        If[m == 3 && (an == 0 && bn == 0 && (dn > 5 ) && monodromyNext == 0 ), Return[False];];
        If[m == 2 && (an == 0 && bn == 0 && (dn > 7 ) && monodromyNext == 0 ), Return[False];];
        If[m == 1 && (an == 0 && bn == 0 && (dn > 9 ) && monodromyNext == 0 ), Return[False];];
    ];
    If[monodromy == 1,
        If[m == 4 && (an == 0 && bn == 0 && (dn > 4) && monodromyNext == 0 ), Return[False];];
        If[m == 3 && (an == 0 && bn == 0 && (dn > 6 ) && monodromyNext == 0 ), Return[False];];
        If[m == 2 && (an == 0 && bn == 0 && (dn > 9) && monodromyNext == 0 ), Return[False];];
        If[m == 1 && (an == 0 && bn == 0 && (dn > 10) && monodromyNext == 0 ), Return[False];];
    ];
];

If[(a == 2 && b == 3 && d == 8),
    If[monodromy == 0,
        If[m == 4 && (an == 0 && bn == 0 && (dn > 7  ) && monodromyNext == 0 ), Return[False];];
        If[m == 3 && (an == 0 && bn == 0 && (dn > 9 ) && monodromyNext == 0 ), Return[False];];
        If[m == 2 && (an == 0 && bn == 0 && (dn > 11 ) && monodromyNext == 0 ), Return[False];];
        If[m == 1 && (an == 0 && bn == 0 && (dn > 13 ) && monodromyNext == 0 ), Return[False];]
  (* one for the pot in the
```

```
            monodromy case.  *) ;
      ];
      If[monodromy == 1,
          If[m == 4 && (an == 0 && bn == 0 && (dn > 8 ) && monodromyNext == 0 ), Return[False];];
          If[m == 3 && (an == 0 && bn == 0 && (dn > 10) && monodromyNext == 0 ), Return[False];];
          If[m == 2 && (an == 0 && bn == 0 && (dn > 12) && monodromyNext == 0 ), Return[False];];
          If[m == 1 && (an == 0 && bn == 0 && (dn > 14) && monodromyNext == 0 ), Return[False];];
      ];
  ];
  If[(a == 2 && b == 3 && d == 9),
      If[monodromy == 0 && (an == 0 && bn == 0 && dn > 11 && monodromyNext == 0 ), Return[False];];
      If[monodromy == 1 && (an == 0 && bn == 0 && dn > 12 && monodromyNext == 0 ), Return[False];];
  ];

  If[(a == 2 && b == 3 && d ≥  10),
      If[monodromy == 0 && (an == 0 && bn == 0 && (dn > 4 (d - 6) - 1) && monodromyNext == 0 ), Return[False];];
      If[monodromy == 1 && (an == 0 && bn == 0 && (dn > 4 (d - 6) ) && monodromyNext == 0 ), Return[False];]
   (* one for the pot in the
           monodromy case.  *);
  ];

  (* Gaugeless curves have transverse algebra in e8;
  we can see these through inspecting residuals; see workbooks.
          On a -1 with I_1, \phi must vanish somewhere else than an intersection with an su(n)
   (M&R and BMM has this stated also), hence I_1 has bound given by I_9 transverse.  Meeting Instar,
  similarly we are capped at I4star or less for the various gaugeless options. *)

  If[m == 1 && ! (d ≥ 2 && (! (a ≥ 1 && b == 1) ) ),
     If[an == 0 && bn == 0 && dn ≥  10 , Return[False]; ] (* A9 not in e8. C5 not in e8. *) ;
     If[an == 2 && bn == 3 && dn ≥  11 , Return[False]; ] (* D9 not in e8. B8 not in e8 *) ;
  ];

  Return[booleanToReturn];
 ];


(* IsValidMonodromyAssignmentToTriplet:
 Test for valid monodromy on a triplet of curves with transverse intersection. *)

IsValidMonodromyAssignmentToTriplet[enhancementWithResiduals_, monodromyAssignmentToQuiver_,
  curvesAreCompact_: True] := Module[{booleanToReturn, length, a, b, d, ap, bp, dp, an, bn,
   dn, monodromy, monodromyPrev, monodromyNext, mp, m, mn, p, q, monMaxPrev, monMaxNext},
  booleanToReturn = True;
  If[Length[monodromyAssignmentToQuiver] ≠ 3 || Length[enhancementWithResiduals] ≠ 3,
   Print["error: monodromy assignment and or enhancement with residuals are not of length 3"];
   Return[False];
   ];

   (* Orders of vanishing along each curve: *)
  a = enhancementWithResiduals[[2, 2, 1]];
  b = enhancementWithResiduals[[2, 2, 2]];
  d = enhancementWithResiduals[[2, 2, 3]];
  ap = enhancementWithResiduals[[1, 2, 1]];
  bp = enhancementWithResiduals[[1, 2, 2]];
  dp = enhancementWithResiduals[[1, 2, 3]];
  an = enhancementWithResiduals[[3, 2, 1]];
  bn = enhancementWithResiduals[[3, 2, 2]];
  dn = enhancementWithResiduals[[3, 2, 3]];

  (* monodromies and self intersections *)
  monodromy = monodromyAssignmentToQuiver[[2]];
  monodromyPrev = monodromyAssignmentToQuiver[[1]];
  monodromyNext = monodromyAssignmentToQuiver[[3]];
  m = enhancementWithResiduals[[2, 1, 1]];
  (* mp = enhancementWithResiduals[[1,1,1]];
  Not called for currently; if needed in edits, note this method is
          called for GS with null self intersection on the outer curves  *)
  If[curvesAreCompact,
   mn = enhancementWithResiduals[[3, 1, 1]];
  ];

  (* Local variables for cases *)
  p = 0;
  q = 0;

  If[length ≠ 3 ,
   Print["error: length of input quiver for triplet method is not three"]; Return[False]; ];
  If[length ≠ Length[enhancementWithResiduals] || length ≠ Length[monodromyAssignmentToQuiver],
   Print["error: argument lengths do not match for triplet"];
  ];
```

```
If[(a == 0 && b == 0 && d > 0),
     If[ monodromy == 1 && (ap > 0 || bp > 0 ) && (an > 0 || bn > 0), Return[False]; ]
 (* su(n) I_n can't have two
              nontrivial f,g neighbors; this is also enforced in some additional needed cases in
                    the intersection contributions method.  *) ;
     If[(d == 1 || d == 2  || (d ≥ 3 && monodromy == 0 )) && (ap > 0 || bp > 0 ) && (an > 0 || bn > 0) &&
                              (ap > 2 || an > 2 || bp > 3 || bn > 3), Return[False];] (* We can't have
  z^3 or beyond divide the constant term in the expansion of f if there are two nontrivial f,
 g neigbors for example. Note the similar pair restrictions in the pair restrictions method. *) ;
 If[curvesAreCompact,
      If[monodromy == 0 && (Mod[d, 2] ≠ 0) && m == 1,
           If[mn == 2 && (an == 0 && bn == 0 && dn > 0),
               If[dn > (12 + d) - 6, Return[False];] (* In the case meeting m==2, we cannot have
                                                         \mu in -
       1 I_n odd guy's expansion with a root at
                                                             the intersection
        so we lose 6 of the available vanishings
                                                                of the 12+
       d delta residuals to mu^3.  *) ;
           ];
      ];
      If[monodromy == 0 && (Mod[d, 2] == 0) && m == 1,
           If[mn == 2 && (an == 0 && bn == 0 && dn > 0),
               If[dn > (12 + d) - 4, Return[False];] (* In the case meeting m==2, we cannot have
                                                         \mu in -
       1 I_n even guy's expansion with a root at
                                                             the intersection
        so we lose 4 of the available vanishings
                                                                of the 12+
       d delta residuals to mu^2.  *) ;
           ];
      ];
 ](* These are the only restrictions here using that we have compact curves.*);
];

(*  rule out a II with transverse bigger than g2 f4...if f4 then not also bigger than g2 *)
If[ a ≥ 1 && b == 1 && d == 2  (* if II in center *)
         && ap ≥ 3 && bp == 4 && dp == 8  (* and IV^* on left *)
            && ( (an == 2 && bn ≥ 3 && dn == 6 ) || (an ≥ 2 && bn == 3 && dn == 6)),
 (* and Izerostar on right *)
               If[  monodromyPrev == 0  (* f4 on IV^* *)
                     && monodromyNext > 0 ,  (* bigger than g2  on I_0^*  *)
                  Return[False];
                 ];
];

If[ (a ≥ 2 && b == 2 && d == 4),
     If[ (Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] ≥ (12 - 4 m) ) && monodromy == 0,

   If[ (EvenQ[  Max[Ceiling[dp / 2], bp]] ) && (EvenQ[  Max[Ceiling[dn / 2], bn]] ), Return[False]; ]
                  (* used all g residuals, then if both even g intersections,
    then IV is without monodromy *) ;];
     If[ (Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] ≥ (12 - 4 m) ) && monodromy == 1,

   If[ (! EvenQ[  Max[Ceiling[dp / 2], bp]] ) || (! EvenQ[  Max[Ceiling[dn / 2], bn]] ), Return[False]; ]
                  (* used all residuals and with larger algebra,
    so each g intersection must be even *) ;];
     If[(Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] > (12 - 4 m) ), Return[False];]
                  (* if used more g residuals then allowed for any monodromy, not allowed *) ;
     If[m == 2 && monodromy == 0,
          If[(ap ≥ 2 && bp == 2 && dp == 4) && (an == 0 && bn == 0 && dn ≥ 2), Return[False]; ]
             (* IV+I_2 are not a valid transverse pair. This gives
              purely even g contributions and hence the monodroy along
               the base IV would give su(3) there. *) ;
     ];
     If[m == 2 && monodromy == 0,
          If[(ap ≥ 2 && bp == 2 && dp == 4) && (an == 0 && bn == 0 && dn ≥ 2),
               Return[False];
             ]
             (* IV+I_≥ 2 are not a valid transverse pair. This gives
              purely even g contributions and hence the monodroy along
               the base IV would give su(3) there. Note the I_2 requires at
                least 0,2,4 contributions. *);
     ];
     If[m == 2 && monodromy == 0,
          If[(ap == 1 && bp ≥ 2 && dp == 3) && (an == 0 && bn == 0 && dn ≥ 2),
               Return[False];
```

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              ]
              (* III+I_≥2 are not a valid transverse pair. This gives
              purely even g contributions and hence the monodroy along
               the base IV would give su(3) there. *);
     ];
     If[m == 1 && monodromy == 0,
          If[(ap == 0 && bp == 0 && dp == 4) && (an == 0 && bn == 0 && dn == 4) ,
              Return[False];
          ];
          If[(ap == 0 && bp == 0 && dp == 6) && (an ≥ 2 && bn == 2 && dn == 4) ,
              Return[False];
          ];
          If[(ap == 0 && bp == 0 && dp == 6) && (an == 1 && bn ≥ 2 && dn == 3) ,
              Return[False];
          ];
          If[(ap == 0 && bp == 0 && dp == 6) && (an == 0 && bn == 0 && dn == 2) ,
              Return[False];
          ];
    ]
   (* For IVns we need to eliminate a few configurations which leave purely even bt contributions.
           See the BMM errata note for details. Note I_n has \bt contribution n. *) ;
];

(* Now the same thing for IV* *)
If[ (a ≥ 3 && b == 4 && d == 8),
     If[ (Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] ≥ (12 - 2 m) ) && monodromy == 0,

   If[ (EvenQ[  Max[Ceiling[dp / 2], bp]] ) && (EvenQ[  Max[Ceiling[dn / 2], bn]] ), Return[False]; ]
                  (* used all g residuals, then if both even g intersections,
    then IV* is without monodromy *) ;];
     If[ (Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] ≥ (12 - 2 m) ) && monodromy == 1,

   If[ (! EvenQ[  Max[Ceiling[dp / 2], bp]] ) || (! EvenQ[  Max[Ceiling[dn / 2], bn]] ), Return[False]; ]
                   (* used all residuals and with larger algebra,
    so each g intersection must be even *)        ;];
     If[(Max[Ceiling[dp / 2], bp] + Max[Ceiling[dn / 2], bn] > (12 - 2 m) ), Return[False];]
                  (* if used more g residuals then allowed for any monodromy, not allowed *) ;
](* Some of the above is taken care of already by intersection
 contributions except when we use all residuals via even contributions to g
 residuals and hence force the IV (or IV* ) to have the larger algebra. *);

(* Next is the analog for Izerostar so(8) with A>0 where g goes as a cube. *)
If[ (a ≥ 3 && b == 3 && d == 6) && monodromy == 2,
     If[ (Max[3 Ceiling[dp / 6], 3 Ceiling[bp / 3]] + Max[3 Ceiling[dn / 6], 3 Ceiling[bn / 3]] > (12 - 3 m) ) ,
                          Return[False];]
   (* we have 12 - 3m g residuals and cannot exceed this; even type here. *) ;
];
If[ (a ≥ 3 && b == 3 && d == 6),
     If[ Mod[dp, 6] == 0 && Mod[dn, 6] == 0 && dp + dn == (24 - 6 m) && monodromy ≠ 2 ,
                         Return[False];]
 (* if we used all delta residuals in multiples of 6 then it must be so(8).
                                             Note: to get here,
 these are then of course I_n curves (Izero also) or
                                                   we would
   have thrown a 4 6 12 error and tossed it.  *) ;
     If[Mod[bp, 3] == 0  && Mod[bn, 3] == 0 && bp + bn == (12 - 3 m) && monodromy ≠ 2,
                         Return[False];] (* g intersections are purely cubic then must be so(8) *) ;
     If[Mod[bp, 3] == 0 && Mod[dn, 6] == 0 && 2 bp + dn == (24 - 6 m) && monodromy ≠ 2,
                         Return[False];](* g intersections are purely cubic then must be so(8) *) ;
];

If[(a ≥ 2 && b ≥ 3 && d == 6) && (m == 3) && (monodromy == 0) &&
   (ap == 0 && bp == 0 && dp ≥ 2) && ((dn ≥ 2) && (! (an ≥ 1 && bn == 1)) ), Return[False]; ]
        (* sp1 is max GS: see BMM. *) ;
If[(a ≥ 2 && b ≥ 3 && d == 6) && (m == 3) && (monodromy == 0) &&
   (ap == 1 && bp ≥ 2 && dp == 3) && ((dn ≥ 2) && (! (an ≥ 1 && bn == 1)) ), Return[False]; ]
        (* sp1 is max GS: see BMM. *) ;
If[(a ≥ 2 && b ≥ 3 && d == 6) && (m == 3) && (monodromy == 0) &&
   (ap ≥ 2 && bp == 2 && dp == 4) && ((dn ≥ 2) && (! (an ≥ 1 && bn == 1)) ), Return[False]; ]
        (* sp1 is max GS: see BMM. *) ;
If[(a == 2 && b == 3 && d == 6) && (m == 3) && (monodromy == 0) &&
   (ap == 0 && bp == 0 && dp == 2) && (an == 1 && bn == 2 && dn == 3) , Return[False]; ]
        (* I2 Izerostar(g2 m=3) I2 is forbidden;
see izerostarRestrictions tex...the monodromy cover partially factors*) ;
If[(a == 2 && b == 3 && d == 6) && (m == 3) && (ap == 0 && bp == 0 && dp == 2) &&
   (an == 1 && bn == 3 && dn == 3) , Return[False]; ]
        (* I2 Izerostar(m=3) III(1,3,3) is forbidden for any monodromy;
see izerostarRestrictions tex...*) ;
If[(a == 2 && b == 3 && d == 6) && (m == 3) && (ap == 0 && bp == 0 && dp == 2) &&
   (an ≥ 2 && bn == 2 && dn == 4) , Return[False]; ]
```

```
        (* I2 Izerostar(m=3) IV is forbidden for any monodromy;
see izerostarRestrictions tex...*) ;
If[(a == 2 && b == 3 && d == 6) && (m == 3) && (ap == 0 && bp == 0 && dp == 2) &&
   (an == 0 && bn == 0 && dn == 4) , Return[False]; ]
        (* I2 Izerostar(m=3) I4 is forbidden; see izerostarRestrictions tex...*) ;

(* BMM Restrictions from known GS maximums (from single transverse curve GS fibers). *)
(* For Izerostar center triplets rules from BMM GS restrictions. *)
If[( (a ≥ 2 && b == 3 && d == 6) || (a == 2 && b ≥ 3 && d == 6) )
     && (ap == 0 && bp == 0 && dp ≥ 2) && (an == 0 && bn == 0 && dn ≥ 2),

   If[monodromy == 0,
      If[ m == 3,
          Return[False];
       ];
      If[ m == 2,
          If[dp ≥ 8 || dn ≥ 8, Return[False]; ];
       ];
      If[ m == 1,
          If[dp ≥ 15 || dn ≥ 15, Return[False];];
       ];
    ];

  If[monodromy == 1,
     If[ m == 3,
        If[dp ≥ 4 || dn ≥ 4, Return[False]; ]
            (* No sp(1)+sp(2) since sp(2) is GS max. *);
      ];
     If[m == 2,
        If[dp ≥ 10 || dn ≥ 10, Return[False];](*No sp(5) summands allowed.
                   TODO: MOVE TO PAIRS METHOD AND DELETE. *);
      ];
     If[m == 2,
        If[dp ≥ 6 && dn ≥ 4, Return[False];]
          (* See BMM section in p. 29 and GS for \Izerostar_ss on a -2.
              We can't carry out such a tuning. sp(3)+sp(2) not possible *);
      ];
     If[m == 1,
        If[dp ≥ 8 && dn ≥ 6, Return[False];]
          (* See BMM section in p. 29 and GS for \Izerostar_ss on a -1.
              We can't carry out such a tuning. sp(4)+sp(3) not possible.*);
      ];
     If[m == 1,
        If[(dp ≥ 14 && dn ≥  4) || (dp ≥ 12 && dn ≥ 6) , Return[False]; ];
      ] (* See BMM section in p. 29 and GS for \Izerostar_ss on a -1.
           This would directly exceed this GS. *);
   ];

   If[monodromy == 2,
       If[m == 3,
              If[dp ≥ 4 || dn ≥ 4, Return[False];](*No sp(2) summands allowed.
                   TODO: MOVE TO PAIRS METHOD AND DELETE. *);
        ];
        If[m == 2,
              If[(dp ≥ 6 || dn ≥ 6) , Return[False];](*No sp(3) summands allowed.
                   TODO: MOVE TO PAIRS METHOD AND DELETE. *);
        ];
        If[m == 1,
              If[(dp ≥ 8 || dn ≥ 8) , Return[False];](*No sp(3) summands allowed.
                   TODO: MOVE TO PAIRS METHOD AND DELETE. *);
        ];

    ](* Note these so(8) restrictions are
  actually only pair restrictions and the more powerful tuplet restriction
             are on transverse triplets and beyond.
             Those have not been implemented for so(8).  *) ;
];

If[( (a ≥ 2 && b == 3 && d == 6) || (a == 2 && b ≥ 3 && d == 6) ),
    If[monodromy == 1,

  If[m == 3 && (ap == 0 && bp == 0 && dp ≥ 4) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
        If[m == 2 && b ≥ 4 && (ap == 2 && an == 2) , Return[False]; ]
   (* see so(7) B>0 form for f in IzerostarRestrictions.pdf. mu has pos. degree. *) ;
        If[m == 1 && b ≥ 4 && (ap == 2 && an == 4) , Return[False]; ]
   (* see so(7) B>0 form for f in IzerostarRestrictions.pdf mu has pos. degree. *) ;
    ] ;
    If[monodromy == 0,

  If[m == 1 && (ap == 0 && bp == 0 && dp ≥ 14) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
```

```
        If[m == 2 && (ap == 0 && bp == 0 && dp ≥ 8 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
   Return[False]; ];
        If[m == 3 && (ap == 0 && bp == 0 && dp ≥ 2 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
   Return[False]; ];
        If[m == 3 && (ap == 1 && bp ≥ 2 && dp == 3 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
   Return[False]; ];
        If[m == 3 && (ap ≥ 2 && bp == 2 && dp == 4 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
   Return[False]; ];
     ];
];


(* Instar restrictions *)

If[(a == 2 && b == 3 && d ≥ 7),
    If[m ≥ 3 && (ap > 0 || bp > 0) && (an > 0 || bn > 0),
     Return[False];
    ](* This is redundant condition also handled by intersection contributions*);
];


If[(a == 2 && b == 3 && d ≥ 10),
    If[monodromy == 0 && ap == 0 && bp == 0 &&
   dp ≥ ( 4 (d - 6) - 2 )   && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False];];
    If[monodromy == 1 && ap == 0 && bp == 0 && dp ≥ ( 4 (d - 6)    )   &&
   (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False];]
    (* We cannot have additional symmetry beyond sp(2n-1) or sp(2n)
  for the Instar{ns} Instar{s} cases respectively *);
    If[monodromy == 1,
        If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
            p = Floor[dp / 2];
            q = Floor[dn / 2];
             If[ p + q > 4 (d - 6), Return[False];](* all GS is in first summand sp(9-m)*);
          ];
     ];
     If[monodromy == 0,
        If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
            p = Floor[dp / 2];
            q = Floor[dn / 2];
             If[ p + q > 4 (d - 6) - 2, Return[False];](* all GS is in first summand sp(9-m)*);
          ];
      ];
  ];


If[(a == 2 && b == 3 && d == 9 ),
    If[monodromy == 1 && (ap == 0 && bp == 0 && dp ≥ 12 ) &&
    (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
    If[monodromy == 0 && (ap == 0 && bp == 0 && dp ≥ 10 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
   Return[False]; ]
       (* We cannot have additional symmetry beyond the max GS from BMM table 6.2 *) ;
    If[monodromy == 0,
        If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
            p = Floor[dp / 2];
            q = Floor[dn / 2];
            If[ p + q > 9 - m, Return[False];](* all GS is in first summand sp(9-m)*);
         ];
     ];
    If[monodromy == 1,
        If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
            p = Floor[dp / 2];
            q = Floor[dn / 2];
            If[ p + q > 10 - m, Return[False];](* all GS is in first summand sp(10-m). Note
                   m is always 4 in this case. We have ruled p+q>6, and GS max algebra sp(6).
                      Note this can be
      removed as it is taken care of by naive intersection contributions.*);
         ];
     ];
];

(* I_2* restrictions *)
If[(a == 2 && b == 3 && d == 8 ),
    If[monodromy == 0,
        If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
           p = Max[Floor[dp / 2], Floor[dn / 2]];
           q = Min[ Floor[dp / 2], Floor[dn / 2]];
           If[ p + q > 7 - m, Return[False];]
               (*  The weaker field theory constraints are:
                     If[m>1&& (p≥ 7-m)&& (q>0), Return[False];]
      (* Second summand is trivial for m>1.*);
```

```
                    If[m==1&& q>1 && p+q> 7-m, Return[False];]
    (* Cannot fit too large a factor in second summand*);
                    If[m>1&&  p+q> 7-m, Return[False];](* all GS is in first summand sp(7-m)*);
               *);
         ];
         If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
            p = Floor[dp / 2];
            If[p == 7 - m,
              Return[False];
            ](* Here as safety to rule out additional III fiber
                for example in case of direct user input skipping pair methods. *);
         ];
      ];
    If[monodromy == 1,
         If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
           p = Max[Floor[dp / 2], Floor[dn / 2]];
           q = Min[ Floor[dp / 2], Floor[dn / 2]];
           If[ p + q > 8 - m, Return[False];]
            (*The weaker field theory constraints are:
                    If[m>1&& (p≥ 8-m)&& (q>0), Return[False];]
      (* Second summand is trivial for m>1.*);
                    If[m==1&& q>1 && p+q> 8-m, Return[False];]
    (* Cannot fit too large a factor in second summand*);
                    If[m>1&&  p+q> 8-m, Return[False];](* all GS is in first summand sp(8-m)*);
             *)
            ;
         ];
         If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
           p = Floor[dp / 2];
            If[p == 8 - m,
               Return[False];
            ](* Here as safety to rule out additional III fiber
                for example in case of direct user input skipping pair methods. *);
         ];
     ];
](* BMM p.34 comments/BMM field theory constraints: Max algs sp(7-m)+sp(Max[2-m,0])
     for I_2*ns and sp(8-m)+sp(Max[2-m,0]) for I_2*s.*);


(* I_1* restrictions: *)
If[(a == 2 && b == 3 && d == 7 ),
    If[monodromy == 0,
         If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
             p = Max[Floor[dp / 2], Floor[dn / 2]];
             q = Min[ Floor[dp / 2], Floor[dn / 2]];
             If[ (p + q >  5 - m), Return[False];];
          ];
      ];
    If[monodromy == 1,
         If[(ap == 0 && bp == 0 && dp ≥ 1 ) && (an == 0 && bn == 0 && dn ≥ 1),
             p = Max[Floor[dp / 2], Floor[dn / 2]];
             q = Min[ Floor[dp / 2], Floor[dn / 2]];
          If[((p + q > 6 - m) ), Return[False]; ]
             (* One may be able to say more here for the second summand in
                 the so(10) case.

             If[m≥ 3&&(p+q>6-m ),Return[False];];
                 If[m≤2&&(q>1&& (p+q>6-m)),Return[False];]
                   (* One factor can at most hold sp1.
                  For m=2, sp>2+sp2 does fit in the first GS summand, sp4.
                  For m=1, sp3+sp3 must cannot lie in the first summand, sp5. *);
              *);
          ];
     ];
](* BMM p.33 comments/BMM Table 5.1 field theory constraints: Max algs sp(5-m)+sp(4-m) for I_1*ns
      and sp(6-m)+su(4-m) for I_1*s.*);

(* IV center restrictions from BMM. *)

If[(a ≥ 2 && b == 2 && d == 4),
   If[monodromy == 0,
     If[m == 2,
          If[((ap ≥ 2 && bp ≥ 3 && dp == 6 ) && monodromyPrev == 1) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ),
     Return[False];](*TODO: irrelevant, remove. *) ;
          If[((ap ≥ 2 && bp ≥ 3 && dp == 6 ) && monodromyPrev == 0) &&
      (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False];]
              (* g2 sp1 is not a subalgebra of so(7) *) ;
     ];
     If[m == 1,
          If[ ((ap ≥ 2 && bp ≥ 3 && dp == 6 ) && monodromyPrev == 1) &&
```

```
                    (an == 0 && bn == 0 && dn ≥ 4 ), Return[False]; ];
          ];
        ];
        If[monodromy == 1,
             If[m == 2,
               If[(ap == 0 && bp == 0 && dp ≥ 4) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
              ];
             If[m == 1,
               If[(ap == 0 && bp == 0 && dp ≥ 8) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ]
                  (* sp(4) transverse symm prevents additional transverse symm curves *) ;
               If[(ap == 0 && bp == 0 && dp ≥ 4) && (an == 0 && bn == 0 && dn ≥ 4), Return[False]; ]

        (* sp(2) transverse symmetry prevents additional transverse symmetry besides su(2) or su(3) *) ;
            ];
        ];
    ];

    If[ (a == 1 && b ≥ 2 && d == 3),
         If[m == 2,
             If[(ap ≥ 2 && bp ≥ 3 && dp == 6) &&  (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ]
              (* so(7)  transverse symmetry prevents additional transverse symmetry  *) ;
             If[((ap ≥ 2 && bp ≥ 3 && dp == 6 ) && monodromyPrev == 0) &&
         (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False];]
                  (* g2 sp1 is not a subalgebra of so(7) *) ;
           ];

         If[m == 1,
               If[(ap == 0 && bp == 0 && dp == 10 ) &&  (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ]
                       (* su(10)  transverse symmetry prevents additional transverse symmetry  *) ;
               If[(ap ≥ 2 && bp ≥ 2 && dp == 6 )  && ( (an == 0 && bn == 0 && dn ≥ 7 && monodromyNext == 1)
                       || (an == 0 && bn == 0 && dn ≥ 8 && monodromyNext == 0)), Return[False];];
         ];
    ];

    If[(a == 0 && b == 0 && d == 6  && monodromy == 1 && m == 1 ),
          If[(ap == 0 && bp == 0 && dp == 15 ) &&  (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
    ] (* su(6)* case *) ;



    (* Case: I_n center triplets.  First for odd n, then for n is 2, then for n even. *)
    If[(a == 0 && b == 0 && d ≥ 3 && (Mod[d, 2] ≠ 0) ) ,
          If[monodromy == 1,
                If[m == 1,

        If[ (ap == 0 && bp == 0 && dp ≥ 8 + d) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
                     If[(ap == 0 && bp == 0 && dp > 1) && (an == 0 && bn == 0 && dn > 1) ,

         If[ dp + dn > 8 + d, Return[False]; ](* su(n+6) is the maximum *);
                       ];
                ];

       If[m == 2 && (ap == 0 && bp == 0 && dp ≥ 2 d) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False];
                ];
           ];
     (* The following cases have m==2 is taken care of by residuals. *)
          If[monodromy == 0 && m == 1,
              If[ (ap ≥ 2 && bp ≥ 3 && dp ≥ 6 ) && (an ≥ 2 && bn ≥ 3 && dn ≥ 6),
                      If[dp == 6 ,
                           If[monodromyPrev == 2,
                               monMaxPrev = 1;
                                ,
                               monMaxPrev = 0;
                             ];
                       ];
                      If[dp > 6 ,
                          monMaxPrev = monodromyPrev;
                       ];
                      If[dn == 6 ,
                           If[monodromyNext == 2,
                               monMaxNext = 1;
                                ,
                               monMaxNext = 0;
                             ];
                       ];
                      If[dn > 6 ,
                          monMaxNext = monodromyNext;
                       ];
                     If[(dp - 6) + (dn - 6) > (d - monMaxPrev - monMaxNext), Return[False]; ];
              ];
```

```
          If[(ap ≥ 2 && bp ≥ 3 && dp ≥ 6 ) && (an == 0 && bn == 0 && dn ≥ 1),
                  If[dp == 6 ,
                       If[monodromyPrev == 2,
                           monMaxPrev = 1;
                            ,
                           monMaxPrev = 0;
                         ];
                   ];
                  If[dp > 6 ,
                      monMaxPrev = monodromyPrev;
                   ];
                  If[(((dp - 6) + dn > d + 3 - monMaxPrev)), Return[False];];
          ];
         If[(ap == 0 && bp == 0 && dp ≥ 1) && (an == 0 && bn == 0 && dn ≥ 1) ,

   If[ dp + dn > d + 6, Return[False]; ](* su(n+6) is the maximum GS; I_1 intersections have
                           contributions to \Deltat of the same form. *);
          ];
          If[d == 3 ,
               If[(ap ≥ 2 && bp == 2 && dp == 4 ) && (an == 0 && bn == 0 && dn > 1),
                       If[ dn > 8, Return[False];](* p.20 BMM transverse IV I_N  restriction. *) ;
                ];
               If[(ap == 1 && bp ≥ 2 && dp == 3 ) && (an == 0 && bn == 0 && dn > 1),
                 If[ dn > 9, Return[False];](* p.20 BMM transverse III I_N  restriction. *) ;
                ];
           ];
      ];
];

(* Case: I_2 center triplets *)

If[(a == 0 && b == 0 && d == 2 ),
     If[m == 1,
       If[ (ap == 2 && bp == 3 && dp ≥ 12 ) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ]
         (* No gauged fibers along with the I_6* max. *);
        If[(ap == 0 && bp == 0 & dp ≥ 2) && (an == 0 && bn == 0 && dn ≥ 2),
            If[dp + dn > 10, Return[False]; ](*I_10 is the BMM GS max for I_n types *);
        ];
       If[(ap == 0 && bp == 0 & dp ≥ 2) && (an ≥ 2 && bn ≥ 3 && dn ≥ 6),

   If[dp + (dn - 6) > 6, Return[False]; ](* I_p* + I_{6-p} is a maximal configuration. BMM p.18 *);
        ];
       If[(ap ≥ 2 && bp ≥ 3 & dp ≥ 6) && (an ≥ 2 && bn ≥ 3 && dn ≥ 6),
            If[(dp - 6) + (dn - 6) > 2, Return[False]; ]
     (* I_p* + I_q* for p+q=2 is a maximal configuration. BMM p.18 *);
        ](* See the transverse triplets method for the remaining BMM p.18 restriction
             from the maximal configurations with three fibers. *);
    ];
     If[m == 2 && ((ap == 0 && bp == 0 && dp ≥ 4 && monodromyPrev == 1) ||
      (ap == 0 && bp == 0 && dp ≥ 5 && monodromyPrev == 0))
                      && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
];

(* More BMM In GS restrictions
 (subgroup restrictions fro so so GS pairs in larger so max GS algebra)
   One might look for other triplet restrictions of this kind. *)
If[(a == 0 && b == 0 && d ≥ 2 && EvenQ[d]),
   If[monodromy == 1,
           If[m == 1,
               If[d == 6,

     If[ (ap == 0 && bp == 0 && dp ≥ 9 + d)  && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
                    If[(ap == 0 && bp == 0 && dp ≥ 2) && ( an == 0 && bn == 0 && dn ≥ 2
                         (* One might review generalizations for any dn>0. *) ),
                           If[dp + dn > 15, Return[False];];
                     ];
               ];
               If[d == 4,
                  If[(ap == 1 && bn ≥ 2 && dp == 3) && (an == 0 && bn == 0 && dn ≥ 2),
                     If[dn > 10, Return[False];];
                     ];
               ](* For d==4 we could also have a III involved. *) ;
               If[d ≠ 6,

     If[ (ap == 0 && bp == 0 && dp ≥ 8 + d)  && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
                    If[(ap == 0 && bp == 0 && dp ≥ 2) && ( an == 0 && bn == 0 && dn ≥ 2 ),
                          If[dp + dn > 8 + d, Return[False];];
                      ];
               ];
```

```
            ];
            If[m == 2,
                If[ (ap == 0 && bp == 0 && dp ≥ 2 d) && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ];
            ];
    ];
    If[monodromy == 0,
            If[m == 1,

    If[(ap == 2 && bp == 3 && (dp ≥ d + 10))  && (dn ≥ 2 && (! (an ≥ 1 && bn == 1) ) ), Return[False]; ]
                (* This is the same condition as in the table: GS bounded by so(16+2n )
                                    on In ns but translated to order in Delta of the transverse *) ;
                If[(ap ≥ 2 && bp ≥ 3 && dp ≥ 6 && an ≥ 2 && bn ≥ 3 && dn ≥ 6),
                            If[(dp - 6) + (dn - 6) > d , Return[False]; ];
                ];
                If[(ap ≥ 2 && bp ≥ 3 && dp ≥ 6 && an == 0 && bn == 0 && dn > 0),
                            If[ (dp - 6) + dn > d + 4 , Return[False]; ]
        (* p.21 BMM GS maximal configs: transverse su so pair. *);
                ];
                If[(ap == 0 && bp == 0 && dp ≥ 2 && an == 0 && bn == 0 && dn > 0) ,
                            If[ dp + dn > 8 + d, Return[False]; ]
                    (* p.21 BMM GS maximal
        configs: transverse su su pair must live in su(8+d) i.e., we must have
                    su(dp)+su(dn)\subset su(d+8). *);
                ](* Note,
    the special case d==6 only has special considerations when without monodromy. *) ;
            ](* The m==2 cases should be taken care of by residuals. *);
    ];
](* End of case: I_n for n even. *);

(* More BMM GS for In restrictions: *)

(* Case Izero center triplets. *)

If[ a ≥ 0 && b ≥ 0 && d == 0  && m == 1 ,
   If[(ap ≥ 2 && bp == 2 && dp == 4 ) && (an == 2 && bn == 3 && dn == 8), Return[False]; ];
];

(* M&R Gaugeless Triplet Restrictions (Izero's are from Persson's list) : *)

(* Izero restrictions read from (what is not in) Persson's list. *)
If[((a ≥ 0 && b ≥ 0 && d == 0) ),
    If[((ap ≥ 4 && bp ≥ 5 && dp ≥ 10) && (an == 0 && bn == 0 && dn == 2)), Return[False];];
    If[((ap == 3 && bp ≥ 5 && dp == 9) && (an == 0 && bn == 0 && dn ≥ 3)) , Return[False];];
    If[((ap == 2 && bp == 3 && dp == 10) && ((an ≥ 1 && bn == 1 && dn == 2) || (an == 0 && bn == 0 && dn ≥ 2 ))),
  Return[False];];
    If[((ap == 0 && bp == 0 && dp == 9) &&
            ((an == 0 && bn == 0 && dn ≥ 2) || (an ≥ 1 && bn == 1 && dn == 2) ||
                    (an ≥ 2 && bn == 2 && dn == 4 ) || (an == 1 && bn ≥ 2 && dn == 3))), Return[False];];
    If[((ap == 2 && bp == 3 && dp == 9) && ((an == 1 && bn ≥ 2 && dn == 3) || (an == 0 && bn == 0 && dn ≥ 2 ))),
  Return[False];];
    If[((ap == 0 && bp == 0 && dp == 8) && ((an == 0 && bn == 0 && dn ≥ 3) ||
       (an ≥ 2 && bn == 2 && dn == 4 ) || (an == 1 && bn ≥ 2 && dn == 3))),
         Return[False];];
    If[((ap ≥ 3 && bp == 4 && dp == 8) && ((an == 0 && bn == 0 && dn ≥ 4))), Return[False];];
    If[((ap == 2 && bp == 3 && dp == 8) && ((an ≥ 2 && bn == 2 && dn ≥ 4 ) || (an == 0 && bn == 0 && dn ≥ 3 ))),
  Return[False];];
    If[((ap == 0 && bp == 0 && dp == 7) && ((an == 0 && bn == 0 && dn ≥ 3) || (an ≥ 2 && bn == 2 && dn == 4))),
  Return[False];];
    If[((ap == 2 && bp == 3 && dp == 7) && ((an == 0 && bn == 0 && dn ≥ 5))), Return[False];];
    If[((ap == 0 && bp == 0 && dp == 6) && ((an == 0 && bn == 0 && dn ≥ 4) || (an ≥ 2 && bn ≥ 3 && dn == 6))),
  Return[False];];
    If[((ap ≥ 2 && bp ≥ 3 && dp == 6) && ((an == 0 && bn == 0 && dn ≥ 5))), Return[False];];
];

(* Case TYPE II CENTER BANNED TRIPLETS *)

  (* M&R II transverse pair restrictions: *)
If[(a ≥ 1 && b == 1 && d == 2),

 If[ (ap ≥ 3 && bp == 4 && dp == 8 && monodromyPrev == 1) && ((a ≥ 2 && b ≥ 3 && d == 6)), Return[False];];
   If[ (ap ≥ 3 && bp == 4 && dp == 8 ) && ((a ≥ 2 && b ≥ 3 && d == 6)) &&
   (monodromyPrev == 1 || monodromyNext ≥ 1 ), Return[False];];
   If[ (ap == 2 && bp == 3 && dp ≥  7) && ((a ≥ 2 && b ≥ 3 && d == 6)) &&
   (monodromyPrev == 1 || monodromyNext ≥ 1 ), Return[False];];
   If[ (ap == 0 && bp == 0 && dp ≥  6 ) && ((a ≥ 2 && b ≥ 3 && d == 6)) &&
   (monodromyPrev == 1 || monodromyNext ≥ 1 ), Return[False];];
   If[ (ap == 0 && bp == 0 && dp ≥  10) && ((a ≥ 2 && b ≥ 3 && d == 6)) &&
   (monodromyPrev ≥ 0 || monodromyNext ≥ 0 ), Return[False];];
   If[ (ap == 0 && bp == 0 && dp ≥  6 ) && (an == 0 && bn == 0 && dn ≥ 3) &&
   (monodromyPrev == 1 && monodromyNext == 1 ), Return[False];];
```

```
    If[(ap == 0 && bp == 0 && dp ≥  7) && (an == 0 && bn == 0 && dn ≥ 3) &&
    (monodromyPrev == 0 && monodromyNext == 1 ),  Return[False];];
  ];
(* Case I_1 CENTER BANNED TRIPLETS *)

(* M&R I_1 Transverse pair restrictions *)
If[(a == 0 && b == 0 && d == 1)
      && (ap ≥ 3 && bp == 4 && dp == 8) && (an == 0 && bn == 0 && dn ≥ 3)
                            && (monodromyPrev == 1 && monodromyNext == 1 ),
           Return[False];];
If[(a == 0 && b == 0 && d == 1)
      && (ap ≥ 3 && bp == 4 && dp == 8) && (an == 0 && bn == 0 && dn ≥ 4),
           Return[False];];
If[(a == 0 && b == 0 && d == 1)
      && ((ap ≥ 2 && bp == 3 && dp == 6 ) || (ap == 2 && bp ≥ 3 && dp == 6))
       && (an == 2 && bn == 3 && dn ≥ 7)
                               && (monodromyPrev == 2 || monodromyNext == 1 || dn > 7) ,
           Return[False];];
If[(a == 0 && b == 0 && d == 1)
      && (ap == 2 && bp == 3 && dp ≥  7) && (an == 0 && bn == 0 && dn ≥ 2)
                               && ((dp + dn > 10 ) || (monodromyPrev == 1 && (dp + dn == 10))) ,
           Return[False];];
If[(a == 0 && b == 0 && d == 1)
      && ((ap ≥ 2 && bp == 3 && dp == 6 ) || (ap == 2 && bp ≥ 3 && dp == 6))
       && (an == 0 && bn == 0 && dn ≥ 2)
                               && ((dp + dn > 10 ) || (monodromyPrev == 2 && (dp + dn == 10))) ,
           Return[False];];




(*

(* The comment here can be enabled to allow direct imposition of the triplets coming from the
   Atomic e_8 gauging condition.
   (This is from p.50 Atomic. Algebras meeting a -
    1 must have sum in e_8 when there is no algebra there. *)
If[m==1 && d≤2 && (!(a==0 && b==0 && d==2)),
       If[(ap==2 && bp==3 && dp ≥ 7) && (an==2 && bn==2 && dn≥ 7), Return[False];]
                  (* B4-B4 and beyond are not subs of e8.  *) ;
       If[(ap==2 && bp==3 && dp ≥ 7) && (an==2 && bn==2 && dn≥ 7), Return[False];]
                  (* B4+B4 not in e8.  Same if either or both B4 is replaced with D5. *) ;
       If[(ap==2 && bp==3 && dp ≥ 7) &&
                       (((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                                     && monodromyNext==2 ), Return[False];]
                  (* B4+D4 not in e8. Note: B4+B3 is in e8.  *) ;
       If[(ap==2 && bp==3 && dp ≥ 8 ) &&
                       (an==0 && bn==0 && dn ≥ 6)  && (monodromyNext==0 ), Return[False];]

  (* C2+B5 is maximal in D8 and this is the only sub of e8 that might hold such an option. *) ;
       If[(ap==2 && bp==3 && dp ≥ 8 ) &&
                       (an==0 && bn==0 && dn ≥ 4)  && (monodromyNext==1 ), Return[False];]
                  (* A3+B5 banned since
                   C2+
   B5 is maximal in D8 and this is the only sub of e8 that might hold such an option. *) ;
       If[(ap==2 && bp==3 && dp ≥ 7 ) &&
                       (an==0 && bn==0 && dn ≥ 5)  &&
    ( monodromyPrev==1 && monodromyNext==1 ), Return[False];]
                  (* A3 D5 is maximal in D8 and this is the only sub
   of e8 that might hold such an option. *) ;
      If[(ap==2 && bp==3 && dp ≥ 8 ) &&
                       (an==0 && bn==0 && dn ≥ 4)  &&
    ( monodromyPrev==1 && monodromyNext==1 ), Return[False];]
                  (* A3 D6 banned purely by rank *) ;
       If[(ap==0 && bp==0 && dp ≥ 3) &&
                       ((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                                 && (monodromyPrev==1 && monodromyNext==2 )
                                             && ((dp-1)+ 4 > 8 ), Return[False];]
                  (* A_n + D4 not in e8 when n+4>8 simply by rank.  *) ;
       If[(ap==0 && bp==0 && dp ≥ 3) &&
                       ((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                                 && (monodromyPrev==1 && monodromyNext==1 )
                                             && ((dp-1)+ 3 > 8 ), Return[False];]
                  (* A_n + B3 not in e8 when n+3>8 simply by rank.  *) ;
       If[(ap==0 && bp==0 && dp ≥ 3) &&
                       ((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                                 && (monodromyPrev==1 && monodromyNext==0)
                                             && ((dp-1)+ 2 > 8 ), Return[False];]
                  (* A_n + g2 not in e8 when n+2>8 simply by rank.  *) ;
       If[(ap==0 && bp==0 && dp ≥ 6) &&
```

```
                    ((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                                   && (monodromyPrev==1 && monodromyNext==0), Return[False];]
                (* A_6 + g2 not in e8 by inspection.  *) ;
        If[(ap==0 && bp==0 && dp ≥ 3 ) && (an==0 && bn==0 && dn≥ 3)
                  && (monodromyPrev==1 && monodromyNext==1 ) && (dp + dn -2 > 8), Return[False];]
                (* A_p + A_q is not in e8 when p + q > 8 simply by rank counting. *);
        If[(ap≥ 3 && bp==4 && dp==8 ) &&
                              ((an==2 && bn≥ 3&& dn==6)  ||(an≥ 2&&bn==3 && dn==6))
                  && (monodromyPrev==1 || monodromyNext ≥ 1 ), Return[False];]

     (* f4 g2 is a maximal subalgebra of e8; neither curve can have the large algebra. *);
        If[(ap≥ 3 && bp==4 && dp==8 ) &&
                              (an==0 && bn==0 && dn ≥ 4)
                  && (monodromyPrev==1 && monodromyNext==1 ), Return[False];]
                   (* e6 A2 is a maximal subalgebra of e8. *);
        If[(ap≥ 3 && bp==4 && dp==8 ) &&
                              (an==0 && bn==0 && dn ≥ 4)
                  && (monodromyPrev==1 && monodromyNext==0 ), Return[False];]
                   (* e6 C2 is not a subalgebra of e8. *);
        If[(ap==3 && bp ≥  4 && dp==9 ) &&
                              (an==0 && bn==0 && dn ≥ 3 && monodromyNext==1), Return[False];]
                (* e7 A1 is a maximal subalgebra of e8. *);
        If[(ap≥ 4 && bp==5 && dp==10 ) &&
                              (dn ≥ 2 && (!(an≥ 1 && bn==1) ) ), Return[False];]
                (* e8 on the left means the right has no algebra. *);
        If[(ap==0 && bp==0 && dp≥ 3 ) && (an==2 && bn==3 && dn≥ 7)
                                                                  &&
      (monodromyPrev==1 && monodromyNext==1)&& (dp + dn > 11),
                 Return[False]; ]  ;
   ];

  *)
  Return[booleanToReturn];
 ];

(* IsValidMonodromyAssignmentToOrderedTransverseTriplet:
 Transverse triplets:(Left,Middle,Right) i.e. quartets (curve,L,M,R) .*)

IsValidMonodromyAssignmentToOrderedTransverseTriplet[
   quiverCurveEnh_, transverseEnhL_, transverseEnhM_, transverseEnhR_,
   monOnCurve_, monL_, monM_, monR_] :=
  Module[ {a, b, d, aL, aM, aR, bL, bM, bR, dL, dM, dR, p, q, r, m, monMaxL, monMaxM},
   m = quiverCurveEnh[[1, 1, 1]];
   a = quiverCurveEnh[[1, 2, 1]];
   b = quiverCurveEnh[[1, 2, 2]];
   d = quiverCurveEnh[[1, 2, 3]];
   aL = transverseEnhL[[1, 2, 1]];
   aM = transverseEnhM[[1, 2, 1]];
   aR = transverseEnhR[[1, 2, 1]];
   bL = transverseEnhL[[1, 2, 2]];
   bM = transverseEnhM[[1, 2, 2]];
   bR = transverseEnhR[[1, 2, 2]];
   dL = transverseEnhL[[1, 2, 3]];
   dM = transverseEnhM[[1, 2, 3]];
   dR = transverseEnhR[[1, 2, 3]];

   (* From the lines after the main Persson's list table. *)
   If[d == 0,
      If[(aL == 0 && bL == 0 && dL ≥ 7) &&
           (aM == 1 && bM ≥ 2 && dM == 3) &&
            (aR ≥ 1 && bR ≥ 1 && dR ≥ 2),
     Return[False];
     ];
      If[(aL == 0 && bL == 0 && dL ≥ 6) &&
           (aM == 1 && bM == 2 && dM ≥ 3) &&
            (aR == 1 && bR == 2 && dR ≥ 3),
     Return[False];
     ];
      If[(aL == 0 && bL == 0 && dL ≥ 5) &&
           (aM ≥ 2 && bM ≥ 2 && dM ≥ 4) &&
            (aR ≥ 1 && bR ≥ 2 && dR ≥ 3),
     Return[False];
     ];
   ];

   (* From the main Persson's list table. *)
   If[d == 0,
     If[(aL == 0 && bL == 0 && dL ≥ 8 ) &&
          (aM == 0 && bM == 0 && dM == 2 ) &&
           (aR == 2 && bR == 2 && dR == 2),
```

```
 Return[False];
 ];
If[(aL ≥ 3 && bL ≥ 4 && dL ≥ 8 ) &&
      (aM == 0 && bM == 0 && dM ≥ 2) &&
        (aR == 0 && bR == 0 && dR ≥ 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 7 ) &&
      (aM == 1 && bM ≥ 2 && dM == 3 ) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL == 2 && bL == 2 && dL == 7 ) &&
      (aM == 0 && bM == 0 && dM == 3 ) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 6 ) &&
      (aM == 0 && bM == 0 && dM == 4 ) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 6 ) &&
      (aM == 0 && bM == 0 && dM == 3 ) &&
        (aR == 0 && bR == 0 && dR == 3),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 6) &&
      (aM ≥ 2 && bM == 2 && dM == 4) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL ≥ 2 && bL ≥ 3 && dL == 6) &&
      (aM ≥ 2 && bM == 2 && dM == 4) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL ≥ 2 && bL ≥ 3 && dL == 6 ) &&
      (aM == 0 && bM == 0 && dM == 3 ) &&
        (aR == 0 && bR == 0 && dR ≥ 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 5 ) &&
      (aM == 0 && bM == 0 && dM == 5 ) &&
        (aR == 0 && bR == 0 && dR == 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 5 ) &&
      (aM == 0 && bM == 0 && dM == 5 ) &&
        (aR ≥ 1 && bR == 1 && dR == 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 5 ) &&
      (aM == 0 && bM == 0 && dM == 4 ) &&
        (aR == 0 && bR == 0 && dR ≥ 2),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 5 ) &&
      (aM ≥ 2 && bM == 2 && dM == 4 ) &&
        (aR == 0 && bR == 0 && dR == 3),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 5 ) &&
      (aM == 0 && bM == 0 && dM ≥ 3 ) &&
        (aR == 0 && bR == 0 && dR == 3),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 4 ) &&
      (aM == 0 && bM == 0 && dM ≥ 3 ) &&
        (aR == 0 && bR == 0 && dR ≥ 3),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 4 ) &&
      (aM == 0 && bM == 0 && dM == 4 ) &&
        (aR ≥ 2 && bR == 2 && dR == 4),
 Return[False];
];
If[(aL == 0 && bL == 0 && dL == 4 ) &&
      (aM == 0 && bM == 0 && dM == 4 ) &&
        (aR == 1 && bR ≥ 2 && dR == 3),
```

```
    Return[False];
   ];
   If[(aL == 0 && bL == 0 && dL == 4 ) &&
        (aM ≥ 2 && bM == 2 && dM == 4 ) &&
          (aR ≥ 2 && bR == 2 && dR == 4),
    Return[False];
   ];
   If[(aL ≥ 2 && bL == 2 && dL == 4 ) &&
        (aM ≥ 2 && bM == 2 && dM == 4 ) &&
          (aR ≥ 2 && bR == 2 && dR == 4),
    Return[False];
   ];
];

(* From BMM Table 6.1 *)
(* I_0* restrictions *)
If[(a ≥ 2 && b ≥ 3 && d == 6),
   If[monOnCurve == 1,
     If[m == 3,
        If[(aL == 0 && bL == 0 && dL ≥ 2 ) &&
            (aM == 0 && bM == 0 && dM ≥ 2 ) &&
             ((dR ≥ 2) && (! (aR ≥ 1 && bR == 1))),
        Return[False](* We can't have three
                        fibers with algebra
                        rank one each since
                       C2 is the GS max. *);
        ];
        If[(aL == 1 && bL ≥ 2 && dL == 3 ) &&
            (aM == 0 && bM == 0 && dM ≥ 2 ) &&
             ((dR ≥ 2) && (! (aR ≥ 1 && bR == 1))),
        Return[False](* Again,
                      C2 is the GS max. *);
        ];
        If[(aL == 1 && bL ≥ 2 && dL == 3 ) &&
            (aM == 1 && bM ≥ 2 && dM == 3 ) &&
             ((dR ≥ 2) && (! (aR ≥ 1 && bR == 1))),
        Return[False](* Again,
                      C2 is the GS max. *);
        ];

     ];
     If[m == 2,
        If[(aL == 0 && bL == 0 && dL == 9 ) &&
             (aM == 0 && bM == 0 && dM == 3 ) &&
               (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
         Return[False];
         ];
     ];
     If[m == 1,
        If[(aL == 0 && bL == 0 && dL == 13 ) &&
             (aM == 0 && bM == 0 && dM == 5 ) &&
               (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
         Return[False];
         ];
     ];
   ](* so(7) restrictions *);
   If[monOnCurve == 2,
     If[m == 2,
        If[(aL == 0 && bL == 0 && dL ≥ 2 ) &&
             (aM == 0 && bM == 0 && dM ≥ 2 ) &&
               (aR == 0 && bR == 0 && dR ≥ 2) &&
                 (dL ≥ 4 && dM ≥ 4 && dR ≥ 4),
         Return[False];
         ];
     ];
     If[m == 1,
        If[(aL == 0 && bL == 0 && dL ≥ 2 ) &&
             (aM == 0 && bM == 0 && dM ≥ 2 ) &&
               (aR == 0 && bR == 0 && dR ≥ 2) &&
                 (dL ≥ 6 && dM ≥ 6 && dR ≥ 4),
         Return[False];
         ];
     ];
   ](* so(8) restrictions *);
];

(* IV restrictions *)
If[(a ≥ 2 && b == 2 && d == 4),
   If[monOnCurve == 0,
     If[m == 1,
```

```
          If[(aL ≥ 2 && bL ≥ 3 && dL == 6) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
          Return[False];
          ];
          If[(aL ≥ 2 && bL == 2 && dL == 4) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (aR ≥ 2 && bR == 2 && dR == 4),
          Return[False];
          ](* IV IV I_4 is ruled out for IVs since all g contributions are then even.*);
          If[(aL == 1 && bL ≥ 2 && dL == 3) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (aR ≥ 2 && bR == 2 && dR == 4),
          Return[False];
          ](* III IV I_4 is ruled out for IVs since all g contributions are then even.*);
          If[(aL == 1 && bL ≥ 2 && dL == 3) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (aR == 1 && bR ≥ 2 && dR == 3),
          Return[False];
          ](* III III I_4 is ruled out for IVs since all g contributions are then even.
                TODO: add a general IV check for treating the compact CY case
                      rather than GS case as follows. Eliminate the transverse quartets with
                           purely even \bt contributions. *);
          If[(aL == 0 && bL == 0 && dL ≥ 2) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (aR ≥ 2 && bR == 2 && dR == 4),
          Return[False];
          ](* I_2 I_4 (or beyond) and a IV give purely even \bt contribtions.*);
          If[(aL == 0 && bL == 0 && dL ≥ 2) &&
              (aM == 0 && bM == 0 && dM ≥ 4) &&
                (aR == 1 && bR ≥ 2 && dR == 3),
          Return[False];
          ](* I_2 I_4 (or beyond) and a III also give purely even \bt contribtions.*);
        ];
    ](* su(2) restrictions. See BMM pp.24-25.*);
    If[monOnCurve == 1,
      If[m == 1,
         If[(aL ≥ 2 && bL == 2 && dL == 4) &&
              (aM == 0 && bM == 0 && dM ≥ 6) &&
                (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
          Return[False];
          ];
      ];
      If[m == 2,
         If[((aL == 0 && bL == 0 && dL == 3 )
                 || (aL == 1 && bL ≥ 2 && dL == 3)
                 || (aL ≥ 2 && bL == 2 && dL == 4)) &&
              ((aM == 0 && bM == 0 && dM == 3 )
                 || (aM == 1 && bM ≥ 2 && dM == 3)
                 || (aM ≥ 2 && bM == 2 && dM == 4)) &&
              (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
          Return[False];
          ](* Any pair of GS fibers from {I_3,III,IV}
              is relatively maximal. See BMM pp.24-25 *);
      ];
    ](* su(3) restrictions *);
](* One might consider tightening the restrictions
    here where all vanishings are used to prevent
    any additional fiber with nontrivial Delta
    vanishing rather than only those supporting
    nontrivial algebra. *);

(* III restrictions *)
If[(a == 1 && b ≥ 2 && d == 3),
    If[m == 1,
      If[(aL ≥ 2 && bL ≥ 3 && dL == 6) &&
            (aM == 0 && bM == 0 && dM ≥ 6) &&
              (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2)),
         Return[False];
      ];
    ];
    If[m == 2,
      If[((aL == 1 && bL ≥ 2 && dL == 3)
           || (aL == 0 && bL == 0 && dL ≥ 2)) &&
          ((aM == 1 && bM ≥ 2 && dM == 3)
             || (aM == 0 && bM == 0 && dM ≥ 2)) &&
             (dR > 0),
        Return[False];
      ];
    ](* This is already taken care of my intersection
```

```
        contributions, but is here as a safety. *);
];
(* I_n and Instar restriction on number nontrivial F,G fibers *)
(* At most two nontrivial F,G fibers with monodromy (a==2&& b==3&&d≥ 7)*)
If[(a == 0 && b == 0 && d ≥ 1),
   If[monOnCurve == 0,
      If[(aL > 0 || bL > 0) && (aM > 0 || bM > 0) && (aR > 0 || bR > 0),
       Return[False];
      ];
    ];
];
If[(a == 2 && b == 3 && d ≥ 7),
    If[m ≥ 2 && (aL > 0 || bL > 0) && (aM > 0 || bM > 0) && (aR > 0 || bR > 0),
     Return[False];
    ](* This is redundant condition also handled by intersection contributions*);
];


(* I_n odd BMM restrictions, starting with n=3. *)
If[(a == 0 && b == 0 && d ≥ 3 && d == 3),
   If[monOnCurve == 1,
      If[(aL ≥ 2 && bL == 2 && dL == 4 && monL == 1) &&
            (aM == 0 && bM == 0 && dM ≥ 8) &&
              (dR > 0 || dM > 8),
       Return[False];
      ];
    ];
];

If[(a == 0 && b == 0 && d ≥ 3 && Mod[d, 2] ≠ 0),
   If[monOnCurve == 1,
     If[(aL == 1 && bL ≥ 2 && dL == 3) &&
           (aM == 0 && bM == 0 && dM ≥ 9) &&
             (dR > 0 || dM > 9),
         Return[False];
      ](* p.21 BMM restriction for I_3. *);
  ];
  If[monOnCurve == 0,
     If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6) &&
          (aM ≥ 2 && bM ≥ 3 && dM ≥ 6) ,
                 If[dL == 6 ,
                      If[monL == 2,
                           monMaxL = 1;
                            ,
                           monMaxL = 0;
                      ];
                  ];
                 If[dL > 6 ,
                     monMaxL = monL;
                  ];
                 If[dM == 6 ,
                      If[monM == 2,
                           monMaxM = 1;
                            ,
                           monMaxM = 0;
                        ];
                  ];
                 If[dM > 6 ,
                     monMaxM = monM;
                  ];
                If[(dL - 6) + (dM - 6) > (d - monMaxL - monMaxM), Return[False]; ];
                If[aR == 0 && bR == 0 && dR ≥ 2,
                   If[(dL - 6) + (dM - 6) > (d - monMaxL - monMaxM - dR), Return[False]; ];
               ](* BMM p. 20 I_n for n odd GS rules. *);
      ];
      If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6) &&
         (aM == 0 && bM == 0 && dM ≥ 2) &&
           (aR == 0 && bR == 0 && dR ≥ 2),
                If[dL == 6 ,
                      If[monL == 2,
                           monMaxL = 1;
                            ,
                           monMaxL = 0;
                      ];
                 ];
                If[dL > 6 ,
                     monMaxL = monL;
                 ];
                If[(dL - 6) > (d + 3 - monMaxL - dM - dR), Return[False]; ];
       ](* Analogue of bottom of p. 20 BMM restriction for I_p* + two I_n fibers. *);
```

```
    ];
];

(* I_n even BMM restrictions, starting with n=2. *)
If[(a == 0 && b == 0 && d == 2),
    If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6) &&
         (aM == 0 && bM == 0 && dM ≥ 1) &&
           ((dR > 0 && (dL + dM ≥ 12)) ||
             (dL + dM > 12)),
      Return[False];
    ];
    If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6) &&
         (aM == 0 && bM == 0 && dM ≥ 2) &&
           (aR == 0 && bR == 0 && dR ≥ 2) &&
           ((dL - 6) + dM + dR > 6),
      Return[False];
    ];
];
If[(a == 0 && b == 0 && d ≥ 4 && Mod[d, 2] == 0),
   If[monOnCurve == 1,
     If[d == 4 &&
          (aL == 1 && bL ≥ 2 && dL == 3) &&
           (aM == 0 && bM == 0 && dM ≥ 10) &&
             ((dR > 0) || (dM > 10)),
        Return[False](* III, I_10 is a maximal GS option. *);
      ];
   ];
   If[monOnCurve == 0,
     If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6) &&
           (aM == 0 && bM == 0 && dM ≥ 1) &&
            (aR == 0 && bR == 0 && dR ≥ 0) &&
             ((dL - 6) + dM + dR > 4 + d),
         Return[False](* Note that the final condition on p.22 BMM for
             (I_p* I_q* I_{n-p-q}) is automatic through residuals
              contributions.*);
      ];
   ];
];

(* Gaugeless curves I_1 and II *)
(* Type I_1 conditions from M&R. *)
If[a == 0 && b == 0 && d == 1,
 If[(aL ≥ 3 && bL == 4 && dL == 8) &&
      (aM == 0 && bM == 0 && dM ≥ 3 && monM == 1) &&
        (    (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
           || (monL == 1)
           || (dM > 3)
         ),
    Return[False];
 ];
 If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6 ) &&
      (aM == 2 && bM == 3 && dM ≥ 7) &&
        (    (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
           || (monL == 1)
           || (monM == 1)
           || (dL > 6)
           || (dM > 7)
         ),
    Return[False];
 ];
 If[(aL == 2 && bL == 3 && dL ≥ 7 ) &&
      (aM == 0 && bM == 0 && dM > 1) &&
       (dL - 6 + dM == 4) &&
        (    (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
           || (monL == 1)
         ),
    Return[False];
 ];
 If[(aL ≥ 2 && bL ≥ 3 && dL == 6 ) &&
      (aM == 0 && bM == 0 && dM > 1) &&
       (dL - 6 + dM == 4) &&
        (    (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
           || (monL == 2)
         ),
    Return[False];
 ];
];

(* Type II conditions from M&R *)
If[a ≥ 1 && b == 1 && d == 2,
```

```
      If[(aL ≥ 3 && bL == 4 && dL == 8 ) &&
          (aM ≥ 2 && bM ≥ 3 && dM ≥ 6) &&
            (  (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
             || (monL == 1)
             || (monM ≥ 1)
             ||  (dM > 6)
            ),
         Return[False];
      ];
      If[(aL == 2 && bL == 2 && dL ≥ 7) &&
          (aM ≥ 2 && bM ≥ 3 && dM ≥ 6) &&
            (  (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
             || (monL == 1)
             || (monM ≥ 1)
             || (dL > 7)
            ),
         Return[False];
      ];
      If[(aL ≥ 2 && bL ≥ 3 && dL ≥ 6 ) &&
          (aM == 0 && bM == 0 && dM ≥ 6) &&
            (  (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
             || (monL ≥ 1)
             || (monM == 1)
             || (dM > 9)
            ),
         Return[False];
      ];
      If[(aL == 0 && bL == 0 && dL ≥ 6 ) &&
          (aM == 0 && bM == 0 && dM == 3 && monM == 1) &&
            (  (dR ≥ 2 && ! (aR ≥ 1 && bR == 1 && dR == 2))
             || (monL == 1)
             || (dL > 6)
            ),
         Return[False];
      ];
    ];


    Return[True](* Otherwise the triplet meeting the compact curve is
                   allowed by this method. *);
  ];

(* ComputeAPosterioriResidualVanishingsFromQuiverAndEnhancementAndMonodromy:
 Finds residuals w/ IntersectionContributions. *)

ComputeAPosterioriResidualVanishingsFromQuiverAndEnhancementAndMonodromy[quiver_,
    enhancement_, monodromy_] := Module[{enhWithResiduals, enhWithAPostResiduals},
    enhWithResiduals = ComputeResidualVanishingsFromQuiverAndEnhancement[quiver, enhancement];
    enhWithAPostResiduals =
     ComputeAPosterioriResidualsConsideringMonodromy[enhWithResiduals, monodromy];
    Return[enhWithAPostResiduals];
  ];

(* IsValidMonodromyAssignmentToQuiver.    Test whether a monodromy assignment to quiver is valid. *)
```

```
IsValidMonodromyAssignmentToQuiver[enhancementWithResiduals_, monodromyAssignmentToQuiver_] :=
  Module[{toReturn, length, enhancementWithResidualsPair,
    quiverPair, enhancementPair, monodromyAssignmentPair, quiverTriplet,
    enhancementWithResidualsTriplet, monodromyAssignmentTriplet, i},
   (* Note that here we do not use the residuals and an plain enhancement will go through. *)
   toReturn = True;
   length = Length[enhancementWithResiduals];
   If[length == 1,
        If[Length[monodromyAssignmentToQuiver] == 1,
             Return[True];
             ,
             Print["error: length of quiver and monodromy assignment don't match"];
        ];
   ];

   If[length ≥ 2,
    For[i = 1, i ≤ length - 1, i ++,
         enhancementWithResidualsPair =
        {enhancementWithResiduals[[i]], enhancementWithResiduals[[i + 1]]};
        monodromyAssignmentPair = { monodromyAssignmentToQuiver[[i]],
         monodromyAssignmentToQuiver[[i + 1]]};

        If[! IsValidMonodromyAssignmentToPair[
                    enhancementWithResidualsPair , monodromyAssignmentPair ],
                   Return[False]; ];
        If[! IsValidMonodromyAssignmentToPair[
                    Reverse[enhancementWithResidualsPair] , Reverse[monodromyAssignmentPair ]],
                   Return[False]; ];
     ];
   ];

   If[length ≥ 3,
    For[i = 2, i ≤ length - 1, i ++,
         enhancementWithResidualsTriplet =
        {enhancementWithResiduals[[i - 1]], enhancementWithResiduals[[i]],
                              enhancementWithResiduals[[i + 1]]};
         monodromyAssignmentTriplet =
        {monodromyAssignmentToQuiver[[i - 1]], monodromyAssignmentToQuiver[[i]],
                              monodromyAssignmentToQuiver[[i + 1]]};

        If[! IsValidMonodromyAssignmentToTriplet[
          enhancementWithResidualsTriplet , monodromyAssignmentTriplet ],
                   Return[False]; ];
        If[! IsValidMonodromyAssignmentToTriplet[
          Reverse[enhancementWithResidualsTriplet ], Reverse[monodromyAssignmentTriplet] ],
                   Return[False]; ];
     ];
   ];

   Return[toReturn];
  ];

(* IsValidMonodromyAssignmentToQuiverFromGivenPosition:
 Test validity of given monodromy position to the quiver RHS. *)

IsValidMonodromyAssignmentToQuiverFromGivenPosition[enhancementWithResiduals_,
   monodromyAssignmentToQuiver_ , position_] := Module[{startPosition, endPosition},
     startPosition = Max[position, 1];
     endPosition = Length[monodromyAssignmentToQuiver];
   If[startPosition > endPosition, Print["error see IsValidMonodromyAssignmentToQuiver"];];
   Return[
    IsValidMonodromyAssignmentToQuiver[Take[enhancementWithResiduals, {startPosition, endPosition}],

     Take[monodromyAssignmentToQuiver, {startPosition, endPosition}]]];
  ];

IsValidMonodromyAssignmentToQuiverFromGivenPositionNoResiduals[
   enhancementNoResiduals_, monodromyAssignmentToQuiver_ , position_] := Module[
   {startPosition, endPosition, enhancementWithResidualsOnTestRegion, monodromyAssignmentOnTestRegion },
     startPosition = Min[position - 1, 1];
     endPosition = Min[Length[enhancementNoResiduals], position + 1];
   If[startPosition > endPosition,
    Print["error see IsValidMonodromyAssignmentToQuiverFromAGivenPositionNoResiduals"];];
     enhancementWithResidualsOnTestRegion = ComputeResidualVanishings[
     Take[enhancementNoResiduals, {startPosition, endPosition}]];
     monodromyAssignmentOnTestRegion =  Take[monodromyAssignmentToQuiver, {startPosition, endPosition}];
   Return[IsValidMonodromyAssignmentToQuiver[enhancementWithResidualsOnTestRegion,
                                           monodromyAssignmentOnTestRegion]];
  ];
```

```
(* IsValidMonodromyAssignmentToSubquiver:   Test given monodromy in the position range specified. *)

IsValidMonodromyAssignmentToSubquiver[enhancementWithResiduals_,
   monodromyAssignmentToQuiver_ , positions_] := Module[{startPosition, endPosition, length},
   length = Min[Length[monodromyAssignmentToQuiver] , Length[enhancementWithResiduals]];
   If[Length[Flatten[{monodromyAssignmentToQuiver} ]] ≤ 1 && Length[enhancementWithResiduals ≠ 1],
       Print["error see IsValidMonodromyAssignmentToSubquiver"];
   ];
     If[Length[Select[Flatten[{1, monodromyAssignmentToQuiver}], (# < 0) & ]] > 0, Return[False]; ];
     startPosition = Min[Max[positions[[1]] - 1, 1], length];
     endPosition = Min[positions[[2]], length ];
     If[startPosition > endPosition, Print["error, see IsValidMonodromyAssignmentToSubquiver"]; ];
   Return[
    IsValidMonodromyAssignmentToQuiver[Take[enhancementWithResiduals, {startPosition, endPosition}],

     Take[monodromyAssignmentToQuiver, {startPosition, endPosition}]]];
  ];

(* The following few methods allow stored values
 from a table that can be written and read from file. *)

(* InitializeStoredQuiversTable: *)

InitializeStoredQuiversTable[] := Module[{j, counter},
   counter = 0;
   For[j = 1, j ≤ 12, j ++, storedQuiversTable[{j}] = 1; counter ++; ];
   Return[counter];
  ];


(* InitializeEnhToMonTable:  *)

InitializeEnhToMonTable[] := Module[{i, j, k, enhOnSingleCurveQuiver, typesOptions, counter},
   counter = 0;
   For[i = 1 , i ≤ Length[enhTypes], i ++,
      For[j = 1, j ≤ Length[enhTypes[[i]]], j ++,
         enhOnSingleCurveQuiver = {{i}, enhTypes[[i, j]] } ;
         typesOptions = MonodromyTypes[enhOnSingleCurveQuiver];
         enhToMonodromyOptionsTable[ {enhOnSingleCurveQuiver} ] =
                Table[{typesOptions[[k]]}, {k, 1, Length[typesOptions]}];
      counter ++;
     ];
   ];
   Return[counter];
  ];

(* We either import cached values or initialize from scratch depending on the user option choices.  *)

Quiet[
  If[toReadFileWithSavedMonodromyHashTable,
      Get[ monodromyHashFileNameIn];
      numberOfQuiversStored = numberOfQuiversStored + Length[hashKeyTable];
      Print["hashed enhancement options read. Total quivers stored so far:",
                Length[hashKeyTable]];
  ];
  Print[
   "Monodromy hash tables initialized, starting with {num quivers, num stored enh mon options} = ",
          If[! toReadFileWithSavedMonodromyHashTable,  {InitializeStoredQuiversTable[],
                                                InitializeEnhToMonTable[]}
                                               ,
                              Length[DownValues[storedQuiversTable]]
                                                ]
    ];

 ](* number quivers, then total type assignments with stored to start if hash not read: *) ;
```

Monodromy hash tables initialized, starting with {num quivers, num stored enh mon options} = {12, 224}

```
(* ReadStoredQuiversTable: The following reads from the stored table that can be written to file.  *)
```

```
ReadStoredQuiversTable[quiver_] := Module[{toReturn},
   If[Length[quiver] > maxLengthToBeHashedEnhToMon, Return[-1];];
   toReturn = -1;

   While[writeHashInProgressStoredQuivers,
               Pause[hashingReadPauseLengthStoredQuivers ];
   ];

   readHashInProgressStoredQuivers = True;
   Pause[hashingReadPauseLengthStoredQuivers ];
     toReturn = storedQuiversTable[quiver];
   readHashInProgressStoredQuivers = False;
   Pause[hashingReadPauseLengthStoredQuivers ];
   If[(ToString[Head[toReturn]] == "storedQuiversTable") || TrueQ[toReturn == Null],
        Return[-1];
    ,
        Return[1];
   ];
  ];

(* WriteStoredQuiversTable:  WRITES HASH. *)


WriteStoredQuiversTable[quiver_] := Module[{},
   If[Length[quiver] > maxLengthToBeHashedEnhToMon, Return[];];
   If[(1 ==  ReadStoredQuiversTable[quiver]), Return[];](*already stored.*);
   While[readHashInProgressStoredQuivers,
      Pause[hashingReadPauseLengthStoredQuivers];
   ];
   writeHashInProgressStoredQuivers = True;
   storedQuiversTable[quiver] = 1;
   writeHashInProgressStoredQuivers = False;
  ](* CHANGE THIS TO TAKE A QUIVER LIST *) ;

WriteListToStoredQuiversTable[quiverList_] := Module[{i, quiver, quiversToWrite},
   quiver = {};
   quiversToWrite =
     Flatten[Reap[
       For[i = 1, i ≤ Length[quiverList], i ++,
          quiver = quiverList[[i]];
          If[     TrueQ[(1 == ReadStoredQuiversTable[quiver])]
               ||  TrueQ[(1 == ReadStoredQuiversTable[Reverse[quiver]])],
              Continue[];
           ](* this quiver is already stored in this case, so try the next one. *);
          If[
          Length[quiver] < minLengthToBeHashedEnhToMon || Length[quiver] > maxLengthToBeHashedEnhToMon,
             Continue[];
           ,
             Sow[quiver];
           ];
        ];][[2]], 1];
   For[i = 1, i ≤ Length[quiversToWrite], i ++,
    quiver = quiversToWrite[[i]];
    WriteStoredQuiversTable[quiver];
   ];
   Return[quiver];
  ];

(* MaxLengthStoredSubquiver:
 Gives the longest stored quiver that is a subquiver of that given and matches the start. *)

MaxLengthStoredSubquiver[quiver_] := Module[{length, maxSoFar, i, storedQuiver, toReverse, subquiver },
   length = Length[quiver];
   maxSoFar = 1;
   toReverse = False;
   For[i = 1, i ≤ length, i ++,
      subquiver = Take[quiver, i];
     If[TrueQ[(ReadStoredQuiversTable[subquiver] == 1)],
      maxSoFar = i;
      toReverse = False;
      ,
      If[TrueQ[(ReadStoredQuiversTable[Reverse[subquiver]] == 1)],
           toReverse = True;
           maxSoFar = i;
        ];
      ];
   ];
   Return[{Take[quiver, maxSoFar], toReverse}];
  ];
```

```
EnhancementToMonodromyFromTable[enhancement_, toReverse_] :=
  Module[{enhancementForLookup, toReturn, valueFromTable},
   toReturn = {};
   If[toReverse,
    enhancementForLookup = Reverse[enhancement];
    ,
    enhancementForLookup = enhancement;
   ];
   While[writeHashInProgressEnhToMon,
      Pause[hashingReadPauseLengthEnhToMon];
   ];
   readHashInProgressEnhToMon = True;
   valueFromTable = enhToMonodromyOptionsTable[ enhancementForLookup ];
   readHashInProgressEnhToMon = False;
   If[ (TrueQ[(ToString[Head[valueFromTable]] == "enhToMonodromyOptionsTable")]) ||
      (TrueQ[valueFromTable == Null]) ,
     toReturn = {{-2}};
    ,
     toReturn = valueFromTable;
   ];
   Return[toReturn];
  ];

ReadSharedMonodromyTable[enhancementList_, toReverse_] :=
  Module[{i, readFromHash, inputListLength, outList},
   inputListLength = Length[enhancementList];
   outList = {};
   readFromHash = {};
   If[toReverse,
                   readFromHash =  Table[EnhancementToMonodromyFromTable[
                                 Reverse[ enhancementList[[i]]] , False ],  {i, 1, inputListLength}];
                   outList = Table[Reverse[readFromHash[[i]], 2], {i, 1, inputListLength}];
                  ,
                   outList = Table[ EnhancementToMonodromyFromTable[
                                 enhancementList[[i]] , False ], {i, 1, inputListLength}];
   ];
   Return[outList];
  ];

ReadEnhancementMonodromyPairsListFromHash[quiver_] :=
  Module[{i, typesList,  typesListLength, startPosition, maxStartSubquiverAndToReverse,
    toReverse, startSubquiver, listOfEnhancementMonodromyListPairs,
    listOfMonOptionsLists, outList, quiverForLookup, isDType},

   listOfMonOptionsLists = {};
   outList = {};
   typesList = {};
   typesListLength = 0;
   maxStartSubquiverAndToReverse = {};
   toReverse = False;
   startSubquiver = {};
   If[Depth[quiver] > 2, isDType = True;, isDType = False;];

   typesList = ListPossibleTypesOnQuiver[quiver]
         (* This is a simple lookup after types already hashed. *);
   typesListLength = Length[typesList];
   maxStartSubquiverAndToReverse = MaxLengthStoredSubquiver[quiver];
   toReverse = maxStartSubquiverAndToReverse[[2]];
   startSubquiver = maxStartSubquiverAndToReverse[[1]] ;
   If[toReverse,
      quiverForLookup = Reverse[quiver];
    ,
      quiverForLookup = quiver;
   ];
   If[ReadStoredQuiversTable[quiverForLookup] == 1,
     listOfMonOptionsLists = ReadSharedMonodromyTable[typesList, toReverse];
     outList = Table[ {typesList[[i]], listOfMonOptionsLists[[i]]}, {i, 1, typesListLength}];
    If[! isDType,
      Return[outList];
     ,
      Return[ Select[outList, ! TrueQ[#[[2]] == {{-2}}] &]];
    ];
    ,
     Return[{{-3}, {-3}}](* Return this as a null value*);
   ];
  ];
```

```
WriteSharedMonodromyTable[quiver_, hashQueue_, isDType_: False] :=
  Module[{i, listToAddLength, singleQuiverToCheck, quiversToAdd, position, enhancement, startWritingTime,
    hashQueueBeforeFlattening, flattenedHashQueue, positionsToKeep, singleQuiverToAdd},
   (*If it's a dType quiver, we add it directly and we skip the preamble.*)

   (* function to list *)
   quiversToAdd = {};
   startWritingTime = AbsoluteTime[];
   position = 1;
   positionsToKeep = {};
   singleQuiverToAdd = {-4} (* Null value. *);
   If[(! (TrueQ[quiver == Null])) && ReadStoredQuiversTable[quiver] == 1, Return[quiver];];
   If[! isDType,
    hashQueueBeforeFlattening = hashQueue;
    If[Depth[hashQueue] > 6 (* The depth of a list of enh monodromy pairs is 6.
                            If a list of lists of these, then it's depth 7. *),
      For[i = 1, i ≤ Length[hashQueueBeforeFlattening], i ++,
        singleQuiverToCheck = Flatten[Take[hashQueueBeforeFlattening[[i, 1, 1]], All, 1]]
                        (* read the quiver from the first enhancement *);
          If[ReadStoredQuiversTable[singleQuiverToCheck] == 1,
             Continue[];
             ,

       If[minLengthToBeHashedEnhToMon ≤ Length[singleQuiverToCheck] ≤ maxLengthToBeHashedEnhToMon,
               AppendTo[positionsToKeep, i];
               AppendTo[quiversToAdd, singleQuiverToCheck];
               ];
          ];
      ];
      If[Length[positionsToKeep] == 0, Return[]; ](* In this case everything is already added. *);
      flattenedHashQueue = Flatten[Reap[
                     For[i = 1, i ≤ Length[positionsToKeep], i ++,
                             Sow[hashQueueBeforeFlattening[[positionsToKeep[[i]]]]];
                       ];
         ][[2]], 1](* Since we flattened twice,
     this is a list of enhancement to monodromy options pairs. *) ;
        flattenedHashQueue = Flatten[flattenedHashQueue, 1];
      ,
      flattenedHashQueue = hashQueueBeforeFlattening (* In this case it was a simple list already. *);
      singleQuiverToAdd = Flatten[Take[hashQueueBeforeFlattening[[1, 1]], All, 1]]
             (* read the quiver from the first enhancement *);
      If[ReadStoredQuiversTable[singleQuiverToAdd] == 1, Return[];];
    ](* This allows the input to be a list of cases to add or a list of lists. *);
    ,
    (* If D-type, then the add the list directly and we know the quiver: *)
    singleQuiverToAdd = quiver;
    flattenedHashQueue = hashQueue;
   ](* The usage for non-DTypes is more involved.
      For DTypes we have a simple list to add. *);
   While[readHashInProgressEnhToMon == True,
      Pause[hashingWritePauseLengthEnhToMon];
   ];
   writeHashInProgressEnhToMon = True;
   (* write the hash table values: *)
   For[i = 1, i ≤ Length[flattenedHashQueue], i ++,
       enhToMonodromyOptionsTable[flattenedHashQueue[[i, 1]]] = flattenedHashQueue[[i, 2]];
   ];

   If[(! (TrueQ[singleQuiverToAdd == {-4}])),
      WriteListToStoredQuiversTable[{singleQuiverToAdd}];
      numberOfQuiversStored ++;
    ,
      WriteListToStoredQuiversTable[quiversToAdd];
   ];
   writeHashInProgressEnhToMon = False(* let the other kernels write and read now *);
   Return[{AbsoluteTime[] - startWritingTime}]
    (* not used; here to be sure the method runs to the end line.*) ;
  ];
```

```
TossNegAPostResiduals[enhancementWithResiduals_, monodromyOptionsList_] :=
  Module[{i, aPosterioriResiduals, outlist},
    outlist = Flatten[Reap[
        For[i = 1, i ≤ Length[monodromyOptionsList], i++ ,
            aPosterioriResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
                                    enhancementWithResiduals, monodromyOptionsList[[i]]
                                                                            ];
            If[TrueQ[Flatten[{aPosterioriResiduals}] == {-1}], Continue[];];
            If[ ResidualsArePositive[ aPosterioriResiduals],
                Sow[monodromyOptionsList[[i]] ];
            ];
          ];
      ][[2]], 1];
   If[Length[outlist] == 0, Return[{{-1}}]; ];
   Return[outlist];
  ];

IsNotNullMonodromyAssignment[monodromyAssignment_] := Module[{flatList},
         flatList = Flatten[{monodromyAssignment}];
         If[Length[Select[flatList, (# < 0) &]] > 0,
                Return[False]; ,
                Return[True];
          ];
  ];

(* The following method assigns all possible monodromies to the first curve,
then proceeds down the line attempting to assign compatible monodromies with the
 choices up to that point.  Previously hashed results are used to help this along. *)

ComputeListOfMonodromyOptionsForGivenEnhancement[quiver_, enhancement_, startPosition_,
   optionsOnPartialQuiver_] := Module[{ monodromyOptionsList, monodromyOptionsWithPosResiduals,
    length, listOfPartialMonodromyAssignments, nextListOfPartialMonodromyAssignments,
    partialMonodromyAssignmentToQuiver, partialMonodromyAssignmentToQuiverToAdd, i, j, k, r,
    monodromyAssignmentOnCurveToAdd,  atLeastOneMonodromyOptionSoFar, residualsArePositive,
    aPosterioriResiduals, readFromHash, enhancementWithResiduals, partialEnhancementWithResiduals,
    partialEnhancement, partialQuiver, partialQuiverList, subquiversToWrite,
    listBeforeElimination, monodromyOptionsForNextCurve, allowedContinuationsOfAssignment,
    nextPosition, maxNextSubquiver, growLength, toTryHashingThisList, null},

   (* If it's in the look up table, use that result. *)
   null = {{-1}}(* We'll write this to the table if it's a banned assignment *);
   atLeastOneMonodromyOptionSoFar = True;
   monodromyOptionsList = {};
   monodromyOptionsWithPosResiduals = {};
   listOfPartialMonodromyAssignments = {};
   nextListOfPartialMonodromyAssignments = {};
   partialEnhancementWithResiduals = {};
   monodromyOptionsForNextCurve = {};
   listBeforeElimination = {};
   aPosterioriResiduals = {};
   partialEnhancement = {};
   partialQuiver = {};
   allowedContinuationsOfAssignment = {};
   maxNextSubquiver = {};
   growLength = 0;
   length = Length[quiver];

   If[TrueQ[optionsOnPartialQuiver == -1]
      || TrueQ[Flatten[optionsOnPartialQuiver] == {-1}],
     Return[{{-1}}];
   ];
   enhancementWithResiduals = ComputeResidualVanishings[enhancement];
   If[! ResidualsArePositive[enhancementWithResiduals], Return[{{-1}}];];
   listOfPartialMonodromyAssignments = optionsOnPartialQuiver;


   For[i = startPosition + 1, i ≤ length , i++ ,
         atLeastOneMonodromyOptionSoFar = False;
         partialQuiver = Take[quiver, i];
         partialEnhancement = Take[enhancement, i];
         partialEnhancementWithResiduals = Take[enhancementWithResiduals, i];
         growLength = 1;
         monodromyOptionsForNextCurve = Table[{MonodromyTypes[enhancement[[i]] ][[j]]},
                                              {j, 1, Length[MonodromyTypes[enhancement[[i]] ]]}];
         If[TrueQ[listOfPartialMonodromyAssignments == -1], Return[{{-1}}];];
         listOfPartialMonodromyAssignments =  Select[listOfPartialMonodromyAssignments,
                                                     IsNotNullMonodromyAssignment];
         If[Length[listOfPartialMonodromyAssignments] == 0, Return[{{-1}}];];
         nextListOfPartialMonodromyAssignments =
```

```
            Flatten[Reap[
              For[k = 1, k ≤ Length[listOfPartialMonodromyAssignments], k ++,
                    partialMonodromyAssignmentToQuiver = listOfPartialMonodromyAssignments[[k]];
                    If[TrueQ[partialMonodromyAssignmentToQuiver  == -1] ||
                                TrueQ[Flatten[partialMonodromyAssignmentToQuiver] == {-1}], Continue[];];
                    listBeforeElimination = Map[Join[ partialMonodromyAssignmentToQuiver, #] &,
                                                      monodromyOptionsForNextCurve, {1}] ;
                    allowedContinuationsOfAssignment =
                      Flatten[Reap[
                         For[j = 1, j ≤ Length[listBeforeElimination] , j ++,
                              If[IsValidMonodromyAssignmentToSubquiver[partialEnhancementWithResiduals,
                                                                        listBeforeElimination[[j]],
                                                                                  {i - 1, i - 1 + growLength}
                                                                        ],
                                   atLeastOneMonodromyOptionSoFar = True;
                                   Sow[listBeforeElimination[[j]]];
                              ];
                          ];
                       ][[2]], 1];
                    If[(! (TrueQ[Flatten[{allowedContinuationsOfAssignment}] == {-1}]))
                          && TrueQ[Length[Flatten[{allowedContinuationsOfAssignment}]] ≠ 0],
                            Sow[allowedContinuationsOfAssignment];
                      ];
                 ];
             ][[2]], 1];
           If[Depth[nextListOfPartialMonodromyAssignments] > 3,
            nextListOfPartialMonodromyAssignments = Apply[Join, nextListOfPartialMonodromyAssignments];
           ];
           If[! atLeastOneMonodromyOptionSoFar || Length[nextListOfPartialMonodromyAssignments] == 0,
                  Return[null];
            ];

          listOfPartialMonodromyAssignments =
                TossNegAPostResiduals[
                  partialEnhancementWithResiduals, nextListOfPartialMonodromyAssignments];
          If[TrueQ[{-1} == Flatten[{listOfPartialMonodromyAssignments}]], Return[null];];
          nextListOfPartialMonodromyAssignments = {};
     ](* Move to next position until finished moving along the quiver. *);
   If[Length[listOfPartialMonodromyAssignments] == 0, Return[null];];
   Return[listOfPartialMonodromyAssignments];
  ];

(***********************************
      GAUGE ALGEBRAS METHODS.
   ***********************************)

(* ClearMemory:   Clears the cache. *)


ClearMemory[] := Module[{}, Unprotect[In, Out];
   Clear[In, Out];
   Protect[In, Out];
   ClearSystemCache[];];

(* TimeUsedForQuiver[] Quiver gauge options timing benchmark method. *)


TimeUsedForQuiver[] := Module[{timeSinceLastCall},
   timeSinceLastCall = AbsoluteTime[] - lastTime;
   lastTime = AbsoluteTime[];
   Return[timeSinceLastCall];
  ];

(* TimeSinceStart[] Total timing benchmark. *)


TimeSinceStart[] := Module[{timeSinceLastCall},
   timeSinceLastCall = AbsoluteTime[] - launchTime;
   Return[timeSinceLastCall];
  ];

(* IsNonemptyAlgebra:   *)


IsNonemptyAlgebra[algebrasOption_] := Module[{},
         Return[! TrueQ[{-1} == Flatten[{algebrasOption}] ]];
  ];
```

```
DeleteDuplicatesQuiverSymm[quiver_, allAlgebras_] := Module[{},
   If[TrueQ[quiver == Reverse[quiver]],
      Return[ DeleteDuplicates[allAlgebras, (TrueQ[Reverse[#1] == #2] || TrueQ[#1 == #2]) &]];
   ];
   Return[DeleteDuplicates[allAlgebras]];
  ];

GaugeAlgebrasOnQuiver[quiver_] :=
  Module[{typesList, typesAssignment, typesOnly, enhancementWithResiduals, monodromyList, monodromy,
    algebrasOption, allAlgebras, i , j, k, listWithSymmetricAssignmnentsRemoved, symmTrim,
    outputAlgebraList, nonemptyAlgebras, algebrasOptions, typesOnlyList, listOfMonodromyLists,
    listLength, listOfEnhancementMonodromyListPairs, positionsQueue, tableQueue,
    maxStartSubquiverAndToReverse , toReverse, startSubquiver, startPosition, startTypesList,
    startOptionsList, typesListLength, writingTimeOrBadQuiver, hashQueue},

   startOptionsList = {};
   listOfMonodromyLists = {};
   hashQueue = {};

   typesList = ListPossibleTypesOnQuiver[quiver];
   If[TrueQ[typesList == {-1}], Print["error ", quiver, " has null type assignment"]; Return[{-1}];];
   (* Print["starting monodromy for", quiver,  "kernel",$KernelID, "total time",TimeSinceStart[]];
   Add this line during timing tests. *)
   typesListLength = Length[typesList];
   startPosition = 1;

   If[numberOfQuiversStored ≥ maxNumberOfQuiversToHash,
      toHash = False;
      Print["toHash=False " , numberOfQuiversStored, " quivers stored"];
   ];

   maxStartSubquiverAndToReverse = MaxLengthStoredSubquiver[quiver];
   toReverse = maxStartSubquiverAndToReverse [[2]];
   startSubquiver = maxStartSubquiverAndToReverse [[1]] ;
   startPosition = Length[startSubquiver];

   If[ReadStoredQuiversTable[quiver] == 1,
      listOfMonodromyLists = ReadSharedMonodromyTable[typesList, toReverse];
      typesOnlyList = Map[EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver, typesList, {1}];
      listOfEnhancementMonodromyListPairs =
                  Flatten[ Table[{typesOnlyList[[i]], listOfMonodromyLists[[i, j]]},
                                              {i, 1, Length[typesOnlyList]} ,
                                              {j, 1, Length[ listOfMonodromyLists[[i]] ]}
                        ], 1];
      listOfEnhancementMonodromyListPairs = Select[listOfEnhancementMonodromyListPairs,
                                               (ToString[#[[2]]] ≠ ToString[{-1}]) & ];
      algebrasOptions = Map[Apply[GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy, #] &,
                                               listOfEnhancementMonodromyListPairs ];

      algebrasOptions = Select[algebrasOptions, IsNonemptyAlgebra];
      algebrasOptions = DeleteDuplicatesQuiverSymm[quiver, algebrasOptions];
      Print[{quiver, "#algs:", Length[algebrasOptions],
          "time", TimeUsedForQuiver[], "kernel",
      $KernelID, "total time", TimeSinceStart[], "from hashed values"}] ;
      Return[algebrasOptions];
   ];


   allAlgebras = {};
   typesOnly  = {};
   algebrasOption = {};
   enhancementWithResiduals = {};
   startTypesList = Table[  Take[typesList[[i]], startPosition], {i, 1, Length[typesList]}] ;
   startOptionsList = ReadSharedMonodromyTable[startTypesList, toReverse];

   allAlgebras = Flatten[Reap [
       For[i = 1, i ≤ Length[typesList], i ++,
           typesAssignment = typesList[[i]];
           typesOnly  = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[typesAssignment];
           monodromyList  = ComputeListOfMonodromyOptionsForGivenEnhancement[quiver,
                                    typesAssignment, startPosition, startOptionsList[[i]]];
           tableQueue[typesList[[i]]] = monodromyList;
           If[Length[Flatten[{monodromyList}]] == 0, Continue[]; ];
           If[TrueQ[{-1} == Flatten[{monodromyList}]] , Continue[];];
           For[ j = 1, j ≤ Length[monodromyList], j ++,
              monodromy = monodromyList[[j]];

          algebrasOption = GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[typesOnly, monodromy ];
              If[! TrueQ[{-1} == Flatten[{algebrasOption}]],
```

```
                Sow[ algebrasOption];
               ];
            ];
        ]; ][[2]], 1];

  (*positionsQueue[Length[quiver]] = 1; *)
  If[quiver == Reverse[quiver],
     allAlgebras = DeleteDuplicates[allAlgebras, (Reverse[#1] == #2 || #1 == #2) &];
     ,
     allAlgebras = DeleteDuplicates[allAlgebras];
  ];
  If[toHash && ( Length[quiver] ≤ maxLengthToBeHashedEnhToMon),
       hashQueue = Table[{typesList[[i]], tableQueue[typesList[[i]]]}, {i, 1, typesListLength}];
       writingTimeOrBadQuiver = WriteSharedMonodromyTable[quiver, hashQueue];
       If[Length[writingTimeOrBadQuiver] > 1,
    Print["quiver", writingTimeOrBadQuiver, " failed monodromy table read"];];
  ];
  allAlgebras = Sort[allAlgebras];
  Print[{quiver, "#algs:", Length[allAlgebras], "time", TimeUsedForQuiver[], "kernel", $KernelID,
    "total time", TimeSinceStart[], "started from cache to position", startPosition}] ;
  Return[allAlgebras];
 ];

(* JoinCompatibleEnhancementsOnOverlap:   This joins them on a single curve overlap. *)


JoinCompatableEnhancementsOnOverlap[left_, right_] := Module[{},
    If[Length[left] == 1,
     Return[right];
    ];
    If[Length[right] == 1,
     Return[left];
    ];
    Return[  Join[left,  Drop[right, 1]
               ]
         ];
 ];

(* JoinCompatableMonOptionsListsOnSingleOverlapFixedEnh:   *)


(*   Sews together a list of options on a LHS quiver to another list
 of options for RHS quiver where we will identify the single joining curve.  *)

JoinCompatableMonOptionsListsOnSingleOverlapFixedEnh[leftOptions_, rightOptions_] :=
  Module[{i, j, k, leftToMatch, rightToMatch,
    sewedOptionsForFixedOverlappingMonodromy, trimmedRight, aSewedOption, allSewedOptions},

   If[  Length[Flatten[{leftOptions}]] == 0
      || Length[Flatten[rightOptions]] == 0
      || TrueQ[Flatten[{leftOptions}] == {-1}]
      || TrueQ[Flatten[{rightOptions}] == {-1}],
    Return[{{-1}}];
   ];
   allSewedOptions = Flatten[Reap[
       For[i = 0, i ≤ 2 (* Mon options are 0,1,2 *), i ++,
               leftToMatch = Select[leftOptions, (Last[#] == i) &];
               rightToMatch = Select[rightOptions, (First[#] == i) &];
               If[Length[leftToMatch] == 0 || Length[rightToMatch] == 0, Continue[];];
         (* We forget all such options if no sewing possible for this option on the sew site.  *);
               trimmedRight = Drop[rightToMatch, 0, 1](* Drop one at level 3. *);
               sewedOptionsForFixedOverlappingMonodromy =
                     Flatten[Reap[
                        For[j = 1, j ≤ Length[leftToMatch], j ++,
                            For[k = 1, k ≤ Length[trimmedRight], k ++,
                              aSewedOption = Join[leftToMatch[[j]], trimmedRight[[k]]];
                              Sow[aSewedOption];
                         ];
                       ];
                            ][[2]], 1];
               Sow[sewedOptionsForFixedOverlappingMonodromy];
        ];
     ][[2]], 1](* Flatten to level two to forget the grouping by monodromy.*);
   allSewedOptions = Flatten[allSewedOptions, 1];
   If[Depth[allSewedOptions] < 3, Return[{{-1}}]; ];
   Return[allSewedOptions];
  ];
```

```
TossNegAPostResidualsNearSew[enhancement_, monodromyOptionsList_, sewSite_] :=
  Module[{i, aPosterioriResiduals, outlist, leftPosition, rightPosition,
    positionsToTakeForTest, trimmedMonodromyOptionsList, trimmedEnhancementWithResiduals},
     outlist = {};
   If[TrueQ[{-1} == Flatten[{monodromyOptionsList}]]
        || (0 == Length[Flatten[{monodromyOptionsList}]]),
      Return[{{-1}}];
   ];
   leftPosition = Min[Max[1, sewSite - 3], Length[enhancement]];
   rightPosition = Min[sewSite + 3, Length[enhancement]];
   positionsToTakeForTest = {leftPosition, rightPosition} (* 5 positions with center sew site. *) ;
   If[leftPosition > rightPosition,
     Print["error: see TossNegAPostResidualsNearSew", leftPosition, rightPosition];
   ];
   trimmedEnhancementWithResiduals = ComputeResidualVanishings[
                                         Take[enhancement, positionsToTakeForTest]];

   outlist = Flatten[Reap[
         For[i = 1, i ≤ Length[monodromyOptionsList], i ++ ,
              If[Length[monodromyOptionsList[[i]]] == 0
                    || (TrueQ[Flatten[{monodromyOptionsList[[i]]}] == {-1} ]), Continue[];];
             trimmedMonodromyOptionsList  = Take[ monodromyOptionsList[[i]], positionsToTakeForTest];
             aPosterioriResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
                                        trimmedEnhancementWithResiduals, trimmedMonodromyOptionsList ];
             If[TrueQ[Flatten[{aPosterioriResiduals}] == {-1}], Continue[];];
             If[ ResidualsArePositive[ aPosterioriResiduals],
                  Sow[monodromyOptionsList[[i]] ];
              ];
           ](* end for loop *);
      ][[2]], 1](* end sow. *);
   If[Length[outlist] == 0, Return[{{-1}}]; ];
   Return[outlist];
  ];

(* ListEnhancementMonodromyPairsOnLinearQuiverUsingSewing
 This method is central to the main workflow. *)

ListEnhancementMonodromyPairsOnLinearQuiverUsingSewing[quiver_] :=
  Module[{i, j, k, r, s, t, w , maxStartSubquiverAndToReverse, toReverse,
    startSubquiver, startPosition, nextPosition, maxNextSubquiverAndToReverse,
    nextSubquiver, lengthOfQuiverToSew, remainderOfQuiverIncludingOverlap, length,
     enhMonPairsLast, enhMonPairsNext, growLength, collectedOptionsOnSewedQuiver, leftPair,
    rightPair, leftEnh, rightEnh, leftMonOptionsList, rightMonOptionsList, sewedEnh,
    sewedMon, sewedListOfMonOptionsForEnh, rightOptionsGroupedByFirstMonValue,
     rightMonOptionsGroupedByFirstCurveOptionNowDelted,
    numRightSewSiteMonOptions, leftOptionsGroupedByMonToSew,
    enhMonPairsToWriteToTableIfShortEnough, quiversToHashTable,
    sewedEnhWithResiduals, monOptionsForPartialEnh, typesOnNext, enhOnNext,
    enhOnFirstCurveOfNext, monOptionsNextEnh, toReturn, previousPosition},

   growLength = 1;
   leftPair = {};
   rightPair = {};
   enhMonPairsNext = {};
   enhMonPairsLast = {};
   quiversToHashTable = {};
   monOptionsForPartialEnh = {};
   monOptionsNextEnh = {};
   enhMonPairsToWriteToTableIfShortEnough = {};
   enhOnNext = {};
   enhOnFirstCurveOfNext = {};

   length = Length[quiver];
   maxStartSubquiverAndToReverse = MaxLengthStoredSubquiver[quiver];
   toReverse = maxStartSubquiverAndToReverse [[2]];
   startSubquiver = maxStartSubquiverAndToReverse [[1]] ;
   startPosition = Length[Flatten[{startSubquiver}]];

   enhMonPairsLast = ReadEnhancementMonodromyPairsListFromHash[startSubquiver];
   enhMonPairsNext = {} (* We will glue these together on the single curve overlaps. One can
                          extend these methods for efficiency to allow by growing sewing on
                           higher multiple overlaps. *);
   If[TrueQ[Flatten[{enhMonPairsLast[[1]]}] == {-3}],
      startPosition = 1;
      toReverse = False;
      enhMonPairsLast = ReadEnhancementMonodromyPairsListFromHash[Take[quiver, 1]];
      Print["read from hash table error; see ListEnhancementMonodromyPairsOnQuiverUsingSewing"];
    (* This blocks any bad reads from the hash table.  *)
     ,
```

```
 If[startPosition == length, Return[enhMonPairsLast];]
   (* If the full answer is hashed, return that.*);
];

toReturn = Catch[

  previousPosition = startPosition;

  enhMonPairsToWriteToTableIfShortEnough = Flatten[Reap[
      For[i = startPosition, i < Length[quiver], i = i + Max[growLength, 1],
        remainderOfQuiverIncludingOverlap = Take[quiver, {i, length} ];
        maxNextSubquiverAndToReverse =
         MaxLengthStoredSubquiver[remainderOfQuiverIncludingOverlap];
        nextSubquiver = maxNextSubquiverAndToReverse[[1]];
        lengthOfQuiverToSew = Length[nextSubquiver];
        growLength = lengthOfQuiverToSew - 1;
        If[( i + growLength ≤ maxLengthToBeHashedEnhToMon),
              AppendTo[quiversToHashTable,
           Take[quiver, Min[length, i + Max[growLength, 1]]]](* We add at least one curve. *);
        ];
        (* If needed since not hashed,
        we get the options on the subquiver with positions {i,i+1} *)
        If[1 + growLength < minLengthToSew(* In this case we glue on a single curve *),
            If[! TrueQ[{i, Min[i + 1, length]} == {1, 1}],
                  AppendTo[quiversToHashTable, Take[quiver, {i, Min[i + 1, length]}]]
                   (* Min is not really needed.  *);
             ];
            growLength = 1;
            nextSubquiver = Take[quiver, {i, Min[i + 1, length]}];
            lengthOfQuiverToSew = 2;
         (* In this case we are going to sew one more curve on at position i+1
                                                with overlap at position i. *);
            typesOnNext = ListPossibleTypesOnQuiver[nextSubquiver];
            enhMonPairsNext  = Flatten[Reap[
                                        For[k = 1, k ≤ Length[typesOnNext], k ++,
                                                enhOnNext = typesOnNext[[k]];
                                                enhOnFirstCurveOfNext = enhOnNext[[1]];
                                                monOptionsNextEnh =

                 ComputeListOfMonodromyOptionsForGivenEnhancement[nextSubquiver,

                  enhOnNext, 1(*start pos.*),

                  Table[{MonodromyTypes[enhOnFirstCurveOfNext ] [[r]]},

                   {r, 1, Length[MonodromyTypes[enhOnFirstCurveOfNext ]  ]}
                                                                                ]
                   (* table of mon options on curve...this let's us avoid

                   asking the main kernel for the information. *)];
                                                Sow[{enhOnNext, monOptionsNextEnh}];
                                           ];
                                  ][[2]], 1];
            Sow[enhMonPairsNext](* We send the newly computed enhancement mon pair
                                on the 2 curve subquiver to the hash table *);
            If[i == 1,
                  If[length == 2,
                       If[minLengthToBeHashedEnhToMon ≤ 2,
                              WriteSharedMonodromyTable[quiver, enhMonPairsNext ];
                          ];
                      Throw[enhMonPairsNext];
                  ](* If the whole quiver has length 2,
          we are done. We set the current pairs as those to return
                       and we line them up to write to the table. *);
                  enhMonPairsLast = enhMonPairsNext;
                  enhMonPairsNext = {};
                  Continue[]
           (* If we are starting out, we don't need to check the glueing is consistent,
                              so we move to the next position. We go
            back to the top of the i loop. *);
             ];
          ,
         (* Otherwise, we can used stored values to move ahead more than one position. *)
           enhMonPairsNext = ReadEnhancementMonodromyPairsListFromHash[nextSubquiver];
        ];
        If[Length[enhMonPairsLast] == 0 || Length[enhMonPairsNext] == 0, Print["error, mon pairs not
            initialized, see ListEnhancementMonodromyPairsOnQuiverUsingSewing"];
         Throw[{}];
        ];
        (* Now we
```

```
sew: Add a subquiver we have hashed monodromy info for by sewing on a single overlap. *)
 collectedOptionsOnSewedQuiver = Flatten[Reap[
          For[j = 1, j ≤ Length[enhMonPairsLast], j ++,
                   leftPair = enhMonPairsLast[[j]];
                   If[Length[leftPair] == 0,
       Print["left length is zero"]; Continue[];](* not needed.*);
                   If[Length[leftPair[[2]] ] == 0, Print["no options here"];
       Continue[];](* not needed.*);
                   leftEnh = leftPair[[1]];
                   leftMonOptionsList = leftPair[[2]];
                   If[Length[leftEnh] < i,
       Print["error in LHS length ListEnhancementMonodromyPairsOnQuiverUsingSewing"];
                    Continue[];
                   ];

                    If[Length[enhMonPairsNext] == 0,

       Print["error in length, see ListEnhancementMonodromyPairsOnQuiverUsingSewing"];
                     Throw[{}];
                   ];
                   For[k = 1, k ≤ Length[enhMonPairsNext], k ++,
                        rightPair = enhMonPairsNext[[k]];
                        If[Length[rightPair == 0],
        Print["right length is zero"]; Continue[];](* not needed.*);
                        If[Length[rightPair[[2]]] == 0, Continue[];](* not needed.*);
                        rightEnh =  rightPair[[1]];
                        rightMonOptionsList = rightPair[[2]];
                        If[! (TrueQ[leftEnh[[i]] == rightEnh[[1]] ])
        (* enhancements must agree at overlap position *) ,
                            Continue[] (* If they don't agree,
         then move to next right side enhancement option *);
                          ,

        sewedEnh = JoinCompatableEnhancementsOnOverlap[leftEnh, rightEnh];
                            If[ ! ResidualsArePositive[
                                   ComputeResidualVanishings[

            Take[sewedEnh, {Max[1, i - 1], Min[i + 1, Length[sewedEnh]]}]]],
                                 Continue[];
                            ];
                            If[   TrueQ[{-1} == Flatten[{ leftMonOptionsList  } ] ]
                               || TrueQ[{-1} == Flatten[{ rightMonOptionsList } ] ]
                               ||   0 == Length[Flatten[    { leftMonOptionsList  } ] ]
                               ||   0 == Length[Flatten[    { rightMonOptionsList  } ] ]

         (* Either is our null means no valid algebra on one of left or right. *) ,
                                   If[minLengthToBeHashedEnhToMon ≤
           i ≤ maxLengthToBeHashedEnhToMon,
                                        Sow[ { sewedEnh, {{-1}}   }];
                                    ]
          (* If short enough to store, we record this subenhancement (as banned). *);
                                   Continue[](* move to next right side option *) ;
                              ]
        (* otherwise, if also algs agree and both non-null lists: *);
                       (*         Print[  leftMonOptionsList,rightMonOptionsList];*)

        sewedListOfMonOptionsForEnh = JoinCompatableMonOptionsListsOnSingleOverlapFixedEnh[

          leftMonOptionsList, rightMonOptionsList];
                            If[(0 == Length[Flatten[{sewedListOfMonOptionsForEnh}]])
                                      &&
          (minLengthToBeHashedEnhToMon ≤ i + growLength ≤ maxLengthToBeHashedEnhToMon),
                              Sow[ { sewedEnh, {{-1}}   }](* If there are
           no compatable optons, record this enhancement as banned. *);
                              ,
                              sewedListOfMonOptionsForEnh  =
          TossNegAPostResidualsNearSew[sewedEnh, sewedListOfMonOptionsForEnh, i];
                               If[(TrueQ[Flatten[{sewedListOfMonOptionsForEnh}] ==
                {-1}] || 0 == Length[Flatten[{sewedListOfMonOptionsForEnh}]])
                                     && (minLengthToBeHashedEnhToMon ≤
             i + growLength ≤ maxLengthToBeHashedEnhToMon),
                                 Sow[ { sewedEnh, {{-1}}   }](* Ban this
           enhancement if there are no options after checking residuals. *);
                                 Continue[];
                                 ,
                                 sewedListOfMonOptionsForEnh =
                                             Select[sewedListOfMonOptionsForEnh,

            IsValidMonodromyAssignmentToQuiverFromGivenPositionNoResiduals[
              sewedEnh, #, i] &];
```

```
                                              If[ (    (0 == Length[Flatten[
                                    {sewedListOfMonOptionsForEnh}]])
                                                 || (TrueQ[{-1} == Flatten[
                                    {sewedListOfMonOptionsForEnh}]])
                                                )
                                               &&
                                (minLengthToBeHashedEnhToMon ≤ i + growLength ≤ maxLengthToBeHashedEnhToMon),
                                                Sow[ { sewedEnh, {{-1}}  }];
                                                Continue[];
                                               ,
                                                Sow[ { sewedEnh,
                                 sewedListOfMonOptionsForEnh }](* These are the ones that pass. *);
                                               ](* Keep only options that pass
                               the valid pair and triplet tests near the sew site. *);
                                           ];
                                        ]
                          (* Now the values we sew are only in the case if the enhancements agree.  *);
                                       ] (* Consider implementing a condition
                         to treat the cases when we have
                                          gone beyond maxLengthToHashEnhToMon
                         to toss the null mon options
                                           and enhancements.
                                        *);
                                  ](* k loop end *);
                              ];][[2]], 1]
                (* j loop end. Collect those that are to be recorded and flatten to a list
                                          of {enh,monOptions} tuples for the sewed subquiver. *);
                         If[Length[collectedOptionsOnSewedQuiver] == 0, Throw[{}]; ];
                         Sow[collectedOptionsOnSewedQuiver]
              (* sew these to write to table if short enough*);
                         enhMonPairsLast = collectedOptionsOnSewedQuiver;
                         enhMonPairsNext = {};
             ](* i loop end.  We've finished sewing quiver together from subquivers
                 Overlapping on curves at position i along the quiver. *);
          ][[2]], 1]
         (* We reap the values and flatten the list to store to enhMonOptionsTable if short enough. *);
      If[Length[collectedOptionsOnSewedQuiver] == 0, Throw[{}];];
       (* Write collected quivers to hash table *)
       WriteSharedMonodromyTable[Null, enhMonPairsToWriteToTableIfShortEnough];
       Throw[collectedOptionsOnSewedQuiver];
      ](* This ends the special case of the Throw and Catch for length 2.  *);
    Return[toReturn](* We return the list of options including banned monodromy assignments.
                         These can
     be quickly transformed into viable algebra options on the quiver.
                              Along the way, we have hashed the information
     for the subquivers we had not previously recorded. *);
   ];

FlattenedNegativeMonElts[enhMonPair_] := Module[{mon, flatListOfNegatives},
    mon = enhMonPair[[2]];
    flatListOfNegatives = Select[Flatten[mon], (# < 0) &];
    Return[flatListOfNegatives];
   ];

ListEnhancementMonodromyPairsOnDTypeQuiverUsingSewing[inputChain_] :=
  Module[{i, j, k, n, r, s, length, branchPositions, branchedQuivers,
     enhMonPairsOnBranchedQuivers, backboneQuiver, numEnhancementsOnBranch,
     newEnhMonPairsIncludingBranchesSoFar, residualsOnBackbone, backboneEnhMonPairs,
     partiallySewnMonOptionsForThisEnh, monOptionToSew, partialEnh, partialEnhMonOptionToSow,
     maxStoredSubquiverAndToReverse, storedSubquiver, storedQuiverLength,
     enhMonPairsStored, pairsListBeforeFirstSelect, pairsListAfterFirstSelect,
     passingMonodromyAssignmentsFirst, numPairs, pairsListBeforeSelect, pairsListAfterSelect,
     enh, mon, passingMonodromyAssignments, aPostResiduals, toSow, enhWithResiduals },
    length = Length[inputChain];

    (* First we check if the whole answer is stored *)
    If[ReadStoredQuiversTable[inputChain] == 1,
     enhMonPairsStored = ReadEnhancementMonodromyPairsListFromHash[inputChain];
     If[(Length[Select[Flatten[{enhMonPairsStored}], (# < 0) &]] > 0),
       Return[enhMonPairsStored](* If the data read is OK,return stored values. *);
     ](* Now we treat the case the data is not stored already. *);
    ];
    branchPositions = Table[If[Depth[inputChain[[i]]] > 1, 1, 0], {i, 1, length}];
    branchedQuivers = Table[If[branchPositions[[i]] == 1, inputChain[[i]], {}], {i, 1, length}];
    enhMonPairsOnBranchedQuivers =
       Table[
      If[branchPositions[[i]] == 1,
        Select[(ListEnhancementMonodromyPairsOnLinearQuiverUsingSewing[branchedQuivers[[i]]]),
       ! TrueQ[#[[2]] == {{-1}}] &]
       ,
```

```
    {}
   ], {i, 1, length}];
backboneQuiver =
 Table[If[branchPositions[[i]] == 1, inputChain[[i, 1]], inputChain[[i]]], {i, 1, length}];
backboneEnhMonPairs = ListEnhancementMonodromyPairsOnLinearQuiverUsingSewing[backboneQuiver];
backboneEnhMonPairs = Select[backboneEnhMonPairs, (Length[FlattenedNegativeMonElts[#]] == 0) &];
(* Now the sewing loop. *)
For[i = 1, i ≤ length, i++,
     If[branchPositions[[i]] == 0,
          Continue[];
      ,
        newEnhMonPairsIncludingBranchesSoFar = Flatten[Reap[
             For[j = 1, j ≤ Length[backboneEnhMonPairs], j++,
                  numEnhancementsOnBranch = Length[enhMonPairsOnBranchedQuivers[[i]]];
                  For[k = 1, k ≤ numEnhancementsOnBranch, k++,

           If[! TrueQ[backboneEnhMonPairs[[j, 1, i]] == enhMonPairsOnBranchedQuivers[[i, k, 1, 1]]],
                      Continue[];
                      ,
                      (* We now know the enhancements agree on the overlap.
                           Next we sew the matching monodromy options. *)
                      partiallySewnMonOptionsForThisEnh = Flatten[Reap[
                            For[r = 1, r ≤ Length[backboneEnhMonPairs[[j, 2]]], r++,
                               For[s = 1, s ≤ Length[enhMonPairsOnBranchedQuivers[[i, k, 2]]], s++,
                                  If[! (backboneEnhMonPairs[[j, 2, r, i]] ==
                                         enhMonPairsOnBranchedQuivers[[i, k, 2, s, 1]]
                                         ),
                                        Continue[];
                                        ,
                                         monOptionToSew =
                                             Table[If[! n == i,
                                                     backboneEnhMonPairs[[j, 2, r, n]]
                                                     ,

                                                     enhMonPairsOnBranchedQuivers[[i, k, 2, s]]
                                                       ]
                                              , {n, 1, length}];
                                         Sow[monOptionToSew];
                                  ];
                               ];
                            ];
                         ][[2]], 1];
                   ];
                    partialEnh = Table[If[branchPositions[[n]] == 0 || n > i,
                                 backboneEnhMonPairs[[j, 1, n]]
                                 ,
                                 enhMonPairsOnBranchedQuivers[[i, k, 1]]
                              ]
                        , {n, 1, length}];
                    partialEnhMonOptionToSow = {partialEnh, partiallySewnMonOptionsForThisEnh};
                    Sow[partialEnhMonOptionToSow];
                   ];
              ];
        ][[2]], 1];
        backboneEnhMonPairs = newEnhMonPairsIncludingBranchesSoFar;
      ];
  ];
(*  Now we do an intersection contributions check: *)
backboneEnhMonPairs = Select[backboneEnhMonPairs, Length[#[[2]]] > 0 &];

pairsListBeforeFirstSelect = backboneEnhMonPairs;
pairsListAfterFirstSelect = Flatten[Reap[
     For[i = 1, i ≤ Length[ pairsListBeforeFirstSelect], i++,
          enh = pairsListBeforeFirstSelect[[i, 1]];
           enhWithResiduals = ComputeResidualVanishings[enh];
           If[Length[Select[Flatten[{enhWithResiduals}], (# < 0) &]] > 0, Continue[];];
          passingMonodromyAssignmentsFirst = Flatten[Reap[
                   For[j = 1, j ≤ Length[ pairsListBeforeFirstSelect[[i, 2]] ], j++,
                        mon = pairsListBeforeFirstSelect[[i, 2, j]] ;
                        If[Length[Select[Flatten[mon], (# < 0) &]] > 0, Continue[];];
                        aPostResiduals = ComputeAPosterioriResidualsConsideringMonodromy[enh, mon];
                        If[Length[Select[Flatten[{aPostResiduals}], (# < 0) &]] == 0,
                            Sow[mon];
                      ];
                    ];
                   ][[2]], 1];
         If[Length[passingMonodromyAssignmentsFirst] > 0,
            Sow[{enh, passingMonodromyAssignmentsFirst}];
         ];
      ];
```

```
    ][[2]], 1];
pairsListBeforeSelect = pairsListAfterFirstSelect
(*  Finally we do quartet check on the junctions. *) ;
numPairs = Length[pairsListBeforeSelect];
pairsListAfterSelect = Flatten[Reap[
     For[i = 1, i ≤ numPairs, i ++,
         enh = pairsListBeforeSelect[[i, 1]];
         passingMonodromyAssignments = Flatten[Reap[
                For[j = 1, j ≤ Length[ pairsListBeforeSelect[[i, 2]] ], j ++,
                    mon = pairsListBeforeSelect[[i, 2, j]] ;
                    toSow = True;
                     For[k = 2, k ≤ (length - 1), k ++,
                            If[branchPositions[[k]] == 1,
                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k - 1]]}, {enh[[k, 2]]}, {enh[[k + 1]]},
                                     mon[[k, 1]], mon[[k - 1]], mon[[k, 2]], mon[[k + 1]]   ],
                                  toSow = False;
                                  Break[];
                             ]
                 (* TODO: Note that here we are using that the branching cannot happen an two
                                    nextdoor neighbor locations. Check
                  this is a fact to be sure!*) ;
                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k - 1]]}, {enh[[k + 1]]}, {enh[[k, 2]]},
                                     mon[[k, 1]], mon[[k - 1]], mon[[k + 1]], mon[[k, 2]]  ],
                                  toSow = False;
                                  Break[];
                             ];
                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k, 2]]}, {enh[[k - 1]]}, {enh[[k + 1]]},
                                     mon[[k, 1]], mon[[k, 2]], mon[[k - 1]], mon[[k + 1]]   ],
                                  toSow = False;
                                  Break[];
                             ];

                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k + 1]]}, {enh[[k - 1]]}, {enh[[k, 2]]},
                                     mon[[k, 1]], mon[[k + 1]], mon[[k - 1]], mon[[k, 2]]  ],
                                  toSow = False;
                                  Break[];
                             ];
                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k, 2]]}, {enh[[k + 1]]}, {enh[[k - 1]]},
                                     mon[[k, 1]], mon[[k, 2]], mon[[k + 1]] , mon[[k - 1]]  ],
                                  toSow = False;
                                  Break[];
                             ];
                              If[! IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                                     {enh[[k, 1]]}, {enh[[k + 1]]}, {enh[[k, 2]]}, {enh[[k - 1]]},
                                     mon[[k, 1]], mon[[k + 1]] , mon[[k, 2]], mon[[k - 1]]  ],
                                  toSow = False;
                                  Break[];
                             ](* That completes the six permutations of the three transverse
                                  curves to the junction curve. *);

                If[! IsValidMonodromyAssignmentToTriplet[{enh[[k - 1]], enh[[k, 1]], enh[[k, 2]]},
                                        {mon[[k - 1]], mon[[k, 1]], mon[[k, 2]]}, True],
                               toSow = False;
                                 Break[];
                             ];

                If[! IsValidMonodromyAssignmentToTriplet[{enh[[k, 2]], enh[[k, 1]], enh[[k - 1]]},
                                        {mon[[k, 2]], mon[[k, 1]], mon[[k - 1]]}, True],
                               toSow = False;
                                  Break[];
                             ];

                If[! IsValidMonodromyAssignmentToTriplet[{enh[[k, 2]], enh[[k, 1]], enh[[k + 1]]},
                                        {mon[[k, 2]], mon[[k, 1]], mon[[k + 1]]}, True],
                               toSow = False;
                                  Break[];
                             ];

                If[! IsValidMonodromyAssignmentToTriplet[{enh[[k + 1]], enh[[k, 1]], enh[[k, 2]]},
                                        {mon[[k + 1]], mon[[k, 1]], mon[[k, 2]]}, True],
                               toSow = False;
                                  Break[];
                             ];
                         ](*This ends the if statement checking if it's a branching point. *);
                      ](*This ends the k loop moving along the quiver backbone. *);
```

```
                    If[toSow, Sow[mon];];
                  ];
                 ][[2]], 1];
           If[Length[passingMonodromyAssignments] > 0,
              Sow[{enh, passingMonodromyAssignments}];
            ,
            Continue[];
           ];
        ];
      ][[2]], 1];
   WriteSharedMonodromyTable[inputChain, pairsListAfterSelect, True]
   (* True tells the table it is a DType. *);
   Return[pairsListAfterSelect];
  ];

(* ListEnhancementMonodromyPairsOnQuiverUsingSewing:
 A wrapper method that combined the linear and D-type methods. *)

ListEnhancementMonodromyPairsOnQuiverUsingSewing[quiver_] := Module[{},
   If[Depth[quiver] ≤ 2, Return[ListEnhancementMonodromyPairsOnLinearQuiverUsingSewing[quiver]];];
   Return[ListEnhancementMonodromyPairsOnDTypeQuiverUsingSewing[quiver]];
  ];

(* GaugeAlgebrasOnQuiverUsingSewing:  This is the main call for the
   gauge algebra calculation when not pushing further to treat global symmetry.
        Note that this method works for D-types also. *)

  (* This is a hopefully more efficient version which uses stored values
 to make computation more effficient by joining subquivers with known sets of
 allowed enhancement and monodromy options rather than computing curve by curve.  *)

GaugeAlgebrasOnQuiverUsingSewing[quiver_] :=
  Module[{i, j, listOfEnhMonOptionsPairs, listOfBareEnhancementMonodromyListPairs,
      algebrasOptions},
      listOfEnhMonOptionsPairs = ListEnhancementMonodromyPairsOnQuiverUsingSewing[quiver];
      If[Length[listOfEnhMonOptionsPairs] == 0, Print[quiver, " has no algebras viable "]; Return[{}];];

   listOfEnhMonOptionsPairs = Select[listOfEnhMonOptionsPairs, (! TrueQ[Flatten[{#[[2]]}] == {-1} ]) &];
      listOfEnhMonOptionsPairs = Select[listOfEnhMonOptionsPairs,
     (! TrueQ[Length[Flatten[{#[[2]]}]] == 0 ]) &](* not really needed*);
      listOfBareEnhancementMonodromyListPairs =
                   Flatten[ Table[{

        EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[ listOfEnhMonOptionsPairs[[i, 1]] ] ,
                             listOfEnhMonOptionsPairs[[i, 2, j]]
                                  },
                                                  {i, 1, Length[listOfEnhMonOptionsPairs]} ,
                                                  {j, 1, Length[listOfEnhMonOptionsPairs[[i, 2]] ]}
                          ], 1];
      listOfBareEnhancementMonodromyListPairs = Select[listOfBareEnhancementMonodromyListPairs,
                                                   (! (Length[FlattenedNegativeMonElts[#]] > 0)) & ]
     (* Toss entries with no valid monondromy assignment. *) ;
      algebrasOptions = Map[Apply[GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy, #] &,
                                                    listOfBareEnhancementMonodromyListPairs, {1}];
      algebrasOptions = Select[algebrasOptions, IsNonemptyAlgebra]
   (* This is not really needed unless we construct

    the enhTypes table at the top of the program

    incorrectly.  This cleans up the output for
                                                                     such a scenario. *);
      algebrasOptions = DeleteDuplicatesQuiverSymm[quiver, algebrasOptions];
      algebrasOptions = Select[algebrasOptions, Length[#] > 0 &];
      quiversFinished ++;
     (*   Print[{quiver, "#algs:",Length[algebrasOptions],
                "time",TimeUsedForQuiver[], "kernel",
       $KernelID, "total time",TimeSinceStart[],"quivers finished", quiversFinished}]


     (* Uncomment this command for printing out times taken to compute gauge algebras. *) ; *)
     algebrasOptions = Sort[algebrasOptions];
     Return[algebrasOptions];
  ];
```

```
TableFormForDTypeGaugeAlgebra[gaugeAlgebra_] :=
  Module[{i, j, branchPositions, maxBranchLength, filledAlgebra, length, trans, tableForm},
   length = Length[gaugeAlgebra];
   branchPositions = Table[If[Depth[gaugeAlgebra[[i]]] > 2, 1, 0], {i, 1, Length[gaugeAlgebra]}];
   maxBranchLength = Max[ Table[Length[gaugeAlgebra[[i]]], {i, 1, length}]];
   filledAlgebra = Table[PadRight[
                         If[branchPositions[[i]] == 1,
                             gaugeAlgebra[[i]]
                              ,
                              {gaugeAlgebra[[i]]}
                           ]
                        , maxBranchLength, ""], {i, 1, length}];
   trans = Transpose[filledAlgebra];
   tableForm = TableForm[trans];
   Return[tableForm];
  ];

TableFormForATypeGaugeAlgebra[gaugeAlgebra_] := Module[{tableForm},
   tableForm = TableForm[gaugeAlgebra];
   Return[tableForm];
  ];

TableFormFromGaugeAlgebra[gaugeAlgebra_] := Module[{},
   If[Depth[gaugeAlgebra] > 3,
        Return[TableFormForDTypeGaugeAlgebra[gaugeAlgebra]];
   ];
   Return[TableFormForATypeGaugeAlgebra[gaugeAlgebra]];
  ];

GaugeAlgebrasFromQuiverInTableFormat[quiver_, toPrint_: False] := Module[{tableFormAlgebras},
   If[toPrint,
    If[Depth[quiver] ≤ 2,
       tableFormAlgebras = TableFormForATypeGaugeAlgebra /@ GaugeAlgebrasOnQuiverUsingSewing[quiver];
        Print[{quiver, tableFormAlgebras}];
       Return[tableFormAlgebras];
    ];
    tableFormAlgebras = TableFormForDTypeGaugeAlgebra /@ GaugeAlgebrasOnQuiverUsingSewing[quiver];
    Print[{TableFormatQuiver@quiver, tableFormAlgebras}];
    Return[tableFormAlgebras];
   ];
   If[Depth[quiver] ≤ 2,
      Return[TableFormForATypeGaugeAlgebra /@ GaugeAlgebrasOnQuiverUsingSewing[quiver]];
   ];
   Return[TableFormForDTypeGaugeAlgebra /@ GaugeAlgebrasOnQuiverUsingSewing[quiver]];
  ];

TableFormatDTypeQuiver[quiver_] :=
  Module[{i, j, branchPositions, maxBranchLength, filledQuiver, length, trans},
   length = Length[quiver];
   branchPositions = Table[If[Depth[quiver[[i]]] > 1, 1, 0], {i, 1, Length[quiver]}];
   maxBranchLength = Max[ Table[Length[quiver[[i]]], {i, 1, length}]];
   filledQuiver = Table[PadRight[
                         If[branchPositions[[i]] == 1,
                             quiver[[i]]
                              ,
                              {quiver[[i]]}
                           ]
                        , maxBranchLength, ""], {i, 1, length}];
   trans = Transpose[filledQuiver];
   Return[TableForm[trans]];
  ];

TableFormatQuiver[quiver_] := Module[{},
   If[Depth[quiver] > 2,
       Return[TableFormatDTypeQuiver[quiver]];
   ];
    Return[TableForm[{quiver}]];
  ];
```

```
TableFormatGaugeAlgebrasOnQuiverList[quiverList_, toPrintProgress_: False] :=
  Module[{i, quiver, quiverAlgebras, listForAllQuivers},
   listForAllQuivers = Flatten[Reap[
       For[i = 1, i ≤ Length[quiverList], i++,
         quiver = quiverList[[i]];
         quiverAlgebras = GaugeAlgebrasFromQuiverInTableFormat[quiver];
         Sow[{TableFormatQuiver[quiver], quiverAlgebras}]
         (* TODO REPLACE QUIVER WITH FORMATTED QUIVER *) ;
         If[toPrintProgress,
          Print[TableFormatQuiver[quiver]];
          Print[quiverAlgebras];
         ];
        ];
      ][[2]], 1];
   Return[listForAllQuivers];
  ];

ConvertDTypeBMQuiverFormatToAtomicFormat[quiver_] :=
  Module[{i, j, length, branchPositions, atomicFormattedQuiver},
   length = Length[quiver];
   branchPositions = Table[If[Depth[quiver[[i]]] > 1, 1, 0], {i, 1, Length[quiver]}];
   atomicFormattedQuiver = Table[If[branchPositions[[i]] == 0,
                                    quiver[[i]]
                                  ,
                                    {quiver[[i, 1]],
                                        If[Length[ Drop[quiver[[i]], 1]] > 1,
                                           Drop[quiver[[i]], 1]
                                            ,
                                           Drop[quiver[[i]], 1][[1]]
                                          ]
                                    }
                                  ]
                                 , {i, 1, length}
                                ];
   Return[atomicFormattedQuiver];
  ];

ConvertAtomicAlgebraToBMFormat[gaugeAlgebra_] := Module[{i, length, branchPositions, editedAlgebra},
   length = Length[gaugeAlgebra];
   branchPositions = Table[If[Depth[gaugeAlgebra[[i]]] > 2, 1, 0], {i, 1, Length[gaugeAlgebra]}];
   editedAlgebra = Table[ If[ branchPositions[[i]] == 0,
                            gaugeAlgebra[[i]]
                             ,
                             Flatten[gaugeAlgebra[[i]], 1]
                            ]
                          , {i, 1, length}
                         ];
   Return[editedAlgebra];
  ];

ConvertSu2ToSp1OnSummand[summand_] := Module[{newSummand},
   newSummand = If[ToString[summand] == ToString[su[2]], sp[1], summand];
   Return[newSummand];
  ];

ConvertSu2ToSp1OnATypeAlgebra[aTypeAlgebra_] := Module[{i},
   If[Length[aTypeAlgebra] == 0, Return[aTypeAlgebra];];
   Return[ ConvertSu2ToSp1OnSummand /@ aTypeAlgebra];
  ];

ConvertSu2ToSp1[algebra_] := Module[{i, branchPositions, table},
   If[Depth[algebra] < 4, Return[ConvertSu2ToSp1OnATypeAlgebra[algebra]]; ];
   branchPositions = Table[If[Depth[algebra[[i]]] > 2, 1, 0], {i, 1, Length[algebra]}];
   table = Table[ If[branchPositions[[i]] == 1,
      ConvertSu2ToSp1OnATypeAlgebra[algebra[[i]]], ConvertSu2ToSp1OnSummand[algebra[[i]]]
                 ],
     {i, 1, Length[algebra]}];
   Return[table];
  ];
```

```
TableFormatGaugeAlgebrasOnQuiverListViaAtomicFromBMFormatQuiverList[
   quiverList_, toPrintProgress_: False] := Module[{i, quiver, quiverAlgebrasInAtomicForm,
    quiverAlgebras, atomicFormOfDTypeQuiver, listForAllQuivers, quiverAlgebrasUnformatted},
   listForAllQuivers = Flatten[Reap[
       For[i = 1, i ≤ Length[quiverList], i++,
         quiver = quiverList[[i]];
         If[toPrintProgress,
          Print[TableFormatQuiver[quiver]];
         ];
         atomicFormOfDTypeQuiver = ConvertDTypeBMQuiverFormatToAtomicFormat[quiver];
         quiverAlgebrasInAtomicForm =
            AssignEnhancements[atomicFormOfDTypeQuiver, leftnodeGauge, rightnodeGauge];
         quiverAlgebrasUnformatted = ConvertAtomicAlgebraToBMFormat /@ quiverAlgebrasInAtomicForm;
         If[! distinguishSu2FromSp1,
          quiverAlgebrasUnformatted = ConvertSu2ToSp1 /@ quiverAlgebrasUnformatted;
         ];
          quiverAlgebrasUnformatted = Sort[DeleteDuplicates@quiverAlgebrasUnformatted];
          quiverAlgebras = TableFormFromGaugeAlgebra /@ quiverAlgebrasUnformatted;
         Sow[{TableFormatQuiver[quiver], quiverAlgebras}]
         (* TODO REPLACE QUIVER WITH FORMATTED QUIVER *) ;
         If[toPrintProgress,
          Print[quiverAlgebras];
         ];
        ];
      ][[2]], 1];
   Return[listForAllQuivers];
  ];

TableFormatAtomicDTypeAlgebraList[algebraList_] := Module[{},
   Return[TableFormFromGaugeAlgebra /@ (ConvertAtomicAlgebraToBMFormat /@ algebraList)];
  ];

(* GaugeAlgebrasAndTypesRealizingThemOnQuiver[quiver_] *)

GaugeAlgebrasAndTypesRealizingThemOnQuiver[quiver_] := Module[{i, length, unpairedAlgebras,
    pairs, unsorted, listOfEnhMonOptionsPairs, listOfBareEnhancementMonodromyListPairs ,
    enhancementAlgebraPairs, rawPairs, enhancementsForFixedAlgebra,
    listOfAlgebraAllEnhRealizingAlgebraPairs},
      unpairedAlgebras = GaugeAlgebrasOnQuiverUsingSewing[quiver];
      length = Length[unpairedAlgebras];

      listOfEnhMonOptionsPairs = ListEnhancementMonodromyPairsOnQuiverUsingSewing[quiver];

   listOfEnhMonOptionsPairs = Select[listOfEnhMonOptionsPairs, (! TrueQ[Flatten[{#[[2]]}] == {-1} ]) &];
      listOfEnhMonOptionsPairs = Select[listOfEnhMonOptionsPairs,
     (! TrueQ[Length[Flatten[{#[[2]]}]] == 0 ]) &](* not really needed*);
      listOfBareEnhancementMonodromyListPairs =
                 Flatten[ Table[{

       EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[ listOfEnhMonOptionsPairs[[i, 1]] ] ,
                             listOfEnhMonOptionsPairs[[i, 2, j]]
                                  },
                                                 {i, 1, Length[listOfEnhMonOptionsPairs]} ,
                                                 {j, 1, Length[listOfEnhMonOptionsPairs[[i, 2]] ]}
                          ], 1];
     listOfBareEnhancementMonodromyListPairs = Select[listOfBareEnhancementMonodromyListPairs,
                                                  (ToString[#[[2]]] ≠ ToString[{-1}]) & ]
     (* not really needed *) ;
     enhancementAlgebraPairs =
    Map[{#[[1]], Apply[GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy, #]} &,
                                                  listOfBareEnhancementMonodromyListPairs ];
    listOfAlgebraAllEnhRealizingAlgebraPairs  = Flatten[Reap[
        For[i = 1, i ≤ length, i++,
           enhancementsForFixedAlgebra =
           Select[enhancementAlgebraPairs, ( TrueQ[ #[[2]] == unpairedAlgebras[[i]]]) &];
           Sow[{unpairedAlgebras[[i]],  Take[enhancementsForFixedAlgebra, All, 1]}];
        ];
      ][[2]], 1];
    Return[listOfAlgebraAllEnhRealizingAlgebraPairs];
  ];

(* GaugeAlgebrasAndTypesOnQuiver:
 Gauge algebras and order of vanishing along each curve are printed for the given quiver. *)
```

```
GaugeAlgebrasAndTypesOnQuiver[quiver_] :=
  Module[{typesList, typesAssignment, enhancementWithResiduals, typesOnly,
    monodromyList, monodromy, algebrasOption, allAlgebrasWithTypesShown, i , j},
   typesList = ListPossibleTypesOnQuiver[quiver];
   allAlgebrasWithTypesShown = {};
   typesOnly  = {};
   algebrasOption = {};
   enhancementWithResiduals = {};

   For[i = 1, i ≤ Length[typesList], i++,
      typesAssignment = typesList[[i]];
      enhancementWithResiduals = ComputeResidualVanishings[typesAssignment];
      typesOnly  = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[typesAssignment];
      monodromyList  =
     ComputeListOfMonodromyOptionsForGivenEnhancement[ quiver, enhancementWithResiduals];

       For[j = 1, j ≤ Length[monodromyList], j++,
          monodromy = monodromyList[[j]];
          algebrasOption = GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[ typesOnly, monodromy ];

         If[! (ToString[algebrasOption[[1]]] == ToString[-1]),
      AppendTo[allAlgebrasWithTypesShown, {algebrasOption, typesOnly} ];];
       ];
   ];
    Return[Sort[DeleteDuplicates[allAlgebrasWithTypesShown]]];
  ];

(*******************************************************
    COMPARISON METHODS FOR OUTPUTS
  ***************************************************)

(* AlgebraListDiff:      Method to compare algebra lists *)


AlgebraListDiff[quiver_, firstList_, secondList_] := Module[{i, diff, reversedEntriesSecondList},
   diff = {};
   reversedEntriesSecondList = {};
   If[Depth[firstList] == 2, Return[{}];];
   If[Depth[secondList] == 2, Return[firstList];];
   If[quiver == Reverse[quiver],
     diff = Complement[firstList, secondList, SameTest → ((#1 == #2 || #1 == Reverse[#2])  &)];
     Return[diff];
   ];
   diff = Complement[firstList, secondList];
   Return[diff];
  ];

(* AssignEnhancementsToListOfQuiversViaBMM:
 Iterate through a list of quivers and assign all possible gauge enhancements *)


AssignEnhancementsToListOfQuiversViaBMM[quiverList_] := Module[{i, quiver,
    enhancementList, quiverEnhancementListPair, listOfQuiversAndAllAllowedEnhancementsOnEach},
    listOfQuiversAndAllAllowedEnhancementsOnEach = {};
   For[i = 1, i ≤ Length[quiverList], i++,
      quiver = quiverList[[i]];
      enhancementList = GaugeAlgebrasOnQuiver[quiver];
      quiverEnhancementListPair = {quiver, enhancementList};
      AppendTo[listOfQuiversAndAllAllowedEnhancementsOnEach, quiverEnhancementListPair];
       Print[{quiver, TimeUsedForQuiver[]}];

   ];
   Return[listOfQuiversAndAllAllowedEnhancementsOnEach];
  ];

(* CompareQuiverListEnhancements:   Quiver list enhancements comparison method *)
```

```
CompareQuiverListEnhancements[quiverList_,
   theBMMListOfEnhancementsOnListOfQuivers_, theAtomicListOfEnhancementsOnListOfQuivers_] := Module[
   {i, quiver, inBMMButNotInAtomic, inAtomicButNotInBMM, tripletOfComparison, returnComparisonList},

   tripletOfComparison = {};
   returnComparisonList = {};
   quiver = {};
   inBMMButNotInAtomic = {};
   inAtomicButNotInBMM = {};


   For[i = 1, i ≤ Length[quiverList], i ++,
     quiver = quiverList[[i]];
     If[(ToString[ theBMMListOfEnhancementsOnListOfQuivers[[i, 1]]] ≠ ToString[quiver] ) ||
      (ToString[theAtomicListOfEnhancementsOnListOfQuivers[[i, 1]]] ≠ ToString[quiver]),
          Print["not a valid comparison; not comparing enhancements on the same quiver list!",
      quiver ];
          Continue[];
       ];
      inBMMButNotInAtomic = AlgebraListDiff[quiver, theBMMListOfEnhancementsOnListOfQuivers[[i, 2]],

       theAtomicListOfEnhancementsOnListOfQuivers[[i, 2]]   ];
      inAtomicButNotInBMM = AlgebraListDiff[quiver, theAtomicListOfEnhancementsOnListOfQuivers[[i, 2]],

       theBMMListOfEnhancementsOnListOfQuivers[[i, 2]]  ];

      tripletOfComparison = {quiver, inBMMButNotInAtomic , inAtomicButNotInBMM };
      AppendTo[returnComparisonList, tripletOfComparison];
   ];

   Return[returnComparisonList];
  ];

(* COMPARISONS FOR OUTPUTS METHODS.   Five
 methods for comparing quiver enhancement outputs to atomic *)

QuiversWhereBMMIsStrictlyTighter[quiverDiffList_] := Module[{i, listWhereStrictlyTighter},
   listWhereStrictlyTighter = {};
   For[i = 1, i ≤ Length[quiverDiffList], i ++,
       If[Length[quiverDiffList[[i, 2]]] == 0 && Length[quiverDiffList[[i, 3]]] > 0 ,
         AppendTo[listWhereStrictlyTighter, quiverDiffList[[i]]];
         ];
   ];

   Return[listWhereStrictlyTighter];
  ];

QuiversWhereAtomicIsStrictlyTighter[quiverDiffList_] := Module[{i, listWhereStrictlyTighter},
   listWhereStrictlyTighter = {};
   For[i = 1, i ≤ Length[quiverDiffList], i ++,
       If[Length[quiverDiffList[[i, 2]]] > 0 && Length[quiverDiffList[[i, 3]]] == 0 ,
         AppendTo[listWhereStrictlyTighter, quiverDiffList[[i]]];
         ];
   ];

   Return[listWhereStrictlyTighter];
  ];

QuiversWithNotComparableGaugeAssignments[quiverDiffList_] := Module[{i, listWhereStrictlyTighter},
   listWhereStrictlyTighter = {};
   For[i = 1, i ≤ Length[quiverDiffList], i ++,
       If[Length[quiverDiffList[[i, 2]]] > 0 && Length[quiverDiffList[[i, 3]]] > 0 ,
         AppendTo[listWhereStrictlyTighter, quiverDiffList[[i]]];
        ];
   ];
   Return[listWhereStrictlyTighter];
  ];

QuiversWithIdenticalGaugeAssignments[quiverDiffList_, ourEnhancementsList_] :=
  Module[{i, listWhereSame},
   listWhereSame = {};
   For[i = 1, i ≤ Length[quiverDiffList], i ++,
       If[Length[quiverDiffList[[i, 2]]] == 0 && Length[quiverDiffList[[i, 3]]] == 0 ,
         AppendTo[listWhereSame, ourEnhancementsList[[i]]];
        ];
   ];
   Return[listWhereSame];
  ];
```

```
QuiversWhereBMMAllowsNoViableAlgebraButAtomicDoes[ourEnhancementsOnListOfQuivers_,
   atomicQuiverListEnhancementsOnListOfQuivers_ ] := Module[{i, listWhereWeHaveNoPossibleAssignment},
    listWhereWeHaveNoPossibleAssignment = {};
   For[i = 1, i ≤ Length[ourEnhancementsOnListOfQuivers], i ++,
       If[Length[ourEnhancementsOnListOfQuivers[[i, 2]]] == 0 &&
        Length[atomicQuiverListEnhancementsOnListOfQuivers[[i, 2]]] > 0 ,
          AppendTo[ listWhereWeHaveNoPossibleAssignment,
         atomicQuiverListEnhancementsOnListOfQuivers[[i]]];
         ];
   ];
   Return[ listWhereWeHaveNoPossibleAssignment];
  ];

(* FundamentalRepDim:  Finds the arguments of so[n], returning n in this example. *)


(* The following method allows an input such as "so[7]" and returns the integer there, eg 7.

 This may of use in studying outputs if paired with SortBy eg
 SortBy[Algs414, Min[FundamentalRepDim[#[[1,1]]]-FundamentalRepDim[#[[1,2]]],

    FundamentalRepDim[#[[1,3]]]-FundamentalRepDim[#[[1,2]]]]& ]
 *)

FundamentalRepDim[algebra_] := Module[{returnInt, algebraString},
   algebraString = ToString[algebra];

   If[(StringTake[algebraString, 2] == "so") || (StringTake[algebraString, 2] == "su")
         || (StringTake[algebraString, 2] == "sp"),
    returnInt = ToExpression[StringTake[algebraString, {4, -2}]];
   ];
   If[(StringTake[algebraString, 1] == "e") || (StringTake[algebraString, 1] == "g")
         || (StringTake[algebraString, 1] == "f") || (StringTake[algebraString, 1] == "n"),
    returnInt = ToExpression[StringTake[algebraString, {3, -2}]];
   ];

   Return[returnInt];
  ];

(*******************************************************
    SIMISIMPLE LIE ALGEBRA METHODS
  ****************************************************)

(* RankOfAlgSummand[summand_] *)


RankOfAlgSummand[summand_] := Module[{},
      Return[summand[[1, 2]] * summand[[2]]];
  ];

(* RankOfProductAlg[alg_] *)


RankOfProductAlg[alg_] := Module[{i, summand, summandRank, rankSoFar},
     rankSoFar = 0;
     For[i = 1, i ≤ Length[alg], i ++,
        summand = alg[[i]];
        summandRank = RankOfAlgSummand[summand];
        rankSoFar = rankSoFar + summandRank;
   ];
   Return[rankSoFar];
  ];

(* AlgebraSimplify[algFactorList_] *)
```

```
AlgebraSimplify[listOfAlgebraFactors_] :=
  Module[{i, j, algFactorList, algsThusFar, algFactor, powerOfGivenAlg, joinedList, renamedList},
   algsThusFar = {};
   algFactorList = listOfAlgebraFactors;
   joinedList = Flatten[Reap[
       For[i = 1, i ≤ Length[algFactorList], i ++,
         algFactor = algFactorList[[i, 1]];
         If[MemberQ[algsThusFar, algFactor],
           Continue[];
          ,
           AppendTo[algsThusFar, algFactor];
          ];
         If[TrueQ[algFactor == {0, 0}], Continue[];];
         powerOfGivenAlg = 0;
         For[j = i, j ≤ Length[algFactorList], j ++,
           If[TrueQ[algFactorList[[j, 1]] == algFactor],
              powerOfGivenAlg = powerOfGivenAlg + algFactorList[[j, 2]];
            ];
          ];
         If[TrueQ[algFactor == {b, 1}]
           || TrueQ[algFactor == {c, 1}]
           || TrueQ[algFactor == {d, 1}],
           algFactor = {a, 1};
          ];
         If[TrueQ[algFactor == {b, 2}],
           algFactor = {c, 2};
          ];
         If[TrueQ[algFactor == {d, 3}],
           algFactor = {a, 3};
          ];
         If[TrueQ[algFactor == {d, 2}],
           algFactor = {a, 1};
           powerOfGivenAlg = 2 * powerOfGivenAlg;
          ];
         Sow[{algFactor, powerOfGivenAlg}];
        ];
      ][[2]], 1];
   If[Length[Flatten[joinedList]] == 0,
    Return[{{{0, 0}, 0}}];
   ];
   algFactorList = joinedList;
   (* Now that we've renamed using the exceptional isomorphisms,
    recount again the same way: *)
   algsThusFar = {};
   joinedList = Flatten[Reap[
       For[i = 1, i ≤ Length[algFactorList], i ++,
         algFactor = algFactorList[[i, 1]];
         If[MemberQ[algsThusFar, algFactor],
           Continue[];
          ,
           AppendTo[algsThusFar, algFactor];
          ];
         If[TrueQ[algFactor == {0, 0}], Continue[]];
         powerOfGivenAlg = 0;
         For[j = i, j ≤ Length[algFactorList], j ++,
           If[TrueQ[algFactorList[[j, 1]] == algFactor],
              powerOfGivenAlg = powerOfGivenAlg + algFactorList[[j, 2]];
            ];
          ];
         Sow[{algFactor, powerOfGivenAlg}];
        ];
      ][[2]], 1];
   Return[SortBy[joinedList, {#[[1, 1]] &, #[[1, 2]] &} ]];
  ];

AlgListSimplify[algList_] := Module[{i, listNoDups},
   listNoDups = DeleteDuplicates[algList];
   listNoDups =
    DeleteDuplicates[Table[AlgebraSimplify[listNoDups[[i]]], {i, 1, Length[listNoDups]}]];
   Return[SortBy[listNoDups, -RankOfProductAlg[#] &]];
  ];

AlgebraSum[alg1_, alg2_] := Module[{sumAlg, simplifiedAlg},
   sumAlg = Join[alg1, alg2];
   simplifiedAlg = AlgebraSimplify[sumAlg];
   Return[simplifiedAlg];
  ];
```

```
CombinedAlgOptionsFromPairOfOptionSets[options1_, options2_] := Module[{},
   Return[AlgListSimplify[Flatten[Outer[AlgebraSum, options1, options2, 1], 1]]];
  ];

AllAlgSumsPossible[listOfAlgOptionsOnEachCurve_] :=
  Module[{i, optionsThusFar, remainingOptions, length, sortedOptions},
   length = Length[listOfAlgOptionsOnEachCurve];
   If[length == 1, Return[listOfAlgOptionsOnEachCurve[[1]]]; ];
   optionsThusFar = listOfAlgOptionsOnEachCurve[[1]];
   For[i = 2, i ≤ Length[listOfAlgOptionsOnEachCurve], i ++,
     optionsThusFar =
      CombinedAlgOptionsFromPairOfOptionSets[optionsThusFar, listOfAlgOptionsOnEachCurve[[i]]];
   ];
   Return[AlgListSimplify[optionsThusFar]];
  ];

SubsequenceAlgQ[productAlgUnsimplified_, possibleSubAlgUnsimplified_] :=
  Module[{i, j, lastIndex, subSummand, prodSummand, productAlg, possibleSubAlg, foundAllIndicesQ},
   productAlg = AlgebraSimplify[productAlgUnsimplified];
   possibleSubAlg = AlgebraSimplify[possibleSubAlgUnsimplified];
   lastIndex  = 0;
   If[TrueQ[
     ToString[Flatten[possibleSubAlgUnsimplified]] == ToString[Flatten[{{0, 0}, 0}]]], Return[True];];
   (* Check the length, then total rank *)
   If[Length[possibleSubAlg] > Length[productAlg], Return[False];];
   If[RankOfProductAlg[possibleSubAlg] > RankOfProductAlg[productAlg],
     Return[False];
   ];
   (* Check for direct subsequence *)
   foundAllIndicesQ = False;
   For[i = 1, i ≤ Length[possibleSubAlg], i ++,
       subSummand = possibleSubAlg[[i]];
       If[TrueQ[ToString[subSummand[[1]]] == ToString[{0, 0}]], Continue[];];
       If[i > Length[productAlg], Return[False];];
       For[j = lastIndex + 1, j ≤ Length[productAlg], j ++,
          (* If found, store index and break; otherwise return False. *)
           prodSummand = productAlg[[j]];
          If[ToString[subSummand[[1, 1]]] == ToString[ prodSummand[[1, 1]]]
               && TrueQ[subSummand[[1, 2]] ≤ prodSummand[[1, 2]] ]
               && TrueQ[subSummand[[2]] ≤ prodSummand[[2]] ]
              ,
              lastIndex = j;
              If[i == Length[possibleSubAlg], foundAllIndicesQ = True;];
              Break[] (* Then this is a summand of the possibly larger algebra,
                        so move to the next summand to see if it's included also. *);
             ,
              If[j ≥ Length[productAlg], Return[False];](* Then we didn't find a summand,
                                                             so it's not a subsequence *);
              lastIndex = j;
              Continue[];
          ];
       ];
   ];
   Return[foundAllIndicesQ];
  ];

AlgEqual[alg1_, alg2_] := Module[{},
   Return[
          TrueQ[SubsequenceAlgQ[alg1, alg2] && SubsequenceAlgQ[alg2, alg1]]
        ];
  ];
```

```
RelMaxAlgsInAlgListIgnoringBranchingRules[algListUnsimplified_] :=
  Module[{i, j, length, isASubsequence, nonSubsequenceAlgs, algList},
   algList = AlgListSimplify[algListUnsimplified];
   length = Length[algList];
   nonSubsequenceAlgs = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
          isASubsequence = False;
           For[j = 1, j ≤ length, j ++,
             If[j == i,
              Continue[];
             ,
              If[SubsequenceAlgQ[algList[[j]], algList[[i]]],
                 isASubsequence = True;
                 Break[];
                ];
              ];
           ];
          If[isASubsequence == False,
              Sow[algList[[i]]];
           ];
        ];
      ][[2]], 1];
   Return[nonSubsequenceAlgs];
  ];

(* Algebra Inclusion Rules for exceptionals begin here. *)

e8MaxAlgs =
  {
   {
    {{ a, 1}, 1 },
    {{ a, 2}, 1 },
    {{ a, 5}, 1 }
   },
   {
    {{a, 1 }, 1 },
    {{e, 7 }, 1 }
   },
   {
    {{a, 2 }, 1 },
    {{e, 6 }, 1 }
   },
   {
    {{a, 3 }, 1 },
    {{d, 5 }, 1 }
   },
   {
    {{a, 4 }, 2 }
   },
   {
    {{a, 1 }, 1 },
    {{a, 7 }, 1 }
   },
   {
    {{ d, 8 }, 1 }
   },
   {
    {{a, 8 }, 1 }
   },
   {
    {{g, 2 }, 1 },
    {{f, 4 }, 1 }
   },
   {
    {{e, 8 }, 1 }
   }
  };
e8MaxAlgs = AlgListSimplify[e8MaxAlgs];
```

```
e7MaxAlgs =
  {
   {
    {{d, 6 }, 1 },
    {{a, 1 }, 1 }
   },
   {
    {{ a, 5 }, 1 },
    {{ a, 2}, 1 }
   },
   {
    {{a, 3 }, 2},
    {{a, 1 }, 1 }
   },
   {
    {{a, 7 }, 1 }
   },
   {
    {{ g, 2 }, 1 },
    {{c, 3 }, 1 }
   },
   {
    {{f, 4 }, 1 },
    {{a, 1 }, 1 }
   },
   {
    {{e, 6 }, 1 }
   },
   {
    {{e, 7 }, 1 }
   }
  };
e7MaxAlgs = AlgListSimplify[e7MaxAlgs];

e6MaxAlgs =
  {
   {
    {{a, 5 }, 1 },
    {{ a, 1}, 1}
   },
   {
    {{ a, 2}, 3}
   },
   {
    {{ g, 2 }, 1 },
    {{ c, 3}, 1 }
   },
   {
    {{g, 2 }, 1 },
    {{ a, 2}, 1}
   },
   {
    {{ f, 4 }, 1 },
    {{ a, 1}, 1 }
   },
   {
    {{c, 4 }, 1 }
   },
   {
    {{d, 5 }, 1 }
   },
   {
    {{e, 6 }, 1 }
   }
  };
e6MaxAlgs = AlgListSimplify[e6MaxAlgs];
```

```
f4MaxAlgs =
  {
   {
    {{a, 1 }, 1 },
    {{ c, 3}, 1}
   },
   {
    {{a, 2 }, 2 }
   },
   {
    {{a, 3 }, 1 },
    {{ a, 1}, 1}
   },
   {
    {{b, 4 }, 1 }
   },
   {
    {{g, 2 }, 1 },
    {{a, 1 }, 1 }
   },
   {
    {{f, 4 }, 1 }
   }
  };
f4MaxAlgs = AlgListSimplify[f4MaxAlgs];


g2MaxAlgs =
  {
   {
    {{ a, 1 }, 2 }
   },
   {
    {{ a, 2 }, 1 }
   },
   {
    {{ g, 2 }, 1 }
   }
  };
g2MaxAlgs = AlgListSimplify[g2MaxAlgs];



DNMaxAlgs[bigN_] := Module[{k, tableSubAlgs, length},
    If[bigN == 2, Return[ {   {{{a, 1}, 2} } }]; ];
    tableSubAlgs = Table[AlgebraSimplify[ { {{d, k}, 1}, {{d, bigN - k}, 1}}], {k, 2, bigN - 2}];
    tableSubAlgs = Join[tableSubAlgs,
        {
          { {{d, bigN  }, 1}},
          { {{d, bigN - 1}, 1}},
          { {{b, bigN - 1}, 1}},
          { {{a, bigN - 1}, 1}}
        }
                     ];
   Return[AlgListSimplify[tableSubAlgs]];
  ];

CNMaxAlgs[bigN_] := Module[{k, tableSubAlgs, length},
    If[bigN == 2, Return[ AlgListSimplify[{  {{{c, 2}, 1} },  {{{a, 1}, 2} } }]]; ];
    tableSubAlgs = Table[AlgebraSimplify[ { {{c, k}, 1}, {{c, bigN - k}, 1}}], {k, 1, bigN - 1}];
   tableSubAlgs = Join[tableSubAlgs,
        {
          { {{c, bigN  }, 1}},
          { {{c, bigN - 1  }, 1}, {{a, 1}, 1}},
          { {{a, bigN - 1}, 1}},
           { {{c, bigN - 1}, 1}}
        }
                     ];
   Return[AlgListSimplify[tableSubAlgs]];
  ];
```

```
BNMaxAlgs[bigN_] := Module[{k, tableSubAlgs, length},
    If[bigN == 2, Return[ {  {{{c, 2}, 1} },  {{{a, 1}, 2} }, {{{a, 2}, 1} } }]; ];
    If[bigN == 3, Return[ {  {{{g, 2}, 1} },
                             {{{b, 3}, 1} },
                             {{{a, 3}, 1} },
                             {{{c, 2}, 1}, {{a, 1}, 1} },
                             {{{a, 1}, 3} },
                             {{{a, 1}, 2} }
                            }
                        ];
      ];
    tableSubAlgs = Table[AlgebraSimplify[ { {{d, k}, 1}, {{b, bigN - k}, 1}}], {k, 2, bigN - 1}];
   tableSubAlgs = Join[tableSubAlgs,
        {
          { {{b, bigN  }, 1}},
          { {{b, bigN - 1}, 1}},
          { {{d, bigN  }, 1}}
        }
                    ];
   Return[AlgListSimplify[tableSubAlgs]];
  ];

ANMaxAlgs[bigN_] := Module[{k, tableSubAlgs, length},
    If[bigN == 1, Return[ {  {{{a, 1}, 1} }, {{{0, 0}, 0} } }]; ] ;
    If[bigN == 2, Return[ {  {{{a, 2}, 1} },  {{{a, 1}, 1} } }]; ];
    If[bigN == 3, Return[ {  {{{a, 3}, 1} },
                              {{{a, 2}, 1} },
                              {{{c, 2}, 1} },
                              {{{a, 1}, 2} }
                            }]; ];
    tableSubAlgs = Table[AlgebraSimplify[ { {{a, k}, 1}, {{a, bigN - 1 - k}, 1}}], {k, 2, bigN - 2}];
    tableSubAlgs = Join[tableSubAlgs,
        {
          { {{a, bigN  }, 1}},
          { {{a, bigN - 1}, 1}}
        }
                    ];
     If[Mod[bigN, 2] == 0,
        AppendTo[tableSubAlgs,  { {{b, bigN / 2  }, 1}}];
     ];
     If[Mod[bigN, 2] == 1,
         AppendTo[tableSubAlgs,  { {{c, Ceiling[bigN / 2]  }, 1}}];
     ];
   Return[AlgListSimplify[tableSubAlgs]];
  ];
```

```
AllMaximalSubalgebrasOfSimple[simpleAlg_] := Module[{rank},
    If[TrueQ[simpleAlg[[1, 1]] == a ],
       rank = simpleAlg[[1, 2]];
        Return[ ANMaxAlgs[rank]];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == b ],
       rank = simpleAlg[[1, 2]];
        Return[ BNMaxAlgs[rank]];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == c ],
       rank = simpleAlg[[1, 2]];
        Return[ CNMaxAlgs[rank]];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == d ],
       rank = simpleAlg[[1, 2]];
        Return[ DNMaxAlgs[rank]];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == g ],
       If[simpleAlg[[1, 2]] == 2,
          Return[ g2MaxAlgs];
      ];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == f ],
       If[simpleAlg[[1, 2]] == 4,
          Return[ f4MaxAlgs];
      ];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == e ],
       If[simpleAlg[[1, 2]] == 8,
         Return[ e8MaxAlgs];
     ];
      If[simpleAlg[[1, 2]] == 7,
         Return[e7MaxAlgs ];
     ];
      If[simpleAlg[[1, 2]] == 6,
         Return[ e6MaxAlgs];
     ];
    ];
    If[TrueQ[simpleAlg[[1, 1]] == 0 ],
       If[simpleAlg[[1, 2]] == 0,
          Return[ {
               {{{0, 0}, 0}}
            }];
      ];
    ];
   ];

AllPossiblyMaximalSubalgebrasOfSemiSimpleAlg[semiSimpleAlg_] := Module[{i, factorList, length, allSums},
    If[TrueQ[{{{0, 0}, 0}} == semiSimpleAlg], Return[{semiSimpleAlg}]; ];
    length = Length[semiSimpleAlg];
    factorList = Flatten[Reap[
         For[i = 1, i ≤ length, i ++,
              Sow[ ConstantArray[
                                  AllMaximalSubalgebrasOfSimple[
                                      {semiSimpleAlg[[i, 1]], 1} ],
                     semiSimpleAlg[[i, 2]]
                                  ]
                  ];
           ];
        ][[2]], 2];
    allSums = AllAlgSumsPossible[factorList];
    Return[allSums];
   ];

(* Now we initialize the algebra cache table. *)

cacheOfAlgebrasToItsMaximalSubAlgebras[] := {{{{0, 0}, 0}}};

cacheOfAlgebrasToAllItsSubAlgebras[] := {{{{0, 0}, 0}}};

cacheOfRelMaxProperSubAlgsOfSemiSimple[] := {{{{0, 0}, 0}}};

indexOfAlgebrasForWhichAllSubsKnown[] := {};

indexOfAlgebrasForWhichAllMaxSubsKnown[] = {};
```

```
ReadAlgebrasFromCache[] := Module[{numStored},
      Get[ algebraCacheFileNameIn];
      numStored = Length[DownValues[cacheOfAlgebrasToItsMaximalSubAlgebras]];
      Print["cached algebras read. Total algebras stored so far:",
                 Length[numStored]];
  ];

(* Now we import the values or initialize from
 scratch. We print the count of stored algebras in either case.  *)

Quiet[
  If[toReadFileWithAlgCache,
      ReadAlgebrasFromCache[];
    ,
    Print["Algebra cache not read from file. To change, see variable 'toReadFileWithAlgCache'."];
   ];
 ];
```

cached algebras read. Total algebras stored so far:0

```
WriteAlgCacheToFile[] := Module[{},
   If[toWriteFileWithAlgCache,
       DumpSave[algebraCacheFileNameOut,
           {cacheOfAlgebrasToItsMaximalSubAlgebras,
                cacheOfAlgebrasToAllItsSubAlgebras,
                  cacheOfRelMaxProperSubAlgsOfSemiSimple,
                     indexOfAlgebrasForWhichAllSubsKnown}
       ];
       Print["algebra cache saved to file: ", algebraCacheFileNameOut ];
       Print["Number algebras with saved subalgebras : ",
               Length[DownValues[ cacheOfAlgebrasToItsMaximalSubAlgebras]]
       ];
   ];
   If[! toWriteFileWithAlgCache,
           Print["Nothing written,
                    see settings for variable 'toWriteFileWithAlgCache' and
                    change setting from 'False.'"
           ];
   ];
   Return["Done."];
  ];

WriteEnhancementHashTablesToFile[] := Module[{},
   If[toWriteFileSavingHashTable,
      DumpSave[enhancementHashFileNameOut,
                 {hashKeyTable, hashTable}
              ];
    Print["Enhancement hash tables saved to file: ", enhancementHashFileNameOut ];
    Print["Number quivers with saved enhancement info : ", Length[hashKeyTable]];
   ];
   If[! toWriteFileSavingHashTable,
           Print["Nothing written,
                   see settings for variable 'toWriteFileSavingHashTable' and
                   change setting from 'False.'"
          ];
   ];
   Return["Done."];
  ];

WriteMonodromyHashTablesToFile[] := Module[{},
   If[toWriteFileWithSavedMonodromyHashTable ,
      DumpSave[monodromyHashFileNameOut,
                 {storedQuiversTable , enhToMonodromyOptionsTable}
              ];
    Print["Monodromy hash tables saved to file: ", monodromyHashFileNameOut];
    Print["Number quivers with saved monodromy info : ", Length[DownValues[ storedQuiversTable]]];
   ];
   If[! toWriteFileWithSavedMonodromyHashTable ,
         Print["Nothing written,
              see settings for variable 'toWriteFileWithSavedMonodromyHashTable' and
              change setting from 'False.'"
         ];
   ];
   Return["Done."];
  ];
```

```
WriteAllEnhancementsMonodromyAndAlgebraTablesToFile[] := Module[{},
   WriteEnhancementHashTablesToFile[];
   WriteMonodromyHashTablesToFile[];
   WriteAlgCacheToFile[];
   Return["Enhancements, monodromy, and algebra tables written to file."];
  ];

(* Algebra cache table functions. *)


ReadAlgToIndexTableForWhichAllSubsAreKnown[alg_] :=
  Module[{toReturn}, If[RankOfProductAlg[alg] > maxAlgRankStoredAllSubAlgs, Return[-1];];
   toReturn = Null;
   While[writeHashInProgressAlgToAllSubsIndex, Pause[hashingReadPauseLengthAlgToAllSubsIndex ];];
   readHashInProgressAlgToAllSubsIndex = True;
   Pause[hashingReadPauseLengthAlgToAllSubsIndex];
   toReturn = indexOfAlgebrasForWhichAllSubsKnown[alg];
   readHashInProgressAlgToAllSubsIndex = False;
   Pause[hashingReadPauseLengthAlgToAllSubsIndex];
   If[(ToString[Head[toReturn]] == "indexOfAlgebrasForWhichAllSubsKnown") ||
     TrueQ[toReturn == Null] || Length[toReturn] == 0, Return[-1];, Return[1];];];

WriteAlgToIndexTableForWhichAllSubsAreKnown[alg_] := Module[{},
   If[RankOfProductAlg[alg] > maxAlgRankStoredAllSubAlgs, Return[];];
   If[(1 == ReadAlgToIndexTableForWhichAllSubsAreKnown[alg]), Return[];](*already stored.*);
   While[readHashInProgressAlgToAllSubsIndex,
     Pause[hashingWritePauseLengthAlgToAllSubsIndex];
   ];
   writeHashInProgressAlgToAllSubsIndex = True;
   indexOfAlgebrasForWhichAllSubsKnown[alg] = 1;
   writeHashInProgressAlgToAllSubsIndex = False;
  ](* CHANGE THIS TO TAKE A QUIVER LIST *) ;

ReadAlgToAllSubalgebrasTable[alg_] := Module[{ toReturn, valueFromTable},
   toReturn  = {};

   While[writeHashInProgressAlgToAllSubs ,
      Pause[hashingReadPauseLengthAlgToAllSubs];
   ];
   readHashInProgressAlgToAllSubs = True;
   valueFromTable  = cacheOfAlgebrasToAllItsSubAlgebras[ alg];
   readHashInProgressAlgToAllSubs = False;
   If[ (TrueQ[(ToString[Head[valueFromTable]] == "cacheOfAlgebrasToAllItsSubAlgebras")]) ||
     (TrueQ[valueFromTable == Null]) ,
    toReturn = Null;
    ,
    toReturn = valueFromTable;
   ];
   Return[toReturn];
  ];

WriteAlgToAllSubalgebrasTable[alg_, subAlgs_] := Module[{ startWritingTime},
   (* function to list *)
   startWritingTime = AbsoluteTime[];
   If[TrueQ[ReadAlgToAllSubalgebrasTable[alg] ≠ Null], Return[];];
   While[readHashInProgressAlgToAllSubs == True,
      Pause[hashingWritePauseLengthAlgToAllSubs];
   ];
   writeHashInProgressAlgToAllSubs = True;
   cacheOfAlgebrasToAllItsSubAlgebras[alg] = subAlgs;
   WriteAlgToIndexTableForWhichAllSubsAreKnown[alg];
   writeHashInProgressAlgToAllSubs = False(* let the other kernels write and read now *);
   Return[{AbsoluteTime[] - startWritingTime}](* not used*) ;
  ];

(* Maximal subalgebra cache table functions. *)


ReadAlgToIndexTableForWhichAllMaxSubsAreKnown[alg_] :=
  Module[{toReturn}, If[RankOfProductAlg[alg] > maxAlgRankStoredAllMaxSubs, Return[-1];];
   toReturn = Null;
   While[writeHashInProgressAlgToAllMaxSubsIndex, Pause[hashingReadPauseLengthAlgToAllMaxSubsIndex ];];
   readHashInProgressAlgToAllMaxSubsIndex = True;
   Pause[hashingReadPauseLengthAlgToAllMaxSubsIndex];
   toReturn = indexOfAlgebrasForWhichAllMaxSubsKnown[alg];
   readHashInProgressAlgToAllMaxSubsIndex = False;
   Pause[hashingReadPauseLengthAlgToAllMaxSubsIndex];
   If[(ToString[Head[toReturn]] == "indexOfAlgebrasForWhichAllMaxSubsKnown") ||
     TrueQ[toReturn == Null] || Length[toReturn] == 0, Return[-1];, Return[1];];];
```

```
WriteAlgToIndexTableForWhichAllMaxSubsAreKnown[alg_] := Module[{},
   If[RankOfProductAlg[alg] > maxAlgRankStoredAllMaxSubs, Return[];];
   If[(1 == ReadAlgToIndexTableForWhichAllMaxSubsAreKnown[alg]), Return[];](*already stored.*);
   While[readHashInProgressAlgToAllMaxSubsIndex,
      Pause[hashingWritePauseLengthAlgToAllMaxSubsIndex];
   ];
   writeHashInProgressAlgToAllMaxSubsIndex = True;
   indexOfAlgebrasForWhichAllMaxSubsKnown[alg] = 1;
   writeHashInProgressAlgToAllMaxSubsIndex = False;
  ](* CHANGE THIS TO TAKE A QUIVER LIST *) ;

ReadAlgToAllMaxSubalgebrasTable[alg_] := Module[{ toReturn, valueFromTable},
   toReturn  = {};

   While[writeHashInProgressAlgToAllMaxSubs ,
      Pause[hashingReadPauseLengthAlgToAllMaxSubs];
   ];
   readHashInProgressAlgToAllMaxSubs = True;
   valueFromTable  = cacheOfRelMaxProperSubAlgsOfSemiSimple[ alg];
   readHashInProgressAlgToAllMaxSubs = False;
   If[ (TrueQ[(ToString[Head[valueFromTable]] == "cacheOfRelMaxProperSubAlgsOfSemiSimple")]) ||
     (TrueQ[valueFromTable == Null]) ,
     toReturn = Null;
    ,
     toReturn  = valueFromTable;
   ];
   Return[toReturn];
  ];

WriteAlgToAllMaxSubalgebrasTable[alg_, subAlgs_] := Module[{ startWritingTime},
   (* function to list *)
   startWritingTime = AbsoluteTime[];
   If[TrueQ[ReadAlgToAllMaxSubalgebrasTable[alg] ≠ Null], Return[];];
   While[readHashInProgressAlgToAllMaxSubs == True,
      Pause[hashingWritePauseLengthAlgToAllMaxSubs];
   ];
   writeHashInProgressAlgToAllMaxSubs = True;
   cacheOfRelMaxProperSubAlgsOfSemiSimple[alg] = subAlgs;
   WriteAlgToIndexTableForWhichAllMaxSubsAreKnown[alg];
   writeHashInProgressAlgToAllMaxSubs = False(* let the other kernels write and read now *);
   Return[{AbsoluteTime[] - startWritingTime}](* not used*) ;
  ];



(* Methods for finding subalgebras *)


AllPossiblyMaximalSubalgebrasOfSemiSimpleAlgWithCaching[alg_] :=
  Module[{i, semiSimpleAlg, factorList, length, allSums, cached, isCached, rank},
   If[TrueQ[{{{0, 0}, 0}} == alg], Return[{alg}]; ];
   semiSimpleAlg = AlgebraSimplify[alg];
   rank = RankOfProductAlg[semiSimpleAlg];
   cached = cacheOfAlgebrasToItsMaximalSubAlgebras[semiSimpleAlg];
   If[(ToString[Head[cached]] == "cacheOfAlgebrasToItsMaximalSubAlgebras")
        || TrueQ[cached == Null]
          || rank > maxAlgRankStoredAllMaxSubs,
       isCached = False;
       ,
       Return[cached];
   ];
   length = Length[semiSimpleAlg];
   factorList = Flatten[Reap[
       For[i = 1, i ≤ length, i++,
           Sow[ ConstantArray[
                             AllMaximalSubalgebrasOfSimple[
                                 {semiSimpleAlg[[i, 1]], 1} ],
                  semiSimpleAlg[[i, 2]]
                             ]
              ];
        ];
      ][[2]], 2];
   allSums = AllAlgSumsPossible[factorList];
   If[isCached == False && (rank ≤  maxAlgRankStoredAllMaxSubs),
    cacheOfAlgebrasToItsMaximalSubAlgebras[semiSimpleAlg] = allSums;];
   Return[allSums];
  ];
```

```
AllSemiSimpleSubalgebrasOfSemiSimple[semiSimpleAlg_] :=
  Module[{i, length, factorList, allSums, lastLength, nextList, nextLength,
         lastListOfSemiSimpleMaxSubAlgs},
   lastListOfSemiSimpleMaxSubAlgs = {semiSimpleAlg};
   lastLength = 1;
   nextLength = 2 (* Anything bigger than 1 *);
   While[lastLength < nextLength,
    lastLength = Length[lastListOfSemiSimpleMaxSubAlgs];
    nextList = Flatten[Reap[
        For[i = 1, i ≤ Length[ lastListOfSemiSimpleMaxSubAlgs], i++,
          Sow[AllPossiblyMaximalSubalgebrasOfSemiSimpleAlg[
                    lastListOfSemiSimpleMaxSubAlgs[[i]]
                                                        ]
             ];
         ];
       ][[2]], 2];
     nextList = AlgListSimplify[nextList];
     nextLength = Length[nextList];
     lastListOfSemiSimpleMaxSubAlgs = nextList;
   ];
   Return[lastListOfSemiSimpleMaxSubAlgs];
  ];

AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[alg_] :=
  Module[{i, length, factorList, allSums, lastLength, nextList, nextLength,
         lastListOfSemiSimpleMaxSubAlgs, semiSimpleAlg, rank, cached, isCached},
   semiSimpleAlg = AlgebraSimplify[alg];
   rank = RankOfProductAlg[alg];
   cached = ReadAlgToIndexTableForWhichAllSubsAreKnown[semiSimpleAlg];
   If[cached == -1 || TrueQ[cached == Null],
       isCached = False;
       ,
       Return[ReadAlgToAllSubalgebrasTable[alg]];
   ];

   lastListOfSemiSimpleMaxSubAlgs = {semiSimpleAlg};
   lastLength = 1;
   nextLength = 2 (* Anything bigger than 1 *) ;
   While[lastLength < nextLength,
    lastLength = Length[lastListOfSemiSimpleMaxSubAlgs];
    nextList = Flatten[Reap[
        For[i = 1, i ≤ Length[ lastListOfSemiSimpleMaxSubAlgs], i++,
          Sow[AllPossiblyMaximalSubalgebrasOfSemiSimpleAlgWithCaching[
                    lastListOfSemiSimpleMaxSubAlgs[[i]]
                                                        ]
             ];
         ];
       ][[2]], 2];
     nextList = AlgListSimplify[nextList];
     nextLength = Length[nextList];
     lastListOfSemiSimpleMaxSubAlgs = nextList;
   ];
   If[isCached == False && (rank ≤ maxAlgRankStoredAllSubAlgs),
    WriteAlgToAllSubalgebrasTable[semiSimpleAlg, lastListOfSemiSimpleMaxSubAlgs];];
   Return[lastListOfSemiSimpleMaxSubAlgs];
  ];

SubsequenceAlgUsingBranchingRulesQ[productAlgUnsimplified_, possibleSubAlgUnsimplified_] :=
  Module[{i, allSubsOfProd, numSubsOfProd, subOfProd, productAlg, possibleSubAlg},
   productAlg = AlgebraSimplify[productAlgUnsimplified];
   possibleSubAlg = AlgebraSimplify[possibleSubAlgUnsimplified];
   (* Next check the total rank, and direct subsequence for efficiency if it gives a conclusion.
                                    Finding all subAlgs is an alternative. *)
   If[SubsequenceAlgQ[productAlg, possibleSubAlg], Return[True];];
   If[RankOfProductAlg[productAlg] < RankOfProductAlg[possibleSubAlg],
     Return[False];
   ];
   (* Find all subAlgs if the above wasn't conclusive: *)
   allSubsOfProd = AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[productAlg];
   numSubsOfProd = Length[allSubsOfProd];
   For[i = 1, i ≤ numSubsOfProd, i++,
     subOfProd = allSubsOfProd[[i]];
     If[SubsequenceAlgQ[subOfProd, possibleSubAlg], Return[True];];
   ];
   Return[False];
  ];
```

```
AlgMemberQ[algList_, alg_] := Module[{},
   If[TrueQ[Total[Flatten[{alg[[1, 1]]}]] == 0], Return[True];];
   Return[MemberQ[algList, alg]];
  ];

SubAlgIgnoringNontrivialInclusionsQ[algUnsimplified_, subAlgUnsimplified_] :=
  Module[{i, allSubsOfAlg, alg, subAlg},
   alg = AlgebraSimplify[algUnsimplified];
   subAlg = AlgebraSimplify[subAlgUnsimplified];
   (* Check the total rank, and direct subsequence for efficiency. *)
   If[SubsequenceAlgQ[alg, subAlg], Return[True];];
   If[RankOfProductAlg[alg] < RankOfProductAlg[subAlg],
     Return[False];
   ];
   Return[False];
  ];

SubAlgQ[algUnsimplified_, subAlgUnsimplified_] := Module[{i, allSubsOfAlg, alg, subAlg},
   alg = AlgebraSimplify[algUnsimplified];
   subAlg = AlgebraSimplify[subAlgUnsimplified];
   (* Check the total rank, and direct subsequence for efficiency. *)
   If[SubsequenceAlgQ[alg, subAlg], Return[True];];
   If[RankOfProductAlg[alg] < RankOfProductAlg[subAlg],
     Return[False];
   ];
   allSubsOfAlg = AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[alg];
   Return[AlgMemberQ[allSubsOfAlg, subAlg]];
  ];

AlgInAlgListIgnoringNontrivialInclusionsMemberQ[algListUnsimplified_, subAlgUnsimplified_] :=
  Module[{i, length, algList, subAlg},
   subAlg = AlgebraSimplify[subAlgUnsimplified];
   algList = AlgListSimplify[algListUnsimplified];
   length = Length[algList];
   If[AlgMemberQ[algList, subAlg], Return[True];];
   For[i = 1, i ≤ length, i ++,
    If[SubAlgIgnoringNontrivialInclusionsQ[algList[[i]], subAlg], Return[True];];
   ];
   Return[False];
  ];

SubAlgMemberQ[algListUnsimplified_, subAlgUnsimplified_] := Module[{i, length, algList, subAlg},
   subAlg = AlgebraSimplify[subAlgUnsimplified];
   algList = AlgListSimplify[algListUnsimplified];
   length = Length[algList];
   If[AlgMemberQ[algList, subAlg], Return[True];];
   For[i = 1, i ≤ length, i ++,
    If[SubsequenceAlgQ[algList[[i]], subAlg], Return[True];];
   ];
   For[i = 1, i ≤ length, i ++,
     If[SubAlgQ[algList[[i]], subAlg], Return[True];];
   ];
   Return[False];
  ];

DeleteAlgFromList[list_, alg_] :=
  Module[{i, simplifiedList, simplifiedAlg, listWithDeletion, length, algOnList},
   simplifiedList = AlgListSimplify[list];
   simplifiedAlg = AlgebraSimplify[alg];
   length = Length[simplifiedList];
   listWithDeletion = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
          algOnList = simplifiedList[[i]];
          If[AlgEqual[simplifiedAlg, algOnList], Continue[];, Sow[algOnList];];
        ];
      ][[2]], 1];
   If[Length[Flatten[listWithDeletion]] == 0, Return[{{ {{0, 0}, 0}}}];];
   Return[AlgListSimplify[listWithDeletion]];
  ];
```

```
RelMaxAlgs[algListUnsimplified_] := Module[{i, j, relMaxes, length, tableAllSubs, algList, alg, otherSubs},
   algList = AlgListSimplify[algListUnsimplified];
   length = Length[algList];
   tableAllSubs = AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching /@ algList;
   relMaxes = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
            alg = algList[[i]];
            otherSubs = Catenate[Delete[tableAllSubs, i]];
            If[SubAlgMemberQ[otherSubs, alg ],
               Continue[];
               ,
               Sow[alg];
            ];
        ];
      ][[2]], 1];
   Return[AlgListSimplify[relMaxes]];
  ];

RelMaxProperSubAlgsOfSemiSimple[semiSimpleAlg_] :=
  Module[{i, alg, allAlgs, length, allAlgsWithoutInputAlg, toReturn},
   alg = AlgebraSimplify[semiSimpleAlg];
   If[TrueQ[alg == { {{a, 1}, 1}}] || TrueQ[alg == { {{0, 0}, 0}}], Return[{{ {0, 0}, 0}}];];
   allAlgs = AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[alg];
   length = Length[allAlgs];
   allAlgsWithoutInputAlg = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
         If[AlgEqual[alg, allAlgs[[i]]],
            Continue[];
           ,
            Sow[allAlgs[[i]]];
          ];
        ];
      ][[2]], 1];
   allAlgsWithoutInputAlg = AlgListSimplify[allAlgsWithoutInputAlg];
   toReturn = DeleteAlgFromList[RelMaxAlgs[allAlgsWithoutInputAlg], alg];
   Return[toReturn];
  ];
```

```
RelMaxProperSubAlgsOfSemiSimpleEfficient[semiSimpleAlg_] :=
  Module[{i, alg, algLength, summand, multiplicity, summandAlgWithMultOne, maxesOfSimpleSummand,
    allAlgs, length, allAlgsWithoutInputAlg, productPossibleMaxesOfSummand, summandMaxes,
    listOfSummandMaxes , possibleMaxes, listWithoutStartingAlg, toSow, toReturn, isCached, cached, rank},
   alg = AlgebraSimplify[semiSimpleAlg];
   If[TrueQ[alg == { {{a, 1}, 1}}] || TrueQ[alg == { {{0, 0}, 0}}], Return[{{ {0, 0}, 0}}];];
   rank = RankOfProductAlg[alg];

   If[rank ≤ maxAlgRankStoredAllMaxProperSubs,
    If[-1 == ReadAlgToIndexTableForWhichAllMaxSubsAreKnown[alg],
      isCached = True
      ,
      isCached = False;
     ];
   ];

   If[isCached,
       Return[ReadAlgToAllMaxSubalgebrasTable@alg];
   ];

   algLength = Length[alg];
   listOfSummandMaxes = Flatten[Reap[
       For[i = 1, i ≤ algLength, i++,
          summand = alg[[i, 1]];
          multiplicity = alg[[i, 2]];
          If[TrueQ[summand == {a, 1}],
             If[multiplicity == 1,
              Sow[{  {{{0, 0}, 0}},  {{{a, 1}, 1}}}];
              ,
              Sow[{{{{a, 1}, multiplicity - 1}}, {{{a, 1}, multiplicity}}}];
          ];
            Continue[];
           ];

          If[multiplicity ≤ 1,
            toSow = AlgListSimplify[Join[RelMaxProperSubAlgsOfSemiSimple[{alg[[i]]}], {{alg[[i]]}}]];
            Sow[toSow];
            Continue[];
           ];
          summandAlgWithMultOne = {summand, 1};
          maxesOfSimpleSummand = AlgListSimplify[Join[
            RelMaxProperSubAlgsOfSemiSimple[{summandAlgWithMultOne}], {{summandAlgWithMultOne}}]] ;
          productPossibleMaxesOfSummand =
               AllAlgSumsPossible[Join[{{{{summand, multiplicity - 1}}}},
                                      {RelMaxProperSubAlgsOfSemiSimple[{summandAlgWithMultOne}]}
                                     ]
                              ];
          summandMaxes = AlgListSimplify[Join[ productPossibleMaxesOfSummand, {{alg[[i]]}}]];
          Sow[DeleteAlgFromList[summandMaxes, alg]];
        ];
      ][[2]], 1];
   possibleMaxes = AllAlgSumsPossible[listOfSummandMaxes];
   listWithoutStartingAlg = DeleteAlgFromList[possibleMaxes, alg];
   If[algLength == 1, Return[listWithoutStartingAlg];];
   toReturn = RelMaxAlgsInAlgListIgnoringBranchingRules[listWithoutStartingAlg];
   toReturn = DeleteAlgFromList[toReturn, alg];
   If[rank ≤ maxAlgRankStoredAllMaxProperSubs && TrueQ[isCached == False],
    WriteAlgToAllMaxSubalgebrasTable[alg, toReturn];
   ];
   Return[toReturn];
  ];

(* A few output formatting methods follow. *)

FormatSummand[summand_] := Module[{let, letter, toReturn},
   If[TrueQ[summand == {}] || TrueQ[summand == {{0, 0}, 0}], Return[0];];
   let = summand[[1, 1]];
   If[MemberQ[{a, b, c, d}, let], let = Capitalize[ToString[let]];, let = ToString[let];];
   If[summand[[2]] == 1,
      toReturn = Subscript[let, summand[[1, 2]]];
     ,
      toReturn = Superscript[Subscript[let, summand[[1, 2]]], summand[[2]]];
   ];
   Return[toReturn];
  ];
```

```
FormatSummandForTex[summand_] := Module[{let, letter, toReturn},
   If[TrueQ[summand == {}] || TrueQ[summand == {{0, 0}, 0}], Return[0];];
   let = summand[[1, 1]];
   If[MemberQ[{a, b, c, d}, let], let = Capitalize[ToString[let]];, let = ToString[let];];
   If[summand[[2]] == 1,
    toReturn = StringJoin[{let, "_{", ToString[summand[[1, 2]] ], "}"}];
    ,
    toReturn = StringJoin[{let, "_{", ToString[summand[[1, 2]] ], "}^", ToString[summand[[2]]  ] }];
   ];
   Return[toReturn];
  ];

FormatSummandForTexPhysGauge[summand_] := Module[{let, letter, arg},
   If[TrueQ[summand == {}] || TrueQ[summand == {{0, 0}, 0}], Return["\\nzero{}"];];
   let = summand[[1, 1]];
   arg = summand[[1, 2]];
   If[MemberQ[{a, b, c, d}, let],
    If[TrueQ[let == a], Return["\\suText<" <> ToString[arg + 1] <> ">{}" ]; ];
    If[TrueQ[let == b], Return["\\soText<" <> ToString[2 * arg + 1] <> ">{}" ];];
    If[TrueQ[let == c], Return["\\spText<" <> ToString[arg] <> ">{}" ]; ];
    If[TrueQ[let == d], Return["\\soText<" <> ToString[2 * arg] <> ">{}" ]; ];
   ];
   If[TrueQ[let == e], Return["\\eText<" <> ToString[arg] <> ">{}" ]; ];
   If[TrueQ[let == f], Return["\\fText{}"];];
   If[TrueQ[let == g], Return["\\gText{}"];];

   Print["error in type, see FormatSummandForTexPhysGauge"];
   Return[-1];
  ];

FormatSemiSimple[alg_] := Module[{},
   Return[( If[Length[#] > 1,
                                CirclePlus @@ Map[FormatSummand, #]
                                ,
                                FormatSummand[Flatten[#, 1]]
                            ]) &
          [ alg]
        ];
  ];

CirclePlusForTex[summands_] := Module[{i, stringList},
   stringList = Flatten[Table[{summands[[i]], " \\oplus "}, {i, 1, Length[summands]}]];
   stringList = Drop[stringList, -1];
   Return[StringJoin[stringList]];
  ];

FormatSemiSimpleForTex[alg_] := Module[{},
   Return[( If[Length[#] > 1,
                                CirclePlusForTex @ Map[FormatSummandForTex, #]
                                ,
                                FormatSummandForTex[Flatten[#, 1]]
                            ]) &
          [ alg]
        ];
  ];

FormatListOfSemiSimple[alg_] := Module[{},
   Return[TableForm[ FormatSemiSimple /@ alg]];
  ];

FormatLinearListOfSemiSimpleTogether[alg_] := Module[{},
   Return[FormatSemiSimple /@ alg];
  ];

FormatListOfSemiSimpleTogether[alg_] := Module[{i, isDType, toReturn},
   If[Depth[alg] > 5, isDType = True; , isDType = False;];
   If[! isDType, Return[FormatLinearListOfSemiSimpleTogether @ alg];];
   toReturn = Table[FormatLinearListOfSemiSimpleTogether @ (alg[[i]]), {i, 1, Length[alg]}];
   Return[toReturn];
  ];

FormatLinearListOfSemiSimpleTogetherForTex[alg_] := Module[{},
   Return[ FormatSemiSimpleForTex /@ alg];
  ];
```

```
FormatListOfSemiSimpleTogetherForTex[alg_] := Module[{i, isDType, toReturn},
   If[Depth[alg] > 5, isDType = True; , isDType = False;];
   If[! isDType, Return[FormatLinearListOfSemiSimpleTogetherForTex@ alg];];
   toReturn = Table[FormatLinearListOfSemiSimpleTogetherForTex@(alg[[i]]), {i, 1, Length[alg]}];
   Return[toReturn];
  ];

AllAlgSubsCachedQ[alg_, rank_: 0] := Module[{},
   If[! ((-1 ≠ ReadAlgToIndexTableForWhichAllSubsAreKnown@alg)
                || TrueQ[alg == Null]
                 || rank > maxAlgRankStoredAllSubAlgs),
     Return[True];
     ,
     Return[False];
    ];
  ];


AllSemiSimpleSubalgebrasOfSemiSimpleWithCachingAndProductLookup[alg_] :=
  Module[{i, j, length, factorList, allSums, lastLength, nextList, nextLength,
    lastListOfSemiSimpleMaxSubAlgs, semiSimpleAlg, rank, cached, isCached, isAProductOfCached,
    numberOfSummands, isFactorCached, summand, factor, factorRank, multiplicity,
    highestMultiplicityStored, subSummandAlgToLookupItsSubs, subSummandPowerupAlgToCacheItsSubs,
    listOfAllSummandAllSubAlgs, maxMultSubsCached, remainingMult, summandSubAlgs,
    allSummandSubAlgs, toReturn, multsForWhichSubAlgsStored, largestMultAllAlgsCached,
    numberOfLargestMultFactorsInSummand, lastTotalExponent, toSow, subSummandRank},
   semiSimpleAlg = AlgebraSimplify[alg]; ;
   rank = RankOfProductAlg[alg];
   isCached = ReadAlgToIndexTableForWhichAllSubsAreKnown[semiSimpleAlg];
   If[isCached == 1,
        Return[ReadAlgToAllSubalgebrasTable[semiSimpleAlg]];
   ];
        numberOfSummands = Length[semiSimpleAlg];
        listOfAllSummandAllSubAlgs = Flatten[Reap[
            For[i = 1, i ≤ numberOfSummands, i ++,
                summand = semiSimpleAlg[[i]];
                isCached = ReadAlgToIndexTableForWhichAllSubsAreKnown[{summand}];
                If[isCached == 1,
                     Sow[ReadAlgToAllSubalgebrasTable@summand];
                     Continue[](* Keep all the summand subalgebras if they're cached. *);
                 ](* Otherwise, compute them. *);
                factor  = summand[[1]];
                factorRank = summand[[1, 2]];
                multiplicity = summand[[2]];
                remainingMult = multiplicity;
               lastTotalExponent = 0;
                multsForWhichSubAlgsStored = Select[Range[multiplicity],
                       AllAlgSubsCachedQ[{ {factor, #}} , factorRank * #] &];
                If[Length[multsForWhichSubAlgsStored] == 0,
                  Sow[AllAlgSumsPossible[ConstantArray[

              Append[AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[{ {factor, 1}}], {{{0, 0}, 0}}]
                                   , multiplicity]]];
                 ,
                 summandSubAlgs = Flatten[Reap[
                      While[remainingMult > 0,
                             multsForWhichSubAlgsStored =
                 Select[multsForWhichSubAlgsStored, (# ≤ remainingMult) &];
                            largestMultAllAlgsCached = Last@multsForWhichSubAlgsStored;
                            If[largestMultAllAlgsCached == 0,
                                 multsForWhichSubAlgsStored = {1};
                                 toSow = AllAlgSumsPossible[ConstantArray[

                     AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[{ {factor, 1}}], multiplicity]];
                                 Sow[toSow];
                                 Break[];
                            ]
                 (* This should never be encountered with typical usage since we should be caching
                              strictly increasing multiplicity factors starting with mult 1. *);

                numberOfLargestMultFactorsInSummand = Floor[remainingMult / largestMultAllAlgsCached];
                            lastTotalExponent = largestMultAllAlgsCached *
                  numberOfLargestMultFactorsInSummand;
                            subSummandPowerupAlgToCacheItsSubs =
                              {{factor,
                                largestMultAllAlgsCached * numberOfLargestMultFactorsInSummand}};
                            subSummandAlgToLookupItsSubs = {{factor, largestMultAllAlgsCached}};
                            toSow = AllAlgSumsPossible[ConstantArray[
```

```
                    AllSemiSimpleSubalgebrasOfSemiSimpleWithCaching[subSummandAlgToLookupItsSubs]
                                                                   ,
                    numberOfLargestMultFactorsInSummand]];
                            subSummandRank =
                              RankOfProductAlg[subSummandPowerupAlgToCacheItsSubs];
                            If[(subSummandRank ≤  maxAlgRankStoredAllSubAlgs),
                               WriteAlgToAllSubalgebrasTable[subSummandPowerupAlgToCacheItsSubs, toSow];
                            ](* Cache the result if small enough rank *);
                            remainingMult = multiplicity - lastTotalExponent;
                            Sow[toSow];
                        ];
                     ][[2]], 1];
                    allSummandSubAlgs = AllAlgSumsPossible@ summandSubAlgs;
                    If[(rank ≤  maxAlgRankStoredAllSubAlgs),
                       WriteAlgToAllSubalgebrasTable[{summand}, allSummandSubAlgs];];
                    Sow[allSummandSubAlgs];
                 ];
          ];
         ][[2]], 1];
       toReturn = AllAlgSumsPossible@ listOfAllSummandAllSubAlgs;
       If[rank ≤  maxAlgRankStoredAllSubAlgs,
            WriteAlgToAllSubalgebrasTable[semiSimpleAlg, toReturn];
       ];
   Return[toReturn];
  ];

AllSubAlgs[alg_] := Module[{},
   Return[AllSemiSimpleSubalgebrasOfSemiSimpleWithCachingAndProductLookup[alg]];
  ];

SubAlgWithCachingQ[algUnsimplified_, subAlgUnsimplified_, alreadySimplifiedAndCached_: False] :=
  Module[{i, allSubsOfAlg, alg, subAlg},
   If[alreadySimplifiedAndCached,
    Return[MemberQ[AllSubAlgs[algUnsimplified], subAlgUnsimplified]];
   ];
   alg = AlgebraSimplify[algUnsimplified];
   subAlg = AlgebraSimplify[subAlgUnsimplified];
   (* Check the total rank, and direct subsequence for efficiency. *)
   If[RankOfProductAlg[alg] < RankOfProductAlg[subAlg],
     Return[False];
   ];
   If[SubsequenceAlgQ[alg, subAlg], Return[True];];
   Return[MemberQ[AllSubAlgs[alg], subAlg]];
  ];

AllSubAlgsOutputFormat[alg_] := Module[{},
   Return[FormatListOfSemiSimpleTogether@AllSubAlgs[alg]];
  ];
```

```
RelMaxAlgsQuicker[algListUnsimplified_] :=
  Module[{i, j, relMaxSubsequenceAlgs, otherAlgs, relMaxes, length, tableAllSubs, algList, alg, otherSubs},
   algList = AlgListSimplify[algListUnsimplified];
   length = Length[algList];
   relMaxSubsequenceAlgs = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
            alg = algList[[i]];
            otherAlgs = Delete[algList, i];
            If[AlgInAlgListIgnoringNontrivialInclusionsMemberQ[otherAlgs, alg ],
               Continue[];
               ,
               Sow[alg];
           ];
        ];
      ][[2]], 1];

   algList = AlgListSimplify[relMaxSubsequenceAlgs];
   length = Length[algList];
   tableAllSubs = AllSubAlgs /@ algList;
   relMaxes = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
            alg = algList[[i]];
            otherSubs = Catenate[Delete[tableAllSubs, i]];
            If[SubAlgMemberQ[otherSubs, alg ],
               Continue[];
               ,
               Sow[alg];
           ];
        ];
      ][[2]], 1];
   Return[AlgListSimplify[relMaxes]];
  ];

RelMaxAlgsFromListOfAlgOptionsOnEachCurve[listOfOptionsOnEachCurve_] :=
  Module[{i, choicesOnCurve, relMaxesOnCurve, possibleMaxList, maximalAlgList,
    listOfRelMaxAlgLists, listOfProductAlgs, relativelyMaximalProductAlgs},
   listOfRelMaxAlgLists = Flatten[Reap [
       For[i = 1, i ≤ Length[ listOfOptionsOnEachCurve], i ++,
          choicesOnCurve = listOfOptionsOnEachCurve[[i]];
          relMaxesOnCurve = RelMaxAlgsQuicker[choicesOnCurve] ;
          Sow[relMaxesOnCurve];
        ];
      ][[2]], 1];
   listOfProductAlgs = AllAlgSumsPossible[listOfRelMaxAlgLists];
   relativelyMaximalProductAlgs = RelMaxAlgsQuicker[listOfProductAlgs];
   Return[relativelyMaximalProductAlgs];
  ];

RelMaxProperSubAlgsOfSemiSimpleQuicker[semiSimpleAlg_] :=
  Module[{i, alg, allAlgs, length, allAlgsWithoutInputAlg, toReturn},
   alg = AlgebraSimplify[semiSimpleAlg];
   If[TrueQ[alg == { {{a, 1}, 1}}] || TrueQ[alg == { {{0, 0}, 0}}], Return[{{ {0, 0}, 0}}];];
   allAlgs = AllSubAlgs[alg];
   length = Length[allAlgs];
   allAlgsWithoutInputAlg = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
         If[AlgEqual[alg, allAlgs[[i]]],
            Continue[];
            ,
            Sow[allAlgs[[i]]];
          ];
        ];
      ][[2]], 1];
   allAlgsWithoutInputAlg = AlgListSimplify[allAlgsWithoutInputAlg];
   toReturn = DeleteAlgFromList[RelMaxAlgsQuicker[allAlgsWithoutInputAlg], alg];
   Return[toReturn];
  ];
```

```
RelMaxProperSubAlgsOfSemiSimpleMoreEfficient[semiSimpleAlg_] :=
  Module[{i, alg, algLength, summand, multiplicity, summandAlgWithMultOne, maxesOfSimpleSummand,
    allAlgs, length, allAlgsWithoutInputAlg, productPossibleMaxesOfSummand, summandMaxes,
    listOfSummandMaxes , possibleMaxes, listWithoutStartingAlg, toSow, toReturn, isCached, cached, rank},
   If[TrueQ[alg == { {{a, 1}, 1}}] || TrueQ[alg == { {{0, 0}, 0}}], Return[{{ {0, 0}, 0}}];];
   alg = AlgebraSimplify[semiSimpleAlg];
   rank = RankOfProductAlg[alg];

   If[rank ≤ maxAlgRankStoredAllMaxProperSubs,
    If[ReadAlgToIndexTableForWhichAllMaxSubsAreKnown[alg] == 1;
      Return[ReadAlgToAllMaxSubalgebrasTable@alg];
      ,
      isCached = False;
     ];
   ];

   algLength = Length[alg];
   listOfSummandMaxes = Flatten[Reap[
       For[i = 1, i ≤ algLength, i ++,
          summand = alg[[i, 1]];
          multiplicity = alg[[i, 2]];
          If[TrueQ[summand == {a, 1}],
              If[multiplicity == 1,
                Sow[{  {{{0, 0}, 0}},  {{{a, 1}, 1}}}];
                ,
                Sow[{{{{a, 1}, multiplicity - 1}}, {{{a, 1}, multiplicity}}}];
          ];
            Continue[];
           ];

          If[multiplicity ≤ 1,
            toSow =
           AlgListSimplify[Join[RelMaxProperSubAlgsOfSemiSimpleQuicker[{alg[[i]]}], {{alg[[i]]}}]];
            Sow[toSow];
            Continue[];
           ];
          summandAlgWithMultOne = {summand, 1};
          maxesOfSimpleSummand = AlgListSimplify[Join[RelMaxProperSubAlgsOfSemiSimpleQuicker[
              {summandAlgWithMultOne}], {{summandAlgWithMultOne}}]] ;
          productPossibleMaxesOfSummand =
                AllAlgSumsPossible[Join[{{{{summand, multiplicity - 1}}}},
                                        {RelMaxProperSubAlgsOfSemiSimpleQuicker[{summandAlgWithMultOne}]}
                                       ]
                                 ];
          summandMaxes = AlgListSimplify[Join[ productPossibleMaxesOfSummand, {{alg[[i]]}}]];
          Sow[DeleteAlgFromList[summandMaxes, alg]];
        ];
      ][[2]], 1];
   possibleMaxes = AllAlgSumsPossible[listOfSummandMaxes];
   listWithoutStartingAlg = DeleteAlgFromList[possibleMaxes, alg];
   If[algLength == 1, Return[listWithoutStartingAlg];];
   toReturn = RelMaxAlgsInAlgListIgnoringBranchingRules[listWithoutStartingAlg];
   toReturn = DeleteAlgFromList[toReturn, alg];
   If[rank ≤ maxAlgRankStoredAllMaxProperSubs && TrueQ[isCached == False],
    WriteAlgToAllMaxSubalgebrasTable[alg, toReturn];
   ];
   Return[toReturn];
  ];

IsSubSummandQ[alg_, semiSimpleSummand_] :=
  Module[{i, summandType, summandPower, matchingSummand, matchingSummandPower},
   summandType = semiSimpleSummand[[1]];
   summandPower = semiSimpleSummand[[2]];
   matchingSummand = Select[alg, TrueQ[#[[1]] == summandType] &];
   If[Length[matchingSummand] == 0, Return[False];];
   matchingSummand = Flatten[matchingSummand, 1];
   matchingSummandPower = matchingSummand[[2]];
   If[matchingSummandPower ≥ summandPower, Return[True];];
   Return[False];
  ];

(* e.g. *)
```

```
MaximalComplementarySubalgebras[alg_, subAlg_] :=
  Module[{i, j, allSubAlgs, subsWithThisSummand, thisSummand, summandType, summandPower,
    complementaryAlgebras, containingAlg, complementaryAlg, termsNotInSubAlg,
    subAlgTermsWithoutPower, termsToMatchFromSubAlg, matchingTermFromContainingAlg,
    matchingTermPower, remainingPower,  remainingTypeAndPower, differenceOfPowersPartialSub},
   allSubAlgs = AllSubAlgs[alg];
   subsWithThisSummand = allSubAlgs;
   For[i = 1, i ≤ Length[subAlg], i ++ ,
    thisSummand = subAlg[[i]];
    subsWithThisSummand = Select[subsWithThisSummand, IsSubSummandQ[#, thisSummand] &];
   ](* Those that remain contain the subAlgInQuestion explicitly. Now we subtract the
    appropriate powers from each corresponding containing algebra term of each of them. *);
   If[Length[subsWithThisSummand] == 0 , Return[False]; ];
   subAlgTermsWithoutPower = Table[subAlg[[i, 1]], {i, 1, Length[subAlg]}];
   complementaryAlgebras = Flatten[Reap[
       For[j = 1, j ≤ Length[subsWithThisSummand], j ++,
          containingAlg = subsWithThisSummand[[j]];
          termsNotInSubAlg = Select[containingAlg, ! MemberQ[subAlgTermsWithoutPower, #[[1]]] &];
          termsToMatchFromSubAlg = Select[containingAlg, MemberQ[subAlgTermsWithoutPower, #[[1]]] &];
         differenceOfPowersPartialSub = Flatten[Reap[
              For[i = 1, i ≤ Length[subAlg], i ++,
                  thisSummand = subAlg[[i]];
                  summandType = thisSummand[[1]];
                  summandPower = thisSummand[[2]];
                  matchingTermFromContainingAlg =
                     Flatten[#, 1] &@ Select[termsToMatchFromSubAlg, TrueQ[#[[1]] == summandType] &];
                  matchingTermPower = matchingTermFromContainingAlg[[2]];
                  remainingPower = matchingTermPower - summandPower;
                  If[remainingPower > 0,
                      Sow[{summandType, remainingPower}];
                  ];
              ];
             ][[2]], 1];
         complementaryAlg = Join[ differenceOfPowersPartialSub , termsNotInSubAlg];
         If[Length[complementaryAlg] == 0, Continue[]; ];
         Sow[complementaryAlg];
        ];
      ][[2]], 1];
   complementaryAlgebras = RelMaxAlgsQuicker[complementaryAlgebras];
   Return[complementaryAlgebras];
  ];

(* e.g. *)

(*******************************************************
    GS FUNCTIONS
  ****************************************************)

(* Here we specify the orders of vanishing permitted for various GS fiber
 Kodaira types. We omit fibers with no gauge algebra and do not involve A or B.  *)

basicKodairaTypesForGSPre = {
    {{2, 3, 7}, {2, 3, 8}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for "-2" curves: *)
    {{2, 3, 7}, {2, 3, 8}, {2, 3, 9}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for "-3" curves: *)
   {{2, 2, 4}, {2, 3, 6}, {2, 3, 7}, {2, 3, 8}, {3, 4, 8}, {3, 5, 9}},
   (* for "-4" curves *)
    {{2, 3, 6}, {3, 4, 8}, {3, 5, 9}},
   (* for "-5" curves *)
   {{3, 4, 8}, {3, 5, 9}},
   (* for "-6" curves *)
   {{3, 4, 8}, {3, 5, 9}},
   (* for "-7" curves *)
   {{3, 5, 9}}  ,
   (* for "-8" curves *)
   {{3, 5, 9}},
   (* for "-9" *)
   {{4, 5, 10}}(* THESE SHOULD NOT BE HERE!!!, NO? *) ,
   (* for "-10" *)
   {{4, 5, 10}},
   (* for "-11" *)
   {{4, 5, 10}},
   (* for "-12"  *)
   {{4, 5, 10}}
           };

basicKodairaTypesForGS = DeleteDuplicates[Flatten[basicKodairaTypesForGSPre, 1]]
```

```
{{2, 3, 7}, {2, 3, 8}, {1, 2, 3}, {2, 2, 4}, {2, 3, 6}, {3, 4, 8}, {3, 5, 9}, {2, 3, 9}, {4, 5, 10}}
```

```
ListGSFiberTypesOnSingleNonCompactCurves[basicKodairaTypes_,
   maxNForIn_, maxNForInstar_, maxAforIzerostar_, maxBforIzerostar_] :=
  Module[ {i, listToAddForIn, listToAddToForSelfIntersectionFourInstar,
     listToAddForIzerostarA, listToAddForIzerostarB, outArray,
     returnList, listToAddForIV, listToAddForIII, listToAddForIVStar},

   (* First, In, Instar *)
   listToAddForIn = Table[{0, 0, i}, {i, 2, maxNForIn}];
   listToAddToForSelfIntersectionFourInstar = Table[{2, 3, 6 + i}, {i, 1, maxNForInstar}];
   listToAddForIzerostarA  = Table[{2 + i, 3, 6}, {i, 1, maxAforIzerostar }];
   listToAddForIzerostarB   = Table[{2, 3 + i, 6}, {i, 1, maxBforIzerostar }];
   listToAddForIII  = Table[{1, 2 + i, 3}, {i, 1, maxBforIIIGS}];
   listToAddForIV  = Table[{2 + i, 2, 4}, {i, 1, maxAforIVGS}];
   listToAddForIVStar  = Table[{3 + i, 4, 8}, {i, 1, maxAforIVStarGS}];
   returnList = Join[ basicKodairaTypes, listToAddForIn,
     listToAddToForSelfIntersectionFourInstar, listToAddForIzerostarA,
     listToAddForIzerostarB, listToAddForIII, listToAddForIV, listToAddForIVStar];
   (* Note: IIistar add-ons only go on for -8. Those for IIstar only for -
     12.Those for Instar only on -4.
                  For -7, we add nothing; only (3,5,9) possible at all.  For -9,
   -10,-11, nothing possible. IIstar only for -12.  *)
   Return[returnList];
  ];

theKodairaTypesForGS  = DeleteDuplicates[ListGSFiberTypesOnSingleNonCompactCurves[
    basicKodairaTypesForGS, maxNForInGS, maxNForInstarGS, maxAforIzerostarGS, maxBforIzerostarGS]];

theKodairaTypesForGS = SortBy[theKodairaTypesForGS, {#[[3]] &, #[[1]] &, #[[2]] &} ]

{{0, 0, 2}, {0, 0, 3}, {1, 2, 3}, {1, 3, 3}, {0, 0, 4}, {2, 2, 4}, {3, 2, 4}, {0, 0, 5}, {0, 0, 6}, {2, 3, 6},
 {2, 4, 6}, {2, 5, 6}, {3, 3, 6}, {4, 3, 6}, {0, 0, 7}, {2, 3, 7}, {0, 0, 8}, {2, 3, 8}, {3, 4, 8},
 {4, 4, 8}, {0, 0, 9}, {2, 3, 9}, {3, 5, 9}, {0, 0, 10}, {2, 3, 10}, {4, 5, 10}, {0, 0, 11}, {2, 3, 11},
 {0, 0, 12}, {2, 3, 12}, {0, 0, 13}, {2, 3, 13}, {0, 0, 14}, {2, 3, 14}, {0, 0, 15}, {2, 3, 15}, {0, 0, 16},
 {2, 3, 16}, {0, 0, 17}, {2, 3, 17}, {0, 0, 18}, {2, 3, 18}, {0, 0, 19}, {2, 3, 19}, {2, 3, 20}}

DistributeDefinitions[theKodairaTypesForGS];


KodairaTypesMeetingResidualsConstraints[at_, bt_, dt_] := Module[{},
    Return[Select[theKodairaTypesForGS, ((#[[1]] ≤ at) && (#[[2]] ≤ bt) && (#[[3]] ≤ dt)) &]];
  ];

DistinctCurveCollections[listOfCollections_] :=
  Module[{i, j, distinctCollections, collectionA, collectionB, length, toSkip},
   length = Length[listOfCollections];
   distinctCollections = Flatten[Reap
        [
          For[i = 1, i ≤ length - 1, i ++,
           toSkip = False;
           collectionA = listOfCollections[[i]];
           For[j = i + 1, j ≤ length, j ++,
             If[j == i,
               Continue[];
              ];
             collectionB = listOfCollections[[j]];
             If[SubsetQ[collectionA, collectionB] && SubsetQ[collectionB, collectionA],
               toSkip = True (* skip if it appears later in the list *);
               ];
           ];
           If[toSkip == False,
              Sow[collectionA];
             ];
          ];
          Sow[listOfCollections[[length]]](* keep the last collection in the list.*);
      ][[2]], 1];
   Return[distinctCollections];
  ];

NaiveIntersectionContribution[fiberCollection_] := Module[{},
   Return[Sum[fiberCollection[[i]], {i, 1, Length[fiberCollection]}]];
  ];

(* Determines the allowed monodromy option on a curve; 0 for ns,
1 for s, except Izerostar: 0 is ns, 1 is ss, 2 is s *)
(* This return value -1 indicates an error in type assignment inconsistent with our old results *)
```

```
MonodromyTypesForGSFiber[ enhancement_ ] := Module[{a, b, d, monodromyTypesOptions},
   a = enhancement[[1]];
   b = enhancement[[2]];
   d = enhancement[[3]];
    monodromyTypesOptions = {-1};
   If[a ≥ 4 && b == 5,  Return[{0}];];
   If[a == 3 && b ≥ 5, Return[{0}];];
   If[a ≥ 3 && b == 4, Return[{0, 1}];];
   If[a == 2  && b == 3 && d ≥ 7, Return[{0, 1}];];
   If[((a == 2 && b ≥ 3) || (a ≥ 2 && b == 3)) && d == 6 ,
       Return[ {0, 1, 2}] ;
   ];
   If[b == 2 && a ≥ 2 && d == 4,
      Return[{0, 1}];
     ];
   If[a == 1 && b ≥ 2 && d == 3, Return[{0}]; ];
   If[a == 0 && b == 0 && d ≤ 2, Return[{0}];];
   If[a == 0 && b == 0 && d ≥ 3, Return[{0, 1}];];
   Print["bad assignment...see MonodromyTypes
      function and refer to Kodaira table for valid Kodaira types"];
   Print[{{a, b, d}}];
   Return[monodromyTypesOptions ];
  ];

AllMonodromyCombinations[listOfMonodromyOptionsOnEachCurve_] := Module[{},
     Return[Flatten[Outer[Join  @@@  Partition[List[##], 1] &, ##, 1], 1] & @@
       listOfMonodromyOptionsOnEachCurve];
  ];

AllPossibleMonodromyAssignmentsToFibersForGS[listOfKodairaTypeCollections_] :=
  Module[{i, j, option, monodromyOptions, algebra, algebraList, typesAlgebrasPair,
    optionsForGS, length, typeCollection, numberOfFibers, fiber, fiberMonodromyOptions,
    listOfPossibilitiesOfFiberWithMonodromy, listOfFiberOptionsLists,
    listOfMonodromyOptionsForThisFiberCollection, numberOfOptions, depthCorrectedOptionsForGS},
   length = Length[listOfKodairaTypeCollections];
   optionsForGS = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
           typeCollection = listOfKodairaTypeCollections[[i]];
           numberOfFibers = Length[typeCollection];
           listOfFiberOptionsLists = Flatten[Reap[
               For[j = 1, j ≤ numberOfFibers, j ++,
                   fiber = typeCollection[[j]];
                   fiberMonodromyOptions = MonodromyTypesForGSFiber[fiber];
                   listOfPossibilitiesOfFiberWithMonodromy = Flatten[Outer[Append, {{fiber}},
                                                              fiberMonodromyOptions, 1], 1];
                   Sow[listOfPossibilitiesOfFiberWithMonodromy];
                ];
             ][[2]], 1];
           listOfMonodromyOptionsForThisFiberCollection =
          AllMonodromyCombinations[listOfFiberOptionsLists];
           Sow[listOfMonodromyOptionsForThisFiberCollection];
        ];
      ][[2]], 1];
   numberOfOptions = Length[optionsForGS];
   optionsForGS =
    Table[ Flatten[ optionsForGS[[i]], Max[0, Depth[optionsForGS[[i]]] - 5]], {i, 1, numberOfOptions}];
   numberOfOptions = Length[optionsForGS];
   depthCorrectedOptionsForGS = Flatten[Reap[
       For[i = 1, i ≤ numberOfOptions, i ++,
         For[j = 1, j ≤ Length[optionsForGS[[i]]], j ++,
            If[Depth[optionsForGS[[i, j]] ] < 4,
                Sow[{optionsForGS[[i, j]]}];
                ,
                Sow[optionsForGS[[i, j]]];
               ];
          ];
        ];
      ][[2]], 1];
   Return[depthCorrectedOptionsForGS];
  ];
```

```
AllPossibleMonodromyPairRestrictionPassingFibersForSummandGS[
   listOfKodairaTypeCollections_, enhWithResidualsOnSummand_: Null, monodromyOnSummand_: Null] :=
  Module[{i, j, option, monodromyOptions, algebra, algebraList, typesAlgebrasPair, optionsForGS, length,
    typeCollection, numberOfFibers, fiber, fiberMonodromyOptions, listOfPossibilitiesOfFiberWithMonodromy,
    listOfFiberOptionsLists, listOfMonodromyOptionsForThisFiberCollection,
    numberOfOptions, depthCorrectedOptionsForGS, skipThisCollection},
   length = Length[listOfKodairaTypeCollections];
   optionsForGS = Flatten[Reap[
       For[i = 1, i ≤ length, i ++,
           typeCollection = listOfKodairaTypeCollections[[i]];
           numberOfFibers = Length[typeCollection];
           skipThisCollection = False;
           listOfFiberOptionsLists = Flatten[Reap[
               For[j = 1, j ≤ numberOfFibers, j ++,
                   fiber = typeCollection[[j]];
                   fiberMonodromyOptions = MonodromyTypesForGSFiber[fiber];
                   If[! TrueQ[monodromyOnSummand == Null],
                          fiberMonodromyOptions =
                             Select[fiberMonodromyOptions,
                                     TrueQ[IsValidMonodromyAssignmentToPair[
                                             Join[enhWithResidualsOnSummand, {{{Null}, fiber, {Null}}}],
                                                        {monodromyOnSummand, #},
                                                                False
                                                                   ]] &
                                  ]
                  (* Choose only those transvers fibers with monodromy pair validity.
                                        'False' here indicates this is a
                 noncompact transverse fiber. *);
                          If[Length[fiberMonodromyOptions] == 0, skipThisCollection = True; Break[]; ];

                   ];
                   listOfPossibilitiesOfFiberWithMonodromy = Flatten[Outer[Append, {{fiber}},
                                                                fiberMonodromyOptions, 1], 1];
                   Sow[listOfPossibilitiesOfFiberWithMonodromy];
                ];
            ][[2]], 1];
          If[skipThisCollection == True, Continue[];]
                (* We skip adding this collection when one of its curves
                    has no valid monodromy meeting pair restrictions rules from
                     IsValidMonodromyAssignmentToPair *) ;
          listOfMonodromyOptionsForThisFiberCollection =
         AllMonodromyCombinations[listOfFiberOptionsLists];
          Sow[listOfMonodromyOptionsForThisFiberCollection];
       ];
     ][[2]], 1];
   numberOfOptions = Length[optionsForGS];
   optionsForGS =
    Table[ Flatten[ optionsForGS[[i]], Max[0, Depth[optionsForGS[[i]]] - 5]], {i, 1, numberOfOptions}];
   numberOfOptions = Length[optionsForGS];
   depthCorrectedOptionsForGS = Flatten[Reap[
       For[i = 1, i ≤ numberOfOptions, i ++,
         For[j = 1, j ≤ Length[optionsForGS[[i]]], j ++,
            If[Depth[optionsForGS[[i, j]] ] < 4,
                Sow[{optionsForGS[[i, j]]}];
                ,
                Sow[optionsForGS[[i, j]]];
              ];
          ];
       ];
     ][[2]], 1];
   Return[depthCorrectedOptionsForGS];
  ];
```

```
AlgFromTypeAndMonodromyForSummandToGS[typeWithMon_] :=
  Module[{ordf, ordg, orddelta, algebraToReturn, type, monodromy},
   algebraToReturn = -1; (* This catches any errors as all
    actual cases should arise before this is returned *)
   type = typeWithMon[[1]];
   monodromy = typeWithMon[[2]];
   If[monodromy == -1, Return[-1]; ];
   ordf = type[[1]];
   ordg = type[[2]];
   orddelta = type[[3]];
   If[ordf == 0 && ordg == 0 && orddelta == 0, Return[{{0, 0}, 0}];];
   If[ordf == 1 && orddelta == 3, Return[{{a, 1}, 1}]; ];
   If[ordf == 0 && ordg == 0 && orddelta ≥ 2,
      If[(monodromy == 0 && orddelta == 2) || (monodromy == 1 && orddelta ≠ 2),
     Return[ {{a, orddelta - 1}, 1} ]; ];
      If[monodromy == 0 ,         Return[ {{c, Floor[orddelta / 2] }, 1}  ]; ];
     ];
   If[ordf ≥ 2 && ordg == 2 && orddelta == 4,
      If[ monodromy == 1, Return[{{a, 2}, 1}];, Return[{{a, 1}, 1}];];
     ];
   If[ orddelta == 6 && ((ordf == 2 && ordg ≥ 3 ) || (ordf ≥ 2 && ordg == 3)) ,
      If[monodromy == 2, Return[{{d, 4}, 1}];];
      If[monodromy == 1, Return[ {{b, 3}, 1}];];
      If[monodromy == 0, Return[ {{g, 2}, 1}]; ];
     ];
   If[ ordf == 2 && ordg == 3 && orddelta ≥ 7,
      If[monodromy == 1, Return[ {{d, orddelta - 2}, 1}];];
      If[monodromy == 0, Return[{{b, orddelta - 3}, 1}];];
     ];
   If[ ordf ≥ 3 && ordg == 4 && orddelta == 8,
      If[monodromy == 1, Return[ {{e, 6}, 1}];, Return[{{f, 4}, 1}]; ];
     ];
   If[ ordf == 3 && ordg ≥ 5 && orddelta == 9, Return[{{e, 7}, 1}];];
   If[ ordf ≥ 4 && ordg == 5 && orddelta == 10, Return[{{e, 8}, 1}];];

   Print["error algebra not in list: see AlgFromTypeAndMonodromyForSummandToGS. typeWithMon = ",
    typeWithMon];
   Return[algebraToReturn];
  ];

SemiSimpleAlgFromFiberSet[fiberSetWithMonodromy_] := Module[{flatFiberSetWithMonodromy},
      If[Depth[fiberSetWithMonodromy] ≥ 5,
           flatFiberSetWithMonodromy = Flatten[fiberSetWithMonodromy, 1];
           ,
           flatFiberSetWithMonodromy = fiberSetWithMonodromy;
      ] ;
      Return[AlgebraSimplify[Map[AlgFromTypeAndMonodromyForSummandToGS, flatFiberSetWithMonodromy]]];
  ];

SemiSimpleAlgebrasFromFiberSetsAndTotalAlgebra[fiberSetWithMonodromy_] := Module[{i, algsList, totalAlg},
      algsList = Table[AlgebraSimplify[Map[
                         AlgFromTypeAndMonodromyForSummandToGS,
                                    fiberSetWithMonodromy[[i]]
                                      ]],
                        {i, 1, Length[fiberSetWithMonodromy]}];
      totalAlg = AlgebraSimplify[Catenate[algsList]];
      Return[{algsList, totalAlg}];
  ];

SemiSimpleAlgebrasFromFiberSetsAndTotalAlgebraDTypes[fiberSetWithMonodromy_] :=
  Module[{i, algsList, totalAlg, gsAlgAssignmentOnDTypeQuiver, totalAlgsInEachCol},
      gsAlgAssignmentOnDTypeQuiver = Table[AlgebraSimplify[Map[
                         AlgFromTypeAndMonodromyForSummandToGS,
                                    fiberSetWithMonodromy[[i, j]]
                                      ]],

     {i, 1, Length[fiberSetWithMonodromy]}, {j, 1, Length[fiberSetWithMonodromy[[i]]]}];
      totalAlgsInEachCol = Table[ If[Length[gsAlgAssignmentOnDTypeQuiver [[i]]] == 1,
                                      Flatten[ gsAlgAssignmentOnDTypeQuiver [[i]], 1]
                                    ,
                                      AlgebraSimplify[Catenate[ gsAlgAssignmentOnDTypeQuiver [[i]] ]]
                                  ],
                              {i, 1, Length[fiberSetWithMonodromy]}];
      totalAlg = AlgebraSimplify[Catenate[totalAlgsInEachCol]];
      Return[{gsAlgAssignmentOnDTypeQuiver, totalAlg}];
  ];
```

```
(* Note that for treating the D-type GS,
we might approach by modifying the following method to include a transverse compact curve
 input (that in the Null case should generate the same result). We would have first assign a
 given all the compact curves fully, and then find the allowed remaining non-compact fibers.

   We are assuming aPosterioriResiduals are passed in as part of the tripletEnh.
*)

AllNaivePossibleTransverseTypeCollectionsGroupedByNumberOfTransverseCurves[
   enhWithResidualsOnSingleCurve_] :=
  Module[{k, i, isPossibleToAssignAnotherFiber, fiberCollectionOptionsForGSPrevious,
    fiberCollectionOptionsForGSNext, fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves,
    fiberCollection, fiberCollectionWithAnotherFiber, residualsContribution,
    residualsRemaining, curveTypesPossibleToAdd, numNewCurveOptions, newFiberCollectionOptions,
    startResidualsOnCurve, oldFiberCollection, enhType, residuals},

   enhType = enhWithResidualsOnSingleCurve[[1, 2]]
   (* There is only one curve. The second spot is the Kodaira type. *);
   residuals = enhWithResidualsOnSingleCurve[[1, 3]](* The third position is the residuals. *);

   startResidualsOnCurve = residuals;
   isPossibleToAssignAnotherFiber = True;
   fiberCollectionOptionsForGSPrevious = {{0, 0, 0}};
   fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves
     = Flatten[Reap[
       While[isPossibleToAssignAnotherFiber,
              fiberCollectionOptionsForGSNext = Flatten[Reap[
                For[i = 1, i ≤ Length[fiberCollectionOptionsForGSPrevious], i ++,
                   fiberCollection = fiberCollectionOptionsForGSPrevious[[i]];
                   residualsContribution = NaiveIntersectionContribution[fiberCollection];
                   residualsRemaining = startResidualsOnCurve - residualsContribution;
                   curveTypesPossibleToAdd =
                KodairaTypesMeetingResidualsConstraints @@ residualsRemaining;
                   curveTypesPossibleToAdd = Select[curveTypesPossibleToAdd,
                 ! BlowdownNeededOnPair[enhType, #] &];
                   numNewCurveOptions = Length[curveTypesPossibleToAdd];
                   If[numNewCurveOptions == 0, Continue[];];
                   If[TrueQ[fiberCollection == {0, 0, 0}],
                       oldFiberCollection = {};
                   ,
                       oldFiberCollection = fiberCollection;
                   ];
                   newFiberCollectionOptions =
                Table[Append[oldFiberCollection, curveTypesPossibleToAdd[[k]]],
                                             {k, 1, numNewCurveOptions}];
                   Sow[newFiberCollectionOptions];
                ];
               ][[2]], 2];
         If[Length[Flatten[ fiberCollectionOptionsForGSNext]] > 0,
          isPossibleToAssignAnotherFiber = True;
          Sow[DistinctCurveCollections[fiberCollectionOptionsForGSNext]];
          fiberCollectionOptionsForGSPrevious = fiberCollectionOptionsForGSNext;
          ,
          isPossibleToAssignAnotherFiber = False;
         ];
        ];
      ][[2]], 1](* We avoid full flatten to a list of options since if we can't use any of
                     n-fiber collections, we can't use any of the n+1 fiber collections.*);
   Return[fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves];
  ];

NumberOfNontrivialFGFibers[fiberCollectionWithMonodromy_] :=
  Module[{i, fNeighbors, gNeighbors, length, neighbors, positionsWithOneOfFGNonzero},
   length = Length[fiberCollectionWithMonodromy];
   If[length == 0, Return[0];];
   fNeighbors = Table[If[fiberCollectionWithMonodromy[[i, 1, 1]] > 0, 1, 0], {i, 1, length}];
   gNeighbors = Table[If[fiberCollectionWithMonodromy[[i, 1, 2]] > 0, 1, 0], {i, 1, length}];
   neighbors = fNeighbors + gNeighbors;
   positionsWithOneOfFGNonzero = Table[If[neighbors[[i]] > 0, 1, 0], {i, 1, length}];
   Return[Total[positionsWithOneOfFGNonzero]];
  ];

NonCompactCollectionIntersectionContributionForLinearQuiver[aPostEnhResidualsOnSummand_,
   fiberCollectionWithMonodromy_, positionInQuiver_, enhWithResidualsOnQuiver_, monodromyOnQuiver_,
   fibersCollectionsForGSOnEntireQuiver_: Null, nonTrivialFGFiberCountVec_: Null] :=
  Module[{i, j, quiverLength, aPostResidualsOnSummand, monodromyOnCurve, isNeg, contributionTotal,
    contribution, numberOfFibers, fiber, enhOnQuiverCurve, transverseFibers, leftNeighbor, rightNeighbor,
    numberOfNontrivialFGNeighbors, nonTrivialFGFiberCountVecRedux, leftContribution, rightContribution},
   If[TrueQ[fiberCollectionWithMonodromy == {}] ||
```

```
        TrueQ[fiberCollectionWithMonodromy == {{{0, 0, 0}, 0}}] , Return[{0, 0, 0}];];
quiverLength = Length[enhWithResidualsOnQuiver];
monodromyOnCurve = monodromyOnQuiver[[positionInQuiver]];
enhOnQuiverCurve = enhWithResidualsOnQuiver[[positionInQuiver]];
aPostResidualsOnSummand = aPostEnhResidualsOnSummand[[1, 3]];
contributionTotal = {0, 0, 0};
leftNeighbor  = If[positionInQuiver == 1, {}, {{enhWithResidualsOnQuiver[[positionInQuiver - 1, 2]],
     monodromyOnQuiver[[positionInQuiver - 1]]}}];
rightNeighbor = If[positionInQuiver == quiverLength,
                                    {},
  {{enhWithResidualsOnQuiver[[positionInQuiver + 1, 2]], monodromyOnQuiver[[positionInQuiver + 1]]}}];

numberOfFibers = Length[fiberCollectionWithMonodromy];
For[i = 1, i ≤ numberOfFibers, i ++,
   transverseFibers = Join[leftNeighbor, rightNeighbor, Drop[fiberCollectionWithMonodromy, {i}]];
   numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
   fiber = fiberCollectionWithMonodromy[[i]];
    If[ TrueQ[fibersCollectionsForGSOnEntireQuiver == Null] ||
         TrueQ[nonTrivialFGFiberCountVec == Null],
      contribution = IntersectionContributionsToResiduals[
                            enhOnQuiverCurve, {{Null}, fiber[[1]], {Null}},
                             monodromyOnCurve, fiber[[2]], enhWithResidualsOnQuiver ,
                              positionInQuiver, -1, numberOfNontrivialFGNeighbors] ;
     ,
      nonTrivialFGFiberCountVecRedux =
          Table[ If[j ≠ positionInQuiver,
                      nonTrivialFGFiberCountVec[[j]]
                    ,
                      numberOfNontrivialFGNeighbors
                    ]
             , {j, 1, quiverLength}];
      contribution = IntersectionContributionsToResiduals[
                           enhOnQuiverCurve, {{Null}, fiber[[1]], {Null} },
                            monodromyOnCurve, fiber[[2]], enhWithResidualsOnQuiver ,
                              positionInQuiver, -1, numberOfNontrivialFGNeighbors,
                              fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux] ;
     ];
    If[(Length[Select[contribution, (# < 0) &]] > 0),
       Return[{-1, -1, -1}];
    ];
   contribution = Table[Max[fiber[[1, j]], contribution[[j]]], {j, 1, 3}]
  (* Safety in case IntersectionContribution
                          method returns less than the naive
   intersection values due to potential error there *);
   contributionTotal = contributionTotal + contribution;
   If[Length[Select[Flatten[aPostResidualsOnSummand - contributionTotal], (# < 0) &]] > 0,
  Return[{-1, -1, -1}];];
];
(* Now we add the contributions from the left and right neighbors in the case where we are doing
 the final check call where GS fibers are assigned on the entire quiver. In other words,
we are computing the final intersection contributions at the site. *)
 If[! TrueQ[nonTrivialFGFiberCountVec == Null],
    If[TrueQ[leftNeighbor == {}],
      leftContribution = {0, 0, 0};
      ,
      transverseFibers = Join[rightNeighbor, fiberCollectionWithMonodromy];
      numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
      nonTrivialFGFiberCountVecRedux =
          Table[ If[j ≠ positionInQuiver,
                      nonTrivialFGFiberCountVec[[j]]
                    ,
                      numberOfNontrivialFGNeighbors
                    ]
             , {j, 1, quiverLength}];
      leftContribution =  IntersectionContributionsToResiduals[
                           enhOnQuiverCurve, {{Null}, Flatten[leftNeighbor, 1][[1]], {Null} },
                            monodromyOnCurve, Flatten[leftNeighbor, 1][[2]], enhWithResidualsOnQuiver ,
                              positionInQuiver, positionInQuiver - 1, numberOfNontrivialFGNeighbors,

      fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
     ];
    If[TrueQ[rightNeighbor == {}],
      rightContribution = {0, 0, 0};
      ,
      transverseFibers = Join[leftNeighbor, fiberCollectionWithMonodromy];
      numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
      nonTrivialFGFiberCountVecRedux =
          Table[ If[j ≠ positionInQuiver,
                      nonTrivialFGFiberCountVec[[j]]
                    ,
```

```
                        numberOfNontrivialFGNeighbors
                      ]
                , {j, 1, quiverLength}];
        rightContribution =  IntersectionContributionsToResiduals[
                            enhOnQuiverCurve, {{Null}, Flatten[rightNeighbor, 1][[1]], {Null} },
                            monodromyOnCurve, Flatten[rightNeighbor, 1][[2]], enhWithResidualsOnQuiver ,
                              positionInQuiver, positionInQuiver + 1, numberOfNontrivialFGNeighbors,

        fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
       ];
      contributionTotal = contributionTotal + leftContribution + rightContribution;
      If[(Length[Select[Flatten[Join[leftContribution, rightContribution]], (# < 0) &]] > 0),
     Return[{-1, -1, -1}];];
   ];
   Return[contributionTotal];
  ];

NonCompactCollectionIntersectionContributionInDType[aPostEnhResidualsOnSummand_,
   fiberCollectionWithMonodromy_, positionInQuiver_, enh_, monodromyOnQuiver_,
   fibersCollectionsForGSOnEntireQuiver_: Null, nonTrivialFGFiberCountVec_: Null] :=
  Module[{i, j, k, quiverLength, aPostResidualsOnSummand, monodromyOnCurve, isNeg,
    contributionTotal, contribution, numberOfFibers, fiber, enhOnQuiverCurve, transverseFibers,
    leftNeighbor, rightNeighbor, numberOfNontrivialFGNeighbors, nonTrivialFGFiberCountVecRedux,
    leftContribution, rightContribution, topContribution, bottomContribution, branchPositions,
    nontrivialFGVecToWrap, wrappedNonTrivialFGVec, xpos, ypos, bottomNeighbor, topNeighbor,
    wrappedEnhWithRes, wrappedNonTrivialFGVecFromArgs, wrappedMonodromyOnQuiver },
   contribution = {0, 0, 0};
   contributionTotal = {0, 0, 0};
   If[(Min @@ Depth /@ monodromyOnQuiver ≤ 1),
      wrappedMonodromyOnQuiver = WrapDTypeMonOnQuiver[monodromyOnQuiver];
      ,
      wrappedMonodromyOnQuiver = monodromyOnQuiver
     (* Otherwise it's already wrapped in the method calling this method. *);
   ];
   quiverLength = Length[enh];
   wrappedEnhWithRes =  WrapDTypeEnhancement[enh];
   branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
   If[TrueQ[nonTrivialFGFiberCountVec == Null],
    If[TrueQ[fiberCollectionWithMonodromy == {}] ||
           TrueQ[fiberCollectionWithMonodromy == {{{0, 0, 0}, 0}}] , Return[{0, 0, 0}];];
   ];
   xpos = positionInQuiver[[1]];
   ypos = positionInQuiver[[2]];
   monodromyOnCurve = wrappedMonodromyOnQuiver[[xpos, ypos]];
   enhOnQuiverCurve = wrappedEnhWithRes[[xpos, ypos]];
   nontrivialFGVecToWrap = TotalNumberOfNontrivialTransverseFGFibersAsVec[enh];
   wrappedNonTrivialFGVec = WrapNonTrivialFGFiberVecOnDTypeQuiver[nontrivialFGVecToWrap];
   If[! TrueQ[nonTrivialFGFiberCountVec == Null],
     wrappedNonTrivialFGVecFromArgs = WrapNonTrivialFGFiberVecOnDTypeQuiver[nonTrivialFGFiberCountVec];
   ];
   aPostResidualsOnSummand = aPostEnhResidualsOnSummand[[1, 3]];
   If[ypos > 1,
     leftNeighbor = {};
     rightNeighbor = {};
     bottomNeighbor =
     {{wrappedEnhWithRes[[xpos, ypos - 1, 2]], wrappedMonodromyOnQuiver[[xpos, ypos - 1]]}};
     If[ypos < Length[wrappedMonodromyOnQuiver[[xpos]]],

     topNeighbor =  {{wrappedEnhWithRes[[xpos, ypos + 1, 2]], wrappedMonodromyOnQuiver[[xpos, ypos + 1]]}};
      ,
       topNeighbor =  {};
      ];
   ];
    If[ypos == 1,
     bottomNeighbor = {};
     If[branchPositions[[xpos]] == 0,
       topNeighbor = {};
       ,

     topNeighbor =  {{wrappedEnhWithRes[[xpos, ypos + 1, 2]], wrappedMonodromyOnQuiver[[xpos, ypos + 1]]}};
    ];
    If[xpos == quiverLength,
      rightNeighbor = {};

     leftNeighbor = {{wrappedEnhWithRes[[xpos - 1, ypos, 2]], wrappedMonodromyOnQuiver[[xpos - 1, ypos]]}};
    ];
    If[xpos == 1,
      leftNeighbor = {};
      rightNeighbor =
      {{wrappedEnhWithRes[[xpos + 1, ypos, 2]], wrappedMonodromyOnQuiver[[xpos + 1, ypos]]}};
```

```
 ];
 If[1 < xpos < quiverLength,
   leftNeighbor =
   {{wrappedEnhWithRes[[xpos - 1, ypos, 2]], wrappedMonodromyOnQuiver[[xpos - 1, ypos]]}};
   rightNeighbor = {{wrappedEnhWithRes[[xpos + 1, ypos, 2]],
     wrappedMonodromyOnQuiver[[xpos + 1, ypos]]}};
 ];
];
numberOfFibers = Length[fiberCollectionWithMonodromy];
For[i = 1, i ≤ numberOfFibers, i ++,
  transverseFibers = Join[leftNeighbor, rightNeighbor, topNeighbor, bottomNeighbor,
                          Drop[fiberCollectionWithMonodromy, {i}]];
  numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
  fiber = fiberCollectionWithMonodromy[[i]];
  If[ TrueQ[fibersCollectionsForGSOnEntireQuiver == Null] ||
       TrueQ[nonTrivialFGFiberCountVec == Null],
    contribution = IntersectionContributionsToResiduals[
                        enhOnQuiverCurve, {{Null}, fiber[[1]], {Null}},
                        monodromyOnCurve, fiber[[2]], wrappedEnhWithRes ,
                         positionInQuiver,
     -1, numberOfNontrivialFGNeighbors, Null, wrappedNonTrivialFGVec, False];
   ,
    nonTrivialFGFiberCountVecRedux =
       Table[ If[TrueQ[{j, k} ≠ positionInQuiver],
                 wrappedNonTrivialFGVec[[j, k]]
                ,
                 numberOfNontrivialFGNeighbors
                ]
          , {j, 1, quiverLength}, {k, 1, Length[wrappedEnhWithRes[[j]] ]}];
    contribution = IntersectionContributionsToResiduals[
                       enhOnQuiverCurve, {{Null}, fiber[[1]], {Null}},
                       monodromyOnCurve, fiber[[2]], wrappedEnhWithRes ,
                         positionInQuiver, -1, numberOfNontrivialFGNeighbors,
                         fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux] ;
   ];
  If[Length[Select[contribution, (# < 0) &]] > 0,
     Return[{-1, -1, -1}];
   ];
  contribution = Table[Max[fiber[[1, j]], contribution[[j]]], {j, 1, 3}]
 (* Safety in case IntersectionContribution
                      method returns less than the naive
  intersection values due to potential error there *);
  contributionTotal = contributionTotal + contribution;
 If[Length[Select[Flatten[aPostResidualsOnSummand - contributionTotal], (# < 0) &]] > 0,
  Return[{-1, -1, -1}];
 ];
];
(* Now we add the contributions from the left and right neighbors in the case where we are doing
 the final check call where GS fibers are assigned on the entire quiver. In other words,
we are computing the final intersection contributions at the site. *)
If[! TrueQ[nonTrivialFGFiberCountVec == Null],
   If[TrueQ[leftNeighbor == {}],
     leftContribution = {0, 0, 0};
     ,

  transverseFibers = Join[rightNeighbor, topNeighbor, bottomNeighbor, fiberCollectionWithMonodromy];
     numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
     nonTrivialFGFiberCountVecRedux =
        Table[ If[TrueQ[{j, k} ≠ positionInQuiver],
                 wrappedNonTrivialFGVecFromArgs[[j, k]]
                ,
                 numberOfNontrivialFGNeighbors
                ]
          , {j, 1, quiverLength}, {k, 1, Length[wrappedEnhWithRes[[j]] ]}];
     leftContribution =  IntersectionContributionsToResiduals[
                       enhOnQuiverCurve, {{Null}, Flatten[leftNeighbor, 1][[1]], {Null} },
                       monodromyOnCurve, Flatten[leftNeighbor, 1][[2]], wrappedEnhWithRes ,
                         positionInQuiver, {xpos - 1, ypos}, numberOfNontrivialFGNeighbors,

     fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
    ];
   If[TrueQ[rightNeighbor == {}],
     rightContribution = {0, 0, 0};
     ,

  transverseFibers = Join[leftNeighbor, topNeighbor, bottomNeighbor, fiberCollectionWithMonodromy];
     numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
     nonTrivialFGFiberCountVecRedux =
        Table[ If[TrueQ[{j, k} ≠ positionInQuiver],
                 wrappedNonTrivialFGVecFromArgs[[j, k]]
```

```
                    ,
                     numberOfNontrivialFGNeighbors
                    ]
               , {j, 1, quiverLength}, {k, 1, Length[wrappedEnhWithRes[[j]] ]}];
     rightContribution =  IntersectionContributionsToResiduals[
                         enhOnQuiverCurve, {{Null}, Flatten[rightNeighbor, 1][[1]], {Null} },
                          monodromyOnCurve, Flatten[rightNeighbor, 1][[2]], wrappedEnhWithRes ,
                            positionInQuiver, {xpos + 1, ypos}, numberOfNontrivialFGNeighbors,

     fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
    ];
    If[TrueQ[topNeighbor == {}],
     topContribution = {0, 0, 0};
     ,
     transverseFibers =
    Join[leftNeighbor, rightNeighbor, bottomNeighbor, fiberCollectionWithMonodromy];
     numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
     nonTrivialFGFiberCountVecRedux =
          Table[ If[TrueQ[{j, k} ≠ positionInQuiver],
                     wrappedNonTrivialFGVecFromArgs[[j, k]]
                    ,
                     numberOfNontrivialFGNeighbors
                    ]
               , {j, 1, quiverLength}, {k, 1, Length[wrappedEnhWithRes[[j]] ]}];
     topContribution =  IntersectionContributionsToResiduals[
                         enhOnQuiverCurve, {{Null}, Flatten[topNeighbor, 1][[1]], {Null} },
                          monodromyOnCurve, Flatten[topNeighbor, 1][[2]], wrappedEnhWithRes ,
                            positionInQuiver, {xpos, ypos + 1}, numberOfNontrivialFGNeighbors,

     fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
    ];
    If[TrueQ[bottomNeighbor == {}],
     bottomContribution = {0, 0, 0};
     ,

   transverseFibers = Join[leftNeighbor, rightNeighbor, topNeighbor, fiberCollectionWithMonodromy];
     numberOfNontrivialFGNeighbors = NumberOfNontrivialFGFibers[transverseFibers];
     nonTrivialFGFiberCountVecRedux =
          Table[ If[TrueQ[{j, k} ≠ positionInQuiver],
                     wrappedNonTrivialFGVecFromArgs[[j, k]]
                    ,
                     numberOfNontrivialFGNeighbors
                    ]
               , {j, 1, quiverLength}, {k, 1, Length[wrappedEnhWithRes[[j]] ]}];
     bottomContribution =  IntersectionContributionsToResiduals[
                         enhOnQuiverCurve, {{Null}, Flatten[bottomNeighbor, 1][[1]], {Null} },
                          monodromyOnCurve, Flatten[bottomNeighbor, 1][[2]], wrappedEnhWithRes ,
                            positionInQuiver, {xpos, ypos - 1}, numberOfNontrivialFGNeighbors,

     fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVecRedux, True];
    ];
    contributionTotal =
   contributionTotal + leftContribution + rightContribution + topContribution + bottomContribution;
    If[(Length[Select[Flatten[Join[leftContribution, rightContribution,
          topContribution, bottomContribution]] , (# < 0) &]
              ] > 0),
     Return[{-1, -1, -1}];
    ];
  ];
  Return[contributionTotal];
 ];


NonCompactCollectionIntersectionContribution[aPostEnhResidualsOnSummand_,
  fiberCollectionWithMonodromy_, positionInQuiver_, enh_, monodromyOnQuiver_,
  fibersCollectionsForGSOnEntireQuiver_: Null, nonTrivialFGFiberCountVec_: Null] := Module[{toReturn},
  If[Depth[positionInQuiver] == 1,
   Return[NonCompactCollectionIntersectionContributionForLinearQuiver[
     aPostEnhResidualsOnSummand, fiberCollectionWithMonodromy, positionInQuiver, enh,
     monodromyOnQuiver, fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVec]];
  ];
  toReturn = NonCompactCollectionIntersectionContributionInDType[
    aPostEnhResidualsOnSummand, fiberCollectionWithMonodromy, positionInQuiver, enh,
    monodromyOnQuiver, fibersCollectionsForGSOnEntireQuiver, nonTrivialFGFiberCountVec];
  Return[toReturn];
 ];
```

```
RemainingResidualsAfterGSFibers[aPostEnhWithResidualsOnSummand_,
   fiberCollectionWithMonodromy_, positionInQuiver_, enhWithResidualsOnQuiver_, monodromyOnQuiver_] :=
  Module[{aPostResidualsOnSummand, residualsRemaining, totalCont, isDType},
   totalCont = {0, 0, 0};
     isDType = False;
     If[Depth[positionInQuiver] > 1, isDType = True;];
     If[isDType,
      totalCont =  NonCompactCollectionIntersectionContributionInDType[aPostEnhWithResidualsOnSummand,
        fiberCollectionWithMonodromy, positionInQuiver, enhWithResidualsOnQuiver, monodromyOnQuiver];
      ,
      totalCont =
      NonCompactCollectionIntersectionContributionForLinearQuiver[aPostEnhWithResidualsOnSummand,
        fiberCollectionWithMonodromy, positionInQuiver, enhWithResidualsOnQuiver, monodromyOnQuiver];
     ];
     aPostResidualsOnSummand = aPostEnhWithResidualsOnSummand[[1, 3]];

     If[Length[Select[totalCont, (# < 0) &]] > 0,
        Return[{-1, -1, -1}];
     ];
     If[Length[Select[aPostResidualsOnSummand, (# < 0) &]] > 0,
        Return[{-1, -1, -1}];
     ];
     residualsRemaining = aPostResidualsOnSummand - totalCont;
     If[Length[Select[residualsRemaining , (# < 0) &]] > 0,
        Return[{-1, -1, -1}];
     ];
     Return[residualsRemaining];
  ];

TossTrivialGSFibersInCollection[fiberCollection_] := Module[{toReturn},
   toReturn = fiberCollection;
       If[Length[fiberCollection] > 1,
           toReturn = Select[fiberCollection, StrictlyPositiveType[#[[1]]] &];
       ](* Toss the {0,0,0} entries if length>1*);
   Return[toReturn];
  ];

TossTrivialGSFibersInCollectionsList[fiberCollectionsListWithMon_] :=
  Module[{i, fibersWithMonodromy, outList},
   outList = Flatten[Reap[
       For[i = 1, i ≤ Length[fiberCollectionsListWithMon], i ++,
            fibersWithMonodromy = fiberCollectionsListWithMon[[i]];
              fibersWithMonodromy = TossTrivialGSFibersInCollection[fibersWithMonodromy];
              Sow[fibersWithMonodromy];
        ];
      ][[2]], 1];
   DeleteDuplicates[outList];
   Return[outList];
  ];

NonNegType[typeTriplet_] := Module[{},
   If[(Length[Select[Flatten[typeTriplet], (# < 0) &]] > 0),
     Return[False];
   ];
   Return[True];
  ];

StrictlyPositiveType[typeTriplet_] := Module[{},
   If[Length[Select[Flatten@typeTriplet, (# < 0) &]] > 0,
     Return[False];
     ,
     If[(Length[Select[Flatten@typeTriplet, (# > 0) &]] > 0),
        Return[True];
      ];
   ];
   Return[False];
  ];

TransverseTypeCollectionsWithMonodromy[enhWithAPostResidualsOnCurve_,
   monodromyOnSummand_, positionInQuiver_, enhWithResidualsOnQuiver_, monodromyOnQuiver_] :=
  Module[{n, k, i, isPossibleToAssignAnotherFiber, fiberCollectionOptionsForGSPrevious,
    fiberCollectionOptionsForGSNext, fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves,
    fiberCollectionWithMonodromy, fiberCollectionWithAnotherFiber, residualsContribution,
    residualsRemaining, curveTypesPossibleToAdd, curveTypesWithMonodromyPossibleToAdd,
    numNewCurveOptions, newFiberCollectionOptions, startResidualsOnCurve, enhType,
    residuals, curveTypesPossibleToAddMonodromyOptions, listOfGSFiberWithMonToAdd},
   (* TODO: Note that the call to this method in the main workflow gives
     a wrapped monodromy but not a wrapped enhWith residuals. *)
   enhType = enhWithAPostResidualsOnCurve[[1, 2]](* There is only one
```

```
 curve. The second spot is the Kodaira type. *);
residuals = enhWithAPostResidualsOnCurve[[1, 3]](* The third position is the residuals. *);
startResidualsOnCurve = residuals
(* These are the aPost residuals when called from the main method. *);
isPossibleToAssignAnotherFiber = True;
fiberCollectionOptionsForGSPrevious = {{{{0, 0, 0}, 0}}};
fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves
  = Flatten[Reap[
    While[isPossibleToAssignAnotherFiber,
           fiberCollectionOptionsForGSNext = Flatten[Reap[
            For[i = 1, i ≤ Length[fiberCollectionOptionsForGSPrevious], i ++,
                fiberCollectionWithMonodromy = fiberCollectionOptionsForGSPrevious[[i]];
                fiberCollectionWithMonodromy =
                   TossTrivialGSFibersInCollection[fiberCollectionWithMonodromy];
                residualsContribution = NonCompactCollectionIntersectionContribution[

             enhWithAPostResidualsOnCurve, fiberCollectionWithMonodromy,
                                             positionInQuiver, enhWithResidualsOnQuiver,
                                                monodromyOnQuiver];
                If[Length[Select[residualsContribution, (# < 0) &]] > 0, Continue[];];
                residualsRemaining = startResidualsOnCurve - residualsContribution;
                curveTypesPossibleToAdd =
            KodairaTypesMeetingResidualsConstraints @@ residualsRemaining;
                If[Length[curveTypesPossibleToAdd] == 0, Continue[];];
                curveTypesPossibleToAdd =
            Select[curveTypesPossibleToAdd, ! BlowdownNeededOnPair[enhType, #] &];
                If[Length[curveTypesPossibleToAdd] == 0, Continue[];];
                numNewCurveOptions = Length[curveTypesPossibleToAdd];
                curveTypesPossibleToAddMonodromyOptions = Table[{curveTypesPossibleToAdd[[k]],

              MonodromyTypesForGSFiber[curveTypesPossibleToAdd[[k]]]
                                                                  }, {k, 1, numNewCurveOptions}];
                curveTypesPossibleToAddMonodromyOptions =
                   Table[{curveTypesPossibleToAddMonodromyOptions [[k, 1]] ,
                            curveTypesPossibleToAddMonodromyOptions [[k, 2, n]]
                          }
                         , {k, 1, Length[curveTypesPossibleToAddMonodromyOptions ]}
                         , {n, 1, Length[curveTypesPossibleToAddMonodromyOptions [[k, 2]]]}
                        ];
               listOfGSFiberWithMonToAdd = Flatten[curveTypesPossibleToAddMonodromyOptions, 1];
               listOfGSFiberWithMonToAdd = DeleteDuplicates[listOfGSFiberWithMonToAdd] ;
               listOfGSFiberWithMonToAdd =
            Select[listOfGSFiberWithMonToAdd, IsValidMonodromyAssignmentToPair[
                                             Join[enhWithAPostResidualsOnCurve,

                {{{Null}, #[[1]], {Null}}}], {monodromyOnSummand, #[[2]]},
                                                              False ] &
                                             ]
             (* Retain only those choices for a new GS fiber with
                                                a valid  pair intersection
            with the summand in the quiver. *);
               newFiberCollectionOptions = Table[Append[fiberCollectionWithMonodromy,
                                                     listOfGSFiberWithMonToAdd[[n]]]
                                               , {n, 1, Length[listOfGSFiberWithMonToAdd]}];
               newFiberCollectionOptions =
                    Table[If[(Length[newFiberCollectionOptions[[k]]] == 1),
                        newFiberCollectionOptions[[k]]
                        ,
                        Select[newFiberCollectionOptions[[k]], (! TrueQ[# == {{0, 0, 0}, 0}]) & ]
                       ],
                     {k, 1, Length[newFiberCollectionOptions]}
                     ];
                newFiberCollectionOptions = Table[SortBy[newFiberCollectionOptions[[k]],
                                                  {#[[1, 3]] &, #[[1, 2]] &,
                                                   #[[1, 1]] &, #[[2]] &}]
                                          , {k, 1 , Length[newFiberCollectionOptions]}];
                newFiberCollectionOptions = DeleteDuplicates[newFiberCollectionOptions];
                numNewCurveOptions = Length[newFiberCollectionOptions];
                If[numNewCurveOptions == 0, Continue[];];
                newFiberCollectionOptions = Select[newFiberCollectionOptions,

             NonNegType[ RemainingResidualsAfterGSFibers[

                 enhWithAPostResidualsOnCurve, #, positionInQuiver,

                 enhWithResidualsOnQuiver, monodromyOnQuiver]] &
                                               ];
                If[Length[Flatten[newFiberCollectionOptions]] == 0, Continue[];];
                newFiberCollectionOptions =
                  TossTrivialGSFibersInCollectionsList[newFiberCollectionOptions];
```

```
                      newFiberCollectionOptions = DeleteDuplicates[newFiberCollectionOptions];
                      Sow[newFiberCollectionOptions];
                    ];
                  ][[2]], 2];
            If[Length[Flatten[ fiberCollectionOptionsForGSNext]] > 0,
              Sow[DistinctCurveCollections[fiberCollectionOptionsForGSNext]];
              fiberCollectionOptionsForGSNext = Select[fiberCollectionOptionsForGSNext,
                                                          (RemainingResidualsAfterGSFibers[

                     enhWithAPostResidualsOnCurve, #, positionInQuiver,

                     enhWithResidualsOnQuiver, monodromyOnQuiver]
                                                              [[3]] ≥ 2) &
                                                   ](* Those with more possible GS
                                                        may have another curve added in
                                                        the next iteration (checkeding dt ≥ 2). *);
              If[Length[fiberCollectionOptionsForGSNext] > 0,
               isPossibleToAssignAnotherFiber = True;
               fiberCollectionOptionsForGSPrevious = fiberCollectionOptionsForGSNext;
               ,
               isPossibleToAssignAnotherFiber = False;
               ];
             ,
             isPossibleToAssignAnotherFiber = False;
            ];
           ];
       ][[2]], 1](* We later flatten to a list of options; some
                      n-fiber collections may be possible while
                      n+1-fiber collections valid a priori may not.*);
   Return[fiberCollectionOptionsForGSAsListGroupedByNumberOfCurves];
  ];

AllPossiblePairsFiberCollectionsAndAlgebrasForCurveGS[enhWithResidualsOnSummand_,
   monodromyOnSummand_: Null] := Module[{i, residuals, possibleTransverseTypesCollections,
    collectionsOfTransverseCurvesWithMonodromyAssignments, collectionSemisimpleAlgebraPairs,
    typeCollectionSemiSimpleAlgPairs, numberOfPossibleFiberSets, monodromy},

   residuals = enhWithResidualsOnSummand[[1, 3]](* There is only one curve. *);
   possibleTransverseTypesCollections =
         AllNaivePossibleTransverseTypeCollectionsGroupedByNumberOfTransverseCurves[
            enhWithResidualsOnSummand];
   possibleTransverseTypesCollections =
         Flatten[ possibleTransverseTypesCollections , 1];
   collectionsOfTransverseCurvesWithMonodromyAssignments =
         AllPossibleMonodromyPairRestrictionPassingFibersForSummandGS[
                    possibleTransverseTypesCollections, enhWithResidualsOnSummand, monodromyOnSummand];
   numberOfPossibleFiberSets = Length[collectionsOfTransverseCurvesWithMonodromyAssignments];
   typeCollectionSemiSimpleAlgPairs = Table[
                          { collectionsOfTransverseCurvesWithMonodromyAssignments[[i]],
      SemiSimpleAlgFromFiberSet[ collectionsOfTransverseCurvesWithMonodromyAssignments[[i]]] },
                          {i, 1, numberOfPossibleFiberSets}];
   Return[typeCollectionSemiSimpleAlgPairs];
  ];

NaiveOptionsForSingleCurveGS[enhWithResidualsOnSummand_, monodromyOnSummand_: Null] := Module[{algs},
   algs = Flatten[
            Drop[
              AllPossiblePairsFiberCollectionsAndAlgebrasForCurveGS[
                                   enhWithResidualsOnSummand, monodromyOnSummand]
                , 0, 1]
            , 1];
   Return[AlgListSimplify[algs]];
  ];


NaiveRelMaxOptionsForSingleCurveGS[enhWithResidualsOnSummand_, monodromyOnSummand_: Null] :=
  Module[{algs, relMaxAlgs},
   algs = NaiveOptionsForSingleCurveGS[enhWithResidualsOnSummand, monodromyOnSummand];
   relMaxAlgs = RelMaxAlgsQuicker[algs];
   If[Length[relMaxAlgs] == 0, Return[{{{{0, 0}, 0}}}];];
   Return[relMaxAlgs];
  ];
```

```
NaiveOptionsForEnhancedQuiverGS[enhWithResidualsOnQuiver_,
   monodromyOnQuiver_: Null, toUseAPostResidualsOnQuiver_: False] :=
  Module[{i, relMaxOptionsOnEachCurve, productAlgs, enhWithResiduals},
   If[toUseAPostResidualsOnQuiver
        && TrueQ[monodromyOnQuiver == Null],
      Print["need to input monodromy to use a post methods"];
   ];
   enhWithResiduals = If[toUseAPostResidualsOnQuiver && (! TrueQ[monodromyOnQuiver == Null]),
                          ComputeAPosterioriResidualsConsideringMonodromy[
                               enhWithResidualsOnQuiver, monodromyOnQuiver]
                          ,
                          enhWithResidualsOnQuiver
                      ];
   relMaxOptionsOnEachCurve = Table[NaiveRelMaxOptionsForSingleCurveGS[{enhWithResidualsOnQuiver[[i]]},
                                              If[TrueQ[monodromyOnQuiver == Null],
                                                 Null
                                                 ,
                                                 monodromyOnQuiver[[i]]]
                                              ]
                                   , {i, 1, Length[enhWithResidualsOnQuiver]}
                                  ];
   productAlgs = AllAlgSumsPossible[relMaxOptionsOnEachCurve];
   productAlgs = RelMaxAlgsQuicker[productAlgs];
   Return[productAlgs](* Check whether RelMaxAlgs is needed to see new inclusions in the product *) ;
  ];

NaiveAPostOptionsForGS[enhWithResidualsOnQuiver_, monodromyChoice_] :=
  Module[{relMaxOptionsOnEachCurve, productAlgs, enhWithAPostResidualsOnQuiver},
   enhWithAPostResidualsOnQuiver =

    ComputeAPosterioriResidualsConsideringMonodromy @@ {enhWithResidualsOnQuiver, monodromyChoice};
   relMaxOptionsOnEachCurve = NaiveRelMaxOptionsForSingleCurveGS[{#}] & /@
     enhWithAPostResidualsOnQuiver;
   productAlgs = AllAlgSumsPossible[relMaxOptionsOnEachCurve];
   Return[productAlgs](* Check whether RelMaxAlgs is needed to see new inclusions in the product *) ;
  ];

(* Finds transverse fibers allowed by pair restrictions and
 enhWithResiduals (using the precomputed aPost residuals on the quiver. *)

TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[enhWithResidualsOnSummand_,
   monodromyOnSummand_: Null, toUseIntCont_: False, positionInQuiver_: Null, enh_: Null,
   monodromyOnQuiver_: Null] := Module[{residuals, possibleTransverseTypesCollections,
    collectionsOfTransverseCurvesWithMonodromyAssignments, collectionSemisimpleAlgebraPairs,
    typeCollectionSemiSimpleAlgPairs, numberOfPossibleFiberSets, monodromy},
   residuals = enhWithResidualsOnSummand[[1, 3]](* This is the enh on
    the curve for which we are finding GS summands. *);
   If[toUseIntCont,
    collectionsOfTransverseCurvesWithMonodromyAssignments =
           TransverseTypeCollectionsWithMonodromy[
                 enhWithResidualsOnSummand,
      monodromyOnSummand, positionInQuiver, enh, monodromyOnQuiver];
      collectionsOfTransverseCurvesWithMonodromyAssignments =
           Flatten[collectionsOfTransverseCurvesWithMonodromyAssignments, 1];
    ,
    possibleTransverseTypesCollections =
           AllNaivePossibleTransverseTypeCollectionsGroupedByNumberOfTransverseCurves[
                                                       enhWithResidualsOnSummand];
    possibleTransverseTypesCollections  = Flatten[ possibleTransverseTypesCollections , 1]
        (*  For the line above, WE ARE DROPPING THE GROUPING BY NUMBER OF TRANSVERSE CURVES. *) ;
    collectionsOfTransverseCurvesWithMonodromyAssignments =
           AllPossibleMonodromyPairRestrictionPassingFibersForSummandGS[
               possibleTransverseTypesCollections, enhWithResidualsOnSummand, monodromyOnSummand];
    ];
   Return[Append[collectionsOfTransverseCurvesWithMonodromyAssignments, {{{0, 0, 0}, 0}}]];
  ];

(* Example *)

TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[{{{3}, {2, 3, 6}, {2, 3, 6}}}, 0]

{{{{0, 0, 2}, 0}}, {{{0, 0, 3}, 0}}, {{{1, 2, 3}, 0}}, {{{1, 3, 3}, 0}},
 {{{2, 2, 4}, 0}}, {{{0, 0, 2}, 0}, {{0, 0, 2}, 0}}, {{{0, 0, 3}, 0}, {{0, 0, 2}, 0}},
 {{{0, 0, 3}, 0}, {{0, 0, 3}, 0}}, {{{1, 2, 3}, 0}, {{0, 0, 2}, 0}},
 {{{1, 2, 3}, 0}, {{0, 0, 3}, 0}}, {{{1, 3, 3}, 0}, {{0, 0, 2}, 0}}, {{{1, 3, 3}, 0}, {{0, 0, 3}, 0}},
 {{{2, 2, 4}, 0}, {{0, 0, 2}, 0}}, {{{0, 0, 2}, 0}, {{0, 0, 2}, 0}, {{0, 0, 2}, 0}}, {{{0, 0, 0}, 0}}}
```

```
WrapDTypeEnhancement[enh_] := Module[{ branchPositions, length, wrappedEnh },
   length = Length[enh];
    branchPositions = Table[If[Depth[enh[[i]]] > 3, 1, 0], {i, 1, length}];
    wrappedEnh = Table[If[branchPositions[[i]] == 0,
                             {enh[[i]]}
                             ,
                             enh[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have the same depth.*);
   Return[wrappedEnh];
  ];

WrapDTypeMonOnQuiver[monodromyOnQuiver_] := Module[{ branchPositions, length, wrappedMon},
   length = Length[monodromyOnQuiver];
    branchPositions = Table[If[Depth[monodromyOnQuiver[[i]]] > 1, 1, 0], {i, 1, length}];
    wrappedMon = Table[If[branchPositions[[i]] == 0,
                             {monodromyOnQuiver[[i]]}
                             ,
                             monodromyOnQuiver[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have length 1 rather than 0.*);
   Return[wrappedMon];
  ];

WrapDTypeQuiver[quiver_] := Module[{},
   Return@WrapDTypeMonOnQuiver[quiver];
  ];

(* UnwrapDTypeQuiver[wrappedQuiver_] *)


UnwrapDTypeQuiver[wrappedQuiver_] := Module[{i, j, length, newQuiver},
   length = Length[wrappedQuiver];
   newQuiver = Table [
      If[Length[wrappedQuiver[[i]]] == 1,
       wrappedQuiver[[i, 1]]
       ,
       wrappedQuiver[[i]]
      ]
      , {i, 1, length}];
   Return[newQuiver];
  ];

(* QuiverFromEnhancement[enh_] *)


QuiverFromEnhancement[enh_] := Module[{wrappedEnh, wrappedQuiver, unwrappedQuiver},
   If[Depth[enh] ≤ 4,
     Return[Flatten@ Take[enh, All, {1}]];
   ];
   wrappedEnh = WrapDTypeEnhancement[enh];
   wrappedQuiver = Take[wrappedEnh, All, All, {1}];
   wrappedQuiver  =
    Table[wrappedQuiver[[i, j, 1, 1]], {i, 1, Length[wrappedQuiver]}, {j, 1, Length[wrappedQuiver[[i]]]}];
   unwrappedQuiver = UnwrapDTypeQuiver[wrappedQuiver];
   Return[unwrappedQuiver];
  ];

WrapNonTrivialFGFiberVecOnDTypeQuiver[nontrivialFGFiberVec_] :=
  Module[{ branchPositions, length, wrappedVec},
   length = Length[nontrivialFGFiberVec];
    branchPositions = Table[If[Depth[nontrivialFGFiberVec[[i]]] > 1, 1, 0], {i, 1, length}];
    wrappedVec = Table[If[branchPositions[[i]] == 0,
                             {nontrivialFGFiberVec[[i]]}
                             ,
                             nontrivialFGFiberVec[[i]]
                            ]
                          , {i, 1, length}]
      (* We wrap the single curve entries so they have length 1 rather than 0.*);
   Return[wrappedVec];
  ];
```

```
OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGSOnLinearQuiver[
   aPostEnhWithResiduals_, mon_, toUseIntCont_: False, enh_: Null, monodromyOnQuiver_: Null] :=
  Module[{i, listOfTransverseFiberOptionListsForEachCurve, length},
   length = Length[mon];
   If[toUseIntCont,
    listOfTransverseFiberOptionListsForEachCurve =
          Table[TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[
                  {aPostEnhWithResiduals[[i]]}, mon[[i]], True, i, enh , monodromyOnQuiver
                   ], {i, 1, length}];
    ,
    listOfTransverseFiberOptionListsForEachCurve =
          Table[TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[
                  {aPostEnhWithResiduals[[i]]}, mon[[i]]
                   ], {i, 1, length}] ;
   ];
   Return[listOfTransverseFiberOptionListsForEachCurve];
  ];

OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGSDType[
   aPostEnhWithResiduals_, mon_, toUseIntCont_, enh_, monodromyOnQuiver_] := Module[
   {i, j, listOfTransverseFiberOptionListsForEachCurve, length, wrappedAPostEnh, wrappedMon, wrappedEnh},
   wrappedAPostEnh = WrapDTypeEnhancement[aPostEnhWithResiduals];
   wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
   length = Length[wrappedMon];
   If[toUseIntCont,
    listOfTransverseFiberOptionListsForEachCurve =
          Table[TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[
                  {wrappedAPostEnh[[i, j]]}, wrappedMon[[i, j]], True, {i, j}, enh , wrappedMon
                   ], {i, 1, length}, {j, 1, Length[wrappedMon[[i]] ]}];
    ,
    listOfTransverseFiberOptionListsForEachCurve =
          Table[TransverseFiberCollectionsMeetingPairRestrictionsAndResidualsForCurveGS[
                  {wrappedAPostEnh[[i, j]]}, wrappedMon[[i, j]]
                   ], {i, 1, length}, {j, 1, Length[wrappedMon[[i]]]}];
   ];
   Return[listOfTransverseFiberOptionListsForEachCurve];
  ];

OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGS[aPostEnhWithResiduals_,
   mon_, toUseIntCont_: False, enh_: Null, monodromyOnQuiver_: Null] := Module[{},
   If[TrueQ[enh == Null] || TrueQ[Depth[enh] ≤ 4],
    Return[ OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGSOnLinearQuiver[
       aPostEnhWithResiduals, mon, toUseIntCont, enh, monodromyOnQuiver]];
   ];
   Return[OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGSDType[
     aPostEnhWithResiduals, mon, toUseIntCont, enh, monodromyOnQuiver]];

  ];

FiberCollectionAlgPairsForCurveFiberCollection[fiberCollection_] := Module[{length, fiberAlgPairs},
   length = Length[fiberCollection];
   If[length == 0, Return[{}];];
   fiberAlgPairs = Table[{ fiberCollection[[i]], SemiSimpleAlgFromFiberSet[ fiberCollection[[i]]] },
                          {i, 1, length}];
   Return[fiberAlgPairs];
  ];

FiberCollectionAlgebrasOnCurvesTotalAlgebraTripleList[fiberCollection_] :=
  Module[{length, fiberAlgPairs},
   length = Length[fiberCollection];
   If[length == 0, Return[{}];];
   fiberAlgPairs = Table[
     { fiberCollection[[i]], SemiSimpleAlgebrasFromFiberSetsAndTotalAlgebra[ fiberCollection[[i]]] },
                          {i, 1, length}];
   Return[fiberAlgPairs];
  ];

FiberCollectionAlgebrasOnCurvesTotalAlgebraTripleListDTypes[fiberCollection_] :=
  Module[{i, j, length, fiberAlgPairs},
   length = Length[fiberCollection];
   If[length == 0, Return[{}];];
   fiberAlgPairs = Table[{ fiberCollection[[i]],
      SemiSimpleAlgebrasFromFiberSetsAndTotalAlgebraDTypes[ fiberCollection[[i]]] },
                          {i, 1, length}];
   Return[fiberAlgPairs];
  ];
```

```
RelMaxAlgsOptionsFiberTuplesFromOptionsForFiberCollectionList[optionsForFiberCollectionList_] :=
  Module[{i, j, length, fiberCollectionsOnCurve, fiberAlgTuplesOnCurve, algOptionsOnCurve,
    relMaxAlgsOnCurve, numRelMaxAlgsOnCurve, collectionsOfFibersOnCurveForGivenAlg,
    listOfEachCurveAlgOptions, listOfEachCurveRelMaxAlgOptions, totalAlgRelMaxes, alg,
    fibersGivingChosenAlg, tuplesOfRelMaxAlgAndFiberCollectionsRealizingThatAlgOnCurve,
    listOfRelMaxAlgFiberTuplesOptionsOnQuiver, fiberAlgTuplesDroppingEmpties,
    algsForPossibleQuiverGS, relMaxAlgsForQuiverGS, numFiberCollectionForCurveGS,
    relMaxAlgOptionsOnEachCurve, algebrasPossible, numCurvesWithNontrivialSummandToGS},
    length = Length[optionsForFiberCollectionList] (* Length of quiver *);
    listOfRelMaxAlgFiberTuplesOptionsOnQuiver = Flatten[Reap[
        For[i = 1, i ≤ length, i ++,
            fiberCollectionsOnCurve = optionsForFiberCollectionList[[i]];
            numFiberCollectionForCurveGS = Length[fiberCollectionsOnCurve];
            If[numFiberCollectionForCurveGS == 0, Sow[{}]; Continue[]; ];
            fiberAlgTuplesOnCurve =
                  FiberCollectionAlgPairsForCurveFiberCollection[fiberCollectionsOnCurve];
            algOptionsOnCurve = Flatten[Drop[fiberAlgTuplesOnCurve, 0, 1], 1];
            relMaxAlgsOnCurve = RelMaxAlgsQuicker[algOptionsOnCurve];
            numRelMaxAlgsOnCurve = Length[relMaxAlgsOnCurve];
            If[numRelMaxAlgsOnCurve == 0, Sow[{}]; Continue[]; ];
            tuplesOfRelMaxAlgAndFiberCollectionsRealizingThatAlgOnCurve = Flatten[Reap[
                For[j = 1, j ≤ numRelMaxAlgsOnCurve, j ++,
                    alg = relMaxAlgsOnCurve[[j]];
                    fibersGivingChosenAlg = Select[fiberAlgTuplesOnCurve, TrueQ[#[[2]] == alg] &];
                    Sow[{alg, fibersGivingChosenAlg}];
                ];
            ][[2]], 1];
            Sow[tuplesOfRelMaxAlgAndFiberCollectionsRealizingThatAlgOnCurve];
        ];
      ][[2]], 1];

   fiberAlgTuplesDroppingEmpties = Select[listOfRelMaxAlgFiberTuplesOptionsOnQuiver, (Length[#] > 0) &];
      numCurvesWithNontrivialSummandToGS = Length[fiberAlgTuplesDroppingEmpties];
      If[numCurvesWithNontrivialSummandToGS == 0, Return[{}];];
      relMaxAlgOptionsOnEachCurve =
    Table[ DeleteDuplicates[Drop[fiberAlgTuplesDroppingEmpties[[i]], 0, -1]],
                                          {i, 1, numCurvesWithNontrivialSummandToGS}];
      relMaxAlgOptionsOnEachCurve = Table[ Flatten[relMaxAlgOptionsOnEachCurve[[i, j]], 1],
                                          {i, 1, numCurvesWithNontrivialSummandToGS},
                                          {j, 1, Length[relMaxAlgOptionsOnEachCurve[[i]]]}];
      algebrasPossible = AllAlgSumsPossible[relMaxAlgOptionsOnEachCurve];
      relMaxAlgsForQuiverGS = RelMaxAlgsQuicker[algebrasPossible];
      Return[{relMaxAlgsForQuiverGS, listOfRelMaxAlgFiberTuplesOptionsOnQuiver}];
  ];

TotalNumberOfNontrivialTransverseFGFibersAsVecLinearQuiverOnly[enhWithResiduals_] :=
  Module[{i, fiberCollectionWithMonodromyListIncludingQuiverTransverses,
    length, enh, nontrivialTransverseFGQiverFibersVec,
    nontrivialFGQiverTransverseFibersCountVec, totalNontrivialTransverseFGFiberCountVec},
    length = Length[enhWithResiduals];
    enh = Drop[enhWithResiduals, 0, 1]
         (* Drop the self intersections for later the NumberOfNontrivialFGFibers method.*);
    nontrivialTransverseFGQiverFibersVec =
          Table[If[length == 1 , {},
                  If[i == 1, {enh[[i + 1]]},
                    If[i == length, {enh[[i - 1]]},
                      {enh[[i - 1]], enh[[i + 1]]}
                     ]
                   ]
                 ]
                , {i, 1, length}];
   nontrivialFGQiverTransverseFibersCountVec  =
          NumberOfNontrivialFGFibers /@ nontrivialTransverseFGQiverFibersVec;
   totalNontrivialTransverseFGFiberCountVec =
          nontrivialFGQiverTransverseFibersCountVec;
   Return[totalNontrivialTransverseFGFiberCountVec];
  ];
```

```
TotalNumberOfNontrivialTransverseFGFibersAsVecDTypesQuiverOnly[enhWithResiduals_] :=
  Module[{i, j, branchPositions, length, enh, backboneEnh, backboneFGVec, totalFGVec},
    length = Length[enhWithResiduals];
    branchPositions = Table[If[Depth[enhWithResiduals[[i]]] > 3, 1, 0], {i, 1, length}];
    backboneEnh = Table[If[branchPositions[[i]] == 0,
                          enhWithResiduals[[i]]
                          ,
                          enhWithResiduals[[i, 1]]
                         ]
                        , {i, 1, length}];
   backboneFGVec = TotalNumberOfNontrivialTransverseFGFibersAsVecLinearQuiverOnly[backboneEnh];
   totalFGVec = Table[ If[branchPositions[[i]] == 0,
                          backboneFGVec[[i]]
                         ,
                          Table[If[j > 1,

         TotalNumberOfNontrivialTransverseFGFibersAsVecLinearQuiverOnly[enhWithResiduals[[i]] ][[j]]
                                  ,
                                  backboneFGVec[[i]] +

          TotalNumberOfNontrivialTransverseFGFibersAsVecLinearQuiverOnly[enhWithResiduals[[i]] ][[j]]
                                  ]
                                 , {j, 1, Length[enhWithResiduals[[i]]]}]
                         ]

                       , {i, 1, length}];
   Return[totalFGVec];
  ];

TotalNumberOfNontrivialTransverseFGFibersAsVec[enhWithResiduals_] := Module[{},
   If[Depth[enhWithResiduals] ≤ 4,
      Return[TotalNumberOfNontrivialTransverseFGFibersAsVecLinearQuiverOnly[enhWithResiduals]];
   ];
    Return[TotalNumberOfNontrivialTransverseFGFibersAsVecDTypesQuiverOnly[enhWithResiduals]];
  ];

(* Not used: *)

ResidualsAfterIntContIncludingGSFibers[optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, nontrivialFGFiberCountVec, aPostEnhWithResiduals,
    contributionsVector, quiverLength, remainingResidualsVector},
   quiverLength = Length[enhWithResiduals];
   aPostEnhWithResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
     enhWithResiduals, monodromyOnQuiver, nontrivialFGFiberCountVec];
   contributionsVector = Table[
                           NonCompactCollectionIntersectionContribution[
                                {aPostEnhWithResiduals[[i]]}, optionForFibersToGS[[i]], i,
                                              enhWithResiduals, monodromyOnQuiver]
                             , {i, 1, quiverLength}];
   If[Length[Select[Flatten[contributionsVector], (# < 0) &]] > 0,
         Return[ConstantArray[{-1, -1, -1}, quiverLength]];];
   remainingResidualsVector =
         Table[enhWithResiduals[[i, 3]] - contributionsVector[[i]], {i, 1, quiverLength}];
   If[Length[Select[Flatten[remainingResidualsVector], (# < 0) &]] > 0,
        Return[ConstantArray[{-1, -1, -1}, quiverLength]];];
   Return[remainingResidualsVector];
  ];
```

```
FiberCollectionListGSIsConsistentWithEnhIntContLinearQuiver[optionForFibersToGS_,
   enhWithResiduals_, monodromyOnQuiver_] := Module[{i, j, nontrivialFGFiberCountVec,
    aPostEnhWithResiduals, contributionsVector, quiverLength, remainingResidualsVector,
    residualsVector, notAPostResidualsVector, totalAvailableVanishingsVector, bareResiduals},
   quiverLength = Length[monodromyOnQuiver];
   nontrivialFGFiberCountVec = TotalNumberOfNontrivialTransverseFGFibersAsVec[
             enhWithResiduals];
   aPostEnhWithResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
            enhWithResiduals, monodromyOnQuiver, nontrivialFGFiberCountVec];
   bareResiduals = TotalAPrioriVanishingsAvailable[enhWithResiduals];
   If[! ResidualsArePositive[aPostEnhWithResiduals], Return[False];];
   contributionsVector = Table[
                  NonCompactCollectionIntersectionContribution[
                     {aPostEnhWithResiduals[[i]]}, optionForFibersToGS[[i]], i,
                         enhWithResiduals, monodromyOnQuiver,
                            optionForFibersToGS, nontrivialFGFiberCountVec]
                          , {i, 1, quiverLength}];
   totalAvailableVanishingsVector = bareResiduals;
   remainingResidualsVector = totalAvailableVanishingsVector - contributionsVector;
   If[Length[Select[Flatten[contributionsVector], (# < 0) &]] > 0, Return[False];];
   If[Length[Select[Flatten[remainingResidualsVector], (# < 0) &]] > 0, Return[False];];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithEnhIntContDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, nontrivialFGFiberCountVec, aPostEnhWithResiduals, contributionsRaggedArray,
    quiverLength, remainingResidualsRaggedArray, wrappedAPostEnh, totalAvailableVanishingsRaggedArray},
   quiverLength = Length[monodromyOnQuiver];
   nontrivialFGFiberCountVec =
    TotalNumberOfNontrivialTransverseFGFibersAsVecDTypesQuiverOnly[enhWithResiduals];
   aPostEnhWithResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
            enhWithResiduals, monodromyOnQuiver];
   wrappedAPostEnh = WrapDTypeEnhancement[aPostEnhWithResiduals];
   totalAvailableVanishingsRaggedArray =
    WrappedTotalAPrioriVanishingsAvailableDTypes[enhWithResiduals];
   (* Table[wrappedAPostEnh[[i,j,3]],{i,1,Length[wrappedAPostEnh]},
     {j,1,Length[wrappedAPostEnh[[i]] ]}]; *)
   If[Length[Select[Flatten[aPostEnhWithResiduals], (# < 0) &]] > 0, Return[False];];
   contributionsRaggedArray = Table[
                  NonCompactCollectionIntersectionContribution[
                     {wrappedAPostEnh[[i, j]]}, optionForFibersToGS[[i, j]], {i, j},
                         enhWithResiduals, monodromyOnQuiver,
                            optionForFibersToGS, nontrivialFGFiberCountVec]

                          , {i, 1, quiverLength}, {j, 1, Max[Length[monodromyOnQuiver[[i]]], 1]}];
   If[Length[Select[Flatten[contributionsRaggedArray], (# < 0) &]] > 0, Return[False];];
   remainingResidualsRaggedArray = totalAvailableVanishingsRaggedArray - contributionsRaggedArray;
   If[Length[Select[Flatten[remainingResidualsRaggedArray], (# < 0) &]] > 0, Return[False];];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithEnhIntCont[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] := Module[{},
   If[Depth[monodromyOnQuiver] > 2,
     Return@FiberCollectionListGSIsConsistentWithEnhIntContDTypes[
       optionForFibersToGS, enhWithResiduals, monodromyOnQuiver];
   ];
   Return@FiberCollectionListGSIsConsistentWithEnhIntContLinearQuiver[
     optionForFibersToGS, enhWithResiduals, monodromyOnQuiver];
  ];
```

```
PairsTransverseToCurvePassTripletRestrictions[quiverCurveEnh_, monOnCurve_, transverseFibers_] :=
  Module[{k, n, numberOfTransverseFibers, transverseFiber1, transverseFiber2,
    transverseEnh1, transverseEnh2, mon1, mon2, monodromyTriplet, triplet},
   numberOfTransverseFibers = Length[transverseFibers];
   If[(numberOfTransverseFibers ≤ 1),
    Return[True];
   ];
   For[k = 1, k ≤ numberOfTransverseFibers, k ++,
     transverseFiber1 = transverseFibers[[k]];
     transverseEnh1 = {{ {Null}, transverseFiber1[[1]], {Null} }};
     mon1 = transverseFiber1[[2]];
     For[n = 1, n ≤ numberOfTransverseFibers, n ++,
       If[TrueQ[k == n], Continue[];];
       transverseFiber2 = transverseFibers[[n]];
       transverseEnh2 = {{ {Null}, transverseFiber2[[1]], {Null} }};
       mon2 = transverseFiber2[[2]];
       monodromyTriplet = {mon1, monOnCurve, mon2};
       triplet = Join[transverseEnh1, quiverCurveEnh, transverseEnh2];
       If[(! (TrueQ[IsValidMonodromyAssignmentToTriplet[triplet, monodromyTriplet, False]])),
         Return[False];
        ];
    ];
   ];
   Return[True];
  ];

TripletsTransverseToCurvePassQuartetRestrictions[
   quiverCurveEnh_, monOnCurve_, transverseFibers_] := Module[
   {k, n, j, numberOfTransverseFibers, transverseFiber1, transverseFiber2, transverseEnh1, transverseEnh2,
    mon1, mon2, monodromyTriplet, triplet, transverseFiber3, transverseEnh3, mon3, quartet},
   numberOfTransverseFibers = Length[transverseFibers];
   If[(numberOfTransverseFibers ≤ 2),
    Return[True];
   ];
   For[k = 1, k ≤ numberOfTransverseFibers, k ++,
     transverseFiber1 = transverseFibers[[k]];
     transverseEnh1 = {{ {Null}, transverseFiber1[[1]], {Null} }};
     mon1 = transverseFiber1[[2]];
     For[n = 1, n ≤ numberOfTransverseFibers, n ++,
       If[k == n, Continue[];];
       transverseFiber2 = transverseFibers[[n]];
       transverseEnh2 = {{ {Null}, transverseFiber2[[1]], {Null} }};
       mon2 = transverseFiber2[[2]];
       For[j = 1, j ≤ numberOfTransverseFibers, j ++,
        If[j == n || j == k, Continue[];];
        transverseFiber3 = transverseFibers[[j]];
        transverseEnh3 = {{ {Null}, transverseFiber3[[1]], {Null} }};
        mon3 = transverseFiber3[[2]];
        quartet = Join[quiverCurveEnh, transverseEnh1, transverseEnh2, transverseEnh3];
        If[(! (TrueQ[IsValidMonodromyAssignmentToOrderedTransverseTriplet[
                 quiverCurveEnh, transverseEnh1, transverseEnh2, transverseEnh3,
                 monOnCurve, mon1, mon2, mon3]])),
          Return[False];
         ];
      ];
    ];
   ];
   Return[True];
  ];
```

```
FiberCollectionListGSIsConsistentWithTripletRestrictions[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
    pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
    nontrivialTransverseFGQiverFibers, tripletRestrictions},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    enh = Drop[Drop[enh, 0, 1], 0, -1];
    enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
    nontrivialTransverseFGQiverFibers =
          Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                 If[i == 1, {enhMonPairs[[i + 1]]},
                   If[i == quiverLength, {enhMonPairs[[i - 1]]},
                     {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                     ]
                   ]
                 ]
                , {i, 1, quiverLength}];
   fibersTransverseToCurves =
            Table[ Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                              , {i, 1, quiverLength}
                             ](* All transverse curves are now in the same data format,
                                   having combined compact and noncompact transverse curves *);
   For[j = 1, j ≤ quiverLength, j ++,
     If[! PairsTransverseToCurvePassTripletRestrictions[
           {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
       Return[False];
     ];
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithTripletRestrictionsDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
    pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
    tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
    wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
    topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
    fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, fibersAtXPos},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
    wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
    wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
    enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                          wrappedMon[[i, j]]
                         },
                 {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
    wrappedEnhWithRes = WrapDTypeEnhancement[enh];
    branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
    fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
        For[xpos = 1, xpos ≤ quiverLength, xpos ++,
          fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
               For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

                 If[ypos > 1,
                    leftNeighbor = {};
                    rightNeighbor = {};
                    bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
                    If[ypos < Length[wrappedMon[[xpos]]],
                       topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                     ,
                       topNeighbor =  {};
                     ];
                 ];
                 If[ypos == 1,
                   bottomNeighbor = {};
                   If[branchPositions[[xpos]] == 0,
                     topNeighbor = {};
                     ,
                     topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                  ];
                  If[xpos == quiverLength,
                    rightNeighbor = {};
                    leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                  ];
                  If[xpos == 1,
                    leftNeighbor = {};
```

```
                    rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
                  ];
                  If[1 < xpos < quiverLength,
                    leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                    rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
                  ];
                 ];
                 fibersTransverseToSingleCurveToSow =
                  Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
                 Sow[fibersTransverseToSingleCurveToSow];
                ];
              ][[2]], 1];
          Sow[fibersTransverseToCurvesAtFixedXPosToSow];
         ];
      ][[2]], 1];

    nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
    fibersTransverseToCurves =
        Flatten[Reap[
              For[i = 1, i ≤ quiverLength, i ++,
                fibersAtXPos = Flatten[Reap[
                      For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                            Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                                ];
                       ];
              ][[2]], 1];
               Sow[fibersAtXPos];
              ];
      ][[2]], 1];

    For[i = 1, i ≤ quiverLength, i ++,
     For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
        If[! PairsTransverseToCurvePassTripletRestrictions[
                {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
         Return[False];
        ];
      ];
    ];
    Return[True];
   ];

FiberCollectionListGSIsConsistentWithQuartetRestrictions[
    optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
   Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
     pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
     nontrivialTransverseFGQiverFibers, tripletRestrictions},
     quiverLength = Length[monodromyOnQuiver];
     enh = enhWithResiduals;
     enh = Drop[Drop[enh, 0, 1], 0, -1];
     enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
     nontrivialTransverseFGQiverFibers =
             Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                     If[i == 1, {enhMonPairs[[i + 1]]},
                       If[i == quiverLength, {enhMonPairs[[i - 1]]},
                         {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                         ]
                       ]
                     ]
                   , {i, 1, quiverLength}];
    fibersTransverseToCurves =
                          Table[Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                                , {i, 1, quiverLength}
                              ]
       (* all transverse curves are now in the same data format, having combined compact
                                and noncompact transverse curves *);
    For[j = 1, j ≤ quiverLength, j ++,
      If[! TripletsTransverseToCurvePassQuartetRestrictions[
           {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
       Return[False];
      ];
    ];
    Return[True];
   ];

FiberCollectionListGSIsConsistentWithQuartetRestrictionsDTypes[
    optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
   Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
     pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
     tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
     wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
```

```
 topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
 fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, fibersAtXPos},
quiverLength = Length[monodromyOnQuiver];
enh = enhWithResiduals;
bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                        wrappedMon[[i, j]]
                       },
                {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
wrappedEnhWithRes = WrapDTypeEnhancement[enh];
branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
    For[xpos = 1, xpos ≤ quiverLength, xpos ++,
      fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
           For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

              If[ypos > 1,
                 leftNeighbor = {};
                 rightNeighbor = {};
                 bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
                 If[ypos < Length[wrappedMon[[xpos]]],
                    topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                  ,
                    topNeighbor =  {};
                  ];
              ];
               If[ypos == 1,
                 bottomNeighbor = {};
                 If[branchPositions[[xpos]] == 0,
                    topNeighbor = {};
                    ,
                    topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
               ];
               If[xpos == quiverLength,
                  rightNeighbor = {};
                   leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
               ];
               If[xpos == 1,
                  leftNeighbor = {};
                   rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
               ];
               If[1 < xpos < quiverLength,
                   leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                   rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
               ];
              ];
              fibersTransverseToSingleCurveToSow =
               Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
              Sow[fibersTransverseToSingleCurveToSow];
             ];
          ][[2]], 1];
      Sow[fibersTransverseToCurvesAtFixedXPosToSow];
     ];
   ][[2]], 1];

nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
fibersTransverseToCurves =
     Flatten[Reap[
          For[i = 1, i ≤ quiverLength, i ++,
            fibersAtXPos = Flatten[Reap[
                  For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                       Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                          ];
                  ];
          ][[2]], 1];
           Sow[fibersAtXPos];
          ];
   ][[2]], 1];

For[i = 1, i ≤ quiverLength, i ++,
 For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
     If[! TripletsTransverseToCurvePassQuartetRestrictions[
           {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
      Return[False];
     ];
   ];
];
Return[True];
```

```
FibersPassInstarGSCheck[m_, d_, quiverCurveEnh_, monOnCurve_, transverseFibers_] :=
  Module[{k, n, j, numberOfFibers, transverseFibersWithoutMon, algRank},
   numberOfFibers = Length[transverseFibers];
   If[(numberOfFibers ≤ 1),
    Return[True];
   ];
   transverseFibersWithoutMon = Flatten[Drop[transverseFibers, 0, -1] , 1];
   transverseFibersWithoutMon = Select[transverseFibersWithoutMon, ! ((#[[2]] == 1 && #[[3]] == 2)
                                 || (#[[3]] == 1)) &](* Drop the type I_1's and II's*);
   numberOfFibers = Length[transverseFibersWithoutMon];
   algRank = 0;
   For[k = 1, k ≤ numberOfFibers, k ++,
     algRank = algRank + Floor[transverseFibersWithoutMon[[k, 3]] / 2];
   ];
   If[d == 7,
     If[ monOnCurve == 0 && ( algRank > 5 - m ) , Return[False];]
    (* We have a larger field theory GS with additional sp1
                                                                 summand for m=1,
    2 which we could write as + 4-m *);
     If[ monOnCurve == 1 && ( algRank > 6 - m ) , Return[False];]
    (* We have a field theory GS with sp1 summand for m=1,2.
                                                                we could write as +
     Ceiling[Max[0,3-m]/2] *);
   ];
   If[d == 8,
     If[ monOnCurve == 0 && ( algRank > 7 - m ) , Return[False];]
    (*We have a field theory GS with sp1 summand for m=1,2.
                                                               we
     could write as  Ceiling[Max[0,2-m]/2] *);
     If[ monOnCurve == 1 && ( algRank > 8 - m ) , Return[False];]
    (*We have a field theory sp1 summand for m=1,2.
                                                                we could write as +
     Ceiling[Max[0,2-m]/2] *);
   ];
   If[d == 9,
     If[ monOnCurve == 0 && algRank > 9 - 4 , Return[False];](* Only m=2,m=4 are possible and
               BMM gives the same sp(5) GS max for both cases. *);
     If[ monOnCurve == 1 &&  algRank > 10 - m , Return[False];]
    (*Only m=4 is possible and sp(5) is the resulting GS max. *);
   ];
   If[d ≥ 10,
     If[ monOnCurve == 0 && algRank > (2 (d - 6)) - 1 , Return[False];];
     If[ monOnCurve == 1 &&  algRank > (2 (d - 6)) , Return[False];];
   ];

   Return[True];
  ];
```

```
FiberCollectionListGSIsConsistentWithInstarInCheck[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, a, b, d, m, quiverLength, enh, enhMonPairs, fibersTransverseToCurves,
    quiverCurve, pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
    nontrivialTransverseFGQiverFibers, tripletRestrictions},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    enh = Drop[Drop[enh, 0, 1], 0, -1];
    enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
    nontrivialTransverseFGQiverFibers =
          Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                   If[i == 1, {enhMonPairs[[i + 1]]},
                     If[i == quiverLength, {enhMonPairs[[i - 1]]},
                       {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                       ]
                     ]
                   ]
                  , {i, 1, quiverLength}];
   fibersTransverseToCurves = Table[

      Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                                 , {i, 1, quiverLength}
                               ]
      (* all transverse curves are now in the same data format, having combined compact
                                    and noncompact transverse curves *);
   For[j = 1, j ≤ quiverLength, j ++,
      a = enhWithResiduals[[j, 2, 1]];
      b = enhWithResiduals[[j, 2, 2]];
      d = enhWithResiduals[[j, 2, 3]];
      If[! (a == 2 && b == 3 && d ≥ 7), Continue[];](* We'll do a check only if curve is type Instar*);
      m = enhWithResiduals[[j, 1, 1]];
      If[! FibersPassInstarGSCheck[m, d,
        {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
       Return[False];
      ];
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithInstarInCheckDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
    pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
    tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
    wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
    topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
    fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, a, b, d, fibersAtXPos},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
    wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
    wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
    enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                          wrappedMon[[i, j]]
                         },
                  {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
    wrappedEnhWithRes = WrapDTypeEnhancement[enh];
    branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
    fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
        For[xpos = 1, xpos ≤ quiverLength, xpos ++,
          fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
               For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

                 If[ypos > 1,
                    leftNeighbor = {};
                    rightNeighbor = {};
                    bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
                    If[ypos < Length[wrappedMon[[xpos]]],
                       topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                      ,
                       topNeighbor =  {};
                      ];
                 ];
                 If[ypos == 1,
                   bottomNeighbor = {};
                   If[branchPositions[[xpos]] == 0,
                     topNeighbor = {};
                     ,
                     topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
```

```
                ];
                If[xpos == quiverLength,
                  rightNeighbor = {};
                   leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                ];
                If[xpos == 1,
                  leftNeighbor = {};
                   rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
                ];
                If[1 < xpos < quiverLength,
                   leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                   rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
                ];
               ];
               fibersTransverseToSingleCurveToSow =
                Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
               Sow[fibersTransverseToSingleCurveToSow];
              ];
            ][[2]], 1];
         Sow[fibersTransverseToCurvesAtFixedXPosToSow];
        ];
      ][[2]], 1];

   nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
   fibersTransverseToCurves =
        Flatten[Reap[
              For[i = 1, i ≤ quiverLength, i ++,
               fibersAtXPos = Flatten[Reap[
                    For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                         Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                            ];
                    ];
            ][[2]], 1];
             Sow[fibersAtXPos];
            ];
      ][[2]], 1];
   For[i = 1, i ≤ quiverLength, i ++,
    For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
        a = wrappedEnhWithRes[[i, j, 2, 1]];
        b = wrappedEnhWithRes[[i, j, 2, 2]];
        d = wrappedEnhWithRes[[i, j, 2, 3]];
        If[! (a == 2 && b == 3 && d ≥ 7), Continue[];](* We'll do a check only if curve is type Instar*);
        If[! TripletsTransverseToCurvePassQuartetRestrictions[
             {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
        Return[False];
      ];
     ];
   ];
   Return[True];
  ];

FibersPassInGSCheck[m_, d_, quiverCurveEnh_, monOnCurve_, transverseFibers_] :=
  Module[{k, n, j, nForInstar, numberOfFibers, transverseFibersWithoutMon,
    totalTransverseInDelta, theInstarFibers, theInstarFibersWithMonInfo,
    numberOfTransverseInstar, monValuesMaxedAtOne, totalMonCount},
   numberOfFibers = Length[transverseFibers];
   If[m == 2, Return[True]; ](* For m==2, the counts are taken care of by residuals. *);
   If[(numberOfFibers ≤ 1),
    Return[True];
   ];
   theInstarFibersWithMonInfo =
    Select[transverseFibers, #[[1, 1]] ≥ 2 && #[[1, 2]] ≥ 3 && #[[1, 3]] ≥ 6 &];
   monValuesMaxedAtOne = Table[
                         If[  theInstarFibersWithMonInfo[[k, 1, 3]] == 6,
                                If[theInstarFibersWithMonInfo[[k, 2]] == 2, 1, 0]
                               ,
                                theInstarFibersWithMonInfo[[k, 2]]
                          ]
                      , {k, 1, Length[theInstarFibersWithMonInfo]}];
   totalMonCount = If[Length[monValuesMaxedAtOne] > 0,
                     Total[monValuesMaxedAtOne]
                    ,
                     0
                  ];

   transverseFibersWithoutMon = Flatten[Drop[transverseFibers, 0, -1] , 1];
   transverseFibersWithoutMon = Select[transverseFibersWithoutMon, ! ((#[[2]] == 1 && #[[3]] == 2)
                               || (#[[3]] == 1)) &](* Drop the type I_1's and II's*);
   numberOfFibers = Length[transverseFibersWithoutMon];
```

```
 totalTransverseInDelta = 0;
 theInstarFibers = Select[transverseFibersWithoutMon,
                             TrueQ[#[[1]] ≥ 2 && #[[2]] ≥ 3 && #[[3]] ≥ 6 ] &];
 numberOfTransverseInstar = Length@ theInstarFibers;
 If[numberOfTransverseInstar == 0,
   nForInstar = 0;
  ,
   nForInstar = Sum[theInstarFibers[[k, 3]] - 6, {k, 1, Length[theInstarFibers]}];
 ];

 For[k = 1, k ≤ numberOfFibers, k ++,
   If[transverseFibersWithoutMon[[k, 1]] == 1
        && transverseFibersWithoutMon[[k, 2]] ≥ 2
          && transverseFibersWithoutMon[[k, 3]] == 3,
      totalTransverseInDelta = totalTransverseInDelta + 3;
      Continue[];
    ](* Type III fibers count 3 towards the total. *);
   If[transverseFibersWithoutMon[[k, 1]] ≥ 2
        && transverseFibersWithoutMon[[k, 2]] == 2
          && transverseFibersWithoutMon[[k, 3]] == 4,
      totalTransverseInDelta = totalTransverseInDelta + 4;
      Continue[];
    ](* Type IV fibers count 3 towards the total. *);
   If[! TrueQ[Drop[transverseFibersWithoutMon[[k]], -1] == {0, 0}], Continue[]; ];
   totalTransverseInDelta = totalTransverseInDelta + transverseFibersWithoutMon[[k, 3]];
 ](* We've totalled the I_n delta orders which are transverse.
        One might also add in the II contributions
       and I_1 contributions after checking these also
    obey the \Deltah restrictions in the BMM p.19-22 discussion.*);
 If[numberOfTransverseInstar == 0,
  If[d ≥ 3 && Mod[d, 2] ≠ 0,
    If[ monOnCurve == 1 && totalTransverseInDelta > 8 + d  , Return[False];];
    If[ monOnCurve == 0 && totalTransverseInDelta > 6 + d , Return[False];];
  ];
  If[d ≥ 2 && Mod[d, 2] == 0,
    If[ monOnCurve == 1 && totalTransverseInDelta > 8 + d + KroneckerDelta[d, 6] , Return[False];];
    If[ monOnCurve == 0 && totalTransverseInDelta > 8 + d , Return[False];];
  ];
 ];
 If[numberOfTransverseInstar > 0,
  If[d ≥ 2 && Mod[d, 2] == 0,
    If[ monOnCurve == 0 && totalTransverseInDelta > d + 4 - nForInstar , Return[False];];
  ];
  If[d ≥ 3 && Mod[d, 2] ≠ 0,
   If[numberOfTransverseInstar == 1,
     If[ monOnCurve == 0 && totalTransverseInDelta > d + 3 - nForInstar - totalMonCount ,  Return[False];];
   ];
   If[numberOfTransverseInstar == 2,
     If[ monOnCurve == 0 && totalTransverseInDelta > d - nForInstar - totalMonCount ,  Return[False];];
   ];
  ];

 ](* Note the Instar cases for the \su(n) along the base curve aren't possible.
       Those are ruled out in the pairs method earlier. *);
 Return[True];
];
```

```
FiberCollectionListGSIsConsistentWithInGSCheck[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, a, b, d, m, quiverLength, enh, enhMonPairs, fibersTransverseToCurves,
    quiverCurve, pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
    nontrivialTransverseFGQiverFibers, tripletRestrictions},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    enh = Drop[Drop[enh, 0, 1], 0, -1];
    enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
    nontrivialTransverseFGQiverFibers =
          Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                 If[i == 1, {enhMonPairs[[i + 1]]},
                   If[i == quiverLength, {enhMonPairs[[i - 1]]},
                     {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                     ]
                   ]
                 ]
                , {i, 1, quiverLength}];
   fibersTransverseToCurves = Table[

     Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                              , {i, 1, quiverLength}
                            ]
     (* all transverse curves are now in the same data format, having combined compact
                                  and noncompact transverse curves *);
   For[j = 1, j ≤ quiverLength, j ++,
     a = enhWithResiduals[[j, 2, 1]];
     b = enhWithResiduals[[j, 2, 2]];
     d = enhWithResiduals[[j, 2, 3]];
     If[! (a == 0 && b == 0 && d ≥ 2), Continue[];](* We'll do a check only if curve is type I_n *);
     m = enhWithResiduals[[j, 1, 1]];
     If[! FibersPassInGSCheck[m, d,
       {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
      Return[False];
     ];
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithInGSCheckDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
    pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
    tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
    wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
    topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
    fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, a, b, d, fibersAtXPos},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
    wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
    wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
    enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                          wrappedMon[[i, j]]
                         },
                 {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
    wrappedEnhWithRes = WrapDTypeEnhancement[enh];
    branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
    fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
        For[xpos = 1, xpos ≤ quiverLength, xpos ++,
          fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
               For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

                 If[ypos > 1,
                   leftNeighbor = {};
                   rightNeighbor = {};
                   bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
                   If[ypos < Length[wrappedMon[[xpos]]],
                     topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                    ,
                     topNeighbor =  {};
                    ];
                 ];
                 If[ypos == 1,
                   bottomNeighbor = {};
                   If[branchPositions[[xpos]] == 0,
                     topNeighbor = {};
                     ,
                     topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
```

```
            ];
            If[xpos == quiverLength,
              rightNeighbor = {};
               leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
            ];
            If[xpos == 1,
              leftNeighbor = {};
               rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
            ];
            If[1 < xpos < quiverLength,
               leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
               rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
            ];
           ];
           fibersTransverseToSingleCurveToSow =
            Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
           Sow[fibersTransverseToSingleCurveToSow];
          ];
        ][[2]], 1];
     Sow[fibersTransverseToCurvesAtFixedXPosToSow];
    ];
  ][[2]], 1];

 nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
 fibersTransverseToCurves =
     Flatten[Reap[
          For[i = 1, i ≤ quiverLength, i ++,
            fibersAtXPos = Flatten[Reap[
                  For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                       Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                           ];
                   ];
          ][[2]], 1];
           Sow[fibersAtXPos];
          ];
  ][[2]], 1];
 For[i = 1, i ≤ quiverLength, i ++,
  For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
     a =  wrappedEnhWithRes [[i, j, 2, 1]];
     b =  wrappedEnhWithRes [[i, j, 2, 2]];
     d =  wrappedEnhWithRes [[i, j, 2, 3]];
     If[! (a == 0 && b == 0 && d ≥ 2), Continue[];](* We'll do a check only if curve is type Instar*);
     If[! FibersPassInGSCheck[
          {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
      Return[False];
    ];
   ];
 ];
 Return[True];
];
```

```
FibersPassIzerostarGSCheck[m_, quiverCurveEnh_, monOnCurve_, transverseFibers_] :=
  Module[{k, n, j, numberOfFibers, transverseFibersWithoutMon, fiberAlgRank, totAlgRank},
   numberOfFibers = Length[transverseFibers];
   If[(numberOfFibers == 1),
    Return[True];
   ];
   transverseFibersWithoutMon = Flatten[Drop[transverseFibers, 0, -1] , 1];
   transverseFibersWithoutMon = Select[transverseFibersWithoutMon, ! ((#[[2]] == 1 && #[[3]] == 2)
                                || (#[[3]] == 1)) &](* Drop the type I_1's and II's*);
   numberOfFibers = Length[transverseFibersWithoutMon];
   totAlgRank = 0;
   For[k = 1, k ≤ numberOfFibers, k ++,
     If[transverseFibersWithoutMon[[k, 1]] ≥ 2 && transverseFibersWithoutMon[[k, 2]] == 2 ,
     fiberAlgRank = 1;
      ,
     fiberAlgRank =  Floor[transverseFibersWithoutMon[[k, 3]] / 2];
    ](* type IV must be type IV{ns} (or we have nonminimal intersection)
     and has rank one. Other types we can use floor(d/2) as the rank *) ;
     totAlgRank = totAlgRank + fiberAlgRank;
   ](* Now the table 6.1 constraints for possibly many transverse fibers *);
   If[monOnCurve == 0,
     If[ m == 3 && (totAlgRank > 1), Return[False];];
     If[ m == 2 && (totAlgRank > 4) , Return[False];];
     If[ m == 1 && (totAlgRank > 7) , Return[False];];
   ];
   If[monOnCurve == 1,
     If[ m == 3 && ( totAlgRank > 2) , Return[False];];
     If[ m == 2 && ( totAlgRank > 5) , Return[False];];
     If[ m == 1 && ( totAlgRank > 8) , Return[False];];
   ];
   If[monOnCurve == 2,
     If[ m == 3 &&  ( totAlgRank > 3) , Return[False];];
     If[ m == 2 && ( totAlgRank > 6) , Return[False];];
     If[ m == 1 &&  ( totAlgRank > 9) , Return[False];];
   ];

   Return[True];
  ];

FiberCollectionListGSIsConsistentWithIzerostarInCheck[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, a, b, d, m, quiverLength, enh, enhMonPairs, fibersTransverseToCurves,
    quiverCurve, pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
    nontrivialTransverseFGQiverFibers, tripletRestrictions},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    enh = Drop[Drop[enh, 0, 1], 0, -1];
    enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
    nontrivialTransverseFGQiverFibers =
          Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                 If[i == 1, {enhMonPairs[[i + 1]]},
                   If[i == quiverLength, {enhMonPairs[[i - 1]]},
                     {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                     ]
                   ]
                 ]
                , {i, 1, quiverLength}];
   fibersTransverseToCurves = Table[

      Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                                 , {i, 1, quiverLength}
                              ]
      (* all transverse curves are now in the same data format, having combined compact
                                   and noncompact transverse curves *);
    For[j = 1, j ≤ quiverLength, j ++,
      a = enhWithResiduals[[j, 2, 1]];
      b = enhWithResiduals[[j, 2, 2]];
      d = enhWithResiduals[[j, 2, 3]];
      If[! (a ≥ 2 && b ≥ 3 && d == 6), Continue[];](* We'll do a check only if curve is type Instar*);
      m = enhWithResiduals[[j, 1, 1]];
      If[! FibersPassIzerostarGSCheck[m,
        {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
       Return[False];
      ];
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithIzerostarInCheckDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
```

```
 pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
 tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
 wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
 topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
 fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, a, b, d, fibersAtXPos},
quiverLength = Length[monodromyOnQuiver];
enh = enhWithResiduals;
bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                     wrappedMon[[i, j]]
                    },
          {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
wrappedEnhWithRes = WrapDTypeEnhancement[enh];
branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
   For[xpos = 1, xpos ≤ quiverLength, xpos ++,
     fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
         For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

           If[ypos > 1,
             leftNeighbor = {};
             rightNeighbor = {};
             bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
             If[ypos < Length[wrappedMon[[xpos]]],
               topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
              ,
               topNeighbor =  {};
              ];
           ];
            If[ypos == 1,
             bottomNeighbor = {};
             If[branchPositions[[xpos]] == 0,
              topNeighbor = {};
              ,
              topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
            ];
            If[xpos == quiverLength,
              rightNeighbor = {};
              leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
            ];
            If[xpos == 1,
              leftNeighbor = {};
              rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
            ];
            If[1 < xpos < quiverLength,
              leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
              rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
            ];
           ];
           fibersTransverseToSingleCurveToSow =
            Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
           Sow[fibersTransverseToSingleCurveToSow];
          ];
        ][[2]], 1];
     Sow[fibersTransverseToCurvesAtFixedXPosToSow];
    ];
  ][[2]], 1];

nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
fibersTransverseToCurves =
    Flatten[Reap[
        For[i = 1, i ≤ quiverLength, i ++,
          fibersAtXPos = Flatten[Reap[
              For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                  Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                     ];
              ];
        ][[2]], 1];
         Sow[fibersAtXPos];
        ];
  ][[2]], 1];
For[i = 1, i ≤ quiverLength, i ++,
 For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
    a =  wrappedEnhWithRes [[i, j, 2, 1]];
    b =  wrappedEnhWithRes [[i, j, 2, 2]];
    d =  wrappedEnhWithRes [[i, j, 2, 3]];
    If[! (a ≥ 2 && b ≥ 3 && d == 6), Continue[];](* We'll do a check only if curve is type Instar*);
    If[! FibersPassIzerostarGSCheck[
```

```
            {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
        Return[False];
       ];
      ];
    ];
    Return[True];
   ];

FibersPassIVnsGSCheck[m_, quiverCurveEnh_, monOnCurve_, transverseFibers_] :=
  Module[{k, n, j, numberOfFibers, transverseFibersWithoutMon, fiberAlgRank, totAlgRank},
   numberOfFibers = Length[transverseFibers];
   If[(numberOfFibers == 1),
    Return[True];
   ];
   transverseFibersWithoutMon = Flatten[Drop[transverseFibers, 0, -1] , 1];
   transverseFibersWithoutMon = Select[transverseFibersWithoutMon, (#[[2]] == 2)
                              || (#[[1]] == 0 && #[[2]] == 0 && #[[3]] ≥ 2) &]
     (* Keep the III, IV and I_n n≥ 2*);
   numberOfFibers = Length[transverseFibersWithoutMon];
   If[monOnCurve == 0,
      If[numberOfFibers ≥ 4, Return[False]; ](* This method is ruled out those configurations giving
        purely even \bt contributions to
      a type IV curve with monodromy not caught by the monodromy checks
       for transverse pairs and triplets,e.g. the transverse configurations
       3IV,I_2 and 3III,I_2.  *) ;
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithIVnsGSCheck[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, a, b, d, m, quiverLength, enh, enhMonPairs, fibersTransverseToCurves,
    quiverCurve, pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets,
    nontrivialTransverseFGQiverFibers, tripletRestrictions},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    enh = Drop[Drop[enh, 0, 1], 0, -1];
    enhMonPairs = Table[{enh[[i, 1]], monodromyOnQuiver[[i]]}, {i, 1, quiverLength}];
    nontrivialTransverseFGQiverFibers =
          Table[If[quiverLength == 1 , {{{0, 0, 0}, 0}},
                 If[i == 1, {enhMonPairs[[i + 1]]},
                   If[i == quiverLength, {enhMonPairs[[i - 1]]},
                     {enhMonPairs[[i - 1]], enhMonPairs[[i + 1]]}
                    ]
                  ]
                ]
               , {i, 1, quiverLength}];
   fibersTransverseToCurves = Table[

     Join[nontrivialTransverseFGQiverFibers[[i]], optionForFibersToGS[[i]] ]
                             , {i, 1, quiverLength}
                           ]
     (* all transverse curves are now in the same data format, having combined compact
                               and noncompact transverse curves *);
   For[j = 1, j ≤ quiverLength, j ++,
     a = enhWithResiduals[[j, 2, 1]];
     b = enhWithResiduals[[j, 2, 2]];
     d = enhWithResiduals[[j, 2, 3]];
     If[! (a ≥ 2 && b == 2 && d == 4), Continue[];](* We'll do a check only if curve is type Instar*);
     m = enhWithResiduals[[j, 1, 1]];
     If[! FibersPassIVnsGSCheck[m,
        {enhWithResiduals[[j]]}, monodromyOnQuiver[[j]], fibersTransverseToCurves[[j]] ],
      Return[False];
     ];
   ];
   Return[True];
  ];

FiberCollectionListGSIsConsistentWithIVnsGSCheckDTypes[
   optionForFibersToGS_, enhWithResiduals_, monodromyOnQuiver_] :=
  Module[{i, j, quiverLength, enh, enhMonPairs, fibersTransverseToCurves, quiverCurve,
    pairOfFibersTransverseToQuiverCurve, triplet, listOfTriplets, nontrivialTransverseFGQiverFibers,
    tripletRestrictions, wrappedTypesOnlyRaggedMat, wrappedMon, bareTypes,
    wrappedEnhWithRes, xpos, ypos, leftNeighbor, rightNeighbor, bottomNeighbor,
    topNeighbor, branchPositions, fibersOfQuiverTransverseToCurvesAtEachQuiverPosition,
    fibersTransverseToCurvesAtFixedXPosToSow, fibersTransverseToSingleCurveToSow, a, b, d, fibersAtXPos},
    quiverLength = Length[monodromyOnQuiver];
    enh = enhWithResiduals;
    bareTypes = EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[enh];
    wrappedTypesOnlyRaggedMat = WrapBareOrders[bareTypes];
```

```
 wrappedMon = WrapDTypeMonOnQuiver[monodromyOnQuiver];
 enhMonPairs = Table[{wrappedTypesOnlyRaggedMat[[i, j]],
                        wrappedMon[[i, j]]
                       },
               {i, 1, quiverLength}, {j, 1, Length[wrappedMon[[i]]]}];
 wrappedEnhWithRes = WrapDTypeEnhancement[enh];
 branchPositions = Table[If[Length[wrappedEnhWithRes[[i]]] > 1, 1, 0], {i, 1, quiverLength}];
 fibersOfQuiverTransverseToCurvesAtEachQuiverPosition = Flatten[Reap[
     For[xpos = 1, xpos ≤ quiverLength, xpos ++,
       fibersTransverseToCurvesAtFixedXPosToSow = Flatten[Reap[
            For[ypos = 1, ypos ≤ Length[wrappedEnhWithRes[[xpos]]], ypos ++,

              If[ypos > 1,
                 leftNeighbor = {};
                 rightNeighbor = {};
                 bottomNeighbor = {enhMonPairs[[xpos, ypos - 1]]};
                 If[ypos < Length[wrappedMon[[xpos]]],
                    topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
                  ,
                    topNeighbor =  {};
                  ];
              ];
              If[ypos == 1,
                bottomNeighbor = {};
                If[branchPositions[[xpos]] == 0,
                   topNeighbor = {};
                   ,
                   topNeighbor =  {enhMonPairs[[xpos, ypos + 1]]};
               ];
               If[xpos == quiverLength,
                 rightNeighbor = {};
                  leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
               ];
               If[xpos == 1,
                 leftNeighbor = {};
                  rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
               ];
               If[1 < xpos < quiverLength,
                  leftNeighbor = {enhMonPairs[[xpos - 1, ypos]]};
                  rightNeighbor = {enhMonPairs[[xpos + 1, ypos]]};
               ];
              ];
              fibersTransverseToSingleCurveToSow =
               Join[leftNeighbor, rightNeighbor, bottomNeighbor, topNeighbor];
              Sow[fibersTransverseToSingleCurveToSow];
             ];
           ][[2]], 1];
       Sow[fibersTransverseToCurvesAtFixedXPosToSow];
      ];
    ][[2]], 1];

 nontrivialTransverseFGQiverFibers = fibersOfQuiverTransverseToCurvesAtEachQuiverPosition;
 fibersTransverseToCurves =
      Flatten[Reap[
           For[i = 1, i ≤ quiverLength, i ++,
             fibersAtXPos = Flatten[Reap[
                  For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]] ], j ++,
                       Sow[ Join[nontrivialTransverseFGQiverFibers[[i, j]], optionForFibersToGS[[i, j]] ]
                          ];
                   ];
           ][[2]], 1];
            Sow[fibersAtXPos];
           ];
    ][[2]], 1];
 For[i = 1, i ≤ quiverLength, i ++,
  For[j = 1, j ≤ Length[wrappedEnhWithRes[[i]]], j ++,
     a =  wrappedEnhWithRes [[i, j, 2, 1]];
     b =  wrappedEnhWithRes [[i, j, 2, 2]];
     d =  wrappedEnhWithRes [[i, j, 2, 3]];
     If[! (a ≥ 2 && b == 2 && d == 4), Continue[];](* We'll do a check only if curve is type Instar*);
     If[! FibersPassIVnsGSCheck[
          {wrappedEnhWithRes[[i, j]]}, wrappedMon[[i, j]], fibersTransverseToCurves[[i, j]] ],
     Return[False];
    ];
   ];
 ];
 Return[True];
];
```

```
GroupByNumberOfTransverseGSCurves[fiberCollectionsList_] := Module[{i, returnList, maxNumCurves},
   maxNumCurves = 0;
   For[i = 1, i ≤ Length[fiberCollectionsList], i ++,
     If[Length[fiberCollectionsList [[i]]] > maxNumCurves,
      maxNumCurves = Length[fiberCollectionsList [[i]]];];
   ];
   returnList = Flatten[Reap[
       For[i = 1, i ≤ maxNumCurves, i ++,
           Sow[Select[fiberCollectionsList, (Length[#] == i) &]];
         ];
      ][[2]], 1];
   Select[returnList, ( Length[#] > 0) &];
   Return[returnList];
  ];

FiberSort[fiberList_] := Module[{empties, nonEmpties},
   If[TrueQ[Flatten[fiberList] == {}] || Total[Flatten[fiberList]] == 0, Return[{{{0, 0, 0}, 0}}];];
   nonEmpties = Select[fiberList, ! (TrueQ[# == {{0, 0, 0}, 0}] || TrueQ[# == {}]) &];
   empties = Select[fiberList, (TrueQ[# == {}] || TrueQ[# == {{0, 0, 0}, 0}]) &];
   If[Length[empties] > 0, empties = {{{0, 0, 0}, 0}};, empties = {};];
   Return[Join[empties, SortBy[nonEmpties, {#[[1, 3]] &, #[[1, 2]] &,
                                              #[[1, 1]] &, #[[2]] &}]]];
  ];

TotalDeltaOfGSFibers[fiberList_] := Module[{},
   Return@Total@Flatten@Drop[#, 0, 2] &@ Flatten[#, 1] &@Drop[#, 0, -1] &@Flatten[#, 1] & @ fiberList;
  ](* This is for linear quivers only. *);

TotalDeltaOfGSFibersOnDTypeQuiver[fiberList_] := Module[{},
   Return@Total@Flatten@Drop[#, 0, 2] &@ Flatten[#, 1] &@Drop[#, 0, -1] &@Flatten[#, 2] & @ fiberList;
  ];

FiberListsSimplify[fibersLists_] := Module[{i, list},
   (* list = SortBy[fibersLists, {TotalDeltaOfGSFibers[#] &}]; TODO: Reinstate. *)
   list = DeleteDuplicates[list];
   Return[list];
  ];

CombinedGSOptions[optionsOnFirstCurves1_, optionsOnCurve2_] := Module[{i, j, length1, length2, fiberSum},
   length1 = Length[optionsOnFirstCurves1];
   length2 = Length[optionsOnCurve2];
   fiberSum = Flatten[Table[
      Append[ optionsOnFirstCurves1[[i]], optionsOnCurve2[[j]]], {i, 1, length1}, {j, 1, length2}], 1];
   Return[fiberSum];
  ];

CombineGSOptionsAddingCurve[optionsOnFirstCurves_, optionsOnFinalCurve_] :=
  Module[{i, j, length1, length2, fiberSums},
   length1 = Length[optionsOnFirstCurves];
   length2 = Length[optionsOnFinalCurve];
   fiberSums = Table[Append[optionsOnFirstCurves[[i]], FiberSort[optionsOnFinalCurve[[j]]]],
     {i, 1, length1}, {j, 1, length2}];
   Return[Flatten[fiberSums, 1]];
  ];

OptionsListFromCurveAtATimeOptionsForGSFibers[listOfFiberOptionsOnEachCurve_] :=
  Module[{i, optionsThusFar, remainingOptions, length, sortedOptions},
   length = Length[listOfFiberOptionsOnEachCurve];
   optionsThusFar = listOfFiberOptionsOnEachCurve[[1]];
   optionsThusFar = Table[{optionsThusFar[[i]]}, {i, 1, Length[optionsThusFar]}];
   For[i = 2, i ≤ Length[listOfFiberOptionsOnEachCurve], i ++,
     optionsThusFar = CombinedGSOptions[optionsThusFar, listOfFiberOptionsOnEachCurve[[i]]];
   ];
   optionsThusFar = DeleteDuplicates[optionsThusFar];
   (* optionsThusFar=SortBy[optionsThusFar,{TotalDeltaOfGSFibers[#]&}]; TODO: Reinstate. *)
   Return[optionsThusFar] ;
  ];
```

```
OptionsListFromCurveAtATimeOptionsForGSFibersDTypes[listOfFiberOptionsOnEachCurve_] :=
  Module[{i, j, k, optionsThusFar, optionsOnNextCol, remainingOptions, length, sortedOptions},
   length = Length[listOfFiberOptionsOnEachCurve];
   optionsThusFar = listOfFiberOptionsOnEachCurve[[1, 1]];
   optionsThusFar = Table[{{optionsThusFar[[k]]}}, {k, 1, Length[optionsThusFar]}];
   For[i = 2, i ≤ Length[listOfFiberOptionsOnEachCurve], i ++,
     optionsOnNextCol =  Table[{listOfFiberOptionsOnEachCurve[[i, 1, k]]},
                              {k, 1, Length[listOfFiberOptionsOnEachCurve[[i, 1]]]}];
     For[j = 2, j ≤ Length[listOfFiberOptionsOnEachCurve[[i]]], j ++,
      optionsOnNextCol = CombinedGSOptions[optionsOnNextCol, listOfFiberOptionsOnEachCurve[[i, j]] ];
    ](* We are again here using that the branching can't happen at the first site. **) ;
    optionsThusFar = CombinedGSOptions[optionsThusFar, optionsOnNextCol ];
   ];
   optionsThusFar = DeleteDuplicates[optionsThusFar];
   optionsThusFar = SortBy[optionsThusFar, {TotalDeltaOfGSFibersOnDTypeQuiver[#] &}];
   Return[optionsThusFar] ;
  ];

ConvertEmptyOptionsToTrivialFiberSet[quiver_, optionsForFibersToGS_] := Module[{k, j, optionsToReturn},
   If[Depth[quiver] == 2,
    optionsToReturn = Table[If[TrueQ[optionsForFibersToGS[[k]] == {}],
                                         {{{{0, 0, 0}, 0}}}
                                         ,
                                         optionsForFibersToGS[[k]]
                                        ]
                                       , {k, 1, Length[optionsForFibersToGS]}];
    Return[optionsToReturn];
    ,
    optionsToReturn = Table[If[TrueQ[optionsForFibersToGS[[k, j]] == {}],
                                          {{{{0, 0, 0}, 0}}}
                                          ,
                                          optionsForFibersToGS[[k, j]]
                                         ]
                                        , {k, 1, Length[optionsForFibersToGS]},
       {j, 1, Length[optionsForFibersToGS[[k]] ]}](* TODO: Delete this duplicate code.*) ;
   ];
   Return[optionsToReturn];
  ];

PairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS[enh_,
   monodromyOnQuiver_, toUseIntCont_: True, toDoAPostIntersectionContCheck_: True,
   toDoTripletRestrictionsCheck_: True, toDoQuartetRestrictionsCheck_: True, toDoInstarInCheck_: True,
   toDoIzerostarInCheck_: True, toDoInGSCheck_: True, toDoIVnsGSCheck_: True] :=
  Module[{enhWithAPostResiduals, isDType, quiver, optionsForFibersToGS, fiberOptions,
    fiberAlgTuplesOnQuiver, algOptionsOnQuiverForGivenEnhancement,
    relMaxAlgsOnQuiverForGivenEnhancement, maximalAlgebraInducingFiberAlgTuples,
    pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS, enhWithResiduals},
       If[Length[enh[[1]]] ≤ 2,
         enhWithResiduals = ComputeResidualVanishings@enh;
         ,
         enhWithResiduals = enh;
        ];
        quiver = QuiverFromEnhancement[enhWithResiduals];
        If[Depth[quiver] > 2,
             isDType = True;, isDType = False;
        ];
       enhWithAPostResiduals =  ComputeAPosterioriResidualsConsideringMonodromy[
                                              enhWithResiduals, monodromyOnQuiver];
       If[Length[Select[Flatten[enhWithAPostResiduals], (# ≤ -1) &]] > 0,
          Return[{}];
        ];
       If[toUseIntCont,
          optionsForFibersToGS =
              OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGS[
                                enhWithAPostResiduals, monodromyOnQuiver, True,
                                                 enhWithResiduals, monodromyOnQuiver];
          If[! isDType,
           optionsForFibersToGS = ConvertEmptyOptionsToTrivialFiberSet[quiver, optionsForFibersToGS];
           ];
          If[isDType,
            fiberOptions = OptionsListFromCurveAtATimeOptionsForGSFibersDTypes[optionsForFibersToGS]
                           (* For DTypes there is an extra level of depth here. *) ;
            ,
           fiberOptions = OptionsListFromCurveAtATimeOptionsForGSFibers[optionsForFibersToGS] ;
           ];
           If[toDoAPostIntersectionContCheck,
                  If[Length[fiberOptions] > 0,
                    If[isDType,
                        fiberOptions =
```

```
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithEnhIntContDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
        fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithEnhIntContLinearQuiver[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoTripletRestrictionsCheck && Length[fiberOptions] > 0,
    If[isDType,
        fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithTripletRestrictionsDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
        fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithTripletRestrictions[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoQuartetRestrictionsCheck && Length[fiberOptions] > 0,
    If[isDType,
        fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithQuartetRestrictionsDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
        fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithQuartetRestrictions[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoInstarInCheck && Length[fiberOptions] > 0,
    If[isDType,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithInstarInCheckDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithInstarInCheck[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoIzerostarInCheck && Length[fiberOptions] > 0,
    If[isDType,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithIzerostarInCheckDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithIzerostarInCheck[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoInGSCheck && Length[fiberOptions] > 0,
    If[isDType,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithInGSCheckDTypes[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
        ,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithInGSCheck[
                        #, enhWithResiduals, monodromyOnQuiver]) &];
    ];
 ];
If[toDoIVnsGSCheck && Length[fiberOptions] > 0,
    If[isDType,
    fiberOptions =
            Select[fiberOptions,
                (FiberCollectionListGSIsConsistentWithIVnsGSCheckDTypes[
```

```
                                #, enhWithResiduals, monodromyOnQuiver]) &];
                      ,
                  fiberOptions =
                        Select[fiberOptions,
                             (FiberCollectionListGSIsConsistentWithIVnsGSCheck[
                                        #, enhWithResiduals, monodromyOnQuiver]) &];
                    ];
                ];
              If[Length[fiberOptions] == 0, Continue[];](* Triplet checks have then banned the enh
                                                  enhancement with monodromy itself. *);
              If[isDType,
                       fiberAlgTuplesOnQuiver =
                           FiberCollectionAlgebrasOnCurvesTotalAlgebraTripleListDTypes[fiberOptions];
                      ,
                       fiberAlgTuplesOnQuiver =
                           FiberCollectionAlgebrasOnCurvesTotalAlgebraTripleList[fiberOptions];
              ];
              If[isDType,

    algOptionsOnQuiverForGivenEnhancement = Flatten[Drop[fiberAlgTuplesOnQuiver, 0, 1, 1], 2](*
                     There is something not quite right with the indices
      here. The previous call should be reviewed: FiberColl..*);
                ,

    algOptionsOnQuiverForGivenEnhancement = Flatten[Drop[fiberAlgTuplesOnQuiver, 0, 1, 1], 2];
                ](*TODO recheck whether we really should only use
    one case. Also recheck the AType case. *);
              relMaxAlgsOnQuiverForGivenEnhancement = RelMaxAlgsQuicker[
                                                    algOptionsOnQuiverForGivenEnhancement];
              If[isDType,
               maximalAlgebraInducingFiberAlgTuples = Select[fiberAlgTuplesOnQuiver,
                     (MemberQ[relMaxAlgsOnQuiverForGivenEnhancement, #[[2, 2]]] &)];
                ,
               maximalAlgebraInducingFiberAlgTuples = Select[fiberAlgTuplesOnQuiver,
                     (MemberQ[relMaxAlgsOnQuiverForGivenEnhancement, #[[2, 2]]] &)];
                ];
              pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS =
                       maximalAlgebraInducingFiberAlgTuples;
            ,
             optionsForFibersToGS = GroupByNumberOfTransverseGSCurves[optionsForFibersToGS];
             optionsForFibersToGS = Select[optionsForFibersToGS, Length[#] > 0 &];
             optionsForFibersToGS = Flatten[optionsForFibersToGS, 1];
             pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS =
                   RelMaxAlgsOptionsFiberTuplesFromOptionsForFiberCollectionList[
                                                         optionsForFibersToGS];
          ];
        ,
           optionsForFibersToGS =
              OptionsForFiberCollectionListGivingEnhancedQuiverWithMonodromyOptionsOfGSOnLinearQuiver[
                                         enhWithAPostResiduals, monodromyOnQuiver];
           pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS =
                 RelMaxAlgsOptionsFiberTuplesFromOptionsForFiberCollectionList[
                                                        optionsForFibersToGS];
      ];
     Return[ pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS];
];
```

```
WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[
   quiver_, toUseIntCont_: False, toDoAPostIntersectionContCheck_: False,
   toDoTripletRestrictionsCheck_: True, toDoQuartetRestrictionsCheck_: True,
   toDoInstarInCheck_: True, toDoIzerostarInCheck_: True, toDoInGSCheck_: True] := Module[
   {i, j, k, enhMonOptionsPairsIncludingNoValidMons, enhMonOptionsPairs, numEnhs, enhMonodromyOptionsPair,
    enh, enhWithResiduals, numMonOptions, monodromyOnQuiver, enhWithAPostResiduals,
    optionsForFibersToGS, fiberOptions, pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS,
    gsListByMonForGivenEnh, listByEnhancementOfQuiverOptionsForEachMonOptionGS, enhWithNoMonViable,
    fiberAlgTuplesOnQuiver, algOptionsOnQuiverForGivenEnhancement, relMaxAlgsOnQuiverForGivenEnhancement,
    maximalAlgebraInducingFibers, maximalAlgebraInducingFiberAlgTuples, isDType},
   If[Depth[quiver] > 2,
     isDType = True;, isDType = False;
   ];
   enhMonOptionsPairsIncludingNoValidMons = ListEnhancementMonodromyPairsOnQuiverUsingSewing[quiver];
   enhMonOptionsPairs = Select[enhMonOptionsPairsIncludingNoValidMons,
                          ! (TrueQ[#[[2]] == {{-1}}] || TrueQ[#[[2]] == {{-2}}]) &];
   enhWithNoMonViable = Select[enhMonOptionsPairsIncludingNoValidMons,
                          (TrueQ[#[[2]] == {{-1}}] || TrueQ[#[[2]] == {{-2}}]) &];
   enhWithNoMonViable = Flatten[Drop[enhWithNoMonViable, 0, -1], 1];
   enhWithNoMonViable =
    Table[{enhWithNoMonViable[[i]], "no valid monodromy"}, {i, 1, Length[enhWithNoMonViable]}];
   numEnhs = Length[enhMonOptionsPairs];
   listByEnhancementOfQuiverOptionsForEachMonOptionGS = Flatten[Reap[
       For[i = 1, i ≤ numEnhs, i ++,
            enhMonodromyOptionsPair = enhMonOptionsPairs[[i]];
            enh = enhMonodromyOptionsPair [[1]];
            enhWithResiduals = ComputeResidualVanishings[enh];
            If[Length[Select[Flatten[enhWithResiduals], (# ≤ -1) &]] > 0, Continue[];];
            numMonOptions = Length[enhMonOptionsPairs[[i, 2]]];
            If[numMonOptions == 0, Continue[];];
            gsListByMonForGivenEnh = Flatten[Reap[
                For[j = 1, j ≤ numMonOptions, j ++,
                   monodromyOnQuiver = enhMonodromyOptionsPair[[2, j]];
                   enhWithAPostResiduals = ComputeAPosterioriResidualsConsideringMonodromy[
                                              enhWithResiduals, monodromyOnQuiver];
                   If[Length[Select[Flatten[enhWithAPostResiduals], (# ≤ -1) &]] > 0,
                     Continue[];
                   ];
                   pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS =
                       PairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS[
                         enhWithAPostResiduals, monodromyOnQuiver,
                         toUseIntCont , toDoAPostIntersectionContCheck ,
                         toDoTripletRestrictionsCheck , toDoQuartetRestrictionsCheck ,
                         toDoInstarInCheck , toDoIzerostarInCheck, toDoInGSCheck ];
                   Sow[{monodromyOnQuiver, enhWithAPostResiduals,
                  pairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS }];
                ];
              ][[2]], 1];
           Sow[{enhWithResiduals, gsListByMonForGivenEnh}];
        ];
      ][[2]], 1];
   Return[{listByEnhancementOfQuiverOptionsForEachMonOptionGS, enhWithNoMonViable}];
  ];

DetailedQuiverGS[quiver_] := Module[{},
   Return[
     WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[quiver, True, True, True, True, True, True, True]];
  ];
```

```
AlgOptionsGroupedByCommonEnhWithDifferentMonodromyForQuiverGS[quiver_, toUseIntCont_: False,
   toDoAPostIntersectionContCheck_: False, toDoTripletRestrictionsCheck_: True,
   toDoQuartetRestrictionsCheck_: True, toDoInstarInCheck_: True, toDoIzerostarInCheck_: True] :=
  Module[{i, j, k, optionsWithFibersByEnhAndChoiceOfMonodromy, algsOptions, numEnhs},
   optionsWithFibersByEnhAndChoiceOfMonodromy = WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[
      quiver, toUseIntCont, toDoAPostIntersectionContCheck, toDoTripletRestrictionsCheck,
      toDoQuartetRestrictionsCheck, toDoInstarInCheck, toDoIzerostarInCheck][[1]];
   numEnhs = Length[optionsWithFibersByEnhAndChoiceOfMonodromy];
   If[toDoAPostIntersectionContCheck,
    algsOptions = Table[
                     AlgListSimplify[
                       Table[
                         optionsWithFibersByEnhAndChoiceOfMonodromy[[i , 2, j, 3, k, 2, 2]],
                        {k, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3]]]} ]]
                         ,
                    {i, 1, numEnhs},
                    {j, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2]]]}
                     ](* replace the k,2 above with k,1 for the fiber set for the given algebra *);
    algsOptions = DeleteDuplicates[Flatten[algsOptions, 1]];
    algsOptions = Table[If[Length[algsOptions[[i]]] == 0,
                        algsOptions[[i]]
                        ,
                        Flatten[algsOptions[[i]], 1]
                       ],
                        {i, 1, Length[algsOptions]}];
    algsOptions = AlgListSimplify[algsOptions];
    ,
    algsOptions =  Table[If[Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3]]] > 0,
                                    optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3, 1]]
                                    ,
                                    {}
                               ]
                     , {i, 1, numEnhs},
                       {j, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2]]]}];
    algsOptions = Flatten[algsOptions, 1];
    algsOptions = Select[algsOptions, (Length[#] > 0) &];
    algsOptions = DeleteDuplicates[algsOptions];
    algsOptions = Table[AlgListSimplify[algsOptions[[i]]], {i, 1, Length[algsOptions]}];
    algsOptions = DeleteDuplicates[algsOptions];
   ];
   Return[algsOptions];
  ];

FormattedEnhancementAndMonodromyDistinctAlgebrasForGS[
   quiver_, toUseIntCont_: True, toDoAPostIntersectionContCheck_: True,
   toDoTripletRestrictionsCheck_: True, toDoQuartetRestrictionsCheck_: True,
   toDoInstarInCheck_: True, toDoIzerostarInCheck_: True] := Module[{},
   Return[FormatListOfSemiSimpleTogether@ AlgOptionsGroupedByCommonEnhWithDifferentMonodromyForQuiverGS[
       quiver, toUseIntCont, toDoAPostIntersectionContCheck, toDoTripletRestrictionsCheck,
       toDoQuartetRestrictionsCheck, toDoInstarInCheck, toDoIzerostarInCheck]];
  ];

DistinctQuiverGSAlgebras[quiver_] := Module[{},
   Return[AlgOptionsGroupedByCommonEnhWithDifferentMonodromyForQuiverGS[
      quiver, True, True, True, True, True, True]];
  ];

FormattedDistinctQuiverGSAlgebras[quiver_] := Module[{},
   Return[FormatListOfSemiSimpleTogether@ DistinctQuiverGSAlgebras@quiver];
  ];

RelMaxQuiverGSAlgebras[quiver_] := Module[{},
   Return[RelMaxAlgsQuicker@DistinctQuiverGSAlgebras@quiver];
  ];

FormattedRelMaxQuiverGSAlgebras[quiver_] := Module[{},
   Return[FormatListOfSemiSimpleTogether@ RelMaxQuiverGSAlgebras@quiver];
  ];

MaximalAlgsForAllEnhsQuiverGS[quiver_, toUseIntCont_: False, toDoAPostIntersectionContCheck_: False,
   toDoTripletRestrictionsCheck_: True, toDoQuartetRestrictionsCheck_: True, toDoInstarInCheck_: True,
   toDoIzerostarInCheck_: True] := Module[{optionsGroupedByCommonEnhDifferentMonodromy},
   optionsGroupedByCommonEnhDifferentMonodromy =
       AlgOptionsGroupedByCommonEnhWithDifferentMonodromyForQuiverGS[
     quiver, toUseIntCont, toDoAPostIntersectionContCheck, toDoTripletRestrictionsCheck,
     toDoQuartetRestrictionsCheck, toDoInstarInCheck, toDoIzerostarInCheck];
   If[Length[optionsGroupedByCommonEnhDifferentMonodromy] == 0, Return[{{{{0, 0}, 0}}}]; ];
   Return[RelMaxAlgsQuicker[AlgListSimplify[Flatten[optionsGroupedByCommonEnhDifferentMonodromy, 1]]]];
  ];
```

```
FormattedMaximalAlgsForAllEnhsQuiverGSConsideringIntersectionContributions[
   quiver_, toUseIntCont_: True, toDoAPostIntersectionContCheck_: True,
   toDoTripletRestrictionsCheck_: True, toDoQuartetRestrictionsCheck_: True,
   toDoInstarInCheck_: True, toDoIzerostarInCheck_: True] := Module[{},
   Return[FormatListOfSemiSimpleTogether@MaximalAlgsForAllEnhsQuiverGS[quiver,
       toUseIntCont, toDoAPostIntersectionContCheck, toDoTripletRestrictionsCheck,
       toDoQuartetRestrictionsCheck, toDoInstarInCheck, toDoIzerostarInCheck]];
  ];

GaugeGSPairs[quiver_] :=
  Module[{i, j, k, optionsWithFibersByEnhAndChoiceOfMonodromy, algsOptions, numEnhs},
   optionsWithFibersByEnhAndChoiceOfMonodromy =
    WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[quiver, True, True, True, True, True, True][[1]];
   numEnhs = Length[optionsWithFibersByEnhAndChoiceOfMonodromy];
   algsOptions = Table[
                  {GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[
                     EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                     optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 2]]],
                     optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 1]]]
                   ,
                    AlgListSimplify[
                      Table[
                        optionsWithFibersByEnhAndChoiceOfMonodromy[[i , 2, j, 3, k, 2, 2]],
                       {k, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3]]]}]
                    ]
                  }
                      ,
                  {i, 1, numEnhs},
                  {j, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2]]]}
                   ](* replace the k,2 above with k,1 for the fiber set for the given algebra *);
   algsOptions = Join @@ algsOptions;
   algsOptions = DeleteDuplicates[algsOptions];
   algsOptions = Table[
                     If[Length[algsOptions[[i, 2]]] == 0,
                          {algsOptions[[i, 1]], { { {{0, 0}, 0} } }}
                          ,
                          algsOptions[[i]]
                      ],
                     {i, 1, Length[algsOptions]}];

   Return[algsOptions];
  ];

FormattedGSAlgebrasInGaugeGSPairs[quiver_] := Module[{i, pairs, table},
   pairs = GaugeGSPairs@quiver;
   table = Table[{pairs[[i, 1]], FormatListOfSemiSimpleTogether@ pairs[[i, 2]]}, {i, 1, Length[pairs]}];
   Return[table];
  ];

(* The following includes those GS options possible in
 certain geometries giving restricted GS with same gauge assignments. *)

TableFormedQuiverGS[quiver_] := Module[{quiverTableForm, tableGaugeGSPair, pairs, table, gaugeGSTable},
   quiverTableForm = TableFormatQuiver[quiver];
   pairs = GaugeGSPairs@quiver;
   gaugeGSTable = Table[{TableFormFromGaugeAlgebra@pairs[[i, 1]],
      FormatListOfSemiSimpleTogether@ pairs[[i, 2]]}, {i, 1, Length[pairs]}];
   Return[{quiverTableForm, gaugeGSTable}];
  ];

ListGaugeGSOptionsFromGaugeGSPairs[gaugeGSPairs_, toDeleteReverseSymm_: False, quiver_: Null ] :=
  Module[{i, j, gaugesNoDups, gaugeAlgsNoDups, optionsForGivenGaugeToGS,
    gaugeAlg, listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg},
   gaugeAlgsNoDups = DeleteDuplicates@ Flatten[#, 1] &@Drop[#, 0, -1] &@gaugeGSPairs;
   If[toDeleteReverseSymm && TrueQ[quiver == Reverse[quiver]],
    gaugeAlgsNoDups = DeleteDuplicates[gaugeAlgsNoDups, #1 == Reverse[#2] &];
   ];
   listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg = Flatten[Reap[
       For[i = 1, i ≤ Length[gaugeAlgsNoDups], i++,
         gaugeAlg = gaugeAlgsNoDups[[i]];
         optionsForGivenGaugeToGS = Select[gaugeGSPairs, TrueQ[#[[1]] == gaugeAlg] & ];
          optionsForGivenGaugeToGS =
           AlgListSimplify@ (Flatten[#, 2] & @Drop[optionsForGivenGaugeToGS, 0, 1]);
         optionsForGivenGaugeToGS = RelMaxAlgsQuicker[optionsForGivenGaugeToGS];
         Sow[{gaugeAlg, optionsForGivenGaugeToGS}];
        ];
      ][[2]], 1];
   Return[listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg];
  ];
```

```
TableFormedMaxQuiverGS[quiver_, toDeleteReverseSymm_: True] :=
  Module[{i, quiverTableForm, tableGaugeGSPair, pairs, table, gaugeGSTable, maxGSPairs, gaugeAlgsNoDups},
   quiverTableForm = TableFormatQuiver[quiver];
   pairs = GaugeGSPairs@quiver;
   maxGSPairs = ListGaugeGSOptionsFromGaugeGSPairs[pairs, toDeleteReverseSymm, quiver];
   gaugeGSTable = Table[{TableFormFromGaugeAlgebra@(maxGSPairs[[i, 1]]),
      FormatListOfSemiSimpleTogether@ (maxGSPairs[[i, 2]])}, {i, 1, Length[maxGSPairs]}];
   Return[{quiverTableForm, gaugeGSTable}];
  ];

ListGaugeGSOptions[quiver_] := Module[{},
   Return[ListGaugeGSOptionsFromGaugeGSPairs[GaugeGSPairs[quiver]]];
  ];

ListGaugeGSOptionsFormatted[quiver_] := Module[{i, pairsList, algsGSOnly, listGSOptions, formattedList},
   pairsList = ListGaugeGSOptionsFromGaugeGSPairs[GaugeGSPairs[quiver]];
   formattedList = Table[
     {pairsList[[i, 1]], FormatListOfSemiSimpleTogether[pairsList[[i, 2]]]}, {i, 1, Length[pairsList]}];
   Return[formattedList];
  ];

ComputeGSForQuiverListFormatted[quiverList_] := Module[{i, gsTable, gsOptionsForQuiver},
   gsTable = Flatten[Reap[
       For[i = 1, i ≤ Length[quiverList], i++,
         Print[quiverList[[i]]];
         gsOptionsForQuiver = ListGaugeGSOptionsFormatted[quiverList[[i]] ];
         Print[{quiverList[[i]], gsOptionsForQuiver}];
         Sow[{quiverList[[i]], gsOptionsForQuiver }];
        ];
      ][[2]], 1];
   Return[gsTable];
  ];

GSFlatData[quiver_] :=
  Module[{optionsWithFibersByEnhAndChoiceOfMonodromy, dataToFlatten, emptiesDropped, flatData },
   optionsWithFibersByEnhAndChoiceOfMonodromy =
    WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[quiver, True, True, True, True, True, True][[1]];
   dataToFlatten = optionsWithFibersByEnhAndChoiceOfMonodromy;
   emptiesDropped = Select[Flatten[
                          Table[{dataToFlatten[[i, 1]],
                                 dataToFlatten[[i, 2, j, 1]] ,
                                  dataToFlatten[[i, 2, j, 3]]},
                     {i, 1, Length[dataToFlatten]},
                     {j, 1, Length[dataToFlatten[[i, 2]] ]}
                           ], 1],
                  Length[#[[3]]] > 0 &];

   flatData = Flatten[ Table[ {emptiesDropped[[i, 1]],
                             emptiesDropped[[i, 2]],
                              GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[
                                       EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                                          emptiesDropped[[i, 1]]                          ],
                                                      emptiesDropped[[i, 2]] ],
                                emptiesDropped[[i, 3, j, 1]],
                                 emptiesDropped[[i, 3, j, 2, 1]],
                                 emptiesDropped[[i, 3, j, 2, 2]]} ,
                     {i, 1, Length[emptiesDropped]},
                     {j, 1, Length[emptiesDropped[[ i, 3 ]]
                  ]}] , 1];
   Return[flatData];
  ];

GaugeGSPairsDetailed[quiver_] :=
  Module[{i, j, k, optionsWithFibersByEnhAndChoiceOfMonodromy, algsOptions, numEnhs, gaugeAlgsNoDups,
    listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg, gaugeAlg, optionsForGivenGaugeToGS,
    gaugeGSPairs, gaugeGSPairsWithCurveByCurveGS, pairsList, formattedList, dataToFlatten,
    emptiesDropped, flatOptionsList, flatData, summandAlgsGivingSomeRelMaxAlgebraForThisGauge,
    flatGaugeMatches, maxGSFlatGaugeMatches, fibersGivingMaxGS, matchingGivenGS,
    waysToGetThatGSFiberByFiber, fibersGivingMaxGSSorted, trivialGS, nonTrivialGS},

   optionsWithFibersByEnhAndChoiceOfMonodromy =
    WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[quiver, True, True, True, True, True, True][[1]];
   numEnhs = Length[optionsWithFibersByEnhAndChoiceOfMonodromy];
   algsOptions = Table[
                  {GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[
                     EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                     optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 2]]],
                     optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 1]]]
                   ,
```

```
                        AlgListSimplify[
                          Table[
                            optionsWithFibersByEnhAndChoiceOfMonodromy[[i , 2, j, 3, k, 2, 2]],
                           {k, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3]]]}]
                        ]
                    }
                        ,
                    {i, 1, numEnhs},
                    {j, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2]]]}
                    ](* replace the k,2 above with k,1 for the fiber set for the given algebra *);
algsOptions = Join @@ algsOptions;
algsOptions = DeleteDuplicates[algsOptions];
algsOptions = Table[
                    If[Length[algsOptions[[i, 2]]] == 0,
                         {algsOptions[[i, 1]], { { {{0, 0}, 0} } }}
                         ,
                         algsOptions[[i]]
                     ],
                    {i, 1, Length[algsOptions]}];
gaugeGSPairs = algsOptions;
gaugeAlgsNoDups = DeleteDuplicates[Flatten[Drop[gaugeGSPairs, 0, -1], 1]];
dataToFlatten = optionsWithFibersByEnhAndChoiceOfMonodromy;
emptiesDropped = Select[Flatten[Table[{dataToFlatten[[i, 1]],
                                       dataToFlatten[[i, 2, j, 1]],
                                        dataToFlatten[[i, 2, j, 3]]},
                                 {i, 1, Length[dataToFlatten]},
                                 {j, 1, Length[dataToFlatten[[i, 2]] ]}]
                              , 1],
                 Length[#[[3]]] > 0 &];
flatData = Flatten[ Table[ {emptiesDropped[[i, 1]],
                              emptiesDropped[[i, 2]],
                               GaugeAlgebrasFromEnhancementOnQuiverAndMonodromy[
                                       EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                                          emptiesDropped[[i, 1]]                         ],
                                                    emptiesDropped[[i, 2]] ],
                                 emptiesDropped[[i, 3, j, 1]],
                                  emptiesDropped[[i, 3, j, 2, 1]],
                                  emptiesDropped[[i, 3, j, 2, 2]]} ,
                    {i, 1, Length[emptiesDropped]},
                    {j, 1, Length[emptiesDropped[[ i, 3 ]]
                 ]}] , 1];


pairsList = Flatten[Reap[
     For[i = 1, i ≤ Length[gaugeAlgsNoDups], i ++,
       gaugeAlg = gaugeAlgsNoDups[[i]];
       optionsForGivenGaugeToGS = Select[gaugeGSPairs, TrueQ[#[[1]] == gaugeAlg] &];
       optionsForGivenGaugeToGS = AlgListSimplify @
                                   (Flatten[#, 2] & @
                                     (Drop[optionsForGivenGaugeToGS, 0, 1]));
       optionsForGivenGaugeToGS = RelMaxAlgsQuicker[optionsForGivenGaugeToGS];
       If[TrueQ[optionsForGivenGaugeToGS == {}], Sow[{gaugeAlg, {}}]; Continue[];];
       flatGaugeMatches = Select[ flatData,
                TrueQ[#[[3]] == gaugeAlg ] &];
       maxGSFlatGaugeMatches = Select[flatGaugeMatches,
                                    MemberQ[optionsForGivenGaugeToGS, #[[6]] ] &];
       fibersGivingMaxGSSorted = Flatten[Reap[
             For[j = 1, j ≤ Length[optionsForGivenGaugeToGS], j ++,
                matchingGivenGS =
                   Select[flatGaugeMatches, TrueQ[optionsForGivenGaugeToGS[[j]] == #[[6]]] &];
                waysToGetThatGSFiberByFiber =
                   DeleteDuplicates[ Table[matchingGivenGS [[k, 5]],
                                     {k, 1, Length[matchingGivenGS ]}]];
                Sow[waysToGetThatGSFiberByFiber];
             ];
          ][[2]], 1];

       (* The format of flat data element i is
        {1 enh, 2 mon, 3 gauge algs, 4 GS fibers, 5 GS algs, 6 total GS alg }
            One can replace indices here with [[j,4]] to retrieve
            Kodaira type data for relMaxGS options. *)

       Sow[{gaugeAlg, optionsForGivenGaugeToGS, fibersGivingMaxGSSorted}];
      ];
    ][[2]], 1];
trivialGS = Select[pairsList, Length[#[[2]]] == 0 &];
nonTrivialGS = Select[pairsList, Length[#[[2]]] > 0 &];

formattedList = Table[{nonTrivialGS[[i, 1]],
                         FormatListOfSemiSimpleTogether[nonTrivialGS[[i, 2]]],
```

```
                        Table[ FormatListOfSemiSimpleTogether /@ nonTrivialGS[[i, 3, j]]
                              , {j, 1, Length[nonTrivialGS[[i, 3]]]}
                              ]
                    }, {i, 1, Length[nonTrivialGS]}];
  Return[Join[trivialGS, formattedList]];
 ];

KodairaGaugePairsOnQuiverFromEnhancementAndMonodromyForTex[typesOnlyOnQuiver_, monOnQuiver_] :=
 Module[{i, texKodairaTypesAndAlgDataPairsOnQuiver, tableReplacingAlgDataWithAlgsForTex},
  texKodairaTypesAndAlgDataPairsOnQuiver =
   GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex[typesOnlyOnQuiver, monOnQuiver];
  tableReplacingAlgDataWithAlgsForTex = Table[
        {texKodairaTypesAndAlgDataPairsOnQuiver[[i, 1]],
             FormatSummandForTex@ ( texKodairaTypesAndAlgDataPairsOnQuiver[[i, 2]])
        }
      , {i, 1, Length[monOnQuiver]}];
  Return[tableReplacingAlgDataWithAlgsForTex];
 ];

KodairaGaugePairsOnLinearQuiverFromEnhancementAndMonodromyForTexPhysGauge[
  typesOnlyOnQuiver_, monOnQuiver_] :=
 Module[{i, texKodairaTypesAndAlgDataPairsOnQuiver, tableReplacingAlgDataWithAlgsForTex},
  texKodairaTypesAndAlgDataPairsOnQuiver =
   GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex[typesOnlyOnQuiver, monOnQuiver];
  tableReplacingAlgDataWithAlgsForTex = Table[
        {texKodairaTypesAndAlgDataPairsOnQuiver[[i, 1]],
             FormatSummandForTexPhysGauge@ ( texKodairaTypesAndAlgDataPairsOnQuiver[[i, 2]])
        }
      , {i, 1, Length[monOnQuiver]}];
  Return[tableReplacingAlgDataWithAlgsForTex];
 ];

KodairaGaugePairsOnDTypeQuiverFromEnhancementAndMonodromyForTexPhysGauge[
  typesOnlyOnQuiver_, monOnQuiver_] :=
 Module[{i, j, wrappedMon, texKodairaTypesAndAlgDataPairsOnQuiver, tableReplacingAlgDataWithAlgsForTex},
  texKodairaTypesAndAlgDataPairsOnQuiver =
   GaugeAlgebrasFromEnhancementOnDTypeQuiverAndMonodromyForTex[typesOnlyOnQuiver, monOnQuiver];
  wrappedMon = WrapDTypeMonOnQuiver[monOnQuiver];
  tableReplacingAlgDataWithAlgsForTex = Table[
        {texKodairaTypesAndAlgDataPairsOnQuiver[[i, j, 1]],
             FormatSummandForTexPhysGauge@ ( texKodairaTypesAndAlgDataPairsOnQuiver[[i, j, 2]])
        }
      , {i, 1, Length[wrappedMon]}, {j, 1, Length[wrappedMon[[i]]]}];
  Return[tableReplacingAlgDataWithAlgsForTex];
 ];

KodairaGaugePairsOnQuiverFromEnhancementAndMonodromyForTexPhysGauge[typesOnlyOnQuiver_, monOnQuiver_] :=
 Module[{i, texKodairaTypesAndAlgDataPairsOnQuiver, tableReplacingAlgDataWithAlgsForTex},
  If[Depth[monOnQuiver] ≤ 2,
   Return@KodairaGaugePairsOnLinearQuiverFromEnhancementAndMonodromyForTexPhysGauge[
      typesOnlyOnQuiver, monOnQuiver];
  ];
  Return@KodairaGaugePairsOnDTypeQuiverFromEnhancementAndMonodromyForTexPhysGauge[
    typesOnlyOnQuiver, monOnQuiver];
 ];

GaugeAlgsOnlyOnLinearQuiverFromEnhancementAndMonodromyForTexPhysGauge[
  typesOnlyOnQuiver_, monOnQuiver_] :=
 Module[{i, texKodairaTypesAndAlgDataPairsOnQuiver, tableReplacingAlgDataWithAlgsForTex},
  texKodairaTypesAndAlgDataPairsOnQuiver =
   GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex[typesOnlyOnQuiver, monOnQuiver];
  tableReplacingAlgDataWithAlgsForTex = Table[
            FormatSummandForTexPhysGauge@
            ( texKodairaTypesAndAlgDataPairsOnQuiver[[i, 2]])
      , {i, 1, Length[monOnQuiver]}];
  Return[tableReplacingAlgDataWithAlgsForTex];
 ];
```

```
GaugeAlgsOnlyOnDTypeQuiverFromEnhancementAndMonodromyForTexPhysGauge[typesOnlyOnQuiver_, monOnQuiver_] :=
  Module[{i, j, texKodairaTypesAndAlgDataPairsOnQuiver,
    tableReplacingAlgDataWithAlgsForTex, wrappedTypesOnlyOnQuiver, wrappedMon, mon, types},
   wrappedTypesOnlyOnQuiver = WrapBareOrders[typesOnlyOnQuiver];
   wrappedMon = WrapDTypeMonOnQuiver[monOnQuiver];
   If[Depth[typesOnlyOnQuiver] < 3,
     mon = wrappedMon;
     types = wrappedTypesOnlyOnQuiver;
     ,
     mon = monOnQuiver;
     types = typesOnlyOnQuiver;
   ];
   texKodairaTypesAndAlgDataPairsOnQuiver =
    GaugeAlgebrasFromEnhancementOnQuiverAndMonodromyForTex[types, mon];
   mon = wrappedMon;
   tableReplacingAlgDataWithAlgsForTex = Table[
              FormatSummandForTexPhysGauge@
              (texKodairaTypesAndAlgDataPairsOnQuiver[[i, j, 2]])
       , {i, 1, Length[mon]}, {j, 1, Length[mon[[i]]]}];
   Return[tableReplacingAlgDataWithAlgsForTex];
  ];

GaugeAlgsOnlyOnQuiverFromEnhancementAndMonodromyForTexPhysGauge[
   typesOnlyOnQuiver_, monOnQuiver_] := Module[{isDType},
   If[Depth[monOnQuiver] > 2, isDType = True, isDType = False;];
   If[! isDType,
    Return[GaugeAlgsOnlyOnLinearQuiverFromEnhancementAndMonodromyForTexPhysGauge[
       typesOnlyOnQuiver, monOnQuiver]];
   ];
   Return @GaugeAlgsOnlyOnDTypeQuiverFromEnhancementAndMonodromyForTexPhysGauge[
     typesOnlyOnQuiver, monOnQuiver];
  ];

WrapFlatDataGaugeAlgDType[gaugeAlg_] := Module[{i, j, table},
   Return@Table[{gaugeAlg[[i, j]]}, {i, 1, Length[gaugeAlg]}, {j, 1, Length[gaugeAlg[[i]]]}];
  ];

GaugeGSPairsDetailedForTex[quiver_] :=
  Module[{i, j, k, optionsWithFibersByEnhAndChoiceOfMonodromy, algsOptions, numEnhs,
    gaugeAlgsKodairaPairsNoDups, listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg,
    gaugeAlgKodairaPair, gaugeAlgOnly, gaugeAlg, optionsForGivenGaugeToGS, gaugeGSPairs,
    gaugeGSPairsWithCurveByCurveGS, pairsList, formattedList, dataToFlatten, emptiesDropped,
    flatOptionsList, flatData, summandAlgsGivingSomeRelMaxAlgebraForThisGauge, flatGaugeMatches,
    maxGSFlatGaugeMatches, fibersGivingMaxGS, matchingGivenGS, waysToGetThatGSFiberByFiber,
    fibersGivingMaxGSSorted, trivialGS, nonTrivialGS, pairsListSortedByGauge,
    gatheredByGauge, tableOfPairsOfGaugeMaxGSForAllKodairaTypesForGivenGauge,
    maxAlgsGivenGauge, gaugesOnly, pairsListByGaugeOnly, matchingGaugeGSLists, nonTrivialTable,
    nonTrivialKodairaPairsForThisGauge, togetherAgain, isDType, isSymmQuiver},
   If[Depth[quiver] > 2,
     isDType = True;
    ,
     isDType = False;
   ];
   If[quiver == Reverse[quiver],
    isSymmQuiver = True;
    ,
    isSymmQuiver = False;
   ];

   optionsWithFibersByEnhAndChoiceOfMonodromy =
       WithShowingEnhAndMonodromyGiveOptionsForQuiverGS[
           quiver, True, True, True, True, True, True][[1]];
   numEnhs = Length[optionsWithFibersByEnhAndChoiceOfMonodromy];
   algsOptions = Table[
                 {KodairaGaugePairsOnQuiverFromEnhancementAndMonodromyForTexPhysGauge[
                    EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                    optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 2]]],
                    optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 1]]]
                   ,
                    AlgListSimplify[
                      Table[
                         optionsWithFibersByEnhAndChoiceOfMonodromy[[i , 2, j, 3, k, 2, 2]],
                        {k, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2, j, 3]]]} ]
                    ]
                 }
                      ,
                 {i, 1, numEnhs},
                 {j, 1, Length[optionsWithFibersByEnhAndChoiceOfMonodromy[[i, 2]]]}
                  ](* One can replace the k,2 above with k,1 to access the
```

```
                        fiber set for the given algebra *);
algsOptions = Join @@ algsOptions;
algsOptions = DeleteDuplicates[algsOptions];
algsOptions = Table[
                    If[Length[algsOptions[[i, 2]]] == 0,
                        {algsOptions[[i, 1]], { { {{0, 0}, 0} } }}
                        ,
                        algsOptions[[i]]
                     ],
                    {i, 1, Length[algsOptions]}];
gaugeGSPairs = algsOptions;
gaugeAlgsKodairaPairsNoDups = DeleteDuplicates[Flatten[Drop[gaugeGSPairs, 0, -1], 1]];
If[isSymmQuiver,
   gaugeAlgsKodairaPairsNoDups = DeleteDuplicates[gaugeAlgsKodairaPairsNoDups, #1 == Reverse[#2] &];
];
dataToFlatten = optionsWithFibersByEnhAndChoiceOfMonodromy;
emptiesDropped = Select[
                   Flatten[
                     Table[{dataToFlatten[[i, 1]],
                              dataToFlatten[[i, 2, j, 1]] ,
                                dataToFlatten[[i, 2, j, 3]]},
                      {i, 1, Length[dataToFlatten]},
                       {j, 1, Length[dataToFlatten[[i, 2]] ]}],
                   1],
                 Length[#[[3]]] > 0 &];
flatData = Flatten[ Table[ {emptiesDropped[[i, 1]],
                             emptiesDropped[[i, 2]],
                             KodairaGaugePairsOnQuiverFromEnhancementAndMonodromyForTexPhysGauge[
                                      EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                                         emptiesDropped[[i, 1]]                            ],
                                                   emptiesDropped[[i, 2]] ],
                                emptiesDropped[[i, 3, j, 1]],
                                 emptiesDropped[[i, 3, j, 2, 1]],
                                  emptiesDropped[[i, 3, j, 2, 2]],
                             GaugeAlgsOnlyOnQuiverFromEnhancementAndMonodromyForTexPhysGauge[
                                      EnhancementWithSelfIntersectionsToTypesOnlyOnQuiver[
                                         emptiesDropped[[i, 1]]                            ],
                                                   emptiesDropped[[i, 2]] ]
                           } ,
                    {i, 1, Length[emptiesDropped]},
                    {j, 1, Length[emptiesDropped[[ i, 3 ]]
                 ]}] , 1];
pairsList = Flatten[Reap[
     For[i = 1, i ≤ Length[gaugeAlgsKodairaPairsNoDups], i ++,
       gaugeAlgKodairaPair = gaugeAlgsKodairaPairsNoDups[[i]];
       If[! isDType,
        gaugeAlgOnly = Flatten[Drop[gaugeAlgKodairaPair, 0, 1], 1];
        ,
        gaugeAlgOnly = Drop[gaugeAlgKodairaPair, 0, 0, 1];
       ];
       If[! isDType,
        optionsForGivenGaugeToGS = Select[gaugeGSPairs, TrueQ[#[[1]] == gaugeAlgKodairaPair] & ]
          (* TODO: recheck change from previous version working for A-
            Types where this read gaugeAlg rather than gaugeAlgOnly *) ;
        ,
        optionsForGivenGaugeToGS =
          Select[gaugeGSPairs, TrueQ[Flatten[#, 1] &@Drop[#, -1] & @# == gaugeAlgKodairaPair] & ];
       ];
       optionsForGivenGaugeToGS =
                  AlgListSimplify@ (Flatten[#, 2] & @ (Drop[optionsForGivenGaugeToGS, 0, 1]));
       optionsForGivenGaugeToGS = RelMaxAlgsQuicker[optionsForGivenGaugeToGS];
       If[TrueQ[optionsForGivenGaugeToGS == {}],
        Sow[{gaugeAlgOnly, gaugeAlgKodairaPair, {}}]; Continue[]; ];
       flatGaugeMatches = Select[ flatData,
                TrueQ[#[[3]] == gaugeAlgKodairaPair ] &];
       maxGSFlatGaugeMatches =
                   Select[flatGaugeMatches,
                     MemberQ[optionsForGivenGaugeToGS, #[[6]] ] &]
           (* These are the options giving the max GS with matching gauge and Kodaira type. *) ;
       fibersGivingMaxGSSorted = Flatten[Reap[
             For[j = 1, j ≤ Length[optionsForGivenGaugeToGS], j ++,
                matchingGivenGS =
                    Select[flatGaugeMatches,
                        TrueQ[optionsForGivenGaugeToGS[[j]] == #[[6]]] &];
                waysToGetThatGSFiberByFiber =
                        DeleteDuplicates[
                            Table[matchingGivenGS [[k, 5]],
                             {k, 1, Length[matchingGivenGS ]}]]];
               Sow[waysToGetThatGSFiberByFiber];
             ];
```

```
         ][[2]], 1];
       (* The format of flat data element i is
        {1 enh, 2 mon, 3 gauge algs with Kodaira types, 4 GS fibers,
         5 GS algs, 6 total GS alg, 7 raw gauge algebra without Kodaira types }
           so replace with [[j,4]] to get types for relMaxGS options. *)
       Sow[{gaugeAlgOnly, gaugeAlgKodairaPair, optionsForGivenGaugeToGS, fibersGivingMaxGSSorted}];
      ];
    ][[2]], 1];
pairsListSortedByGauge = SortBy[pairsList, {#[[1]] &}];
trivialGS = Select[pairsListSortedByGauge,
          Length[#[[3]]] == 0 &];
trivialGS = SortBy[trivialGS, {#[[1]] &}];
nonTrivialGS = Select[pairsListSortedByGauge,
          Length[#[[3]]] > 0 &];
nonTrivialGS = SortBy[nonTrivialGS, {#[[1]] &}];
If[! isDType,
 gaugesOnly = DeleteDuplicates@(Flatten[Take[nonTrivialGS, All, {1}], 1]);
 ,
 gaugesOnly = DeleteDuplicates@Flatten[#, 1] &@ Take[#, All, {1}] & @nonTrivialGS;
];
pairsListByGaugeOnly = Flatten[Reap[
    For[i = 1, i ≤ Length[gaugesOnly], i ++,
      gaugeAlg = gaugesOnly[[i]](* This is the raw gauge algebra without the Kodaira type *) ;
      If[! isDType,
       optionsForGivenGaugeToGS = Select[gaugeGSPairs,
                TrueQ[Flatten[Drop[#[[1]], 0, 1], 1] == gaugeAlg] & ];
       ,
       optionsForGivenGaugeToGS = Select[gaugeGSPairs,
          TrueQ[Drop[#, 0, 0, 1] & @Flatten[#, 1] &@Drop[#, -1] & @# == gaugeAlg] & ];
      ];
      optionsForGivenGaugeToGS =
              AlgListSimplify@ (Flatten[#, 2] & @ (Drop[optionsForGivenGaugeToGS, 0, 1]));
      optionsForGivenGaugeToGS = RelMaxAlgsQuicker[optionsForGivenGaugeToGS];
      If[TrueQ[optionsForGivenGaugeToGS == {}], Sow[{gaugeAlg, gaugeAlg, {}}]; Continue[]; ];
      If[! isDType,
       flatGaugeMatches = Select[ flatData,
              TrueQ[#[[7]] == gaugeAlg ] &];
       ,
       flatGaugeMatches = Select[ flatData,
              TrueQ[WrapFlatDataGaugeAlgDType[#[[7]]] == gaugeAlg ] &];
      ];
      maxGSFlatGaugeMatches =
              Select[flatGaugeMatches,
                MemberQ[optionsForGivenGaugeToGS, #[[6]] ] &];
      fibersGivingMaxGSSorted = Flatten[Reap[
           For[j = 1, j ≤ Length[optionsForGivenGaugeToGS], j ++,
             matchingGivenGS =
                Select[flatGaugeMatches,
                   TrueQ[optionsForGivenGaugeToGS[[j]] == #[[6]]] &];
             waysToGetThatGSFiberByFiber =
                   DeleteDuplicates[
                      Table[matchingGivenGS [[k, 5]],
                       {k, 1, Length[matchingGivenGS ]}]];
             Sow[waysToGetThatGSFiberByFiber];
           ];
         ][[2]], 1];
       (* The format of flat data element i is
        {1 enh, 2 mon, 3 gauge algs, 4 GS fibers, 5 GS algs, 6 total GS alg, 7 raw gauge alg}
           so replace with [[j,4]] to get types for relMaxGS options. *)
       Sow[{gaugeAlg, gaugeAlg, optionsForGivenGaugeToGS, fibersGivingMaxGSSorted}];
      ];
    ][[2]], 1];

pairsListByGaugeOnly = Select[pairsListByGaugeOnly, Length[#[[3]]] > 0 &];
nonTrivialTable = Flatten[Reap[
    For[i = 1, i ≤ Length[gaugesOnly], i ++,
         gaugeAlg = gaugesOnly[[i]];
         nonTrivialKodairaPairsForThisGauge =
            Select[nonTrivialGS, TrueQ[#[[1]] == gaugeAlg] &];
         If[Length[nonTrivialKodairaPairsForThisGauge] > 1,
            AppendTo[nonTrivialKodairaPairsForThisGauge, pairsListByGaugeOnly[[i]]];
         ];
         Sow[nonTrivialKodairaPairsForThisGauge];
      ];
    ][[2]], 2];

nonTrivialGS = nonTrivialTable;
togetherAgain = Join[trivialGS, nonTrivialGS];
nonTrivialGS = Select[togetherAgain, Length[#[[3]]] > 0 &];
trivialGS = Select[togetherAgain, Length[#[[3]]] == 0 &];
```

```
  trivialGS = Drop[trivialGS, 0, 1];
  nonTrivialGS = Drop[nonTrivialGS, 0, 1];
  formattedList = Table[{nonTrivialGS[[i, 1]],
                        FormatListOfSemiSimpleTogetherForTex[nonTrivialGS[[i, 2]]],
                            Table[ FormatListOfSemiSimpleTogetherForTex /@ nonTrivialGS[[i, 3, j]]
                                , {j, 1, Length[nonTrivialGS[[i, 3]]]}
                                ]
                        }, {i, 1, Length[nonTrivialGS]}];

  Return[Join[trivialGS, formattedList]];
 ];

GSPairsDetailedOnQuiverList[quiverList_] := Module[{i, gsTable, gsOptionsForQuiver},
  gsTable = Flatten[Reap[
      For[i = 1, i ≤ Length[quiverList], i++,
        Print[quiverList[[i]]];
        gsOptionsForQuiver = GaugeGSPairsDetailed[quiverList[[i]] ];
        Print[{quiverList[[i]], gsOptionsForQuiver}];
        Sow[{quiverList[[i]], gsOptionsForQuiver }];
       ];
     ][[2]], 1];
  Return[gsTable];
 ];

GSPairsDetailedOnQuiverListTableForm[quiverList_, toPrint_: True] :=
 Module[{i, j, k, m, gsTable, gsOptionsForQuiver, noEmpties, quiver, quiverLength},
  gsTable = Flatten[Reap[
      For[i = 1, i ≤ Length[quiverList], i++,
        quiver = quiverList[[i]];
        Print[quiver];
        quiverLength = Length [quiver];
        gsOptionsForQuiver = GaugeGSPairsDetailed[quiverList[[i]] ];
        noEmpties = Table[If[Length[gsOptionsForQuiver[[j, 2]]] == 0,
                              {Append[gsOptionsForQuiver[[j, 1]], "GS Tot:"],
            Join[ConstantArray[" ", quiverLength], {"0"}]}
                          ,
                              Join[ {Append[gsOptionsForQuiver[[j, 1]], "GS Tot:"]},
                                    Flatten[
                                          Table[Flatten[

               {gsOptionsForQuiver[[j, 3, k, m]],  gsOptionsForQuiver[[j, 2, k]] }
                                                      ]
                                           , {k, 1, Length[gsOptionsForQuiver[[j, 3]]]  ] }
                                           , {m, 1, Length[ gsOptionsForQuiver[[j, 3, k]]   ]}
                                             ]
                                       , 1]
                                  ]
                          ]
                    , {j, 1 , Length[gsOptionsForQuiver]}
                  ];
        Do[Print[TableForm[noEmpties[[j]]  ]  ]; , {j, 1, Length[noEmpties]}];
        Print[""];
        Sow[{quiver, gsOptionsForQuiver}];
       ];
     ][[2]], 1];
  If[toPrint, Return[gsTable]; ];
 ];
```

```
TableOfDataForLinearQuiver[gsOptionsForQuiver_, quiverLength_] := Module[{table, j, k, m},
    table = Table[If[Length[gsOptionsForQuiver[[j, 2]]] == 0,

       {Append[gsOptionsForQuiver[[j, 1]], "GS Tot:"], Join[ConstantArray[" ", quiverLength], {"0"}]}
                         ,
                                 Join[ {Append[gsOptionsForQuiver[[j, 1]], "GS Tot:"]},
                                         Flatten[
                                                Table[Flatten[
                                                         {gsOptionsForQuiver[[j, 3, k, m]],
                                                          gsOptionsForQuiver[[j, 2, k]]
                                                          }
                                                              ]
                                                 , {k, 1, Length[gsOptionsForQuiver[[j, 3]]  ] }
                                                 , {m, 1, Length[ gsOptionsForQuiver[[j, 3, k]]   ]}
                                                    ]
                                              , 1]
                                     ]
                          ]
                 , {j, 1 , Length[gsOptionsForQuiver]}
               ];
   Return[table];
  ];
```

```
WriteTableForLinearQuiverToFile[quiver_, fileStream_, noEmpties_] :=
  Module[{i, j, k, length, tableArgString, quiverAsTableEntry, lineOfTableData, tableEntry},
   length = Length[quiver];
   If[length < 8,
    WriteString[fileStream , "{ \\small\\setlength{\\tabcolsep}{3pt} "];
    ,
    If[length < 10,
      WriteString[fileStream , "{ \\footnotesize\\setlength{\\tabcolsep}{2pt} "];
      ,
      If[length < 12,
         WriteString[fileStream , "{ \\scriptsize\\setlength{\\tabcolsep}{1pt} "];
        ,
        WriteString[fileStream , "{ \\tiny\\setlength{\\tabcolsep}{1pt} "];
       ];
     ];
   ];
   tableArgString = StringJoin[{"\\begin{longtable} " <> "{" ,
                                StringJoin[ ConstantArray["c", length + 1 ]
               ], "}"}];
    WriteString[fileStream , tableArgString];
     quiverAsTableEntry = Flatten[Table[{ToString[quiver[[k]]], " & "}
                             , {k, 1, length}]];
     AppendTo[quiverAsTableEntry, "GS Total: " <> StringJoin["\\", "\\"] <> "\n  "];
     quiverAsTableEntry = StringJoin[quiverAsTableEntry];
     WriteString[fileStream, quiverAsTableEntry];


    For[i = 1, i ≤ Length[noEmpties], i ++,
      For[j = 1, j ≤ Length[noEmpties[[i]]], j ++,
        If[j == 1,
           lineOfTableData = noEmpties[[i, j]];
           lineOfTableData = Table[
                {
                       If[Depth [lineOfTableData[[k]]] < 2,
                          "(" <> lineOfTableData[[k]] <> ")"
                        ,
                            "(" <> lineOfTableData[[k, 1]] <> ","
                               <> lineOfTableData[[k, 2]] <>
                             ")"
                        ]
                , " & "}
                                 , {k, 1, Length[lineOfTableData] - 1}];
            AppendTo[lineOfTableData, " "];
            lineOfTableData = Flatten[lineOfTableData];
            AppendTo[lineOfTableData, StringJoin["\\", "\\"] <> "\n  "];
            tableEntry = StringJoin[lineOfTableData];
            WriteString[fileStream, tableEntry];
        ,
            lineOfTableData = noEmpties[[i, j]];
            lineOfTableData = Flatten[Table[
                        {" $ " <> ToString[lineOfTableData[[k]]] <> " $ ", " & "}
                                  , {k, 1, Length[lineOfTableData]}]];
            lineOfTableData = Drop[lineOfTableData, -1];
            AppendTo[lineOfTableData, StringJoin["\\", "\\"] <> "\n  "];
            tableEntry = StringJoin[lineOfTableData];
            WriteString[fileStream, tableEntry];
         ];
       ];
   ];
   WriteString[fileStream , "\\end{longtable} } \n"];
   Return[Null];
  ];

PadDataToMaxBranchLength[gsData_, quiver_] :=
  Module[{i, maxBranchLength, length, branchPositions, filledData, trans},
   length = Length[quiver];
   branchPositions = Table[If[Depth[quiver[[i]]] > 1, 1, 0], {i, 1, Length[quiver]}];
   maxBranchLength = Max[ Table[Length[quiver[[i]]], {i, 1, length}]];
   filledData = Table[PadRight[
                         gsData[[i]]
                        , maxBranchLength, ""], {i, 1, Length[gsData]}];
   trans = Transpose[filledData];
   Return[trans];
  ];

WriteTableForDTypeQuiverToFile[quiver_, fileStream_, noEmpties_] := Module[
   {i, j, k, n, length, tableArgString, quiverAsTableEntry, lineOfTableData, tableEntry, maxBranchLength,
    filledQuiver, branchPositions, transposedFilledQuiver, enhSubtable, gsSubtable, singleEntry},
   length = Length[quiver];
```

```
branchPositions = Table[If[Depth[quiver[[i]]] > 1, 1, 0], {i, 1, Length[quiver]}];
maxBranchLength = Max[ Table[Length[quiver[[i]]], {i, 1, length}]];
filledQuiver = Table[PadRight[
                        If[branchPositions[[i]] == 1,
                            quiver[[i]]
                             ,
                            {quiver[[i]]}
                          ]
                       , maxBranchLength, ""], {i, 1, length}];
transposedFilledQuiver = Transpose[filledQuiver];
If[length < 8,
 WriteString[fileStream , "{ \\small\\setlength{\\tabcolsep}{3pt} "];
 ,
 If[length < 10,
   WriteString[fileStream , "{ \\footnotesize\\setlength{\\tabcolsep}{2pt} "];
   ,
   If[length < 12,
      WriteString[fileStream , "{ \\scriptsize\\setlength{\\tabcolsep}{1pt} "];
      ,
      WriteString[fileStream , "{ \\tiny\\setlength{\\tabcolsep}{1pt} "];
     ];
  ];
];
tableArgString = StringJoin[{"\\begin{longtable} " <> "{" ,
                                  StringJoin[ ConstantArray["c", length + 1 ]
             ], "}"}];
WriteString[fileStream , tableArgString];

quiverAsTableEntry = Table[
                                StringJoin@Flatten@ Table[
                                  If[TrueQ[{k, j} == {1, length}],
                                    {ToString[transposedFilledQuiver[[k, j]]], " & ",
                                           "GS Total: " <> StringJoin["\\", "\\"] <> "\n  "}
                                    ,
                                    If[TrueQ[j == length],
                                        {ToString[transposedFilledQuiver[[k, j]]], " & ",
                                              StringJoin["\\", "\\"] <> "\n  "}
                                         ,
                                          {ToString[transposedFilledQuiver[[k, j]]], " & "}
                                      ]
                                    ]
                                  , {j, 1, length}], {k, 1, maxBranchLength}];
For[k = 1, k ≤ maxBranchLength, k ++,
   WriteString[fileStream, quiverAsTableEntry[[k]]];
];
For[i = 1, i ≤ Length[noEmpties], i ++,
   For[j = 1, j ≤ Length[noEmpties[[i]]], j ++,
     If[j ≤ maxBranchLength,
     enhSubtable = noEmpties[[i, j]];
     lineOfTableData = Flatten[Reap[
           For[n = 1, n ≤ Length[enhSubtable] - 1, n ++,
               singleEntry =
                          {
                          If[Length [enhSubtable[[n]]] == 1,
                              "(" <> enhSubtable[[n]] <> ")"
                           ,
                            If[Length [enhSubtable[[n]]] ≥ 2,
                                 "(" <> enhSubtable[[n, 1]] <> ","
                                      <> enhSubtable[[n, 2]] <>
                                  ")"
                               ,
                              ""
                            ]
                          ]
                   , " & "};
            Sow[singleEntry];
          ];
          ][[2]], 1];
         AppendTo[lineOfTableData, Last@enhSubtable];
         lineOfTableData = Flatten[lineOfTableData];
         AppendTo[lineOfTableData, StringJoin["\\", "\\"] <> "\n  "];
         tableEntry = StringJoin[lineOfTableData];
         WriteString[fileStream, tableEntry];
      ](* This ends the case that we are printing the enhancement. *) ;
    If[j > maxBranchLength,
     gsSubtable =  noEmpties[[i, j]];
     If[Depth[gsSubtable] ≤ 2,
      lineOfTableData = Flatten[Table[
                        {" $ " <> ToString[gsSubtable[[k]]] <> " $ ", " & "}
                                , {k, 1, Length[gsSubtable]}]];
```

```
            If[Length[lineOfTableData] == 0,
              Continue[];
            ];
             lineOfTableData = Drop[lineOfTableData, -1](* Drop the final ampersand. *) ;
             AppendTo[lineOfTableData, StringJoin["\\", "\\"] <> "\n  "];
             tableEntry = StringJoin[lineOfTableData];
             WriteString[fileStream, tableEntry];
        ,
        gsSubtable = Flatten[gsSubtable, 1];
        For[n = 1, n ≤ Length[gsSubtable], n ++,
         lineOfTableData = Flatten[Table[
                          {" $ " <> ToString[gsSubtable[[n, k]]] <> " $ ", " & "}
                                      , {k, 1, Length[gsSubtable[[n]]]}]];
         If[Length[lineOfTableData] == 0,
           Continue[];
         ];
             lineOfTableData = Drop[lineOfTableData, -1](* Drop the final ampersand. *) ;
             AppendTo[lineOfTableData, StringJoin["\\", "\\"] <> "\n  "];
             tableEntry = StringJoin[lineOfTableData];
             WriteString[fileStream, tableEntry];
        ];
       ];
        ](* This ends the case that we are printing a GS entry *) ;
     ](* Ends j loop. *);
   ](* Ends i loop. *);
   WriteString[fileStream , "\\end{longtable} } \n"];
   Return[Null];
  ];

TableOfDataForDTypeQuiver[gsOptionsForQuiver_, quiver_] :=
  Module[{table, i, j, k, m, maxBranchLength, length, quiverLength, toAppend, firstRows, toSow,
    paddedData, transposedPaddedData, toBeFinalColumnAfterATranspose, dataAfterMiniAppend},
   quiverLength = Length[quiver];
   maxBranchLength = Max[ Table[Length[quiver[[i]]], {i, 1, quiverLength}]];
   table = Flatten[Reap[
       For[j = 1, j ≤ Length[gsOptionsForQuiver], j ++,
          If[Length[gsOptionsForQuiver[[j, 2]]] == 0,
               toSow =  Append[Transpose@Append[
                         Transpose@PadDataToMaxBranchLength[
                          gsOptionsForQuiver[[j, 1]], quiver]
                                     ,
                         PadRight[{"GS Tot:"}, maxBranchLength, ""]
                                      ]
                                ,
                         Join[ConstantArray[" ", quiverLength], {"0"}]
                             ];
               Sow[toSow];
            ,
               paddedData = PadDataToMaxBranchLength[gsOptionsForQuiver[[j, 1]], quiver];
               transposedPaddedData = Transpose[paddedData];
               toBeFinalColumnAfterATranspose = PadRight[{"GS Tot:"}, maxBranchLength, ""];
               dataAfterMiniAppend = Append[transposedPaddedData, toBeFinalColumnAfterATranspose];
               firstRows = Transpose[dataAfterMiniAppend];
               toAppend =  Table[{Transpose@Append[
                                  Transpose@PadDataToMaxBranchLength[
                                   gsOptionsForQuiver[[j, 3, k, m]], quiver
                                                                        ]
                                              ,
                                  PadRight[ {gsOptionsForQuiver[[j, 2, k]]} ,
                                  maxBranchLength, ""]
                                                    ]
                                 }
                              , {k, 1, Length[gsOptionsForQuiver[[j, 3]]  ] }
                              , {m, 1, Length[ gsOptionsForQuiver[[j, 3, k]]   ]}
                              ];
            toAppend = Flatten[toAppend, 2];
            toSow = Append[firstRows, toAppend];
            Sow[toSow];
           ];
        ]; ][[2]], 1];
   Return[table](* Now for testing this,
   but the writing to file loop for DTypes also requires modification for the quiver. *) ;
  ];
```

```
WriteGSTableForQuiverToFile[quiver_, fileStream_, noEmpties_] := Module[{isDType},
   If[Depth[quiver] > 2,
     isDType = True;
     ,
     isDType = False;
   ];
   If[! isDType,
      Return[WriteTableForLinearQuiverToFile[quiver, fileStream, noEmpties]];
   ];
   Return[WriteTableForDTypeQuiverToFile[quiver, fileStream, noEmpties]];
  ];

GSPairsDetailedOnQuiverListTableFormToTex[quiverList_,
   templateFileName_: outputTemplateForTexFileName, fileNameOut_: outputTexFileName, toPrint_: False] :=
  Module[{i, j, k, m, gsTable, gsOptionsForQuiver, noEmpties, quiver, isDType,
    quiverLength, maxQuiverLength, fileStream, texHeader, outList},
   maxQuiverLength = Max @@ Table[Length[quiverList[[i]]], {i, 1, Length[quiverList]}];
   Quiet[DeleteFile[fileNameOut]];
   fileStream = OpenAppend[fileNameOut, PageWidth → Infinity];
   texHeader = Import[templateFileName];
   WriteString[fileStream, texHeader ];

   gsTable = Flatten[Reap[
       For[i = 1, i ≤ Length[quiverList], i ++,
         quiver = quiverList[[i]];
         Print[quiver];
         If[Depth[quiver] ≤ 2,
           isDType = False;
          ,
          isDType = True;
         ];
         quiverLength = Length [quiver];
         gsOptionsForQuiver = GaugeGSPairsDetailedForTex[quiverList[[i]] ];
         If[! isDType,
          noEmpties = TableOfDataForLinearQuiver[gsOptionsForQuiver, quiverLength];
          ,
          noEmpties = TableOfDataForDTypeQuiver[gsOptionsForQuiver, quiver];
         ];
         WriteGSTableForQuiverToFile[quiver, fileStream, noEmpties];
         If[toPrint,
          Do[Print[TableForm[noEmpties[[j]]  ]  ]; , {j, 1, Length[noEmpties]}];
          Print[noEmpties];
         ];
         If[toPrint, Print[""]; ];
         Sow[{quiver, gsOptionsForQuiver}];
        ];
      ][[2]], 1];

   WriteString[fileStream, "\\end{document}"];
   Print["tables written to ", Close[fileStream] ];
   If[toPrint, Return[gsTable]; ];
  ];

ListGaugeGSOptionsFromGaugeGSPairsDetailed[gaugeGSPairs_] :=
  Module[{i, j, gaugesNoDups, gaugeAlgsNoDups, optionsForGivenGaugeToGS,
    gaugeAlg, listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg},
   gaugeAlgsNoDups = Catenate[DeleteDuplicates[Drop[gaugeGSPairs, 0, -1]]];
   listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg = Flatten[Reap[
       For[i = 1, i ≤ Length[gaugeAlgsNoDups], i ++,
         gaugeAlg = gaugeAlgsNoDups[[i]];
         If[TrueQ[gaugeAlg == {-1}], Continue[];];
         optionsForGivenGaugeToGS = Select[gaugeGSPairs, TrueQ[#[[1]] == gaugeAlg] & ];
         optionsForGivenGaugeToGS = Drop[optionsForGivenGaugeToGS, 0, 1];
         optionsForGivenGaugeToGS = Table[
                                            Flatten[optionsForGivenGaugeToGS[[j]], 1]
                                             , {j, 1, Length[optionsForGivenGaugeToGS]}
                                          ];
         optionsForGivenGaugeToGS = AlgListSimplify[optionsForGivenGaugeToGS];
         optionsForGivenGaugeToGS = RelMaxAlgsQuicker[optionsForGivenGaugeToGS];
         Sow[{gaugeAlg, optionsForGivenGaugeToGS}];
        ];
      ][[2]], 1];
   Return[listOfGaugeAlgsWithCatenatedGSOptionsForGivenGaugeAlg];
  ];
```

```
FormattedListGaugeGSOptionsDetailed[quiver_] :=
  Module[{i, pairsList, algsGSOnly, listGSOptions, formattedList},
   pairsList = ListGaugeGSOptionsFromGaugeGSPairsDetailed[GaugeGSPairsDetailed[quiver]];
   formattedList = Table[
     {pairsList[[i, 1]], FormatListOfSemiSimpleTogether[pairsList[[i, 2]]]}, {i, 1, Length[pairsList]}];
   Return[formattedList];
  ];

ComputeGSForQuiverListDetailed[quiverList_] := Module[{i, gsTable, gsOptionsForQuiver},
   gsTable = Flatten[Reap[
       Do[
         Print[quiverList[[i]]];
         gsOptionsForQuiver = FormattedListGaugeGSOptionsDetailed[quiverList[[i]] ];
         Print[{quiverList[[i]], gsOptionsForQuiver}];
         Sow[{quiverList[[i]], gsOptionsForQuiver }];
         ,

         {i, 1, Length[quiverList]}
        ];
      ][[2]], 1];
   Return[gsTable];
  ];

SetOptions[EvaluationNotebook[], PageHeaders → {{"", "", ""}, {"", "", ""}},
 PrintingOptions → {"PrintingMargins" → {{90, 72}, {72, 72}}}]

SetOptions[EvaluationNotebook[], ShowPageBreaks → False]

notebookDirectory = NotebookDirectory[];

ParallelEvaluate[SetDirectory[notebookDirectory]];

ParallelComputeGSOnQuiverListsAndWriteToFile[quiverLists_, templateFileName_, outputFileNameStub_] :=
  Module[{i, aQuiverList, inputArgs, inputs},
   inputs = Flatten[Reap[
       For[i = 1, i ≤ Length[quiverLists], i ++,
         aQuiverList = quiverLists[[i]];
         inputArgs = {aQuiverList, templateFileName,
                outputFileNameStub <> ToString[i] <> ".tex"};
         Sow[inputArgs];
        ];
      ][[2]], 1];
   Parallelize[
    Do[
      Print[inputs[[i, 1]]];
      GSPairsDetailedOnQuiverListTableFormToTex @@ (inputs[[i]]);
      , {i, 1, Length[inputs]}
     ];,
    Method → "FinestGrained"
   ];
   Return[];
  ];

(*******************************************************
     Now we write the cache tables to file for future runs.
  *******************************************************)

WriteAllEnhancementsMonodromyAndAlgebraTablesToFile[]

Nothing written,
                 see settings for variable 'toWriteFileSavingHashTable' and
                 change setting from 'False.'

Nothing written,
             see settings for variable 'toWriteFileWithSavedMonodromyHashTable' and
             change setting from 'False.'

algebra cache saved to file: cacheTablesAlgs06_03_2017v1.mx

Number algebras with saved subalgebras : 2727

Enhancements, monodromy, and algebra tables written to file.

(* Examples for single Kodaira type GS computation: *)
```

```
PairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS[{{{2}, {2, 2, 4}}}, {0}]

{{{{{{2, 3, 6}, 0}}}, {{{{{g, 2}, 1}}}, {{{g, 2}, 1}}}}},
 {{{{{3, 3, 6}, 0}}}, {{{{{g, 2}, 1}}}, {{{g, 2}, 1}}}}}, {{{{{4, 3, 6}, 0}}}, {{{{{g, 2}, 1}}}, {{{g, 2}, 1}}}}}}

PairOfQuiverAlgMaxesAndCurveAtATimeAlgsWithFibersGivingAlgToGS[
 {{{3}, {2, 3, 6}}, {{1}, {1, 1, 2}}, {{3}, {2, 3, 6}}}, {0, 0, 1}]

{{{{{{0, 0, 2}, 0}}, {{{0, 0, 0}, 0}}, {{{0, 0, 4}, 0}}},
   {{{{{a, 1}, 1}}, {{{0, 0}, 0}}, {{{c, 2}, 1}}}, {{{a, 1}, 1}, {{c, 2}, 1}}}}},
 {{{{{0, 0, 3}, 0}}, {{{0, 0, 0}, 0}}, {{{0, 0, 4}, 0}}},
   {{{{{a, 1}, 1}}, {{{0, 0}, 0}}, {{{c, 2}, 1}}}, {{{a, 1}, 1}, {{c, 2}, 1}}}}},
 {{{{{1, 2, 3}, 0}}, {{{0, 0, 0}, 0}}, {{{0, 0, 4}, 0}}},
   {{{{{a, 1}, 1}}, {{{0, 0}, 0}}, {{{c, 2}, 1}}}, {{{a, 1}, 1}, {{c, 2}, 1}}}}}}
```

Appendix D

Tables of 6D SCFT global symmetries

In the following sections, we collect tables for the (relatively) maximal global symmetries which can be realized for each possible gauge enhancement and viable choice of Kodaira assignment realizing that gauge enhancement on the indicated quivers. Potentially realizable relatively maximal global symmetry algebra summands coming from each curve are listed below the corresponding curve entry with the total global symmetry algebra in the rightmost column. First row entries for a given enhancement give the gauge algebra summands from each curve and the Kodaira type realizing that gauge summand. Note that since we include notation for the Kodaira types indicating whether the fiber has split, semi-split, or non-split monodromy cover polynomial, including the gauge algebra summands is often redundant; we keep it for ease of reference and to distinguish the gauge summary entries when there is more than one Kodaira assignment on the quiver realizing a given gauge algebra.

In an attempt to increase readability of the tables we present here and excerpts elsewhere in the text (and to provide ease of matching with conventions in the literature), we write global symmetry algebras (and their summands) in a notation distinct from that in which we express gauge algebra summands. For gauge algebra terms, we use $\mathfrak{su}(n), \mathfrak{so}(2n+1), \mathfrak{sp}(n), \mathfrak{so}(2n)$ notations in place of the more condensed corresponding notation typical in referring to the Dynkin diagrams for these algebras given by A_{n-1}, B_n, C_n, D_n, respectively. We make no distinction in notation for the algebras $\mathfrak{e}_8, \mathfrak{e}_7, \mathfrak{e}_6, \mathfrak{f}_4$, and $\mathfrak{g}_2$.

Data presentation for global symmetries of gauge theories arising on interior and end links can be prohibitively lengthy due to the often large number of degrees of freedom of the outermost -1 curves. We thus collect only a select few here[1].[2] We collect the links here by q/p as in (2.2) in the following way. Consider the link given by the string α. This determines q/p via its Hirzebruch-Jung continued fraction giving p/q. However, since the reverse of this string, $\bar{\alpha}$, gives an isomorphic orbifold, we group by both $\alpha, \bar{\alpha}$ as follows. Define $A(\alpha) := q/p \mod 1$. We then group our data by the pair of rationals given by $(a, \bar{a})(\alpha) := (\min(A(\alpha), A(\bar{\alpha})), \max(A(\alpha), A(\bar{\alpha}))$. Note that in some cases, orbifold isomorphisms occur for quivers that are not grouped together by the first of these entries, though this often does not occur. In the cases with a base blowing down to a D-type endpoint, there is a natural analogy since the action is similarly generated, though there is another generator with action $(z_1, z_2) \mapsto, (z_2, -z_1)$. Since the bases blowing-down to E-type endpoints are few in number and the analogy is less direct, we omit the relation to orbifold angles in collecting results for these bases.

The subtables are sorted by gauge algebra. In cases where more than one Kodaira assignment is possible to realize this algebra, the resulting global symmetry maxima across all Kodaira assignments appears below the maxima which can occur for each separately. In a few cases, we include some length two quivers which are not links since they contain a DE-node. (These include $61, 71$ and 81.)

D.1 Linear bases including all 0-links, selected end-links, and selected short bases

D.1.1 $(a, \bar{a}) = (19/24, 19/24)$

2	2	3	1	5	1	3	2	2	GS Total:
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n$_0$)	
									0

D.1.2 $(a, \bar{a}) = (14/19, 15/19)$

2	2	3	1	5	1	3	2	GS Total:
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	

[1] Tables for the remaining interior links, end links, and arbitrary quivers (with the lone exception of the handful of 4-valent bases which can be studied by inspection trivially via dropping the bottom-most -1 curve) are readily procured via single command in the accompanying computer algebra routines (in many cases with significant computation time requirements).

[2] In the case of rogue bases, a user must specify the maximum allowable gauge and global symmetry summands permissible so that the search is finite.

								0
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
								0

D.1.3 $(a, \bar{a}) = (11/15, 11/15)$

2	3	1	5	1	3	2	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
							0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
							0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
							0

D.1.4 $(a, \bar{a}) = (9/14, 11/14)$

3	1	5	1	3	2	2	GS Total:
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
							0
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	0	0	A_1

D.1.5 $(a, \bar{a}) = (4/13, 10/13)$

2	2	3	1	5	GS Total:
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
					0

D.1.6 $(a, \bar{a}) = (7/11, 8/11)$

3	1	5	1	3	2	GS Total:
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
						0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
						0
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	A_1
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	

A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	0	0	0	0	A_1

D.1.7 $(a,\bar{a}) = (3/10, 7/10)$

2	3	1	5	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
				0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
				0

D.1.8 $(a,\bar{a}) = (4/9, 7/9)$

2	3	1	5	1	3	1	5	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1

D.1.9 $(a,\bar{a}) = (8/9, 8/9)$

2	2	3	1	5	1	3	1	5	1	3	2	2	GS Total:
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
													0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	0	0	0	0	A_1	0	0	0	0	0	0	A_1

D.1.10 $(a,\bar{a}) = (3/8, 3/8)$

5	1	3	1	5	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

					0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
					0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
					0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	A_1

D.1.11 $(a,\bar{a}) = (5/8, 5/8)$

2	3	2	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
			0

3	1	5	1	3	GS Total:
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
					0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
					0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	A_1	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	A_1	A_1

D.1.12 $(a,\bar{a}) = (1/7, 1/7)$

8	1	GS Total:
$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_1	A_1

D.1.13 $(a,\bar{a}) = (2/7, 4/7)$

3	1	5	GS Total:
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
			0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
			0

$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	A_1

D.1.14 $(a,\bar{a}) = (3/7, 5/7)$

2	2	3	GS Total:
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
			0

2	3	2	1	5	GS Total:
$(\text{III},\mathfrak{su}(2))$	$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
					0
$(\text{III},\mathfrak{su}(2))$	$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{III}^*,\mathfrak{e}_7)$	
					0
$(\text{III},\mathfrak{su}(2))$	$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
					0

3	1	5	1	3	1	5	GS Total:
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
							0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
							0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
							0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
							0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	A_1	0	0	A_1^2
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	A_1
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	A_1
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	A_1	0	0	A_1

D.1.15 $(a,\bar{a}) = (6/7, 6/7)$

2	2	3	1	3	2	2	GS Total:
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
							0
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
							0

2	3	2	1	5	1	3	2	2	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
									0

2	3	1	5	1	3	1	5	1	3	2	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
											0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
											0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
											0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	0	0	0	0	A_1	0	0	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	0	0	0	0	A_1	0	0	0	0	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	0	0	0	0	A_1	0	0	0	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
0	0	0	0	0	A_1	0	0	0	0	0	A_1

D.1.16 $(a, \bar{a}) = (1/6, 1/6)$

7	1	GS Total:
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	
0	A_1	A_1

8	1	2	GS Total:
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_0,n_0)	
0	A_1	0	A_1
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	(II,n_0)	
0	0	A_1	A_1
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_1,n_0)	
0	A_1	A_1	A_1^2
($\mathfrak{e}_7$)	(n_0)	(n_0)	
0	A_1	A_1	A_1^2
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	

0	0	A_3	A_3
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	B_3	B_3
$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	
0	0	B_3	B_3

D.1.17 $(a, \bar{a}) = (5/6, 5/6)$

2	2	3	1	3	2	GS Total:
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
						0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
						0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
						0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
						0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
						0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	0	0	A_1	0	A_1

2	2	3	1	5	1	2	3	GS Total:
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
								0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
								0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
								0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	0	0	0	0	A_1	A_1

2	3	2	1	5	1	3	2	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
								0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
								0

2	2	3	1	5	1	3	2	1	5	GS Total:

(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
										0
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
										0
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{III}^*,\mathfrak{e}_7)$	
										0
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
										0
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
										0
(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
										0

3	1	5	1	3	1	5	1	3	2	GS Total:
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	
										0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	
										0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	
										0
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	
										0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	
A_1	0	0	0	A_1	0	0	0	0	0	A_1^2
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	0	0	A_1	0	0	0	0	0	A_1^2
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	0	0	A_1	0	0	0	0	0	A_1^2
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	0	0	A_1
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	0	0	A_1
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	
0	0	0	0	A_1	0	0	0	0	0	A_1
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	
0	0	0	0	A_1	0	0	0	0	0	A_1
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	0	0	0	A_1	0	0	0	0	0	A_1

D.1.18 $(a,\bar{a}) = (1/5, 1/5)$

6	1	GS Total:
$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	
0	A_2	A_2
$(\text{III}^*,\mathfrak{e}_7)$	(I_0,n_0)	
0	A_1	A_1

7	1	2	GS Total:
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_0,n_0)	
0	A_1	0	A_1
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(II,n_0)	
0	0	A_1	A_1
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_1,n_0)	
0	A_1	A_1	A_1^2
($\mathfrak{e}_7$)	(n_0)	(n_0)	
0	A_1	A_1	A_1^2
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	
0	0	A_3	A_3
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	
0	0	B_3	B_3
($\mathfrak{e}_7$)	(n_0)	($\mathfrak{su}(2)$)	
0	0	B_3	B_3

D.1.19 $(a, \bar{a}) = (2/5, 3/5)$

2	3	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
		0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
		0
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_1	A_1

5	1	2	3	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
				0
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
				0
(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
				0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
				0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
				0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	

				0
(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	
0	0	0	A_1	A_1
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	
0	0	0	A_1	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	
0	0	0	A_1	A_1

5	1	3	2	1	5	GS Total:
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
						0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
						0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(III*,$\mathfrak{e}_7$)	
						0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
						0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
						0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
						0

D.1.20 $(a,\bar{a}) = (4/5, 4/5)$

2	3	1	3	2	GS Total:
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
					0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
					0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
					0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
					0
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
					0
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
					0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	

					0
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
					0
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_1	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_1	0	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
0	A_1	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
0	A_1	0	A_1	0	A_1^2
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_1,n_0)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
0	A_1	0	A_1	0	A_1^2
($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	(n_0)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	
0	A_1	0	A_1	0	A_1^2
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
0	0	A_1	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
0	0	A_2	0	0	A_2
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
0	0	A_1	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	
0	0	A_2	0	0	A_2

3	1	3	2	2	GS Total:
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
					0
(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
					0
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	A_1
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	A_1
($\mathfrak{g}_2$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_1	0	0	0	0	A_1
(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
C_2	0	0	0	0	C_2
(I_0^{*ss},$\mathfrak{so}(7)$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	

C_2	0	0	0	0	C_2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_2	0	0	0	0	C_2
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1^3	0	0	0	0	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	0	0	0	C_2

2	1	5	1	3	2	2	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
g_2	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2	0	0	0	0	0	0	A_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
B_3	0	0	0	0	0	0	B_3

$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_4	0	0	0	0	0	0	A_4
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_5	0	0	0	0	0	0	A_5

2	3	2	1	5	1	3	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
							0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
							0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	0	A_1	A_1

3	2	1	5	1	3	2	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	A_1
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	A_1

2	3	1	5	1	3	2	1	5	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0

3	1	5	1	3	1	5	1	3	GS Total:
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	0
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	0	0	0	A_1	0	0	0	A_1	A_1^3
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	0	0	0	0	0	0	0	A_1	A_1^2
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	0	0	0	0	0	A_1	A_1
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	0	A_1	0	0	0	A_1	A_1^2
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
0	0	0	0	A_1	0	0	0	0	A_1

$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	0	0	0	A_1	A_1

D.1.21 $(a, \bar{a}) = (1/4, 1/4)$

5	1	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_2	A_2
$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_1	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_2	A_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	
0	C_3	C_3
0	g_2	g_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	
0	A_2	A_2
$(\mathfrak{f}_4)$	(n_0)	
0	C_3	C_3
0	g_2	g_2

6	1	2	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_2	0	A_2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	A_1	A_1^2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathfrak{e}_6)$	(n_0)	(n_0)	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	B_3	B_3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	
0	0	A_3	A_3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
0	0	g_2	g_2
$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	

0	0	B_3	B_3
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(I$_3^s$,$\mathfrak{su}(3)$)	
0	0	A_5	A_5
(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	
0	0	A_2^2	A_2^2
0	0	C_2	C_2
($\mathfrak{e}_6$)	(n$_0$)	($\mathfrak{su}(3)$)	
0	0	A_5	A_5
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	
0	A_1	0	A_1
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(II,n$_0$)	
0	0	A_1	A_1
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	
0	A_1	A_1	A_1^2
($\mathfrak{e}_7$)	(n$_0$)	(n$_0$)	
0	A_1	A_1	A_1^2
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	
0	0	A_3	A_3
(III*,$\mathfrak{e}_7$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	
0	0	B_3	B_3
($\mathfrak{e}_7$)	(n$_0$)	($\mathfrak{su}(2)$)	
0	0	B_3	B_3

D.1.22 $(a, \bar{a}) = (3/4, 3/4)$

2	3	1	3	GS Total:
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	
				0
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	0	0	A_1	A_1
(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	

0	0	0	A_1	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_1	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_1	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	
0	0	0	A_1	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1^3	A_1^3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1^3	A_1^3
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(8))$	
0	0	0	A_1^3	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	A_1	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_1	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	
0	A_1	0	A_1	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	

0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	
0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_1	0	A_1^3	A_1^4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_1	0	A_1^2	A_1^3
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(8))$	
0	A_1	0	A_1^3	A_1^4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_2	0	A_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
0	0	A_2	0	A_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	0	C_2	C_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	0	C_2	C_2
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
0	0	0	C_2	C_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	0	C_3	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	0	C_3	C_3
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
0	0	0	C_3	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	0	C_4	C_4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	0	C_4	C_4
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
0	0	0	C_4	C_4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	

0	0	A_1	A_1	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_1	A_1	A_1^2
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
0	0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1	A_1^2	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1	A_1	A_1^2
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
0	0	A_1	A_1^2	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	A_1	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	A_1	A_1^2
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
0	0	A_1	A_1	A_1^2

2	1	5	1	3	2	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	C_4
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	

A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_1	0	0	0	0	A_1
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_1	0	0	0	0	A_1
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_1	0	0	0	0	A_1
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_1	0	0	0	0	A_1
(n_0)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
B_3	0	0	0	0	0	B_3
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
B_3	0	0	0	0	0	B_3
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_3	0	0	0	0	0	A_3
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_3	0	0	0	0	0	A_3
(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
g_2	0	0	0	0	0	g_2
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
g_2	0	0	0	0	0	g_2
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	A_2
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	

B_3	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_5	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_5	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_4	0	0	0	0	0	A_4
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_4	0	0	0	0	0	A_4
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_5	0	0	0	0	0	A_5

3	1	5	1	2	3	GS Total:
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
						0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
						0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
						0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
						0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
						0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	0	0	A_1	A_1
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	0	0	A_1	A_1

3	1	5	1	3	2	1	5	GS Total:
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
								0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1

D.1.23 $(a,\bar{a}) = (1/3, 1/3)$

5	1	2	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	

0	A_2	0	A_2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	A_1	A_1^2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathfrak{e}_6)$	(n_0)	(n_0)	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	B_3	B_3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	
0	0	A_3	A_3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
0	0	g_2	g_2
$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	
0	0	B_3	B_3
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	
0	0	A_5	A_5
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
0	0	A_2^2	A_2^2
0	0	C_2	C_2
$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
0	0	A_5	A_5
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	
0	A_1	0	A_1
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	A_1
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	
0	A_1	A_1	A_1^2
$(\mathfrak{e}_7)$	(n_0)	(n_0)	
0	A_1	A_1	A_1^2
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	
0	0	A_3	A_3
$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	B_3	B_3
$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	
0	0	B_3	B_3

(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	0	C_4	C_4
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	
0	A_2	0	A_2
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(II,n$_0$)	
0	A_1	A_1	A_1^2
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	
0	A_2	A_1	$A_1 \oplus A_2$
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(II,n$_0$)	
0	A_1	0	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_1$,n$_0$)	
0	A_1	0	A_1
($\mathfrak{f}_4$)	(n$_0$)	(n$_0$)	
0	A_2	A_1	$A_1 \oplus A_2$
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	
0	0	B_3	B_3
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	
0	0	A_3	A_3
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(III,$\mathfrak{su}(2)$)	
0	0	A_1	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	0	g_2	g_2
(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	
0	0	A_2	A_2
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
0	0	g_2	g_2
($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(2)$)	
0	0	B_3	B_3
0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2$
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_3^s$,$\mathfrak{su}(3)$)	
0	0	A_5	A_5
(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_3^s$,$\mathfrak{su}(3)$)	
0	0	A_4	A_4
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	
0	0	A_2^2	A_2^2
0	0	C_2	C_2
($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(3)$)	

0	0	A_5	A_5

D.1.24 $(a, \bar{a}) = (2/3, 2/3)$

3	1	3	GS Total:
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
			0
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
			0
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
A_1	0	0	A_1
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_1	0	A_1	A_1^2
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_1	0	A_1	A_1^2
($\mathfrak{g}_2$)	(n$_0$)	($\mathfrak{g}_2$)	
A_1	0	A_1	A_1^2
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	
A_1	0	C_2	$A_1 \oplus C_2$
($\mathfrak{g}_2$)	(n$_0$)	($\mathfrak{so}(7)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_0$,n$_0$)	(I$_0^{*s}$,$\mathfrak{so}(8)$)	
A_1	0	A_1^3	A_1^4
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(I$_1^{*ns}$,$\mathfrak{so}(9)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	A_1	0	A_1
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	A_2	0	A_2
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	A_1	0	A_1
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_3^{ns}$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	A_2	0	A_2
($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
0	A_2	0	A_2
(I$_0^{*ns}$,$\mathfrak{g}_2$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	

0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	C_2	C_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_4	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_4	C_4
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
0	0	C_4	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_1	A_1^2	A_1^3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_1	A_1	A_1^2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
0	A_1	A_1^2	A_1^3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_1	A_1	A_1^2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	

A_1	A_1	A_1	A_1^3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	A_1	A_1^2
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_3	0	A_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	0	C_2	C_2^2
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

A_1	A_1	A_1	A_1^3
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_7^{ns}, \mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_6^{ns}, \mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_7^{ns}, \mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_8^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	
0	A_3	0	A_3
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_9^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	
0	A_4	0	A_4
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
0	A_4	0	A_4
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_8^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	$(\mathrm{I}_9^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_4^{ns}, \mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
C_3	0	C_3	C_3^2
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_6^{ns}, \mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_7^{ns}, \mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
C_2	0	C_2	C_2^2
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_8^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_9^{ns}, \mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	$(\mathrm{I}_{10}^{ns}, \mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	

0	A_5	0	A_5
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2	0	C_2	C_2^2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_2	0	A_1^3	$A_1^3 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_2	0	A_1^2	$A_1^2 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(8))$	
C_2	0	A_1^3	$A_1^3 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_1	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_1	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_3	0	A_3
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_2	0	A_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_4	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	

A_1	0	C_4	$A_1 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
A_1	0	C_4	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	A_1	A_1^2	A_1^4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	A_1	A_1	A_1^3
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_1	A_1	A_1^2	A_1^4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	A_1	A_1	A_1^3
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1^3	0	A_1^3	A_1^6
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1^2	0	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_1^2	0	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1^2	0	C_3	$A_1^2 \oplus C_3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
A_1^2	0	C_3	$A_1^2 \oplus C_3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	

A_1^2	0	C_4	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1^2	A_1	A_1^2	A_1^5
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	0	A_1	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_1^2	A_1	A_1^2	A_1^5
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1^2	0	A_1	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_1^2	0	A_1	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_1	A_1	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_2	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_1	C_2	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
0	A_1	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_3	C_3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_3	C_3
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
0	0	C_3	C_3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_1	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_2	0	A_2
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
0	A_2	0	A_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	

A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	C_3	C_3
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	C_2	C_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1^3	A_1^3
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	C_2	C_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1^2	0	A_1^2
0	A_2	0	A_2

5	1	2	2	3	GS Total:
$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
					0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	A_1

5	1	3	1	5	1	3	1	5	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
									0

(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
									0
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
									0
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
									0
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
									0
(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
									0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	A_1	0	0	0	A_1	0	0	A_1^2
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
0	0	A_1	0	0	0	0	0	0	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	A_1	0	0	0	0	0	0	A_1

D.1.25 $(a, \bar{a}) = (1/2, 1/2)$

2	2	1	5	GS Total:
(II,n$_0$)	(II,n$_0$)	(I$_0$,n$_0$)	(III*,$\mathfrak{e}_7$)	
				0
(II,n$_0$)	(III,$\mathfrak{su}(2)$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(III,$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(III*,$\mathfrak{e}_7$)	
				0
(III,$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(III,$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(IV^{ns},$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	
				0

$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
				0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_3	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_3	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_3	0	0	0	C_3
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	A_2	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	A_1	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	A_2	0	A_2
(n_0)	(n_0)	(n_0)	$(\mathfrak{e}_6)$	
0	0	A_2	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	0	A_1	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	0	A_1	0	A_1
(n_0)	(n_0)	(n_0)	$(\mathfrak{e}_7)$	
0	0	A_1	0	A_1
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_2	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	A_1

$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_2	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	A_1
(n_0)	(n_0)	(n_0)	$(\mathfrak{f}_4)$	
0	0	A_2	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	g_2	0	0	g_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	A_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
0	g_2	0	0	g_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
0	A_1	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
0	A_2	0	0	A_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
0	A_2	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	g_2	0	0	g_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	A_1
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
0	g_2	0	0	g_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_4	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	C_4

(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	C_3	0	0	C_3
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	C_3	0	0	C_3
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	
0	C_3	0	0	C_3
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_1	0	A_1	0	A_1^2
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
g_2	0	A_1	0	$A_1 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_2	0	A_2	0	A_2^2
($\mathfrak{su}(2)$)	(n_0)	(n_0)	($\mathfrak{e}_6$)	
A_2	0	A_2	0	A_2^2
g_2	0	A_1	0	$A_1 \oplus g_2$
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	
A_1	0	0	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	
g_2	0	0	0	g_2
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	
A_2	0	A_1	0	$A_1 \oplus A_2$
($\mathfrak{su}(2)$)	(n_0)	(n_0)	($\mathfrak{e}_7$)	
A_2	0	A_1	0	$A_1 \oplus A_2$
g_2	0	0	0	g_2
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_1	0	A_1	0	A_1^2
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
g_2	0	A_1	0	$A_1 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_2	0	A_2	0	A_2^2
($\mathfrak{su}(2)$)	(n_0)	(n_0)	($\mathfrak{f}_4$)	
A_2	0	A_2	0	A_2^2
g_2	0	A_1	0	$A_1 \oplus g_2$
(III,$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_1	A_1	0	0	A_1^2
(I_2,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_1	A_1	0	0	A_1^2
($\mathfrak{su}(2)$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{e}_6$)	

A_1	A_1	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	A_1	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	A_1	0	0	A_1^2
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	A_1	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	A_1
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_1	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_3	0	0	A_3
$(\mathfrak{su}(2))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
0	A_3	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_3	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_2
$(\mathfrak{su}(2))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
0	A_3	0	0	A_3
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	0	0	0	A_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

A_2	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_3	0	0	0	A_3
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_3	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_2	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_3	0	0	0	A_3
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_3	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	A_2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	A_2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	A_3
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	A_3
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_3	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	A_2	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	A_2	0	0	A_2^2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_2	A_2	0	0	A_2^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	A_2	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	A_2	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	A_1	0	0	$A_1 \oplus A_2$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
A_2	A_2	0	0	A_2^2
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

A_5	0	0	0	A_5
(I_4^s,$\mathfrak{su}(4)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	
A_5	0	0	0	A_5
(I_4^s,$\mathfrak{su}(4)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_5	0	0	0	A_5
(I_4^s,$\mathfrak{su}(4)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_4	A_1	0	0	$A_1 \oplus A_4$
(I_4^s,$\mathfrak{su}(4)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_4	A_1	0	0	$A_1 \oplus A_4$
(I_4^s,$\mathfrak{su}(4)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_4	0	0	0	A_4
($\mathfrak{su}(4)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	
A_4	A_1	0	0	$A_1 \oplus A_4$
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_6	0	0	0	A_6
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_6	0	0	0	A_6
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_6	0	0	0	A_6
($\mathfrak{su}(5)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	
A_6	0	0	0	A_6
(I_6^s,$\mathfrak{su}(6)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
A_8	0	0	0	A_8
(I_6^s,$\mathfrak{su}(6)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_8	0	0	0	A_8

5	1	3	1	3	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
					0
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
					0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
					0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
					0
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	0	A_1	A_1
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	

0	0	0	0	C_3	C_3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	0	A_1^3	A_1^3
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	0	A_1^2	0	A_1^2
0	0	0	A_2	0	A_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	A_1	A_1^2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	A_1	A_1^2
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	
0	0	A_1	0	A_1	A_1^2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(7))$	
0	0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1	0	A_1^3	A_1^4
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_1	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_1	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_2	0	A_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	A_2	0	A_2

$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
0	0	0	A_2	0	A_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	0	0	C_2	C_2
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	0	0	C_3	C_3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	0	0	C_3	C_3
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
0	0	0	0	C_3	C_3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	0	0	C_4	C_4
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	0	0	C_4	C_4
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
0	0	0	0	C_4	C_4
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	A_1	A_1	A_1^2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	A_1	A_1	A_1^2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	A_2	A_1	$A_1 \oplus A_2$
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
0	0	0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1	A_1^2	A_1^3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1	A_1	A_1^2
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
0	0	0	A_1	A_1^2	A_1^3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	0	A_1	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	A_1	A_1	A_1^2

$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
0	0	0	A_1	A_1	A_1^2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	A_1	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	0	0	0	C_3	C_3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	0	A_1^3	A_1^3
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	0	C_2	C_2
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
0	0	0	A_1^2	0	A_1^2
0	0	0	A_2	0	A_2

2	1	5	1	3	1	5	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	A_1	0	0	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	

A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	A_1	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	A_1	0	0	A_1^3
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	A_1	0	0	$A_1^2 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	A_1	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	A_1	0	0	A_1^2
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_2	0	0	A_1	0	0	$A_1^2 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	A_1	0	0	$A_1 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	A_1	0	0	$A_1 \oplus A_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	A_1	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
g_2	0	0	0	A_1	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	A_1	0	0	$A_1 \oplus A_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	A_1	0	0	$A_1^2 \oplus A_2$
g_2	0	0	0	A_1	0	0	$A_1 \oplus g_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
B_3	0	0	0	A_1	0	0	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	A_1	0	0	$A_1^2 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

A_3	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
g_2	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
g_2	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	A_5

$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	A_1	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_4	0	0	0	A_1	0	0	$A_1 \oplus A_4$
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	A_1	0	0	$A_1 \oplus A_2^2$
C_2	0	0	0	A_1	0	0	$A_1 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
A_5	0	0	0	A_1	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_4	0	0	0	0	0	0	A_4
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_4	0	0	0	0	0	0	A_4
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	A_5

5	1	3	2	2	1	5	GS Total:
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
							0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	0	0	0	A_1	0	A_1

(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	0	0	0	A_1	0	A_1

D.1.26 $(a, \bar{a}) = (0, 0)$

2	1	3	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
			0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
			0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	A_1	A_1
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
C_4	0	0	C_4
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
C_4	0	A_1	$A_1 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
C_4	0	A_1	$A_1 \oplus C_4$
($\mathfrak{g}_2$)	(n_0)	($\mathfrak{g}_2$)	
C_4	0	A_1	$A_1 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*ss},$\mathfrak{so}(7)$)	
C_4	0	C_2	$C_2 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(I_0^{*ss},$\mathfrak{so}(7)$)	
C_4	0	C_2	$C_2 \oplus C_4$
($\mathfrak{g}_2$)	(n_0)	($\mathfrak{so}(7)$)	
C_4	0	C_2	$C_2 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
C_4	0	A_1^3	$A_1^3 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
C_4	0	C_2	$C_2 \oplus C_4$
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
C_4	0	0	C_4
(I_0^{*ns},$\mathfrak{g}_2$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$
(I_0^{*ns},$\mathfrak{g}_2$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$
($\mathfrak{g}_2$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(10)$)	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_2	C_2	$A_2 \oplus C_2^2$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
C_2	A_2	C_2	$A_2 \oplus C_2^2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_1	C_3	$A_1 \oplus C_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_1	C_3	$A_1 \oplus C_2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
C_2	A_1	C_3	$A_1 \oplus C_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_2	A_2	0	$A_2 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	A_4	0	$A_1 \oplus A_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_4	A_1	$A_1^2 \oplus A_4$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_1	A_4	A_1	$A_1^2 \oplus A_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_5	0	A_5
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_6	0	A_6

$(\mathfrak{g}_2)$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
0	A_6	0	A_6
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_5	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_5	A_1	$A_1 \oplus A_5$
$(\mathfrak{g}_2)$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
0	A_5	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	A_1	0	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	A_2	0	$A_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	A_1	0	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	A_2	0	$A_2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_3	A_2	0	$A_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_3	0	C_3	C_3^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_3	0	C_3	C_3^2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
C_3	0	C_3	C_3^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	0	C_4	$C_3 \oplus C_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	0	C_4	$C_3 \oplus C_4$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
C_3	0	C_4	$C_3 \oplus C_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_3	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_3	A_1	A_1^2	$A_1^3 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
C_3	A_1	A_1^2	$A_1^3 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_2	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	A_1	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_1	0	A_1^2
(n_0)	(n_0)	$(\mathfrak{e}_7)$	
A_1	A_1	0	A_1^2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	A_1^2

(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	A_1	0	A_1
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
A_1	A_2	0	$A_1 \oplus A_2$
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	A_1	0	A_1
(n_0)	(n_0)	($\mathfrak{f}_4$)	
A_1	A_2	0	$A_1 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	D_4	A_1	$A_1 \oplus D_4$
0	$A_1 \oplus A_3$	A_1	$A_1^2 \oplus A_3$
(II,n_0)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
(II,n_0)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	C_3	A_1	$A_1 \oplus C_3$
0	g_2	A_1	$A_1 \oplus g_2$
(I_1,n_0)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
(I_1,n_0)	(I_1,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2	A_1	$A_1 \oplus A_2$
(n_0)	(n_0)	($\mathfrak{g}_2$)	
0	D_4	A_1	$A_1 \oplus D_4$
0	$A_1 \oplus A_3$	A_1	$A_1^2 \oplus A_3$
A_1	A_3	A_1	$A_1^2 \oplus A_3$
0	C_3	A_1	$A_1 \oplus C_3$
(I_0,n_0)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	A_3	C_3	$A_3 \oplus C_3$
0	$A_1 \oplus A_2$	C_3	$A_1 \oplus A_2 \oplus C_3$
(II,n_0)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	A_2	C_3	$A_1 \oplus A_2 \oplus C_3$
(II,n_0)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	C_3	C_3
(I_1,n_0)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	A_3	C_3	$A_1 \oplus A_3 \oplus C_3$
(I_1,n_0)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	A_1^2	C_3	$A_1^2 \oplus C_3$
(n_0)	(n_0)	($\mathfrak{so}(10)$)	
A_1	A_3	C_3	$A_1 \oplus A_3 \oplus C_3$

$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_1^2	C_4	$A_1^2 \oplus C_4$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_1	C_4	$A_1^2 \oplus C_4$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_4	C_4
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_1	C_4	$A_1^2 \oplus C_4$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_1	C_4	$A_1 \oplus C_4$
(n_0)	(n_0)	$(\mathfrak{so}(11))$	
0	A_1^2	C_4	$A_1^2 \oplus C_4$
A_1	A_1	C_4	$A_1^2 \oplus C_4$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_1^2	C_5	$A_1^2 \oplus C_5$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_1	C_5	$A_1^2 \oplus C_5$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_5	C_5
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_1	C_5	$A_1^2 \oplus C_5$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_5	C_5
(n_0)	(n_0)	$(\mathfrak{so}(12))$	
0	A_1^2	C_5	$A_1^2 \oplus C_5$
A_1	A_1	C_5	$A_1^2 \oplus C_5$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	D_4	C_2	$C_2 \oplus D_4$
0	$A_1 \oplus A_3$	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	C_2	$A_1 \oplus C_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	

0	A_2^2	C_2	$A_2^2 \oplus C_2$
0	B_3	C_2	$B_3 \oplus C_2$
(n_0)	(n_0)	($\mathfrak{so}(7)$)	
0	A_2^2	C_2	$A_2^2 \oplus C_2$
0	D_4	C_2	$C_2 \oplus D_4$
0	$A_1 \oplus A_3$	C_2	$A_1 \oplus A_3 \oplus C_2$
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
(I_0,n_0)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
0	D_4	A_1^3	$A_1^3 \oplus D_4$
0	$A_1 \oplus A_3$	A_1^3	$A_1^4 \oplus A_3$
(II,n_0)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_1	A_3	A_1^3	$A_1^4 \oplus A_3$
(II,n_0)	(I_1,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
0	A_1	A_1^2	A_1^3
(I_1,n_0)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_1	A_3	A_1^3	$A_1^4 \oplus A_3$
(I_1,n_0)	(I_1,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
0	$A_1 \oplus A_2$	A_1^2	$A_1^3 \oplus A_2$
(n_0)	(n_0)	($\mathfrak{so}(8)$)	
0	D_4	A_1^3	$A_1^3 \oplus D_4$
0	$A_1 \oplus A_3$	A_1^3	$A_1^4 \oplus A_3$
A_1	A_3	A_1^3	$A_1^4 \oplus A_3$
(I_0,n_0)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	A_3	C_2	$A_3 \oplus C_2$
0	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2 \oplus C_2$
(II,n_0)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
(II,n_0)	(II,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	A_1	C_2	$A_1 \oplus C_2$
(II,n_0)	(I_1,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	A_1	C_2	$A_1 \oplus C_2$
(I_1,n_0)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$
(I_1,n_0)	(I_1,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	A_1^2	C_2	$A_1^2 \oplus C_2$
0	A_2	C_2	$A_2 \oplus C_2$
(n_0)	(n_0)	($\mathfrak{so}(9)$)	
A_1	A_3	C_2	$A_1 \oplus A_3 \oplus C_2$

(I_0,n_0)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
0	e_6	0	e_6
0	$A_2 \oplus A_3$	0	$A_2 \oplus A_3$
(II,n_0)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
A_1	A_5	0	$A_1 \oplus A_5$
A_1	D_4	0	$A_1 \oplus D_4$
(I_1,n_0)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
A_1	A_5	0	$A_1 \oplus A_5$
A_1	D_5	0	$A_1 \oplus D_5$
A_1	$A_1 \oplus A_4$	0	$A_1^2 \oplus A_4$
A_1	$A_2 \oplus A_3$	0	$A_1 \oplus A_2 \oplus A_3$
(n_0)	(n_0)	($\mathfrak{su}(3)$)	
0	e_6	0	e_6
A_1	D_5	0	$A_1 \oplus D_5$
A_1	$A_1 \oplus A_4$	0	$A_1^2 \oplus A_4$
A_1	$A_2 \oplus A_3$	0	$A_1 \oplus A_2 \oplus A_3$
(II,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	B_3	0	B_3
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2^2	0	A_2^2
0	C_2	0	C_2
0	g_2	0	g_2
(II,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2^2	0	A_2^2
0	A_4	0	A_4
0	$A_1 \oplus A_3$	0	$A_1 \oplus A_3$
(II,n_0)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_5	0	A_5
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	D_5	0	D_5
0	A_2^2	0	A_2^2
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
0	A_5	0	A_5
0	D_5	0	D_5
(II,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	A_3	C_2	$A_3 \oplus C_2$
0	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2 \oplus C_2$
(II,n_0)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	

0	A_3	C_2	$A_3 \oplus C_2$
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	D_4	C_2	$C_2 \oplus D_4$
0	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2 \oplus C_2$
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(10)$)	
0	D_4	C_2	$C_2 \oplus D_4$
0	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2 \oplus C_2$
(II,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
0	A_1^2	C_3	$A_1^2 \oplus C_3$
0	A_2	C_3	$A_2 \oplus C_3$
(II,n_0)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
0	A_3	C_3	$A_3 \oplus C_3$
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
0	A_1^2	C_3	$A_1^2 \oplus C_3$
0	A_2	C_3	$A_2 \oplus C_3$
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(11)$)	
0	A_3	C_3	$A_3 \oplus C_3$
(II,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
0	A_1^2	C_4	$A_1^2 \oplus C_4$
0	A_2	C_4	$A_2 \oplus C_4$
(II,n_0)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
0	A_2	C_4	$A_2 \oplus C_4$
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
0	A_1^2	C_4	$A_1^2 \oplus C_4$
0	A_2	C_4	$A_2 \oplus C_4$
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(12)$)	
0	A_1^2	C_4	$A_1^2 \oplus C_4$
0	A_2	C_4	$A_2 \oplus C_4$
(II,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	B_3	A_1	$A_1 \oplus B_3$
(II,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_2^2	A_1	$A_1 \oplus A_2^2$
0	A_4	A_1	$A_1 \oplus A_4$
0	$A_1 \oplus A_3$	A_1	$A_1^2 \oplus A_3$
(II,n_0)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_5	A_1	$A_1 \oplus A_5$
(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	D_5	A_1	$A_1 \oplus D_5$

0	A_2^2	A_1	$A_1 \oplus A_2^2$
(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
0	A_5	A_1	$A_1 \oplus A_5$
0	D_5	A_1	$A_1 \oplus D_5$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_2^2	A_1^2	$A_1^2 \oplus A_2^2$
0	A_4	A_1^2	$A_1^2 \oplus A_4$
0	$A_1 \oplus A_3$	A_1^2	$A_1^3 \oplus A_3$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_4	A_1	$A_1 \oplus A_4$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	D_5	A_1^2	$A_1^2 \oplus D_5$
0	A_2^2	A_1^2	$A_1^2 \oplus A_2^2$
(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
0	D_5	A_1^2	$A_1^2 \oplus D_5$
0	A_2^2	A_1^2	$A_1^2 \oplus A_2^2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_3	A_1	$A_1 \oplus A_3$
0	$A_1 \oplus A_2$	A_1	$A_1^2 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_4	A_1	$A_1 \oplus A_4$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	D_4	A_1	$A_1 \oplus D_4$
0	$A_1 \oplus A_2$	A_1	$A_1^2 \oplus A_2$
(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
0	A_4	A_1	$A_1 \oplus A_4$
0	D_4	A_1	$A_1 \oplus D_4$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
C_4	0	0	C_4
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

C_2	0	C_2	C_2^2
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	
C_2	A_2	0	$A_2 \oplus C_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
A_1	A_3	0	$A_1 \oplus A_3$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_7^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_7^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_8^{ns}$,$\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
0	A_4	0	A_4
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_9^{ns}$,$\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
0	A_4	0	A_4
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
0	A_4	0	A_4
$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	$(\mathrm{I}_8^{ns}$,$\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	

0	A_4	A_1	$A_1 \oplus A_4$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathfrak{so}(10))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
0	A_4	A_1	$A_1 \oplus A_4$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	C_3
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_3	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_3	0	A_1^2	$A_1^2 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
C_3	0	A_1^2	$A_1^2 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	0	C_3	C_3^2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	0	C_3	C_3^2
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
C_3	0	C_3	C_3^2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_3	A_1	0	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_3	A_1	0	$A_1 \oplus C_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
C_3	A_1	0	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	A_2	0	$A_2 \oplus C_2$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
C_2	A_1	C_2	$A_1 \oplus C_2^2$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_3	0	$A_1 \oplus A_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_4	0	$A_1 \oplus A_4$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	

A_1	A_4	0	$A_1 \oplus A_4$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathfrak{so}(11))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_1	A_3	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_5	0	A_5
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	C_4
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	C_4
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_4	0	0	C_4
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_4	0	A_1^2	$A_1^2 \oplus C_4$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
C_4	0	A_1^2	$A_1^2 \oplus C_4$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_4	0	C_2	$C_2 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

C_4	0	C_2	$C_2 \oplus C_4$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
C_4	0	C_2	$C_2 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_4	0	C_3	$C_3 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_4	A_1	0	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_4	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_4	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
C_4	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_3	A_2	0	$A_2 \oplus C_3$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
C_3	A_1	A_1	$A_1^2 \oplus C_3$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	A_1	C_2	$A_1 \oplus C_2 \oplus C_3$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
C_3	A_1	C_2	$A_1 \oplus C_2 \oplus C_3$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
C_2	A_3	0	$A_3 \oplus C_2$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_3	A_1	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	A_2	A_1	$A_1 \oplus A_2 \oplus C_2$
$(\mathfrak{so}(12))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	

C_2	A_3	A_1	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_{10}^{ns}$,$\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
A_1	A_5	0	$A_1 \oplus A_5$
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_2$,$\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns}$,$\mathfrak{g}_2)$	
C_5	0	0	C_5
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns}$,$\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns}$,$\mathfrak{g}_2)$	
C_5	0	0	C_5
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_5	0	0	C_5
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_2$,$\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	
C_5	0	A_1	$A_1 \oplus C_5$
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns}$,$\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	
C_5	0	A_1	$A_1 \oplus C_5$
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_5	0	A_1	$A_1 \oplus C_5$
$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	$(\mathrm{I}_2$,$\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s}$,$\mathfrak{so}(8))$	
C_5	0	A_1^2	$A_1^2 \oplus C_5$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
C_3	0	C_2	$C_2 \oplus C_3$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	
C_3	0	0	C_3
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
C_3	0	0	C_3
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
C_3	0	0	C_3
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
C_3	0	0	C_3
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_3^{*ns}$,$\mathfrak{so}(13))$	$(\mathrm{I}_7^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	

C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
C_2	0	C_2	C_2^2
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_2	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_3	0	$A_1 \oplus A_3$
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_1	A_3	0	$A_1 \oplus A_3$
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(13))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_4	0	A_4
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	C_4
$(\mathrm{I}_3^{*ns},\mathfrak{so}(13))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus C_4$	0	A_1	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus C_4$	0	A_1	$A_1^2 \oplus C_4$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	
$A_1 \oplus C_4$	0	A_1	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
$A_1 \oplus C_4$	0	C_2	$A_1 \oplus C_2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
$A_1 \oplus C_4$	0	C_2	$A_1 \oplus C_2 \oplus C_4$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	

$A_1 \oplus C_4$	0	C_2	$A_1 \oplus C_2 \oplus C_4$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{I}_0^{*s}$,$\mathfrak{so}(8))$	
$A_1 \oplus C_4$	0	A_1^3	$A_1^4 \oplus C_4$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_1$,$\mathrm{n}_0)$	$(\mathrm{I}_0^{*s}$,$\mathfrak{so}(8))$	
$A_1 \oplus C_4$	0	A_1^2	$A_1^3 \oplus C_4$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(8))$	
$A_1 \oplus C_4$	0	A_1^3	$A_1^4 \oplus C_4$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^s$,$\mathfrak{su}(3))$	
$A_1 \oplus C_4$	0	0	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
$A_1 \oplus C_2$	A_2	A_1	$A_1^2 \oplus A_2 \oplus C_2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
$A_1 \oplus C_2$	A_2	A_1	$A_1^2 \oplus A_2 \oplus C_2$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
$A_1 \oplus C_2$	A_2	A_1	$A_1^2 \oplus A_2 \oplus C_2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
$A_1 \oplus C_2$	A_1	C_2	$A_1^2 \oplus C_2^2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	
$A_1 \oplus C_2$	A_2	C_2	$A_1 \oplus A_2 \oplus C_2^2$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
$A_1 \oplus C_2$	A_2	C_2	$A_1 \oplus A_2 \oplus C_2^2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
$A_1 \oplus C_2$	A_1	C_3	$A_1^2 \oplus C_2 \oplus C_3$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s}$,$\mathfrak{so}(12))$	
$A_1 \oplus C_2$	A_1	C_3	$A_1^2 \oplus C_2 \oplus C_3$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
$A_1 \oplus C_2$	A_1	C_3	$A_1^2 \oplus C_2 \oplus C_3$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	
$A_1 \oplus C_2$	A_3	0	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_4^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
$A_1 \oplus C_2$	A_2	0	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_5^{ns}$,$\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns}$,$\mathfrak{so}(9))$	
$A_1 \oplus C_2$	A_3	0	$A_1 \oplus A_3 \oplus C_2$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
$A_1 \oplus C_2$	A_3	0	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s}$,$\mathfrak{so}(10))$	
A_1^2	A_4	0	$A_1^2 \oplus A_4$
$(\mathrm{I}_0^{*ss}$,$\mathfrak{so}(7))$	$(\mathrm{I}_6^{ns}$,$\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns}$,$\mathfrak{so}(11))$	

A_1^2	A_3	A_1	$A_1^3 \oplus A_3$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1^2	A_4	A_1	$A_1^3 \oplus A_4$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_1^2	A_4	A_1	$A_1^3 \oplus A_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1^2	A_3	C_2	$A_1^2 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1^2	A_3	C_2	$A_1^2 \oplus A_3 \oplus C_2$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_1^2	A_3	C_2	$A_1^2 \oplus A_3 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_5	0	$A_1 \oplus A_5$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_6	0	$A_1 \oplus A_6$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_1	A_6	0	$A_1 \oplus A_6$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_5	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_5	A_1	$A_1^2 \oplus A_5$
$(\mathfrak{so}(7))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_1	A_5	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	A_1	0	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	A_1	0	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	A_2	0	$A_2 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_4	A_2	0	$A_2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_4	0	C_2	$C_2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_4	0	C_2	$C_2 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
C_4	0	C_2	$C_2 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

C_4	0	C_3	$C_3 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_3^{ns}, \mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns}, \mathfrak{so}(11))$	
C_4	0	C_3	$C_3 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
C_4	0	C_3	$C_3 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_2, \mathfrak{su}(2))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
C_4	0	C_4	C_4^2
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_3^{ns}, \mathfrak{su}(2))$	$(\mathrm{I}_2^{*s}, \mathfrak{so}(12))$	
C_4	0	C_4	C_4^2
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
C_4	0	C_4	C_4^2
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{III}, \mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	
C_4	A_1	A_1	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_2, \mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	
C_4	A_1	A_1	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_3^{ns}, \mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	
C_4	A_2	A_1	$A_1 \oplus A_2 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_4	A_2	A_1	$A_1 \oplus A_2 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_2, \mathfrak{su}(2))$	$(\mathrm{I}_0^{*s}, \mathfrak{so}(8))$	
C_4	A_1	A_1^2	$A_1^3 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_3^{ns}, \mathfrak{su}(2))$	$(\mathrm{I}_0^{*s}, \mathfrak{so}(8))$	
C_4	A_1	A_1	$A_1^2 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
C_4	A_1	A_1^2	$A_1^3 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_2, \mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns}, \mathfrak{so}(9))$	
C_4	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	$(\mathrm{I}_3^{ns}, \mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns}, \mathfrak{so}(9))$	
C_4	A_1	A_1	$A_1^2 \oplus C_4$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
C_4	A_1	A_1	$A_1^2 \oplus C_4$
$(\mathrm{I}_0^{*s}, \mathfrak{so}(8))$	$(\mathrm{I}_0, \mathrm{n}_0)$	$(\mathrm{I}_0^{*ns}, \mathfrak{g}_2)$	
$A_1^2 \oplus C_2^2$	0	A_1	$A_1^3 \oplus C_2^2$
$(\mathrm{I}_0^{*s}, \mathfrak{so}(8))$	$(\mathrm{I}_0, \mathrm{n}_0)$	$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	
$A_1^2 \oplus C_2^2$	0	C_2	$A_1^2 \oplus C_2^3$
$(\mathrm{I}_0^{*s}, \mathfrak{so}(8))$	$(\mathrm{I}_1, \mathrm{n}_0)$	$(\mathrm{I}_0^{*ss}, \mathfrak{so}(7))$	
$A_1 \oplus C_2^2$	0	C_2	$A_1 \oplus C_2^3$
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{so}(7))$	

$A_1^2 \oplus C_2^2$	0	C_2	$A_1^2 \oplus C_2^3$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
$A_1^2 \oplus C_2^2$	0	A_1^3	$A_1^5 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
$A_1^2 \oplus C_2^2$	0	0	$A_1^2 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
$A_1^2 \oplus C_2$	A_2	A_1	$A_1^3 \oplus A_2 \oplus C_2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
$A_1 \oplus C_2$	A_1	A_1	$A_1^3 \oplus C_2$
($\mathfrak{so}(8)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(10)$)	
$A_1^2 \oplus C_2$	A_2	A_1	$A_1^3 \oplus A_2 \oplus C_2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
$A_1^2 \oplus C_2$	A_1	C_2	$A_1^3 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
$A_1 \oplus C_2$	A_1	C_2	$A_1^2 \oplus C_2^2$
($\mathfrak{so}(8)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(11)$)	
$A_1^2 \oplus C_2$	A_1	C_2	$A_1^3 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
$A_1^2 \oplus C_2$	A_1	C_3	$A_1^3 \oplus C_2 \oplus C_3$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
$A_1 \oplus C_2$	0	C_3	$A_1 \oplus C_2 \oplus C_3$
($\mathfrak{so}(8)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(12)$)	
$A_1^2 \oplus C_2$	A_1	C_3	$A_1^3 \oplus C_2 \oplus C_3$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
$A_1^2 \oplus C_2$	A_3	0	$A_1^2 \oplus A_3 \oplus C_2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
$A_1^2 \oplus C_2$	A_2	0	$A_1^2 \oplus A_2 \oplus C_2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
$A_1 \oplus C_2$	A_2	0	$A_1 \oplus A_2 \oplus C_2$
($\mathfrak{so}(8)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(9)$)	
$A_1^2 \oplus C_2$	A_2	0	$A_1^2 \oplus A_2 \oplus C_2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
$A_1 \oplus C_2^2$	A_1	0	$A_1^2 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
C_2^2	A_1	0	$A_1 \oplus C_2^2$
($\mathfrak{so}(8)$)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
$A_1 \oplus C_2^2$	A_1	0	$A_1^2 \oplus C_2^2$
(I_0^{*s},$\mathfrak{so}(8)$)	(I_2,$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	

$A_1 \oplus C_2^2$	0	C_2	$A_1 \oplus C_2^3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2^2	0	C_2	C_2^3
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
$A_1 \oplus C_2^2$	0	C_2	$A_1 \oplus C_2^3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
$A_1 \oplus C_2^2$	0	C_3	$A_1 \oplus C_2^2 \oplus C_3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2^2	0	C_3	$C_2^2 \oplus C_3$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
$A_1 \oplus C_2^2$	0	C_3	$A_1 \oplus C_2^2 \oplus C_3$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
$A_1 \oplus C_2^2$	0	C_4	$A_1 \oplus C_2^2 \oplus C_4$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
$A_1 \oplus C_2^2$	A_1	A_1	$A_1^3 \oplus C_2^2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2^2	A_1	A_1	$A_1^2 \oplus C_2^2$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
$A_1 \oplus C_2^2$	A_1	A_1	$A_1^3 \oplus C_2^2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
$A_1 \oplus C_2^2$	A_1	A_1^2	$A_1^4 \oplus C_2^2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_2^2	0	A_1	$A_1 \oplus C_2^2$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
$A_1 \oplus C_2^2$	A_1	A_1^2	$A_1^4 \oplus C_2^2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
$A_1 \oplus C_2^2$	0	A_1	$A_1^2 \oplus C_2^2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_2^2	0	A_1	$A_1 \oplus C_2^2$
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
$A_1 \oplus C_2^2$	0	A_1	$A_1^2 \oplus C_2^2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	C_3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	

A_1	A_1	A_1	A_1^3
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_2	0	$A_1 \oplus A_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	A_1	0	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	A_2	0	$A_1 \oplus A_2$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
A_1	A_2	0	$A_1 \oplus A_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_3	0	A_3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_2	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	

C_2	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
C_2	A_1	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
C_2	0	C_2	C_2^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
C_2	0	C_2	C_2^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
C_2	A_1	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_2	0	A_1^2	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
C_2	0	A_1^2	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_{10}^{s},\mathfrak{su}(10))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_{10}^{s},\mathfrak{su}(10))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_{10}	0	0	A_{10}
$(\mathfrak{su}(10))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_{10}^{s},\mathfrak{su}(10))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	

A_{11}	0	A_1	$A_1 \oplus A_{11}$
$(\mathrm{I}_{10}^{s},\mathfrak{su}(10))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_{2}^{*s},\mathfrak{so}(12))$	
A_9	A_1	0	$A_1 \oplus A_9$
$(\mathrm{I}_{11}^{s},\mathfrak{su}(11))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_{2}^{*s},\mathfrak{so}(12))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_{12}^{s},\mathfrak{su}(12))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_{2}^{*s},\mathfrak{so}(12))$	
A_{13}	0	0	A_{13}
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
B_3	0	0	B_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
$A_1 \oplus A_2$	0	0	$A_1 \oplus A_2$
g_2	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_3	0	0	A_3
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
B_3	0	0	B_3
$A_1 \oplus A_2$	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
B_3	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_3	0	0	A_3
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
B_3	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
g_2	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	$A_1 \oplus A_2$
g_2	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	A_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
B_3	0	0	B_3

$A_1 \oplus A_2$	0	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
B_3	A_2	A_1	$A_1 \oplus A_2 \oplus B_3$
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_1	A_1	A_1^3
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
$A_1 \oplus A_2$	0	A_1	$A_1^2 \oplus A_2$
g_2	0	A_1	$A_1 \oplus g_2$
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
g_2	A_1	A_1	$A_1^2 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_3	A_3	A_1	$A_1 \oplus A_3^2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_2	A_1	A_1	$A_1^2 \oplus A_2$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{g}_2$)	
A_3	A_3	A_1	$A_1 \oplus A_3^2$
B_3	A_2	A_1	$A_1 \oplus A_2 \oplus B_3$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
B_3	A_1	C_3	$A_1 \oplus B_3 \oplus C_3$
(III,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	0	C_3	$A_1 \oplus C_3$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
$A_1 \oplus A_2$	0	C_3	$A_1 \oplus A_2 \oplus C_3$
g_2	0	C_3	$C_3 \oplus g_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
g_2	0	C_3	$C_3 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_3	A_2	C_3	$A_2 \oplus A_3 \oplus C_3$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_2	A_1	C_3	$A_1 \oplus A_2 \oplus C_3$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{so}(10)$)	
A_3	A_2	C_3	$A_2 \oplus A_3 \oplus C_3$
B_3	A_1	C_3	$A_1 \oplus B_3 \oplus C_3$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_2^{*ns},$\mathfrak{so}(11)$)	
B_3	0	C_4	$B_3 \oplus C_4$
(III,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_1	0	C_4	$A_1 \oplus C_4$
(IV^{ns},$\mathfrak{su}(2)$)	(I_1,n_0)	(I_2^{*ns},$\mathfrak{so}(11)$)	

g_2	0	C_4	$C_4 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_3	A_1	C_4	$A_1 \oplus A_3 \oplus C_4$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	0	C_4	$A_2 \oplus C_4$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(11))$	
A_3	A_1	C_4	$A_1 \oplus A_3 \oplus C_4$
B_3	0	C_4	$B_3 \oplus C_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
B_3	0	C_5	$B_3 \oplus C_5$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	A_1	C_5	$A_1 \oplus A_3 \oplus C_5$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(12))$	
A_3	A_1	C_5	$A_1 \oplus A_3 \oplus C_5$
B_3	0	C_5	$B_3 \oplus C_5$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
B_3	A_2	C_2	$A_2 \oplus B_3 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	0	C_2	$A_1 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
$A_1 \oplus A_2$	0	C_2	$A_1 \oplus A_2 \oplus C_2$
g_2	0	C_2	$C_2 \oplus g_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
g_2	0	C_2	$C_2 \oplus g_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
g_2	A_1	C_2	$A_1 \oplus C_2 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_3	A_3	C_2	$A_3^2 \oplus C_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2^2 \oplus C_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(7))$	
A_3	A_3	C_2	$A_3^2 \oplus C_2$
A_2	$A_1 \oplus A_2$	C_2	$A_1 \oplus A_2^2 \oplus C_2$
B_3	A_2	C_2	$A_2 \oplus B_3 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
B_3	A_2	A_1^3	$A_1^3 \oplus A_2 \oplus B_3$

(III,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_1	0	A_1^2	A_1^3
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
$A_1 \oplus A_2$	0	A_1^3	$A_1^4 \oplus A_2$
g_2	0	A_1^3	$A_1^3 \oplus g_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
g_2	0	A_1^2	$A_1^2 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_3	A_3	A_1^3	$A_1^3 \oplus A_3^2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_2	A_2	A_1^2	$A_1^2 \oplus A_2^2$
A_2	A_1^2	A_1^2	$A_1^4 \oplus A_2$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{so}(8)$)	
A_3	A_3	A_1^3	$A_1^3 \oplus A_3^2$
B_3	A_2	A_1^3	$A_1^3 \oplus A_2 \oplus B_3$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
B_3	A_1	C_2	$A_1 \oplus B_3 \oplus C_2$
(III,$\mathfrak{su}(2)$)	(II,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(III,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
$A_1 \oplus A_2$	0	C_2	$A_1 \oplus A_2 \oplus C_2$
g_2	0	C_2	$C_2 \oplus g_2$
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
g_2	0	C_2	$C_2 \oplus g_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
g_2	0	C_2	$C_2 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_3	A_2	C_2	$A_2 \oplus A_3 \oplus C_2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_2	A_1^2	C_2	$A_1^2 \oplus A_2 \oplus C_2$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{so}(9)$)	
A_3	A_2	C_2	$A_2 \oplus A_3 \oplus C_2$
B_3	A_1	C_2	$A_1 \oplus B_3 \oplus C_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
B_3	A_3	0	$A_3 \oplus B_3$
B_3	$A_1 \oplus A_2$	0	$A_1 \oplus A_2 \oplus B_3$

$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	A_2	0	$A_1 \oplus A_2^2$
$A_1 \oplus A_2$	A_1^2	0	$A_1^3 \oplus A_2$
g_2	A_1^2	0	$A_1^2 \oplus g_2$
g_2	A_2	0	$A_2 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	$A_1 \oplus A_3$	0	$A_1 \oplus A_3^2$
A_3	A_2^2	0	$A_2^2 \oplus A_3$
A_3	A_4	0	$A_3 \oplus A_4$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{su}(3))$	
A_3	$A_1 \oplus A_3$	0	$A_1 \oplus A_3^2$
A_3	A_2^2	0	$A_2^2 \oplus A_3$
A_3	A_4	0	$A_3 \oplus A_4$
B_3	A_3	0	$A_3 \oplus B_3$
B_3	$A_1 \oplus A_2$	0	$A_1 \oplus A_2 \oplus B_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_3	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	D_5	A_1	$A_1 \oplus D_5$
$(\mathfrak{su}(2))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
0	D_5	A_1	$A_1 \oplus D_5$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	D_4	C_2	$C_2 \oplus D_4$
$(\mathfrak{su}(2))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
0	D_4	C_2	$C_2 \oplus D_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	A_2	C_3	$A_2 \oplus C_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	D_4	C_3	$C_3 \oplus D_4$
$(\mathfrak{su}(2))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
0	D_4	C_3	$C_3 \oplus D_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_4	0	A_4
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	D_6	0	D_6
$(\mathfrak{su}(2))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(7))$	

0	D_6	0	D_6
(III,$\mathfrak{su}(2)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	A_3	0	A_3
(I_2,$\mathfrak{su}(2)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
0	D_5	0	D_5
($\mathfrak{su}(2)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(9)$)	
0	D_5	0	D_5
(III,$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	B_3	0	$A_1 \oplus B_3$
(III,$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	g_2	0	g_2
(III,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_2	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	g_2	0	g_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_1	0	A_1^2
(IV^{ns},$\mathfrak{su}(2)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	A_1
(I_2,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	D_4	0	$A_1 \oplus D_4$
(I_2,$\mathfrak{su}(2)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	B_4	0	B_4
($\mathfrak{su}(2)$)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
A_1	D_4	0	$A_1 \oplus D_4$
0	B_4	0	B_4
(III,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	A_1	C_2	$A_1^2 \oplus C_2$
(III,$\mathfrak{su}(2)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	C_2	C_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	0	C_2	$A_1 \oplus C_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	C_2	C_2

$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
0	A_1	C_2	$A_1 \oplus C_2$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	C_3	$A_1 \oplus C_3$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	A_1	C_3	$A_1^2 \oplus C_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	A_1	C_3	$A_1 \oplus C_3$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
A_1	A_1	C_3	$A_1^2 \oplus C_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_4	$A_1 \oplus C_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_4	C_4
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	C_4	$A_1 \oplus C_4$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	A_1	C_4	$A_1^2 \oplus C_4$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	C_4	C_4
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
A_1	A_1	C_4	$A_1^2 \oplus C_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	B_3	A_1	$A_1^2 \oplus B_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_2	A_1	$A_1 \oplus A_2$

$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	B_4	A_1	$A_1 \oplus B_4$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
0	B_4	A_1	$A_1 \oplus B_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	A_2	A_1^2	$A_1^3 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	A_1	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	A_1	A_1^2	A_1^4
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1	A_1
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	D_4	A_1^2	$A_1^3 \oplus D_4$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	B_3	A_1	$A_1 \oplus B_3$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_1	D_4	A_1^2	$A_1^3 \oplus D_4$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	A_1	A_1	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	A_1	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	A_1
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	

0	B_3	A_1	$A_1 \oplus B_3$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
0	B_3	A_1	$A_1 \oplus B_3$
A_1	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2^2	0	0	A_2^2
C_2	0	0	C_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_5	0	0	A_5
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	
A_5	0	0	A_5
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	A_2^2
C_2	0	0	C_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_5	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_4	0	0	A_4
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	
A_5	0	0	A_5
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	A_1	$A_1 \oplus A_2^2$
C_2	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{g}_2)$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_2^2	0	C_3	$A_2^2 \oplus C_3$
C_2	0	C_3	$C_2 \oplus C_3$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_5	A_1	C_3	$A_1 \oplus A_5 \oplus C_3$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{so}(10))$	
A_5	A_1	C_3	$A_1 \oplus A_5 \oplus C_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2^2	0	C_2	$A_2^2 \oplus C_2$

C_2	0	C_2	C_2^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	0	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_4	A_1^2	C_2	$A_1^2 \oplus A_4 \oplus C_2$
A_4	A_2	C_2	$A_2 \oplus A_4 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{so}(7))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
A_4	A_1^2	C_2	$A_1^2 \oplus A_4 \oplus C_2$
A_4	A_2	C_2	$A_2 \oplus A_4 \oplus C_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_2^2	0	A_1^3	$A_1^3 \oplus A_2^2$
C_2	0	A_1^3	$A_1^3 \oplus C_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_2	0	A_1^2	$A_1^2 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_5	A_1	A_1^3	$A_1^4 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_4	A_2	A_1^2	$A_1^2 \oplus A_2 \oplus A_4$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{so}(8))$	
A_5	A_1	A_1^3	$A_1^4 \oplus A_5$
A_4	A_2	A_1^2	$A_1^2 \oplus A_2 \oplus A_4$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_2^2	0	C_2	$A_2^2 \oplus C_2$
C_2	0	C_2	C_2^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_2	0	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_4	0	C_2	$A_4 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{so}(9))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2^2	A_2	0	A_2^3
A_2^2	A_1^2	0	$A_1^2 \oplus A_2^2$

C_2	A_1^2	0	$A_1^2 \oplus C_2$
C_2	A_2	0	$A_2 \oplus C_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
A_5	A_3	0	$A_3 \oplus A_5$
A_5	$A_1 \oplus A_2$	0	$A_1 \oplus A_2 \oplus A_5$
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{su}(3)$)	
A_5	A_3	0	$A_3 \oplus A_5$
A_5	$A_1 \oplus A_2$	0	$A_1 \oplus A_2 \oplus A_5$
(IV^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	A_2	A_1	$A_1 \oplus A_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
(I_3^s,$\mathfrak{su}(3)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	B_3	A_1	$A_1 \oplus B_3$
($\mathfrak{su}(3)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(10)$)	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
(IV^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
0	A_1	C_2	$A_1 \oplus C_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
0	B_3	C_2	$B_3 \oplus C_2$
($\mathfrak{su}(3)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(11)$)	
0	B_3	C_2	$B_3 \oplus C_2$
A_1	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
(IV^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
0	A_1	C_3	$A_1 \oplus C_3$
(I_3^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_1	A_2	C_3	$A_1 \oplus A_2 \oplus C_3$
(I_3^s,$\mathfrak{su}(3)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
0	A_1	C_3	$A_1 \oplus C_3$
($\mathfrak{su}(3)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(12)$)	
A_1	A_2	C_3	$A_1 \oplus A_2 \oplus C_3$
(IV^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_3	0	A_3
(I_3^s,$\mathfrak{su}(3)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_1	D_5	0	$A_1 \oplus D_5$
($\mathfrak{su}(3)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(7)$)	

A_1	D_5	0	$A_1 \oplus D_5$
(IV^s,$\mathfrak{su}(3)$)	(I$_4^{ns}$,$\mathfrak{sp}(2)$)	(I$_1^{*ns}$,$\mathfrak{so}(9)$)	
0	A_2	0	A_2
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_4^{ns}$,$\mathfrak{sp}(2)$)	(I$_1^{*ns}$,$\mathfrak{so}(9)$)	
A_1	D_4	0	$A_1 \oplus D_4$
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_5^{ns}$,$\mathfrak{sp}(2)$)	(I$_1^{*ns}$,$\mathfrak{so}(9)$)	
0	B_4	0	B_4
($\mathfrak{su}(3)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(9)$)	
A_1	D_4	0	$A_1 \oplus D_4$
0	B_4	0	B_4
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_6^{ns}$,$\mathfrak{sp}(3)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	
0	D_6	0	D_6
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_6^{ns}$,$\mathfrak{sp}(3)$)	(I$_2^{*ns}$,$\mathfrak{so}(11)$)	
0	D_5	A_1	$A_1 \oplus D_5$
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_6^{ns}$,$\mathfrak{sp}(3)$)	(I$_2^{*s}$,$\mathfrak{so}(12)$)	
0	D_5	C_2	$C_2 \oplus D_5$
(IV^s,$\mathfrak{su}(3)$)	(III,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_2	A_1	0	$A_1 \oplus A_2$
(IV^s,$\mathfrak{su}(3)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_2	g_2	0	$A_2 \oplus g_2$
(IV^s,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_2	A_1	0	$A_1 \oplus A_2$
(IV^s,$\mathfrak{su}(3)$)	(I$_3^{ns}$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
0	A_1	0	A_1
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_3	A_2	0	$A_2 \oplus A_3$
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_3^{ns}$,$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	
A_2	B_3	0	$A_2 \oplus B_3$
($\mathfrak{su}(3)$)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
A_2	B_3	0	$A_2 \oplus B_3$
(IV^s,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	
A_2	0	C_2	$A_2 \oplus C_2$
(IV^s,$\mathfrak{su}(3)$)	(I$_3^{ns}$,$\mathfrak{su}(2)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	
0	0	C_2	C_2
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	
A_3	A_1	C_2	$A_1 \oplus A_3 \oplus C_2$
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_3^{ns}$,$\mathfrak{su}(2)$)	(I$_1^{*s}$,$\mathfrak{so}(10)$)	
A_2	0	C_2	$A_2 \oplus C_2$

$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_3	A_1	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	0	C_3	$A_2 \oplus C_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	C_3	C_3
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_3	0	C_3	$A_3 \oplus C_3$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	0	C_3	$A_2 \oplus C_3$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
A_3	0	C_3	$A_3 \oplus C_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_2	0	C_4	$A_2 \oplus C_4$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	0	C_4	$A_3 \oplus C_4$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_2	0	C_4	$A_2 \oplus C_4$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(12))$	
A_3	0	C_4	$A_3 \oplus C_4$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	A_1	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	A_1	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	A_1	A_1^2
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_3	A_2	A_1	$A_1 \oplus A_2 \oplus A_3$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	B_3	A_1	$A_1 \oplus A_2 \oplus B_3$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_2	B_3	A_1	$A_1 \oplus A_2 \oplus B_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_2	A_1	A_1^2	$A_1^3 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_1	A_1
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_3	A_2	A_1^2	$A_1^2 \oplus A_2 \oplus A_3$

$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_2	A_1	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_3	A_2	A_1^2	$A_1^2 \oplus A_2 \oplus A_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_2	0	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	A_1
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_3	A_1	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_2	A_1	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_3	A_1	A_1	$A_1^2 \oplus A_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_7	A_1	A_1	$A_1^2 \oplus A_7$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(4))$	(n_0)	$(\mathfrak{g}_2)$	
A_7	A_1	A_1	$A_1^2 \oplus A_7$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_7	0	C_3	$A_7 \oplus C_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_7	A_1	C_2	$A_1 \oplus A_7 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_6	0	C_2	$A_6 \oplus C_2$
$(\mathfrak{su}(4))$	(n_0)	$(\mathfrak{so}(7))$	
A_7	A_1	C_2	$A_1 \oplus A_7 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_7	A_1	A_1^3	$A_1^4 \oplus A_7$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_7	0	C_2	$A_7 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_7	A_1^2	0	$A_1^2 \oplus A_7$
A_7	A_2	0	$A_2 \oplus A_7$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_3	A_2	A_1	$A_1 \oplus A_2 \oplus A_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	

A_2	A_1	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
A_3	A_2	A_1	$A_1 \oplus A_2 \oplus A_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_3	A_1	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	A_1	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
A_3	A_1	C_2	$A_1 \oplus A_3 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	A_1	C_3	$A_1 \oplus A_3 \oplus C_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_2	0	C_3	$A_2 \oplus C_3$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
A_3	A_1	C_3	$A_1 \oplus A_3 \oplus C_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_3	D_4	0	$A_3 \oplus D_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_3	A_2	0	$A_2 \oplus A_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_2	B_3	0	$A_2 \oplus B_3$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
A_2	B_3	0	$A_2 \oplus B_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	D_5	0	$A_1 \oplus D_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	B_4	A_1	$A_1 \oplus B_4$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_1	D_4	A_1	$A_1^2 \oplus D_4$
0	B_4	A_1	$A_1 \oplus B_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	D_4	C_2	$A_1 \oplus C_2 \oplus D_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	B_3	C_2	$B_3 \oplus C_2$
$(\mathfrak{su}(4))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_1	D_4	C_2	$A_1 \oplus C_2 \oplus D_4$

$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	D_6	0	D_6
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	D_6	A_1	$A_1 \oplus D_6$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	A_1	0	$A_1 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	A_1	0	$A_1 \oplus A_4$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_5	A_1	0	$A_1 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_5	0	C_2	$A_5 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_4	0	C_2	$A_4 \oplus C_2$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(10))$	
A_5	0	C_2	$A_5 \oplus C_2$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_5	0	C_3	$A_5 \oplus C_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_4	0	C_3	$A_4 \oplus C_3$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(11))$	
A_5	0	C_3	$A_5 \oplus C_3$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_5	0	C_4	$A_5 \oplus C_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_4	A_1	A_1	$A_1^2 \oplus A_4$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_5	A_1	A_1^2	$A_1^3 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_4	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_5	A_1	A_1^2	$A_1^3 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_5	0	A_1	$A_1 \oplus A_5$

$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_4	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_5	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_9	A_1	0	$A_1 \oplus A_9$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_4	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_5	0	C_2	$A_5 \oplus C_2$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_4	0	C_2	$A_4 \oplus C_2$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
A_5	0	C_2	$A_5 \oplus C_2$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_5	0	C_3	$A_5 \oplus C_3$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_4	0	C_3	$A_4 \oplus C_3$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(12))$	
A_5	0	C_3	$A_5 \oplus C_3$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_5	A_2	0	$A_2 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_5	A_1	0	$A_1 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_4	A_1	0	$A_1 \oplus A_4$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
A_5	A_1	0	$A_1 \oplus A_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_3	D_4	0	$A_3 \oplus D_4$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_3	A_2	A_1	$A_1 \oplus A_2 \oplus A_3$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	B_3	A_1	$A_1 \oplus A_2 \oplus B_3$

$(\mathfrak{su}(5))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_2	B_3	A_1	$A_1 \oplus A_2 \oplus B_3$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	A_2	C_2	$A_2 \oplus A_3 \oplus C_2$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_2	A_1	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_3	A_2	C_2	$A_2 \oplus A_3 \oplus C_2$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	D_5	0	$A_1 \oplus D_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	B_5	0	B_5
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_1	D_5	0	$A_1 \oplus D_5$
0	B_5	0	B_5
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	D_5	A_1	$A_1^2 \oplus D_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	B_4	A_1	$A_1 \oplus B_4$
$(\mathfrak{su}(5))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_1	D_5	A_1	$A_1^2 \oplus D_5$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	D_7	0	D_7
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_7	0	0	A_7
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_6	0	0	A_6
$(\mathfrak{su}(5))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_7	0	0	A_7
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_7	0	C_2	$A_7 \oplus C_2$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(5))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	

A_7	0	A_1^2	$A_1^2 \oplus A_7$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(5))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(8))$	
A_7	0	A_1^2	$A_1^2 \oplus A_7$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(5))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(9))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(10))$	
A_7	0	A_1	$A_1 \oplus A_7$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_7	0	C_2	$A_7 \oplus C_2$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_6	0	C_2	$A_6 \oplus C_2$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(11))$	
A_7	0	C_2	$A_7 \oplus C_2$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_7	0	C_3	$A_7 \oplus C_3$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_7	A_1	0	$A_1 \oplus A_7$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_7	0	0	A_7
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_5^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_6	0	0	A_6
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(2))$	$(\mathfrak{so}(9))$	
A_7	0	0	A_7
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_5	A_2	0	$A_2 \oplus A_5$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	

A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_4	A_1	A_1	$A_1^2 \oplus A_4$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_5	A_1	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_4	0	C_2	$A_4 \oplus C_2$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(12))$	
A_5	A_1	C_2	$A_1 \oplus A_5 \oplus C_2$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_3	D_4	0	$A_3 \oplus D_4$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_2	B_4	0	$A_2 \oplus B_4$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_3	D_4	0	$A_3 \oplus D_4$
A_2	B_4	0	$A_2 \oplus B_4$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	D_4	A_1	$A_1 \oplus A_3 \oplus D_4$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_2	B_3	A_1	$A_1 \oplus A_2 \oplus B_3$
$(\mathfrak{su}(6))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_3	D_4	A_1	$A_1 \oplus A_3 \oplus D_4$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	D_6	0	$A_1 \oplus D_6$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_9	0	0	A_9
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_8	0	0	A_8
$(\mathfrak{su}(6))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_9	0	0	A_9
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_9	0	A_1	$A_1 \oplus A_9$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_8	0	A_1	$A_1 \oplus A_8$
$(\mathfrak{su}(6))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_9	0	A_1	$A_1 \oplus A_9$

(I_6^s,$\mathfrak{su}(6)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*s},$\mathfrak{so}(8)$)	
A_9	0	A_1^2	$A_1^2 \oplus A_9$
(I_7^s,$\mathfrak{su}(7)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_9	0	A_1	$A_1 \oplus A_9$
(I_7^s,$\mathfrak{su}(7)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_9	0	0	A_9
(I_7^s,$\mathfrak{su}(7)$)	(I_4^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_9	0	0	A_9
(I_7^s,$\mathfrak{su}(7)$)	(I_5^{ns},$\mathfrak{sp}(2)$)	(I_1^{*ns},$\mathfrak{so}(9)$)	
A_8	0	0	A_8
($\mathfrak{su}(7)$)	($\mathfrak{sp}(2)$)	($\mathfrak{so}(9)$)	
A_9	0	0	A_9
(I_7^s,$\mathfrak{su}(7)$)	(I_6^{ns},$\mathfrak{sp}(3)$)	(I_1^{*s},$\mathfrak{so}(10)$)	
A_7	A_1	0	$A_1 \oplus A_7$
(I_7^s,$\mathfrak{su}(7)$)	(I_6^{ns},$\mathfrak{sp}(3)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_7	0	A_1	$A_1 \oplus A_7$
(I_7^s,$\mathfrak{su}(7)$)	(I_7^{ns},$\mathfrak{sp}(3)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_6	0	A_1	$A_1 \oplus A_6$
($\mathfrak{su}(7)$)	($\mathfrak{sp}(3)$)	($\mathfrak{so}(11)$)	
A_7	0	A_1	$A_1 \oplus A_7$
(I_7^s,$\mathfrak{su}(7)$)	(I_6^{ns},$\mathfrak{sp}(3)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_7	0	C_2	$A_7 \oplus C_2$
(I_7^s,$\mathfrak{su}(7)$)	(I_7^{ns},$\mathfrak{sp}(3)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_6	0	C_2	$A_6 \oplus C_2$
($\mathfrak{su}(7)$)	($\mathfrak{sp}(3)$)	($\mathfrak{so}(12)$)	
A_7	0	C_2	$A_7 \oplus C_2$
(I_7^s,$\mathfrak{su}(7)$)	(I_8^{ns},$\mathfrak{sp}(4)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_5	A_2	0	$A_2 \oplus A_5$
(I_7^s,$\mathfrak{su}(7)$)	(I_9^{ns},$\mathfrak{sp}(4)$)	(I_2^{*ns},$\mathfrak{so}(11)$)	
A_4	B_3	0	$A_4 \oplus B_3$
($\mathfrak{su}(7)$)	($\mathfrak{sp}(4)$)	($\mathfrak{so}(11)$)	
A_5	A_2	0	$A_2 \oplus A_5$
A_4	B_3	0	$A_4 \oplus B_3$
(I_7^s,$\mathfrak{su}(7)$)	(I_8^{ns},$\mathfrak{sp}(4)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_5	A_2	A_1	$A_1 \oplus A_2 \oplus A_5$
(I_7^s,$\mathfrak{su}(7)$)	(I_9^{ns},$\mathfrak{sp}(4)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_4	A_1	A_1	$A_1^2 \oplus A_4$
($\mathfrak{su}(7)$)	($\mathfrak{sp}(4)$)	($\mathfrak{so}(12)$)	

A_5	A_2	A_1	$A_1 \oplus A_2 \oplus A_5$
$(\mathrm{I}_7^s,\mathfrak{su}(7))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_3	D_5	0	$A_3 \oplus D_5$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_4^{ns},\mathfrak{sp}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_9	0	0	A_9
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_9	0	A_1	$A_1 \oplus A_9$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_7^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_8	0	A_1	$A_1 \oplus A_8$
$(\mathfrak{su}(8))$	$(\mathfrak{sp}(3))$	$(\mathfrak{so}(11))$	
A_9	0	A_1	$A_1 \oplus A_9$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_9	0	C_2	$A_9 \oplus C_2$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_7	A_1	0	$A_1 \oplus A_7$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_6	A_1	0	$A_1 \oplus A_6$
$(\mathfrak{su}(8))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_7	A_1	0	$A_1 \oplus A_7$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_7	A_1	A_1	$A_1^2 \oplus A_7$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_6	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(8))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(12))$	
A_7	A_1	A_1	$A_1^2 \oplus A_7$
$(\mathrm{I}_8^s,\mathfrak{su}(8))$	$(\mathrm{I}_{10}^{ns},\mathfrak{sp}(5))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_5	D_4	0	$A_5 \oplus D_4$
$(\mathrm{I}_9^s,\mathfrak{su}(9))$	$(\mathrm{I}_6^{ns},\mathfrak{sp}(3))$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_{11}	0	0	A_{11}
$(\mathrm{I}_9^s,\mathfrak{su}(9))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_9	0	0	A_9
$(\mathrm{I}_9^s,\mathfrak{su}(9))$	$(\mathrm{I}_9^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_8	0	0	A_8
$(\mathfrak{su}(9))$	$(\mathfrak{sp}(4))$	$(\mathfrak{so}(11))$	
A_9	0	0	A_9
$(\mathrm{I}_9^s,\mathfrak{su}(9))$	$(\mathrm{I}_8^{ns},\mathfrak{sp}(4))$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	

A_9	0	A_1	$A_1 \oplus A_9$
(I_9^s,$\mathfrak{su}(9)$)	(I_9^{ns},$\mathfrak{sp}(4)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_8	0	A_1	$A_1 \oplus A_8$
($\mathfrak{su}(9)$)	($\mathfrak{sp}(4)$)	($\mathfrak{so}(12)$)	
A_9	0	A_1	$A_1 \oplus A_9$
(I_9^s,$\mathfrak{su}(9)$)	(I_{10}^{ns},$\mathfrak{sp}(5)$)	(I_2^{*s},$\mathfrak{so}(12)$)	
A_7	A_2	0	$A_2 \oplus A_7$

3	2	1	3	GS Total:
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
				0
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	A_2	A_1	$A_1 \oplus A_2$
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	A_1	A_1
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	A_1	A_1	A_1^2
($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{g}_2$)	
0	0	A_2	A_1	$A_1 \oplus A_2$
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	A_1	C_3	$A_1 \oplus C_3$
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	0	C_3	C_3
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_1,n_0)	(I_1^{*s},$\mathfrak{so}(10)$)	
0	0	0	C_3	C_3
($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{so}(10)$)	
0	0	A_1	C_3	$A_1 \oplus C_3$
(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_2^{*ns},$\mathfrak{so}(11)$)	

0	0	0	C_4	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
0	0	0	C_4	C_4
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(11))$	
0	0	0	C_4	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
0	0	0	C_5	C_5
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	0	A_1	C_2	$A_1 \oplus C_2$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(7))$	
0	0	A_2	C_2	$A_2 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	A_2	A_1^3	$A_1^3 \oplus A_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1^3	A_1^3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
0	0	0	A_1^2	A_1^2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(8))$	
0	0	A_2	A_1^3	$A_1^3 \oplus A_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	A_1	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
0	0	0	C_2	C_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{so}(9))$	
0	0	A_1	C_2	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_3	0	A_3
0	0	$A_1 \oplus A_2$	0	$A_1 \oplus A_2$

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1^2	0	A_1^2
0	0	A_2	0	A_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{su}(3))$	
0	0	A_3	0	A_3
0	0	$A_1 \oplus A_2$	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	A_1
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	0	A_1
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	A_1
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	A_2	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	
A_1	0	A_1	C_3	$A_1^2 \oplus C_3$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	
A_1	0	0	C_4	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	
A_1	0	0	C_5	$A_1 \oplus C_5$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	0	A_2	C_2	$A_1 \oplus A_2 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	
A_1	0	A_2	A_1^3	$A_1^4 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	
A_1	0	A_1	C_2	$A_1^2 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	A_3	0	$A_1 \oplus A_3$
A_1	0	$A_1 \oplus A_2$	0	$A_1^2 \oplus A_2$

2	1	5	1	2	3	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	C_4
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	

C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
C_4	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_1	0	0	0	A_1	A_1^3
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	A_2	0	0	0	A_1	$A_1^2 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_1	A_2	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1^2
(n_0)	(n_0)	$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
0	A_1	0	0	0	A_1	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	

A_1	A_1	0	0	0	A_1	A_1^3
(n_0)	(n_0)	($\mathfrak{e}_7$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	
A_1	A_1	0	0	0	A_1	A_1^3
(I_0,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2	0	0	0	0	A_2
(I_0,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2	0	0	0	0	A_2
(I_0,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_2	0	0	0	0	A_2
(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_1	0	0	0	0	A_1^2
(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_1	0	0	0	0	A_1^2
(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_1	0	0	0	0	A_1^2
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	A_1	0	0	0	0	A_1
(n_0)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	

A_1	A_1	0	0	0	A_1	A_1^3
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_1	0	0	0	A_1	A_1^2
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_1	A_2	0	0	0	A_1	$A_1^2 \oplus A_2$
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
0	A_1	0	0	0	A_1	A_1^2
(n_0)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	
A_1	A_2	0	0	0	A_1	$A_1^2 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
B_3	0	0	0	0	0	B_3
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
B_3	0	0	0	0	0	B_3
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_3	0	0	0	0	0	A_3
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_3	0	0	0	0	0	A_3
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
B_3	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
$A_1 \oplus A_2$	0	0	0	0	A_1	$A_1^2 \oplus A_2$
g_2	0	0	0	0	A_1	$A_1 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	0	A_1	$A_1^2 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III*,$\mathfrak{e}_7$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
B_3	0	0	0	0	0	B_3

$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	0	A_3
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
B_3	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	

A_3	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	0	A_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	
B_3	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
g_2	0	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
$A_1 \oplus A_2$	0	0	0	0	A_1	$A_1^2 \oplus A_2$
g_2	0	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
B_3	0	0	0	0	A_1	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	0	0	0	0	0	A_5

(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_2^2	0	0	0	0	A_1	$A_1 \oplus A_2^2$
C_2	0	0	0	0	A_1	$A_1 \oplus C_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_5	0	0	0	0	A_1	$A_1 \oplus A_5$
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	
A_5	0	0	0	0	A_1	$A_1 \oplus A_5$
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_2^2	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	C_2
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_4	0	0	0	0	0	A_4
(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_4	0	0	0	0	0	A_4
(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	
A_4	0	0	0	0	0	A_4
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	
A_5	0	0	0	0	0	A_5
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	
A_2^2	0	0	0	0	A_1	$A_1 \oplus A_2^2$
C_2	0	0	0	0	A_1	$A_1 \oplus C_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	

A_5	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	
A_4	0	0	0	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	
A_5	0	0	0	0	A_1	$A_1 \oplus A_5$

2	2	1	5	1	3	GS Total:
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
						0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	

C_3	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
(n_0)	(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_2	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	0	A_1	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_2	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	0	A_1	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	0	A_1	A_1^2
(n_0)	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
0	0	A_2	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

0	0	A_1	0	0	0	A_1
(n_0)	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
0	0	A_2	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	g_2	0	0	0	0	g_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
0	g_2	0	0	0	0	g_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	A_1	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	A_1	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	g_2	0	0	0	A_1	$A_1 \oplus g_2$
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
0	g_2	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	g_2	0	0	0	0	g_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
0	g_2	0	0	0	0	g_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_4	0	0	0	0	0	C_4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	C_3	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	C_3	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
0	C_3	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	C_3	0	0	0	0	C_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	C_3	0	0	0	0	C_3
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
0	C_3	0	0	0	0	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
g_2	0	A_1	0	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	A_2	0	0	0	A_2^2
$(\mathfrak{su}(2))$	(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_2	0	A_2	0	0	0	A_2^2
g_2	0	A_1	0	0	0	$A_1 \oplus g_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	A_1	0	0	A_1	A_1^3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	A_1	0	0	A_1	$A_1^2 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	A_2	0	0	A_1	$A_1 \oplus A_2^2$
$(\mathfrak{su}(2))$	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_2	0	A_2	0	0	A_1	$A_1 \oplus A_2^2$
g_2	0	A_1	0	0	A_1	$A_1^2 \oplus g_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

g_2	0	A_1	0	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	A_2	0	0	0	A_2^2
$(\mathfrak{su}(2))$	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_2	0	A_2	0	0	0	A_2^2
g_2	0	A_1	0	0	0	$A_1 \oplus g_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	A_1	A_1^3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	A_1	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	0	0	0	A_1	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_1	0	0	0	A_1	A_1^3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	A_1

$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_3	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathfrak{su}(2))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
0	A_3	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_3	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathfrak{su}(2))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
0	A_3	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_3	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	A_2
$(\mathfrak{su}(2))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
0	A_3	0	0	0	0	A_3
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	0	0	0	0	A_2
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	0	0	A_3
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	0	0	0	0	A_2

$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_3	0	0	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_3	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	0	0	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_3	0	0	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	0	0	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_3	0	0	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	0	0	0	0	0	A_2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_3	0	0	0	0	0	A_3
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	A_2	0	0	0	A_1	$A_1 \oplus A_2^2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	A_1	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	A_2	0	0	0	A_1	$A_1 \oplus A_2^2$

$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_2	A_2	0	0	0	A_1	$A_1 \oplus A_2^2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	A_1	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_2	A_2	0	0	0	0	A_2^2
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	0	0	A_5
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	0	0	A_5
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	A_1	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	A_1	0	0	0	A_1	$A_1^2 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	0	0	0	0	A_1	$A_1 \oplus A_4$
$(\mathfrak{su}(4))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_4	A_1	0	0	0	A_1	$A_1^2 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	A_1	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	0	0	0	0	0	A_4
$(\mathfrak{su}(4))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_4	A_1	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_6	0	0	0	0	0	A_6
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_6	0	0	0	0	A_1	$A_1 \oplus A_6$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_6	0	0	0	0	A_1	$A_1 \oplus A_6$
$(\mathfrak{su}(5))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_6	0	0	0	0	A_1	$A_1 \oplus A_6$

$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_6	0	0	0	0	0	A_6
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_6	0	0	0	0	0	A_6
$(\mathfrak{su}(5))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_6	0	0	0	0	0	A_6
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_8	0	0	0	0	0	A_8
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_8	0	0	0	0	A_1	$A_1 \oplus A_8$
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_8	0	0	0	0	0	A_8

2	3	1	3	1	5	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
						0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
						0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	A_1	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	A_1	0	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
0	A_1	0	A_1	0	0	A_1^2

(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	
0	A_1	0	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	A_1	0	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_2,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	A_1	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_3^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	A_2	0	0	0	A_2
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	
0	0	A_1	0	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	
0	0	A_2	0	0	0	A_2

3	1	3	2	1	5	GS Total:
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	0

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	A_1	0	0	0	A_1^2

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	0	A_1	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	A_1
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
0	A_1	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
0	A_1	0	0	0	0	A_1
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	A_1
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
0	A_2	0	0	0	0	A_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2

$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_3	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_3	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_3	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_3	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_4	0	0	0	0	0	C_4

$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_4	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_4	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2

$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_2	0	A_1	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$

$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1^3	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1^3	0	A_1	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1^2	0	A_1	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1^3	0	A_1	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1^3	0	A_1	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1^2	0	A_1	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1^3	0	A_1	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^3	0	A_1	0	0	0	A_1^4

$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^2	0	A_1	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1^3	0	A_1	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1^2	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(9))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2
$(\mathfrak{so}(9))$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_2	0	0	0	0	0	C_2

$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	A_1	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	0	A_1	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	0	A_1

2	1	5	1	2	3	2	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1

$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
(n_0)	(n_0)	$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
B_3	0	0	0	0	0	0	B_3
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
B_3	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3

(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
g_2	0	0	0	0	0	0	g_2
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	g_2
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	
B_3	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	$A_1 \oplus A_2$
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_5	0	0	0	0	0	0	A_5
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	
A_5	0	0	0	0	0	0	A_5
(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_5	0	0	0	0	0	0	A_5
(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_4	0	0	0	0	0	0	A_4
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	
A_2^2	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	C_2
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{so}(7)$)	($\mathfrak{su}(2)$)	
A_5	0	0	0	0	0	0	A_5

2	2	1	5	1	3	2	GS Total:
(II,n_0)	(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	0
(II,n_0)	(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	0
(IV^{ns},$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	0

$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
							0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
							0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_2	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1

$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_2	0	0	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	A_1
(n_0)	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	0	A_2	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	g_2	0	0	0	0	0	g_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	g_2	0	0	0	0	0	g_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	g_2	0	0	0	0	0	g_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	C_4

$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	C_4
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	C_3	0	0	0	0	0	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	C_3	0	0	0	0	0	C_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	C_3	0	0	0	0	0	C_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	C_3	0	0	0	0	0	C_3
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	C_3	0	0	0	0	0	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
g_2	0	A_1	0	0	0	0	$A_1 \oplus g_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
g_2	0	A_1	0	0	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2	0	A_2	0	0	0	0	A_2^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2	0	A_2	0	0	0	0	A_2^2
$(\mathfrak{su}(2))$	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_2	0	A_2	0	0	0	0	A_2^2
g_2	0	A_1	0	0	0	0	$A_1 \oplus g_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	

A_1	0	0	0	0	0	0	A_1
(III,$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	0	A_1
(I$_2$,$\mathfrak{su}(2)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_1$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	0	A_1
(I$_2$,$\mathfrak{su}(2)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_1$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_1	0	0	0	0	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
A_1	A_1	0	0	0	0	0	A_1^2
(I$_2$,$\mathfrak{su}(2)$)	(I$_3^s$,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_3	0	0	0	0	0	A_3
(I$_2$,$\mathfrak{su}(2)$)	(I$_3^s$,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_3	0	0	0	0	0	A_3
(I$_2$,$\mathfrak{su}(2)$)	(I$_3^s$,$\mathfrak{su}(3)$)	(I$_1$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
(I$_2$,$\mathfrak{su}(2)$)	(I$_3^s$,$\mathfrak{su}(3)$)	(I$_1$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
(III,$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
(III,$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
0	A_2	0	0	0	0	0	A_2
($\mathfrak{su}(2)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
0	A_3	0	0	0	0	0	A_3
(IV^s,$\mathfrak{su}(3)$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	0	A_2
(IV^s,$\mathfrak{su}(3)$)	(III,$\mathfrak{su}(2)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	0	A_2
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_3	0	0	0	0	0	0	A_3
(I$_3^s$,$\mathfrak{su}(3)$)	(I$_2$,$\mathfrak{su}(2)$)	(I$_0$,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_3	0	0	0	0	0	0	A_3
(IV^s,$\mathfrak{su}(3)$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_2	0	0	0	0	0	0	A_2
(IV^s,$\mathfrak{su}(3)$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	

A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2	0	0	0	0	0	0	A_2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2	0	0	0	0	0	0	A_2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_3	0	0	0	0	0	0	A_3
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2	A_2	0	0	0	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2	A_2	0	0	0	0	0	A_2^2
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2	A_1	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2	A_1	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_2	A_2	0	0	0	0	0	A_2^2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_2	A_2	0	0	0	0	0	A_2^2
$(\mathfrak{su}(3))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_2	A_2	0	0	0	0	0	A_2^2
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_5	0	0	0	0	0	0	A_5
$(\mathfrak{su}(4))$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_5	0	0	0	0	0	0	A_5
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_4	A_1	0	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_4	A_1	0	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_4	0	0	0	0	0	0	A_4
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	

A_4	0	0	0	0	0	0	A_4
($\mathfrak{su}(4)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
A_4	A_1	0	0	0	0	0	$A_1 \oplus A_4$
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_6	0	0	0	0	0	0	A_6
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_6	0	0	0	0	0	0	A_6
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_6	0	0	0	0	0	0	A_6
(I_5^s,$\mathfrak{su}(5)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_6	0	0	0	0	0	0	A_6
($\mathfrak{su}(5)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
A_6	0	0	0	0	0	0	A_6
(I_6^s,$\mathfrak{su}(6)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	
A_8	0	0	0	0	0	0	A_8
(I_6^s,$\mathfrak{su}(6)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	
A_8	0	0	0	0	0	0	A_8
($\mathfrak{su}(6)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	
A_8	0	0	0	0	0	0	A_8

3	1	3	1	5	1	3	GS Total:
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
							0
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
							0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
							0
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
							0
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	0	0	0	A_1	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	A_1	0	0	0	A_1	A_1^2
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	
0	0	A_1	0	0	0	0	A_1
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	
0	0	0	0	0	0	A_1	A_1
(I_0^{*ns},$\mathfrak{g}_2$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	

A_1	0	A_1	0	0	0	A_1	A_1^3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	A_1	0	0	0	A_1	A_1^3
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	0	A_1	0	0	0	A_1	A_1^3
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	A_1	0	0	0	0	A_1^2
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	0	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	0	A_1	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
0	A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	A_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	A_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
C_2	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_3	0	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
C_3	0	0	0	0	0	A_1	$A_1 \oplus C_3$
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_3	0	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
C_3	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
C_4	0	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

C_4	0	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
C_4	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
C_2	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	A_1	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	A_1	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
C_2	0	A_1	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1	A_1^3
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1	A_1^3
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	

A_1^3	0	A_1	0	0	0	A_1	A_1^5
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1^3	0	A_1	0	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1^3	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1^3	0	0	0	0	0	A_1	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1^3	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1^2	A_1	0	0	0	0	A_1	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1	A_1^3
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1^2	A_1	0	0	0	0	A_1	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1^2	A_1	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1^2	A_1	0	0	0	0	0	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	A_1	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
C_2	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
C_2	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	A_1	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1	A_1^3
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	A_1	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	0	A_1	0	0	0	A_1	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	0	A_1	0	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1^2	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	A_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1^2	0	0	0	0	A_1	A_1^3
0	A_2	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1^2	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	A_2

2	1	5	1	3	2	1	5	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	A_1	0	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	A_1	0	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$

$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	A_1	0	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$

g_2	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
B_3	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	
B_3	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_3	0	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	

g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	g_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
B_3	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_4	0	0	0	0	0	0	0	A_4
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_4	0	0	0	0	0	0	0	A_4
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	
A_5	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_5	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_4	0	0	0	0	0	0	0	A_4
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^*,\mathfrak{e}_7)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_7)$	

A_5	0	0	0	0	0	0	0	A_5
$(I_3^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(III,\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	0	A_5
$(I_3^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	0	A_5
$(I_3^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	0	A_5
$(I_3^s,\mathfrak{su}(3))$	(I_1,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(III,\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_4	0	0	0	0	0	0	0	A_4
$(I_3^s,\mathfrak{su}(3))$	(I_1,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_4	0	0	0	0	0	0	0	A_4
$(I_3^s,\mathfrak{su}(3))$	(I_1,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_4	0	0	0	0	0	0	0	A_4
$(IV^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(III,\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(IV^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(IV^s,\mathfrak{su}(3))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	
A_2^2	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	
A_5	0	0	0	0	0	0	0	A_5

2	2	1	5	1	3	2	2	GS Total:
(II,n_0)	$(III,\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(IV^{ns},\mathfrak{su}(2))$	$(III,\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(III,\mathfrak{su}(2))$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(IV^{ns},\mathfrak{su}(2))$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(III,\mathfrak{su}(2))$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(IV^{ns},\mathfrak{su}(2))$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	0
$(I_0^{*ns},\mathfrak{g}_2)$	$(III,\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	
C_3	0	0	0	0	0	0	0	C_3
$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	
C_3	0	0	0	0	0	0	0	C_3
$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(IV^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(I_0^{*ns},\mathfrak{g}_2)$	$(IV^{ns},\mathfrak{su}(2))$	(II,n_0)	

C_3	0	0	0	0	0	0	0	C_3
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	0	0	0	0	0	A_1
(n_0)	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
0	0	A_2	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	g_2	0	0	0	0	0	0	g_2
(n_0)	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
0	g_2	0	0	0	0	0	0	g_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	C_3	0	0	0	0	0	0	C_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	C_3	0	0	0	0	0	0	C_3
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
0	C_3	0	0	0	0	0	0	C_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
g_2	0	A_1	0	0	0	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2	0	A_2	0	0	0	0	0	A_2^2
$(\mathfrak{su}(2))$	(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_2	0	A_2	0	0	0	0	0	A_2^2

g_2	0	A_1	0	0	0	0	0	$A_1 \oplus g_2$
(III,$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	A_1	0	0	0	0	0	0	A_1^2
(I_2,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	A_1	0	0	0	0	0	0	A_1^2
(III,$\mathfrak{su}(2)$)	(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	0	0	0	A_1
(I_2,$\mathfrak{su}(2)$)	(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	0	0	0	A_1
($\mathfrak{su}(2)$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_1	A_1	0	0	0	0	0	0	A_1^2
(I_2,$\mathfrak{su}(2)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_3	0	0	0	0	0	0	A_3
(I_2,$\mathfrak{su}(2)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_2	0	0	0	0	0	0	A_2
(III,$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_2	0	0	0	0	0	0	A_2
(IV^{ns},$\mathfrak{su}(2)$)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_2	0	0	0	0	0	0	A_2
($\mathfrak{su}(2)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
0	A_3	0	0	0	0	0	0	A_3
(IV^s,$\mathfrak{su}(3)$)	(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	0	0	0	0	0	0	0	A_2
(I_3^s,$\mathfrak{su}(3)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_3	0	0	0	0	0	0	0	A_3
(IV^s,$\mathfrak{su}(3)$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	0	0	0	0	0	0	0	A_2
(I_3^s,$\mathfrak{su}(3)$)	(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_3	0	0	0	0	0	0	0	A_3
(IV^s,$\mathfrak{su}(3)$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	0	0	0	0	0	0	0	A_2
($\mathfrak{su}(3)$)	($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_3	0	0	0	0	0	0	0	A_3
(I_3^s,$\mathfrak{su}(3)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	A_2	0	0	0	0	0	0	A_2^2
(I_3^s,$\mathfrak{su}(3)$)	(I_3^s,$\mathfrak{su}(3)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
(IV^s,$\mathfrak{su}(3)$)	(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	A_2	0	0	0	0	0	0	A_2^2
($\mathfrak{su}(3)$)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_2	A_2	0	0	0	0	0	0	A_2^2
(I_4^s,$\mathfrak{su}(4)$)	(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_5	0	0	0	0	0	0	0	A_5

$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_4	A_1	0	0	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_4^s,\mathfrak{su}(4))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_4	0	0	0	0	0	0	0	A_4
$(\mathfrak{su}(4))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_4	A_1	0	0	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_6	0	0	0	0	0	0	0	A_6
$(\mathrm{I}_5^s,\mathfrak{su}(5))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_6	0	0	0	0	0	0	0	A_6
$(\mathfrak{su}(5))$	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_6	0	0	0	0	0	0	0	A_6
$(\mathrm{I}_6^s,\mathfrak{su}(6))$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_8	0	0	0	0	0	0	0	A_8

3	1	3	1	5	1	3	2	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
								0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
								0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
								0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
								0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	A_1	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	

A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathfrak{so}(10))$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3

$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_3	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_4	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	

A_1	A_2	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1^3	0	A_1	0	0	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1^3	0	A_1	0	0	0	0	0	A_1^4
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1^3	0	A_1	0	0	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1^3	0	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1^3	0	0	0	0	0	0	0	A_1^3
$(\mathfrak{so}(8))$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1^3	0	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1^2	A_1	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1^2	A_1	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1^2	A_1	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(9))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_2	0	A_1	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathfrak{so}(9))$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
C_2	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2

$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
A_1	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	0	A_1	0	0	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	
0	A_1^2	0	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	
0	A_1^2	0	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	0	A_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	
0	A_1^2	0	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	0	A_2

3	1	5	1	2	3	1	5	GS Total:
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1

3	2	1	5	1	3	1	5	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
								0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	
0	0	0	0	0	A_1	0	0	A_1
$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1
$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	A_1	0	0	A_1^2
$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	0	A_1
$(\text{I}_0^{*ss},\mathfrak{so}(7))$	$(\text{III},\mathfrak{su}(2))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	0	A_1

3	1	5	1	3	2	2	1	5	GS Total:
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{III}^{*},\mathfrak{e}_7)$	
									0
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
A_1	0	0	0	0	0	0	A_1	0	A_1^2
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{III}^{*},\mathfrak{e}_7)$	
A_1	0	0	0	0	0	0	0	0	A_1

$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
A_1	0	0	0	0	0	0	A_1	0	A_1^2
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	
0	0	0	0	0	0	0	A_1	0	A_1
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	$(\text{IV}^{ns},\mathfrak{su}(2))$	(II,n_0)	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	0	0	A_1	0	A_1

2	1	5	1	3	1	5	1	3	GS Total:
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_4$
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
C_4	0	0	0	A_1	0	0	0	0	$A_1 \oplus C_4$
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
C_4	0	0	0	0	0	0	0	0	C_4
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
C_4	0	0	0	0	0	0	0	A_1	$A_1 \oplus C_4$
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
C_4	0	0	0	0	0	0	0	0	C_4
(I_0,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	0	0	A_2
(II,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
(I_1,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	0	0	A_1	$A_1 \oplus A_2$
(II,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	0	0	A_1	A_1^3
(I_1,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	0	0	A_2
(II,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
(I_1,n_0)	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(n_0)	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	

0	A_2	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	A_1	0	0	0	A_1	A_1^4
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	A_1	0	0	0	A_1	$A_1^3 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	A_1	0	0	0	A_1	A_1^3
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	A_1	0	0	0	A_1	A_1^3
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_2	0	0	A_1	0	0	0	A_1	$A_1^3 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	A_1	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	A_1	0	0	0	0	A_1^3
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	A_1	0	0	0	0	$A_1^2 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	A_1	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	A_1	0	0	0	0	A_1^2
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	A_1	0	0	0	0	$A_1^2 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_2	0	0	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_1	0	0	0	0	0	0	A_1	A_1^3
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	0	0	0	A_1	A_1^2

$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
0	A_1	0	0	0	0	0	0	A_1	A_1^2
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_1	A_2	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
0	A_1	0	0	0	0	0	0	0	A_1
(n_0)	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	0	0	0	0	0	A_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
g_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
B_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	0	0	0	0	0	A_3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	g_2

$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	A_1	0	0	0	A_1	A_1^3
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	A_1	$A_1^3 \oplus A_2$
g_2	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus g_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
B_3	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	A_1	$A_1^3 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
B_3	0	0	0	A_1	0	0	0	0	$A_1 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	A_1	0	0	0	0	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
g_2	0	0	0	A_1	0	0	0	0	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	0	$A_1^2 \oplus A_2$
g_2	0	0	0	A_1	0	0	0	0	$A_1 \oplus g_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
B_3	0	0	0	A_1	0	0	0	0	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	0	$A_1^2 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
g_2	0	0	0	0	0	0	0	0	g_2

$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	0	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
B_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus B_3$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_3$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_1	0	0	0	0	0	0	0	A_1	A_1^2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
g_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_2$
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
g_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus g_2$
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
B_3	0	0	0	0	0	0	0	A_1	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	0	0	0	0	A_1	$A_1^2 \oplus A_2$
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_3	0	0	0	0	0	0	0	0	A_3
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_1	0	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
g_2	0	0	0	0	0	0	0	0	g_2
$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_2	0	0	0	0	0	0	0	0	A_2
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
B_3	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	

A_2^2	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_2^2$
C_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_5	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2^2	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_4$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_2^2$
C_2	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_5	0	0	0	A_1	0	0	0	A_1	$A_1^2 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2^2	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_2^2$
C_2	0	0	0	A_1	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_5	0	0	0	A_1	0	0	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	0	0	0	0	0	0	0	0	A_4
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2^2	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	C_2

$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_5	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_4	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_4$
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	
A_2^2	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_2^2$
C_2	0	0	0	0	0	0	0	A_1	$A_1 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	
A_5	0	0	0	0	0	0	0	A_1	$A_1 \oplus A_5$
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^s,\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_4	0	0	0	0	0	0	0	0	A_4
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	
A_2^2	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	
A_5	0	0	0	0	0	0	0	0	A_5

3	1	3	1	5	1	3	2	2	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
									0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
									0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	A_1	0	0	0	0	0	0	A_1^2
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_1	0	A_1	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	

0	A_2	0	0	0	0	0	0	0	A_2
$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
0	A_2	0	0	0	0	0	0	0	A_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_3	0	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*s},\mathfrak{so}(10))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{so}(10))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_3	0	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*ns},\mathfrak{so}(11))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_3	0	0	0	0	0	0	0	0	C_3
$(\mathfrak{so}(11))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_3	0	0	0	0	0	0	0	0	C_3
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_4	0	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_2^{*s},\mathfrak{so}(12))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_4	0	0	0	0	0	0	0	0	C_4
$(\mathfrak{so}(12))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_4	0	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{so}(7))$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
C_2	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_0^{*ss},\mathfrak{so}(7))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathfrak{so}(7))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_1	A_2	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1^3	0	A_1	0	0	0	0	0	0	A_1^4
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1^3	0	0	0	0	0	0	0	0	A_1^3

$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1^2	A_1	0	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_0^{*s},\mathfrak{so}(8))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathfrak{so}(8))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_1^2	A_1	0	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
C_2	0	0	0	0	0	0	0	0	C_2
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_2,\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	0	0	0	0	0	0	0	0	A_1
$(\mathrm{I}_1^{*ns},\mathfrak{so}(9))$	$(\mathrm{I}_3^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathfrak{so}(9))$	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_1	A_1	0	0	0	0	0	0	0	A_1^2
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	0	A_1	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
0	A_1^2	0	0	0	0	0	0	0	A_1^2
0	A_2	0	0	0	0	0	0	0	A_2

5	1	3	1	5	1	3	2	1	5	GS Total:
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	0
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	

0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{n}_0)$	$(\mathfrak{e}_6)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	A_1	0	0	0	0	0	0	0	A_1
$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	
0	0	A_1	0	0	0	0	0	0	0	A_1

2	3	1	5	1	3	2	2	1	5	GS Total:
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
										0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
										0
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{n}_0)$	$(\mathfrak{n}_0)$	$(\mathfrak{e}_6)$	
0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{n}_0)$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	
0	0	0	0	0	0	0	0	A_1	0	A_1

2	1	5	1	3	1	5	1	3	2	2	GS Total:
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
C_4	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_4$
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
C_4	0	0	0	0	0	0	0	0	0	0	C_4
$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
0	A_2	0	0	0	0	0	0	0	0	0	A_2
$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
A_1	A_1	0	0	0	0	0	0	0	0	0	A_1^2
$(\mathrm{I}_1,\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
A_1	A_2	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathfrak{n}_0)$	$(\mathfrak{n}_0)$	$(\mathfrak{e}_6)$	$(\mathfrak{n}_0)$	$(\mathfrak{su}(3))$	$(\mathfrak{n}_0)$	$(\mathfrak{f}_4)$	$(\mathfrak{n}_0)$	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	$(\mathfrak{n}_0)$	
A_1	A_2	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
0	A_2	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
A_1	A_1	0	0	A_1	0	0	0	0	0	0	A_1^3
$(\mathrm{I}_1,\mathfrak{n}_0)$	$(\mathrm{I}_0,\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathfrak{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathfrak{n}_0)$	
A_1	A_2	0	0	A_1	0	0	0	0	0	0	$A_1^2 \oplus A_2$

(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_1	0	0	A_1	0	0	0	0	0	0	A_1^2
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_1	0	0	A_1	0	0	0	0	0	0	A_1^2
(n_0)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_1	A_2	0	0	A_1	0	0	0	0	0	0	$A_1^2 \oplus A_2$
(I_0,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_2	0	0	0	0	0	0	0	0	0	A_2
(II,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	A_1	0	0	0	0	0	0	0	0	0	A_1^2
(I_1,n_0)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	A_2	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(II,n_0)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_1	0	0	0	0	0	0	0	0	0	A_1
(I_1,n_0)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
0	A_1	0	0	0	0	0	0	0	0	0	A_1
(n_0)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
A_1	A_2	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
B_3	0	0	0	0	0	0	0	0	0	0	B_3
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_3	0	0	0	0	0	0	0	0	0	0	A_3
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	0	0	g_2
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{e}_6$)	(n_0)	($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
B_3	0	0	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
B_3	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus B_3$
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_3	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_3$
(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	A_1	0	0	0	0	0	0	A_1^2
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
g_2	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	0	0	0	$A_1^2 \oplus A_2$
g_2	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
($\mathfrak{su}(2)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	(n_0)	
B_3	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus B_3$
$A_1 \oplus A_2$	0	0	0	A_1	0	0	0	0	0	0	$A_1^2 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
B_3	0	0	0	0	0	0	0	0	0	0	B_3
(I_2,$\mathfrak{su}(2)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_3	0	0	0	0	0	0	0	0	0	0	A_3
(III,$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_1	0	0	0	0	0	0	0	0	0	0	A_1
(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
g_2	0	0	0	0	0	0	0	0	0	0	g_2
(I_2,$\mathfrak{su}(2)$)	(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	
A_2	0	0	0	0	0	0	0	0	0	0	A_2

$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
g_2	0	0	0	0	0	0	0	0	0	0	g_2
$(\mathfrak{su}(2))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
B_3	0	0	0	0	0	0	0	0	0	0	B_3
$A_1 \oplus A_2$	0	0	0	0	0	0	0	0	0	0	$A_1 \oplus A_2$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_5	0	0	0	0	0	0	0	0	0	0	A_5
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2^2	0	0	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_5	0	0	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_5	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_4	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_4$
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2^2	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_2^2$
C_2	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus C_2$
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_5	0	0	0	A_1	0	0	0	0	0	0	$A_1 \oplus A_5$
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_5	0	0	0	0	0	0	0	0	0	0	A_5
$(\mathrm{I}_3^{s},\mathfrak{su}(3))$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_4	0	0	0	0	0	0	0	0	0	0	A_4
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	
A_2^2	0	0	0	0	0	0	0	0	0	0	A_2^2
C_2	0	0	0	0	0	0	0	0	0	0	C_2
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	(n_0)	
A_5	0	0	0	0	0	0	0	0	0	0	A_5

2	2	3	1	5	1	3	2	2	1	5	GS Total:
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	
											0
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	
0	0	0	0	0	0	0	0	0	A_1	0	A_1
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	
0	0	0	0	0	0	0	0	0	A_1	0	A_1

D.2 Branching links, including all branching 0-links and 1-links

D.2.1 $\mathfrak{e}_7$-type endpoint

2	2	3	1	5	1	3	2	GS Total:
				1				
				3				

(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)				
				(IV^s,$\mathfrak{su}(3)$)				
								0

D.2.2 $\mathfrak{e}_6$-type endpoint

2	3	1	5	1	3	2	GS Total:
			1				
			3				
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)				
			(IV^s,$\mathfrak{su}(3)$)				
							0
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)				
			(IV^s,$\mathfrak{su}(3)$)				
							0

D.2.3 $a = 7/8$

2	2	3	1	5	1	3	2	2	GS Total:
				1					
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	GS Tot:
				(I_0,n_0)					
0	0	0	0	0	0	0	0	0	A_2
				A_2					

D.2.4 $a = 6/7$

2	2	3	1	5	1	3	2	GS Total:
				1				
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)				
0	0	0	0	0	0	0	0	A_2
				A_2				
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)				
0	0	0	0	0	0	0	0	A_2
				A_2				

(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	GS Tot:
				(n_0)				
0	0	0	0	0	0	0	0	A_2
				A_2				

D.2.5 $a = 5/6$

2	2	3	1	5	1	3	GS Total:
				1			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	0	A_1	$A_1 \oplus A_2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	0	0	A_2
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	0	0	A_2
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
				$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	0	0	C_3
				C_3			
0	0	0	0	0	0	0	g_2
				g_2			
(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	GS Tot:
				(n_0)			
0	0	0	0	0	0	0	C_3
				C_3			
0	0	0	0	0	0	0	g_2
				g_2			

2	3	1	5	1	3	2	GS Total:
			1				
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$				

0	0	0	0	0	0	0	A_2
			A_2				
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$				
0	0	0	0	0	0	0	A_2
			A_2				
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$				
0	0	0	0	0	0	0	A_2
			A_2				
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	GS Tot:
			(n_0)				
0	0	0	0	0	0	0	A_2
			A_2				

D.2.6 $a = 3/4$

1	5	1	3	2	GS Total:
	1				
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$				
A_2	0	0	0	0	A_2^2
	A_2				
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
	$(\mathrm{II},\mathrm{n}_0)$				
A_2	0	0	0	0	$A_2 \oplus C_3$
	C_3				
A_2	0	0	0	0	$A_2 \oplus g_2$
	g_2				
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
	$(\mathrm{I}_1,\mathrm{n}_0)$				
A_2	0	0	0	0	A_2^2
	A_2				
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$				
C_3	0	0	0	0	$A_2 \oplus C_3$
	A_2				
g_2	0	0	0	0	$A_2 \oplus g_2$

	A_2				
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
	(I_0,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
	(I_1,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(I_0,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(II,n_0)				
A_2	0	0	0	0	$A_2 \oplus C_3$
	C_3				
A_2	0	0	0	0	$A_2 \oplus g_2$
	g_2				
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(I_1,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(I_0,n_0)				
C_3	0	0	0	0	$A_2 \oplus C_3$
	A_2				
g_2	0	0	0	0	$A_2 \oplus g_2$
	A_2				
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(I_0,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
	(I_1,n_0)				
A_2	0	0	0	0	A_2^2
	A_2				
(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	GS Tot:

	(n_0)				
A_2	0	0	0	0	$A_2 \oplus C_3$
	C_3				
C_3	0	0	0	0	$A_2 \oplus C_3$
	A_2				
A_2	0	0	0	0	$A_2 \oplus g_2$
	g_2				
g_2	0	0	0	0	$A_2 \oplus g_2$
	A_2				

3	1	5	1	3	GS Total:
		1			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	A_1	$A_1^2 \oplus A_2$
		A_2			
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_2
		A_2			
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	A_1	$A_1 \oplus A_2$
		A_2			
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	A_1	$A_1 \oplus C_3$
		C_3			
0	0	0	0	A_1	$A_1 \oplus g_2$
		g_2			
$(\mathrm{IV}^{s},\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	A_1	$A_1 \oplus A_2$
		A_2			
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	GS Tot:
		(n_0)			
0	0	0	0	A_1	$A_1 \oplus C_3$
		C_3			

0	0	0	0	A_1	$A_1 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	0	A_2
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(II,n$_0$)			
0	0	0	0	0	C_3
		C_3			
0	0	0	0	0	g_2
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	0	0	A_2
		A_2			
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(3)$)	GS Tot:
		(n$_0$)			
0	0	0	0	0	C_3
		C_3			
0	0	0	0	0	g_2
		g_2			

2	3	1	3	2	2	GS Total:
	2					
(III,$\mathfrak{su}(2)$)	(I$_0^{*ss}$,$\mathfrak{so}(7)$)	(I$_0$,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n$_0$)	GS Tot:
	(III,$\mathfrak{su}(2)$)					
						0

2	2	3	1	5	1	3	GS Total:
				1			
				3			
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
				(I$_0$,n$_0$)			
				(IV^s,$\mathfrak{su}(3)$)			
							0
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
				(II,n$_0$)			

				$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$			
0	0	0	0	0	0	0	A_1
				0			
				A_1			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
				$(\mathrm{IV}^{s},\mathfrak{su}(3))$			
0	0	0	0	0	0	A_1	A_1
				0			
				0			

D.2.7 $a = 2/3$

1	5	1	3	GS Total:
	1			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
A_2	0	0	0	A_2^2
	A_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
A_2	0	0	A_1	$A_1 \oplus A_2^2$
	A_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{II},\mathrm{n}_0)$			
A_2	0	0	A_1	$A_1 \oplus A_2 \oplus C_3$
	C_3			
A_2	0	0	A_1	$A_1 \oplus A_2 \oplus g_2$
	g_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{I}_1,\mathrm{n}_0)$			
A_2	0	0	A_1	$A_1 \oplus A_2^2$
	A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
C_3	0	0	A_1	$A_1 \oplus A_2 \oplus C_3$
	A_2			
g_2	0	0	A_1	$A_1 \oplus A_2 \oplus g_2$

	A_2			
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
A_2	0	0	A_1	$A_1 \oplus A_2^2$
	A_2			
$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
	$(\mathrm{I}_1,\mathrm{n}_0)$			
A_2	0	0	A_1	$A_1 \oplus A_2^2$
	A_2			
(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	GS Tot:
	(n_0)			
A_2	0	0	A_1	$A_1 \oplus A_2 \oplus C_3$
	C_3			
C_3	0	0	A_1	$A_1 \oplus A_2 \oplus C_3$
	A_2			
A_2	0	0	A_1	$A_1 \oplus A_2 \oplus g_2$
	g_2			
g_2	0	0	A_1	$A_1 \oplus A_2 \oplus g_2$
	A_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
A_2	0	0	0	A_2^2
	A_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
	$(\mathrm{II},\mathrm{n}_0)$			
A_2	0	0	0	$A_2 \oplus C_3$
	C_3			
A_2	0	0	0	$A_2 \oplus g_2$
	g_2			
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
	$(\mathrm{I}_1,\mathrm{n}_0)$			
A_2	0	0	0	A_2^2
	A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{s},\mathfrak{su}(3))$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$			
C_3	0	0	0	$A_2 \oplus C_3$
	A_2			
g_2	0	0	0	$A_2 \oplus g_2$

	A_2			
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
	(II,n_0)			
C_3	0	0	0	C_3^2
	C_3			
C_3	0	0	0	$C_3 \oplus g_2$
	g_2			
g_2	0	0	0	$C_3 \oplus g_2$
	C_3			
g_2	0	0	0	g_2^2
	g_2			
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
	(I_1,n_0)			
C_3	0	0	0	$A_2 \oplus C_3$
	A_2			
g_2	0	0	0	$A_2 \oplus g_2$
	A_2			
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
	(I_0,n_0)			
A_2	0	0	0	A_2^2
	A_2			
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
	(II,n_0)			
A_2	0	0	0	$A_2 \oplus C_3$
	C_3			
A_2	0	0	0	$A_2 \oplus g_2$
	g_2			
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
	(I_1,n_0)			
A_2	0	0	0	A_2^2
	A_2			
(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	GS Tot:
	(n_0)			
C_3	0	0	0	C_3^2
	C_3			
C_3	0	0	0	$C_3 \oplus g_2$
	g_2			
g_2	0	0	0	$C_3 \oplus g_2$

	C_3			
g_2	0	0	0	g_2^2
	g_2			

2	3 2	1	3	2	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$) (III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
					0
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$) (III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
					0
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$) (III,$\mathfrak{su}(2)$)	(I_0,n_0)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	GS Tot:
0	0 0	0	A_1	0	A_1

2	3	1	5 1	1	3	GS Total:
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$) (I_0,n_0)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
0	0	0	0 A_2	0	A_1	$A_1 \oplus A_2$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$) (I_0,n_0)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
0	0	0	0 A_2	0	A_1	$A_1 \oplus A_2$
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$) (n_0)	(n_0)	($\mathfrak{g}_2$)	GS Tot:
0	0	0	0 A_2	0	A_1	$A_1 \oplus A_2$
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$) (I_0,n_0)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	GS Tot:
0	0	0	0 A_2	0	0	A_2
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$) (II,n_0)	(I_0,n_0)	(IV^{s},$\mathfrak{su}(3)$)	GS Tot:
0	0	0	0	0	0	C_3

			C_3			
0	0	0	0	0	0	g_2
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	0	A_2
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	0	A_2
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	0	A_2
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	0	C_3
			C_3			
0	0	0	0	0	0	g_2
			g_2			
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	GS Tot:
			(n_0)			
0	0	0	0	0	0	C_3
			C_3			
0	0	0	0	0	0	g_2
			g_2			

3	1	5	1	3	2	GS Total:
				1		
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)		
A_1	0	0	0	0	0	$A_1 \oplus D_4$
				D_4		
A_1	0	0	0	0	0	$A_1^2 \oplus A_3$
				$A_1 \oplus A_3$		
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)		

A_1	0	0	0	0	0	$A_1 \oplus D_4$
				D_4		
A_1	0	0	0	0	0	$A_1^2 \oplus A_3$
				$A_1 \oplus A_3$		
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	GS Tot:
				(n_0)		
A_1	0	0	0	0	0	$A_1 \oplus D_4$
				D_4		
A_1	0	0	0	0	0	$A_1^2 \oplus A_3$
				$A_1 \oplus A_3$		
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$		
0	0	0	0	0	0	D_4
				D_4		
0	0	0	0	0	0	$A_1 \oplus A_3$
				$A_1 \oplus A_3$		
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$		
0	0	0	0	0	0	D_4
				D_4		
0	0	0	0	0	0	$A_1 \oplus A_3$
				$A_1 \oplus A_3$		
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	$(\mathfrak{su}(2))$	GS Tot:
				(n_0)		
0	0	0	0	0	0	D_4
				D_4		
0	0	0	0	0	0	$A_1 \oplus A_3$
				$A_1 \oplus A_3$		

2	3	1	5	1	3	GS Total:
			1			
			3			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
			$(\mathrm{IV}^s,\mathfrak{su}(3))$			
						0
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			

$(\text{IV}^{s},\mathfrak{su}(3))$

0

$(\text{III},\mathfrak{su}(2))$ $(\text{I}_0^{*ns},\mathfrak{g}_2)$ (II,n_0) $(\text{IV}^{*ns},\mathfrak{f}_4)$ (I_0,n_0) $(\text{IV}^{s},\mathfrak{su}(3))$ GS Tot:

(II,n_0)

$(\text{I}_0^{*ns},\mathfrak{g}_2)$

0 0 0 0 0 0 A_1

0

A_1

$(\text{IV}^{ns},\mathfrak{su}(2))$ $(\text{I}_0^{*ns},\mathfrak{g}_2)$ (II,n_0) $(\text{IV}^{*ns},\mathfrak{f}_4)$ (I_0,n_0) $(\text{IV}^{s},\mathfrak{su}(3))$ GS Tot:

(II,n_0)

$(\text{I}_0^{*ns},\mathfrak{g}_2)$

0 0 0 0 0 0 A_1

0

A_1

$(\mathfrak{su}(2))$ $(\mathfrak{g}_2)$ (n_0) $(\mathfrak{f}_4)$ (n_0) $(\mathfrak{su}(3))$ GS Tot:

(n_0)

$(\mathfrak{g}_2)$

0 0 0 0 0 0 A_1

0

A_1

$(\text{III},\mathfrak{su}(2))$ $(\text{I}_0^{*ns},\mathfrak{g}_2)$ (II,n_0) $(\text{IV}^{*ns},\mathfrak{f}_4)$ (II,n_0) $(\text{I}_0^{*ns},\mathfrak{g}_2)$ GS Tot:

(I_0,n_0)

$(\text{IV}^{s},\mathfrak{su}(3))$

0 0 0 0 0 A_1 A_1

0

0

$(\text{IV}^{ns},\mathfrak{su}(2))$ $(\text{I}_0^{*ns},\mathfrak{g}_2)$ (II,n_0) $(\text{IV}^{*ns},\mathfrak{f}_4)$ (II,n_0) $(\text{I}_0^{*ns},\mathfrak{g}_2)$ GS Tot:

(I_0,n_0)

$(\text{IV}^{s},\mathfrak{su}(3))$

0 0 0 0 0 A_1 A_1

0

0

$(\mathfrak{su}(2))$ $(\mathfrak{g}_2)$ (n_0) $(\mathfrak{f}_4)$ (n_0) $(\mathfrak{g}_2)$ GS Tot:

(n_0)

$(\mathfrak{su}(3))$

0 0 0 0 0 A_1 A_1

0

0

D.2.8 $a = 1/2$

1	5	1	GS Total:
	1		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*s},\mathfrak{e}_6)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$		
A_2	0	A_2	A_2^3
	A_2		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III}^{*},\mathfrak{e}_7)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$		
A_1	0	A_1	A_1^3
	A_1		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$		
A_2	0	A_2	A_2^3
	A_2		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
	$(\mathrm{II},\mathrm{n}_0)$		
A_2	0	A_2	$A_2^2 \oplus C_3$
	C_3		
A_2	0	A_2	$A_2^2 \oplus g_2$
	g_2		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
	$(\mathrm{I}_1,\mathrm{n}_0)$		
A_2	0	A_2	A_2^3
	A_2		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
	$(\mathrm{I}_0,\mathrm{n}_0)$		
A_2	0	C_3	$A_2^2 \oplus C_3$
	A_2		
A_2	0	g_2	$A_2^2 \oplus g_2$
	A_2		
$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
	$(\mathrm{II},\mathrm{n}_0)$		
A_2	0	C_3	$A_2 \oplus C_3^2$
	C_3		
A_2	0	g_2	$A_2 \oplus C_3 \oplus g_2$
	C_3		

A_2	0	C_3	$A_2 \oplus C_3 \oplus g_2$
	g_2		
A_2	0	g_2	$A_2 \oplus g_2^2$
	g_2		
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	GS Tot:
	(I_1,n_0)		
A_2	0	C_3	$A_2^2 \oplus C_3$
	A_2		
A_2	0	g_2	$A_2^2 \oplus g_2$
	A_2		
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_0,n_0)		
A_2	0	A_2	A_2^3
	A_2		
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(II,n_0)		
A_2	0	A_2	$A_2^2 \oplus C_3$
	C_3		
A_2	0	A_2	$A_2^2 \oplus g_2$
	g_2		
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_1,n_0)		
A_2	0	A_2	A_2^3
	A_2		
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	GS Tot:
	(I_0,n_0)		
C_3	0	C_3	$A_2 \oplus C_3^2$
	A_2		
C_3	0	g_2	$A_2 \oplus C_3 \oplus g_2$
	A_2		
g_2	0	C_3	$A_2 \oplus C_3 \oplus g_2$
	A_2		
g_2	0	g_2	$A_2 \oplus g_2^2$
	A_2		
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_0,n_0)		
C_3	0	A_2	$A_2^2 \oplus C_3$
	A_2		

g_2	0	A_2	$A_2^2 \oplus g_2$
	A_2		
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_1,n_0)		
C_3	0	A_2	$A_2^2 \oplus C_3$
	A_2		
g_2	0	A_2	$A_2^2 \oplus g_2$
	A_2		
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_0,n_0)		
A_2	0	A_2	A_2^3
	A_2		
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(II,n_0)		
A_2	0	A_2	$A_2^2 \oplus C_3$
	C_3		
A_2	0	A_2	$A_2^2 \oplus g_2$
	g_2		
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	GS Tot:
	(I_1,n_0)		
A_2	0	A_2	A_2^3
	A_2		
(n_0)	($\mathfrak{f}_4$)	(n_0)	GS Tot:
	(n_0)		
A_2	0	C_3	$A_2 \oplus C_3^2$
	C_3		
C_3	0	A_2	$A_2 \oplus C_3^2$
	C_3		
C_3	0	C_3	$A_2 \oplus C_3^2$
	A_2		
A_2	0	g_2	$A_2 \oplus C_3 \oplus g_2$
	C_3		
A_2	0	C_3	$A_2 \oplus C_3 \oplus g_2$
	g_2		
C_3	0	A_2	$A_2 \oplus C_3 \oplus g_2$
	g_2		
g_2	0	A_2	$A_2 \oplus C_3 \oplus g_2$
	C_3		

C_3	0	g_2	$A_2 \oplus C_3 \oplus g_2$
	A_2		
g_2	0	C_3	$A_2 \oplus C_3 \oplus g_2$
	A_2		
A_2	0	g_2	$A_2 \oplus g_2^2$
	g_2		
g_2	0	A_2	$A_2 \oplus g_2^2$
	g_2		
g_2	0	g_2	$A_2 \oplus g_2^2$
	A_2		

3	2	1	GS Total:
	2		
(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0,n_0)	GS Tot:
	(II,n_0)		
0	0	e_6	e_6
	0		
0	0	$A_2 \oplus A_3$	$A_2 \oplus A_3$
	0		

5	1	3	2	GS Total:
		1		
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
		(I_0,n_0)		
0	0	0	0	D_4
		D_4		
0	0	0	0	$A_1 \oplus A_3$
		$A_1 \oplus A_3$		
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I_0,n_0)		
0	0	0	0	D_4
		D_4		
0	0	0	0	$A_1 \oplus A_3$
		$A_1 \oplus A_3$		
($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	GS Tot:
		(n_0)		
0	0	0	0	D_4
		D_4		

0 0 0 0 $A_1 \oplus A_3$

$A_1 \oplus A_3$

2 3 1 3 GS Total:

2

(III,$\mathfrak{su}(2)$) (I_0^{*ss},$\mathfrak{so}(7)$) (I_0,n_0) (IV^s,$\mathfrak{su}(3)$) GS Tot:

(III,$\mathfrak{su}(2)$)

0

(III,$\mathfrak{su}(2)$) (I_0^{*ss},$\mathfrak{so}(7)$) (I_0,n_0) (I_0^{*ns},$\mathfrak{g}_2$) GS Tot:

(III,$\mathfrak{su}(2)$)

0 0 0 A_1 A_1

0

(III,$\mathfrak{su}(2)$) (I_0^{*ss},$\mathfrak{so}(7)$) (I_0,n_0) (I_0^{*ss},$\mathfrak{so}(7)$) GS Tot:

(III,$\mathfrak{su}(2)$)

0 0 0 C_2 C_2

0

(III,$\mathfrak{su}(2)$) (I_0^{*ss},$\mathfrak{so}(7)$) (I_0,n_0) (I_0^{*s},$\mathfrak{so}(8)$) GS Tot:

(III,$\mathfrak{su}(2)$)

0 0 0 A_1^3 A_1^3

0

3 1 5 1 3 GS Total:

1

3

(IV^s,$\mathfrak{su}(3)$) (I_0,n_0) (IV^{*s},$\mathfrak{e}_6$) (I_0,n_0) (IV^s,$\mathfrak{su}(3)$) GS Tot:

(I_0,n_0)

(IV^s,$\mathfrak{su}(3)$)

0

(IV^s,$\mathfrak{su}(3)$) (I_0,n_0) (IV^{*ns},$\mathfrak{f}_4$) (I_0,n_0) (IV^s,$\mathfrak{su}(3)$) GS Tot:

(I_0,n_0)

(IV^s,$\mathfrak{su}(3)$)

0

(I_0^{*ns},$\mathfrak{g}_2$) (II,n_0) (IV^{*ns},$\mathfrak{f}_4$) (I_0,n_0) (IV^s,$\mathfrak{su}(3)$) GS Tot:

(II,n_0)

(I_0^{*ns},$\mathfrak{g}_2$)

A_1 0 0 0 0 A_1^2

0

A_1

(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
		(I_0,n_0)			
		(IV^s,$\mathfrak{su}(3)$)			
A_1	0	0	0	A_1	A_1^2
		0			
		0			

(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_0,n_0)			
		(IV^s,$\mathfrak{su}(3)$)			
A_1	0	0	0	0	A_1
		0			
		0			

(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(II,n_0)			
		(I_0^{*ns},$\mathfrak{g}_2$)			
0	0	0	0	0	A_1
		0			
		A_1			

D.2.9 $a = 0$

2	3	1	GS Total:
	2		
(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(I_0,n_0)	GS Tot:
	(III,$\mathfrak{su}(2)$)		
0	0	D_4	D_4
	0		
0	0	$A_1 \oplus A_3$	$A_1 \oplus A_3$
	0		

3	1	5	1	2	GS Total:
		1			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
		(I_0,n_0)			
A_1	0	0	0	$C_2 \oplus C_3$	$A_1 \oplus A_2 \oplus C_2 \oplus C_3$
		A_2			
A_1	0	0	0	C_4	$A_1 \oplus A_2 \oplus C_4$
		A_2			

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	A_2	0	$A_1 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	A_2	0	$A_1 \oplus A_2 \oplus C_3$
		C_3			
A_1	0	0	A_2	0	$A_1 \oplus A_2 \oplus g_2$
		g_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	A_2	0	$A_1 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	A_1	A_1	$A_1^3 \oplus A_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	A_1	A_1	$A_1^3 \oplus C_3$
		C_3			
A_1	0	0	A_1	A_1	$A_1^3 \oplus g_2$
		g_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	A_1	A_1	$A_1^3 \oplus A_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2 \oplus C_3$
		C_3			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2 \oplus g_2$
		g_2			

$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	A_1	0	$A_1^2 \oplus A_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	A_1	0	$A_1^2 \oplus A_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	A_1	0	$A_1^2 \oplus A_2$
		A_2			
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	(n_0)	GS Tot:
		(n_0)			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2 \oplus C_3$
		C_3			
A_1	0	0	A_2	A_1	$A_1^2 \oplus A_2 \oplus g_2$
		g_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	B_3	$A_1 \oplus A_2 \oplus B_3$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	0	B_3	$A_1 \oplus B_3 \oplus C_3$
		C_3			
A_1	0	0	0	B_3	$A_1 \oplus B_3 \oplus g_2$
		g_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	0	B_3	$A_1 \oplus A_2 \oplus B_3$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			

A_1	0	0	0	A_3	$A_1 \oplus A_2 \oplus A_3$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	0	A_3	$A_1 \oplus A_3 \oplus C_3$
		C_3			
A_1	0	0	0	A_3	$A_1 \oplus A_3 \oplus g_2$
		g_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	0	A_3	$A_1 \oplus A_2 \oplus A_3$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	A_1	$A_1^2 \oplus A_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	g_2	$A_1 \oplus A_2 \oplus g_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	A_2	$A_1 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
A_1	0	0	0	A_2	$A_1 \oplus A_2^2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2^2$
		A_2			
A_1	0	0	0	g_2	$A_1 \oplus A_2 \oplus g_2$
		A_2			
$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2 \oplus C_3$
		C_3			

A_1	0	0	0	g_2	$A_1 \oplus C_3 \oplus g_2$
		C_3			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2 \oplus g_2$
		g_2			
A_1	0	0	0	g_2	$A_1 \oplus g_2^2$
		g_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I_1,n_0)			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2^2$
		A_2			
A_1	0	0	0	g_2	$A_1 \oplus A_2 \oplus g_2$
		A_2			
($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	GS Tot:
		(n_0)			
A_1	0	0	0	B_3	$A_1 \oplus B_3 \oplus C_3$
		C_3			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2 \oplus C_3$
		C_3			
A_1	0	0	0	B_3	$A_1 \oplus B_3 \oplus g_2$
		g_2			
A_1	0	0	0	$A_1 \oplus A_2$	$A_1^2 \oplus A_2 \oplus g_2$
		g_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_0,n_0)			
A_1	0	0	0	A_5	$A_1 \oplus A_2 \oplus A_5$
		A_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
		(II,n_0)			
A_1	0	0	0	A_5	$A_1 \oplus A_5 \oplus C_3$
		C_3			
A_1	0	0	0	A_5	$A_1 \oplus A_5 \oplus g_2$
		g_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_1,n_0)			
A_1	0	0	0	A_5	$A_1 \oplus A_2 \oplus A_5$
		A_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_0,n_0)			

A_1	0	0	0	A_4	$A_1 \oplus A_2 \oplus A_4$
		A_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_1,n_0)			
A_1	0	0	0	A_4	$A_1 \oplus A_2 \oplus A_4$
		A_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_0,n_0)			
A_1	0	0	0	A_2^2	$A_1 \oplus A_2^3$
		A_2			
A_1	0	0	0	C_2	$A_1 \oplus A_2 \oplus C_2$
		A_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(II,n_0)			
A_1	0	0	0	A_2^2	$A_1 \oplus A_2^2 \oplus C_3$
		C_3			
A_1	0	0	0	A_2^2	$A_1 \oplus A_2^2 \oplus g_2$
		g_2			
A_1	0	0	0	C_2	$A_1 \oplus C_2 \oplus C_3$
		C_3			
A_1	0	0	0	C_2	$A_1 \oplus C_2 \oplus g_2$
		g_2			
(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I_1,n_0)			
A_1	0	0	0	A_2^2	$A_1 \oplus A_2^3$
		A_2			
A_1	0	0	0	C_2	$A_1 \oplus A_2 \oplus C_2$
		A_2			
($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	GS Tot:
		(n_0)			
A_1	0	0	0	A_5	$A_1 \oplus A_5 \oplus C_3$
		C_3			
A_1	0	0	0	A_5	$A_1 \oplus A_5 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
		(I_0,n_0)			
0	0	0	A_2	0	A_2^2
		A_2			

(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(II,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
		A_2			
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{e}_6$)	(n$_0$)	(n$_0$)	GS Tot:
		(n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(III,$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	B_3	$A_2 \oplus B_3$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	A_3	$A_2 \oplus A_3$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
		A_2			
0	0	0	0	g_2	$A_2 \oplus g_2$
		A_2			
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{e}_6$)	(n$_0$)	($\mathfrak{su}(2)$)	GS Tot:
		(n$_0$)			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
		A_2			
0	0	0	0	B_3	$A_2 \oplus B_3$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(I$_3^s$,$\mathfrak{su}(3)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	A_5	$A_2 \oplus A_5$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
		(I$_0$,n$_0$)			

0	0	0	0	A_2^2	A_2^3
		A_2			
0	0	0	0	C_2	$A_2 \oplus C_2$
		A_2			
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	(n_0)	$(\mathfrak{su}(3))$	GS Tot:
		(n_0)			
0	0	0	0	A_5	$A_2 \oplus A_5$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
		A_2			
0	0	0	0	C_4	$A_2 \oplus C_4$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	$C_2 \oplus C_3$	$C_2 \oplus C_3^2$
		C_3			
0	0	0	0	C_4	$C_3 \oplus C_4$
		C_3			
0	0	0	0	$C_2 \oplus C_3$	$C_2 \oplus C_3 \oplus g_2$
		g_2			
0	0	0	0	C_4	$C_4 \oplus g_2$
		g_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
		A_2			
0	0	0	0	C_4	$A_2 \oplus C_4$
		A_2			
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	GS Tot:
		(n_0)			
0	0	0	0	$C_2 \oplus C_3$	$C_2 \oplus C_3^2$
		C_3			
0	0	0	0	C_4	$C_3 \oplus C_4$
		C_3			
0	0	0	0	$C_2 \oplus C_3$	$C_2 \oplus C_3 \oplus g_2$
		g_2			

0	0	0	0	C_4	$C_4 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_2	0	A_2^2
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	GS Tot:
		(II,n$_0$)			
0	0	0	A_2	0	$A_2 \oplus C_3$
		C_3			
0	0	0	A_2	0	$A_2 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	A_2	0	A_2^2
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(II,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(II,n$_0$)	GS Tot:
		(II,n$_0$)			
0	0	0	A_1	A_1	$A_1^2 \oplus C_3$
		C_3			
0	0	0	A_1	A_1	$A_1^2 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(II,n$_0$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(II,n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
		C_3			

0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(II,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(II,n$_0$)	GS Tot:
		(II,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus C_3$
		C_3			
0	0	0	A_1	0	$A_1 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(II,n$_0$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(II,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus C_3$
		C_3			
0	0	0	A_1	0	$A_1 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_1$,n$_0$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	A_1	0	$A_1 \oplus A_2$
		A_2			
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{f}_4$)	(n$_0$)	(n$_0$)	GS Tot:
		(n$_0$)			
0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
		C_3			

0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
		g_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
		(I_0,n_0)			
0	0	0	0	B_3	$A_2 \oplus B_3$
		A_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
		(II,n_0)			
0	0	0	0	B_3	$B_3 \oplus C_3$
		C_3			
0	0	0	0	B_3	$B_3 \oplus g_2$
		g_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
		(I_1,n_0)			
0	0	0	0	B_3	$A_2 \oplus B_3$
		A_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{I}_2,\mathfrak{su}(2))$	GS Tot:
		(I_0,n_0)			
0	0	0	0	A_3	$A_2 \oplus A_3$
		A_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{I}_2,\mathfrak{su}(2))$	GS Tot:
		(II,n_0)			
0	0	0	0	A_3	$A_3 \oplus C_3$
		C_3			
0	0	0	0	A_3	$A_3 \oplus g_2$
		g_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{I}_2,\mathfrak{su}(2))$	GS Tot:
		(I_1,n_0)			
0	0	0	0	A_3	$A_2 \oplus A_3$
		A_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
		(I_0,n_0)			
0	0	0	0	A_1	$A_1 \oplus A_2$
		A_2			
$(\text{IV}^{s},\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
		(II,n_0)			
0	0	0	0	A_1	$A_1 \oplus C_3$
		C_3			

0	0	0	0	A_1	$A_1 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(III,$\mathfrak{su}(2)$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	0	A_1	$A_1 \oplus A_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	g_2	$A_2 \oplus g_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(II,n$_0$)			
0	0	0	0	g_2	$C_3 \oplus g_2$
		C_3			
0	0	0	0	g_2	g_2^2
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	0	g_2	$A_2 \oplus g_2$
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	A_2	A_2^2
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	GS Tot:
		(II,n$_0$)			
0	0	0	0	A_2	$A_2 \oplus C_3$
		C_3			
0	0	0	0	A_2	$A_2 \oplus g_2$
		g_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_1$,n$_0$)	(I$_2$,$\mathfrak{su}(2)$)	GS Tot:
		(I$_1$,n$_0$)			
0	0	0	0	A_2	A_2^2
		A_2			
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
		(I$_0$,n$_0$)			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
		A_2			

0	0	0	0	g_2	$A_2 \oplus g_2$
		A_2			
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
		(II,n_0)			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
		C_3			
0	0	0	0	g_2	$C_3 \oplus g_2$
		C_3			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
		g_2			
0	0	0	0	g_2	g_2^2
		g_2			
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
		(I_1,n_0)			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
		A_2			
0	0	0	0	g_2	$A_2 \oplus g_2$
		A_2			
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(2))$	GS Tot:
		(n_0)			
0	0	0	0	B_3	$B_3 \oplus C_3$
		C_3			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
		C_3			
0	0	0	0	B_3	$B_3 \oplus g_2$
		g_2			
0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
		g_2			
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		(I_0,n_0)			
0	0	0	0	A_5	$A_2 \oplus A_5$
		A_2			
$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		(II,n_0)			
0	0	0	0	A_5	$A_5 \oplus C_3$
		C_3			
0	0	0	0	A_5	$A_5 \oplus g_2$
		g_2			

$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	A_5	$A_2 \oplus A_5$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	A_4	$A_2 \oplus A_4$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	A_4	$A_4 \oplus C_3$
		C_3			
0	0	0	0	A_4	$A_4 \oplus g_2$
		g_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_3^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	A_4	$A_2 \oplus A_4$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	A_2^2	A_2^3
		A_2			
0	0	0	0	C_2	$A_2 \oplus C_2$
		A_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	A_2^2	$A_2^2 \oplus C_3$
		C_3			
0	0	0	0	A_2^2	$A_2^2 \oplus g_2$
		g_2			
0	0	0	0	C_2	$C_2 \oplus C_3$
		C_3			
0	0	0	0	C_2	$C_2 \oplus g_2$
		g_2			
$(\mathrm{IV}^s,\mathfrak{su}(3))$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^s,\mathfrak{su}(3))$	GS Tot:
		$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	A_2^2	A_2^3
		A_2			

0	0	0	0	C_2	$A_2 \oplus C_2$
		A_2			
($\mathfrak{su}(3)$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	GS Tot:
		(n_0)			
0	0	0	0	A_5	$A_5 \oplus C_3$
		C_3			
0	0	0	0	A_5	$A_5 \oplus g_2$
		g_2			

5	1	2	3	2	GS Total:
			1		
(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)		
0	0	0	0	0	D_4
			D_4		
0	0	0	0	0	$A_1 \oplus A_3$
			$A_1 \oplus A_3$		
(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)		
0	0	0	0	0	D_4
			D_4		
0	0	0	0	0	$A_1 \oplus A_3$
			$A_1 \oplus A_3$		
(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(III,$\mathfrak{su}(2)$)	(I_0^{*ss},$\mathfrak{so}(7)$)	(III,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)		
0	0	0	0	0	D_4
			D_4		
0	0	0	0	0	$A_1 \oplus A_3$
			$A_1 \oplus A_3$		

5	1	3	2	2	GS Total:
		1			
(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(IV^{ns},$\mathfrak{su}(2)$)	(II,n_0)	GS Tot:
		(I_0,n_0)			
0	0	0	0	0	D_4
		D_4			
0	0	0	0	0	$A_1 \oplus A_3$
		$A_1 \oplus A_3$			

3	1	5	1	3	1	5	GS Total:
		1					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	GS Tot:
		(I_0,n_0)					
A_1	0	0	0	A_1	0	0	$A_1^2 \oplus A_2$
		A_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	GS Tot:
		(I_0,n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
		A_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	GS Tot:
		(II,n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus C_3$
		C_3					
A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
		g_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*s},\mathfrak{e}_6)$	GS Tot:
		(I_1,n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
		A_2					
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{e}_6)$	GS Tot:
		(n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus C_3$
		C_3					
A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
		g_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	GS Tot:
		(I_0,n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
		A_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	GS Tot:
		(II,n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus C_3$
		C_3					
A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
		g_2					
$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^s,\mathfrak{su}(3))$	(I_0,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	GS Tot:
		(I_1,n_0)					

A_1	0	0	0	0	0	0	$A_1 \oplus A_2$
		A_2					
$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	GS Tot:
		(n_0)					
A_1	0	0	0	0	0	0	$A_1 \oplus C_3$
		C_3					
A_1	0	0	0	0	0	0	$A_1 \oplus g_2$
		g_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*s}$,$\mathfrak{e}_6)$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*s}$,$\mathfrak{e}_6)$	GS Tot:
		$(\mathrm{I}_0$,$\mathrm{n}_0)$					
0	0	0	0	0	0	0	A_2
		A_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*s}$,$\mathfrak{e}_6)$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	GS Tot:
		$(\mathrm{I}_0$,$\mathrm{n}_0)$					
0	0	0	0	0	0	0	A_2
		A_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns}$,$\mathfrak{g}_2)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	GS Tot:
		$(\mathrm{I}_0$,$\mathrm{n}_0)$					
0	0	0	0	A_1	0	0	$A_1 \oplus A_2$
		A_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns}$,$\mathfrak{g}_2)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	GS Tot:
		$(\mathrm{II}$,$\mathrm{n}_0)$					
0	0	0	0	A_1	0	0	$A_1 \oplus C_3$
		C_3					
0	0	0	0	A_1	0	0	$A_1 \oplus g_2$
		g_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{I}_0^{*ns}$,$\mathfrak{g}_2)$	$(\mathrm{II}$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	GS Tot:
		$(\mathrm{I}_1$,$\mathrm{n}_0)$					
0	0	0	0	A_1	0	0	$A_1 \oplus A_2$
		A_2					
$(\mathfrak{su}(3))$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	GS Tot:
		(n_0)					
0	0	0	0	A_1	0	0	$A_1 \oplus C_3$
		C_3					
0	0	0	0	A_1	0	0	$A_1 \oplus g_2$
		g_2					
$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*ns}$,$\mathfrak{f}_4)$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{s}$,$\mathfrak{su}(3))$	$(\mathrm{I}_0$,$\mathrm{n}_0)$	$(\mathrm{IV}^{*s}$,$\mathfrak{e}_6)$	GS Tot:
		$(\mathrm{I}_0$,$\mathrm{n}_0)$					

0	0	0	0	0	0	0	A_2
		A_2					
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	GS Tot:
		(II,n$_0$)					
0	0	0	0	0	0	0	C_3
		C_3					
0	0	0	0	0	0	0	g_2
		g_2					
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*s},$\mathfrak{e}_6$)	GS Tot:
		(I$_1$,n$_0$)					
0	0	0	0	0	0	0	A_2
		A_2					
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{e}_6$)	GS Tot:
		(n$_0$)					
0	0	0	0	0	0	0	C_3
		C_3					
0	0	0	0	0	0	0	g_2
		g_2					
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	GS Tot:
		(I$_0$,n$_0$)					
0	0	0	0	0	0	0	A_2
		A_2					
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	GS Tot:
		(II,n$_0$)					
0	0	0	0	0	0	0	C_3
		C_3					
0	0	0	0	0	0	0	g_2
		g_2					
(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^s,$\mathfrak{su}(3)$)	(I$_0$,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	GS Tot:
		(I$_1$,n$_0$)					
0	0	0	0	0	0	0	A_2
		A_2					
($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(3)$)	(n$_0$)	($\mathfrak{f}_4$)	GS Tot:
		(n$_0$)					
0	0	0	0	0	0	0	C_3
		C_3					
0	0	0	0	0	0	0	g_2
		g_2					

2	3	1	5	1	2	GS Total:
			1			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
			A_2			
0	0	0	0	0	C_4	$A_2 \oplus C_4$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
			A_2			
0	0	0	0	0	C_4	$A_2 \oplus C_4$
			A_2			
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	GS Tot:
			(n_0)			
0	0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
			A_2			
0	0	0	0	0	C_4	$A_2 \oplus C_4$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_2	0	A_2^2
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_2	0	$A_2 \oplus C_3$
			C_3			
0	0	0	0	A_2	0	$A_2 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_2	0	A_2^2
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
			A_2			

(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus C_3$
			C_3			
0	0	0	0	A_1	A_1	$A_1^2 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(II,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_1,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_1,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(I_0,n_0)			

0	0	0	0	A_2	0	A_2^2
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_2	0	$A_2 \oplus C_3$
			C_3			
0	0	0	0	A_2	0	$A_2 \oplus g_2$
			g_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_0,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_2	0	A_2^2
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus C_3$
			C_3			
0	0	0	0	A_1	A_1	$A_1^2 \oplus g_2$
			g_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(II,n_0)	GS Tot:
			(I_1,n_0)			
0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(I_0,n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(II,n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
			g_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_1,n_0)	GS Tot:
			(I_1,n_0)			

0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
			$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	A_1	0	$A_1 \oplus A_2$
			A_2			
$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	(n_0)	GS Tot:
			(n_0)			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
			g_2			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	B_3	$A_2 \oplus B_3$
			A_2			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	B_3	$B_3 \oplus C_3$
			C_3			
0	0	0	0	0	B_3	$B_3 \oplus g_2$
			g_2			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	B_3	$A_2 \oplus B_3$
			A_2			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_3	$A_2 \oplus A_3$
			A_2			

(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	A_3	$A_3 \oplus C_3$
			C_3			
0	0	0	0	0	A_3	$A_3 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_3	$A_2 \oplus A_3$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(III,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_1	$A_1 \oplus A_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_2	A_2^2
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_2	A_2^2
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
			A_2			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	0	g_2	$C_3 \oplus g_2$
			C_3			

0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
			g_2			
0	0	0	0	0	g_2	g_2^2
			g_2			
$(\mathrm{III},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
			A_2			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	B_3	$A_2 \oplus B_3$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	B_3	$B_3 \oplus C_3$
			C_3			
0	0	0	0	0	B_3	$B_3 \oplus g_2$
			g_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	B_3	$A_2 \oplus B_3$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_3	$A_2 \oplus A_3$
			A_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	A_3	$A_3 \oplus C_3$
			C_3			
0	0	0	0	0	A_3	$A_3 \oplus g_2$
			g_2			
$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
			$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	A_3	$A_2 \oplus A_3$
			A_2			

$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{III},\mathfrak{su}(2))$	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_1	$A_1 \oplus A_2$
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(II,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_1,n_0)	$(\text{I}_2,\mathfrak{su}(2))$	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_2	A_2^2
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_1,n_0)	$(\text{I}_2,\mathfrak{su}(2))$	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_2	A_2^2
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
			A_2			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
			A_2			
0	0	0	0	0	g_2	$A_2 \oplus g_2$
			A_2			
$(\text{IV}^{ns},\mathfrak{su}(2))$	$(\text{I}_0^{*ns},\mathfrak{g}_2)$	(II,n_0)	$(\text{IV}^{*ns},\mathfrak{f}_4)$	(I_0,n_0)	$(\text{IV}^{ns},\mathfrak{su}(2))$	GS Tot:
			(II,n_0)			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	0	g_2	$C_3 \oplus g_2$
			C_3			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
			g_2			
0	0	0	0	0	g_2	g_2^2
			g_2			

($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	GS Tot:
			(n_0)			
0	0	0	0	0	B_3	$B_3 \oplus C_3$
			C_3			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
			C_3			
0	0	0	0	0	B_3	$B_3 \oplus g_2$
			g_2			
0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_5	$A_2 \oplus A_5$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	A_5	$A_5 \oplus C_3$
			C_3			
0	0	0	0	0	A_5	$A_5 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_5	$A_2 \oplus A_5$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_4	$A_2 \oplus A_4$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_4	$A_2 \oplus A_4$
			A_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_2^2	A_2^3
			A_2			
0	0	0	0	0	C_2	$A_2 \oplus C_2$
			A_2			

(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	A_2^2	$A_2^2 \oplus C_3$
			C_3			
0	0	0	0	0	A_2^2	$A_2^2 \oplus g_2$
			g_2			
0	0	0	0	0	C_2	$C_2 \oplus C_3$
			C_3			
0	0	0	0	0	C_2	$C_2 \oplus g_2$
			g_2			
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_2^2	A_2^3
			A_2			
0	0	0	0	0	C_2	$A_2 \oplus C_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_5	$A_2 \oplus A_5$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(II,n_0)			
0	0	0	0	0	A_5	$A_5 \oplus C_3$
			C_3			
0	0	0	0	0	A_5	$A_5 \oplus g_2$
			g_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_5	$A_2 \oplus A_5$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_0,n_0)			
0	0	0	0	0	A_4	$A_2 \oplus A_4$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
			(I_1,n_0)			
0	0	0	0	0	A_4	$A_2 \oplus A_4$
			A_2			

(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	GS Tot:
			(I$_0$,n$_0$)			
0	0	0	0	0	A_2^2	A_2^3
			A_2			
0	0	0	0	0	C_2	$A_2 \oplus C_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	GS Tot:
			(I$_1$,n$_0$)			
0	0	0	0	0	A_2^2	A_2^3
			A_2			
0	0	0	0	0	C_2	$A_2 \oplus C_2$
			A_2			
(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(IV^{s},$\mathfrak{su}(3)$)	GS Tot:
			(II,n$_0$)			
0	0	0	0	0	A_2^2	$A_2^2 \oplus C_3$
			C_3			
0	0	0	0	0	A_2^2	$A_2^2 \oplus g_2$
			g_2			
0	0	0	0	0	C_2	$C_2 \oplus C_3$
			C_3			
0	0	0	0	0	C_2	$C_2 \oplus g_2$
			g_2			
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n$_0$)	($\mathfrak{f}_4$)	(n$_0$)	($\mathfrak{su}(3)$)	GS Tot:
			(n$_0$)			
0	0	0	0	0	A_5	$A_5 \oplus C_3$
			C_3			
0	0	0	0	0	A_5	$A_5 \oplus g_2$
			g_2			

2	2	3	1	5	1	2	GS Total:
				1			
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n$_0$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	GS Tot:
				(I$_0$,n$_0$)			
0	0	0	0	0	0	$C_2 \oplus C_3$	$A_2 \oplus C_2 \oplus C_3$
				A_2			
0	0	0	0	0	0	C_4	$A_2 \oplus C_4$
				A_2			
(II,n$_0$)	(IV^{ns},$\mathfrak{su}(2)$)	(I$_0^{*ns}$,$\mathfrak{g}_2$)	(II,n$_0$)	(IV^{*ns},$\mathfrak{f}_4$)	(I$_0$,n$_0$)	(I$_0$,n$_0$)	GS Tot:

				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_2	0	A_2^2
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	A_2	0	$A_2 \oplus C_3$
				C_3			
0	0	0	0	0	A_2	0	$A_2 \oplus g_2$
				g_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_0,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	A_2	0	A_2^2
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
				$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	A_1	A_1	$A_1^2 \oplus C_3$
				C_3			
0	0	0	0	0	A_1	A_1	$A_1^2 \oplus g_2$
				g_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	A_1	A_1	$A_1^2 \oplus A_2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
				C_3			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
				g_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:

				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2^2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{II},\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_1	0	$A_1 \oplus A_2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	A_1	0	$A_1 \oplus A_2$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_1,\mathrm{n}_0)$	$(\mathrm{I}_1,\mathrm{n}_0)$	GS Tot:
				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	A_1	0	$A_1 \oplus A_2$
				A_2			
(n_0)	$(\mathfrak{su}(2))$	$(\mathfrak{g}_2)$	(n_0)	$(\mathfrak{f}_4)$	(n_0)	(n_0)	GS Tot:
				(n_0)			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus C_3$
				C_3			
0	0	0	0	0	A_2	A_1	$A_1 \oplus A_2 \oplus g_2$
				g_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	0	B_3	$A_2 \oplus B_3$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{II},\mathrm{n}_0)$			
0	0	0	0	0	0	B_3	$B_3 \oplus C_3$
				C_3			
0	0	0	0	0	0	B_3	$B_3 \oplus g_2$
				g_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{III},\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{I}_1,\mathrm{n}_0)$			
0	0	0	0	0	0	B_3	$A_2 \oplus B_3$
				A_2			
$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{ns},\mathfrak{su}(2))$	$(\mathrm{I}_0^{*ns},\mathfrak{g}_2)$	$(\mathrm{II},\mathrm{n}_0)$	$(\mathrm{IV}^{*ns},\mathfrak{f}_4)$	$(\mathrm{I}_0,\mathrm{n}_0)$	$(\mathrm{I}_2,\mathfrak{su}(2))$	GS Tot:
				$(\mathrm{I}_0,\mathrm{n}_0)$			
0	0	0	0	0	0	A_3	$A_2 \oplus A_3$

				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
				(II,n_0)			
0	0	0	0	0	0	A_3	$A_3 \oplus C_3$
				C_3			
0	0	0	0	0	0	A_3	$A_3 \oplus g_2$
				g_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	A_3	$A_2 \oplus A_3$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(III,$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	A_1	$A_1 \oplus A_2$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	g_2	$A_2 \oplus g_2$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	A_2	A_2^2
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_2,$\mathfrak{su}(2)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	A_2	A_2^2
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
				A_2			
0	0	0	0	0	0	g_2	$A_2 \oplus g_2$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2^2$
				A_2			
0	0	0	0	0	0	g_2	$A_2 \oplus g_2$

				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
				(II,n_0)			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
				C_3			
0	0	0	0	0	0	g_2	$C_3 \oplus g_2$
				C_3			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
				g_2			
0	0	0	0	0	0	g_2	g_2^2
				g_2			
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(2)$)	GS Tot:
				(n_0)			
0	0	0	0	0	0	B_3	$B_3 \oplus C_3$
				C_3			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus C_3$
				C_3			
0	0	0	0	0	0	B_3	$B_3 \oplus g_2$
				g_2			
0	0	0	0	0	0	$A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus g_2$
				g_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	A_5	$A_2 \oplus A_5$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
				(II,n_0)			
0	0	0	0	0	0	A_5	$A_5 \oplus C_3$
				C_3			
0	0	0	0	0	0	A_5	$A_5 \oplus g_2$
				g_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	A_5	$A_2 \oplus A_5$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	A_4	$A_2 \oplus A_4$

				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	(I_3^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	A_4	$A_2 \oplus A_4$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_0,n_0)			
0	0	0	0	0	0	A_2^2	A_2^3
				A_2			
0	0	0	0	0	0	C_2	$A_2 \oplus C_2$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
				(I_1,n_0)			
0	0	0	0	0	0	A_2^2	A_2^3
				A_2			
0	0	0	0	0	0	C_2	$A_2 \oplus C_2$
				A_2			
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	(IV^s,$\mathfrak{su}(3)$)	GS Tot:
				(II,n_0)			
0	0	0	0	0	0	A_2^2	$A_2^2 \oplus C_3$
				C_3			
0	0	0	0	0	0	A_2^2	$A_2^2 \oplus g_2$
				g_2			
0	0	0	0	0	0	C_2	$C_2 \oplus C_3$
				C_3			
0	0	0	0	0	0	C_2	$C_2 \oplus g_2$
				g_2			
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{su}(3)$)	GS Tot:
				(n_0)			
0	0	0	0	0	0	A_5	$A_5 \oplus C_3$
				C_3			
0	0	0	0	0	0	A_5	$A_5 \oplus g_2$
				g_2			

2	3	1	5	1	3	2	GS Total:
					1		
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$)	(III,$\mathfrak{su}(2)$)	GS Tot:
					(I_0,n_0)		

0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$
(III,$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$) (I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$) (I_0,n_0)	(III,$\mathfrak{su}(2)$)	GS Tot:
0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$
(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$) (I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:
0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$
($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$) (n_0)	($\mathfrak{su}(2)$)	GS Tot:
0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$

2	2	3	1	5	1	3 1	2	GS Total:
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$) (I_0,n_0)	(III,$\mathfrak{su}(2)$)	GS Tot:
0	0	0	0	0	0	0 D_4	0	D_4
0	0	0	0	0	0	0 $A_1 \oplus A_3$	0	$A_1 \oplus A_3$
(II,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	(I_0^{*ns},$\mathfrak{g}_2$)	(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	(I_0^{*ns},$\mathfrak{g}_2$) (I_0,n_0)	(IV^{ns},$\mathfrak{su}(2)$)	GS Tot:

0	0	0	0	0	0	0	0	D_4
						D_4		
0	0	0	0	0	0	0	0	$A_1 \oplus A_3$
						$A_1 \oplus A_3$		
(n_0)	($\mathfrak{su}(2)$)	($\mathfrak{g}_2$)	(n_0)	($\mathfrak{f}_4$)	(n_0)	($\mathfrak{g}_2$)	($\mathfrak{su}(2)$)	GS Tot:
						(n_0)		
0	0	0	0	0	0	0	0	D_4
						D_4		
0	0	0	0	0	0	0	0	$A_1 \oplus A_3$
						$A_1 \oplus A_3$		

D.3 Selected interior links

D.3.1 $(a, \bar{a}) = (1/3, 1/3)$

1	5	1	GS Total:
(I_0,n_0)	(IV^{*s},$\mathfrak{e}_6$)	(I_0,n_0)	
A_2	0	A_2	A_2^2
(I_0,n_0)	(III^*,$\mathfrak{e}_7$)	(I_0,n_0)	
A_1	0	A_1	A_1^2
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_0,n_0)	
A_2	0	A_2	A_2^2
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	
A_2	0	C_3	$A_2 \oplus C_3$
A_2	0	g_2	$A_2 \oplus g_2$
(I_0,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	
A_2	0	A_2	A_2^2
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(II,n_0)	
C_3	0	C_3	C_3^2
C_3	0	g_2	$C_3 \oplus g_2$
g_2	0	C_3	$C_3 \oplus g_2$
g_2	0	g_2	g_2^2
(II,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	
C_3	0	A_2	$A_2 \oplus C_3$
g_2	0	A_2	$A_2 \oplus g_2$
(I_1,n_0)	(IV^{*ns},$\mathfrak{f}_4$)	(I_1,n_0)	
A_2	0	A_2	A_2^2
(n_0)	($\mathfrak{f}_4$)	(n_0)	
C_3	0	C_3	C_3^2

C_3	0	g_2	$C_3 \oplus g_2$
g_2	0	C_3	$C_3 \oplus g_2$
g_2	0	g_2	g_2^2

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