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by

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Abstract

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The framework of this thesis is Voiculescu's free probability theory. The main theme is the application of bialgebras, particularly Woronowicz-Kac C*-algebraic compact quantum groups, in free probability. A large part of this thesis is concerned with the class of "easy" compact quantum groups, introduced by Banica and Speicher. After a brief background section, we construct two new series $H_n^{(s)}, H_n^{[s]}$ of easy quantum groups and establish some classification results. In Chapter 4, we present a unified approach to de Finetti type results for the class of easy quantum groups. In this way we recover the classical results of de Finetti and Freedman on *exchangeable* and *rotatable* sequences, and the recent free probability analogues of Köstler-Speicher and Curran for *quantum exchangeable* and *quantum rotatable* sequences, within a common framework. In Chapter 5 we introduce a notion of *quantum spreadability*, defined as invariance under certain objects $A_i(k, n)$ which we call *quantum increasing sequence spaces*, and establish a free analogue of a famous theorem of Ryll-Nardzewski. We then consider some well-known results of Diaconis-Shahshahani on the limiting distribution of $\mathrm{Tr}(U^k)$, where U is uniformly chosen from O_n or S_n, within the context of easy quantum groups. We recover their results and establish some surprising free analogues. In Chapter 7 we consider the limiting distribution of $U_N A_N U_N^*$ and B_N, where A_N and B_N are matrices with entries in an arbitrary C*-algebra $\mathcal{B}$ and U_N is a *quantum* Haar unitary random matrix. We show that these are asymptotically free with amalgamation over $\mathcal{B}$ if A_N and B_N have limiting distributions as N goes to infinity, and that this may fail for classical Haar unitaries if $\mathcal{B}$ is infinite-dimensional. In the final chapter we use (non-coassociative) *infinitesimal* bialgebras to prove analytic subordination results in free probability.

To Vidya.

Contents

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Chapter 1

Introduction

The framework of this thesis is free probability theory, which was developed by Voiculescu in the 1980's as a tool for studying free products of operator algebras. Remarkably, there are many deep parallels between classical and free probability, evident for example in the free central limit theorem in which the role of the Gaussian distribution is played by Wigner's semicircle law. This theory has important connections with random matrices, and has led to deep results about the structure of the von Neumann algebras associated to free groups.

The theme of this thesis is the application of bialgebras in free probability. For the most part, these will be Woronowic-Kac C*-algebras [63]. These objects generalize the algebras $C(G)$, where G is a compact group, and can thus be thought of as compact *quantum* groups. The key examples which we consider in this thesis are the universal compact quantum groups S_n^+, O_n^+, U_n^+ of Wang [60, 61]. In many ways, these objects are the natural free probability analogues of the classical groups S_n, O_n and U_n. This is perhaps most apparent in the study of distributional symmetries, beginning with the free de Finetti theorem of Köstler and Speicher [40] which characterizes freeness in terms of invariance under S_n^+. Part of this thesis is devoted to extending this result, to a larger class of quantum groups in Chapter 4 and to a (seemingly) weaker invariance condition in Chapter 5. Other applications of these objects will be considered in Chapters 6 and 7. In the final chapter we will use *infinitesimal* bialgebras to establish some analytic subordination results in free probability. These objects, which first appeared in work of Joni and Rota [35], are at the center of the non-microstates approach to entropy in free probability [52] and of Voiculescu's "free analysis" [53, 57, 58].

The thesis is organized as follows. The next chapter contains preliminaries and notations, here we will recall the basic concepts in free probability. We will also recall the "easiness" condition of Banica and Speicher [10] for a compact orthogonal quantum group. The main examples are S_n, O_n and their free versions S_n^+, O_n^+. These will be the central objects of Chapters 3-6.

In Chapter 3 we present some classification results for easy quantum groups. In the "non-hyperoctahedral" case we are able to give a complete classification. The hyperoctahedral case appears to be quite difficult, and here we will introduce some new examples but leave the classification problem open.

In Chapter 4 we present a unified approach to de Finetti theorems for easy quantum groups. In this way we recover the classical results of de Finetti and Freedman and the recent free probability results from [40, 23, 24] within a common framework. We also develop a new notion of *half-independence*, corresponding to the de Finetti theorems for the the *half-liberated* quantum groups H_n^*, O_n^*.

In Chapter 5 we construct *quantum increasing sequence spaces* $A_i(k,n)$, which provide a free analogue of the space of increasing sequences $(1 \leq l_1 < \cdots < l_k \leq n)$. We define a notion of *quantum spreadability* in terms of these objects, and establish a free analogue of a well-known theorem of Ryll-Nardzewski [44].

We consider the joint distribution of $(\mathrm{Tr}(U^k))_{k \in \mathbb{N}}$, where U is chosen uniformly from an easy quantum group G, in Chapter 6. In this way we recover some well-known results of Diaconis and Shahshahani [29], and establish some surprising free analogues.

In Chapter 7 we consider the joint distribution of $U_N A_N U_N^*$ and B_N, where A_N, B_N are matrices with entries in a unital C*-algebra $\mathcal{B}$ and U_N is Haar distributed on the quantum unitary group U_N^+. We show that if A_N and B_N have limiting distributions as $N \to \infty$, then $U_N A_N U_N^*$ and B_N are asymptotically freely independent with amalgamation over $\mathcal{B}$. We show that this may fail for classical Haar unitary random matrices if the algebra $\mathcal{B}$ is infinite dimensional.

Chapters 3, 4 and 6 are based on a series of joint papers with Teodor Banica and Roland Speicher. Chapter 3 is based on [7], to appear in the Pacific Journal of Mathematics. Chapter 4 is based on the preprint [8]. Chapter 5 is based on my preprint [25]. Chapter 6 is based on [9], to appear in Probability Theory and Related Fields. Chapter 7 is based on the preprint [26], which is joint work with Roland Speicher. Chapter 8 is based on my preprints [21, 22].

Chapter 2

Background and notations

2.1 Combinatorics of classical and free probability

We begin by recalling the basic notions of noncommutative probability spaces and distributions of random variables. The reader is referred to the texts [59, 43] for further details.

Definition 2.1.1.

(1) A *noncommutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$. Elements of $\mathcal{A}$ are called *random variables*.

(2) A *W^*-probability space* is a pair (M, φ), where M is a von Neumann algebra and φ is a faithful, normal state on M. We say that (M, φ) is *tracial* if

$$\varphi(xy) = \varphi(yx), \qquad (x, y \in M).$$

Example 2.1.2. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a (classical) probability space.

(1) Let $M = L^\infty(\Omega)$ be the algebra of bounded, complex $\mathcal{F}$-measurable functions on Ω. Let $\mathbb{E} : M \to \mathbb{C}$ be the expectation functional, then $(M, \mathbb{E})$ is a tracial W*-probability space.

(2) Let $\mathcal{A} = \bigcap_{p \geq 1} L^p(\Omega)$ be the algebra of complex $\mathcal{F}$-measurable functions with finite moments of all orders. Then $(\mathcal{A}, \mathbb{E})$ is a noncommutative probability space.

Given an index set I, we let $\mathbb{C}\langle t_i, t_i^* | i \in I\rangle$ denote the $*$-algebra of polynomials in noncommuting indeterminants $(t_i)_{i \in I}$. Given random variables $(x_i)_{i \in I}$ in a noncommutative probability space $(\mathcal{A}, \varphi)$, there is a unique unital $*$-homomorphism $\text{ev}_x : \mathbb{C}\langle t_i, t_i^* | i \in I\rangle \to \mathcal{A}$ sending $t_i \mapsto x_i$ for $i \in I$. For $p \in \mathbb{C}\langle t_i, t_i^* | i \in I\rangle$ we also denote $\text{ev}_x(p)$ by $p(x)$.

Definition 2.1.3. Let $(x_i)_{i \in I}$ be a family of random variables in the noncommutative probability space $(\mathcal{A}, \varphi)$. The *joint distribution* of $(x_i)_{i \in I}$ is the linear functional $\varphi_x : \mathbb{C}\langle t_i, t_i^* | i \in I\rangle \to \mathbb{C}$ defined by

$$\varphi_x(p) = \varphi(p(x)).$$

Note that the joint distribution of $(x_i)_{i\in I}$ is determined by the collection of *joint $*$-moments*

$$\varphi_x(t_{i_1}^{\epsilon_1}\cdots t_{i_k}^{\epsilon_k}) = \varphi(x_{i_1}^{\epsilon_1}\cdots x_{i_k}^{\epsilon_k}),$$

for $i_1,\ldots,i_k \in I$ and $\epsilon_1,\ldots,\epsilon_k \in \{1,*\}$.

Remark 2.1.4. These definitions have natural "operator-valued" extensions, given by replacing $\mathbb{C}$ by a more general algebra of "scalars". This is the right setting for the notion of freeness with amalgamation, which plays a central role in this thesis.

Definition 2.1.5. An *operator-valued probability* space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ consists of a unital algebra $\mathcal{A}$, a subalgebra $1 \in \mathcal{B} \subset \mathcal{A}$, and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$, i.e. E is a linear map such that $E[1] = 1$ and

$$E[b_1ab_2] = b_1E[a]b_2$$

for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. Elements of $\mathcal{A}$ are called *$\mathcal{B}$-valued random variables*, or just random variables.

Example 2.1.6.

(1) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{G} \subset \Sigma$ be a σ-subalgebra. Let $\mathcal{A} = L^\infty(\mu)$, and let $\mathcal{B} = L^\infty(\mu|_\mathcal{G})$ be the subalgebra of bounded, $\mathcal{G}$-measurable functions on Ω. Then $(\mathcal{A}, \mathbb{E}[\cdot|\mathcal{G}])$ is an operator-valued probability space.

(2) Let (M, τ) be a tracial W*-probability space, and let $1 \in B \subset M$ be a unital W*-subalgebra. Then there is a unique conditional expectation $E : M \to B$ which preserves the state τ. Note that it is essential here that τ is assumed to be a trace.

To define the joint distribution of a family $(x_i)_{i\in I}$ in an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$, we will use the $*$-algebra $\mathcal{B}\langle t_i, t_i^* : i \in I\rangle$ of noncommutative polynomials with coefficients in $\mathcal{B}$. This algebra is spanned by monomials of the form $b_0t_{i_1}^{\epsilon_1}\cdots t_{i_k}^{\epsilon_k}b_k$, for $b_0,\ldots,b_k \in \mathcal{B}$, $i_1,\ldots,i_k \in I$ and $\epsilon_1,\ldots,\epsilon_k \in \{1,*\}$. There is a unique homomorphism from $\mathcal{B}\langle t_i, t_i^* : i \in I\rangle$ into $\mathcal{A}$ which acts as the identity on $\mathcal{B}$ and sends t_i to x_i, which we denote by $p \mapsto p(x)$.

Definition 2.1.7. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i\in I}$ be a family in $\mathcal{A}$. The *B-valued joint distribution* of the family $(x_i)_{i\in I}$ is the linear map $E_x : \mathcal{B}\langle t_i, t_i^* : i \in I\rangle \to \mathcal{B}$ defined by

$$E_x[p] = E[p(x)].$$

Observe that the joint distribution is determined by the *$\mathcal{B}$-valued joint $*$-moments*

$$E_x[b_0t_{i_1}^{\epsilon_1}\cdots t_{i_k}^{\epsilon_k}b_k] = E[b_0x_{i_1}^{\epsilon_1}\cdots x_{i_k}^{\epsilon_k}b_k]$$

for $b_0,\ldots,b_k \in \mathcal{B}$, $i_1,\ldots,i_k \in I$ and $\epsilon_1,\ldots,\epsilon_k \in \{1,*\}$.

Definition 2.1.8. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be a $\mathcal{B}$-valued probability space, and let $(\mathcal{A}_i)_{i\in I}$ be a family of subalgebras of A, each containing $\mathcal{B}$. The algebras $(\mathcal{A}_i)_{i\in I}$ are *conditionally independent given $\mathcal{B}$* if they commute with each other and we have

$$E[a_1\cdots a_m] = E[a_1]\cdots E[a_m]$$

whenever $a_j \in \mathcal{A}_{i_j}$ for *distinct* indices $i_1, \ldots, i_m$. When $\mathcal{B} = \mathbb{C}$ we say simply that the algebras $(\mathcal{A}_i)_{i \in i}$ are *independent.* If $(x_i)_{i \in I}$ is a family of random variables in $\mathcal{A}$, we say that they are conditionally independent given $\mathcal{B}$ if the algebras $\mathcal{A}_i$ generated by x_i and $\mathcal{B}$ are conditionally independent.

Example 2.1.9. Given $\mathcal{B}$-valued probability spaces $(\mathcal{A}_i, E_i : \mathcal{A}_i \to \mathcal{B})$ for $i \in I$, we can form the tensor product

$$\mathcal{A} = \bigotimes_{i \in I} \mathcal{A}_i,$$

where the tensor product is taken with respect to the natural $\mathcal{B} - \mathcal{B}$ bimodule structure on the algebras $\mathcal{A}_i$. Let $E : \mathcal{A} \to \mathcal{B}$ be the conditional expectation $E = \bigotimes E_i$. Under the natural inclusions $\mathcal{A}_i \hookrightarrow \mathcal{A}$, we have $E|_{\mathcal{A}_i} = E_i$ and the algebras $(\mathcal{A}_i)_{i \in I}$ are conditionally independent given $\mathcal{B}$.

Remark 2.1.10. The above example demonstrates the relationship between tensor products and independence. In the noncommutative context, if $(\mathcal{A}_i, E_i : \mathcal{A}_i \to \mathcal{B})$ is a family of $\mathcal{B}$-valued probability spaces, one can also construct the free product with amalgamation over $\mathcal{B}$,

$$\mathcal{A} = \mathop{*}_{i \in I} \mathcal{A}_i.$$

It was shown by Voiculescu that there is a natural conditional expectation $E : \mathcal{A} \to \mathcal{B}$, $E = *_{i \in I} E_i$. E is determined by the conditions $E|_{\mathcal{A}_i} = E_i$, and that the algebras $(\mathcal{A}_i)_{i \in I}$ are free with amalgamation over $\mathcal{B}$, in the sense of the following definition.

Definition 2.1.11 ([49])**.** Let $(\mathcal{A}, E)$ be a $\mathcal{B}$-valued probability space, and let $(\mathcal{A}_i)_{i \in I}$ be a family of subalgebras, each containing $\mathcal{B}$. The algebras $(\mathcal{A}_i)_{i \in I}$ are called *freely independent with amalgamation over* $\mathcal{B}$, or *free with respect to* E, if

$$E[a_1 \cdots a_m] = 0$$

whenever $E[a_j] = 0$ for $j = 1, \ldots, m$ and $a_j \in A_{i_j}$ for indices $i_1, \ldots, i_m \in I$ such that $i_1 \neq i_2 \neq \cdots \neq i_m$. When $\mathcal{B} = \mathbb{C}$ we simply say that the algebras $\mathcal{A}_i$ are *freely independent.*

Remark 2.1.12. Conditional independence and freeness with amalgamation also have rich combinatorial theories, which we now recall. In the free case this is due to Speicher [48], see also the text [43].

Definition 2.1.13.

(1) A *partition* π of a set S is a collection of disjoint, non-empty sets $V_1, \ldots, V_r$ such that $V_1 \cup \cdots \cup V_r = S$. $V_1, \ldots, V_r$ are called the *blocks* of π, and we set $|\pi| = r$. The collection of partitions of S will be denoted $P(S)$, or in the case that $S = \{1, \ldots, k\}$ by $P(k)$.

(2) Given $\pi, \sigma \in P(S)$, we say that $\pi \leq \sigma$ if each block of π is contained in a block of σ. There is a least element of $P(S)$ which is larger than both π and σ, which we denote by $\pi \vee \sigma$.

(3) If S is ordered, we say that $\pi \in P(S)$ is *noncrossing* if whenever V, W are blocks of π and $s_1 < t_1 < s_2 < t_2$ are such that $s_1, s_2 \in V$ and $t_1, t_2 \in W$, then $V = W$. The set of noncrossing partitions of S is denoted by $NC(S)$, or by $NC(k)$ in the case that $S = \{1, \ldots, k\}$.

(4) The noncrossing partitions can also be defined recursively, a partition $\pi \in P(S)$ is noncrossing if and only if it has a block V which is an interval, such that $\pi \setminus V$ is a noncrossing partition of $S \setminus V$.

(5) Given $i_1, \ldots, i_k$ in some index set I, we denote by $\ker \mathbf{i}$ the element of $P(k)$ whose blocks are the equivalence classes of the relation

$$s \sim t \Leftrightarrow i_s = i_t.$$

Note that if $\pi \in P(k)$, then $\pi \leq \ker \mathbf{i}$ is equivalent to the condition that whenever s and t are in the same block of π, i_s must equal i_t.

Definition 2.1.14. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space.

(1) A $\mathcal{B}$*-functional* is a n-linear map $\rho : \mathcal{A}^k \to \mathcal{B}$ such that

$$\rho(b_0 a_1 b_1, a_2 b_2 \ldots, a_n b_n) = b_0 \rho(a_1, b_1 a_2, \ldots, b_{n-1} a_n) b_n$$

for all $b_0, \ldots, b_n \in \mathcal{B}$ and $a_1, \ldots, a_n$. Equivalently, ρ is a linear map from $\mathcal{A}^{\otimes_B n}$ to $\mathcal{B}$, where the tensor product is taken with respect to the natural $\mathcal{B} - \mathcal{B}$-bimodule structure on $\mathcal{A}$.

(2) Suppose that $\mathcal{B}$ is commutative. For $k \in \mathbb{N}$ let $\rho^{(k)}$ be a $\mathcal{B}$-functional. Given $\pi \in P(n)$, we define a $\mathcal{B}$-functional $\rho^{(\pi)} : \mathcal{A}^n \to \mathcal{B}$ by the formula

$$\rho^{(\pi)}[a_1, \ldots, a_n] = \prod_{V \in \pi} \rho(V)[a_1, \ldots, a_n],$$

where if $V = (i_1 < \ldots < i_s)$ is a block of π then

$$\rho(V)[a_1, \ldots, a_n] = \rho_s(a_{i_1}, \ldots, a_{i_s}).$$

If $\mathcal{B}$ is noncommutative, there is no natural order in which to compute the product appearing in the above formula for $\rho^{(\pi)}$. However, the nesting property of noncrossing partitions allows for a natural definition of $\rho^{(\pi)}$ for $\pi \in NC(n)$, which we now recall from [48].

Definition 2.1.15. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and for $k \in \mathbb{N}$ let $\rho^{(k)} : \mathcal{A}^k \to \mathcal{B}$ be a $\mathcal{B}$-functional. Given $\pi \in NC(n)$, define a $\mathcal{B}$-functional $\rho^{(n)} : \mathcal{A}^n \to \mathcal{B}$ recursively as follows:

(1) If $\pi = 1_n$ is the partition containing only one block, define $\rho^{(\pi)} = \rho^{(n)}$.

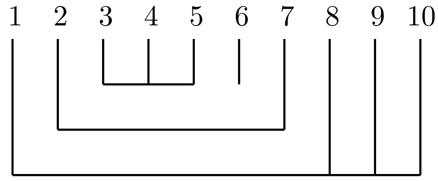
(2) Otherwise, let $V = \{l + 1, \ldots, l + s\}$ be an interval of π and define

$$\rho^{(\pi)}[a_1, \ldots, a_n] = \rho^{(\pi \setminus V)}[a_1, \ldots, a_l \cdot \rho^{(s)}(a_{l+1}, \ldots, a_{l+s}), a_{l+s+1}, \ldots, a_n]$$

for $a_1, \ldots, a_n \in \mathcal{A}$.

Example 2.1.16. Let

$$\pi = \{\{1, 8, 9, 10\}, \{2, 7\}, \{3, 4, 5\}, \{6\}\} \in NC(10),$$



then the corresponding $\rho^{(\pi)}$ is given by

$$\rho^{(\pi)}[a_1, \ldots, a_{10}] = \rho^{(4)}(a_1 \cdot \rho^{(2)}(a_2 \cdot \rho^{(3)}(a_3, a_4, a_5), \rho^{(1)}(a_6) \cdot a_7), a_8, a_9, a_{10}).$$

Definition 2.1.17. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$.

(1) The *operator-valued classical cumulants* $c_E^{(k)} : \mathcal{A}^k \to \mathcal{B}$ are the $\mathcal{B}$-functionals defined by the *classical moment-cumulant formula*

$$E[a_1 \cdots a_n] = \sum_{\pi \in P(n)} c_E^{(\pi)}[a_1, \ldots, a_n].$$

Note that the right hand side of the equation is equal to $c_E^{(n)}[a_1, \ldots, a_n]$ plus lower order terms, and hence $c_E^{(n)}$ can be solved for recursively.

(2) The *operator-valued free cumulants* $\kappa_E^{(k)} : \mathcal{A}^k \to \mathcal{B}$ are the $\mathcal{B}$-functionals defined by the *free moment-cumulant formula*

$$E[a_1, \ldots, a_n] = \sum_{\pi \in NC(n)} \kappa_E^{(\pi)}[a_1, \ldots, a_n].$$

As above, this equation can be solved recursively for $\kappa_E^{(n)}$.

While the definitions of conditional independence and freeness with amalgamation given above appear at first to be quite different, they have very similar expressions in terms of cumulants. In the free case, the following theorem is due to Speicher [48].

Theorem 2.1.18. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and $(x_i)_{i \in I}$ a family of random variables in $\mathcal{A}$.*

(1) If the algebra generated by $\mathcal{B}$ and $(x_i)_{i \in I}$ is commutative, then the variables are conditionally independent given B if and only if

$$c_E^{(n)}[b_0 x_{i_1} b_1, \ldots, x_{i_n} b_n] = 0$$

whenever there are $1 \leq k, l \leq n$ such that $i_k \neq i_l$.

(2) *The variables are free with amalgamation over $\mathcal{B}$ if and only if*

$$\kappa_E^{(n)}[b_0 x_{i_1} b_1, \ldots, x_{i_n} b_n] = 0$$

whenever there are $1 \leq k, l \leq n$ such that $i_k \neq i_l$.

Note that the condition in (1) is equivalent to the statement that if $\pi \in P(n)$, then

$$c_E^{(\pi)}[b_0 x_{i_1} b_1, \ldots, x_{i_n} b_n] = 0$$

unless $\pi \leq \ker \mathbf{i}$, and likewise in (2) for $\pi \in NC(n)$. Stronger characterizations of the joint distribution of $(x_i)_{i \in I}$ can be given by specifying what types of partitions may contribute nonzero cumulants:

Theorem 2.1.19. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$.*

(1) *Suppose that B and $(x_i)_{i \in I}$ generate a commutative algebra. The $\mathcal{B}$-valued joint distribution of $(x_i)_{i \in I}$ has the property corresponding to D in the table below if and only if for any $\pi \in P(n)$*

$$c_E^{(\pi)}[b_0 x_{i_1} b_1, \ldots, x_{i_n} b_n] = 0$$

unless $\pi \in D(n)$ and $\pi \leq \ker \mathbf{i}$.

Partitions D	*Joint distribution*
P: All partitions	*Independent*
P_h: Partitions with even block sizes	*Independent and even*
P_b: Partitions with block size ≤ 2	*Independent Gaussian*
P_2: Pair partitions	*Independent centered Gaussian*

(2) *The $\mathcal{B}$-valued joint distribution of $(x_i)_{i \in I}$ has the property corresponding to D in the the table below if and only if for any $\pi \in NC(n)$*

$$\kappa_E^{(\pi)}[b_0 x_{i_1} b_1, \ldots, x_{i_n} b_n] = 0$$

unless $\pi \in D(n)$ and $\pi \leq \ker \mathbf{i}$.

Noncrossing partitions D	*Joint distribution*
NC: Noncrossing partitions	*Freely independent*
NC_h: NC partitions with even block sizes	*Freely independent and even*
NC_b: NC partitions with block size ≤ 2	*Freely independent semicircular*
NC_2: Noncrossing pair partitions	*Freely independent centered semicircular*

Remark 2.1.20. It is clear from the definitions that the classical and free cumulants can be solved for from the joint moments. In fact, a combinatorial formula for the cumulants in terms of the moments can be given via Möbius inversion. First we recall the definition of the Möbius function on a partially ordered set.

Definition 2.1.21. Let $(P, <)$ be a finite partially ordered set. The *Möbius function* $\mu_P : P \times P \to \mathbb{Z}$ is defined by

$$\mu_P(p,q) = \begin{cases} 0, & p \not\leq q \\ 1, & p = q \\ -1 + \sum_{l\geq 1}(-1)^{l+1}\#\{(p_1,\dots,p_l) \in P^l : p < p_1 < \dots < p_l < q\}, & p < q \end{cases}.$$

Theorem 2.1.22. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i\in I}$ be a family of random variables. Define the $\mathcal{B}$-valued moment functionals $E^{(n)}$ by*

$$E^{(n)}[a_1,\dots,a_n] = E[a_1 \cdots a_n].$$

(1) Suppose that $\mathcal{B}$ is commutative. Then for any $\sigma \in P(n)$ and $a_1,\dots,a_n \in \mathcal{A}$ we have

$$c_E^{(\sigma)}[a_1,\dots,a_n] = \sum_{\substack{\pi\in P(n)\\ \pi\leq\sigma}} \mu_{P(n)}(\pi,\sigma)E^{(\pi)}[a_1,\dots,a_n].$$

(2) For any $\sigma \in NC(n)$ and $a_1,\dots,a_n \in \mathcal{A}$ we have

$$\kappa_E^{(\sigma)}[a_1,\dots,a_n] = \sum_{\substack{\pi\in NC(n)\\ \pi\leq\sigma}} \mu_{NC(n)}(\pi,\sigma)E^{(\pi)}[a_1,\dots,a_n].$$

2.2 Easy quantum groups

Consider a compact group $G \subset O_n$. By the Stone-Weierstrauss theorem, $C(G)$ is generated by the n^2 coordinate functions u_{ij} sending a matrix in G to its (i,j) entry. The structure of G as a compact group is captured by the commutative Hopf C*-algebra $C(G)$ together with comultiplication, counit and antipode determined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$
$$\epsilon(u_{ij}) = \delta_{ij}$$
$$S(u_{ij}) = u_{ji}.$$

Dropping the condition of commutativity we obtain the following definition, adapted from the fundamental paper of Woronowicz [63].

Definition 2.2.1. An *orthogonal Hopf algebra* is a unital C*-algebra A generated by n^2 self-adjoint elements u_{ij}, such that the following conditions hold:

(1) The inverse of $u = (u_{ij}) \in M_n(A)$ is the transpose $u^t = (u_{ji})$.

(2) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ determines a morphism $\Delta : A \to A \otimes A$.

(3) $\epsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\epsilon : A \to \mathbb{C}$.

(4) $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

It follows from the definitions that Δ, ϵ, S satisfy the usual Hopf algebra axioms. If A is an orthogonal Hopf algebra, we use the heuristic formula "$A = C(G)$", where G is an *compact orthogonal quantum group*. Of course if A is noncommutative then G cannot exist as a concrete object, and all statements about G must be interpreted in terms of the Hopf algebra A.

The following two examples, constructed by Wang in [60, 61], are fundamental to our considerations.

Definition 2.2.2.

(1) $A_o(n)$ is the universal C*-algebra generated by n^2 self-adjoint elements u_{ij}, such that $u = (u_{ij}) \in M_n(A)$ is orthogonal.

(2) $A_s(n)$ is the universal C*-algebra generated by n^2 projections u_{ij}, such that the sum along any row or column of $u = (u_{ij}) \in M_n(A_s(n))$ is the identity.

As discussed above, we use the notations $A_o(n) = C(O_n^+)$, $A_s(n) = C(S_n^+)$, and call O_n^+ and S_n^+ the *free orthogonal group* and *free permutation group*, respectively.

We now recall the "easiness" condition from [10] for a compact orthogonal quantum group $S_n \subset G \subset O_n^+$. Let u, v be the fundamental representations of G, S_n on $\mathbb{C}^n$, respectively. By functoriality, the space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ of intertwining operators is contained in $\mathrm{Hom}(v^{\otimes k}, v^{\otimes l})$ for any k, l. But the Hom-spaces for v are well-known: they are spanned by operators T_π with π belonging to the set $P(k, l)$ of partitions between k upper and l lower points. Explicitly, if $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$, then the formula for T_π is given by

$$T_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\cdots,j_l} \delta_\pi \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_l \end{pmatrix} e_{j_1} \otimes \cdots e_{j_l}.$$

Here the δ symbol appearing on the right hand side is 1 when the indices "fit", i.e. if each block of π contains equal indices, and 0 otherwise.

It follows from the above discussion that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ consists of certain linear combinations of the operators T_π, with $\pi \in P(k, l)$. We call G "easy" if these spaces are spanned by partitions:

Definition 2.2.3. A compact orthogonal quantum group $S_n \subset G \subset O_n^+$ is called *easy* if there exist set $D(k, l) \subset P(k, l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_\pi | \pi \in D(k, l))$, for any $k, l \in \mathbb{N}$. If we have $D(k, l) \subset NC(k, l)$ for each $k, l \in \mathbb{N}$, we say that G is a *free quantum group*.

There are four natural examples of classical groups which are easy:

Group	Partitions
Permutation group S_n	P: All partitions
Orthogonal group O_n	P_2: Pair partitions
Hyperoctahedral group H_n	P_h: Partitions with even block sizes
Bistochastic group B_n	P_b: Partitions with block size ≤ 2

There are also the 2 trivial modifications $S_n' = S_n \times \mathbb{Z}_2$ and $B_n' = B_n \times \mathbb{Z}_2$, and it was shown in [10] that these 6 examples are the only ones.

There is a one to one correspondence between classical easy groups and free quantum groups, which on a combinatorial level corresponds to restricting to noncrossing partitions:

Quantum group	Partitions
S_n^+	NC: All noncrossing partitions
O_n^+	NC_2: Noncrossing pair partitions
H_n^+	NC_h: NC partitions with even block sizes
B_n^+	NC_b: NC partitions with block size ≤ 2

There are also free versions of S_n', B_n'.

In general the class of easy quantum groups appears to be quite rigid, as will be discussed in Chapter 3. However, two more examples can be obtained as "half-liberations". The idea is that instead of removing the commutativity relations from the generators u_{ij} of $C(G)$ for a classical easy group G, which would produce $C(G^+)$, we instead require that the the generators "half-commute", i.e. $abc = cba$ for $a, b, c \in \{u_{ij}\}$. More precisely, we define $C(G^*) = C(G^+)/I$, where I is the ideal generated by the relations $abc = cba$ for $a, b, c \in \{u_{ij}\}$. For $G = S_n, S_n', B_n, B_n'$ we have $G^* = G$, however for O_n, H_n we obtain new quantum groups O_n^*, H_n^*. The corresponding partition categories consist of all pair partitions, respectively all partitions, which are *balanced* in the sense that each block contains as many odd as even legs.

Remark 2.2.4. It is a fundamental result of Woronowicz [63] that if G is a compact orthogonal quantum group, then there is a unique state $\int : C(G) \to \mathbb{C}$, called the *Haar state*, which is left and right invariant in the sense that

$$(\textstyle\int \otimes \mathrm{id})\Delta(f) = \textstyle\int(f) \cdot 1_{C(G)} = (\mathrm{id} \otimes \textstyle\int)\Delta(f), \qquad (f \in C(G)).$$

If $G \subset O_n$ is a compact group, then the Haar state on $C(G)$ is given by integrating against the Haar measure on G.

One of the most useful aspects of the easiness condition for a compact orthogonal quantum group is that it leads to a combinatorial *Weingarten formula* for computing the Haar state, which we now recall.

Definition 2.2.5. Let $D(k) \subset P(k)$ be a collection of partitions. For $n \in \mathbb{N}$, define the *Gram matrix* $(G_{kn}(\pi, \sigma))_{\pi,\sigma \in D(k)}$ by the formula

$$G_{kn}(\pi, \sigma) = n^{|\pi \vee \sigma|}.$$

G_{kn} is invertible for n sufficiently large, define the *Weingarten matrix* W_{kn} to be its inverse.

Theorem 2.2.6. *Let $G \subset O_n^+$ be an easy quantum group and let $D(k) \subset P(0, k)$ be the corresponding collection of partitions having no upper points. If G_{kn} is invertible, then*

$$\int u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\substack{\pi,\sigma \in D(k) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{kn}(\pi, \sigma).$$

Remark 2.2.7. The statement of the theorem above is from [10], but goes back to work of Weingarten [62] and was developed in a series of papers [19, 20, 5, 6]. Note that this reduces the problem of evaluating integrals over G to computing the entries of the Weingarten matrix. In Chapter 4 we will give an estimate on the entries of this matrix, which will be fundamental to our results there and in Chapter 6. We will improve this estimate for O_n^+ in Chapter 7.

Chapter 3

Classification results for easy quantum groups

3.1 Introduction

One of the strengths of the theory of compact Lie groups comes from the fact that these objects can be classified. It is indeed extremely useful to know that the symmetry group of a classical or a quantum mechanical system falls into an advanced classification machinery, and applications of this method abound in mathematics and physics.

The quantum groups were introduced by Drinfel′d [30] and Jimbo [34], in order to deal with quite complicated systems, basically coming from number theory or quantum mechanics, whose symmetry group is not "classical". There are now available several extensions and generalizations of the Drinfel′d-Jimbo construction, all of them more or less motivated by the same philosophy. A brief account of the whole story, focusing on constructions which are of interest for the present considerations, is as follows:

(1) Let $G \subset U_n$ be a compact group, and consider the algebra $A = C(G)$. The matrix coordinates $u_{ij} \in A$ satisfy the commutation relations $ab = ba$. The original idea of Drinfel′d-Jimbo, further processed by Woronowicz in [63], was that these commutation relations are in fact the $q = 1$ particular case of the q-commutation relations $ab = qba$, where $q > 0$ is a parameter. The algebra A itself appears then as the $q = 1$ particular case of a certain algebra A_q. While A_q is no longer commutative, we can formally write $A = C(G_q)$, where G_q is a quantum group.

(2) An interesting modification of the above construction was proposed by Wang in [60, 61]. His idea was to construct a new algebra A^+, by somehow "removing" the commutation relations $ab = ba$. Once again we can formally write $A^+ = C(G^+)$, where G^+ is a so-called free quantum group. This construction, while originally coming only with a vague motivation from mathematical physics, was intensively studied in the last 15 years. Among the partial conclusions that we have so far is the fact that the combinatorics of G^+ is definitely interesting,

and should have something to do with physics. In other words, G^+, while being by definition a quite abstract object, is probably the symmetry group of "something" very concrete.

(3) Several variations of Wang's construction have appeared in recent years, notably in connection with the construction and classification of intermediate quantum groups $G \subset G^* \subset G^+$. For instance in the case $G = O_n$, it was shown in [10] that the commutation relations $ab = ba$ can be successfully replaced with the so-called half-commutation relations $abc = cba$, in order to obtain a new quantum group, O_n^*. Some other commutation-type relations, for instance of type $(ab)^s = (ba)^s$, will be described in this chapter.

(4) As a conclusion, the general idea that tends to emerge from the above considerations is that a "very large class" of compact quantum groups should appear in the following way: (a) start with a compact Lie group $G \subset U_n$, (b) build a noncommutative version of $C(G)$, by replacing the commutation relations $ab = ba$ by some weaker relations, (c) deform this latter algebra, by using a positive parameter $q > 0$, or more generally a whole family of such positive parameters.

This was for the motivating story. In practice, now, while the construction (1) is now basically understood, thanks to about 25 years of efforts of many mathematicians, (2) is just at the very beginning of an axiomatization, (3) is still at the level of pioneering examples, and (4) is just a dream. As for the possible applications to physics, basically nothing is known so far, but the hope for such an application increases, as more and more interesting formulae emerge from the study of compact quantum groups.

In this chapter we will advance on the classification work started in [10], and will present a detailed study of the new quantum groups that we find. The objects of interest will be the compact quantum groups $S_n \subset G \subset O_n^+$ which are "easy", as defined in the previous chapter. There are 14 natural examples of easy quantum groups, which were introduced Chapter 2. In addition, there are at least two infinite series, to be introduced in this chapter. The list is as follows:

(1) Groups: $O_n, S_n, H_n, B_n, S_n', B_n'$.

(2) Free versions: $O_n^+, S_n^+, H_n^+, B_n^+, S_n'^+, B_n'^+$.

(3) Half-liberations: O_n^*, H_n^*.

(4) Hyperoctahedral series: $H_n^{(s)}, H_n^{[s]}$.

This list doesn't cover all the easy quantum groups, but we will present here some partial classification results, with the conjectural conclusion that the full list should consist of (1,2,3), and of a multi-parameter series unifying (4). We will also investigate the new quantum groups that we find, by using various techniques from [2, 3, 10, 12, 11].

As already mentioned, we expect the above list to be a useful, fundamental "input" for a number of representation theory and probability considerations. We will present two such applications in the following chapters. We also expect that the new quantum groups that we find can lead in this way to some other interesting applications.

This chapter is organized as follows. In the next section we recall the notion of a "category of partitions" from [10]. In Section 3.3 we construct a new series of easy quantum groups $H_n^{(s)}$. We construct another series $H_n^{[s]}$ in Section 3.4. In the last two sections we state and prove our classification results, relying heavily on the "capping" method from [10, 11].

3.2 Categories of partitions

Recall from Chapter 2 that to each easy quantum group G there are associated collections of partitions $D(k,l) \subset P(k,l)$ for $k,l \in \mathbb{N}$. The collections of partitions which can appear here, called "categories of partitions", were axiomatized in [10]. In this section we will recall some basic results about these objects.

Definition 3.2.1. The *tensor product*, *composition*, and *involution* of partitions are obtained by horizontal and vertical concatenation and upside-down turning.

Definition 3.2.2. A *category of partitions* is a collection of subsets $D(k,l) \subset P(k,l)$ for $k,l \in \mathbb{N}$ such that:

(1) D is stable under tensor product.

(2) D is stable under composition.

(3) D is stable under involution.

(4) D contains the "unit" partition $|$.

(5) D contains the "duality" partition $\sqcap$.

It follows from the axioms that any category of partitions is also closed under rotations, which will be used later in the chapter.

Theorem 3.2.3. *Let D be a category of partitions, and let $p \in D(k,l)$. Let $\overline{p} \in P(0,k+l)$ be the partition obtained by rotating the k upper legs of p counterclockwise. Then $\overline{p} \in D(0,k+l)$.* □

The key result, coming from the Tannaka-Krein duality results of Woronowicz [64], is that there is a one-to-one correspondence betwen easy quantum groups and categories of partitions. This allows us to translate the classification problem for easy quantum groups to the combinatorial problem of classifying the categories of partitions.

Theorem 3.2.4 ([10])**.** *If G is an easy quantum group with associated partitions $D(k,l) \subset P(k,l)$, then D is a category of partitions. Conversely, given a partition category D there is for each $n \in \mathbb{N}$ a unique easy quantum $S_n \subset G \subset O_n^+$ whose associated partitions are precisely $D(k,l)$.*

3.3 The hyperoctahedral series

In this section we introduce a new series of quantum groups, $H_n^{(s)}$ with $s \in \{2, 3, \dots, \infty\}$. These will "interpolate" between $H_n^{(2)} = H_n$ and $H_n^{(\infty)} = H_n^*$.

The quantum group $H_n^{(s)}$ is obtained from H_n^* by imposing the "s-commutation" condition $abab \dots = baba \dots$ (length s words) to the basic coordinates u_{ij}. It is convenient to write down the complete definition of $H_n^{(s)}$, which is as follows.

Definition 3.3.1. $C(H_n^{(s)})$ is the universal C^*-algebra generated by n^2 self-adjoint variables u_{ij}, subject to the following relations:

(1) Orthogonality: $uu^t = u^t u = 1$, where $u = (u_{ij})$ and $u^t = (u_{ji})$.

(2) Cubic relations: $u_{ij} u_{ik} = u_{ji} u_{ki} = 0$, for any i and any $j \neq k$.

(3) Half-commutation: $abc = cba$, for any $a, b, c \in \{u_{ij}\}$.

(4) s-mixing relation: $abab \dots = baba \dots$ (length s words), for any $a, b \in \{u_{ij}\}$.

The fact that $H_n^{(s)}$ is indeed a quantum group follows from the elementary fact that the cubic relations are of "Hopf type", i.e. that they allow the construction of the Hopf algebra maps Δ, ε, S. This can be checked indeed by a routine computation.

Observe that at $s = 2$ the s-mixing is the usual commutation $ab = ba$. This relation being stronger than the half-commutation $abc = cba$, we are led to the algebra generated by n^2 commuting self-adjoint variables satisfying (1,2), which is $C(H_n)$.

As for the case $s = \infty$, the s-mixing relation disappears here by definition. Thus we are led to the algebra defined by the relations (1,2,3), which is $C(H_n^*)$.

Summarizing, we have $H_n^{(2)} = H_n$ and $H_n^{(\infty)} = H_n^*$, as previously claimed. In what follows we present a detailed study of $H_n^{(s)}$, our first technical result being as follows.

Lemma 3.3.2. *For a compact quantum group $G \subset H_n^*$, the following are equivalent:*

(1) The basic coordinates u_{ij} satisfy $abab \dots = baba \dots$ (length s words).

(2) We have $T_p \in End(u^{\otimes s})$, where $p = (135 \dots 2'4'6' \dots)(246 \dots 1'3'5' \dots)$.

Proof. According to the definition of T_p given in Section 2.2, the operator associated to the partition in the statement is given by the following formula:

$$T_p(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes e_{b_2} \otimes \dots) = \delta(a)\delta(b) e_b \otimes e_a \otimes e_b \otimes e_a \otimes \dots$$

Here we use the convention $\delta(a) = 1$ if all the indices a_i are equal, and $\delta(a) = 0$ if not, along with a similar convention for $\delta(b)$. As for the indices a, b appearing on the right, these are the common values of the a indices and b indices, respectively, in the case $\delta(a) = \delta(b) = 1$, and are irrelevant quantities in the remaining cases.

This gives the following formulae:

$$\begin{aligned} T_p u^{\otimes s}(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes \ldots) &= \sum_{ij} e_i \otimes e_j \otimes e_i \otimes \ldots \otimes u_{ia_1} u_{jb_1} u_{ia_2} \ldots \\ u^{\otimes s} T_p(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes \ldots) &= \delta(a)\delta(b) \sum_{ij} e_{i_1} \otimes e_{j_1} \otimes e_{i_2} \otimes \ldots \otimes u_{i_1 b} u_{j_1 a} u_{i_2 b} \ldots \end{aligned}$$

Here the upper sum is over all indices i, j, and the lower sum is over all multi-indices $i = (i_1, \ldots, i_s), j = (j_1, \ldots, j_s)$. The identification of the right terms, after a suitable relabeling of indices, gives the equivalence in the statement. □

We will now show that $H_n^{(s)}$ is indeed an easy quantum group.

Theorem 3.3.3. *$H_n^{(s)}$ is an easy quantum group, and its associated category E_h^s is that of the "s-balanced" partitions, i.e. partitions satisfying the following conditions:*

(1) The total number of legs is even.

(2) In each block, the number of odd legs equals the number of even legs, modulo s.

Proof. As a first remark, at $s = 2$ the first condition implies the second one, so here we simply get the partitions having an even number of legs, corresponding to H_n. Observe also that at $s = \infty$ we get the partitions which are balanced, which correspond to the quantum group H_n^*.

Our first claim is that E_h^s is indeed a category. But this follows from the definitions, as it is easy to see that the s-balancing condition is preserved under the categorical operations.

It remains to prove that this category corresponds indeed to $H_n^{(s)}$. But this follows from the fact that the partition p appearing in Lemma 3.3.2 generates the category E_h^s, as one can check by routine computation. Indeed, we then have that the category of partitions associated to $H_n^{(s)}$ contains E_h^s by Lemma 3.3.2. On the other hand, if G is the easy quantum group with partition category E_h^s given by Theorem 3.2.4, then we have $G \subset H_n^{(s)}$ by Lemma 3.3.2 and Definition 3.3.1. But this implies that the category of partitions associated to $H_n^{(s)}$ is contained in $E_h^{(s)}$, which completes the proof. □

3.4 The higher hyperoctahedral series

In this section we introduce a second one-parameter series of quantum groups, $H_n^{[s]}$ with $s \in \{2, 3, \ldots, \infty\}$, having as main particular case the group $H_n^{[2]} = H_n$.

Definition 3.4.1. $C(H_n^{[s]})$ is the universal C^*-algebra generated by n^2 self-adjoint variables u_{ij}, subject to the following relations:

(1) Orthogonality: $uu^t = u^t u = 1$, where $u = (u_{ij})$ and $u^t = (u_{ji})$.

(2) Ultracubic relations: $acb = 0$, for any $a \neq b$ on the same row or column of u.

(3) s-mixing relation: $abab\ldots = baba\ldots$ (length s words), for any $a, b \in \{u_{ij}\}$.

The fact that $H_n^{[s]}$ is indeed a quantum group follows from the elementary fact that the ultracubic relations are of "Hopf type", i.e. that they allow the construction of the Hopf algebra maps Δ, ε, S. This can be checked indeed by a routine computation.

Our first task is to compare the defining relations for $H_n^{[s]}$ with those for $H_n^{(s)}$. In order to deal at the same time with the cubic and ultracubic relations, it is convenient to use a statement regarding a certain unifying notion, of "k-cubic" relations.

Lemma 3.4.2. *For a compact quantum group $G \subset O_n^+$, the following are equivalent:*

(1) The basic coordinates u_{ij} satisfy the k-cubic relations $ac_1 \ldots c_k b = 0$, for any $a \neq b$ on the same row or column of u, and for any $c_1, \ldots, c_k$.

(2) We have $T_p \in End(u^{\otimes k+2})$, where $p = (1, 1', k+2, k+2')(2, 2') \ldots (k+1, k+1')$.

Proof. According to the definition of T_p given in Section 2.2, the operator associated to the partition in the statement is given by the following formula:

$$T_p(e_a \otimes e_{c_1} \otimes \ldots \otimes e_{c_k} \otimes e_b) = \delta_{ab} e_a \otimes e_{c_1} \otimes \ldots \otimes e_{c_k} \otimes e_a$$

This gives the following formulae:

$$T_p u^{\otimes k+2}(e_a \otimes e_{c_1} \otimes \ldots \otimes e_{c_k} \otimes e_b) = \sum_{ij} e_i \otimes e_{j_1} \otimes \ldots \otimes e_{j_k} \otimes e_i \otimes u_{ia} u_{j_1 c_1} \ldots u_{j_k c_k} u_{ib}$$

$$u^{\otimes k+2} T_p(e_a \otimes e_{c_1} \otimes \ldots \otimes e_{c_k} \otimes e_b) = \delta_{ab} \sum_{ijl} e_i \otimes e_{j_1} \otimes \ldots \otimes e_{j_k} \otimes e_l \otimes u_{ia} u_{j_1 c_1} \ldots u_{j_k c_k} u_{la}$$

Here the sums are over all indices i, l, and over all multi-indices $j = (j_1, \ldots, j_k)$. The identification of the right terms gives the equivalence in the statement. □

We can now establish the precise relationship between $H_n^{[s]}$ and $H_n^{(s)}$, and also show that no further series can appear in this way.

Proposition 3.4.3. *For $k \geq 1$ the k-cubic relations are all equivalent to the ultracubic relations, and they imply the cubic relations.*

Proof. This follows from the following two observations:

(a) The k-cubic relations imply the $2k$-cubic relations. Indeed, one can connect two copies of the partition p in Lemma 3.4.2, by gluing them with two semicircles in the middle, and the resulting partition is the one implementing the $2k$-cubic relations.

(b) The k-cubic relations imply the $(k-1)$-cubic relations. Indeed, by capping the partition p in Lemma 3.4.2 with a semicircle at bottom right, we get a certain partition $p' \in P(k+2, k)$, and by rotating the upper right leg of this partition we get the partition $p'' \in P(k+1, k+1)$ implementing the $(k-1)$-cubic relations. □

The above statement shows that replacing in Definition 3.4.1 the ultracubic condition by any of the k-cubic conditions, with $k \geq 2$, won't change the resulting quantum group. The other consequences of Proposition 3.4.3 are summarized as follows.

Proposition 3.4.4. *The quantum groups $H_n^{[s]}$ have the following properties:*

(1) We have $H_n^{(s)} \subset H_n^{[s]} \subset H_n^+$.

(2) At $s = 2$ we have $H_n^{[2]} = H_n^{(s)} = H_n$.

(3) At $s \geq 3$ we have $H_n^{(s)} \neq H_n^{[s]}$.

Proof. All the assertions basically follow from Lemma 3.4.2:

(1) For the first inclusion, we need to show that half-commutation + cubic implies ultracubic, and this can be done by placing the half-commutation partition next to the cubic partition, then using 2 semicircle cappings in the middle.

The second inclusion follows from Proposition 3.4.3, because the ultracubic relations (1-cubic relations) imply the cubic relations (0-cubic relations).

(2) Observe first that at $s = 2$ the s-commutation is the usual commutation $ab = ba$. Thus we are led here to the algebra generated by n^2 commuting self-adjoint variables satisfying the cubic condition, which is $C(H_n)$.

(3) Finally, $H_n^{(s)} \neq H_n^{[s]}$ will be a consequence of Theorem 3.4.5 below, because at $s \geq 3$ the half-commutation partition $p = (14)(25)(36)$ is s-balanced but not locally s-balanced. □

Theorem 3.4.5. *$H_n^{[s]}$ is an easy quantum group, and its associated category is that of the "locally s-balanced" partitions, i.e. partitions having the property that each of their subpartitions (i.e. partitions obtained by removing certain blocks) are s-balanced.*

Proof. As a first remark, at $s = 2$ the locally s-balancing condition is automatic for a partition having blocks of even size, so we get indeed the category corresponding to H_n.

In the general case now, our first claim is that the locally s-balanced partitions do indeed form a category. But this follows from the observation that the local s-balancing condition is preserved under the categorical operations.

It remains to prove that this category corresponds indeed to $H_n^{[s]}$. But as in the proof of Theorem 3.3.3, this follows from Lemma 3.4.2 and from the fact that the partition generating the category of locally balanced partitions, namely $p = (1346)(25)$, is nothing but the one implementing the ultracubic relations, as one can check by a routine computation. □

3.5 Classification: General strategy

In this section and in the next one we advance on the classification work started in [10]. We will prove that the easy quantum groups constructed so far are the only ones, modulo a conjectured multi-parameter "hyperoctahedral series", unifying the series constructed in the previous sections, and still waiting to be constructed.

Let G be an easy quantum group, with category of partitions denoted P_g. It follows from definitions that $P_g \cap NC$ is a category of noncrossing partitions, and by the results in Section 3.2, this latter category must come from a certain free quantum group K^+. Observe that since $NC_k = P_g \cap NC$ is included into P_g, we have $G \subset K^+$.

Definition 3.5.1. Associated to an easy quantum group G is the easy group K given by the equality of categories $P_g \cap NC = NC_k$.

According now to the easy group classification from [10], discussed in Section 2.2, there are 6 cases to be investigated. We will split the study into two parts: 5 cases will be investigated in the next section, and the remaining case, $K = H_n$, will be eventually left open.

The point with this splitting comes from the following question: do we have $K \subset G$? In the reminder of this section we will try to answer this question.

We begin with the technical lemma, valid in the general case. Let $\Lambda_g, \Lambda_k \subset \mathbb{N}$ be the set of the possible sizes of blocks of elements of P_g, NC_k.

Lemma 3.5.2. *Let G, K be as above.*

(1) $\Lambda_k \subset \Lambda_g \subset \Lambda_k \cup (\Lambda_k - 1)$.

(2) $1 \in \Lambda_g$ *implies* $1 \in \Lambda_k$.

(3) If NC_k is even, so is P_g.

Proof. We will heavily use the various abstract notions and results in [10].

(1) Here the first inclusion follows from $NC_k \subset P_g$. As for the second inclusion, this is equivalent to the following statement: "If b is a block of a partition $p \in P_g$, then there exists a certain block b' of a certain partition $p' \in P_g \cap NC$, having size $\#b$ or $\#b - 1$".

But this latter statement follows by using the "capping" method in [10]. Indeed, we can cap p with semicircles, as for b to remain unchanged, and we end up with a certain partition p' consisting of b and of some extra points, at most one point between any two legs of b, which might be connected or not. Note that the semicircle capping being a categorical operation, this partition p' remains in P_g.

Now by further capping p' with semicircles, as to get rid of the extra points, the size of b can only increase, and we end up with a one-block partition having size at least that of b. This one-block partition is obviously noncrossing, and by capping it again with semicircles we can reduce the number of legs up to $\#b$ or $\#b - 1$, and we are done.

(2) The condition $1 \in \Lambda_g$ means that there exists $p \in P_g$ having a singleton. By capping p with semicircles outside this singleton, we can obtain a singleton, or a double singleton. Since both these partitions are noncrossing, and have a singleton, we are done.

(3) Indeed, assume that P_g is not even, and consider a partition $p \in P_g$ having an odd number of legs. By capping p with enough semicircles we can arrange for ending up with a singleton, and since this singleton is by definition in $P_g \cap NC$, we are done. □

We are now in position of splitting the classification. We have the following key result.

Proposition 3.5.3. *Let G, K be as above.*

(1) If $K \neq H_n$ then $K \subset G \subset K^+$.

(2) If $K = H_n$ then $S_n' \subset G \subset H_n^+$.

Proof. We recall that the inclusion $G \subset K^+$ follows from definitions. For the other inclusion, we have 6 cases, depending on the exact value of the easy group K:

(1.1) $K = O_n$. Here $\Lambda_k = \{2\}$, so by Lemma 3.5.2 (1) we get $\{2\} \subset \Lambda_g \subset \{1, 2\}$. Moreover, from Lemma 3.5.2 (2), we get $\Lambda_g = \{2\}$. Thus $P_g \subset P_o$, which gives $O_n \subset G$.

(1.2) $K = S_n$. Here there is nothing to prove, since $S_n \subset G$ by definition.

(1.3) $K = B_n$. Here $\Lambda_k = \{1, 2\}$, so by Lemma 3.5.2 (1) we get $\Lambda_g = \{1, 2\}$. Thus we have $P_g \subset P_b$, which gives $B_n \subset G$.

(1.4) $K = S_n'$. Here we have $P_g \subset P_s$ by definition, and by using Lemma 3.5.2 (3) we deduce that we have $P_g \subset P_{s'}$, which gives $S_n' \subset G$.

(1.5) $K = B_n'$. Here we have $\Lambda = \{1, 2\}$, so by Lemma 3.5.2 (1) we get $\Lambda_g = \{1, 2\}$. This gives $P_g \subset P_b$, and by Lemma 3.5.2 (3) we get $P_g \subset P_{b'}$, which gives $B_n' \subset G$.

(2) $K = H_n$. Here we have $P_g \subset P_s$ by definition, and by using Lemma 3.5.2 (3) we deduce that we have $P_g \subset P_{s'}$, which gives $S_n' \subset G$. □

With a little more care, one can prove that the easy group K in the above statement (1) is nothing but the "classical version" of G, obtained as dual object to the commutative Hopf algebra $C(G)/I$, where $I \subset C(G)$ is the commutator ideal.

Observe also that the above statement (2) cannot be improved. The point is that for the quantum group $H_n^{(s)}$ with s odd we have $K = H_n$, and $K \not\subset G$.

3.6 The non-hyperoctahedral case

In this section we classify the easy quantum groups, under the non-hyperoctahedral assumption $K \neq H_n$. Here K is as usual the easy group from Definition 3.5.1.

We know from Proposition 3.5.3 that our easy quantum group G appears as an intermediate quantum group, $K \subset G \subset K^+$. In order to classify these intermediate quantum groups, we use the method in [11], where the problem was solved in the case $G = O_n$. For uniformity reasons, we will include as well the case $G = O_n$ in our study.

We will need a number of technical ingredients.

Definition 3.6.1. Let $p \in P(k, l)$ be a partition, with the points counted modulo $k + l$, counterclockwise starting from bottom left.

(1) We call semicircle capping of p any partition obtained from p by connecting with a semicircle a pair of consecutive neighbors.

(2) We call singleton capping of p any partition obtained from p by capping one of its legs with a singleton.

(3) We call doubleton capping of p any partition obtained from p by capping two of its legs with singletons.

In other words, the semicircle, singleton and doubleton cappings are elementary operations on partitions, which lower the total number of legs by $2, 1, 2$ respectively. Observe that there are $k+l$ possibilities for placing the semicircle or the singleton, and $(k+l)(k+l-1)/2$ possibilities for placing the double singleton. Observe also that in the case of 2 particular "semicircle cappings", namely those at left or at right, the semicircle in question is rather a vertical bar; but we will still call it semicircle.

The various cappings of p will be generically denoted p'.

Consider now the $5+5+1 = 11$ categories of partitions P_x, NC_x, E_x, with $x = o, s, b, s', b'$ described in sections 1 and 2. We have the following technical lemma.

Lemma 3.6.2. *Let p be a partition, having j legs.*

(1) If $p \in P_o - E_o$ and $j > 4$, there exists a semicircle capping $p' \in P_o - E_o$.

(2) If $p \in E_o - NC_o$ and $j > 6$, there exists a semicircle capping $p' \in E_o - NC_o$.

(3) If $p \in P_s - NC_s$ and $j > 4$, there exists a singleton capping $p' \in P_s - NC_s$.

(4) If $p \in P_b - NC_b$ and $j > 4$, there exists a singleton capping $p' \in P_b - NC_b$.

(5) If $p \in P_{s'} - NC_{s'}$ and $j > 4$, there exists a doubleton capping $p' \in P_{s'} - NC_{s'}$.

(6) If $p \in P_{b'} - NC_{b'}$ and $j > 4$, there exists a doubleton capping $p' \in P_{b'} - NC_{b'}$.

Proof. We write $p \in P(k, l)$, so that the number of legs is $j = k+l$. In the cases where our partition is a pairing, we use as well the number of strings, $s = j/2$.

Let us agree that all partitions are drawn as to have a minimal number of crossings.

We use the same idea for all the proofs, namely to "isolate" a block of p having a crossing, or an odd number of crossings, then to "cap" p as in the statement, as for this block to remain crossing, or with an odd number of crossings. Here we use of course the observation that the "balancing" condition which defines the categories E_o, E_h can be interpreted as saying that each block has an even number of crossings, when the picture of the partition is drawn such that this number of crossings is minimal.

(1) The assumption $p \notin E_o$ means that p has certain strings having an odd number of crossings. We fix such an "odd" string, and we try to cap p, as for this string to remain odd in the resulting partition p'. An examination of all possible pictures shows that this is possible, provided that our partition has $s > 2$ strings, and we are done.

(2) The assumption $p \notin NC_o$ means that p has certain crossing strings. We fix such a pair of crossing strings, and we try to cap p, as for these strings to remain crossing in p'. Once again, an examination of all possible pictures shows that this is possible, provided that our partition has $s > 3$ strings, and we are done.

(3) Indeed, since p is crossing, we can choose two of its blocks which are intersecting. If there are some other blocks left, we can cap one of their legs with a singleton, and we are done.

If not, this means that our two blocks have a total of $j' \geq j > 4$ legs, so at least one of them has $j'' > 2$ legs. One of these j'' legs can always be capped with a singleton, as for the capped partition to remain crossing, and we are done.

(4) Here we can simply cap with a singleton, as in (3).

(5,6) Here we can cap with a doubleton, by proceeding twice as in (3). □

For a collection of subsets $X(k,l) \subset P(k,l)$ we denote by $\langle X \rangle \subset P$ the category of partitions generated by X. In other words, the elements of $\langle X \rangle$ come from those of X via the categorical operations for the categories of partitions, which are the vertical and horizontal concatenation and the upside-down turning.

Lemma 3.6.3. *Let p be a partition.*

(1) If $p \in P_o - E_o$ then $\langle p, NC_o \rangle = P_o$.

(2) If $p \in E_o - NC_o$ then $\langle p, NC_o \rangle = E_o$.

(3) If $p \in P_s - NC_s$ then $\langle p, NC_s \rangle = P_s$.

(4) If $p \in P_b - NC_b$ then $\langle p, NC_b \rangle = P_b$.

(5) If $p \in P_{s'} - NC_{s'}$ then $\langle p, NC_{s'} \rangle = P_{s'}$.

(6) If $p \in P_{b'} - NC_{b'}$ then $\langle p, NC_{b'} \rangle = P_{b'}$.

Proof. We use Lemma 3.6.2, together with the observation that the "capping partition" appearing there is always in the good category.

That is, we use the fact that the semicircle is in $NC_o, NC_{s'}$, the singleton is in NC_s, NC_b, and the doubleton is in $NC_{b'}$. This observation tells us that, in each of the cases under consideration, the category to be computed can only decrease when replacing p by one of its cappings p'. Indeed, for the singleton and doubleton cappings this is clear from definitions, and for the semicircle capping this is clear as well from definitions, unless in the case where the "capping semicircle" is actually a "bar" added at left or at right, where we can use a categorical rotation operation as in Theorem 3.2.3.

(1,2) These assertions can be proved by recurrence on the number of strings, $s = (k+l)/2$. Indeed, by using Lemma 3.6.2 (1,2), for $s > 3$ we have a descent procedure $s \to s-1$, and this leads to the situation $s \in \{1,2,3\}$, where the statement is clear.

(3) We can proceed by recurrence on the number of legs of p. If the number of legs is $j = 4$, then p is a basic crossing, and we have $\langle p \rangle = P_s$. If the number of legs is $j > 4$ we can apply Lemma 3.6.2 (3), and the result follows from $\langle p \rangle \supset \langle p' \rangle = P_s$.

(4,5,6) This is similar to the proof of (1,3,2), by using Lemma 3.6.2 (4,5,6). □

Lemma 3.6.4. *Let p be a partition.*

(1) If $p \in P_o$ then $\langle p, NC_o \rangle \in \{P_o, E_o, NC_o\}$.

(2) If $p \in P_s$ then $\langle p, NC_s \rangle \in \{P_s, NC_s\}$.

(3) If $p \in P_b$ *then* $\langle p, NC_b \rangle \in \{P_b, NC_b\}$.

(4) If $p \in P_{s'}$ *then* $\langle p, NC_{s'} \rangle \in \{P_{s'}, NC_{s'}\}$.

(5) If $p \in P_{b'}$ *then* $\langle p, NC_{b'} \rangle \in \{P_{b'}, NC_{b'}\}$.

Proof. This follows by rearranging the various technical results in Lemma 3.6.3. □

We are now in position of stating the main result in this paper. Let us call "non-hyperoctahedral" any easy quantum group G such that $K \neq H_n$.

Theorem 3.6.5. *There are exactly* 11 *non-hyperoctahedral easy quantum groups, namely:*

(1) O_n, O_n^*, O_n^+*: the orthogonal quantum groups.*

(2) S_n, S_n^+*: the symmetric quantum groups.*

(3) B_n, B_n^+*: the bistochastic quantum groups.*

(4) $S_n', S_n'^+$*: the modified symmetric quantum groups.*

(5) $B_n', B_n'^+$*: the modified bistochastic quantum groups.*

Proof. We know from Proposition 3.5.3 that what we have to do is to classify the easy quantum groups satisfying $K \subset G \subset K^+$. More precisely, we have to prove that for $K = S_n, B_n, S_n', B_n'$ there is no such partial liberation, and that for $K = O_n$ there is only one partial liberation, namely the above-mentioned quantum group K^*. But this follows from Lemma 3.6.4, via Theorem 3.2.4. □

As for the classification in the hyperoctahedral case, this seems to be a quite difficult problem, that we have to leave open.

Chapter 4

De Finetti theorems for easy quantum groups

4.1 Introduction

In the study of probabilistic symmetries, the classical groups S_n and O_n play central roles. De Finetti's fundamental theorem states that an infinite sequence of random variables whose joint distribution is invariant under finite permutations must be conditionally independent and identically distributed. In [32], Freedman considered sequences of real-valued random variables whose joint distribution is invariant under orthogonal transformations, and proved that any infinite sequence with this property must form a conditionally independent Gaussian family with mean zero and common variance. Although these results fail for finite sequences, approximation results may still be obtained (see [27, 28]). For a thorough treatment of the study of probabilistic symmetries, the reader is referred to the recent text of Kallenberg [37].

In [40], Köstler and Speicher discovered that de Finetti's theorem has a natural free analogue: an infinite sequence of noncommutative random variables has a joint distribution which is invariant under "quantum permutations" coming from S_n^+ if and only if the variables are freely independent and identically distributed with amalgamation, i.e., with respect to a conditional expectation. This was further studied in [23], where this result was extended to more general sequences and an approximation result was given for finite sequences. The free analogue of Freedman's result was obtained in [24], where it was shown that an infinite sequence of self-adjoint noncommutative random variables has a joint distribution which is invariant under "quantum orthogonal transformations" if and only if the variables form an operator-valued free semicircular family with mean zero and common variance.

In this chapter, we present a unified approach to de Finetti theorems for the class of easy quantum groups. If G is an easy quantum group, there is a natural notion of G-invariance for a sequence of noncommutative random variables, which agrees with the usual definition when G is a classical group. Our main result is the following de Finetti type theorem, which characterizes the joint distributions of infinite G-invariant sequences for the 10 natural easy quantum groups introduced in Chapter 2:

Theorem 4.1.1. *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of self-adjoint random variables in a W^*-probability space (M,φ), and suppose that the sequence is G-invariant, where G is one of O,S,H,B, O^*, H^*, O^+,S^+,H^+,B^+. Then there is a W^*-subalgebra $1 \subset \mathcal{B} \subset M$ and a φ-preserving conditional expectation $E : M \to \mathcal{B}$ such that the following hold:*

(1) Free case:

(a) If $G = S^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$.

(b) If $G = H^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent, and have even and identical distributions, with amalgamation over $\mathcal{B}$.

(c) If $G = O^+$, then $(x_i)_{i\in\mathbb{N}}$ form a $\mathcal{B}$-valued free semicircular family with mean zero and common variance.

(d) If $G = B^+$, then $(x_i)_{i\in\mathbb{N}}$ form a $\mathcal{B}$-valued free semicircular family with common mean and variance.

(2) Half-liberated case: Suppose that $x_ix_jx_k = x_kx_jx_i$ for any $i,j,k \in \mathbb{N}$.

(a) If $G = H^*$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally half-independent and identically distributed given $\mathcal{B}$.

(b) If $G = O^*$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally half-independent, and have symmetrized Rayleigh distributions with common variance, given $\mathcal{B}$.

(3) Classical case: Suppose that $(x_i)_{i\in\mathbb{N}}$ commute.

(a) If $G = S$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and identically distributed given $\mathcal{B}$.

(b) If $G = H$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have even and identical distributions, given $\mathcal{B}$.

(c) If $G = O$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have Gaussian distributions with mean zero and common variance, given $\mathcal{B}$.

(d) If $G = B$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have Gaussian distributions with common mean and variance, given $\mathcal{B}$.

The notion of half-independence, appearing in (2) above, will be introduced in the next section. The basic example of a half-independent family of noncommutative random variables is $(x_i)_{i\in I}$,

$$x_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi_i} & 0 \end{pmatrix},$$

where $(\xi_i)_{i\in I}$ are independent, complex-valued random variables and $\mathbb{E}[\xi_i^n\overline{\xi_i}^m] = 0$ unless $n = m$ (see Example 4.2.4). Note that in particular, if $(\xi_i)_{i\in\mathbb{N}}$ are independent and identically distributed complex Gaussian random variables, then x_i has a symmetrized Rayleigh distribution $(\xi_i\overline{\xi_i})^{1/2}$ and we obtain the joint distribution in (2) corresponding to the half-liberated orthogonal group O_n^*. Since the complex Gaussian distribution is known to be characterized by unitary invariance, this appears to be closely related to the connection between U_n and O_n^* observed in [11, 7], see also Section 6.8.

This chapter is organized as follows. In Section 4.2 we introduce half-independence and develop its basic combinatorial theory. In Section 4.3 we give a new estimate on the entries of the Weingarten matrices, which will be used throughout this thesis. In Section 4.4 we introduce the notion of quantum invariance, prove a converse to Theorem 4.1.1, and give approximation results for finite sequences. Section 4.5 contains the proof of Theorem 4.1.1.

4.2 Half independence

In this section we introduce a new kind of independence which appears in the de Finetti theorems for the half-liberated quantum groups H^* and O^*. To define this notion, we require that the variables have a certain degree of commutativity.

Definition 4.2.1. Let $(x_i)_{i\in I}$ be a family of noncommutative random variables. We say that the variables *half-commute* if

$$x_i x_j x_k = x_k x_j x_i$$

for all $i, j, k \in I$.

Observe that if $(x_i)_{i\in I}$ half-commute, then in particular x_i^2 commutes with x_j for any $i, j \in I$.

Definition 4.2.2. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Let $(x_i)_{i\in I}$ be a family of random variables in $\mathcal{A}$ which half-commute. We say that $(x_i)_{i\in I}$ are *conditionally half-independent given* $\mathcal{B}$, or *half-independent with respect to* E, if the following conditions are satisfied:

(1) The variables $(x_i^2)_{i\in I}$ are conditionally independent given $\mathcal{B}$.

(2) For any $i_1, \ldots, i_k \in I$, we have

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

unless for each $i \in I$ the set of $1 \leq j \leq k$ such that $i_j = i$ contains as many odd as even numbers, i.e., unless $\ker \mathbf{i}$ is balanced.

If $B = \mathbb{C}$, then the variables are simply called *half-independent.*

Remark 4.2.3. As a first remark, we note that half-independence is defined only between random variables and not at the level of algebras, in contrast with classical and free independence. In fact, it is known from [47] there are no other good notions of independence between unital algebras other than classical and free.

The conditions may appear at first to be somewhat artificial, but are motivated by the following natural example.

Example 4.2.4. Let (Ω, Σ, μ) be a (classical) probability space, and let $L(\mu)$ denote the algebra of complex-valued random variables on Ω with moments of all orders.

(1) Let $(\xi_i)_{i\in I}$ be a family of independent random variables in $L(\mu)$. Suppose that for each $i \in I$, the distribution of ξ_i is such that

$$\mathbb{E}[\xi_i^n \overline{\xi_i^m}] = 0$$

unless $n = m$. Define random variables

$$x_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi_i} & 0 \end{pmatrix}.$$

A simple computation shows that the variables $(x_i)_{i\in I}$ half-commute. Since

$$x_i^2 = |\xi_i|^2 I_2,$$

it is clear that $(x_i^2)_{i\in I}$ are independent with respect to $\mathbb{E} \circ \mathrm{tr}$. Moreover, the assumption on the distributions of the ξ_i clearly implies that $\mathbb{E}[\mathrm{tr}[x_{i_1} \cdots x_{i_k}]] = 0$ unless k is even and $\ker \mathbf{i}$ is balanced. So $(x_i)_{i\in I}$ are half-independent.

Observe also that the distribution of x_i is equal to that of $(\xi_i\overline{\xi_i})^{1/2}$, where the square root is chosen such that the distribution is even. We call this the *squeezed version* of the complex distribution ξ_i (cf. [10]).

(2) Of particular interest is the case that the $(\xi_i)_{i\in I}$ have complex Gaussian distributions. Here the distribution of x_i is the squeezed version of the complex Gaussian ξ_i, which is a symmetrized Rayleigh distribution.

A fundamental property of free and classical independence is that the joint distribution of a family of random variables $(x_i)_{i\in I}$ which are freely or classically independent is determined by the distributions of x_i for $i \in I$. We will now show that half-independence shares this property. It is convenient to first introduce the following family of permutations which are related to the half-commutation relation.

Definition 4.2.5. We say that a permutation $\omega \in S_n$ *preserves parity* if $\omega(i) \equiv i \pmod 2$ for $1 \leq i \leq n$.

The collection of parity preserving partitions in S_n clearly form a subgroup, which is simply $S(\{1, 3, \dots\}) \times S(\{2, 4, \dots\})$. Moreover, this subgroup is generated by the transpositions $(i\; i+2)$ for $1 \leq i \leq n-2$. It follows that if $(x_i)_{i\in I}$ half-commute, then

$$x_{i_1} \cdots x_{i_n} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(n)}}$$

whenever $\omega \in S_n$ preserves parity.

Lemma 4.2.6. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space such that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Suppose that $(x_i)_{i\in I}$ is a family of random variables in $\mathcal{A}$ which are conditionally half-independent given $\mathcal{B}$. Then the $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ is uniquely determined by the $\mathcal{B}$-valued distributions of x_i for $i \in I$.*

Proof. Let $i_1, \dots, i_k \in I$. We know that

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

unless we have that for each $i \in I$, the set of $1 \leq j \leq k$ such that $i_j = i$ has as many odd as even elements. So suppose that this the case. By the remark above, we know that $x_{i_1} \cdots x_{i_k} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(k)}}$ whenever $\omega \in S_k$ is parity preserving. With an appropriate choose of ω, it follows that

$$x_{i_1} \cdots x_{i_k} = x_{j_1}^{2(k_1)} \cdots x_{j_m}^{2(k_m)}$$

for some $j_1, \dots, j_m \in I$ and $k_1, \dots, k_m \in \mathbb{N}$ such that $k = 2(k_1 + \cdots + k_m)$. Since the joint distribution of $(x_i^2)_{i \in I}$ is clearly determined by the distributions of x_i for $i \in I$, the result follows. □

We will now develop a combinatorial theory for half-independence, based on the family E of balanced partitions.

Definition 4.2.7. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Let $(x_i)_{i \in I}$ be a family of random-variables in $\mathcal{A}$, and suppose that

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

for any odd k and $i_1, \dots, i_k \in I$. Define the *half-liberated cumulants* $\gamma_E^{(n)}$ by the *half-liberated moment-cumulant formula*

$$E[x_{i_1} \cdots x_{i_k}] = \sum_{\substack{\pi \in E_h(k) \\ \pi \leq \ker \mathbf{i}}} \gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}],$$

where $\gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}]$ is defined, as in the classical case, by the formula

$$\gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}] = \prod_{V \in \pi} \gamma_E(V)[x_{i_1}, \dots, x_{i_k}].$$

Observe that both sides of the moment-cumulant formula above are equal to zero for odd values of k, and for even values the right hand side is equal to $\gamma_E^{(k)}[x_{i_1}, \dots, x_{i_k}]$ plus products of lower ordered terms and hence $\gamma_E^{(k)}$ may be solved for recursively. As in the free and classical cases, we may apply the Möbius inversion formula to obtain the following equation for $\gamma_E^{(\pi)}$, $\pi \in E_h(k)$:

$$\gamma_E^{(\pi)}(x_{i_1}, \dots, x_{i_k}) = \sum_{\substack{\sigma \in E_h(k) \\ \sigma \leq \pi}} \mu_{E_h(k)}(\sigma, \pi) E^{(\pi)}[x_{i_1}, \dots, x_{i_k}].$$

Theorem 4.2.8. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Suppose $(x_i)_{i \in \mathbb{N}}$ is a family of variables in $\mathcal{A}$ which half-commute. Then the following conditions are equivalent:*

(1) $(x_i)_{i \in \mathbb{N}}$ are half-independent with respect to E.

(2) $E[x_{i_1} \cdots x_{i_k}] = 0$ *whenever* k *is odd, and*

$$\gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_k}] = 0$$

for any $\pi \in E_h(k)$ *such that* $\pi \not\leq \ker \mathbf{i}$.

Proof. First suppose that condition (2) holds. From the moment-cumulant formula, we have

$$E[x_{i_1} \cdots x_{i_k}] = \sum_{\substack{\pi \in E_h(k) \\ \pi \leq \ker \mathbf{i}}} \gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_k}]$$

for any $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$. Observe that if $\ker \mathbf{i}$ is not balanced then there is no $\pi \in E_h(k)$ such that $\pi \leq \ker \mathbf{i}$, so it follows that $E[x_{i_1} \cdots x_{i_k}] = 0$. It remains to show that $(x_i^2)_{i \in I}$ are independent. Choose $k_1, \ldots, k_m \in \mathbb{N}$, distinct $i_1, \ldots, i_m \in I$ and let $k = 2(k_1 + \cdots + k_m)$. Let $\tau \in E_h(k)$ be the partition with blocks $\{1, \ldots, 2k_1\}, \ldots, \{2(k_1 + \cdots + k_{m-1}) + 1, \ldots, 2k\}$. Then

$$\begin{aligned} E\big[x_{i_1}^{(2k_1)} \cdots x_{i_m}^{(2k_m)}\big] &= \sum_{\substack{\pi \in E_h(k) \\ \pi \leq \tau}} \gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_1}, x_{i_2}, \ldots, x_{i_m}, \ldots, x_{i_m}] \\ &= \prod_{1 \leq j \leq m} \sum_{\pi \in E(2k_j)} \gamma_E^{(\pi)}[x_{i_j}, \ldots, x_{i_j}] \\ &= \prod_{1 \leq j \leq m} E[x_{i_j}^{(2k_j)}], \end{aligned}$$

so that $(x_i^2)_{i \in I}$ are independent and hence $(x_i)_{i \in I}$ are half-independent.

The implication (1)⇒(2) actually follows from (2)⇒(1). Indeed, suppose that $(x_i)_{i \in I}$ are half-independent. Consider the algebra $\mathcal{A}' = \mathcal{B}\langle y_i : i \in I\rangle / \langle y_i y_j y_k = y_k y_j y_i \rangle$ of polynomials in half-commuting indeterminates $(y_i)_{i \in I}$ and coefficients in $\mathcal{B}$. Define a conditional expectation $E' : \mathcal{A}' \to \mathcal{B}$ by

$$E'[y_{i_1} \cdots y_{i_k}] = \sum_{\substack{\pi \in E_h(k) \\ \pi \leq \ker \mathbf{i}}} \gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_k}].$$

(It is easy to see that E' is well-defined, i.e., compatible with the half-commutation relations). Since the half-liberated cumulants are uniquely determined by the moment-cumulant formula, it follows that

$$\gamma_{E'}^{(\pi)}[y_{i_1}, \ldots, y_{i_k}] = \begin{cases} \gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_k}], & \pi \leq \ker \mathbf{i} \\ 0, & \text{otherwise} \end{cases}.$$

By the first part, it follows that $(y_i)_{i \in I}$ are half-independent with respect to E'. Since y_i has the same $\mathcal{B}$-valued distribution as x_i, it follows from Lemma 4.2.6 that $(y_i)_{i \in I}$ have the same joint distribution as $(x_i)_{i \in I}$. It then follows from the moment-cumulant formula that these families have the same half-liberated cumulants, and hence $\gamma_E^{(\pi)}[x_{i_1}, \ldots, x_{i_k}] = 0$ unless $\pi \leq \ker \mathbf{i}$. □

Recall that (centered) Gaussian and semicircular distributions are characterized by the property that their non-vanishing cumulants are those corresponding to pair and noncrossing pair partitions, respectively. We will now show that for half-independence, it is the symmetrized Rayleigh distribution which has this property. This follows from the considerations in [10], but we include here a direct proof.

Proposition 4.2.9. *Let x be a random variable in $(\mathcal{A}, \varphi)$ which has an even distribution. Then x has a symmetrized Rayleigh distribution if and only if*

$$\gamma_E^{(\pi)}[x, \dots, x] = 0$$

for any $\pi \in E_h(k)$ such that $\pi \notin E_o(k)$.

Proof. Since the distribution of x is determined uniquely by its half-liberated cumulants, it suffices to show that if the cumulants have the stated property then x has a symmetrized Rayleigh distribution. Suppose that this is the case, then

$$\begin{aligned}\varphi(x^k) &= \sum_{\pi \in E_o(k)} \gamma^{(\pi)}[x, \dots, x] \\ &= \gamma^{(2)}[x, x] \# \{\pi \in E_o(k)\}.\end{aligned}$$

It is easy to see that the number of partitions in $E_o(k)$ is $m!$ if $k = 2m$ is even and is zero if k is odd. Since these agree with the moments of a symmetrized Rayleigh distribution, the result follows. □

4.3 Weingarten estimate

Recall from Theorem 2.2.6 that integrals over easy quantum groups can be evaluated as sums of the entries in the corresponding Weingarten matrix. In this section we will give an estimate on the entries, which will be required in the next two sections.

Proposition 4.3.1. *Let $k \in \mathbb{N}$ and $D(k) \subset P(k)$. For n sufficiently large, the Gram matrix G_{kn} is invertible. Moreover, the entries of the Weingarten matrix $W_{kn} = G_{kn}^{-1}$ satisfy the following:*

(1) $W_{kn}(\pi, \sigma) = O(n^{|\pi \vee \sigma| - |\pi| - |\sigma|})$.

(2) If $\pi \leq \sigma$, then

$$n^{|\pi|} W_{kn}(\pi, \sigma) = \mu_{D(k)}(\pi, \sigma) + O(n^{-1}),$$

where $\mu_{D(k)}$ is the Möbius function on the partially ordered set $D(k)$ under the restriction of the order on $P(k)$.

Proof. We use a standard method from [19, 20], further developed in [5, 6, 23].

First note that

$$G_{kn} = \Theta_{kn}^{1/2}(1 + B_{kn})\Theta_{kn}^{1/2},$$

where

$$\Theta_{kn}(\pi,\sigma) = \begin{cases} n^{|\pi|} & \pi = \sigma, \\ 0 & \pi \neq \sigma, \end{cases}$$

$$B_{kn}(\pi,\sigma) = \begin{cases} 0 & \pi = \sigma, \\ n^{|\pi \vee \sigma| - \frac{|\pi|+|\sigma|}{2}} & \pi \neq \sigma. \end{cases}$$

Note that the entries of B_{kn} are $O(n^{-1/2})$, it follows that for n sufficiently large $1+B_{kn}$ is invertible and

$$(1+B_{kn})^{-1} = 1 - B_{kn} + \sum_{l\geq 1}(-1)^{l+1}B_{kn}^{l+1}.$$

G_{kn} is then invertible, and

$$W_{kn}(\pi,\sigma) = \sum_{l\geq 1}(-1)^{l+1}(\Theta_{kn}^{-1/2}B_{kn}^{l+1}\Theta_{kn}^{-1/2})(\pi,\sigma) + \begin{cases} n^{-|\pi|}, & \pi = \sigma \\ -n^{|\pi\vee\sigma|-|\pi|-|\sigma|}, & \pi \neq \sigma \end{cases}.$$

Now for $l \geq 1$ we have

$$(\Theta_{kn}^{-1/2}B_{kn}^{l+1}\Theta_{kn}^{-1/2})(\pi,\sigma) = \sum_{\substack{\nu_1,\ldots,\nu_l\in D(k)\\ \pi\neq\nu_1\neq\cdots\neq\nu_l\neq\sigma}} n^{|\pi\vee\nu_1|+|\nu_1\vee\nu_2|+\cdots+|\nu_l\vee\sigma|-|\nu_1|-\cdots-|\nu_l|-|\pi|-|\sigma|}.$$

So to prove (1), it suffices to show that if $\nu_1,\ldots,\nu_l \in D(k)$, then

$$|\pi\vee\nu_1| + |\nu_1\vee\nu_2| + \cdots + |\nu_l\vee\sigma| \leq |\pi\vee\sigma| + |\nu_1| + \cdots + |\nu_l|.$$

We will use the fact that $P(k)$ is a *semi-modular lattice* ([17, §I.8, Example 9]): if $\nu,\tau \in P(k)$ then

$$|\nu| + |\tau| \leq |\nu\vee\tau| + |\nu\wedge\tau|.$$

We will now prove the claim by induction on l, for $l = 1$ we may apply the formula above to find

$$\begin{aligned} |\pi\vee\nu| + |\nu\vee\sigma| &\leq |(\pi\vee\nu)\vee(\nu\vee\sigma)| + |(\pi\vee\nu)\wedge(\nu\vee\sigma)| \\ &\leq |\pi\vee\sigma| + |\nu|. \end{aligned}$$

Now let $l > 1$, by induction we have

$$|\pi\vee\nu_1| + |\nu_1\vee\nu_2| + \cdots + |\nu_{l-1}\vee\nu_l| \leq |\pi\vee\nu_l| + |\nu_1| + \cdots + |\nu_{l-1}|.$$

Also $|\nu_l\vee\sigma| \leq |\pi\vee\sigma| + |\nu_l| - |\pi\vee\nu_l|$, and the result follows.

To prove (2), suppose $\pi,\sigma \in D(k)$ and $\pi \leq \sigma$. The terms which contribute to order $n^{-|\pi|}$ in the expansion come from sequences $\nu_1,\ldots,\nu_l \in D(k)$ such that $\pi\neq\nu_1\neq\cdots\neq\nu_l\neq\sigma$ and

$$|\pi\vee\nu_1| + \cdots + |\nu_l\vee\sigma| = |\sigma| + |\nu_1| + \cdots + |\nu_l|.$$

Since $|\pi \vee \nu_1| \leq |\nu_1|$, $|\nu_1 \vee \nu_2| \leq |\nu_2|, \ldots, |\nu_l \vee \sigma| \leq \sigma$, it follows that each of these must be an equality, which implies $\pi < \nu_1 < \cdots < \nu_l < \sigma$. Conversely, any $\nu_1, \ldots, \nu_l \in D(k)$ such that $\pi < \nu_1 < \cdots < \nu_l < \sigma$ clearly satisfy this equation. Therefore the coefficient of $n^{-|\pi|}$ in $W_{kn}(\pi, \sigma)$ is

$$\begin{cases} 1, & \pi = \sigma \\ -1 + \sum_{l=1}^{\infty} (-1)^{l+1} \#\{\nu_1, \ldots, \nu_l \in D(k) : \pi < \nu_1 < \cdots < \nu_l < \sigma\}, & \pi < \sigma \end{cases}.$$

which is precisely $\mu_{D(k)}(\pi, \sigma)$. □

Remark 4.3.2. Recall that the classical, free and half-liberated cumulants are obtained from the moments by using the Möbius functions on $P(k), NC(k)$ and $E_h(k)$, respectively. To show that this is compatible with the Proposition above, we will need the following result:

Proposition 4.3.3.

(1) If $D = P_o, P_s, P_b, P_h$, then

$$\mu_{D(k)}(\pi, \sigma) = \mu_{P(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.

(2) If $D = E_o, E_h$, then

$$\mu_{D(k)}(\pi, \sigma) = \mu_{E_h(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.

(3) If $D = NC_o, NC_s, NC_b, NC_h$, then

$$\mu_{D(k)}(\pi, \sigma) = \mu_{NC(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.

Proof. Let $Q = P, E_h, NC$ according to cases (1), (2), (3). It is easy to see in each case that $D(k)$ is closed under taking intervals in $Q(k)$, i.e., if $\pi_1, \pi_2 \in D(k)$, $\sigma \in Q(k)$ and $\pi_1 < \sigma < \pi_2$ then $\sigma \in D(k)$. The result now follows immediately from the definition of the Möbius function. □

4.4 Finite quantum invariant sequences

We begin this section by defining the notion of quantum invariance for a sequence of noncommutative random variables under "transformations" coming from an orthogonal quantum group $G_n \subset O_n^+$.

Let $\mathscr{P}_n = \mathbb{C}\langle t_1, \ldots, t_n \rangle$, and let $\alpha_n : \mathscr{P}_n \to \mathscr{P}_n \otimes C(G_n)$ be the unique unital homomorphism such that

$$\alpha_n(t_j) = \sum_{i=1}^{n} t_i \otimes u_{ij}.$$

It is easily verified that α_n is an action of G_n, i.e.,

$$(\mathrm{id} \otimes \Delta) \circ \alpha_n = (\alpha_n \otimes \mathrm{id}) \circ \alpha_n,$$

and

$$(\mathrm{id} \otimes \epsilon) \circ \alpha_n = \mathrm{id}.$$

Definition 4.4.1. Let $(x_1, \ldots, x_n)$ be a sequence of random variables in a noncommutative probability space $(\mathcal{B}, \varphi)$. We say that the joint distribution of this sequence is *invariant under* G_n, or that the sequence is G_n*-invariant*, if the distribution functional $\varphi_x : \mathscr{P}_n \to \mathbb{C}$ is invariant under the coaction α_n, i.e.

$$(\varphi_x \otimes \mathrm{id})\alpha_n(p) = \varphi_x(p)$$

for all $p \in \mathscr{P}_n$. More explicitly, the sequence $(x_1, \ldots, x_n)$ is G_n-invariant if

$$\varphi(x_{j_1} \cdots x_{j_k}) 1_{C(G_n)} = \sum_{1 \le i_1, \ldots, i_k \le n} \varphi(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k}$$

as an equality in $C(G_n)$, for all $k \in \mathbb{N}$ and $1 \le j_1, \ldots, j_k \le n$.

Remark 4.4.2. Suppose that $G_n \subset O_n$ is a compact group. By evaluating both sides of the above equation at $g \in G_n$, we see that a sequence $(x_1, \ldots, x_n)$ is G_n-invariant if and only if

$$\varphi(x_{j_1} \cdots x_{j_k}) = \sum_{1 \le i_1, \ldots, i_k \le n} g_{i_1 j_1} \cdots g_{i_k j_k} \varphi(x_{i_1} \cdots x_{i_k})$$

for each $k \in \mathbb{N}$, $1 \le j_1, \ldots, j_k \le n$ and $g = (g_{ij}) \in G_n$, which coincides with the usual notion of G_n-invariance for a sequence of classical random variables.

We will now prove a converse to Theorem 4.1.1, which holds also for finite sequences and in a purely algebraic context. The proof is adapted from the method of [40, Proposition 3.1].

Proposition 4.4.3. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital subalgebra and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation which preserves φ. Let $(x_1, \ldots, x_n)$ be a sequence in $\mathcal{A}$.*

(1) Free case:

- *(a) If $x_1, \ldots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, then the sequence is S_n^+-invariant.*
- *(b) If $x_1, \ldots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have even distributions with respect to E, then the sequence is H_n^+-invariant.*
- *(c) If $x_1, \ldots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have semicircular distributions with respect to E, then the sequence is B_n^+-invariant.*
- *(d) If $x_1, \ldots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have centered semicircular distributions with respect to E, then the sequence is O_n^+-invariant.*

(2) Half-liberated case: Suppose that $(x_1, \ldots, x_n)$ half-commute, and that $\mathcal{B}$ is central in $\mathcal{A}$.

- *(a) If $x_1, \ldots, x_n$ are half-independent and identically distributed given $\mathcal{B}$, then the sequence is H_n^*-invariant.*

(b) *If $x_1, \ldots, x_n$ are half-independent and identically distributed given $\mathcal{B}$, and have symmetrized Rayleigh distributions with respect to E, then the sequence is O_n^*-invariant.*

(3) *Suppose that $\mathcal{B}$ and $x_1, \ldots, x_n$ generate a commutative algebra.*

(a) *If $x_1, \ldots, x_n$ are conditionally independent and identically distributed given $\mathcal{B}$, then the sequence is S_n-invariant.*

(b) *If $x_1, \ldots, x_n$ are conditionally independent and identically distributed given $\mathcal{B}$, and have even distributions with respect to E, then the sequence is H_n-invariant.*

(c) *If $x_1, \ldots, x_n$ are conditionally independent and identically distributed given $\mathcal{B}$, and have Gaussian distributions with respect to E, then the sequence is B_n-invariant.*

(d) *If $x_1, \ldots, x_n$ are conditionally independent and identically distributed given $\mathcal{B}$, and have centered Gaussian distributions with respect to E, then the sequence is O_n-invariant.*

Proof. Suppose that the joint distribution of $(x_1, \ldots, x_n)$ satisfies one of the conditions specified in the statement of the Proposition, and let D be the partition family associated to the corresponding easy quantum group. By Propositions 2.1.19 and 4.2.8, and the moment-cumulant formulae, for any $k \in \mathbb{N}$ and $1 \leq j_1, \ldots, j_k \leq n$ we have

$$\begin{aligned}
\sum_{1 \leq i_1, \ldots, i_k \leq n} \varphi(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k} &= \sum_{1 \leq i_1, \ldots, i_k \leq n} \varphi(E[x_{j_1} \cdots x_{j_k}]) u_{i_1 j_1} \cdots u_{i_k j_k} \\
&= \sum_{1 \leq i_1, \ldots, i_k \leq n} \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{i}}} \varphi(\xi_E^{(\pi)}[x_1, \ldots, x_1]) u_{i_1 j_1} \cdots u_{i_k j_k} \\
&= \sum_{\pi \in D(k)} \varphi(\xi_E^{(\pi)}[x_1, \ldots, x_1]) \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} u_{i_1 j_1} \cdots u_{i_k j_k},
\end{aligned}$$

where ξ denotes the classical, half or free cumulants in cases (1), (2) and (3) respectively. It follows from the considerations in [10], or by direct computation, that if $\pi \in D(k)$ then

$$\sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} u_{i_1 j_1} \cdots u_{i_k j_k} = \begin{cases} 1_{C(G_n)}, & \pi \leq \ker \mathbf{j} \\ 0, & \text{otherwise} \end{cases}.$$

Applying this above, we find

$$\begin{aligned}
\sum_{1 \leq i_1, \ldots, i_k \leq n} \varphi(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k} &= \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{j}}} \varphi(\xi_E^{(\pi)}[x_1, \ldots, x_1]) 1_{C(G_n)} \\
&= \varphi(x_{j_1} \cdots x_{j_k}) 1_{C(G_n)},
\end{aligned}$$

which completes the proof. □

Remark 4.4.4. To prove the approximation result for finite sequences, we will require more analytic structure. Throughout the rest of the section, we will assume that $G_n \subset O_n^+$ is a compact quantum group, (M, φ) is a W*-probability space and $(x_1, \dots, x_n)$ is a sequence of self-adjoint random variables in M. We denote the von Neumann algebra generated by $(x_1, \dots, x_n)$ by M_n, and define the G_n*-invariant subalgebra* by

$$\mathcal{B}_n = \mathrm{W}^*\big(\{p(x) : p \in \mathscr{P}_n^{\alpha_n}\}\big),$$

where $\mathscr{P}_n^{\alpha_n}$ denotes the fixed point algebra of the action α_n, i.e.,

$$\mathscr{P}_n^{\alpha_n} = \{p \in \mathscr{P}_n : \alpha_n(p) = p \otimes 1_{C(G_n)}\}.$$

We now begin the technical preparations for our approximation result. First we will need to extend the action α_n to the von Neumann algebra context. $L^\infty(G_n)$ will denote the von Neumann algebra obtained by taking the weak closure of $\pi_n(C(G_n))$, where π_n is the GNS representation of $C(G_n)$ on the GNS Hilbert space $L^2(G_n)$ for the Haar state. $L^\infty(G_n)$ is a *Hopf von Neumann algebra*, with the natural structure induced from $C(G_n)$. We note that if $G_n \subset O_n$ is a compact group, then $L^\infty(G_n), L^2(G_n)$ agree with the usual definitions.

Proposition 4.4.5. *Suppose that $(x_1, \dots, x_n)$ is G_n-invariant. Then there is a right coaction $\widetilde{\alpha}_n : M_n \to M_n \otimes L^\infty(G_n)$ determined by*

$$\widetilde{\alpha}_n(p(x)) = (\mathrm{ev}_x \otimes \pi_n)\alpha_n(p)$$

for $p \in \mathscr{P}_n$. Moreover, the fixed point algebra of $\widetilde{\alpha}_n$ is precisely the G_n-invariant subalgebra $\mathcal{B}_n$.

Proof. This follows from [23, Theorem 3.3], after identifying the GNS representation of $\mathscr{P}_n$ for the state φ_x with the homomorphism $\mathrm{ev}_x : \mathscr{P}_n \to M_n$. □

There is a natural conditional expectation $E_n : M_n \to \mathcal{B}_n$ given by integrating the coaction $\widetilde{\alpha}_n$ with respect to the Haar state, i.e.,

$$E_n[m] = (\mathrm{id} \otimes \textstyle\int)\widetilde{\alpha}_n(m).$$

By using the Weingarten calculus, we can give a simple combinatorial formula for the moment functionals with respect to E_n if G_n is one of the easy quantum groups under consideration. In the half-liberated case, we must first show that $\mathcal{B}_n$ is central in M_n.

Lemma 4.4.6. *Suppose that $(x_1, \dots, x_n)$ half-commute. If $H_n^* \subset G_n$ then the G_n-invariant subalgebra $\mathcal{B}_n$ is contained in the center of M_n.*

Proof. Since the G_n-invariant subalgebra is clearly contained in the H_n^*-invariant subalgebra, it suffices to prove the result for $G_n = H_n^*$. Observe that the representation of G_n on the subspace of $\mathscr{P}_n$ consisting of homogeneous noncommutative polynomials of degree k, given by the restriction of α_n, is naturally identified with $u^{\otimes k}$, where u is the fundamental representation of G_n. As discussed in Section 2.2, $\mathrm{Fix}(u^{\otimes k})$ is spanned by the operators T_π for $\pi \in E_h(k)$. It follows that the fixed point algebra of α_n is spanned by

$$p_\pi = \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} t_{i_1} \cdots t_{i_k},$$

for $k \in \mathbb{N}$ and $\pi \in E_h(k)$. Therefore $\mathcal{B}_n$ is generated by $p_\pi(x)$, for $k \in \mathbb{N}$ and $\pi \in E_h(k)$. Recall from Section 4.2 that if $\omega \in S_k$ is a parity preserving permutation, then $x_{i_1} \cdots x_{i_k} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(k)}}$ for any $1 \leq i_1, \ldots, i_k \leq n$. It follows that $p_\pi(x) = p_{\omega(\pi)}(x)$, where $\omega(\pi)$ is given by the usual action of permutations on set partitions. Now if $\pi \in E_h(k)$, it is easy to see that there is a parity preserving permutation $\omega \in S_k$ such that

$$\omega(\pi) = \big\{(1, \ldots, 2k_1), \ldots, (2(k_1 + \cdots + k_{l-1}) + 1, \ldots, 2(k_1 + \ldots + k_l))\big\}$$

is an interval partition. We then have

$$p_\pi(x) = p_{\omega(\pi)}(x) = \left(\sum_{i_1=1}^n x_{i_1}^{2k_1}\right) \cdots \left(\sum_{i_l=1}^n x_{i_l}^{2k_l}\right).$$

Since x_i^2 is central in M_n for $1 \leq i \leq n$, the result follows. □

Proposition 4.4.7. *Suppose that $(x_1, \ldots, x_n)$ is G_n-invariant, and that one of the following conditions is satisfied:*

(1) G_n is a free quantum group O_n^+, S_n^+, H_n^+ or B_n^+.

(2) G_n is a half-liberated quantum group O_n^* or H_n^* and $(x_1, \ldots, x_n)$ half-commute.

(3) G_n is an easy group O_n, S_n, H_n or B_n and $(x_1, \ldots, x_n)$ commute.

Then for any π in the partition category $D(k)$ for the easy quantum group G_n, and any $b_0, \ldots, b_k \in \mathcal{B}_n$, we have

$$E_n^{(\pi)}[b_0 x_1 b_1, \ldots, x_1 b_k] = \frac{1}{n^{|\pi|}} \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k,$$

which holds if n is sufficiently large that the Gram matrix G_{kn} is invertible.

Proof. We prove this by induction on the number of blocks of π. First suppose that $\pi = 1_k$ is the partition with only one block. Then

$$\begin{aligned} E_n^{(1_k)}[b_0 x_1 b_1, \ldots, x_1 b_k] &= E_n[b_0 x_1 \cdots x_1 b_k] \\ &= \sum_{1 \leq i_1, \ldots, i_k \leq n} b_0 x_{i_1} \cdots x_{i_k} b_k \int u_{i_1 1} \cdots u_{i_k 1}, \end{aligned}$$

where we have used the fact that $b_0, \ldots, b_k$ are fixed by the coaction $\widetilde{\alpha}_n$. Applying the Weingarten integration formula in Proposition 2.2.6, we have

$$\begin{aligned} E_n[b_0 x_1 \cdots x_1 b_k] &= \sum_{1 \le i_1, \ldots, i_k \le n} b_0 x_{i_1} \cdots x_{i_k} b_k \sum_{\substack{\sigma, \pi \in D(k) \\ \pi \le \ker \mathbf{i}}} W_{kn}(\pi, \sigma) \\ &= \sum_{\pi \in D(k)} \Bigg(\sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) \Bigg) \sum_{\substack{1 \le i_1, \ldots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k. \end{aligned}$$

Observe that $G_{kn}(\sigma, 1_k) = n^{|\sigma \vee 1_k|} = n$ for any $\sigma \in D(k)$. It follows that for any $\pi \in D(k)$, we have

$$\begin{aligned} n \cdot \sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) &= \sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) G_{kn}(\sigma, 1_k) \\ &= \delta_{\pi 1_k}. \end{aligned}$$

Applying this above, we find

$$\begin{aligned} E_n[b_0 x_1 \cdots x_1 b_k] &= \sum_{\pi \in D(k)} n^{-1} \delta_{\pi 1_k} \sum_{\substack{1 \le i_1, \ldots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k \\ &= \frac{1}{n} \sum_{i=1}^{n} b_0 x_i \cdots x_i b_k, \end{aligned}$$

as desired.

If condition (2) or (3) is satisfied, then the general case follows from the formula

$$E_n^{(\pi)}[b_0 x_1 b_1, \ldots, x_1 b_k] = b_1 \cdots b_k \prod_{V \in \pi} E_n(V)[x_1, \ldots, x_1],$$

where in the half-liberated case we are applying the previous lemma. The one thing we must check here is that if $\pi \in D(k)$ and V is a block of π with s elements, then $1_s \in D(s)$. This is easily verified, in each case, for $D = P_o, P_s, P_h, P_b, E_h, E_o$.

Suppose now that condition (1) is satisfied. Let $\pi \in D(k)$. Since π is noncrossing, π contains an interval $V = \{l+1, \ldots, l+s+1\}$. We then have

$$E_n^{(\pi)}[b_0 x_1 b_1, \ldots, x_1 b_k] = E_n^{(\pi \setminus V)}[b_0 x_1 b_1, \ldots, E_n[x_1 b_{l+1} \cdots x_1 b_{l+s}] x_1, \ldots, x_1 b_k].$$

To apply induction, we must check that $\pi \setminus V \in D(k-s)$ and $1_s \in D(s)$. Indeed, this is easily verified for NC_o, NC_s, NC_h and NC_b. Applying induction, we have

$$\begin{aligned} E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k] &= \frac{1}{n^{|\pi|-1}} \sum_{\substack{1 \le i_1, \dots, i_l, \\ i_{l+s+1}, \dots, i_k \le n \\ (\pi \setminus V) \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots b_l \Big(E_n[x_1 b_{l+1} \cdots x_1 b_{l+s}] \Big) x_{i_{l+s}} \cdots x_{i_k} b_k \\ &= \frac{1}{n^{|\pi|-1}} \sum_{\substack{1 \le i_1, \dots, i_l, \\ i_{l+s+1}, \dots, i_k \le n \\ (\pi \setminus V) \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots b_l \left(\frac{1}{n} \sum_{i=1}^n x_i b_{l+1} \cdots x_i b_{l+s} \right) x_{i_{l+s}} \cdots x_{i_k} b_k \\ &= \frac{1}{n^{|\pi|}} \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k., \end{aligned}$$

which completes the proof. □

We are now prepared to prove the approximation result for finite sequences.

Theorem 4.4.8. *Suppose that $(x_1, \dots, x_n)$ is G_n-invariant, and that one of the following conditions is satisfied:*

(1) G_n is a free quantum group O_n^+, S_n^+, H_n^+ or B_n^+.

(2) G_n is a half-liberated quantum group O_n^* or H_n^* and $(x_1, \dots, x_n)$ half-commute.

(3) G_n is an easy group O_n, S_n, H_n or B_n and $(x_1, \dots, x_n)$ commute.

Let $(y_1, \dots, y_n)$ be a sequence of $\mathcal{B}_n$-valued random variables with $\mathcal{B}_n$-valued joint distribution determined as follows:

- *$G = O^+$: Free semicircular, centered with same variance as x_1.*
- *$G = S^+$: Freely independent, y_i has same distribution as x_1.*
- *$G = H^+$: Freely independent, y_i has same distribution as x_1.*
- *$G = B^+$: Free semicircular, same mean and variance as x_1.*
- *$G = O^*$: Half-liberated Gaussian, same variance as x_1.*
- *$G = H^*$: Half-independent, y_i has same distribution as x_1.*
- *$G = O$: Independent Gaussian, centered with same variance as x_1.*
- *$G = S$: Independent, y_i has same distribution as x_1.*
- *$G = H$: Independent, y_i has same distribution as x_1.*
- *$G = B$: Independent Gaussian, same mean and variance as x_1.*

If $1 \leq j_1, \ldots, j_k \leq n$ *and* $b_0, \ldots, b_k \in \mathcal{B}_n$*, then*

$$\big\|E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] - E[b_0 y_{j_1} \cdots y_{j_k} b_k]\big\| \leq \frac{C_k(G)}{n} \|x_1\|^k \|b_0\| \cdots \|b_k\|,$$

where $C_k(G)$ *is a universal constant which depends only on* k *and the easy quantum group* G*.*

Proof. First we note that it suffices to prove the statement for n sufficiently large, in particular we will assume throughout that n is sufficiently large for the Gram matrix G_{kn} to be invertible (see Proposition 4.3.1).

Let $1 \leq j_1, \ldots, j_k \leq n$ and $b_0, \ldots, b_k \in \mathcal{B}_n$. We have

$$\begin{aligned} E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] &= \sum_{1 \leq i_1, \ldots, i_k \leq n} b_0 x_{i_1} \cdots x_{i_k} b_k \int u_{i_1 j_1} \cdots u_{i_k j_k} \\ &= \sum_{1 \leq i_1, \ldots, i_k \leq n} b_0 x_{i_1} \cdots x_{i_k} b_k \sum_{\substack{\pi, \sigma \in D(k) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{kn}(\pi, \sigma) \\ &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} W_{kn}(\pi, \sigma) \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k. \end{aligned}$$

On the other hand, it follows from the assumptions on $(y_1, \ldots, y_n)$ and the various moment-cumulant formulae that

$$\begin{aligned} E[b_0 y_{j_1} \cdots y_{j_k} b_k] &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \xi_{E_n}^{(\sigma)}[b_0 x_1 b_1, \ldots, x_1 b_k] \\ &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) E_n^{(\pi)}[b_0 x_1 b_1, \ldots, x_1 b_k], \end{aligned}$$

where ξ denotes the relevant free, classical or half-liberated cumulants and we have used Proposition 4.3.3 in the second line to replace the Möbius function on $NC(k)$, $E_h(k)$ or $P(k)$ by $\mu_{D(k)}$. Note that for $G = H, H^*, H^+$ we need to first check that x_1 has an even distribution with respect to E, but this can be seen by computing

$$E[b_0 x_1 b_1 \cdots x_1 b_{2k+1}] = \sum_{1 \leq i_1, \ldots, i_{2k+1} \leq n} b_0 x_{i_1} \cdots x_{i_k} b_{2k+1} \int u_{i_1 j_1} \cdots u_{i_{2k+1} j_{2k+1}},$$

which is equal to zero since the coordinate functions u_{ij} have an even joint distribution if $G = H, H^*, H^+$.

Applying Lemma 4.4.7, we have

$$E[b_0 y_{j_1} \cdots y_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) n^{-|\pi|} \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k.$$

Comparing these two equations, we find that

$$E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] - E[b_0 y_{j_1} \cdots y_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \left(W_{kn}(\pi, \sigma) - \mu_{D(k)}(\pi, \sigma) n^{-|\pi|} \right) \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k.$$

Now since $x_1, \ldots, x_n$ are identically distributed with respect to the faithful state φ, it follows that these variables have the same norm. Therefore

$$\Bigg\| \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k \Bigg\| \leq n^{|\pi|} \|x_1\|^k \|b_0\| \cdots \|b_k\|$$

for any $\pi \in D(k)$. Combining this with former equation, we have

$$\big\| E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] - E[b_0 y_{j_1} \cdots y_{j_k} b_k] \big\| \leq \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} \left| W_{kn}(\pi, \sigma) n^{|\pi|} - \mu_{D(k)}(\pi, \sigma) \right| \|x_1\|^k \|b_0\| \cdots \|b_k\|.$$

Setting

$$C_k(G) = \sup_{n \in \mathbb{N}} n \cdot \sum_{\sigma, \pi \in D(k)} |W_{kn}(\pi, \sigma) n^{|\pi|} - \mu_{D(k)}(\pi, \sigma)|,$$

which is finite by Proposition 4.3.1, completes the proof. □

4.5 Infinite quantum invariant sequences

In this section we will prove Theorem 4.1.1. Throughout this section, we will assume that G is one of the easy quantum groups $O, S, H, B, O^*, H^*, O^+, S^+, H^+$ or B^+. We will make use of the inclusions $G_n \hookrightarrow G_m$ for $n < m$, which correspond to the Hopf algebra morphisms $\omega_{n,m} : C(G_m) \to C(G_n)$ determined by

$$\omega_{n,m}(u_{ij}) = \begin{cases} u_{ij}, & 1 \leq i, j \leq n \\ \delta_{ij} 1_{C(G_n)}, & \max\{i, j\} > n \end{cases}.$$

The existence of $\omega_{n,m}$ may be verified in each case by using the universal relations of $C(G_n)$.

We begin by extending the notion of G_n-invariance to infinite sequences.

Definition 4.5.1. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence in a noncommutative probability space $(\mathcal{A}, \varphi)$. We say that the joint distribution of $(x_i)_{i \in \mathbb{N}}$ is *invariant under* G, or that the sequence is G*-invariant*, if $(x_1, \ldots, x_n)$ is G_n-invariant for each $n \in \mathbb{N}$.

This means that the joint distribution functional of $(x_1, \dots, x_n)$ is invariant under the action $\alpha_n : \mathscr{P}_n \to \mathscr{P}_n \otimes C(G_n)$ for each $n \in \mathbb{N}$. It will be convenient to extend these actions to $\mathscr{P}_\infty = \mathbb{C}\langle t_i | i \in \mathbb{N}\rangle$, by defining $\beta_n : \mathscr{P}_\infty \to \mathscr{P}_\infty \otimes C(G_n)$ to be the unique unital homomorphism such that

$$\beta_n(t_j) = \begin{cases} \sum_{i=1}^n t_i \otimes u_{ij}, & 1 \leq j \leq n \\ t_j \otimes 1_{C(G_n)}, & j > n \end{cases}.$$

It is clear that β_n is an action of G_n, moreover we have the relations

$$(\mathrm{id} \otimes \omega_{n,m}) \circ \beta_m = \beta_n$$

and

$$(\iota_n \otimes \mathrm{id}) \circ \alpha_n = \beta_n \circ \iota_n,$$

where $\iota_n : \mathscr{P}_n \to \mathscr{P}_\infty$ is the natural inclusion.

Lemma 4.5.2. *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of noncommutative random variables in $(\mathcal{A}, \varphi)$. Then the sequence is G-invariant if and only if the joint distribution functional $\varphi_x : \mathscr{P}_\infty \to \mathbb{C}$ is invariant under the actions β_n for all $n \in \mathbb{N}$.*

Proof. Let $\varphi_x^{(n)} : \mathscr{P}_n \to \mathbb{C}$ denote the joint distribution functional of $(x_1, \dots, x_n)$, and note that $\varphi_x^{(n)} = \varphi_x \circ \iota_n$. First suppose that φ_x is invariant under β_n for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and $p \in \mathscr{P}_n$, then

$$\begin{aligned}(\varphi_x^{(n)} \otimes \mathrm{id})\alpha_n(p) &= (\varphi_x \otimes \mathrm{id})(\iota_n \otimes \mathrm{id})\alpha_n(p) \\ &= (\varphi_x \otimes \mathrm{id})\beta_n(\iota_n(p)) \\ &= \varphi_x^{(n)}(p)1_{C(G_n)},\end{aligned}$$

so that $(x_1, \dots, x_n)$ is G_n-invariant.

Conversely, suppose that $(x_i)_{i\in\mathbb{N}}$ is G-invariant, i.e., $\varphi_x^{(n)}$ is invariant under α_n for all $n \in \mathbb{N}$. Fix $p \in \mathscr{P}_\infty$ and $n \in \mathbb{N}$. Then $p = \iota_m(q)$ for some $m \geq n$ and $q \in \mathscr{P}_m$. We then have

$$\begin{aligned}(\varphi_x \otimes \mathrm{id})\beta_n(p) &= (\varphi_x \otimes \omega_{n,m})\beta_m(\iota_m(q)) \\ &= (\varphi_x^{(m)} \otimes \omega_{n,m})\alpha_m(q) \\ &= \varphi_x(p)1_{C(G_n)},\end{aligned}$$

so that φ_x is invariant under β_n. □

Throughout the rest of the section, (M, φ) will be a W*-probability space and $(x_i)_{i\in\mathbb{N}}$ a sequence of self-adjoint random variables in (M, φ). For $n \in \mathbb{N}$, we let M_n denote the von Neumann algebra generated by the variables $(x_1, \dots, x_n)$. We let M_∞ denote the von Neumann algebra generated by the variables $(x_i)_{i\in\mathbb{N}}$, and set $\varphi_\infty = \varphi|_{M_\infty}$. $L^2(M_\infty, \varphi_\infty)$ will denote the GNS Hilbert space, with inner product $\langle m_1, m_2\rangle = \varphi(m_1^* m_2)$. The strong topology on M_∞ will be taken with respect to the faithful representation on $L^2(M_\infty, \varphi_\infty)$. We set

$$\mathcal{B}_n = \mathrm{W}^*\big(\{p(x) : p \in \mathscr{P}_\infty^{\beta_n}\}\big),$$

where $\mathscr{P}_\infty^{\beta_n}$ is the fixed point algebra of the action β_n. Since

$$(\mathrm{id} \otimes \omega_{n,n+1}) \circ \beta_{n+1} = \beta_n,$$

it follows that $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ for all $n \geq 1$. We then define the *G-invariant subalgebra* by

$$\mathcal{B} = \bigcap_{n \geq 1} \mathcal{B}_n.$$

Remark 4.5.3. If $(x_i)_{i\in\mathbb{N}}$ is G-invariant, then as in Proposition 4.4.5, for each $n \in \mathbb{N}$ there is a right coaction $\widetilde{\beta}_n : M_\infty \to M_\infty \otimes L^\infty(G_n)$ determined by

$$\widetilde{\beta}_n(p(x)) = (\mathrm{ev}_x \otimes \pi_n)\beta_n(p)$$

for $p \in \mathscr{P}_\infty$, and moreover the fixed point algebra of $\widetilde{\beta}_n$ is $\mathcal{B}_n$. For each $n \in \mathbb{N}$, there is then a φ_∞-preserving conditional expectation $E_n : M_\infty \to \mathcal{B}_n$ given by integrating the action $\widetilde{\beta}_n$, i.e.

$$E_n[m] = (\mathrm{id} \otimes \textstyle\int)\widetilde{\beta}_n(m)$$

for $m \in M_\infty$. By taking the limit as $n \to \infty$, we may obtain a φ_∞-preserving conditional expectation onto the G-invariant subalgebra.

Proposition 4.5.4. *Suppose that $(x_i)_{i\in\mathbb{N}}$ is G-invariant. Then:*

(1) For any $m \in M_\infty$, the sequence $E_n[m]$ converges in $|\ |_2$ and the strong topology to a limit $E[m]$ in $\mathcal{B}$. Moreover, E is a φ-preserving conditional expectation of M_∞ onto $\mathcal{B}$.

(2) Fix $\pi \in NC(k)$ and $m_1, \ldots, m_k \in M_\infty$, then

$$E^{(\pi)}[m_1 \otimes \cdots \otimes m_k] = \lim_{n\to\infty} E_n^{(\pi)}[m_1 \otimes \cdots \otimes m_k],$$

with convergence in the strong topology.

Proof. Let $\varphi_n = \varphi|_{\mathcal{B}_n}$ and let $L^2(\mathcal{B}_n, \varphi_n)$ denote the GNS Hilbert space, which can be viewed as a closed subspace of $L^2(M, \varphi)$. Let $P_n \in B(L^2(M, \varphi))$ be the orthogonal projection onto $L^2(\mathcal{B}_n, \varphi_n)$. Since $E_n : M \to \mathcal{B}_n$ is a conditional expectation such that $\varphi_n \circ E_n = \varphi$, it follows that

$$E_n[m] = P_n m P_n$$

for $m \in M$. Since P_n converges strongly as $n \to \infty$ to P, where

$$P = \bigwedge_{n\geq 1} P_n$$

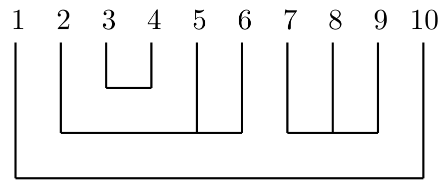
is the orthogonal projection onto $L^2(\mathcal{B}, \varphi|_{\mathcal{B}})$, it follows that

$$E_n[m] \to PmP$$

in $| \ |_2$ and the strong operator topology as $n \to \infty$. Set $E[m] = PmP$, then since $E_n[m]$ converges strongly to $E[m]$ it follows that $E[m] \in \mathcal{B}$, and it is then easy to see that E is a φ-preserving conditional expectation.

To prove (2), observe that if $\pi \in NC(k)$ and $m_1, \ldots, m_k \in M$, then $E_n^{(\pi)}[m_1 \otimes \cdots \otimes m_k]$ is a word in $m_1, \ldots, m_k$ and P_n. For example, if

$$\pi = \{\{1, 10\}, \{2, 5, 6\}, \{3, 4\}, \{7, 8, 9\}\} \in NC(10),$$



then the corresponding expression is

$$E_n^{(\pi)}[m_1 \otimes \cdots \otimes m_{10}] = P_n m_1 P m_2 P m_3 m_4 P_n m_5 m_6 P_n m_7 m_8 m_9 P_n m_{10} P_n.$$

Since multiplication is jointly continuous on bounded sets in the strong topology, this converges as n goes to infinity to the expression obtained by replacing P_n by P, which is exactly $E^{(\pi)}[m_1 \otimes \cdots \otimes m_k]$. □

We are now prepared to prove our main theorem, which we first reformulate in terms of cumulants as follows:

Theorem 4.5.5. *Suppose that $(x_i)_{i \in \mathbb{N}}$ is G-invariant and that one of the following conditions is satisfied:*

(1) G is a free quantum group O^+, S^+, B^+ or H^+.

(2) G is a half-liberated quantum group O^* or H^* and the variables half-commute.

(3) G is a classical easy group O, S, B or H and the variables commute.

Then for any $j_1, \ldots, j_k \in \mathbb{N}$ and $b_0, \ldots, b_k \in \mathcal{B}$, we have

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \xi_E^{(\sigma)}[b_0 x_1 b_1, \ldots, x_1 b_k],$$

where ξ denotes the relevant free, half-liberated or classical cumulants and $D(k)$ is the partition category corresponding to the easy quantum group G.

Proof. Let $j_1, \dots, j_k \in \mathbb{N}$ and $b_0, \dots, b_k \in \mathcal{B}$. As in the proof of Theorem 4.4.8, we have

$$\begin{aligned} E[b_0 x_{j_1} \cdots x_{j_k} b_k] &= \lim_{n\to\infty} E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] \\ &= \lim_{n\to\infty} \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} W_{kn}(\pi, \sigma) \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k \\ &= \lim_{n\to\infty} \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) n^{-|\pi|} \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k. \end{aligned}$$

By Proposition 4.4.7, and using the compatibility

$$(\widetilde{\iota}_n \otimes \mathrm{id}) \circ \widetilde{\alpha}_n = \widetilde{\beta}_n \circ \widetilde{\iota}_n,$$

where $\widetilde{\iota}_n : M_n \to M_\infty$ is the obvious inclusion and $\widetilde{\alpha}_n$ is as in the previous section, we have

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \lim_{n\to\infty} \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k].$$

By (2) of Proposition 4.5.4, we obtain

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) E^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k],$$

and the result now follows from the formulas for cumulants in terms of moments via Möbius inversion, and Proposition 4.3.3. □

Theorem 4.1.1 follows easily:

Proof of Theorem 4.1.1. From Theorem 4.5.5 above, we have

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \xi_E^{(\sigma)}[b_0 x_1 b_1, \dots, x_1 b_k]$$

for any $j_1, \dots, j_k \in \mathbb{N}$ and $b_0, \dots, b_k \in \mathcal{B}$, where ξ denotes the relevant free, classical or half-liberated cumulants and $D(k)$ is the partition category corresponding to the easy quantum group G. Since these cumulants are uniquely determined by the relevant moment-cumulant formula, it follows that

$$\xi_E^{(\sigma)} = \begin{cases} \xi_E^{(\sigma)}[b_0 x_1 b_1, \dots, x_1 b_k], & \sigma \in D(k) \text{ and } \sigma \leq \ker \mathbf{j} \\ 0, & \text{otherwise} \end{cases}.$$

The theorem now follows from the characterizations of these joint distributions in terms of cumulants from Propositions 2.1.19, 4.2.8 and 4.2.9. □

Chapter 5

A characterization of freeness by invariance under quantum spreading

5.1 Introduction

Consider a sequence $(\xi_1, \xi_2, \dots)$ of random variables. Such a sequence is called *exchangeable* if its distribution is invariant under finite permutations, and *spreadable* if it is invariant under taking subsequences, i.e., if

$$(\xi_1, \dots, \xi_k) \overset{d}{\sim} (\xi_{l_1}, \dots, \xi_{l_k})$$

for all $k \in \mathbb{N}$ and $l_1 < \dots < l_k$. In the 1930's, de Finetti gave his famous characterization of infinite exchangeable sequences of random variables taking values in $\{0, 1\}$ as conditionally i.i.d. This was extended to variables taking values in a compact Hausdorff space by Hewitt and Savage [33]. It was later discovered by Ryll-Nardzewski that de Finetti's theorem in fact holds under the apparently weaker condition of spreadability [44].

In the previous chapter we gave de Finetti type theorems for sequences of noncommutative random variables whose joint distribution is invariant under the action of an easy quantum group. The starting point for this work was the free de Finetti theorem of Köstler and Speicher [40], which characterizes *quantum exchangeable* sequences as freely independent and identically distributed with amalgalmation over the tail algebra.

In this chapter we will develop a notion of *quantum spreadability* for sequences of noncommutative random variables. The first problem is to find a suitable quantum analogue of an increasing sequence. The answer which we suggest here is similar to Wang's notion of a quantum permutation. For natural numbers $k \leq n$ we construct certain universal C*-algebras $A_i(k, n)$, which we call *quantum increasing sequence spaces*, whose spectrum is naturally identified with the space of increasing sequences $1 \leq l_1 < \dots < l_k \leq n$. These objects form *quantum families of maps*, in the sense of Sołtan [46], from $\{1, \dots, k\}$ into $\{1, \dots, n\}$. Quantum spreadability is naturally defined as invariance under these familes of quantum transformations. This approach is justified by our main result, which is a free analogue of the Ryll-Nardzewski theorem for quantum spreadable sequences (see Sections 5.2 and 5.4 for definitions and motivating examples):

Theorem 5.1.1. *Let $(\rho_i)_{i\in\mathbb{N}}$ be an infinite sequence of unital $*$-homomorphisms from a unital $*$-algebra C into a tracial W^*-probability space (M, τ). Then the following are equivalent:*

(1) $(\rho_i)_{i\in\mathbb{N}}$ is quantum exchangeable.

(2) $(\rho_i)_{i\in\mathbb{N}}$ is quantum spreadable.

(3) $(\rho_i)_{i\in\mathbb{N}}$ is freely independent and identically distributed with respect to the conditional expectation E onto the tail algebra

$$B = \bigcap_{n\geq 1} W^*\big(\{\rho_i(c) : c \in C, i \geq n\}\big).$$

In the case $C = \mathbb{C}[t]$, the equivalence of (1) and (3) is the main result of [40], and was proved in the previous chapter in the context of easy quantum groups. For general C, the equivalence of (1) and (3) was shown in [23]. The implication (3) $\Rightarrow$ (1) for general C is proved similarly to the case $C = \mathbb{C}[t]$, which was presented in the previous chapter, the details are left to the reader.

Observe that Theorem 5.1.1 holds only for infinite sequences. In the previous chapter we gave an approximation to how far a finite quantum exchangeable sequence is from being free with amalgamation. As in the classical case, finite quantum spreadable sequences are more difficult, and we will not attempt an analysis here. For a treatment of classical finite spreadable sequences, see [36].

This chapter is organized as follows. Section 5.2 contains notations and preliminaries. In Section 5.3, we introduce the algebras $A_i(k, n)$ and prove some basic results. In particular we show that $A_i(k, n)$ is a quotient of $A_s(n)$. In Section 5.4, we introduce the notion of quantum spreadability, and prove the implication (1) $\Rightarrow$ (2) of Theorem 5.1.1. This implication holds in fact for finite sequences, and in a purely algebraic context. We complete the proof of Theorem 5.1.1 in Section 5.5, by showing the implication (2) $\Rightarrow$ (3).

5.2 Background and notations

In the previous chapter we worked only with sequences of self-adjoint operators, which correspond in the classical setting to real-valued random variables. In this chapter we will work with more general sequences, which correspond in the classical setting to allowing our random variables to take value in a more general state space.

5.2.1. **Notations**. Let C be a unital $*$-algebra. Given an index set I, we let

$$C_I = \mathop{*}_{i\in I} C^{(i)}$$

denote the free product (with amalgamation over $\mathbb{C}$), where for each $i \in I$, $C^{(i)}$ is an isomorphic copy of C. For $c \in C$ and $i \in I$ we denote the image of c in $C^{(i)}$ as $c^{(i)}$. The universal property of the free product is that given a unital $*$-algebra A and a family $(\rho_i)_{i\in I}$ of unital $*$-homomorphisms from C to A, there is a unique unital $*$-homomorphism from C_I to A, which we denote by ρ, such that $\rho(c^{(i)}) = \rho_i(c)$ for $c \in C$ and $i \in I$. We will mostly be interested in the case that $I = \{1, \dots, n\}$, in which case we denote C_I by C_n, and $I = \mathbb{N}$ in which case we denote $C_I = C_\infty$.

Definition 5.2.2. Let C be a unital $*$-algebra, (A, φ) a noncommutative probability space and $(\rho_i)_{i \in I}$ a family of unital $*$-homomorphisms from C to A. The *joint distribution* of the family $(\rho_i)_{i \in I}$ is the state φ_ρ on C_I defined by $\varphi_\rho = \varphi \circ \rho$. φ_ρ is determined by the *moments*

$$\varphi_\rho(c_1^{(i_1)} \cdots c_k^{(i_k)}) = \varphi(\rho_{i_1}(c_1) \cdots \rho_{i_k}(c_k)),$$

where $c_1, \dots, c_k \in C$ and $i_1, \dots, i_k \in I$.

5.2.3. *Examples.*

(1) Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, let $(S, \mathcal{S})$ be a measure space and $(\xi)_{i \in I}$ a family of S-valued random variables on Ω. Let $A = L^\infty(\Omega)$, and let $\varphi : A \to \mathbb{C}$ be the expectation functional

$$\varphi(f) = \mathbb{E}[f].$$

Let C be the algebra of bounded, complex-valued, $\mathcal{S}$-measurable functions on S. For $i \in I$, define $\rho_i : C \to A$ by $\rho_i(f) = f \circ \xi_i$. Then φ_ρ is determined by

$$\varphi_\rho(f_1^{(i_1)} \cdots f_k^{(i_k)}) = \mathbb{E}[f_1(\xi_{i_1}) \cdots f_k(\xi_{i_k})]$$

for $f_1, \dots, f_k \in C$ and $i_1, \dots, i_k \in I$.

(2) Let $C = \mathbb{C}[t]$, and let $(x_i)_{i \in I}$ be a family of self-adjoint random variables in A. Define $\rho_i : C \to A$ to be the unique unital $*$-homomorphism such that $\rho_i(t) = x_i$. Then $C_I = C\langle t_i : i \in I \rangle$, and we recover the usual definitions of the joint distribution and moments of the family $(x_i)_{i \in I}$.

Definition 5.2.4. Let C be a unital $*$-algebra, (A, E) a B-valued probability space and $(\rho_i)_{i \in I}$ a family of unital $*$-homomorphisms from C into A.

(1) We let C_I^B denote the free product over $i \in I$, with amalgamation over B, of $C^{(i)} * B$, which is naturally isomorphic to $C_I * B$. For each $i \in I$, we extend ρ_i to a unital $*$-homomorphism $\widetilde{\rho}_i : C * B \to A$ by setting $\widetilde{\rho}_i = \rho_i * \mathrm{id}$. We then let $\widetilde{\rho}$ denote the induced unital $*$-homomorphism from C_I^B into A, which is naturally identified with $\rho * \mathrm{id}$. Explicitly, we have

$$\widetilde{\rho}(b_0 c_1^{(i_1)} b_1 \cdots c_k^{(i_k)} b_k) = b_0 \rho_{i_1}(c_1) b_1 \cdots \rho_{i_k}(c_k) b_k$$

for $b_0, \dots, b_k \in B$, $c_1, \dots, c_k \in C$ and $i_1, \dots, i_k \in I$.

(2) The *B-valued joint distribution* of the family $(\rho_i)_{i \in I}$ is the linear map $E_\rho : C_I * B \to B$ defined by $E_\rho = E \circ \widetilde{\rho}$. E_ρ is determined by the *B-valued moments*

$$E_\rho[b_0 c_1^{(i_1)} \cdots c_k^{(i_k)} b_k] = E[b_0 \rho_{i_1}(c_1) \cdots \rho_{i_k}(c_k) b_k]$$

for $c_1, \dots, c_k \in C$, $b_0, \dots, b_k \in B$ and $i_1, \dots, i_k \in I$.

(3) The family $(\rho_i)_{i\in I}$ is called *identically distributed with respect to* E if $E\circ\widetilde{\rho}_i = E\circ\widetilde{\rho}_j$ for all $i,j\in I$. This is equivalent to the condition that

$$E[b_0\rho_i(c_1)\cdots\rho_i(c_k)b_k] = E[b_0\rho_j(c_1)\cdots\rho_j(c_k)b_k]$$

for any $i,j\in I$ and $c_1,\dots,c_k\in C$, $b_0,\dots,b_k\in B$.

(4) The family $(\rho_i)_{i\in I}$ is called *freely independent with respect to* E, or *free with amalgamation over* B, if

$$E[\widetilde{\rho}_{i_1}(\beta_1)\cdots\widetilde{\rho}_{i_k}(\beta_k)] = 0$$

whenever $i_1\neq\dots\neq i_k\in I$, $\beta_1,\dots,\beta_k\in C*B$ and $E[\widetilde{\rho}_{i_l}(\beta_l)]=0$ for $1\leq l\leq k$.

Recall from Section 2.1 that freeness with amalgamation can be characterized by the vanishing of mixed cumulants. With the notations of this section, we can restate this result as follows:

Theorem 5.2.5. ([48]) *Let C be a unital $*$-algebra, (A,E) be a B-valued probability space and $(\rho_i)_{i\in I}$ a family of unital $*$-homomorphisms from C into A. Then the family $(\rho_i)_{i\in I}$ is free with amalgamation over B if and only if*

$$\kappa_E^{(\pi)}[\widetilde{\rho}_{i_1}(\beta_1)\otimes\cdots\otimes\widetilde{\rho}_{i_k}(\beta_k)] = 0$$

whenever $i_1,\dots,i_k\in I$, $\beta_1,\dots,\beta_k\in C*B$ and $\pi\in NC(k)$ is such that $\pi\not\leq\ker\mathbf{i}$.

□

Corollary 5.2.6. *Let C be a unital $*$-algebra, (A,E) a B-valued probability space and $(\rho_i)_{i\in\mathbb{N}}$ a family of unital $*$-homomorphisms from C into A. Then $(\rho_i)_{i\in\mathbb{N}}$ is freely independent and identically distributed with respect to E if and only if*

$$E[\widetilde{\rho}_{i_1}(\beta_1)\cdots\widetilde{\rho}_{i_k}(\beta_k)] = \sum_{\substack{\pi\in NC(k)\\ \pi\leq\ker\mathbf{i}}}\kappa_E^{(\pi)}[\widetilde{\rho}_1(\beta_1)\otimes\cdots\otimes\widetilde{\rho}_1(\beta_k)]$$

for every $\beta_1,\dots,\beta_k\in C*B$ and $i_1,\dots,i_k\in I$. □

5.3 Quantum increasing sequences

In this section we introduce objects $A_i(k,n)$ which we call *quantum increasing sequence spaces*. As with Wang's quantum permutation group, the idea is to find a natural family of coordinates on the space of increasing sequences $1\leq l_1<\cdots<l_k\leq n$ and "remove commutativity".

Definition 5.3.1. For $k,n\in\mathbb{N}$ with $k\leq n$, we define the *quantum increasing sequence space* $A_i(k,n)$ to be the universal unital C*-algebra generated by elements $\{u_{ij}: 1\leq i\leq n, 1\leq j\leq k\}$ such that

(1) u_{ij} is an orthogonal projection: $u_{ij}^* = u_{ij} = u_{ij}^2$.

(2) each column of the rectangular matrix $u = (u_{ij})$ forms a partition of unity: for $1 \leq j \leq k$ we have
$$\sum_{i=1}^{n} u_{ij} = 1.$$

(3) increasing sequence condition:
$$u_{ij}u_{i'j'} = 0$$
if $j < j'$ and $i \geq i'$.

Remark 5.3.2. We note that the algebra $A_i(k,n)$, together with the morphism $\alpha : \mathbb{C}^n \to \mathbb{C}^k \otimes A_i(k,n)$ defined by
$$\alpha(e_i) = \sum_{j=1}^{k} e_j \otimes u_{ij},$$
gives a *quantum family of maps* from $\{1,\dots,k\}$ to $\{1,\dots,n\}$, in the sense of Sołtan [46].

The motivation for the above definition is as follows. Consider the space $I_{k,n}$ of increasing sequences $\mathbf{l} = (1 \leq l_1 < \cdots < l_k \leq n)$. For $1 \leq i \leq n$, $1 \leq j \leq k$, define $f_{ij} : I_{k,n} \to \mathbb{C}$ by
$$f_{ij}(\mathbf{l}) = \begin{cases} 1, & l_j = i \\ 0, & l_j \neq i \end{cases}.$$
The functions f_{ij} generate $C(I_{k,n})$ by the Stone-Weierstrass theorem, and clearly satisfy the defining relations among the u_{ij} above. Moreover, it can be seen from the Gelfand theory that $C(I_{k,n})$ is the universal *commutative* C*-algebra generated by $\{f_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ satisfying these relations. In other words, $C(I_{k,n})$ is the abelianization of $A_i(k,n)$.

Remark 5.3.3. A first question is whether $A_i(k,n)$ can be larger than $C(I_{k,n})$, i.e., "do quantum increasing sequences exist"? Clearly $A_i(k,n)$ is commutative and hence equal to $C(I_{k,n})$ for $k = 1$. Using Lemma 5.3.4 below, it is not hard to see that $A_i(k,n)$ is also commutative at $k = n$ and $n - 1$. In particular we have $A_i(k,n) = C(I_{k,n})$ whenever $n \leq 3$.

However, if p, q are arbitrary projections in any unital C*-algebra then the following gives a representation of $A_i(2,4)$:
$$\begin{pmatrix} p & 0 \\ 1-p & 0 \\ 0 & q \\ 0 & 1-q \end{pmatrix}$$
In particular, the free product $C(\mathbb{Z}_2) * C(\mathbb{Z}_2)$ is a quotient of $A_i(2,4)$ and hence $A_i(2,4)$ is infinite-dimensional.

Observe that if $(1 \leq l_1 < \cdots < l_k \leq n)$ then we must have $l_{j'} - l_j \geq j' - j$ for $1 \leq j \leq j' \leq k$. In terms of the coordinates f_{ij} on $C(I_{k,n})$, this means that $f_{ij}f_{i'j'} = 0$ if $i' - i < j' - j$. This relation also holds for the coordinates u_{ij} on $A_i(k,n)$, which will be useful to our further analysis.

Lemma 5.3.4. *Fix $k, n \in \mathbb{N}$ with $k \leq n$, and let $\{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ be the standard generators of $A_i(k,n)$. Then*

(1) $u_{ij}u_{i'j'} = 0$ if $1 \leq j \leq j' \leq k$ and $i' - i < j' - j$.

(2) $u_{ij} = 0$ unless $j \leq i \leq n - k + j$, or equivalently $k + i - n \leq j \leq i$.

Proof. (1) is trivial for $j = j'$, so fix $1 \leq j < j' \leq k$ and set $m = j' - j - 1 \geq 0$. Then we have

$$u_{ij}u_{i'j'} = u_{ij}\left(\prod_{l=1}^{m}\sum_{i_l=1}^{n} u_{i_l(j+l)}\right)u_{i'j'} = \sum_{1 \leq i_1,\ldots,i_m \leq n} u_{ij}u_{i_1(j+1)} \cdots u_{i_m(j+m)}u_{i'(j+m+1)}.$$

From the increasing sequence condition, each term in the sum is zero unless $i < i_1 < \cdots < i_m < i'$, which implies $i' - i \geq m + 1 = j' - j$.

For (2), note that from (1) we have $u_{l1}u_{ij} = 0$ if $i - l < j - 1$, or equivalently $l > i - j + 1$. So if $i < j$ then $u_{l1}u_{ij} = 0$ for $l = 1, \ldots, n$ and we then have

$$u_{ij} = \left(\sum_{l=1}^{n} u_{l1}\right) \cdot u_{ij} = 0.$$

Likewise we have $u_{ij}u_{lk} = 0$ if $l < k + i - j$, so if $i > n - k + j$ then this holds for $l = 1, \ldots, n$ and

$$u_{ij} = u_{ij} \cdot \left(\sum_{l=1}^{n} u_{lk}\right) = 0,$$

which completes the proof. □

Now observe that any increasing sequence $1 \leq l_1 < \cdots < l_k \leq n$ can be extended to a permutation in S_n which sends j to l_j for $1 \leq j \leq k$. One way to create such an extension is to set $\pi(j) = l_j$ for $1 \leq j \leq k$, then inductively define $\pi(k+m)$, for $m = 1, \ldots, n-k$, by setting $\pi(k+m)$ to be the least element of $\{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(k+m-1)\}$. After a moment's thought, one sees that $m \leq \pi(k+m) \leq m+k$ and that $\pi(k+m) = m+p$ exactly when $l_p < m+p$ but $l_{p+1} > m+p$ for $1 \leq m \leq n-k$ and $0 \leq p \leq k$, where we set $l_0 = -\infty, l_{k+1} = \infty$.

This gives an inclusion of the space $I_{k,n}$ of increasing sequences into S_n, which dualizes to a unital $*$-homomorphism $C(S_n) \to C(I_{k,n})$. Consider the natural coordinates $\{f_{ij} : 1 \leq i, j \leq n\}$ on S_n and $\{g_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ on $I_{k,n}$. Clearly this map sends f_{ij} to g_{ij} for $1 \leq i \leq n$, $1 \leq j \leq k$. From the remark at the end of the previous paragraph, it follows that $f_{i(k+m)}$ is sent to 0 unless $i = m + p$ for some $0 \leq p \leq k$, and that

$$f_{(m+p)(k+m)} \mapsto \sum_{i=0}^{m+p-1} g_{ip} - g_{(i+1)(p+1)},$$

where we set $g_{00} = 1$ and $g_{i0} = g_{0i} = g_{i(k+1)} = 0$ for $i \geq 1$.

For example, when $k = 2$ and $n = 4$ the matrix (f_{ij}) is as follows:

$$\begin{pmatrix} g_{11} & 0 & 1 - g_{11} & 0 \\ g_{21} & g_{22} & g_{11} - g_{22} & 1 - g_{11} - g_{21} \\ g_{31} & g_{32} & g_{22} & g_{11} + g_{21} - g_{22} - g_{32} \\ 0 & g_{42} & 0 & g_{22} + g_{32} \end{pmatrix}$$

We can now use this formula to define a $*$-homomorphism from $A_s(n)$ to $A_i(k, n)$, which we might think of as "extending quantum increasing sequences to quantum permutations".

Proposition 5.3.5. *Fix natural numbers $k < n$. Let $\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$, $\{u_{ij} : 1 \leq i, j \leq n\}$ be the standard generators of $A_i(k, n)$, $A_s(n)$, respectively. Then there is a unique unital $*$-homomorphism from $A_s(n)$ to $A_i(k, n)$ determined by*

- *$u_{ij} \mapsto v_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq k$.*
- *$u_{i(k+m)} \mapsto 0$ for $1 \leq m \leq n - k$ and $i < m$ or $i > m + k$.*
- *For $1 \leq m \leq n - k$ and $0 \leq p \leq k$,*

$$u_{(m+p)(k+m)} \mapsto \sum_{i=0}^{m+p-1} v_{ip} - v_{(i+1)(p+1)},$$

where we set $v_{00} = 1$ and $v_{i0} = v_{0i} = v_{i(k+1)} = 0$ for $i \geq 1$.

Proof. Let (v_{ij}) be the standard generators of $A_i(k, n)$, and define $\{u_{ij} : 1 \leq i, j \leq n\}$ in $A_i(k, n)$ by

- $u_{ij} = v_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq k$.
- $u_{i(k+m)} = 0$ for $1 \leq m \leq n - k$ and $i < m$ or $i > m + k$.
- For $1 \leq m \leq n - k$ and $0 \leq p \leq k$,

$$u_{(m+p)(k+m)} = \sum_{i=0}^{m+p-1} v_{ip} - v_{(i+1)(p+1)},$$

where we set $v_{00} = 1$ and $v_{i0} = v_{0i} = v_{i(k+1)} = 0$ for $i \geq 1$.

We need to show that $(u_{ij})_{1 \leq i,j \leq n}$ satisfies the magic unitary condition, and the result will then follow from the universal property of $A_s(n)$.

First let us check that u_{ij} is an orthogonal projection for $1 \leq i, j \leq n$. The only non-trivial case is $u_{(m+p)(k+m)}$ for $1 \leq m \leq n - k$ and $0 \leq p \leq k$. Here we just need to check that

$$v_{l(p+1)} \leq \sum_{i=0}^{m+p-1} v_{ip}$$

for $1 \leq l \leq m+p$. The cases $p = 0, k$ are trivial, so let $0 < p < k$. We have

$$v_{l(p+1)} = v_{l(p+1)} \cdot \sum_{i=1}^{n} v_{ip} = v_{l(p+1)} \cdot \sum_{i=1}^{l-1} v_{ip},$$

where we have applied the increasing sequence condition $v_{l(p+1)}v_{ip} = 0$ for $i \geq l$. So we have

$$v_{l(p+1)} \leq \sum_{i=1}^{l-1} v_{ip} \leq \sum_{i=0}^{m+p-1} v_{ip}$$

as desired.

Now we need to check that the sum along any row or column of (u_{ij}) gives the identity. For the first k columns, this follows from the defining relations of v_{ij}. For $m = 1, \ldots, n-k$, the sum along column $k+m$ gives

$$\begin{aligned} \sum_{l=1}^{n} u_{l(k+m)} &= \sum_{p=0}^{k} u_{(m+p)(k+m)} \\ &= \sum_{p=0}^{k} \sum_{i=0}^{m+p-1} v_{ip} - v_{(i+1)(p+1)} \end{aligned}$$

Now since $v_{ip} = v_{(i+1)(p+1)} = 0$ if $i < p$, we continue with

$$\begin{aligned} \sum_{p=0}^{k} \sum_{i=p}^{m+p-1} v_{ip} - v_{(i+1)(p+1)} &= \sum_{p=0}^{k} \sum_{i=0}^{m-1} v_{(i+p)p} - v_{(i+p+1)(p+1)} \\ &= \sum_{i=0}^{m-1} \sum_{p=0}^{k} v_{(i+p)p} - v_{(i+p+1)(p+1)} \\ &= \sum_{i=0}^{m-1} v_{i0} - v_{(i+k+1)(k+1)} \\ &= 1, \end{aligned}$$

since the only nonzero term in the last sum is $v_{00} = 1$.

It now remains only to show that the sum along any row of (u_{ij}) gives the identity. We have

$$\begin{aligned}
\sum_{j=1}^{n} u_{ij} &= \sum_{j=1}^{k} u_{ij} + \sum_{m=1}^{n-k} \sum_{\substack{0\le p\le k\\ m+p=i}} u_{i(k+m)} \\
&= \sum_{j=1}^{k} u_{ij} + \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} u_{i(k+m)} \\
&= \sum_{j=1}^{k} v_{ij} + \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} \sum_{l=0}^{i-1} v_{l(i-m)} - v_{(l+1)(i-m+1)} \\
&= \sum_{j=1}^{k} v_{ij} + \left(\sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{0(i-m)} - v_{i(i-m+1)} \right) + \sum_{l=1}^{i-1} \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{l(i-m)} - v_{l(i-m+1)} \\
&= \sum_{j=1}^{k} v_{ij} + \left(\sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{0(i-m)} - v_{i(i-m+1)} \right) + \sum_{l=1}^{i-1} v_{l\max\{0,k+i-n\}} - v_{l\min\{k+1,i\}}.
\end{aligned}$$

Now note that if $1 \le l \le i-1$ then $v_{l\min\{k+1,i\}} = 0$, indeed this is true by definition if $\min\{k+1,i\} = k+1$, and if $\min\{k+1,i\} = i$ then $v_{li} = 0$ since $l < i$. Also we have $v_{ij} = 0$ unless $k+i-n \le j \le i$. Plugging this in above and rearranging terms, we have

$$\sum_{j=\max\{1,k+i-n\}}^{\min\{k,i\}} v_{ij} - \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{i(i-m+1)} + \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{0(i-m)} + \sum_{l=1}^{i-1} v_{l\max\{0,k+i-n\}}.$$

After reindexing the second sum and combining with the first, we obtain

$$\sum_{j=\max\{1,k+i-n\}}^{\max\{1,k+i+1-n\}-1} v_{ij} + \sum_{m=\max\{i-k,1\}}^{\min\{i,n-k\}} v_{0(i-m)} + \sum_{l=1}^{i-1} v_{l\max\{0,k+i-n\}}.$$

Now if $i \le n-k$, then the first and third sums are zero while the second is 1. If $i > n-k$ then the second sum is zero and the first and third combine as

$$\sum_{l=1}^{i} v_{l(k+i-n)}.$$

Now since $v_{l(k+i-n)} = 0$ if $l > n-k+(k+i-n) = i$, we have

$$\sum_{l=1}^{i} v_{l(k+i-n)} = \sum_{l=1}^{n} v_{l(k+i-n)} = 1.$$

So (u_{ij}) does indeed satisfy the magic unitary condition, which completes the proof. □

5.4 Quantum invariant sequences of random variables

In this section we introduce the notions of quantum exchangeability and quantum spreadability for sequences of algebras, and prove the implications (1)⇒ (2) of Theorem 5.1.1. First let us extend the notion of quantum exchangeability from Chapter 4 to sequences of algebras.

Let C be a unital $*$-algebra. For each $n \in \mathbb{N}$ there is a unique unital $*$-homomorphism $\alpha_n : C_n \to C_n \otimes A_s(n)$ determined by

$$\alpha_n(c^{(j)}) = \sum_{i=1}^{n} c^{(i)} \otimes u_{ij}$$

for $c \in C$ and $1 \leq j \leq n$, indeed this follows from the relations in $A_s(n)$ and the universal property of the free product $C_n = C^{(1)} * \cdots * C^{(n)}$. Moreover α_n is a right coaction of $A_s(n)$ in the sense that

$$(\alpha_n \otimes \mathrm{id}) \circ \alpha_n = (\mathrm{id} \otimes \alpha_n) \circ \alpha_n$$
$$(\mathrm{id} \otimes \epsilon) \circ \alpha_n = \mathrm{id},$$

see [23] for details. The coaction α_n may be regarded as "quantum permuting" the n copies of C inside C_n.

Definition 5.4.1. Let C be a unital $*$-algebra, and let $(\rho_1, \rho_2, \ldots, \rho_n)$ be a sequence of unital $*$-homomorphisms from C into a noncommutative probability space (A, φ). We say that the distribution φ_ρ is *invariant under quantum permutations*, or that the sequence is *quantum exchangeable*, if φ_ρ is invariant under the coaction α_n, i.e.,

$$(\varphi_\rho \otimes \mathrm{id})\alpha_n(c) = \varphi_\rho(c)1_{A_s(n)}$$

for any $c \in C_n$.

This is extended to infinite sequences $(\rho_i)_{i\in\mathbb{N}}$ by requiring that $(\rho_1, \ldots, \rho_n)$ is quantum exchangeable for each $n \in \mathbb{N}$.

5.4.2. *Remarks.*

(1) More explicitly, this amounts to the condition that

$$\sum_{1\leq i_1,\ldots,i_k\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_k}(c_k))u_{i_1j_1}\cdots u_{i_kj_k} = \varphi(\rho_{j_1}(c_1)\cdots\rho_{j_k}(c_k))\cdot 1$$

for any $c_1, \ldots, c_k \in C$ and $1 \leq j_1, \ldots, j_k \leq n$, where u_{ij} are the standard generators of $A_s(n)$.

(2) By the universal property of $A_s(n)$, the sequence $(\rho_1, \ldots, \rho_n)$ is quantum exchangeable if and only if the equation in (1) holds for any family $\{u_{ij} : 1 \leq i, j \leq n\}$ of projections in a unital C*-algebra B such that $(u_{ij}) \in M_n(B)$ is a magic unitary matrix.

(3) For $1 \leq i, j \leq n$, define $f_{ij} \in C(S_n)$ by $f_{ij}(\pi) = \delta_{i\pi(j)}$. The matrix (f_{ij}) is a magic unitary, and the equation in (1) becomes

$$\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_k}(c_k))1_{C(S_n)} = \sum_{1\leq i_1,\ldots,i_k\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_n}(c_n))f_{i_1j_1}\cdots f_{i_kj_k}.$$

Evaluating both sides at $\pi \in S_n$, we find

$$\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_k}(c_k)) = \varphi(\rho_{\pi(j_1)}(c_1)\cdots\rho_{\pi(j_k)}(c_k)),$$

so that quantum exchangeability implies invariance under classical permutations.

We will now introduce the quantum spreadability condition. Let C be a unital $*$-algebra, then for any natural numbers $k \leq n$ there is a unique unital $*$-homomorphism $\alpha_{k,n} : C_k \to C_n \otimes A_i(k,n)$ determined by

$$\alpha_{k,n}(c^{(j)}) = \sum_{i=1}^{n} c^{(i)} \otimes u_{ij}$$

for $c \in C$ and $1 \leq j \leq k$, indeed this follows as above from the relations in $A_i(k,n)$ and the universal property of C_k.

Definition 5.4.3. Let C be a unital $*$-algebra, and let $(\rho_1, \rho_2, \ldots, \rho_n)$ be a sequence of unital $*$-homomorphisms from C into a noncommutative probability space (A, φ). We say that the distribution is *invariant under quantum spreading*, or that sequence is *quantum spreadable*, if for each $k = 1, \ldots, n$ the distribution φ_ρ is invariant under $\alpha_{k,n}$ in the sense that

$$(\varphi_\rho \otimes \mathrm{id})\alpha_{k,n}(c) = \varphi_\rho(c)1_{A_i(k,n)}$$

for any $c \in C_k$.

An infinite sequence $(\rho_1, \rho_2, \ldots)$ is called *quantum spreadable* if $(\rho_1, \ldots, \rho_n)$ is quantum spreadable for each n.

Remark 5.4.4.

(1) Explicitly, the condition is that for each $k = 1, \ldots, n$ we have

$$\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m)) \cdot 1 = \sum_{1\leq i_1,\ldots,i_m\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)) \cdot u_{i_1j_1}\cdots u_{i_mj_m}$$

for all $1 \leq j_1, \ldots, j_m \leq k$ and $c_1, \ldots, c_m \in C$, where (u_{ij}) denote the standard generators of $A_i(k,n)$.

(2) From the universal property of $A_i(k,n)$, the sequence $(\rho_1, \ldots, \rho_n)$ is quantum spreadable if and only if for each $1 \leq k \leq n$, equation (1) holds for any family $\{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ of projections in a unital C*-algebra B which satisfy the definining relations of $A_i(k,n)$.

(3) Let (f_{ij}) denote the generators of $C(I_{k,n})$ introduced in Section 5.3. Plugging f_{ij} into equation (1) and applying both sides to $\mathbf{l} = (1 \leq l_1 < \cdots < l_k \leq n)$, we have

$$\begin{aligned}\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m)) &= \sum_{1\leq i_1,\ldots,i_m\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m) f_{i_1j_1}(\mathbf{l})\cdots f_{i_mj_m}(\mathbf{l})\\ &= \varphi(\rho_{l_{j_1}}(c_1)\cdots\rho_{l_{j_m}}(c_m))\end{aligned}$$

for any $1 \leq j_1, \ldots, j_m \leq k$. So $(\rho_1, \ldots, \rho_k)$ has the same distribution as $(\rho_{l_1}, \ldots, \rho_{l_k})$, and hence quantum spreadability implies classical spreadability. In particular, quantum spreadable sequences are identically distributed.

We can now prove the implication (1) ⇒ (2) of Theorem 5.1.1, this holds in fact for finite sequences and in a purely algebraic context:

Proposition 5.4.5. *Let C be a unital $*$-algebra, and let $(\rho_1, \rho_2, \ldots, \rho_n)$ be a sequence of unital $*$-homomorphisms from C into a noncommutative probability space (A, φ). If the sequence $(\rho_1, \ldots, \rho_n)$ is quantum exchangeable, then it is quantum spreadable.*

Proof. Fix $1 \leq k \leq n$ and let $\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ and $\{u_{ij} : 1 \leq i, j \leq n\}$ be the standard generators of $A_i(k, n)$ and $A_s(n)$, respectively. Assume $(\rho_1, \ldots, \rho_n)$ is quantum exchangeable, and fix $1 \leq j_1, \ldots, j_m \leq k$ and $c_1, \ldots, c_m \in C$. We have

$$\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m))1_{A_s(n)} = \sum_{1\leq i_1,\ldots,i_m\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)) \cdot u_{i_1j_1}\cdots u_{i_mj_m}.$$

By Proposition 5.3.5, there is a unital $*$-homomorphism from $A_s(n)$ to $A_i(k, n)$ which sends u_{ij} to v_{ij} for $1 \leq i \leq n$, $1 \leq j \leq k$. Applying this map to both sides of the above equation, we obtain

$$\varphi(\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m))1_{A_i(k,n)} = \sum_{1\leq i_1,\ldots,i_m\leq n} \varphi(\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)) \cdot v_{i_1j_1}\cdots v_{i_mj_m},$$

so that $(\rho_1, \ldots, \rho_n)$ is quantum spreadable as desired. □

5.5 Quantum spreadability implies freeness with amalgamation

In this section we will complete the proof of Theorem 5.1.1. Throughout this section we will assume that C is a unital $*$-algebra, and that $(\rho_i)_{i\in\mathbb{N}}$ is an infinite sequence of unital $*$-homomorphisms from C into a tracial W*-probability space (M, τ). B will denote the tail algebra:

$$B = \bigcap_{n\geq 1} W^*\big(\{\rho_i(c) : c \in C, i \geq n\}\big).$$

$L^2(M)$ will denote the Hilbert space given by the GNS-representation for τ. Since τ is a trace, there is a unique conditional expectation $E : M \to B$ given by $E[m] = P(m)$, where P is the orthogonal projection of $L^2(M)$ onto $L^2(B)$.

We will assume without loss of generality that M is generated by $\rho_\infty(C_\infty)$, i.e.,

$$M = W^*\big(\{\rho_i(c) : i \in I, c \in C\}\big).$$

Observe that if the sequence $(\rho_i)_{i\in\mathbb{N}}$ is spreadable and hence stationary, the linear map determined by

$$U(\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)) = \rho_{i_1+1}(c_1)\cdots\rho_{i_m+1}(c_m)$$

for $i_1,\ldots,i_m \in \mathbb{N}$ and $c_1,\ldots,c_m \in C$, is well-defined and extends to an isometry $U : L^2(M) \to L^2(M)$.

Recall from Definition 5.2.4 that we set $\widetilde{\rho}_i = \rho_i * \mathrm{id} : C * B \to M$. We will begin by showing that if $(\rho_i)_{i\in\mathbb{N}}$ is quantum spreadable, then the B-valued distribution of $(\widetilde{\rho}_i)_{i\in\mathbb{N}}$ is also invariant under quantum spreading. By this we mean that the joint distribution E_ρ is invariant under the $*$-homomorphisms $\widetilde{\alpha}_{k,n} : C_k * B \to (C_n * B) \otimes A_i(k,n)$ determined by

$$\widetilde{\alpha}_{k,n}(b_0 c_1^{(j_1)} b_1 \cdots c_m^{(j_m)} b_m) = \sum_{1\le i_1,\ldots,i_m\le n} b_0 c_1^{(i_1)} b_1 \cdots c_m^{(i_m)} b_m \otimes u_{i_1 j_1}\cdots u_{i_m j_m}$$

for all $k \le n$, $1 \le j_1,\ldots,j_m \le k$, $b_0,\ldots,b_m \in B$ and $c_1,\ldots,c_m \in C$.

Note that if $1 \le j \le k$, $b_0,\ldots,b_m \in B$ and $c_1,\ldots,c_m \in C$ then

$$\begin{aligned}\widetilde{\alpha}_{k,n}(b_0 c_1^{(j)} \cdots c_m^{(j)} b_m) &= \sum_{1\le i_1,\ldots,i_m\le n} b_0 c_1^{(i_1)} \cdots c_m^{(i_m)} b_m \otimes u_{i_1 j}\cdots u_{i_m j}\\ &= \sum_{i=1}^{n} b_0 c_1^{(i)} \cdots c_m^{(i)} b_m \otimes u_{ij},\end{aligned}$$

from which it follows that if $\beta \in C * B$ then

$$\widetilde{\alpha}_{k,n}(\beta^{(j)}) = \sum_{i=1}^{n} \beta^{(i)} \otimes u_{ij}.$$

Proposition 5.5.1. *Suppose that the sequence $(\rho_i)_{i\in\mathbb{N}}$ is quantum spreadable. Then the joint distribution of $(\widetilde{\rho}_i)_{i\in\mathbb{N}}$ with respect to E is invariant under quantum spreading. Explicitly, for each $k \le n$, $1 \le j_1,\ldots,j_m \le k$ and $\beta_1,\ldots,\beta_m \in C * B$ we have*

$$E[\widetilde{\rho}_{j_1}(\beta_1)\cdots\widetilde{\rho}_{j_m}(\beta_m)] \otimes 1_{A_i(k,n)} = \sum_{1\le i_1,\ldots,i_m\le n} E[\widetilde{\rho}_{i_1}(\beta_1)\cdots\widetilde{\rho}_{i_m}(\beta_m)] \otimes u_{i_1 j_1}\cdots u_{i_m j_m},$$

where the equality holds in $B \otimes A_i(k,n)$.

Proof. We need to show that $1 \le j_1,\ldots,j_m \le n$, $b_0,\ldots,b_m \in B$ and $c_1,\ldots,c_m \in C$ then

$$E[b_0\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m)b_m] = \sum_{1\le i_1,\ldots,i_m\le n} E[b_0\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)b_m] \otimes u_{i_1 j_1}\cdots u_{i_m j_m}.$$

Since E preserves the faithful state τ, it suffices to show that

$$\tau(b_0\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m)b_m) = \sum_{1\le i_1,\ldots,i_m\le n} \tau(b_0\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)b_m)\cdot u_{i_1j_1}\cdots u_{i_mj_m}.$$

We will show that this in fact holds for $b_0,\ldots,b_m$ in $W^*(\{\rho_i(c) : i > n, c \in C\})$. By Kaplansky's density theorem, it suffices to consider the case that $b_0,\ldots,b_m$ are elements of the form $\rho_{l_1}(d_1)\ldots\rho_{l_r}(d_r)$ for $n < l_1,\ldots,l_r \le N$ and $d_1,\ldots,d_r \in C$.

To show this, we extend (u_{ij}) to a $(k+N)\times(n+N)$ matrix by setting

$$v_{ij} = \begin{cases} u_{ij}, & 1\le i\le n, 1\le j\le k\\ \delta_{(i-n)(j-k)}, & i>n, j>k\\ 0, & \text{otherwise}\end{cases}$$

Observe if $b = \rho_{l_1}(d_1)\ldots\rho_{l_r}(d_r)$ is as above, then

$$\begin{aligned}\sum_{1\le i_1,\ldots,i_r\le n+N} \rho_{i_1}(d_1)\cdots\rho_{i_m}(d_r)\otimes v_{i_1l_1}\cdots v_{i_rl_r} &= \rho_{l_1+(n-k)}(d_1)\cdots\rho_{l_r+(n-k)}(d_r)\otimes 1_{A_i(k,n)}\\ &= U^{(n-k)}(b)\otimes 1_{A_i(k,n)}.\end{aligned}$$

Now it is clear that (v_{ij}) satisfies the defining relations of $A_i(k+N, n+N)$, so applying the quantum spreadability condition with (v_{ij}), we have

$$\begin{aligned}&\tau(b_0\rho_{j_1}(c_1)\cdots\rho_{j_m}(c_m)b_m)\\ &\qquad= \sum_{1\le i_1,\ldots,i_m\le n} \tau\big(U^{(n-k)}(b_0)\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)U^{(n-k)}(b_m)\big)\otimes u_{i_1j_1}\cdots u_{i_mj_m}.\end{aligned}$$

But since $(\rho_i)_{i\in\mathbb{N}}$ is spreadable, the right hand side is equal to

$$\sum_{1\le i_1,\ldots,i_m\le n} \tau\big(b_0\rho_{i_1}(c_1)\cdots\rho_{i_m}(c_m)b_m\big)\otimes u_{i_1j_1}\cdots u_{i_mj_m},$$

which completes the proof.

□

The key ingredient in our proof that an infinite quantum spreadable sequence is free with amalgamation is a "measure" on the space of quantum increasing sequences, i.e., a state on $A_i(k,n)$. Unlike in the classical case, there does not appear to be a good notion of "uniform" measure on this quantum space. Instead, we will use the measures induced by a certain representation of $A_i(k, k\cdot n)$.

Proposition 5.5.2. *Fix $k, n \in \mathbb{N}$. Then there is a state $\psi_{k,n} : A_i(k, k\cdot n) \to \mathbb{C}$ such that:*

(1)

$$\psi_{k,n}(u_{l_1j_1}\cdots u_{l_mj_m}) = 0$$

unless $(j_r - 1)\cdot n < l_r \le j_r\cdot n$ for $r = 1,\ldots,m$.

(2)

$$\psi_{k,n}(u_{((j_1-1)\cdot n+i_1)j_1}\cdots u_{((j_m-1)\cdot n+i_m)j_m}) = \sum_{\substack{\pi\in NC(m)\\ \pi\le\ker\mathbf{j}}}\sum_{\substack{\sigma\in NC(m)\\ \sigma\le\pi\wedge\ker\mathbf{i}}}\mu_m(\sigma,\pi)n^{-|\sigma|}$$

for all $1\le j_1,\dots,j_m\le k$ *and* $1\le i_1,\dots,i_m\le n$.

Proof. Let $\{p_{ij} : 1\le i\le n, 1\le j\le k\}$ be projections in a C*-probability space (A,φ) such that

(1) The families $(\{p_{i1} : 1\le i\le n\},\dots,\{p_{ik} : 1\le i\le n\})$ are freely independent.

(2) For $j=1,\dots,k$, we have

$$\sum_{i=1}^n p_{ij} = 1,$$

and $\varphi(p_{ij}) = n^{-1}$ for $1\le i\le n$.

Define $\{u_{lj} : 1\le l\le kn, 1\le j\le k\}$ by $u_{lj}=0$ unless $(j-1)\cdot n < l\le j\cdot n$, and

$$u_{((j-1)\cdot n+i)j} = p_{ij}$$

for $1\le i\le n$, so that (u_{lj}) is given by the following matrix:

$$\begin{pmatrix} p_{11} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ p_{1n} & 0 & \cdots & 0\\ 0 & p_{21} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & p_{2n} & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & p_{k1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & p_{kn}\end{pmatrix}$$

Clearly (u_{lj}) satisfies the defining relations of $A_i(k,k\cdot n)$ and so we obtain a unital $*$-homorphism from $A_i(k,k\cdot n)$ into A. Composing with φ gives a state $\psi_{k,n} : A_i(k,k\cdot n)\to\mathbb{C}$, and we need only show that (u_{lj}) in (A,φ) has the distribution appearing in the statement.

(1) is trivial, as $u_{l_1j_1}\cdots u_{l_mj_m}=0$ unless $(j_r-1)\cdot n < l_r\le j_r\cdot n$ for $r=1,\dots,m$. For (2), we need to show that

$$\varphi(p_{i_1j_1}\cdots p_{i_mj_m}) = \sum_{\substack{\pi\in NC(m)\\ \pi\le\ker\mathbf{j}}}\sum_{\substack{\sigma\in NC(m)\\ \sigma\le\pi\wedge\ker\mathbf{i}}}\mu_m(\sigma,\pi)n^{-|\sigma|}.$$

Now by freeness, we have

$$\begin{aligned}\varphi(p_{i_1j_1}\cdots p_{i_mj_m}) &= \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}} \kappa^{(\pi)}[p_{i_1j_1},\dots,p_{i_mj_m}]\\ &= \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}}\sum_{\substack{\sigma\in NC(m)\\ \sigma\leq\pi}} \mu_m(\sigma,\pi)\varphi^{(\sigma)}[p_{i_1j_1},\dots,p_{i_mj_m}].\end{aligned}$$

Now since

$$\varphi(p_{i_1j}\cdots p_{i_mj}) = \begin{cases} n^{-1}, & i_1=\cdots=i_m\\ 0, & \text{otherwise}\end{cases},$$

it follows that if $\sigma\leq\ker\mathbf{j}$ then

$$\varphi^{(\sigma)}[p_{i_1j_1},\dots,p_{i_mj_m}] = \begin{cases} n^{-|\sigma|}, & \sigma\leq\ker\mathbf{i}\\ 0, & \sigma\not\leq\ker\mathbf{i}\end{cases}.$$

Combining this with the previous equation yields the desired result.

□

Remark 5.5.3. Observe that the formula in (2) above has a very similar structure to the highest order expansion of the *Weingarten formula* for evaluating integrals over the quantum permutation group $A_s(n)$ with respect to its Haar state, see Section 4.3.

The final tool which we require to complete the proof of Theorem 5.1.1 is von Neumann's mean ergodic theorem. This will allow us to give a formula for the expectation functionals $E^{(\sigma)}$ as certain weighted averages, compare with the reversed martingale arguments used in the previous chapter. We note that the unpleasant indices which appear are chosen so to correspond to the formula in Proposition 5.5.2.

Lemma 5.5.4. *Suppose that the sequence $(\rho_i)_{i\in\mathbb{N}}$ is quantum spreadable. Then for any $j\in\mathbb{N}$ and $\beta\in C*B$, we have*

$$E[\widetilde{\rho}_1(\beta)] = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\widetilde{\rho}_{(j-1)\cdot n+i}(\beta),$$

with convergence in $|\ |_2$.

Proof. Since $(\rho_i)_{i\in\mathbb{N}}$ is spreadable, we have

$$\tau(m_1m_2) = \tau(m_1U(m_2))$$

whenever $m_1\in W^*(\{\rho_i(c): 1\leq i\leq n, c\in C\})$ and $m_2\in W^*(\{\rho_i(c): i>n, c\in C\})$. It follows that

$$\tau(mb) = \tau(mU(b))$$

for $m\in M$ and $b\in B$, hence $b=U(b)$. It follows easily that

$$U(\widetilde{\rho}_i(\beta)) = \widetilde{\rho}_{i+1}(\beta)$$

for any $i \in \mathbb{N}$ and $\beta \in C * B$.

Since it is clear that any vector fixed by U must lie in $L^2(B)$, we have in fact the equality

$$L^2(B) = \{\xi \in L^2(M) : U\xi = \xi\}.$$

By von Neumann's mean ergodic theorem, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i = P,$$

where P is the orthogonal projection of $L^2(M)$ onto $L^2(B)$ and the limit holds in the strong operator topology. Therefore for any $m \in M$ we have

$$E[m] = P(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i(m),$$

with the limit holding in $| \ |_2$. Since U is contractive in $| \ |_2$, we have also for any $j \in \mathbb{N}$ that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^{(j-1)\cdot n+i}(m) &= \lim_{n\to\infty} U^{(j-1)\cdot n}\left(\frac{1}{n}\sum_{i=0}^{n-1} U^i(m)\right) \\ &= \lim_{n\to\infty} U^{(j-1)\cdot n} P(m) \\ &= E[m], \end{aligned}$$

since $U \cdot P = P$. Applying this to $m = \widetilde{\rho}_1(\beta)$ gives the desired result. □

Proposition 5.5.5. *Suppose that the sequence $(\rho_i)_{i\in\mathbb{N}}$ is quantum spreadable. Fix $j_1, \dots, j_m \in \mathbb{N}$ and choose $\sigma \in NC(m)$ such that $\sigma \leq \ker \mathbf{j}$. Then for any $\beta_1, \dots, \beta_m \in C * B$, we have*

$$E^{(\sigma)}[\widetilde{\rho}_1(\beta_1) \otimes \cdots \otimes \widetilde{\rho}_1(\beta_m)] = \lim_{n\to\infty} n^{-|\sigma|} \sum_{\substack{1\leq i_1,\dots,i_m\leq n \\ \sigma\leq\ker\mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1)\cdots\widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m),$$

with convergence in $| \ |_2$.

Proof. We will use induction on the number of blocks of σ. If $\sigma = 1_m$ has only one block, then $\sigma \leq \ker \mathbf{j}$ implies $j_1 = \cdots = j_m$ and we have

$$\lim_{n\to\infty} n^{-|\sigma|} \sum_{\substack{1\leq i_1,\dots,i_m\leq n \\ \sigma\leq\ker\mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1)\cdots\widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widetilde{\rho}_{(j_1-1)\cdot n+i}(\beta_1\beta_2\cdots\beta_m).$$

By Lemma 5.5.4, this converges in $| \ |_2$ to

$$E[\widetilde{\rho}_1(\beta_1\beta_2\cdots\beta_m)] = E^{(\sigma)}[\widetilde{\rho}_1(\beta_1) \otimes \cdots \otimes \widetilde{\rho}_1(\beta_m)].$$

Now let $\sigma \in NC(m)$ and let $V = \{l+1, \dots, l+s\}$ be an interval of σ, and let j be the common value of $j_{l+1}, \dots, j_{l+s}$. We have

$$n^{-|\sigma|} \sum_{\substack{1 \le i_1, \dots, i_m \le n \\ \sigma \le \ker \mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1) \cdots \widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m)$$

$$= n^{-|\sigma \setminus V|} \sum_{\substack{1 \le i_1, \dots, i_l, \\ i_{l+s+1}, \dots, i_m \le n \\ \sigma \setminus V \le \ker \mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1) \cdots \Big(\frac{1}{n} \sum_{i=1}^{n} \widetilde{\rho}_{(j-1)\cdot n+i}(\beta_{l+1} \cdots \beta_{l+s}) \Big) \cdots \widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m)$$

As above, the interior sum converges to $E[\widetilde{\rho}_1(\beta_{l+1} \cdots \beta_{l+s})]$ in $|\ |_2$ as $n \to \infty$. Now for any $\beta \in C*B$, since the variables $\widetilde{\rho}_i(\beta)$ are identically $*$-distributed with respect to the faithful trace τ, it follows that $\|\widetilde{\rho}_i(\beta)\|$ is independent of i. Therefore there is a constant D such that

$$|\widetilde{\rho}_{i_1}(\beta_1) \cdots \widetilde{\rho}_{i_l}(\beta_l) \cdot \xi \cdot \widetilde{\rho}_{i_{l+s+1}}(\beta_{l+s+1}) \cdots \widetilde{\rho}_{i_m}(\beta_m)|_2 \le D|\xi|_2$$

for any $\xi \in L^2(M)$ and $i_1, \dots, i_m \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} n^{-|\sigma \setminus V|} \sum_{\substack{1 \le i_1, \dots, i_l, \\ i_{l+s+1}, \dots, i_m \le n \\ \sigma \setminus V \le \ker \mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1) \cdots \Big(\frac{1}{n} \sum_{i=1}^{n} \widetilde{\rho}_j(\beta_{l+1} \cdots \beta_{l+s}) \Big) \cdots \widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m)$$

$$= \lim_{n\to\infty} n^{-|\sigma \setminus V|} \sum_{\substack{1 \le i_1, \dots, i_l, \\ i_{l+s+1}, \dots, i_m \le n \\ \sigma \setminus V \le \ker \mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1) \cdots E[\widetilde{\rho}_1(\beta_{l+1} \cdots \beta_{l+s})] \cdots \widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m).$$

By induction, this converges in $|\ |_2$ to

$$E^{(\sigma \setminus V)}[\widetilde{\rho}_1(\beta_1) \otimes \cdots \otimes \widetilde{\rho}_1(\beta_l) \cdot E[\widetilde{\rho}_1(\beta_{l+1} \cdots \beta_{l+s})] \otimes \cdots \otimes \widetilde{\rho}_1(c_m)],$$

which is precisely $E^{(\sigma)}[\widetilde{\rho}_1(\beta_1) \otimes \cdots \otimes \widetilde{\rho}_1(\beta_m)]$, as desired.

□

We can now complete the proof of Theorem 5.1.1.

Proof of (2)⇒(3). Fix $\beta_1, \dots, \beta_m \in C * B$ and $1 \le j_1, \dots, j_m \le k$. By Proposition 5.5.1, for each $n \in \mathbb{N}$ we have

$$E[\widetilde{\rho}_{j_1}(\beta_1) \cdots \widetilde{\rho}_{j_m}(\beta_m)] \otimes 1_{A_i(k,k\cdot n)} = \sum_{1 \le l_1, \dots, l_m \le kn} E[\widetilde{\rho}_{l_1}(\beta_1) \cdots \widetilde{\rho}_{l_m}(\beta_m)] \otimes u_{l_1 j_1} \cdots u_{l_m j_m}.$$

Applying $(\mathrm{id} \otimes \psi_{k,n})$ to each side of the above equation, we obtain

$$\begin{aligned}
&E[\widetilde{\rho}_{j_1}(\beta_1)\cdots\widetilde{\rho}_{j_m}(\beta_m)] \\
&\quad= \sum_{1\leq i_1,\ldots,i_m\leq n} E[\widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1)\cdots\widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m)] \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}} \sum_{\substack{\sigma\in NC(m)\\ \sigma\leq\pi\wedge\ker\mathbf{i}}} \mu_m(\sigma,\pi)n^{-|\sigma|} \\
&\qquad= \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}} \sum_{\substack{\sigma\in NC(m)\\ \sigma\leq\pi}} \mu_m(\sigma,\pi) E\Big[n^{-|\sigma|} \sum_{\substack{1\leq i_1,\ldots,i_m\leq n\\ \sigma\leq\ker\mathbf{i}}} \widetilde{\rho}_{(j_1-1)\cdot n+i_1}(\beta_1)\cdots\widetilde{\rho}_{(j_m-1)\cdot n+i_m}(\beta_m)\Big].
\end{aligned}$$

Letting $n \to \infty$ and applying Proposition 5.5.5, we have

$$\begin{aligned}
E[\widetilde{\rho}_{j_1}(\beta_1)\cdots\widetilde{\rho}_{j_m}(\beta_m)] &= \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}} \sum_{\substack{\sigma\in NC(m)\\ \sigma\leq\pi}} \mu_m(\sigma,\pi) E^{(\sigma)}[\widetilde{\rho}_1(\beta_1)\otimes\cdots\otimes\widetilde{\rho}_1(\beta_m)] \\
&= \sum_{\substack{\pi\in NC(m)\\ \pi\leq\ker\mathbf{j}}} \kappa_E^{(\pi)}[\widetilde{\rho}_1(\beta_1)\otimes\cdots\otimes\widetilde{\rho}_1(\beta_m)],
\end{aligned}$$

and the result now follows from Corollary 5.2.6. □

Chapter 6

Stochastic aspects of easy quantum groups

6.1 Introduction

In Chapter 4, we showed that the class of easy quantum groups provides a good framework for understanding de Finetti type results in classical and free probability. It was suggested in [10] that another application of this formalism might come from the results of Diaconis and Shahshahani [29], regarding the groups O_n, S_n. We will show in this chapter that this is indeed the case:

(1) The problem makes sense for all easy quantum groups.

(2) There is a global approach to it, by using partitions and cumulants.

(3) The new computations lead to a number of interesting conclusions.

As a first example, consider the orthogonal group O_n, with fundamental representation denoted u. The results in [29], that we will recover as well by using our formalism, state that the asymptotic variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are real Gaussian and independent, with variance k and mean 0 or 1, depending on whether k is odd or even.

In the case of O_n^+, however, the situation is quite different: the variables u_k are free, as one could expect, but they are semicircular at $k = 1, 2$, and circular at $k \geq 3$.

Summarizing, in the orthogonal case we have the following table:

Variable	O_n	O_n^+
u_1	real Gaussian	semicircular
u_2	real Gaussian	semicircular
u_k $(k \geq 3)$	real Gaussian	circular

In the symmetric case the situation is even more surprising, with the Poisson variables from the classical case replaced by several types of variables:

Variable	S_n	S_n^+
u_1	Poisson	free Poisson
$u_2 - u_1$	Poisson	semicircular
$u_k - u_1$ $(k \geq 3)$	sum of Poissons	circular

We will present as well similar computations for the groups H_n, B_n, for their free analogues H_n^+, B_n^+, for the half-liberated quantum groups O_n^*, H_n^*, as well as for the series $H_n^{(s)}$. The calculations in the latter case rely essentially on Diaconis-Shahshahani type results for the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$.

The challenging question, that will eventually be left open, is to find a formal "eigenvalue" interpretation for all the quantum group results.

The chapter is organized as follows. In Section 6.2 we use the Weingarten formula to compute the moments of traces of powers. In Section 6.3, this is refined to a formula for corresponding cumulants, in the classical and the free cases. In the next three sections this will be used to study the orthogonal, bistochastic, symmetric, and hyperoctahedral classical and quantum groups, respectively. Section 6.8 deals with the half-liberated quantum groups O_n^* and H_n^*. The results in these cases will rely on the observation that these half-liberated quantum groups are in some sense orthogonal versions of classical groups, U_n for O_n^* and H^∞ for H_n^*. The main calculations will take place for these classical groups. The same ideas work actually for the half-liberated series $H_n^{(s)}$, by considering those as orthogonal versions of the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$. One of the main results in Section 6.8 is a Diaconis-Shahshahani type result for those classical reflection groups.

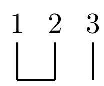
6.2 Moments of powers

In this section we discuss the computation of the asymptotic joint distribution of the variables $\mathrm{Tr}(u^k)$, generalizing the fundamental character $\chi = \mathrm{Tr}(u)$. In the classical case these laws, computed by Diaconis and Shahshahani in [29], can be of course understood in terms of the asymptotic behavior of the eigenvalues of the random matrices $u \in G$.

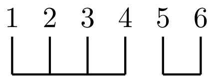
As in [29], we will be actually interested in the more general problem consisting in computing the joint asymptotic law of the variables $\mathrm{Tr}(u^k)$, with $k \in \mathbb{N}$ varying. In order to deal with these joint laws, it is convenient to use the following notation.

Notation: For given $k_1, \ldots, k_r \in \mathbb{N}$, $k := \sum k_i$, we will denote by $\gamma \in S_k$ the permutation with cycles $(1, \ldots, k_1)$, $(k_1 + 1, \ldots, k_1 + k_2)$, $\ldots$,$(k - k_s + 1, \ldots, k)$. If we have, in addition, a partition $\sigma \in P_r$, then σ^δ will denote the canonical lift of σ from P_r to P_k, associated to γ. Thus, σ^γ is that partition which we get from σ by replacing each $j \in \{1, \ldots, r\}$ by the j-th cycle of γ, i.e., $\sigma^\gamma \geq \gamma$ and the i-th and the j-th cycle of γ are in the same block of σ^γ if and only if i and j are in the same block of σ.

As an example, let $\gamma = (1)(2, 3, 4)(5, 6)$. Consider now $\sigma = \{(1, 2), (3)\} \in P_3$:



Then σ^γ is given by making the replacements $1 \to 1$, $2 \to 2, 3, 4$ and $3 \to 5, 6$,



thus $\sigma^\gamma = \{(1,2,3,4),(5,6)\} \in P_6$.

We will also denote by $\gamma(q)$ the partition given by $i \sim_q j$ iff $\gamma(i) \sim_{\gamma(q)} \gamma(j)$.

Our first general result concerns the joint moments of the variables $\mathrm{Tr}(u^k)$, and is valid for any easy quantum group.

Theorem 6.2.1. *Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \ldots, k_s \in \mathbb{N}$, $k := \sum_{i=1}^s k_i$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_s$. Then we have, for any n such that G_{kn} is invertible,*

$$\int_G \mathrm{Tr}(u^{k_1}) \ldots \mathrm{Tr}(u^{k_s})\, du = \#\{p \in D_k | p = \gamma(p)\} + O(1/n). \tag{6.1}$$

If G is a classical easy group, then (6.1) *is exact, without any lower order corrections in n.*

Proof. We denote by I the integral to be computed. According to the definition of γ, we have the following formula:

$$\begin{aligned} I &= \int_G \mathrm{Tr}(u^{k_1}) \ldots \mathrm{Tr}(u^{k_s})\, du \\ &= \sum_{i_1 \ldots i_k} \int_G (u_{i_1 i_2} \ldots u_{i_k i_1}) \ldots (u_{i_{k-k_s+1} i_{k-k_s+2}} \ldots u_{i_k i_{k-k_s+1}}) \\ &= \sum_{i_1 \ldots i_k} \int_G u_{i_1 i_{\gamma(1)}} \ldots u_{i_k i_{\gamma(k)}} \end{aligned}$$

We use now the Weingarten formula from Theorem 2.2.6. We get:

$$\begin{aligned} I &= \sum_{i_1 \ldots i_k = 1}^n \sum_{\substack{p,q \in D_k \\ p \leq \ker \mathbf{i}, q \leq \ker \mathbf{i} \circ \gamma}} W_{kn}(p,q) \\ &= \sum_{i_1 \ldots i_k = 1}^n \sum_{\substack{p,q \in D_k \\ p \leq \ker \mathbf{i}, \gamma(q) \leq \ker \mathbf{i}}} W_{kn}(p,q) \\ &= \sum_{p,q \in D_k} n^{|p \vee \gamma(q)|} W_{kn}(p,q) \\ &= \sum_{p,q \in D_k} n^{|p \vee \gamma(q)|} n^{|p \vee q| - |p| - |q|} (1 + O(1/n)) \end{aligned}$$

The leading order of $n^{|p\vee\gamma(q)|+|p\vee q|-|p|-|q|}$ is n^0, which is achieved if and only if $q \geq p$ and $p \geq \gamma(q)$, or equivalently $p = q = \gamma(q)$. This gives the formula (6.1).

In the classical case, instead of using the approximation for $W_{nk}(p,q)$, we can write $n^{|p\vee\gamma(q)|}$ as $G_{kn}(\gamma(q),p)$. (Note that this only makes sense if we know that $\gamma(q)$ is also an element in D_k; and this is only the case for the classical partition lattices.) Then one can continue as follows:

$$I = \sum_{p,q\in D_k} G_{kn}(\gamma(q),p)W_{kn}(p,q) = \sum_{q\in D_k} \delta(\gamma(q),q) = \#\{q \in D_k | q = \gamma(p)\}.$$

□

We discuss now the computation of the asymptotic joint $*$-distribution of the variables $\mathrm{Tr}(u^k)$. Observe that this is of relevance only in the non-classical context, where the variables $\mathrm{Tr}(u^k)$ are in general (for $k \geq 3$) not self-adjoint.

If c is a cycle we use the notation $c^1 = c$, and c^*= cycle opposite to c.

Definition 6.2.2. Associated to any $k_1, \dots, k_s \in \mathbb{N}$ and any $e_1, \dots, e_s \in \{1, *\}$ is the *trace permutation* $\gamma \in S_k$, with $k = \Sigma k_i$, having as cycles $(1, \dots, k_1)^{e_1}$, $(k_1 + 1, \dots, k_1 + k_2)^{e_2}$, $\dots$, $(k - k_s + 1, \dots, k)^{e_s}$.

Observe that with $e_1, \dots, e_s = 1$ we recover the permutation γ from the beginning of this section. With this notation, we have the following slight generalization of Theorem 6.2.1.

Theorem 6.2.3. *Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \dots, k_s \in \mathbb{N}$, $e_1, \dots, e_s \in \{1, *\}$, $k := \sum_{i=1}^s k_i$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \dots, k_s$ and $e_1, \dots, e_s$. Then we have, for any n such that G_{kn} is invertible,*

$$\int_G \mathrm{Tr}(u^{k_1})^{e_1} \dots \mathrm{Tr}(u^{k_s})^{e_s}\, du = \#\{p \in D_k | p = \gamma(p)\} + O(1/n).$$

If G is a classical easy group, then this formula is valid without any lower order corrections.

Proof. This is similar to the proof of Theorem 6.2.1. □

6.3 Cumulants of powers

The formula for the moments of the variables $\mathrm{Tr}(u^k)$ contains in principle all information about their distribution. However, in order to specify this more explicitly, in particular, to recognize independence/freeness between those (or suitable modifications), it is more advantageous to look at the cumulants of these variables. For this we will restrict to the classical and free cases in this section. We will calculate the classical cumulants (denoted by c_r) for the classical easy groups and the free cumulants (denoted by κ_r) for the free easy groups. Actually we will restrict to the cases

(1) Classical groups: O_n, S_n, H_n, B_n.

(2) Free quantum groups: $O_n^+, S_n^+, H_n^+, B_n^+$.

The reason for this is that we need some kind of multiplicativity for the underlying partition lattice in our calculations, as specified in the next proposition.

Proposition 6.3.1. *Assume that G is one of the easy groups O_n, S_n, H_n, B_n or free quantum groups $O_n^+, S_n^+, H_n^+, B_n^+$ and denote by D_k the corresponding category of partitions. Then we have the following property: let $p \in D_k$ be a partition, and let $q \in P_l$ with $l \leq k$ be a partition arising from p by deleting some blocks. Then $b \in D_l$.*

Proof. This follows from the explicit description of the full categories of partitions for the various easy quantum groups, given in Section 2.2. □

Theorem 6.3.2.

(1) Let G be one of the easy classical groups O_n, S_n, H_n, B_n with D_k as corresponding category of partitions. Consider $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$, $k := \sum_{i=1}^r k_i$ and $e_1, \ldots, e_r \in \{1, *\}$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_r$ and $e_1, \ldots, e_r$. Then we have, for any n such that G_{kn} is invertible, the classical cumulants

$$c_r(\mathrm{Tr}(u^{k_1})^{e_1}, \ldots, \mathrm{Tr}(u^{k_r})^{e_r}) = \#\{p \in D_k | p \vee \gamma = 1_k,\, p = \gamma(p)\}.$$

(2) Let G be one of the easy free groups $O_n^+, S_n^+, H_n^+, B_n^+$ with D_k as corresponding category of non-crossing partitions. Consider $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$, $k := \sum_{i=1}^r k_i$ and $e_1, \ldots, e_r \in \{1, *\}$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_r$ and $e_1, \ldots, e_r$. Then we have, for any n such that G_{kn} is invertible, the free cumulants

$$\kappa_r(\mathrm{Tr}(u^{k_1})^{e_1}, \ldots, \mathrm{Tr}(u^{k_r})^{e_r}) = \#\{p \in D_k | p \vee \gamma = 1_k,\, p = \gamma(p)\} + O(1/n).$$

Proof. (1) Let us denote by c_r the considered cumulant. We write

$$D_\sigma := \{p \in P_k \mid p|_v \in D_{|v|}\, \forall v \in \sigma\}$$

for those partitions p in P_k such that the restriction of p to a block of σ is an element in the corresponding set $D_{|v|}$. Clearly, one has that a $p \in D_\sigma$ is in D_k and must satisfy $p \leq \sigma^\gamma$. Our exclusion of the primed classical groups guarantees, by Proposition 6.3.1, that this is actually a characterization, i.e., we have

$$D_\sigma = \{p \in D_k \mid p \leq \sigma^\gamma\}. \tag{6.2}$$

Then, by the definition of the classical cumulants via Möbius inversion of the moments, we get from (6.1):

$$\begin{aligned} c_r &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \cdot \#\{p \in D_\sigma : p = \gamma(p)\} \\ &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \cdot \#\{p \in D_k : p \leq \sigma^\gamma, p = \gamma(p)\} \\ &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \sum_{\substack{p \in D_k \\ p \leq \sigma^\gamma, p = \gamma(p)}} 1 \end{aligned}$$

In order to exchange the two summations, we first have to replace the summation over $\sigma \in P(r)$ by a summation over $\tau := \sigma^\gamma \in P(k)$. Note that the condition on the latter is exactly $\tau \geq \gamma$ and that we have $\mu(\sigma, 1_r) = \mu(\sigma^\gamma, 1_k)$. Thus:

$$c_r = \sum_{\substack{\tau \in P(k) \\ \tau \geq \gamma}} \mu(\tau, 1_k) \sum_{\substack{p \in D_k \\ p \leq \tau, p = \gamma(p)}} 1 = \sum_{\substack{p \in D_k \\ p = \gamma(p)}} \sum_{\substack{\tau \in P(k) \\ p \vee \gamma \leq \tau}} \mu(\tau, 1_k)$$

The definition of the Möbius function (see (10.11) in [43]) gives for the second summation

$$\sum_{\substack{\tau \in P(k) \\ p \vee \gamma \leq \tau}} \mu(\tau, 1_k) = \begin{cases} 1, & p \vee \gamma = 1_k \\ 0, & \text{otherwise} \end{cases}$$

and the assertion follows.

(2) In the free case, the proof runs in the same way, by using free cumulants and the corresponding Möbius function on non-crossing partitions. Note that we have the analogue of (6.2) in this case only for non-crossing σ. □

6.4 The orthogonal case

In this section we discuss what Theorem 6.3.2 implies for the asymptotic distribution of traces in the case of the orthogonal quantum groups. For the classical orthogonal group we will in this way recover the theorem of Diaconis and Shahshahani [29].

Theorem 6.4.1. *The variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) For O_n, the u_k are real Gaussian variables, with variance k and mean 0 or 1, depending on whether k is odd or even. The u_k's are independent.

(2) For O_n^+, at $k = 1, 2$ we get semicircular variables of variance 1 and mean 0 for u_1 and mean 1 for u_2, and at $k \geq 3$ we get circular variables of mean 0 and covariance 1. The u_k's are $*$-free.

Proof. (1) In this case D_k consists of all pairings of k elements. We have to count all pairings p with the properties that $p \vee \gamma = 1_k$ and $p = \gamma(p)$.

Note that if p connects two different cycles of γ, say c_i and c_j, then the property $p = \gamma(p)$ implies that each element from c_i must be paired with an element from c_j; thus those cycles cannot be connected to other cycles and they must contain the same number of elements. This means that for $s \geq 3$ there are no p with the required properties. Thus all cumulants of order 3 and higher vanish asymptotically and all traces are asymptotically Gaussian.

Since in the case $s = 2$ we only have permissible pairings if the two cycles have the same number of elements, i.e., both powers of u are the same, we also see that the covariance between traces of different powers vanishes and thus different powers are asymptotically independent. The variance of u_k is given by the number of matchings between $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ which are invariant under rotations. Since such a matching is determined by the partner of the first element 1, for which we have k possibilities, the variance of u_k is k. For the mean, if k is odd there is clearly no pairing at all, and if $k = 2p$ is even then the only pairing of $\{1, \dots, 2p\}$ which is invariant under rotations is $(1, p+1), (2, p+2), \dots, (p, 2p)$. Thus the mean of u_k is zero if k is odd and 1 if k is even.

(2) In the quantum case D_k consists of non-crossing pairings. We can essentially repeat the arguments from above but have to take care that only non-crossing pairings are counted. We also have to realize that for $k \geq 3$, the u_k are not selfadjoint any longer, thus we have to consider also u_k^* in these cases. This means that in our arguments we have to allow cycles which are rotated "backwards" under γ.

By the same reasoning as before we see that free cumulants of order three and higher vanish. Thus we get a (semi)circular family. The pairing which gave mean 1 in the classical case is only in the case $k = 2$ a non-crossing one, thus the mean of u_2 is 1, all other means are zero. For the variances, one has again that different powers allow no pairings at all and are asymptotically $*$-free. For the matchings between $\{1, \dots, k\}$ and $\{k+1, \dots, 2k\}$ one has to observe that there is only one non-crossing possibility, namely $(1, 2k), (2, 2k-1), \dots, (k, k+1)$ and this satisfies $p = \gamma(p)$ only if γ rotates both cycles in different directions.

For $k = 1$ and $k = 2$ there is no difference between both directions, but for $k \geq 3$ this implies that we get only a non-vanishing covariance between u_k and u_k^* (with value 1). This shows that u_1 and u_2 are semicircular, whereas the higher u_k are circular. □

6.5 The bistochastic case

In the bistochastic case we have the following version of Theorem 6.4.1.

Theorem 6.5.1. *The variables $u_k = \lim_{n \to \infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) For B_n, the u_k are real Gaussian variables, with variance k and mean 1 or 2, depending on whether k is odd or even. The u_k's are independent.

(2) For B_n^+, at $k = 1, 2$ we get semicircular variables of variance 1 and mean 1 for u_1 and mean 2 for u_2, and at $k \geq 3$ we get circular variables of mean 1 and covariance 1. The u_k's are $*$-free.

Proof. When replacing O_n and O_n^+ by B_n and B_n^+, we also have to allow singletons in p. Note however that the condition $p = \gamma(p)$ implies that if p has a singleton, then the corresponding cycle of γ must consist only of singletons of p , which means in particular that this cycle cannot be connected via p to other cycles. Thus singletons are not allowed for permissible p, unless we only have one cycle of γ, i.e., we are looking at the mean. In this case there is one additional p, consisting just of singletons, which makes a contribution. So the results for B_n and B_n^+ are the same as those for O_n and O_n^+, respectively, with the only exception that all means are shifted by 1. □

6.6 The symmetric case

Let us now consider the case of the symmetric groups. In this case we have to consider all partitions instead of just pairings and the arguments are getting a bit more involved. Nevertheless one can treat these cases still in a quite straightforward way. For the classical permutation groups, one recovers in this way the corresponding result of Diaconis and Shahshahani [29].

Proposition 6.6.1. *The cumulants of* $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) For S_n*, the classical cumulants are given by:*

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{q|k_i \forall i=1,\ldots,r} q^{r-1}$$

(2) For S_n^+*, the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = \begin{cases} 2, & r=1,\ k_1 \geq 2 \\ 2, & r=2,\ k_1=k_2,\ e_1=e_2^* \\ 2, & r=2,\ k_1=k_2=2 \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. (1) Now D_k consists of all partitions. We have to count partitions p which have the properties that $p \vee \gamma = 1_k$ and $p = \gamma(p)$.

Consider a partition p which connects different cycles of γ. Consider the restriction of p to one cycle. Let k be the number of elements in this cycle and t be the number of the points in the restriction. Then the orbit of those t points under γ must give a partition of that cycle; this means that t is a divisor of k and that the t points are equally spaced. The same must be true for all cycles of γ which are connected via p, and the ratio between t and k is the same for all those cycles. This means that if one block of p connects some cycles then the orbit under γ of this block connects exactly those cycles and exhausts all points of those cycles. So if we want to connect all cycles of γ then this can only happen in the way that we have one (and thus all) block of p intersecting each of the cycles of γ. To be more precise, let us consider $c_r(u_{k_1}, \ldots, u_{k_r})$. We have then to look for a common divisor q of all $k_1, \ldots, k_r$; a contributing p is then one the blocks of which are of the following form: k_1/q points in the first cycle (equally spaced), ... k_r/q points in the last cycle (equally spaced). We can specify this by saying to which points in the other cycles the first point in the first cycle is connected. There are q^{r-1} possibilities for such choices. Thus:

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{q|k_i \forall i=1,\ldots,r} q^{r-1}$$

(2) In the quantum permutation case we have to consider non-crossing partitions instead of all partitions. Most of the contributing partitions from the classical case are crossing, so do not count for the quantum case. Actually, whenever a restriction of a block to one cycle has two or more elements then the corresponding partition is crossing, unless the restriction exhausts the

whole group. This is the case $q = 1$ from the considerations above (corresponding to the partition which has only one block), giving a contribution 1 to each cumulant $c_r(u_{k_1}, \dots, u_{k_r})$. For cumulants of order 3 or higher there are no other contributions. For cumulants of second order one might also have contributions coming from pairings (where each restriction of a block to a cycle has one element). This is the same problem as in the O_n^+ case; i.e., we only get an additional contribution for the second order cumulants $c_2(u_k, u_k^*)$. For first order cumulants, singletons can also appear and make an additional contribution. Taking this all together gives the formula in the statement. □

In contrast to the two previous cases, the different traces are now not independent/free any more. Actually, one knows in the classical case that some more fundamental random variables, counting the number of different cycles, are independent. We can recover this result, and its free analogue, from Proposition 6.6.1 in a straightforward way.

Theorem 6.6.2. *The variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) For S_n we have a decomposition of type

$$u_k = \sum_{l|k} lC_l$$

with the variables C_k being Poisson of parameter $1/k$, and independent.

(2) For S_n^+ we have a decomposition of the type

$$u_1 = C_1, \qquad u_k = C_1 + C_k \quad (k \geq 2)$$

where the variables C_l are $*$-free; C_1 is free Poisson, whereas C_2 is semicircular and C_k, for $k \geq 3$, are circular.

Let us first note that the first statement is the result of Diaconis and Shahshahani in [29]. Indeed, the matrix coefficients for S_n are given by $u_{ij} = \chi(\sigma|\sigma(j) = i)$, and it follows that the variable C_l defined by the decomposition of u_k in the statement is nothing but the number of l-cycles. For a direct proof for the fact that these variables C_k are indeed independent and Poisson of parameter $1/k$, see [29]. In what follows we present a global proof for (1) and (2), by using Proposition 6.6.1.

Proof. (1) Let C_k be the number of cycles of length k. Instead of writing this in terms of traces of powers of u, it is more clear to do it the other way round. We have $u_k = \sum_{l|k} lC_l$. We are claiming now that the C_k are independent and each is a Poisson variable of parameter $1/k$, i.e., that $c_r(C_{l_1}, \dots, C_{l_r})$ is zero unless all the l_i's are the same, say $= l$, in which case it is $1/l$ (independent of r). This is compatible with the cumulants for the u_k, according to:

$$c_r(u_{k_1}, \dots, u_{k_r}) = \sum_{l_1|k_1} \cdots \sum_{l_r|k_r} l_1 \cdots l_r c_r(C_{l_1}, \dots, C_{l_r}) = \sum_{l|k_i \forall i} l^r \frac{1}{l}$$

Since the C_k's are uniquely determined by the u_k's, via some kind of Möbius inversion, this shows that also the other way round the formula for the cumulants of the u_k's implies the above stated formula for the cumulants of the C_k's; i.e., we get the result that the C_k are independent and C_k is Poisson with parameter $1/k$.

(2) This follows easily from Proposition 6.6.1. □

Remark 6.6.3. 1) In the classical case the random variable C_l can be defined by

$$C_l = \frac{1}{l} \sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}. \tag{6.3}$$

Note that we divide by l because each term appears actually l-times, in cyclically permuted versions (which are all the same because our variables commute).

Note that, by using commutativity and the monomial condition, in general the expression $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_k i_1}$ has to be zero unless the indices $(i_1, \dots, i_k)$ are of the form $(i_1, \dots, i_l, i_1, \dots, i_l, \dots)$ where l divides k and $i_1, \dots, i_l$ are distinct. This yields then the relation

$$\mathrm{Tr}(u^k) = \sum_{i_1,\dots,i_l=1}^{n} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1} = \sum_{l|k} \sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} (u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1})^{k/l} = \sum_{l|k} l C_l,$$

which we used before to define the C_l. [Note that each $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}$ is an idempotent, thus the power k/l does not matter.] This explicit form (6.3) of the C_l in terms of the u_{ij} can be used to give a direct proof, by using the Weingarten formula, of the fact that the C_l are independent and Poisson. We will not present this calculation here, but will come back to this approach in the case of the hyperoctahedral group in the next section.

2) In the free case we define the "cycle" C_l by requiring neighboring indices to be different,

$$C_l = \sum_{i_1 \neq i_2 \neq \cdots \neq i_l \neq i_1} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}. \tag{6.4}$$

Note that if two adjacent indices are the same in $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}$ then, because of the relation $u_{ij} u_{ik} = 0$ for $j \neq k$, all must be the same or the term vanishes. For the case where all indices are the same we have

$$\sum_i u_{ii} u_{ii} \cdots u_{ii} = \sum_i u_{ii} = C_1.$$

This gives then the relation

$$\mathrm{Tr}(u^k) = C_k + C_1.$$

Again, the C_l are uniquely determined by the $\mathrm{Tr}(u^k)$ and thus our calculations also show that the C_l defined by (6.4) are $*$-free and have the distributions as stated.

6.7 The hyperoctahedral case

The methods in the previous section apply, modulo a grain of salt, as well to the hyperoctahedral case.

Proposition 6.7.1. *The cumulants of* $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) For H_n*, the classical cumulants are given by:*

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{\substack{q|k_i \forall i=1,\ldots,r \\ 2|(\Sigma k_i/q)}} q^{r-1}$$

(2) For H_n^+*, the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = 2, \qquad \text{if } r=2,\ k_1=k_2,\ e_1=e_2^* \text{ or if } r=2,\ k_1=k_2=2$$

and otherwise by

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = \begin{cases} 1, & \sum_l k_l \text{ even} \\ 0, & \sum_l k_l \text{ odd} \end{cases}$$

Proof. This follows similarly as the proof of Proposition 6.6.1, by taking into account that we have to restrict attention to the partitions having even blocks only. □

Theorem 6.7.2. *The variables* $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) For H_n *we have a decomposition of type*

$$u_k = \sum_{l|k} l[C_l^+ + (-1)^{k/l} C_l^-]$$

with the variables C_l^+ *and* C_l^- *being Poisson of parameter* $1/2l$*, and all* C_l^+, C_l^- $(l \in \mathbb{N})$ *being independent.*

(2) For H_n^+ *we have a decomposition of type*

$$u_1 = C_1^+ - C_1^- \qquad u_k = C_1^+ + (-1)^k C_1^- + C_k \quad (k \geq 2)$$

where C_1^+*,* C_1^- *and* C_k $(k \geq 2)$ *are* $*$*-free and* C_1^+*,* C_1^- *are free Poisson elements of parameter* $1/2$*,* C_2 *is a semicircular element, and* C_k $(k \geq 3)$ *are circular elements.*

Proof. (1) This follows in the same way as in the proof of Theorem 6.6.2.
(2) This follows by direct computation. □

In the classical case the random variables C_l^+ and C_l^- should count the number of positive cycles of length l and the number of negative cycles of length l, respectively, and should be given by $C_l^+ = Z_l^+$ and $C_l^- = Z_l^-$ with

$$Z_l^+ = \frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} 1_{\{1\}}(u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}) \tag{6.5}$$

and

$$Z_l^- = -\frac{1}{l} \sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} 1_{\{-1\}}(u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1}). \tag{6.6}$$

Note that $u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1}$ is either -1, 0, 1. $1_{\{1\}}$ denotes the characteristic function on 1 and $1_{\{-1\}}$ the characteristic function on -1. As in Remark 6.6.3 it follows that one has the decomposition

$$\mathrm{Tr}(u^k) = \sum_{l|k} l[Z_l^+ + (-1)^{k/l} Z_l^-] \tag{6.7}$$

Again one expects that the random variables defined by (6.5) and (6.6) are all independent and are Poisson, but because now the Z_l^+, Z_l^- are not uniquely determined by the relations (6.7), we cannot argue that the C_l^+ and C_l^- showing up in the decomposition in Theorem 6.7.2 are the same as Z_l^+ and Z_l^- defined by (6.5) and (6.6). In order to see that this is actually the case, we will now, in the following, calculate the cumulants of the random variables defined by (6.5) and (6.6).

Note first that we can replace the characteristic functions in the following way:

$$2Z_l^+ = \frac{1}{l} \sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} (u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1})^2 + u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1} \tag{6.8}$$

and

$$2Z_l^- = \frac{1}{l} \sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} (u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1})^2 - u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1}. \tag{6.9}$$

Furthermore, we have

$$\sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1} = \sum_{\ker \mathbf{i}=1_l} u_{i_1i_2}\cdots u_{i_li_1}$$

and

$$\sum_{\substack{i_1,\dots,i_l \\ \text{distinct}}} (u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1})^2 = \sum_{\ker \mathbf{i}=\tau_l^{2l}} u_{i_1i_2}\cdots u_{i_{2l}i_1},$$

where $\tau_l^{2l} \in P(2l)$ is the pairing $\{(1, l+1), (2, l+2), \dots, (l, 2l)\}$.

So what we need are cumulants for general variables of the form

$$Z(\sigma) := \sum_{\ker \mathbf{i}=\sigma} u_{i_1i_2}\cdots u_{i_li_1},$$

for arbitrary $l \in \mathbb{N}$ and $\sigma \in P(l)$. Note that $Z(\sigma)$ can only be different from zero if σ is of the form $\sigma = \tau_l^k$ for $k, l \in \mathbb{N}$ with $l|k$, where

$$\tau_l^k = \{(1, l+1, 2l+1, \dots, k-l+1), (2, l+2, 2l+2, \dots, k-l+2), \dots, (l, 2l, \dots, k)\}.$$

For $k = l$, we have $\tau_l^l = 1_l$.

Theorem 6.7.3. *For all $s \in \mathbb{N}$, $k_1, \dots, k_s \in \mathbb{N}$, $\sigma_1 \in P(k_1), \dots, \sigma_s \in P(k_s)$ we have*

$$c_r[Z(\sigma_1), \dots, Z(\sigma_r)] = \#\{q \in D_k \mid q = \gamma(q), q \vee \gamma = 1_k,\\ q \text{ restricted to the } i\text{-th cycle of } \gamma \text{ is } \sigma_i (i = 1, \dots, r)\},$$

where $k = \sum_{i=1}^r k_i$ and γ is the trace permutation associated to $k_1, \dots, k_r$

Note that also the right hand side of the equation is, by the condition $q = \gamma(q)$, zero unless all σ are of the form τ_l^k

Proof. Let us first calculate the corresponding moment. For this we note that one has

$$\sum_{\ker \mathbf{i} \geq \pi} u_{i_1 i_2} \cdots u_{i_l i_1} = \sum_{\substack{\sigma \in P(l) \\ \sigma \geq \pi}} Z(\sigma),$$

and thus, by Möbius inversion on $P(l)$

$$Z(\sigma) = \sum_{\substack{\pi \in P(l) \\ \pi \geq \sigma}} \mu(\sigma, \pi) \sum_{\ker \mathbf{i} \geq \pi} u_{i_1 i_2} \cdots u_{i_l i_1}.$$

With this we can calculate

$$\begin{aligned}
&\int_{H_n} Z(\sigma_1) \cdots Z(\sigma_r) du = \sum_{\pi_1, \dots, \pi_r} \mu(\sigma_1, \pi_1) \cdots \mu(\sigma_r, \pi_r) \sum_{\pi_1 \circ \cdots \circ \pi_r \leq \ker \mathbf{i}} \int_{H_n} u_{i_1 i_{\gamma(1)}} \cdots u_{i_k i_{\gamma(k)}} \\
&= \sum_{\pi_1, \dots, \pi_r} \mu(\sigma_1, \pi_1) \cdots \mu(\sigma_r, \pi_r) \sum_{\pi_1 \circ \cdots \circ \pi_r \leq \ker \mathbf{i}} \sum_{\substack{q, p \in D_k \\ p \leq \ker \mathbf{i}, \gamma(q) \leq \ker \mathbf{i}}} W_{kn}(p, q) \\
&= \sum_{\pi_1, \dots, \pi_r} \mu(\sigma_1, \pi_1) \cdots \mu(\sigma_r, \pi_r) \sum_{q, p \in D_k} G_{kn}(\gamma(q) \vee \pi_1 \circ \cdots \circ \pi_r, p) W_{kn}(p, q) \\
&= \sum_{\pi_1, \dots, \pi_r} \mu(\sigma_1, \pi_1) \cdots \mu(\sigma_r, \pi_r) \sum_{q \in D_k} \delta(\gamma(q) \vee \pi_1 \circ \cdots \circ \pi_r, q)
\end{aligned}$$

In the third line, it looks as if we might have a problem because $\pi_1 \circ \cdots \circ \pi_r$ is in P_k, but not necessarily in D_k. However, our category D_k has the nice property that, for $\pi \in P_k$ and $\sigma \in D_k$, $\pi \geq \sigma$ implies that also $\pi \in D_k$. Thus in particular, $\pi \vee \sigma \in D_k$ for any $\pi \in P_k$ and $\sigma \in D_k$, and we have in our case that always $\gamma(q) \vee \pi_1 \circ \cdots \circ \pi_r \in D_k$. Now note further that $\gamma(q) \vee \pi_1 \circ \cdots \circ \pi_r = q$ is actually equivalent to $\gamma(q) = q$ and $\pi_1 \circ \cdots \circ \pi_r \leq q$. One direction is clear, for the other one has to observe that $\gamma(q) \leq q$ implies $\gamma(q) = q$. With this, we get finally

$$\int_{H_n} Z(\sigma_1) \cdots Z(\sigma_r) du \\ = \#\{q \in D_k \mid q = \gamma(q), q \text{ restricted to the } i\text{-th cycle of } \gamma \text{ is } \sigma_i (i = 1, \dots, r)\} \quad (6.10)$$

From this, we get for the cumulants

$$\begin{aligned} & c_r[Z(\sigma_1), \dots, Z(\sigma_r)] \\ & = \sum_{\pi \in P(r)} \mu(\pi, 1_r) \cdot \#\{q \in D_k \mid q = \gamma(q), q \vee \gamma \leq \pi^\gamma, \\ & \qquad q \text{ restricted to the } i\text{-th cycle of } \gamma \text{ is } \sigma_i (i = 1, \dots, r)\}. \end{aligned}$$

The result follows then from Möbius inversion, as in the proof of Theorem 6.3.2. □

This shows in particular that $c_r[Z(\sigma_1), \dots, Z(\sigma_r)]$ vanishes unless all $\sigma_1, \dots, \sigma_r$ have the same number of blocks. This implies that the sets $\{Z_l^+, Z_l^-\}$ are independent for different l. For fixed l, we have for all $e_1, \dots, e_r \in \mathbb{N}$:

$$c_r[Z(\tau_l^{e_1 l}), \dots, Z(\tau_l^{e_r l})] = \begin{cases} l^{r-1}, & \text{if } \sum_i e_i \text{ is even} \\ 0, & \text{otherwise} \end{cases}.$$

Thus we get in particular

$$\begin{aligned} & c_r[Z(\tau_{l_1}^{2l_1}) \pm Z(1_{l_1}), \dots, Z(\tau_{l_r}^{2l_r}) \pm Z(1_{l_r})] \\ & \qquad = \begin{cases} d_{r,l}, & \text{if } l_1 = \dots = l_r = l \text{ and all signs are either + or all are } - \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where

$$d_{r,l} = l^{r-1} \sum_{\substack{t=0 \\ t \text{ even}}}^{r} \binom{r}{t} = l^{r-1} 2^{r-1}.$$

This shows that also Z_l^+ and Z_l^- are independent, and each of them is Poisson of parameter $1/(2l)$.

Corollary 6.7.4. *In H_n the random variables Z_l^+, Z_l^- ($l \in \mathbb{N}$), defined by* (6.5) *and* (6.6)*, are independent Poisson variables of parameter* $1/(2l)$*.*

Remark 6.7.5. In the free case H_n^+, the variables u_{ii} have also spectrum $\{-1, 0, 1\}$ and we can consider a positive/negative decomposition for C_1, i.e.,

$$C_1^+ = \sum_i 1_{\{1\}}(u_{ii})$$

and

$$C_1^- = -\sum_i 1_{\{-1\}}(u_{ii});$$

the other C_l, $l \geq 2$, are just as in the case of S_n^+. Similarly as for H_n, one can show that these variables are the ones showing up in the decomposition for H_n^+ in Theorem 6.7.2.

6.8 The half-liberated cases

The half-liberated quantum groups O_n^* and $H_n^{(s)}$ are neither classical nor free groups, so both classical and free cumulants are inadequate tools for getting information on the distribution of their traces. In Chapter 4, we introduced half-liberated cumulants to deal with half-independence, but one has to realize that we do not get an analogue of Theorem 6.3.2 for them, because the underlying "balanced" partition lattices do not share the multiplicativity property from Proposition 6.3.1. In order to investigate the distribution of traces in the half-liberated cases we will thus have to proceed via another route. The key insight is here that the half-liberated situations are actually "orthogonal" versions of classical unitary groups and that the main computations can be done over these unitary groups instead.

6.8.1 The half-liberated orthogonal group

Let $u = (u_{ij})_{i,j=1}^n$ be the fundamental representation of O_n^*, and let $v = (v_{ij})_{i,j=1}^n$ be the fundamental representation of the unitary group U_n. Then we can "orthogonalize" U_n by considering

$$w_{ij} := \begin{pmatrix} 0 & v_{ij} \\ \overline{v_{ij}} & 0 \end{pmatrix}. \tag{6.11}$$

Then $w = (w_{ij})_{i,j=1}^n$ is an orthogonal matrix and a simple calculation shows that the w_{ij} half-commute. It is also easy to see (by invoking the Weingarten formula for U_n, see below, Eq. (6.13)), that under this map the Haar state on O_n^* goes to $\int_{U_n} \otimes \mathrm{tr}_2$. Since the Haar state on O_n^* is faithful [11] , the mapping $u_{ij} \mapsto w_{ij}$ is actually an isomorphism.

So we have

$$\mathrm{Tr}(u^{2k+1}) = \begin{pmatrix} 0 & \mathrm{Tr}\big((v\bar{v})^k v\big) \\ \mathrm{Tr}\big((\bar{v}v)^k \bar{v}\big) & 0 \end{pmatrix}$$

and

$$\mathrm{Tr}(u^{2k}) = \begin{pmatrix} \mathrm{Tr}\big((v\bar{v})^k\big) & 0 \\ 0 & \mathrm{Tr}\big((\bar{v}v)^k\big) \end{pmatrix}$$

So what we need the understand is the distribution of the variables

$$v_{2k+1} := \lim_{n\to\infty} \mathrm{Tr}\big((v\bar{v})^k v\big), \qquad v_{2k} := \lim_{n\to\infty} \mathrm{Tr}\big((v\bar{v})^k\big). \tag{6.12}$$

Proposition 6.8.1. *Let $(v_k)_{k\geq 1}$ be as in (6.12), where $v = (v_{ij})_{i,j=1}^n$ are the coordinates of the classical unitary group U_n. Then we have: the variables $(v_k)_{k\geq 1}$ are independent; for k even, v_k is a real Gaussian with mean 0 or 1, depending on whether $k/2$ is odd or even, and variance equal to $k/2$; for k odd, v_k is a complex Gaussian with mean 0 and variance 1.*

Proof. For $\epsilon = (e_1, \ldots, e_k)$ a string of 1's and $*$'s, let us denote by $P_2(\mathbf{e})$ the pairings in P_k such that each block joins a 1 and a $*$. Then the Weingarten formula for U_n [19] says that with the notation

$$u_{ij}^e := \begin{cases} u_{ij}, & \text{if } e = 1 \\ \overline{u_{ij}}, & \text{if } e = * \end{cases}$$

we have

$$\int_{U_n} u_{i_1j_1}^{e_1}\cdots u_{i_kj_k}^{e_k}\,du = \sum_{\substack{p,q\in P_2(\mathbf{e})\\ p\le\ker\mathbf{i},q\le\ker\mathbf{i}}} W(p,q), \tag{6.13}$$

where $\mathbf{e}=(e_1,\ldots,e_k)$.

As in the proof of Theorem 6.2.1 this implies that

$$\int v_{k_1}^{e_1}\cdots v_{k_s}^{e_s} = \#\{p\in P_2(\mathbf{e}) \mid p=\gamma(p)\} + O(1/n). \tag{6.14}$$

The condition $p=\gamma(p)$ implies that the pairing p cannot join two cycles of different lengths, which shows that such an expectation factorizes according to the cycle lengths, which implies the independence of the v_k. The statements on the distribution of v_k follow also immediately from (6.14).

□

Transferring these results from the v_k to

$$u_{2k+1} = \begin{pmatrix} 0 & v_{2k+1}\\ \overline{v_{2k+1}} & 0\end{pmatrix}, \qquad u_{2k} = \begin{pmatrix} v_{2k} & 0\\ 0 & \overline{v_{2k}}\end{pmatrix} \tag{6.15}$$

and noting that the distribution of u_{2k+1} is equal to that of $\sqrt{|v_{2k+1}|^2}$ (which is a symmetrized Rayleigh distribution) yields then the following result. (Recall the notion of "half-independence" from Chapter 4).

Theorem 6.8.2. *For O_n^*, the variables $u_k=\lim_{n\to\infty}\mathrm{Tr}(u^k)$ are as follows. The sets $\{u:k \mid k \text{ odd}\}$ and $\{u_k \mid k \text{ even}\}$ are independent; for k even, the u_k are independent real Gaussian of mean 0 or 1, depending on whether $k/2$ is even or odd, and variance $k/2$; for k odd, the u_k are half-independent symmetrized Rayleigh variables with variance 1.* □

6.8.2 The hyperoctahedral series

The hyperoctahedral series $H_n^{(s)}$ (for $s=2,3,\ldots,\infty$) is determined by the partition lattice of all s-balanced partitions, see Theorem 3.3.3. This series includes the classical hyperoctrahedral group for $s=2$, $H_n^{(2)}=H_n$, and the half-liberated hyperoctahedral group for $s=\infty$, $H_n^{(\infty)}=H_n^*$. As for O_n^*, these groups can be considered as orthogonal versions of classical unitary groups. Namely, let $H_n^s=\mathbb{Z}_s\wr S_n$ be the complex reflection group consisting of monomial matrices having the s-roots of unity as nonzero entries. (Note that for $s=2$, $H_n^{(2)}=H_n^2$.) Then the relation between $H_n^{(s)}$ and H_n^s is the same as the one between O_n^* and U_n, i.e., we can represent the coordinates u_{ij} of $H_n^{(s)}$ by w_{ij} according to (6.11), where v_{ij} are the coordinates of H_n^s. So again, we can realize the asymptotic traces u_k of $H_n^{(s)}$ in the form (6.15), where the v_k are now the asymptotic traces in H_n^s according to (6.12). So our main task will be the determination of the distribution of these v_k.

Actually, we can treat more generally asymptotic traces with arbitrary pattern of the conjugates. So let us consider for an arbitrary string $\mathbf{e}=(e_1,\ldots,e_k)$ of 1 and $*$ the variable

$$v(\mathbf{e}) := \lim_{n\to\infty}\mathrm{Tr}(v^{e_1}\cdots v^{e_k}).$$

For $\mathbf{e} = (e_1, \dots, e_k)$ a string of 1's and $*$'s, we denote by $P^s(\mathbf{e})$ the partitions in P_k such that each block joins the same number, modulo s, of 1 and $*$. Then the Weingarten formula for H^s says that with the notation

$$v_{ij}^e := \begin{cases} v_{ij}, & \text{if } e = 1 \\ \overline{v_{ij}}, & \text{if } e = * \end{cases}$$

we have for the coordinate functions $v = (v_{ij})_{i,j=1}^n$ of H_n^s that

$$\int_{H_n^s} v_{i_1 j_1}^{e_1} \cdots v_{i_k j_k}^{e_k} du = \sum_{\substack{\pi,\sigma \in P^s(\mathbf{e}) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{i}}} W_{\mathbf{e},n}(\pi, \sigma),$$

where $\mathbf{e} = (e_1, \dots, e_k)$ and $W_{\mathbf{e},n}$ is the inverse of the Gram matrix $G_{\mathbf{e},n} = (n^{|p \vee q|})_{p,q \in P^s(\mathbf{e})}$. The leading order in n of the Weingarten function $W_{\mathbf{e},n}$ is given by

$$W_{\mathbf{e},n}(p, q) = n^{|p \vee q| - |p| - |q|}\big(1 + O(1/n)\big)$$

Theorem 6.8.3. *Fix $s \in \{2, 3, \dots, \infty\}$ and consider H^s. Consider $r \in \mathbb{N}$, $k_1, \dots, k_r \in \mathbb{N}$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \dots, k_r$. Then, for any strings $\mathbf{e}_1, \dots, \mathbf{e}_r$ of respective lengths $k_1, \dots, k_r$ we have the classical cumulants*

$$c_r(v(\mathbf{e}_1), \dots, v(\mathbf{e}_r)) = \#\{p \in P^s(\mathbf{e}_1 \cdots \mathbf{e}_r) | p \vee \gamma = 1_k,\, p = \gamma(p)\},$$

where the product of strings is just given by their concatenation.

Proof. As in the proof of Theorem 6.2.1 one gets for the moments

$$\int_{H_n^s} \mathrm{Tr}(v(\mathbf{e}_1)) \cdots \mathrm{Tr}(v(\mathbf{e}_r)) dv = \#\{p \in P^s(\mathbf{e}_1 \cdots \mathbf{e}_r) |\, p = \gamma(p)\} + O(1/n).$$

Note that γ does not necessarily map $P^s(\mathbf{e}_1 \cdots \mathbf{e}_r)$ into itself, and thus we only get the asymptotic version with lower order corrections.

We can then repeat the proof of Theorem 6.3.2. Let us write $\mathbf{e}$ for $\mathbf{e}_1 \cdots \mathbf{e}_r$; then we only have to note that $P^s(\mathbf{e}|_v)$ (for a block $v \in \sigma$) records the information about the original positions of the 1 and $*$ in $\mathbf{e}$, and thus the multiplicativity issue which prevented us from extending Theorem 6.3.2 to all easy classical groups, is not a problem here. Indeed, we have the analogue of (6.2),

$$P^s(\mathbf{e})_\sigma := \{p \in P_k \mid p|_v \in P^s(\mathbf{e}|_v)\, \forall v \in \sigma\} = \{p \in P^s(\mathbf{e}) \mid p \leq \sigma^\gamma\}.$$

□

Again, one can reduce the traces to more basic "cycle" variables. As before, we denote, for $l|k$, by $\tau_l^k \in P_k$ the partition

$$\tau_l^k = \{(1, l+1, 2l+1, \dots, k-l+1), (2, l+2, 2l+2, \dots, k-2+2), \dots, (l, 2l, \dots, k)\}$$

Then we have

$$v(\mathbf{e}) = \sum_{l|k} Z(\tau_l^k, \mathbf{e}), \tag{6.16}$$

where

$$Z(\tau_l^k, \mathbf{e}) := \sum_{\ker \mathbf{i} = \tau_l^k} v_{i_1 i_2}^{e_1} \cdots v_{i_k i_1}^{e_k}, \tag{6.17}$$

for arbitrary $l, k \in \mathbb{N}$ with $l|k$, and $\mathbf{e} = (e_1, \ldots, e_k)$.

As in the proof of Theorem 6.7.3 we can show

$$\begin{aligned} &c_r[Z(\tau_{l_1}^{k_1}, \mathbf{e}_1), \ldots, Z(\tau_{l_r}^{k_r}, \mathbf{e}_r)] \\ &\qquad = \#\{p \in P^s(\mathbf{e}_1 \cdots \mathbf{e}_r) \mid p = \gamma(p), p \vee \gamma = 1_k, \\ &\qquad\qquad p \text{ restricted to the } i\text{-th cycle of } \gamma \text{ is } \tau_{l_i}^{k_i} (i = 1, \ldots, r)\}. \end{aligned} \tag{6.18}$$

Clearly, this is only different from zero if $l_1 = \cdots = l_r$.

Let us define random variables $C_p(\tau_l^k, \mathbf{e})$ by specifying their distribution as

$$\begin{aligned} &c_r[C_p(\tau_{l_1}^{k_1}, \mathbf{e}_1), \ldots, C_p(\tau_{l_r}^{k_r}, \mathbf{e}_r)] \\ &\qquad = \begin{cases} 1/l, & \text{if } l_1 = \cdots = l_r = l \text{ and } p(\tau_l^{k_1}, \ldots, \tau_l^{k_r}) \in P^s(\mathbf{e}_1 \cdots \mathbf{e}_r) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{6.19}$$

where $p(\tau_l^{k_1}, \ldots, \tau_l^{k_r})$ is that partition in P_k whose i-th block consists of the union of the i-th blocks of all the τ's, i.e., it is equal to

$$\tau_l^{k_1} \circ \cdots \circ \tau_l^{k_r} \vee \{(1, k_1 + 1, k_1 + k_2 + 1, \ldots, k - k_r + 1), \ldots, (l, k_1 + l, k_1 + k_2 + l, \ldots, k - k_r + l)\}.$$

Then we can express our variables $Z(\tau_l^k, \mathbf{e})$ in terms of the $C_p(\tau_l^k, \mathbf{e})$ by

$$Z(\tau_l^k, \mathbf{e}) = \sum_{t=1}^{l} C_p(\tau_l^k, \mathbf{e}^{(t)}),$$

where $\mathbf{e}^{(t)}$ is the t-fold cyclic shift of the string $\mathbf{e}$, i.e.,

$$\mathbf{e}^{(t)} = (e_{t+1}, e_{t+2}, \ldots, e_t)$$

The definition (6.19), on the other hand, shows that the variables $C_p(\tau_l^k, \mathbf{e})$ are compound Poisson elements, which are independent for different l. Namely, we can associate to $C_p(\tau_l^k, \mathbf{e})$ a random variable

$$a(\tau_l^k, \mathbf{e}) = \left(\prod_{\substack{i_1 \text{ in first} \\ \text{block of } \tau_l^k}} \omega^{e_{i_1}} \right) \otimes \left(\prod_{\substack{i_2 \text{ in second} \\ \text{block of } \tau_l^k}} \omega^{e_{i_2}} \right) \otimes \cdots \otimes \left(\prod_{\substack{i_l \text{ in } l\text{-th} \\ \text{block of } \tau_l^k}} \omega^{e_{i_l}} \right) \in C(\mathbb{T})^{\otimes l}.$$

Then we have

$$c_r[C_p(\tau_{l_1}^{k_1}, \mathbf{e}_1), \ldots, C_p(\tau_{l_r}^{k_r}, \mathbf{e}_r)] = \frac{1}{l_1}\psi\big(a(\tau_{l_1}^{k_1}, \mathbf{e}_1)\cdots a(\tau_{l_r}^{k_r}, \mathbf{e}_r)\big) \tag{6.20}$$

where ψ is $\bigoplus_l \varphi^{\otimes l}$ on $\bigoplus_l C(\mathbb{T})^{\otimes l}$, with φ denoting integration with respect to the Haar measure on $\mathbb{Z}_s$ (where the latter is being embedded into the unit circle $\mathbb{T}$).

The equation (6.20) shows that the cumulants of the variables C_p are given, up to some factor, as the corresponding moments of some variables a; this is the characterizing property of compound Poisson variables.

If we put now

$$\mathbf{e}_k := (\underbrace{1, *, 1, *, \ldots}_{k}),$$

then we have for our asymptotic traces the decomposition

$$v_k = \sum_{l|k} Z(\tau_l^k, \mathbf{e}_k) = \sum_{l|k}\sum_{t=1}^{l} C_p(\tau_l^k, \mathbf{e}_k^{(t)}). \tag{6.21}$$

Thus we have written the v_k as a sum of compound Poisson variables. These are independent for different l; however, for fixed l, the relation between the C_p for various k is more complicated, according to the $\mathbf{e}$-strings. For k even this reduces again essentially to a sum of independent Poisson variables, whereas for k odd the situation is getting more involved. As we do not see a nice more explicit description, we refrain from working out the details in this case.

Chapter 7

Asymptotic infinitesimal freeness for Haar quantum unitary random matrices

7.1 Introduction

One of the most important results in free probability theory is Voiculescu's asymptotic freeness for random matrices [50]. One simple form of this result is the following. Let A_N and B_N be (deterministic) $N \times N$ matrices with complex entries, and suppose that A_N and B_N have limiting distributions as $N \to \infty$ with respect to the normalized trace on $M_N(\mathbb{C})$. Let $(U_N)_{N\in\mathbb{N}}$ be a sequence of $N \times N$ unitary random matrices, distributed according to Haar measure. Then $U_N A_N U_N^*$ and B_N are asymptotically freely independent as $N \to \infty$. Moreover, freeness holds "up to $O(N^{-2})$", which can be interpreted as *infinitesimal freeness* in the sense of Belinschi-Shlyakhtenko [14].

On the other hand, it is becoming increasingly apparent that in free probability, the roles of the classical groups are played by the "free" quantum groups. This is seen most clearly in the study of quantum distributional symmetries, as has been discussed in the previous chapters.

In this chapter we will consider the limiting distribution of $U_N A_N U_N^*$ and B_N, where A_N and B_N are as above, but U_N is now a Haar distributed $N \times N$ *quantum* unitary random matrix, in the sense of Wang [60]. We will show that asymptotic (infinitesimal) freeness now holds even if the entries of A_N and B_N are allowed to take values in an arbitrary unital C*-algebra $\mathcal{B}$:

Theorem 7.1.1. *Let $\mathcal{B}$ be a unital C*-algebra and let $A_N, B_N \in M_N(\mathcal{B})$ for $N \in \mathbb{N}$. Assume that there is a finite constant C such that $\|A_N\| \leq C$, $\|B_N\| \leq C$ for all $N \in \mathbb{N}$. For each $N \in \mathbb{N}$, let U_N be a Haar distributed $N \times N$ quantum unitary random matrix, with entries independent from $\mathcal{B}$.*

(1) *Suppose that there are linear maps* $\mu_A, \mu_B : \mathcal{B}\langle t\rangle \to \mathcal{B}$ *such that for any* $b_0, \dots, b_k \in \mathcal{B}$,

$$\lim_{N\to\infty} \|(\mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}})[b_0 A_N b_1 \cdots A_N b_k] - \mu_A[b_0 t b_1 \cdots t b_k]\| = 0$$
$$\lim_{N\to\infty} \|(\mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}})[b_0 B_N b_1 \cdots B_N b_k] - \mu_B[b_0 t b_1 \cdots t b_k]\| = 0,$$

where tr_N *denotes the normalized trace on* $M_N(\mathbb{C})$. *Then* $U_N A_N U_N^*$ *and* B_N *are asymptotically free with amalgamation over* $\mathcal{B}$.

(2) *Suppose that in addition, the limits*

$$\lim_{N\to\infty} N\Big\{(\mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}})[b_0 A_N b_1 \cdots A_N b_k] - \mu_A[b_0 t b_1 \cdots t b_k]\Big\}$$
$$\lim_{N\to\infty} N\Big\{(\mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}})[b_0 B_N b_1 \cdots B_N b_k] - \mu_B[b_0 t b_1 \cdots t b_k]\Big\}$$

converge in norm for any $b_0, \dots, b_k \in \mathcal{B}$. *Then* $U_N A_N U_N^*$ *and* B_N *are asymptotically infinitesimally free with amalgamation over* $\mathcal{B}$.

We will present more general asymptotic freeness results in Section 7.5, in particular Theorem 7.1.1 will be a special case of Corollary 7.5.10.

For finite-dimensional $\mathcal{B}$, we show in Proposition 7.5.11 that classical Haar unitary random matrices are sufficient to obtain such a result. However, classical unitaries are in general insufficient for asymptotic freeness with amalgamation, even within the class of *approximately finite dimensional* C*-algebras, and so it is indeed necessary to allow quantum unitary transformations. We will discuss this further in the second part of Section 7.5, see in particular Example 7.5.13 and the remarks which follow.

This chapter is organized as follows. The next section contains notations and preliminaries. Here we collect the basic notions from infinitesimal free probability and introduce the quantum unitary group $A_u(N)$. Section 7.3 contains some combinatorial results, related to the "fattening" operation on noncrossing partitions, which will be required in the sequel. In Section 7.4 we recall the Weingarten formula for computing integrals over $A_u(N)$, and prove a new estimate on the entries of the corresponding Weingarten matrix. Section 7.5 contains our main results, and a discussion of their failure for classical Haar unitaries.

7.2 Preliminaries and notations

Infinitesimal free probability. We will now introduce the notions of operator-valued infinitesimal probability spaces and infinitesimal freeness. This is a straightforward generalization of the framework of [14], and we refer the reader to that paper for further discussion of infinitesimal freeness and its relation to the type B free independence of Biane, Nica and Goodman [16]. See [31] for a more combinatorial treatment of infinitesimal freeness.

Definition 7.2.1.

(1) If $\mathcal{B}$ is a unital algebra, a *$\mathcal{B}$-valued infinitesimal probability space* is a triple $(\mathcal{A}, E, E')$ where $\mathcal{A}$ is a unital algebra which contains $\mathcal{B}$ as a unital subalgebra and E, E' are $\mathcal{B}$-linear maps from $\mathcal{A}$ to $\mathcal{B}$ such that $E[1] = 1$ and $E'[1] = 0$.

(2) Let $(\mathcal{A}, E, E')$ be a $\mathcal{B}$-valued infinitesimal probability space, and let $(\mathcal{A}_i)_{i\in I}$ be a collection of subalgebras $\mathcal{B} \subset \mathcal{A}_i \subset \mathcal{A}$. The algebras are said to be *infinitesimally free with amalgamation over $\mathcal{B}$*, or *infinitesimally free with respect to* (E, E'), if

   (a) $(\mathcal{A}_i)_{i\in I}$ are freely independent with respect to E.

   (b) For any $a_1, \ldots, a_k$ so that $a_j \in \mathcal{A}_{i_j}$ for $1 \le j \le k$ with $i_j \ne i_{j+1}$, we have

$$E'\Big[\big(a_1 - E[a_1]\big)\cdots\big(a_k - E[a_k]\big)\Big] = \sum_{j=1}^{k} E\Big[\big(a_1 - E[a_1]\big)\cdots\big(E'[a_j]\big)\cdots\big(a_k - E[a_k]\big)\Big].$$

We say that subsets $(\Omega_i)_{i\in I}$ are infinitesimally free with amalgamation over $\mathcal{B}$ if the subalgebras $\mathcal{A}_i$ generated by $\mathcal{B}$ and Ω_i are infinitesimally free with respect to (E, E').

Remark 7.2.2. The motivating example is given by a family $(A_i(s))_{i\in I}$ of $\mathcal{B}$-valued random variables for $s > 0$ which are free "up to $o(s)$" as $s \to 0$. This is made precise in the next proposition. Note that there we make the notion "free up to $o(s)$" precise by comparing the family $(A_i(s))_{i\in I}$ with a family $(a_i(s))_{i\in I}$ which is free for all s. Infinitesimal freeness will then occur at $s = 0$ (both for the A_i and the a_i). Since 0 is not necessarily in K, we define the states E and E' on the free algebra $\mathcal{A} := \mathcal{B}\langle A_i | i \in I\rangle$ generated by non-commuting indeterminates $A_i \hat{=} A_i(0) \hat{=} a_i(0)$.

Proposition 7.2.3. *Let $\mathcal{B}$ be a unital C*-algebra and K a subset of $\mathbb{R}$ for which 0 is an accumulation point. Suppose that for each $s \in K$ we have a $\mathcal{B}$-valued probability space $(\mathcal{A}(s), E_s : \mathcal{A}(s) \to \mathcal{B})$ where $\mathcal{A}(s)$ is a unital C*-algebra which contains $\mathcal{B}$ as a unital subalgebra and E_s is contractive. Furthermore, suppose that, for each $s \in K$, there are variables $(A_i(s))_{i\in I}$ belonging to $\mathcal{A}(s)$ such that the following hold:*

(1) There are $\mathcal{B}$-linear maps $E, E' : \mathcal{B}\langle A_i | i \in I\rangle \to \mathcal{B}$ such that

$$E[p(A)] = \lim_{s\to 0} E_s\big[p(A(s))\big]$$
$$E'[p(A)] = \lim_{s\to 0} \frac{1}{s}\Big\{E_s[p(A(s))] - E[p]\Big\}$$

for $p \in \mathcal{B}\langle t_i | i \in I\rangle$, where the limits hold in norm.

(2) For each $i \in I$,

$$\limsup_{s\to 0} \|A_i(s)\| < \infty.$$

Let $I = \bigcup_{j\in J} I_j$ be a partition of I. For $s \in K$, let $(a_i(s))_{i\in I}$ be a family in some $\mathcal{B}$-valued probability space $(\mathcal{C}, F : \mathcal{C} \to \mathcal{B})$ and suppose that

(1) For any $j \in J$, $p \in \mathcal{B}\langle t_i | i \in I_j\rangle$, *and* $s \in K$,

$$E_s[p(A(s))] = F[p(a(s))].$$

(2) The sets $(\{a_i(s) | s \in K, i \in I_j\})_{j \in J}$ *are free with respect to* F.

(3) For any $p \in \mathcal{B}\langle t_i | i \in I\rangle$ *we have*

$$\big\| E_s[p(A(s))] - F[p(a(s))] \big\| = o(s) \quad (\text{as } s \to 0).$$

Then the sets $(\{A_i | i \in I_j\})_{j \in J} \subset \mathcal{B}\langle A_i | i \in I\rangle$ *are infinitesimally free with respect to* (E, E').

Proof. Since E, E' only depend on the distribution of the variables $A_i(s)$ up to first order, it clearly suffices to assume that the sets $(\{A_i(s) : i \in I_j\})_{j \in J}$ are freely independent with respect to E_s for all $s \in K$. It is then clear that the sets $(\{A_i : i \in I_j\})_{j \in J} \subset \mathcal{B}\langle A_i | i \in I\rangle$ are free with respect to E, so it suffices to show that E' satisfies condition (2) of Definition 7.2.1. Let $j_1 \neq \cdots \neq j_k$ in J and $p_l \in \mathcal{B}\langle t_i | i \in I_{j_l}\rangle$ for $1 \leq l \leq k$, and consider

$$\begin{aligned}
E'\Big[\big(p_1(A) - E[p_1(A)]\big) \cdots \big(p_k(A) - E[p_k(A)]\big)\Big] \\
= \lim_{s \to 0} \frac{1}{s}\Big\{ E_s\Big[\big(p_1(A(s)) - E[p_1(A)]\big) \cdots \big(p_k(A(s)) - E[p_k(A)]\big)\Big] \\
- E\Big[\big(p_1(A) - E[p_1(A)]\big) \cdots \big(p_k(A) - E[p_k(A)]\big)\Big]\Big\} \\
= \lim_{s \to 0} \frac{1}{s}\Big\{ E_s\Big[\big(p_1(A(s)) - E[p_1(A)]\big) \cdots \big(p_k(A(s)) - E[p_k(A)]\big)\Big]\Big\},
\end{aligned}$$

where we have used freeness with respect to E. Rewrite this expression as

$$\begin{aligned}
\lim_{s \to 0} \frac{1}{s}\Big\{ E_s\Big[\big((p_1(A(s)) - E_s[p_1(A(s))]) + (E_s[p_1(A(s))] - E[p_1(A)])\big) \\
\cdots \big((p_k(A(s)) - E_s[p_k(A(s))]) + (E_s[p_k(A(s))] - E[p_k(A)])\big)\Big]\Big\},
\end{aligned}$$

and consider the terms which appear in the expansion. First observe that $\|E_s[p_l(A(s))] - E[p_l(A)]\|$ is $O(s)$ for $1 \leq l \leq k$. By the boundedness assumption on the norms of $A_i(s)$, and the contractivity of E_s, it follows that those terms involving more than one expression $(E_s[p_l(A(s))] - E[p_l(A)])$ vanish in the limit.

The term involving none of these expressions is

$$E_s\Big[\big(p_1(A(s)) - E_s[p_1(A(s))]\big) \cdots \big(p_k(A(s)) - E_s[p_k(A(s))]\big)\Big]$$

which is zero by freeness.

So we are left to consider only the terms involving one such expression, which gives the sum from $l = 1$ to k of

$$\lim_{s \to 0} \frac{1}{s}\Big\{ E_s\Big[\big(p_1(A(s)) - E_s[p_1(A(s))]\big) \cdots \big(E_s[p_l(A(s))] - E[p_l(A)]\big) \cdots \big(p_k(A(s)) - E_s[p_k(A(s))]\big)\Big]\Big\},$$

which again by invoking the boundedness assumptions on $A_i(s)$ and contractivity of E_s, converges to

$$\sum_{l=1}^{k} E\Big[\big(p_1(A) - E[p_1(A)]\big)\cdots E'[p_l(A)]\cdots\big(p_k(A) - E[p_k]\big)\Big]$$

as desired. □

Quantum unitary group. We now recall the definition of the quantum unitary group from [60], which is a compact quantum group in the sense of Woronowicz [63].

Definition 7.2.4. $A_u(n)$ is the universal C*-algebra generated by $\{U_{ij} : 1 \leq i,j \leq n\}$ such that the matrix $U = (U_{ij}) \in M_n(A_u(n))$ is unitary. $A_u(n)$ is a C*-Hopf algebra with comultiplication, counit and antipode given by

$$\begin{aligned}\Delta(U_{ij}) &= \sum_{k=1}^{n} U_{ik} \otimes U_{kj}\\ \epsilon(U_{ij}) &= \delta_{ij}\\ S(U_{ij}) &= U_{ji}^*.\end{aligned}$$

The existence of these maps is given by the the universal property of $A_u(n)$.

Remark 7.2.5. It is often useful to consider the heuristic formula "$A_u(n) = C(U_n^+)$", where U_n^+ is the *free unitary group*, as have done in the previous chapters for the easy quantum groups. However we will stay with the Hopf algebra notation in this chapter, which is better suited for our purposes.

Remark 7.2.6. Recall that there is a unique *Haar state* $\psi_n : A_u(n) \to \mathbb{C}$ which is left and right invariant in the sense that

$$(\psi_n \otimes \mathrm{id})\Delta(a) = \psi_n(a)1_{A_u(n)} = (\mathrm{id} \otimes \psi_n)\Delta(a)$$

for $a \in A_u(n)$. We will discuss this further in Section 7.4.

Wang also introduced the free product operation on compact quantum groups in [60]. We will use $A_u(n)^{*\infty}$ to denote the free product of countably many copies of $A_u(n)$. The reader is referred to [60] for details, the only properties that we will need are that

(1) $A_u(n)^{*\infty}$ is generated (as a C*-algebra) by elements $\{U(l)_{ij} : l \in \mathbb{N}, 1 \leq i,j \leq n\}$, such that $U(l) \in M_n(A_u(n)^{*\infty})$ is unitary.

(2) The sets $(\{U(l)_{ij} : 1 \leq i,j \leq n\})_{l\in\mathbb{N}}$ are freely independent with respect to the Haar state $\psi_n^{*\infty}$ on $A_u(n)^{*\infty}$, and for each $l \in \mathbb{N}$, $(U(l)_{ij})$ has the same joint distribution in $(A_u(n)^{*\infty}, \psi_n^{*\infty})$ as (U_{ij}) in $(A_u(n), \psi_n)$.

7.3 The fattening operation on noncrossing partitions

In this section we introduce several operations on partitions and prove some basic results which will be required throughout the remainder of the chapter.

Notation 7.3.1.

(1) Given $\pi \in NC(m)$, we define $\widetilde{\pi} \in NC_2(2m)$ as follows: for each block $V = (i_1, \dots, i_s)$ of π, we add to $\widetilde{\pi}$ the pairings $(2i_1 - 1, 2i_s), (2i_1, 2i_2 - 1), \dots, (2i_{s-1}, 2i_s - 1)$.

(2) Given $\pi \in NC(m)$, we define $\hat{\pi} \in NC(2m)$ by partitioning the m-pairs $(1, 2), (3, 4), \dots, (2m-1, 2m)$ according to π.

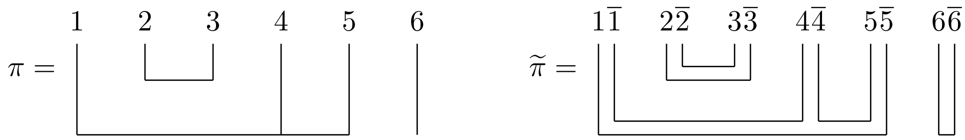
(3) Given $\pi, \sigma \in \mathcal{P}(m)$, we define $\pi \wr \sigma \in \mathcal{P}(2m)$ to be the partition obtained by partitioning the odd numbers $\{1, 3, \dots, 2m-1\}$ according to π and the even numbers $\{2, 4, \dots, 2m\}$ according to σ.

(4) Given $\pi \in \mathcal{P}(m)$, let $\overleftarrow{\pi}$ denote the partition obtained by shifting k to $k-1$ for $1 < k \leq m$ and sending 1 to m, i.e.,

$$s \sim_{\overleftarrow{\pi}} t \qquad \Longleftrightarrow \qquad (s+1) \sim_{\pi} (t+1),$$

where we count modulo m on the right hand side. Likewise we let $\overrightarrow{\pi}$ denote the partition obtained by shifting k to $k+1$ for $1 \leq k < m$ and sending m to 1.

Remark 7.3.2. The map $\pi \mapsto \widetilde{\pi}$ is easily seen to be a bijection, and corresponds to the well-known "fattening" operation. The following example shows this for $\pi = \{\{1, 4, 5\}, \{2, 3\}, \{6\}\}$.



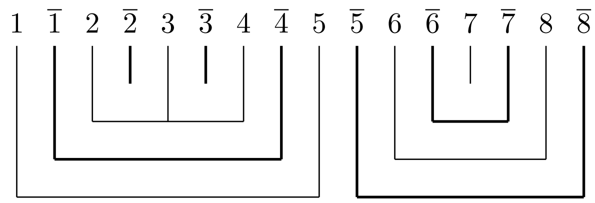
There is a simple description of the inverse, it sends $\sigma \in NC_2(2m)$ to the partition $\tau \in NC(m)$ such that $\sigma \vee \hat{0}_m = \hat{\tau}$, where $\hat{0}_m = \{\{1, 2\}, \dots, \{2m-1, 2m\}\}$. Thus we have for $\pi \in NC(m)$

$$\hat{\pi} = \widetilde{\pi} \vee \hat{0}_m.$$

Note also that $\hat{0}_m = \widetilde{0}_m$ and that $\hat{1}_m = 1_{2m}$.

Definition 7.3.3. Let $\pi \in NC(m)$. The *Kreweras complement* $K(\pi)$ is the largest partition in $NC(m)$ such that $\pi \wr K(\pi) \in NC(2m)$.

Example 7.3.4. If $\pi = \{\{1, 5\}, \{2, 3, 4\}, \{6, 8\}, \{7\}\}$ then $K(\pi) = \{\{1, 4\}, \{2\}, \{3\}, \{5, 8\}, \{6, 7\}\}$, which can be seen follows:



The following lemma provides the relationship between the Kreweras complement on $NC(m)$ and the map $\pi \mapsto \widetilde{\pi}$.

Lemma 7.3.5. *If $\pi \in NC(m)$, then*

$$\widetilde{K(\pi)} = \overleftarrow{\widetilde{\pi}}.$$

Proof. We will prove this by induction on the number of blocks of π. If $\pi = 1_m$ has one block, the result is trivial from the definitions.

Suppose now that $V = \{l+1, \ldots, l+s\}$ is a block of π, $l \geq 1$. First note that $\widetilde{\pi}$ is obtained by taking $\widetilde{\pi \setminus V}$ then adding the pairs $(2l+1, 2(l+s)), (2l+2, 2l+3), \ldots, (2(l+s)-2, 2(l+s)-1)$.

Observe that $K(\pi)$ is obtained by taking $K(\pi \setminus V)$, adding singletons $\{l+1\}, \ldots, \{l+s-1\}$, then placing $l+s$ in the block containing l. It follows that $\widetilde{K(\pi)}$ is the partition obtained by taking $\widetilde{K(\pi \setminus V)}$, which by induction is $\overleftarrow{\widetilde{\pi \setminus V}}$, then moving the leg connected to $2l$ to $2(l+s)$ and adding the pairs $(2l, 2(l+s)-1)$, $(2l+1, 2l+2), \ldots, (2(l+s)-3, 2(l+s)-2)$. The result now follows. □

We will also need the following relationship between $\pi \mapsto \widetilde{\pi}$ and the Kreweras complement on $NC(2m)$. This is a generalization of the relation

$$K(\hat{\pi}) = K(\widetilde{0}_m \vee \widetilde{\pi}) = 0_m \wr K(\pi) \qquad (\pi \in NC(m)),$$

which is obvious from the definition of $\hat{\pi}$.

Lemma 7.3.6. *If $\pi, \sigma \in NC(m)$ and $\sigma \leq \pi$, then $\widetilde{\sigma} \vee \widetilde{\pi} \in NC(2m)$ and*

$$K(\widetilde{\sigma} \vee \widetilde{\pi}) = \sigma \wr K(\pi).$$

Proof. We will prove this by induction on the number of blocks of π. First suppose that $\pi = 1_m$, then we have

$$\widetilde{\sigma} \vee \widetilde{\pi} = \overrightarrow{\overleftarrow{\widetilde{\sigma}} \vee \overleftarrow{\widetilde{\pi}}} = \overrightarrow{\widetilde{K(\sigma)} \vee \hat{0}_m} = \overrightarrow{\widehat{K(\sigma)}}$$

is noncrossing. Moreover,

$$K(\widetilde{\sigma} \vee \widetilde{\pi}) = K\left(\overrightarrow{\widehat{K(\sigma)}}\right) = \overrightarrow{0_m \wr K^2(\sigma)},$$

where for the last equality we used the equation for $K(\hat{\pi})$ mentioned before the Lemma 7.3.6 and the fact that the Kreweras complement commutes with shifting. But, by [43, Exercise 9.23], we have that $K^2(\sigma) = \overleftarrow{\sigma}$ and thus we finally get

$$K(\widetilde{\sigma} \vee \widetilde{\pi}) = \overrightarrow{0_m \wr \overleftarrow{\sigma}} = \sigma \wr 0_m.$$

Now suppose that $V = \{l+1, \dots, l+s\}$, $l \geq 1$ is an interval of π. Observe that $\widetilde{\sigma} \vee \widetilde{\pi}$ is the partition obtained by partitioning $\{1, \dots, 2l\} \cup \{2(l+s)+1, \dots, 2m\}$ according to $\widetilde{\sigma \setminus \sigma|_V} \vee \widetilde{\pi \setminus V}$, and $\{2l+1, \dots, 2(l+s)\}$ according to $\widetilde{\sigma|_V} \vee \widetilde{1_V}$. It follows that $\widetilde{\sigma} \vee \widetilde{\pi}$ is noncrossing and that $K(\widetilde{\sigma} \vee \widetilde{\pi})$ is the partition obtained by partitioning $\{1, \dots, 2l\} \cup \{2(l+s)+1, \dots, 2m\}$ according to $K(\widetilde{\sigma \setminus \sigma|_V} \vee \widetilde{\pi \setminus V})$ and $\{2l+1, \dots, 2(l+s)\}$ according to $K(\widetilde{\sigma|_V} \vee \widetilde{1_V})$, then joining the blocks containing $2l$ and $2(l+s)$. On the other hand, $K(\pi)$ is equal to the partition obtained by taking $K(\pi \setminus V)$ then adding $\{l+1\}, \dots, \{l+s-1\}$ and joining $l+s$ to l, and the result now follows by induction. □

We will need to compare the number of blocks in the join of two partitions before and after fattening. For this purpose we will use the following *linearization lemma* of Kodiyalam-Sunder [38]. Note that the notation $S \mapsto \widetilde{S}$ used in their paper corresponds to the inverse of the fattening procedure $\pi \mapsto \widetilde{\pi}$ used here.

Theorem 7.3.7 ([40]). *Let $\pi, \sigma \in NC(m)$. Then*

$$\left|\widetilde{\pi} \vee \widetilde{\sigma}\right| = m + 2|\pi \vee \sigma| - |\pi| - |\sigma|.$$

In particular, if $\sigma \leq \pi$ then

$$\left|\widetilde{\pi} \vee \widetilde{\sigma}\right| = m + |\pi| - |\sigma|.$$

□

Remark 7.3.8. This is closely related to a "twisting" result relating the "projective version" of the quantum orthogonal group $A_o(n)$ with the quantum permutation group $A_s(n^2)$. This leads to an interesting relationship between free hypergeometric and hyperspherical laws, see [4].

We now introduce some special classes of noncrossing partitions and prove some basic results. These are related to integration on the quantum unitary group via the Weingarten formula to be discussed in the next section.

Notation 7.3.9. Let $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$.

(1) $NC_h^\epsilon(2m)$ denote the set of partitions $\pi \in NC(2m)$ such that each block V of π has an even number of elements, and $\epsilon|_V$ is alternating, i.e., $\epsilon|_V = 1 * 1 * \cdots 1*$ or $*1 * 1 \cdots * 1$.

(2) $NC_2^\epsilon(2m)$ will denote the collection of $\pi \in NC_2(2m)$ such that each pair in π connects a 1 with a $*$, i.e.,

$$s \sim_\pi t \Rightarrow \epsilon_s \neq \epsilon_t.$$

(3) $NC^\epsilon(m)$ will denote the collection of $\pi \in NC(m)$ such that $\widetilde{\pi} \in NC_2^\epsilon(m)$.

Lemma 7.3.10. *Let $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$. If $\sigma, \pi \in NC^\epsilon(m)$ and $\sigma \leq \pi$, then $\widetilde{\sigma} \vee \widetilde{\pi}$ is in $NC_h^\epsilon(2m)$. Conversely, if $\tau \in NC_h^\epsilon(2m)$ then there are unique $\sigma, \pi \in NC^\epsilon(m)$ such that $\sigma \leq \pi$ and $\tau = \widetilde{\sigma} \vee \widetilde{\pi}$.*

Proof. First suppose that $\tau \in NC_h^\epsilon(2m)$. Since each block of τ has an even number of elements, we have $K(\tau) = \sigma \wr K(\pi)$ for some $\sigma, \pi \in NC(m)$ such that $\sigma \leq \pi$. By Lemma 7.3.6 we have $\tau = \widetilde{\sigma} \vee \widetilde{\pi}$, and this clearly determines σ and π uniquely. If V is a block of τ, then $\epsilon|_V$ is alternating and hence $\widetilde{\pi}|_V, \widetilde{\sigma}|_V \in NC_2^\epsilon(V)$. It follows that $\pi, \sigma \in NC^\epsilon(m)$.

Conversely, let $\sigma, \pi \in NC^\epsilon(m)$ with $\sigma \leq \pi$. Let $\hat{\epsilon} = (\epsilon_1, \epsilon_1, \epsilon_2, \epsilon_2, \ldots, \epsilon_{2m}, \epsilon_{2m})$. Observe that if $\tau \in NC(2m)$, then $\tau \in NC_h^\epsilon(2m)$ if and only if $\widetilde{\tau} \in NC_2^{\hat{\epsilon}}(4m)$.

So let $\tau = \widetilde{\sigma} \vee \widetilde{\pi}$, we need to show $\widetilde{\tau} \in NC_2^{\hat{\epsilon}}(4m)$. Now

$$\overleftarrow{\widetilde{\tau}} = \widetilde{K(\tau)} = \sigma \wr \widetilde{K(\pi)},$$

where we have applied Lemmas 7.3.5 and 7.3.6. In other words, $\overleftarrow{\widetilde{\tau}}$ is the partition given by partitioning $\{1, 2, 5, 6, \ldots, 4m-3, 4m-2\}$ according to $\widetilde{\sigma}$ and $\{3, 4, 7, 8, \ldots, 4m-1, 4m\}$ according to $\widetilde{K(\pi)} = \overleftarrow{\widetilde{\pi}}$. Now since $\sigma, \pi \in NC^\epsilon(m)$, it follows that $\overleftarrow{\widetilde{\tau}} \in NC_2^{\overleftarrow{\hat{\epsilon}}}(4m)$, where $\overleftarrow{\hat{\epsilon}} = (\epsilon_1, \epsilon_2, \epsilon_2, \ldots, \epsilon_{2m}, \epsilon_{2m}, \epsilon_1)$, and hence $\widetilde{\tau} \in NC_2^{\hat{\epsilon}}(4m)$. □

Lemma 7.3.11. *$NC^\epsilon(m)$ is closed under taking intervals in $NC(m)$, i.e., if $\sigma, \pi \in NC^\epsilon(m)$ and $\tau \in NC(m)$ is such that $\sigma < \tau < \pi$, then $\tau \in NC^\epsilon(m)$.*

Proof. Let $\sigma, \pi \in NC^\epsilon(m)$, and $\tau \in NC(m)$ such that $\sigma < \tau < \pi$. From the inductive definition of $\widetilde{\tau}$, to show that $\tau \in NC^\epsilon(m)$ it suffices to consider $\pi = 1_m$. Now by the previous lemma, we have $\widetilde{\sigma} \vee \widetilde{1_m} \in NC_h^\epsilon(2m)$. By Lemma 7.3.5,

$$\overleftarrow{\widetilde{\sigma} \vee \widetilde{1_m}} = \widetilde{K(\sigma)} \vee \hat{0}_m = \widehat{K(\sigma)}.$$

Since $\sigma \leq \tau$, we have $\widehat{0}_m \leq \widehat{K(\tau)} \leq \widehat{K(\sigma)}$. Let $\delta = (\epsilon_2, \ldots, \epsilon_{2m}, \epsilon_1)$, and suppose that $\widehat{K(\tau)} \notin NC_h^\delta(2m)$. Let V be a block of $\widehat{K(\tau)}$, and note that V is of the form $(2i_1 - 1, 2i_1, \ldots, 2i_s - 1, 2i_s)$ for some $i_1 < \cdots < i_s$. Since $\hat{0}_m \in NC_h^\delta(2m)$, it follows that there is a $1 \leq l < s$ with $\delta_{2i_l} = \delta_{2i_{l+1}-1}$. Now since $\widehat{0}_m \leq \widehat{K(\tau)} \leq \widehat{K(\sigma)}$, it follows that the block W of $\widehat{K(\sigma)}$ which contains V must have an even number of elements between $2i_l$ and $2i_{l+1} - 1$. But then $\delta|_W$ cannot be alternating, which contradicts $\widehat{K(\sigma)} \in NC_h^\delta(2m)$.

So we have shown that $\widehat{K(\tau)} \in NC_h^\delta(2m)$, and since

$$\overrightarrow{\widehat{K(\tau)}} = \overrightarrow{\widetilde{K(\tau)} \vee \hat{0}_m} = \widetilde{\tau} \vee \widetilde{1}_m,$$

we have $\widetilde{\tau} \vee \widetilde{1_m} \in NC_h^\epsilon(2m)$. But then by the previous lemma, there is a $\gamma \in NC^\epsilon(m)$ with $\widetilde{\gamma} \vee \widetilde{1_m} = \widetilde{\tau} \vee \widetilde{1_m}$, and by Lemma 7.3.6 this implies $\tau = \gamma$ is in $NC^\epsilon(m)$ as claimed. □

7.4 Integration on the quantum unitary group

We begin by recalling the *Weingarten formula* from [5] for computing integrals with respect to the Haar state on $A_u(n)$.

Let $\epsilon_1, \ldots, \epsilon_{2m} \in \{1, *\}$ and define, for $n \in \mathbb{N}$, the *Gram matrix*

$$G_{\epsilon n}(\pi, \sigma) = n^{|\pi \vee \sigma|} \qquad (\pi, \sigma \in NC_2^{\epsilon}(2m)).$$

It is shown in [5] that $G_{\epsilon n}$ is invertible for $n \geq 2$, let $W_{\epsilon n}$ denote its inverse.

Theorem 7.4.1 ([5])**.** *The Haar state on $A_u(n)$ is given by*

$$\psi_n(U_{i_1 j_1}^{\epsilon_1} \cdots U_{i_{2m} j_{2m}}^{\epsilon_{2m}}) = \sum_{\substack{\pi,\sigma \in NC_2^{\epsilon}(2m) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{\epsilon n}(\pi, \sigma)$$

$$\psi_n(U_{i_1 j_1}^{\epsilon_1} \cdots U_{i_{2m+1} j_{2m+1}}^{\epsilon_{2m+1}}) = 0,$$

for $1 \leq i_1, j_1, \ldots, i_{2m+1}, j_{2m+1} \leq n$ and $\epsilon_1, \ldots, \epsilon_{2m+1} \in \{1, *\}$. □

Remark 7.4.2. Note that the Weingarten formula above is effective for computing integrals of products of the entries in U and its conjugate $\overline{U}$, the matrix with (i,j)-entry U_{ij}^*. We will also need to compute integrals of products of entries from U and its adjoint U^*, whose (i,j)-entry we denote $(U^*)_{ij}$ to distinguish from the conjugate $\overline{U}$. To do this we will use the following proposition, which allows us to reduce to the former case. Note that such a formula clearly fails for the classical unitary group.

Proposition 7.4.3. *Let $1 \leq i_1, i_2, \ldots, i_{4m} \leq n$ and $\epsilon_1, \ldots, \epsilon_{2m} \in \{1, *\}$. Then*

$$\psi_n\big((U^{\epsilon_1})_{i_1 i_2} (U^{\epsilon_2})_{i_3 i_4} \cdots (U^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}\big) = \psi_n\big(U_{i_1 i_2}^{\epsilon_1} U_{i_4 i_3}^{\epsilon_2} \cdots U_{i_{4m} i_{4m-1}}^{\epsilon_{2m}}\big).$$

Proof. We will use the fact from [1] that the joint $*$-distribution of $(U_{ij})_{1 \leq i,j \leq n}$ with respect to ψ_n is the same as that of $(zO_{ij})_{1 \leq i,j \leq n}$, where z and (O_{ij}) are random variables in a $*$-probability space (M, τ) such that:

(1) z is $*$-freely independent from $\{O_{ij} : 1 \leq i,j \leq n\}$.

(2) z has a Haar unitary distribution.

(3) (O_{ij}) are self-adjoint, and have the same joint distribution as the generators of the quantum orthogonal group $A_o(n)$.

The joint distribution of (O_{ij}) can also be computed via a Weingarten formula, see [5] for details. The only fact that we will use is that the joint distribution is invariant under transposition, i.e., the families $(O_{ij})_{1 \leq i,j \leq n}$ and $(O_{ji})_{1 \leq i,j \leq n}$ have the same joint distribution.

Now let $\epsilon_1, \ldots, \epsilon_{2m} \in \{1, *\}$. Let $A = \{j : j \text{ is even and } \epsilon_j = *\} \cup \{j : j \text{ is odd and } \epsilon_j = 1\}$, and $B = \{1, \ldots, 2m\} \setminus A$. Let $1 \leq i_1, j_1, \ldots, i_{2m}, j_{2m} \leq n$. For $1 \leq k \leq 2m$, define

$$i_k' = \begin{cases} i_k, & k \in A \\ j_k, & k \in B \end{cases}, \qquad\qquad j_k' = \begin{cases} j_k, & k \in A \\ i_k, & k \in B \end{cases}.$$

We claim that

$$\psi_n\big(U_{i_1j_1}^{\epsilon_1}\cdots U_{i_{2m}j_{2m}}^{\epsilon_{2m}}\big) = \psi_n\big(U_{i'_1j'_1}^{\epsilon_1}\cdots U_{i'_{2m}j'_{2m}}^{\epsilon_{2m}}\big),$$

from which the formula in the statement follows immediately.

As discussed above, we have

$$\psi_n\big(U_{i_1j_1}^{\epsilon_1}\cdots U_{i_{2m}j_{2m}}^{\epsilon_{2m}}\big) = \tau\big((zO_{i_1j_1})^{\epsilon_1}\cdots(zO_{i_{2m}j_{2m}})^{\epsilon_{2m}}\big).$$

Note that the expression $(zO_{i_1j_1})^{\epsilon_1}\cdots(zO_{i_{2m}j_{2m}})^{\epsilon_{2m}}$ can be written as a product of terms of the form $zO_{i_kj_k}$ or $O_{i_kj_k}z^*$, depending if ϵ_k is 1 or $*$. After rewriting the expression in this form, let C be the subset of $\{1,\dots,4m\}$ consisting of those indices corresponding to z or z^*, and let D be its complement. Explicitly, if $\epsilon_k = 1$ then $2k-1$ is in C and $2k$ is in D, and if $\epsilon_k = *$ then $2k$ is in C and $2k-1$ is in D. Given partitions $\alpha,\beta \in NC(2m)$, let $\Theta(\alpha,\beta) \in P(4m)$ be given by partitioning C according to α and D according to β. By freeness, we have

$$\tau\big((zO_{i_1j_1})^{\epsilon_1}\cdots(zO_{i_{2m}j_{2m}})^{\epsilon_{2m}}\big) = \sum_{\substack{\alpha,\beta\in NC(2m)\\ \Theta(\alpha,\beta)\in NC(4m)}} \kappa_\alpha[z^{\epsilon_1},\dots,z^{\epsilon_{2m}}]\kappa_\beta[O_{i_1j_1},\dots,O_{i_{2m}j_{2m}}].$$

Now since Haar unitaries are R-diagonal, we have in particular that $\kappa_\alpha[z^{\epsilon_1},\dots,z^{\epsilon_{2m}}] = 0$ unless each block of α contains an even number of elements. So assume that α has this property, we claim that if β is such that $\Theta(\alpha,\beta)$ is noncrossing, then β does not join any element of A with an element of B. Indeed, suppose that β joins $k_1 < k_2$ and that one of k_1,k_2 is in A and the other is in B. If k_1,k_2 have the same parity, then it follows that one of $\epsilon_{k_1},\epsilon_{k_2}$ is a 1 while the other is a $*$. Suppose that $\epsilon_{k_1} = 1, \epsilon_{k_2} = *$, the other case is similar. Then we have $2k_1$ connected to $2k_2-1$ in $\Theta(\alpha,\beta)$. Since $\Theta(\alpha,\beta)$ is noncrossing, α cannot join any element of $\{k_1+1,\dots,k_2-1\}$ to an element outside of this set. But since this set contains an odd number of elements, we obtain a contradiction to the choice of α.

If k_1,k_2 have different parity, then it follows that $\epsilon_{k_1} = \epsilon_{k_2}$. Suppose that $\epsilon_{k_1} = \epsilon_{k_2} = 1$, the other case is similar. Then $2k_1$ is connected to $2k_2$ in $\Theta(\alpha,\beta)$. It follows that α cannot connect any element of $\{k_1+1,\dots,k_2\}$ to an element outside of this set, and again this set has an odd number of elements which contradicts the choice of α.

So the only nonzero terms appearing in the expression above come from $\beta \in NC(2m)$ which split into noncrossing partitions π of A and σ of B. In this case, if $A = (a_1 < \cdots < a_s)$ and $B = (b_1 < \cdots < b_r)$, we have

$$\begin{aligned}\kappa_\beta[O_{i_1j_1},\dots,O_{i_{2m}j_{2m}}] &= \kappa_\pi[O_{i_{a_1}j_{a_1}},\dots,O_{i_{a_s}j_{a_s}}]\kappa_\sigma[O_{i_{b_1}j_{b_1}},\dots,O_{i_{b_r}j_{b_r}}]\\ &= \kappa_\pi[O_{i_{a_1}j_{a_1}},\dots,O_{i_{a_s}j_{a_s}}]\kappa_\sigma[O_{j_{b_1}i_{b_1}},\dots,O_{j_{b_r}i_{b_r}}]\\ &= \kappa_\beta[O_{i'_1j'_1},\dots,O_{i'_{2m}j'_{2m}}],\end{aligned}$$

where we have used the invariance of the distribution of (O_{ij}) under transposition.

Putting this all together, we have

$$\begin{aligned}\psi_n\big(U^{\epsilon_1}_{i_1j_1}\cdots U^{\epsilon_{2m}}_{i_{2m}j_{2m}}\big) &= \tau\big((zO_{i_1j_1})^{\epsilon_1}\cdots(zO_{i_{2m}j_{2m}})^{\epsilon_{2m}}\big)\\ &= \sum_{\substack{\alpha,\beta\in NC(2m)\\ \Theta(\alpha,\beta)\in NC(4m)}} \kappa_\alpha[z^{\epsilon_1},\ldots,z^{\epsilon_{2m}}]\kappa_\beta[O_{i_1j_1},\ldots,O_{i_{2m}j_{2m}}]\\ &= \sum_{\substack{\alpha,\beta\in NC(2m)\\ \Theta(\alpha,\beta)\in NC(4m)}} \kappa_\alpha[z^{\epsilon_1},\ldots,z^{\epsilon_{2m}}]\kappa_\beta[O_{i_1'j_1'},\ldots,O_{i_{2m}'j_{2m}'}]\\ &= \tau\big((zO_{i_1'j_1'})^{\epsilon_1}\cdots(zO_{i_{2m}'j_{2m}'})^{\epsilon_{2m}}\big)\\ &= \psi_n\big(U^{\epsilon_1}_{i_1'j_1'}\cdots U^{\epsilon_{2m}}_{i_{2m}'j_{2m}'}\big)\end{aligned}$$

as desired.

□

We can now extend this result to the free product $A_u(n)^{*\infty}$.

Corollary 7.4.4. *Let $l_1,\ldots,l_{2m}\in\mathbb{N}$, $\epsilon_1,\ldots,\epsilon_{2m}\in\{1,*\}$ and $1\le i_1,j_1,\ldots,i_{2m},j_{2m}\le n$. In $A_u(n)^{*\infty}$, we have*

$$\psi_n^{*\infty}\big((U(l_1)^{\epsilon_1})_{i_1i_2}(U(l_2)^{\epsilon_2})_{i_3i_4}\cdots(U(l_{2m})^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}\big) = \psi_n^{*\infty}\big(U(l_1)^{\epsilon_1}_{i_1i_2}U(l_2)^{\epsilon_2}_{i_4i_3}\cdots U(l_{2m})^{\epsilon_{2m}}_{i_{4m}i_{4m-1}}\big).$$

Proof. First we claim that in $A_u(n)$, we have

$$\kappa^{(2m)}[(U^{\epsilon_1})_{i_1i_2},(U^{\epsilon_2})_{i_3i_4},\ldots,(U^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}] = \kappa^{(2m)}[U^{\epsilon_1}_{i_1i_2},U^{\epsilon_2}_{i_4i_3},\ldots,U^{\epsilon_{2m}}_{i_{4m}i_{4m-1}}].$$

(Note that any cumulant of odd length is zero by Theorem 7.4.1).

Indeed, we have

$$\begin{aligned}&\kappa^{(2m)}[(U^{\epsilon_1})_{i_1i_2},(U^{\epsilon_2})_{i_3i_4},\ldots,(U^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}]\\ &\qquad= \sum_{\sigma\in NC(2m)}\mu_{2m}(\sigma,1_{2m})\prod_{V\in\sigma}\psi_n(V)[(U^{\epsilon_1})_{i_1i_2},(U^{\epsilon_2})_{i_3i_4},\ldots,(U^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}].\end{aligned}$$

Now it is clear from Theorem 7.4.1 that

$$\psi_n(V)[(U^{\epsilon_1})_{i_1i_2},(U^{\epsilon_2})_{i_3i_4},\ldots,(U^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}] = 0$$

unless V has an even number of elements. So the nonzero terms in the expression above come from those $\sigma\in NC(2m)$ for which every block has as even number of elements. For such a σ, the noncrossing condition implies that each block $V=(l_1<\cdots<l_s)$ must be alternating in parity. By Proposition 7.4.3 we have

$$\begin{aligned}\psi_n(V)[(U^{\epsilon_1})_{i_1i_2},(U^{\epsilon_2})_{i_3i_4},\ldots,(U^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}] &= \psi_n\big((U^{\epsilon_{l_1}})_{i_{2l_1-1}i_{2l_1}}(U^{\epsilon_{l_2}})_{i_{2l_2-1}i_{2l_2}}\cdots(U^{\epsilon_{l_s}})_{i_{2l_s-1}i_{2l_s}}\big)\\ &= \psi_n\big(U^{\epsilon_{l_1}}_{i_{2l_1-1}i_{2l_1}}U^{\epsilon_{l_2}}_{i_{2l_2}i_{2l_2-1}}\cdots U^{\epsilon_{l_s}}_{i_{2l_s}i_{2l_s-1}}\big)\\ &= \psi_n\big(U^{\epsilon_{l_1}}_{i_{2l_1}i_{2l_1-1}}U^{\epsilon_{l_2}}_{i_{2l_2-1}i_{2l_2}}\cdots U^{\epsilon_{l_s}}_{i_{2l_s-1}i_{2l_s}}\big),\end{aligned}$$

where the last equation follows from the invariance of the joint $*$-distribution of (U_{ij}) under transposition. It follows that

$$\begin{aligned}
&\kappa^{(2m)}[(U^{\epsilon_1})_{i_1 i_2}, (U^{\epsilon_2})_{i_3 i_4}, \dots, (U^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}] \\
&\quad = \sum_{\sigma \in NC(2m)} \mu_{2m}(\sigma, 1_{2m}) \prod_{V \in \sigma} \psi_n(V)[(U^{\epsilon_1})_{i_1 i_2}, (U^{\epsilon_2})_{i_3 i_4}, \dots, (U^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}] \\
&\quad = \sum_{\sigma \in NC(2m)} \mu_{2m}(\sigma, 1_{2m}) \prod_{V \in \sigma} \psi_n(V)[U^{\epsilon_1}_{i_1 i_2}, U^{\epsilon_2}_{i_4 i_3}, \dots, U^{\epsilon_{2m}}_{i_{4m} i_{4m-1}}] \\
&\quad = \kappa^{(2m)}[U^{\epsilon_1}_{i_1 i_2}, U^{\epsilon_2}_{i_4 i_3}, \dots, U^{\epsilon_{2m}}_{i_{4m} i_{4m-1}}]
\end{aligned}$$

as claimed.

Now by free independence, in $A_u(n)^{*\infty}$ we have

$$\begin{aligned}
&\psi_n^{*\infty}\big((U(l_1)^{\epsilon_1})_{i_1 i_2} (U(l_2)^{\epsilon_2})_{i_3 i_4} \cdots (U(l_{2m})^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}\big) \\
&\quad = \sum_{\substack{\sigma \in NC(2m) \\ \sigma \leq \ker \mathbf{l}}} \prod_{V \in \sigma} \kappa(V)[(U(l_1)^{\epsilon_1})_{i_1 i_2}, (U(l_2)^{\epsilon_2})_{i_3 i_4}, \dots, (U(l_{2m})^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}].
\end{aligned}$$

Since $\kappa(V)$ is zero unless V has an even number of elements, the only terms which contribute to the sum above come again from $\sigma \in NC(2m)$ for which each block has an even number of elements. From the previous claim, we have

$$\begin{aligned}
&\kappa(V)[(U(l_1)^{\epsilon_1})_{i_1 i_2}, (U(l_2)^{\epsilon_2})_{i_3 i_4}, \dots, (U(l_{2m})^{\epsilon_{2m}})_{i_{4m-1} i_{4m}}] \\
&\quad = \kappa(V)[U(l_1)^{\epsilon_1}_{i_1 i_2}, U(l_2)^{\epsilon_2}_{i_4 i_3}, \dots, U(l_{2m})^{\epsilon_{2m}}_{i_{4m} i_{4m-1}}]
\end{aligned}$$

for each block $V \in \sigma$, and the result follows immediately.

□

Remark 7.4.5. We will now give an estimate on the asymptotic behavior of the entries of $W_{\epsilon n}$ as $n \to \infty$. This improves the estimate given in [5]. Note that by taking $\epsilon = 1 * \cdots 1*$, this estimate also applies to the quantum orthogonal group, see [5].

Theorem 7.4.6. *Let $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$. Let $\pi, \sigma \in NC^\epsilon(m)$. Then*

$$W_{\epsilon n}(\widetilde{\pi}, \widetilde{\sigma}) = O(n^{2|\pi \vee \sigma| - |\pi| - |\sigma| - m}).$$

Moreover,

$$n^{m + |\sigma| - |\pi|} W_{\epsilon n}(\widetilde{\pi}, \widetilde{\sigma}) = \mu_m(\sigma, \pi) + O(n^{-2}),$$

where μ_m is the Möbius function on $NC(m)$.

Proof. We use the method from Theorem 4.3.1.

First observe that

$$G_{\epsilon n} = \Theta_{\epsilon n}^{1/2}(1 + B_{\epsilon n})\Theta_{\epsilon n}^{1/2},$$

where

$$\Theta_{\epsilon n}(\pi,\sigma)=\begin{cases} n^m, & \pi=\sigma \\ 0, & \pi\neq\sigma \end{cases},$$

$$B_{\epsilon n}(\pi,\sigma)=\begin{cases} 0, & \pi=\sigma \\ n^{|\pi\vee\sigma|-m}, & \pi\neq\sigma \end{cases}.$$

Note that the entries of $B_{\epsilon n}$ are $O(n^{-1})$, in particular for n large we have the geometric series expansion

$$(1+B_{\epsilon n})^{-1}=1-B_{\epsilon n}+\sum_{l\geq 1}(-1)^{l+1}B_{\epsilon n}^{l+1}.$$

Hence

$$W_{\epsilon n}(\widetilde{\pi},\widetilde{\sigma})=\sum_{l\geq 1}(-1)^{(l+1)}(\Theta_{\epsilon n}^{-1/2}B_{\epsilon n}^{l+1}\Theta_{\epsilon n}^{-1/2})(\widetilde{\pi},\widetilde{\sigma})+\begin{cases} n^{-m} & \pi=\sigma, \\ -n^{|\widetilde{\pi}\vee\widetilde{\sigma}|-2m} & \pi\neq\sigma. \end{cases}$$

Now for $l\geq 1$, we have

$$(\Theta_{\epsilon n}^{-1/2}B_{\epsilon n}^{l+1}\Theta_{\epsilon n}^{-1/2})(\widetilde{\pi},\widetilde{\sigma})=\sum_{\substack{\nu_1,\dots,\nu_l\in NC^\epsilon(m)\\ \pi\neq\nu_1\neq\cdots\neq\nu_l\neq\sigma}} n^{|\widetilde{\pi}\vee\widetilde{\nu}_1|+|\widetilde{\nu}_1\vee\widetilde{\nu}_2|+\cdots+|\widetilde{\nu}_l\vee\widetilde{\sigma}|-(l+2)m}.$$

Now we claim that

$$\begin{aligned}|\widetilde{\pi}\vee\widetilde{\nu}_1|+\cdots+|\widetilde{\nu}_l\vee\widetilde{\sigma}| &\leq |\widetilde{\pi}\vee\widetilde{\sigma}|+|\widetilde{\nu}_1|+\cdots+|\widetilde{\nu}_l| \\ &\leq |\widetilde{\pi}\vee\widetilde{\sigma}|+l\cdot m,\end{aligned}$$

from which the first statement follows from the above equation and Theorem 7.3.7.

Indeed, the case $l=1$ follows from the semi-modular condition:

$$\begin{aligned}|\widetilde{\pi}\vee\widetilde{\nu}_1|+|\widetilde{\nu}_1\vee\widetilde{\sigma}| &\leq |(\widetilde{\pi}\vee\widetilde{\nu}_1)\vee(\widetilde{\nu}_1\vee\widetilde{\sigma})|+|(\widetilde{\pi}\vee\widetilde{\nu}_1)\wedge(\widetilde{\nu}_1\vee\widetilde{\sigma})| \\ &\leq |\widetilde{\pi}\vee\widetilde{\sigma}|+|\widetilde{\nu}_1| \\ &= |\widetilde{\pi}\vee\widetilde{\sigma}|+m.\end{aligned}$$

The general case follows easily from induction on l.

For the second part, apply Theorem 7.3.7 to find that

$$\begin{aligned}|\widetilde{\pi}\vee\widetilde{\nu}_1|+\cdots+|\widetilde{\nu}_l\vee\widetilde{\sigma}| &= 2(|\nu_1\vee\nu_2|+\cdots+|\nu_l\vee\sigma|-|\nu_1|-\cdots-|\nu_l|) \\ &\quad +2|\pi\vee\nu_1|-|\pi|-|\sigma|+(l+1)m \\ &\leq |\pi|-|\sigma|+(l+1)m,\end{aligned}$$

where equality holds if $\sigma<\nu_l<\cdots<\nu_1<\pi$ and otherwise the difference is at least 2. It then follows from the equation above that $n^{m+|\sigma|-|\pi|}W_{\epsilon n}(\widetilde{\pi},\widetilde{\sigma})$ is equal to

$$\begin{cases} 0, & \sigma\not\leq\pi \\ 1, & \pi=\sigma \\ -1+\sum_{l=1}^{\infty}(-1)^{l+1}|\{(\nu_1,\dots,\nu_l)\in(NC^\epsilon(m))^l:\sigma<\nu_l<\cdots<\nu_1<\pi\}|, & \sigma<\pi \end{cases},$$

up to $O(n^{-2})$. Since $NC^\epsilon(m)$ is closed under taking intervals in $NC(m)$, this is equal to $\mu_m(\sigma,\pi)$. □

As a corollary, we can give an estimate on the free cumulants of the generators U_{ij} of $A_u(n)$. (Note that the cumulants of odd length are all zero since the generators have an even joint distribution).

Corollary 7.4.7. *Let $\epsilon_1,\dots,\epsilon_{2m}\in\{1,*\}$ and $i_1,j_1,\dots,i_{2m},j_{2m}\in\mathbb{N}$. For $\omega\in NC(2m)$, we have for the moment functions*

$$\psi_n^{(\omega)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}]=\sum_{\substack{\sigma,\pi\in NC^\epsilon(m)\\ \widetilde{\pi}\le\ker\mathbf{i}\wedge\omega\\ \widetilde{\sigma}\le\ker\mathbf{j}\wedge\omega}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2})),$$

and for the cumulant functions

$$\kappa^{(\omega)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}]=\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \widetilde{\pi}\le\ker\mathbf{i}\\ \widetilde{\sigma}\le\ker\mathbf{j}\\ \widetilde{\pi}\vee_{nc}\widetilde{\sigma}=\omega}} n^{|\pi|-|\sigma|-m}(\mu_m(\pi,\sigma)+O(n^{-2})).$$

Proof. First note that $\psi_n^{(\omega)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}]=0$ unless $\omega\in NC_h(2m)$, i.e., unless each block of ω has an even number of elements. So suppose this is the case, then by Lemma 7.3.10 we have $\omega=\widetilde{\alpha}\vee\widetilde{\beta}$ for some $\alpha,\beta\in NC(m)$ with $\alpha\le\beta$. By the Weingarten formula, we have

$$\psi_n^{(\omega)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}]=\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \widetilde{\pi}\le\ker\mathbf{i}\wedge\omega\\ \widetilde{\sigma}\le\ker\mathbf{j}\wedge\omega}}\prod_{V\in\omega}W_{\epsilon|_V n}(\widetilde{\pi}|_V,\widetilde{\sigma}|_V).$$

Let $V=(l_1<\dots<l_s)$ be a block of ω. In order to apply Theorem 7.4.6 we have to write $\widetilde{\pi}|_V$ and $\widetilde{\sigma}|_V$ as $\widetilde{\pi_V}$ and $\widetilde{\sigma_V}$, respectively, for some $\pi_V,\sigma_V\in NC(|V|/2)$. Since $\mu_{|V|/2}(\sigma_V,\pi_V)=\mu_{|V|}(\widehat{\sigma_V},\widehat{\pi_V})$, it suffices to recover the doubled versions $\widehat{\sigma_V},\widehat{\pi_V}$ from $\widetilde{\pi}|_V$ and $\widetilde{\sigma}|_V$. But this can be achieved as follows.

$$\widehat{\pi_V}=\widetilde{\pi_V}\vee\hat{0}_{|V|/2}=\widetilde{\pi}|_V\vee\{(l_1,l_2),\dots,(l_{s-1},l_s)\}.$$

So it remains to write $\{(l_1,l_2),\dots,(l_{s-1},l_s)\}$ intrinsically in terms of ω.

Recall from Lemma 7.3.6 that we have $K(\omega)=\alpha\wr K(\beta)$. It follows that for $1\le r\le s$ such that l_r is odd, α has a block whose least element is $\frac{l_r+1}{2}$ and greatest element is $\frac{l_{r+1}}{2}$. Therefore l_r is joined to l_{r+1} in $\widetilde{\alpha}$. So if l_1 is odd, then $\widetilde{\alpha}|_V$ is equal to $\{(l_1,l_2),(l_3,l_4),\dots,(l_{s-1},l_s)\}$. In this case, from Theorem 7.4.6 we have

$$W_{\epsilon|_V n}(\widetilde{\pi}|_V,\widetilde{\sigma}|_V)=n^{\left|\widetilde{\pi}|_V\vee\widetilde{\alpha}|_V\right|-\left|\widetilde{\sigma}|_V\vee\widetilde{\alpha}|_V\right|-|V|/2}(\mu_{|V|}(\widetilde{\sigma}|_V\vee\widetilde{\alpha}|_V,\widetilde{\pi}|_V\vee\widetilde{\alpha}|_V)+O(n^{-2})).$$

On the other hand, if l_1 is even then $\widetilde{\alpha}|_V=\{(l_1,l_s),(l_2,l_3),\dots,(l_{s-2},l_{s-1})\}$. In this case we have

$$\begin{aligned}W_{\epsilon|_V n}(\widetilde{\pi}|_V,\widetilde{\sigma}|_V)&=n^{\left|\widetilde{\sigma}|_V\vee\overrightarrow{\widetilde{\alpha}|_V}\right|-\left|\widetilde{\pi}|_V\vee\overrightarrow{\widetilde{\alpha}|_V}\right|-|V|/2}(\mu_{|V|}(\widetilde{\pi}|_V\vee\overrightarrow{\widetilde{\alpha}|_V},\widetilde{\sigma}|_V\vee\overrightarrow{\widetilde{\alpha}|_V})+O(n^{-2}))\\&=n^{\left|\overleftarrow{\widetilde{\sigma}|_V}\vee\widetilde{\alpha}|_V\right|-\left|\overleftarrow{\widetilde{\pi}|_V}\vee\widetilde{\alpha}|_V\right|-|V|/2}(\mu_{|V|}(\overleftarrow{\widetilde{\pi}|_V}\vee\widetilde{\alpha}|_V,\overleftarrow{\widetilde{\sigma}|_V}\vee\widetilde{\alpha}|_V)+O(n^{-2})),\end{aligned}$$

where here the arrows act on the legs of V. Since this corresponds, by Lemma 7.3.5, to the Kreweras complement on $NC_{|V|/2}$, we have

$$\left|\overleftarrow{\widetilde{\sigma}|_V} \vee \widetilde{\alpha}|_V\right| = |V|/2 + 1 - \left|\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V\right|$$

and

$$\mu_{|V|}(\overleftarrow{\widetilde{\pi}|_V} \vee \widetilde{\alpha}|_V, \overleftarrow{\widetilde{\sigma}|_V} \vee \widetilde{\alpha}|_V) = \mu_{|V|}(\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V, \widetilde{\pi}|_V \vee \widetilde{\alpha}|_V).$$

So it follows that, as in previous case, we have

$$W_{\epsilon|_V n}(\widetilde{\pi}|_V, \widetilde{\sigma}|_V) = n^{\left|\widetilde{\pi}|_V \vee \widetilde{\alpha}|_V\right| - \left|\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V\right| - |V|/2}(\mu_{|V|}(\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V, \widetilde{\pi}|_V \vee \widetilde{\alpha}|_V) + O(n^{-2})).$$

Therefore,

$$\begin{aligned}
&\psi_n^{(\omega)}[U_{i_1 j_1}^{\epsilon_1}, \ldots, U_{i_{2m} j_{2m}}^{\epsilon_{2m}}] \\
&\quad = \sum_{\substack{\sigma,\pi \in NC^\epsilon(m) \\ \widetilde{\pi} \leq \ker \mathbf{i} \wedge \omega \\ \widetilde{\sigma} \leq \ker \mathbf{j} \wedge \omega}} \prod_{V \in \omega} n^{\left|\widetilde{\pi}|_V \vee \widetilde{\alpha}|_V\right| - \left|\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V\right| - |V|/2}(\mu_{|V|}(\widetilde{\sigma}|_V \vee \widetilde{\alpha}|_V, \widetilde{\pi}|_V \vee \widetilde{\alpha}|_V) + O(n^{-2})) \\
&\quad = \sum_{\substack{\sigma,\pi \in NC^\epsilon(m) \\ \widetilde{\pi} \leq \ker \mathbf{i} \wedge \omega \\ \widetilde{\sigma} \leq \ker \mathbf{j} \wedge \omega}} n^{|\widetilde{\pi} \vee \widetilde{\alpha}| - |\widetilde{\sigma} \vee \widetilde{\alpha}| - m}(\mu_{2m}(\widetilde{\sigma} \vee \widetilde{\alpha}, \widetilde{\pi} \vee \widetilde{\alpha}) + O(n^{-2})),
\end{aligned}$$

where we have used the multiplicativity of the Möbius function on $NC(2m)$.

Now since $\widetilde{\sigma} = \widetilde{\sigma} \vee \widetilde{\sigma} \leq \widetilde{\alpha} \vee \widetilde{\beta}$, taking the Kreweras complement and applying Lemma 7.3.6 gives $\alpha \wr K(\beta) \leq \sigma \wr K(\sigma)$. So we have $\alpha \leq \sigma \leq \beta$. By Theorem 7.3.7, we then have $|\widetilde{\sigma} \vee \widetilde{\alpha}| = |\sigma| + m - |\alpha|$. Also, we have

$$\begin{aligned}
\mu_{2m}(\widetilde{\sigma} \vee \widetilde{\alpha}, \widetilde{\pi} \vee \widetilde{\alpha}) &= \mu_{2m}(K(\widetilde{\pi} \vee \widetilde{\alpha}), K(\widetilde{\sigma} \vee \widetilde{\alpha})) \\
&= \mu_{2m}(\alpha \wr K(\pi), \alpha \wr K(\sigma)) \\
&= \mu_m(K(\pi), K(\sigma)) \\
&= \mu_m(\sigma, \pi).
\end{aligned}$$

Plugging this into the equation above, we have

$$\psi_n^{(\omega)}[U_{i_1 j_1}^{\epsilon_1}, \ldots, U_{i_{2m} j_{2m}}^{\epsilon_{2m}}] = \sum_{\substack{\sigma,\pi \in NC^\epsilon(m) \\ \widetilde{\pi} \leq \ker \mathbf{i} \wedge \omega \\ \widetilde{\sigma} \leq \ker \mathbf{j} \wedge \omega}} n^{|\pi| - |\sigma| - m}(\mu_m(\sigma, \pi) + O(n^{-2})).$$

For the cumulant function this gives

$$\begin{aligned}
\kappa^{(\tau)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}] &= \sum_{\substack{\omega\in NC(2m)\\ \omega\leq\tau}} \mu_{2m}(\omega,\tau)\psi_n^{(\omega)}[U_{i_1j_1}^{\epsilon_1},\dots,U_{i_{2m}j_{2m}}^{\epsilon_{2m}}]\\
&= \sum_{\substack{\omega\in NC(2m)\\ \omega\leq\tau}} \mu_{2m}(\omega,\tau) \sum_{\substack{\sigma,\pi\in NC^{\epsilon}(m)\\ \widetilde{\pi}\leq\ker\mathbf{i}\wedge\omega\\ \widetilde{\sigma}\leq\ker\mathbf{j}\wedge\omega}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2}))\\
&= \sum_{\substack{\sigma,\pi\in NC^{\epsilon}(m)\\ \widetilde{\pi}\leq\ker\mathbf{i}\\ \widetilde{\sigma}\leq\ker\mathbf{j}}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2})) \sum_{\substack{\omega\in NC(2m)\\ \widetilde{\pi}\vee_{nc}\widetilde{\sigma}\leq\omega\leq\tau}} \mu_{2m}(\omega,\tau).
\end{aligned}$$

Since

$$\sum_{\substack{\omega\in NC(2m)\\ \widetilde{\pi}\vee_{nc}\widetilde{\sigma}\leq\omega\leq\tau}} \mu_{2m}(\omega,\tau) = \begin{cases}1, & \widetilde{\pi}\vee_{nc}\widetilde{\sigma}=\tau\\ 0, & \text{otherwise}\end{cases},$$

the result follows.

□

As a corollary, we can give an estimate on the Haar state on the free product $A_u(n)^{*\infty}$.

Corollary 7.4.8. *Let $l_1,\dots,l_{2m}\in\mathbb{N}$, $\epsilon_1,\dots,\epsilon_{2m}\in\{1,*\}$ and $i_1,j_1,\dots,i_{2m},j_{2m}\in\mathbb{N}$. In $A_u(n)^{*\infty}$, we have*

$$\psi_n^{*\infty}\Big(U(l_1)_{i_1j_1}^{\epsilon_1}\cdots U(l_{2m})_{i_{2m}j_{2m}}^{\epsilon_{2m}}\Big) = \sum_{\substack{\pi,\sigma\in NC^{\epsilon}(m)\\ \widetilde{\pi}\leq\ker\mathbf{i}\wedge\ker\mathbf{l}\\ \widetilde{\sigma}\leq\ker\mathbf{j}\wedge\ker\mathbf{l}}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2})).$$

Proof. Since the families $(\{U(l)_{ij}\})_{l\in\mathbb{N}}$ are freely independent, we have by the vanishing of mixed cumulants

$$\psi_n^{*\infty}\Big(U(l_1)_{i_1j_1}^{\epsilon_1}\cdots U(l_{2m})_{i_{2m}j_{2m}}^{\epsilon_{2m}}\Big) = \sum_{\substack{\tau\in NC(2m)\\ \tau\leq\ker\mathbf{l}}} \kappa^{(\tau)}[U(l_1)_{i_1j_1}^{\epsilon_1},\dots,U(l_{2m})_{i_{2m}j_{2m}}^{\epsilon_{2m}}].$$

Since the families $(\{U(l)_{ij}\})_{l\in\mathbb{N}}$ are identically distributed, we have

$$\kappa^{(\tau)}[U(l_1)_{i_1j_1}^{\epsilon_1},\dots,U(l_{2m})_{i_{2m}j_{2m}}^{\epsilon_{2m}}] = \kappa^{(\tau)}[U(1)_{i_1j_1}^{\epsilon_1},\dots,U(1)_{i_{2m}j_{2m}}^{\epsilon_{2m}}]$$

for any $\tau\in NC(2m)$ such that $\tau\leq\ker\mathbf{l}$. Applying the previous corollary, we have

$$\begin{aligned}
\psi_n^{*\infty}\Big(U(l_1)_{i_1j_1}^{\epsilon_1}\cdots U(l_{2m})_{i_{2m}j_{2m}}^{\epsilon_{2m}}\Big) &= \sum_{\substack{\tau\in NC(2m)\\ \tau\leq\ker\mathbf{l}}} \sum_{\substack{\pi,\sigma\in NC^{\epsilon}(m)\\ \widetilde{\pi}\leq\ker\mathbf{i}\\ \widetilde{\sigma}\leq\ker\mathbf{j}\\ \widetilde{\pi}\vee_{nc}\widetilde{\sigma}=\tau}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2}))\\
&= \sum_{\substack{\pi,\sigma\in NC^{\epsilon}(m)\\ \widetilde{\pi}\leq\ker\mathbf{i}\wedge\ker\mathbf{l}\\ \widetilde{\sigma}\leq\ker\mathbf{j}\wedge\ker\mathbf{l}}} n^{|\pi|-|\sigma|-m}(\mu_m(\sigma,\pi)+O(n^{-2})).
\end{aligned}$$

□

7.5 Asymptotic freeness results

Remark 7.5.1. Throughout the first part of this section, the framework will be as follows: $\mathcal{B}$ will be a fixed unital C*-algebra, and $(D_N(i))_{i\in I}$ will be a family of matrices in $M_N(\mathcal{B})$ for $N \in \mathbb{N}$, which is a $\mathcal{B}$-valued probability space with conditional expectation $E_N = \mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}}$. Consider the free product $A_u(N)^{*\infty}$, generated by the entries in the matrices $(U_N(l))_{l\in\mathbb{N}} \in M_N(A_u(N)^{*\infty})$. By a family of freely independent Haar quantum unitary random matrices, independent from $\mathcal{B}$, we will mean the family $(U_N(l) \otimes 1_{\mathcal{B}})_{l\in\mathbb{N}}$ in $M_N(A_u(N)^{*\infty} \otimes \mathcal{B}) = M_N(\mathbb{C}) \otimes A_u(N)^{*\infty} \otimes \mathcal{B}$, which we will still denote by $(U_N(l))_{l\in\mathbb{N}}$. We also identify $D_N(i) = D_N(i) \otimes 1_{A_u(N)^{*\infty}}$ for $i \in I$. We will consider the $\mathcal{B}$-valued joint distribution of the family of sets $(\{U_N(1), U_N(1)^*\}, \{U_N(2), U_N(2)^*\}, \dots, \{D_N(i)|i \in I\})$ with respect to the conditional expectation

$$\psi_N^{*\infty} \otimes E_N = \mathrm{tr}_N \otimes \psi_N^{*\infty} \otimes \mathrm{id}_{\mathcal{B}}.$$

We can now state our main result.

Theorem 7.5.2. *Let $\mathcal{B}$ be a unital C*-algebra, and let $(D_N(i))_{i\in I}$ be a family of matrices in $M_N(\mathcal{B})$ for $N \in \mathbb{N}$. Suppose that there is a finite constant C such that $\|D_N(i)\| \leq C$ for all $i \in I$ and $N \in \mathbb{N}$. Let $(U_N(l))_{l\in\mathbb{N}}$ be a family of freely independent $N \times N$ Haar quantum unitary random matrices, independent from $\mathcal{B}$. Let $(u(l), u(l)^*)_{l\in\mathbb{N}}$ and $(d_N(i))_{i\in I, N\in\mathbb{N}}$ be random variables in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ such that*

(1) $(u(l), u(l)^*)_{l\in\mathbb{N}}$ is free from $(d_N(i))_{i\in I}$ with respect to E for each $N \in \mathbb{N}$..

(2) $(\{u(l), u(l)^*\})_{l\in\mathbb{N}}$ is a free family with respect to E, and $u(l)$ is a Haar unitary, independent from $\mathcal{B}$ for each $l \in \mathbb{N}$.

(3) $(d_N(i))_{i\in I}$ has the same $\mathcal{B}$-valued joint distribution with respect to E as $(D_N(i))_{i\in I}$ has with respect to E_N.

Then for any polynomials $p_1, \dots, p_{2m} \in \mathcal{B}\langle t(i)|i \in I\rangle$, $l_1, \dots, l_{2m} \in \mathbb{N}$ and $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$,

$$\left\|(\psi_N^{*\infty} \otimes E_N)[U_N(l_1)^{\epsilon_1} p_1(D_N) \cdots U_N(l_{2m})^{\epsilon_{2m}} p_{2m}(D_N)] - E[u(l_1)^{\epsilon_1} p_1(d_N) \cdots u(l_{2m})^{\epsilon_{2m}} p_{2m}(d_N)]\right\|$$

is $O(N^{-2})$ as $N \to \infty$.

Observe that Theorem 7.5.2 makes no assumption on the existence of a limiting distribution for $(D_N(i))_{i\in I}$. If one assumes also the existence of a limiting (infinitesimal) $\mathcal{B}$-valued joint distribution, then asymptotic (infinitesimal) freeness follows easily. We will state this as Theorem 7.5.4 below, let us first recall the relevant notions.

Definition 7.5.3. Let $\mathcal{B}$ be a unital C*-algebra, and for each $N \in \mathbb{N}$ let $(D_N(i))_{i\in I}$ be a family of noncommutative random variables in a $\mathcal{B}$-valued probability space $(\mathcal{A}(N), E_N : \mathcal{A}(N) \to \mathcal{B})$.

(1) We say that the joint distribution of $(D_N(i))_{i\in I}$ *converges weakly in norm* if there is a $\mathcal{B}$-linear map $E : \mathcal{B}\langle D(i)|i \in I\rangle \to \mathcal{B}$ such that

$$\lim_{N\to\infty} \big\| E_N[b_0 D_N(i_1)\cdots D_N(i_k)b_k] - E[b_0 D(i_1)\cdots D(i_k)b_k]\big\| = 0$$

for any $i_1,\ldots,i_k \in I$ and $b_0,\ldots,b_k \in \mathcal{B}$. If $\mathcal{B}$ is a von Neumann algebra with faithful, normal trace state τ, we say the the joint distribution of $(D_N(i))_{i\in I}$ *converges weakly in* L^2 if the equation above holds with respect to $|\ |_2$.

(2) If $I = \bigcup_{j\in J} I_j$ is a partition of I, we say that the sequence of sets of random variables $(\{D_N(i)|i \in I_j\})_{j\in J}$ are *asymptotically free with amalgamation over* $\mathcal{B}$ if the sets $(\{D(i)|i \in I_j\})_{j\in J}$ are freely independent with respect to E.

(3) We say that the joint distribution of $(D_N(i))_{i\in I}$ *converges infinitesimally in norm* if there is a $\mathcal{B}$-linear map $E' : \mathcal{B}\langle D(i)|i \in I\rangle \to \mathcal{B}$ such that

$$\lim_{N\to\infty} N\Big\{E_N[b_0 D_N(i_1)\cdots D_N(i_k)b_k] - E[b_0 D(i_1)\cdots D(i_k)b_k]\Big\} = E'[b_0 D(i_1)\cdots D(i_k)b_k]$$

with convergence in norm, for any $b_0,\ldots,b_k \in \mathcal{B}$ and $i_1,\ldots,i_k \in I$. If $\mathcal{B}$ is a von Neumann algebra with faithful, normal trace state τ, we say the the joint distribution of $(D_N(i))_{i\in I}$ *converges infinitesimally in* L^2 if the equation above holds with respect to $|\ |_2$.

(4) If $I = \bigcup_{j\in J} I_j$ is a partition of I, we say that the sequence of sets of random variables $(\{D_N(i)|i \in I_j\})_{j\in J}$ are *asymptotically infinitesimally free with amalgamation over* $\mathcal{B}$ if the sets $(\{D(i)|i \in I_j\})_{j\in J}$ are infinitesimally freely independent with respect to (E, E').

Theorem 7.5.4. *Let $\mathcal{B}$ be a unital C*-algebra, and let $(D_N(i))_{i\in I}$ be a family of matrices in $M_N(\mathcal{B})$ for $N \in \mathbb{N}$. Suppose that there is a finite constant C such that $\|D_N(i)\| \le C$ for all $i \in I$ and $N \in \mathbb{N}$. For each $N \in \mathbb{N}$, let $(U_N(l))_{l\in\mathbb{N}}$ be a family of freely independent $N \times N$ Haar quantum unitary random matrices, independent from $\mathcal{B}$.*

(1) If the joint distribution of $(D_N(i))_{i\in I}$ converges weakly (in norm or in L^2 with respect to a faithful trace), then the sets

$$(\{U_N(1), U_N(1)^*\}, \{U_N(2), U_N(2)^*\}, \ldots, \{D_N(i)|i \in I\})$$

are asymptotically free with amalgamation over $\mathcal{B}$ as $N \to \infty$.

(2) If the joint distribution of $(D_N(i))_{i\in I}$ converges infinitesimally (in norm or in L^2 with respect to a faithful trace), then the sets

$$(\{U_N(1), U_N(1)^*\}, \{U_N(2), U_N(2)^*\}, \ldots, \{D_N(i)|i \in I\})$$

are asymptotically infinitesimally free with amalgamation over $\mathcal{B}$ as $N \to \infty$.

Remark 7.5.5. Theorem 7.5.4 follows immediately from Theorem 7.5.2 and Proposition 7.2.3. The proof of Theorem 7.5.2 will require some preparation, we begin by computing the limiting distribution appearing in the statement.

Proposition 7.5.6. *Let $(u(l), u(l)^*)_{l\in\mathbb{N}}$ and $(d_N(i))_{i\in I, N\in\mathbb{N}}$ be random variables in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ such that*

(1) $(u(l), u(l)^*)_{l\in\mathbb{N}}$ is free from $(d_N(i))_{i\in I}$ with respect to E for each $N \in \mathbb{N}$..

(2) $(\{u(l), u(l)^*\})_{l\in\mathbb{N}}$ is a free family with respect to E, and $u(l)$ is a Haar unitary, independent from $\mathcal{B}$ for each $l \in \mathbb{N}$.

Let $a(1), \dots, a(2m)$ be in the algebra generated by $\mathcal{B}$ and $\{d(i)|i \in I\}$, and let $l_1, \dots, l_{2m} \in \mathbb{N}$ and $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$. Then

$$E[u(l_1)^{\epsilon_1} a(1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m)] = \sum_{\substack{\pi,\sigma\in NC^{\epsilon}(m)\\ \sigma\leq\pi \\ \widetilde{\pi}\vee\widetilde{\sigma}\leq\ker\mathbf{l}}} \mu_m(\sigma, \pi) E^{(\sigma \wr K(\pi))}[a(1), \dots, a(2m)].$$

Note that elements of the form appearing in the statement of the proposition span the algebra generated by $(u(l), u(l)^*)_{l\in\mathbb{N}}$ and $(d(i))_{i\in I}$, and so this indeed determines the joint distribution.

Proof. We have

$$E[u(l_1)^{\epsilon_1} a(1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m)] = \sum_{\alpha\in NC(4m)} \kappa_E^{\alpha}[u(l_1)^{\epsilon_1}, a(1), \dots, a(2m)].$$

By freeness, the only non-vanishing cumulants appearing above are those of the form $\tau \wr \gamma$, where $\tau, \gamma \in NC(2m)$, $\tau \leq \ker\mathbf{l}$ and $\gamma \leq K(\tau)$. So we have

$$E[u(l_1)^{\epsilon_1} a(1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m)] = \sum_{\substack{\tau\in NC(2m)\\ \tau\leq\ker\mathbf{l}}} \sum_{\substack{\gamma\in NC(2m)\\ \gamma\leq K(\tau)}} \kappa_E^{(\tau\wr\gamma)}[u(l_1)^{\epsilon_1}, a(1), \dots, a(2m)].$$

Since the expectation of any polynomial in $(u(l), u(l)^*)_{l\in\mathbb{N}}$ with complex coefficients is scalar-valued, it follows that

$$\begin{aligned} &E[u(l_1)^{\epsilon_1} a(1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m)] \\ &\quad= \sum_{\substack{\tau\in NC(2m)\\ \tau\leq\ker\mathbf{l}}} \kappa_E^{(\tau)}[u(l_1)^{\epsilon_1}, \dots, u(l_{2m})^{\epsilon_{2m}}] \sum_{\substack{\gamma\in NC(2m)\\ \gamma\leq K(\tau)}} \kappa_E^{(\gamma)}[a(1), \dots, a(2m)] \\ &\quad= \sum_{\substack{\tau\in NC(2m)\\ \tau\leq\ker\mathbf{l}}} \kappa_E^{(\tau)}[u(l_1)^{\epsilon_1}, \dots, u(l_{2m})^{\epsilon_{2m}}] E^{(K(\tau))}[a(1), \dots, a(2m)]. \end{aligned}$$

Since Haar unitaries are R-diagonal ([43, Example 15.4]), we have

$$\kappa_E^{(\tau)}[u(l_1)^{\epsilon_1}, \dots, u(l_{2m})^{\epsilon_{2m}}] = 0$$

unless $\tau \in NC_h^\epsilon(2m)$. By Lemmas 7.3.10 and 7.3.6, we have

$$E[u(l_1)^{\epsilon_1}a(1)\cdots u(l_{2m})^{\epsilon_{2m}}a(2m)] = \sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \sigma\leq\pi\\ \widetilde{\sigma}\vee\widetilde{\pi}\leq\ker\mathbf{l}}} \kappa_E^{(\widetilde{\sigma}\vee\widetilde{\pi})}[u(l_1)^{\epsilon_1},\dots,u(l_{2m})^{\epsilon_{2m}}]E^{(\sigma\wr K(\pi))}[a(1),\dots,a(2m)].$$

So it remains only to show that if $\sigma,\pi \in NC^\epsilon(m)$ and $\sigma \leq \pi$ then

$$\mu_m(\sigma,\pi) = \kappa_E^{(\widetilde{\sigma}\vee\widetilde{\pi})}[u(l_1)^{\epsilon_1},\dots,u(l_{2m})^{\epsilon_{2m}}].$$

Since the Möbius function is multiplicative on $NC(m)$, we have

$$\mu_m(\sigma,\pi) = \prod_{W\in\pi}\mu_{|W|}(\sigma|_W, 1_W),$$

and so it suffices to consider the case $\pi = 1_m$.

By [43, Proposition 15.1],

$$\kappa_E^{(\widetilde{\sigma}\vee\widetilde{1_m})}[u(l_1)^{\epsilon_1},\dots,u(l_{2m})^{\epsilon_{2m}}] = \prod_{V\in\widetilde{\sigma}\vee\widetilde{1_m}}(-1)^{|V|/2-1}C_{|V|/2-1},$$

where C_n is the n-th Catalan number. Since

$$\widetilde{\sigma}\vee\widetilde{1_m} = \overrightarrow{\overleftarrow{\widetilde{\sigma}}\vee\overleftarrow{\widetilde{1}_m}} = \overrightarrow{\widetilde{K(\sigma)}\vee\widehat{0_m}} = \overrightarrow{\widehat{K(\sigma)}},$$

we have

$$\kappa_E^{(\widetilde{\sigma}\vee\widetilde{1_m})}[u(l_1)^{\epsilon_1},\dots,u(l_{2m})^{\epsilon_{2m}}] = \prod_{W\in K(\sigma)}(-1)^{|W|-1}C_{|W|-1}.$$

On the other hand, we have

$$\begin{aligned}\mu_m(\sigma,1_m) &= \mu_m(0_m,K(\sigma))\\ &= \prod_{W\in K(\sigma)}\mu_{|W|}(0_W,1_W)\\ &= \prod_{W\in K(\sigma)}(-1)^{|W|-1}C_{|W|-1},\end{aligned}$$

where we have used the formula for $\mu_m(0_m,1_m)$ from [43, Proposition 10.15]. □

Proposition 7.5.7. *Let $\mathcal{B}$ be a unital algebra, $A(1),\dots,A(2m)\in M_N(\mathcal{B})$ and $\pi,\sigma\in NC(m)$. Let $E_N = \mathrm{tr}_N\otimes\mathrm{id}_{\mathcal{B}}$. If $\sigma\leq\pi$, then*

$$N^{|\sigma|+|K(\pi)|}E_N^{(\sigma\wr K(\pi))}[A(1),\dots,A(2m)] = \sum_{\substack{1\leq j_1,\dots,j_{2m}\leq N\\ \widetilde{\sigma}\leq\ker\mathbf{j}}}\ \sum_{\substack{1\leq i_1,\dots,i_{2m}\leq N\\ \widetilde{K(\pi)}\leq\ker\mathbf{i}}} A(1)_{j_1j_2}A(2)_{i_1i_2}\cdots A(2m)_{i_{2m-1}i_{2m}}.$$

Proof. First observe that the sum above can be rewritten as

$$\sum_{\substack{1\le i_1,\dots,i_{4m}\le N\\ \widetilde{\sigma\wr K(\pi)}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots A(2m)_{i_{4m-1}i_{4m}}.$$

So this will follow from the formula

$$\sum_{\substack{1\le i_1,\dots,i_{2m}\le N\\ \widetilde{\sigma}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots A(m)_{i_{2m-1}i_{2m}} = N^{|\sigma|}E_N^{(\sigma)}[A(1),\dots,A(m)]$$

for any $\sigma\in NC(m)$.

We will prove this by induction on the number of blocks of m. If $\sigma=1_m$ has only one block, then we have

$$\begin{aligned}\sum_{\substack{1\le i_1,\dots,i_{2m}\le N\\ \widetilde{\sigma}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots A(m)_{i_{2m-1}i_{2m}} &= \sum_{1\le i_1,\dots,i_m\le N} A(1)_{i_1 i_2}A(2)_{i_2 i_3}\cdots A(m)_{i_m i_1}\\ &= N\cdot E_N(A(1)\cdots A(m)).\end{aligned}$$

Suppose now that $V=\{l+1,\dots,l+s\}$ is an interval of σ. Then

$$\begin{aligned}&\sum_{\substack{1\le i_1,\dots,i_{2m}\le N\\ \widetilde{\sigma}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots A(m)_{i_{2m-1}i_{2m}}\\ &= \sum_{\substack{1\le i_1,\dots,i_{2l-2},\\ i_{2(l+s)+1},\dots,i_{2m}\le N\\ \widetilde{\sigma\setminus V}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots\left(\sum_{1\le j_1,\dots,j_s\le N} A(l+1)_{j_1 j_2}\cdots A(l+s)_{j_s j_1}\right)\cdots A(m)_{i_{2m-1}i_{2m}}\\ &= \sum_{\substack{1\le i_1,\dots,i_{2l-2},i_{2(l+s)+1},\dots,i_{2m}\le N\\ \widetilde{\sigma\setminus V}\le \ker \mathbf{i}}} A(1)_{i_1 i_2}\cdots\big(N\cdot E_N(A(l+1)\cdots A(l+s))\big)\cdots A(m)_{i_{2m-1}i_{2m}},\end{aligned}$$

which by induction is equal to

$$N^{|\sigma|}E_N^{(\sigma\setminus V)}[A(1),\dots,A(l)E_N(A(l+1)\cdots A(l+s)),\dots,A(m)] = N^{|\sigma|}E_N^{(\sigma)}[A(1),\dots,A(m)].$$

□

Remark 7.5.8. We will also need to control the sum appearing in the proposition above for $\sigma,\pi\in$

$NC(m)$ with $\sigma \not\leq \pi$. If $\mathcal{B}$ is commutative this poses no difficulty, as then

$$\begin{aligned}&\sum_{\substack{1\leq j_1,\ldots,j_{2m}\leq N\\ \widetilde{\sigma}\leq\ker\mathbf{j}}}\sum_{\substack{1\leq i_1,\ldots,i_{2m}\leq N\\ \widetilde{K(\pi)}\leq\ker\mathbf{i}}} A(1)_{j_1j_2}A(2)_{i_1i_2}\cdots A(2m)_{i_{2m-1}i_{2m}}\\ &=\Bigg(\sum_{\substack{1\leq j_1,\ldots,j_{2m}\leq N\\ \widetilde{\sigma}\leq\ker\mathbf{j}}} A(1)_{j_1j_2}\cdots A(2m-1)_{j_{2m-1}j_{2m}}\Bigg)\Bigg(\sum_{\substack{1\leq i_1,\ldots,i_{2m}\leq N\\ \widetilde{K(\pi)}\leq\ker\mathbf{i}}} A(2)_{i_1i_2}\cdots A(2m)_{i_{2m-1}i_{2m}}\Bigg)\\ &= N^{|\sigma|+|K(\pi)|}E_N^{(\sigma)}[A(1),\ldots,A(2m-1)]E_N^{(K(\pi))}[A(2),\ldots,A(2m)].\end{aligned}$$

However, when $\mathcal{B}$ is noncommutative it is not clear how to express this sum in terms of expectation functionals. Instead, we will use the following bound on the norm:

Proposition 7.5.9. *Let $\mathcal{B}$ be a unital C^*-algebra, and let $A(1),\ldots,A(2m)\in M_N(\mathcal{B})$. If $\sigma,\pi\in NC(m)$ then*

$$\Bigg\|\sum_{\substack{1\leq j_1,\ldots,j_{2m}\leq N\\ \widetilde{\sigma}\leq\ker\mathbf{j}}}\sum_{\substack{1\leq i_1,\ldots,i_{2m}\leq N\\ \widetilde{K(\pi)}\leq\ker\mathbf{i}}} A(1)_{j_1j_2}A(2)_{i_1i_2}\cdots A(2m)_{i_{2m-1}i_{2m}}\Bigg\| \leq N^{|\sigma|+|K(\pi)|}\|A(1)\|\cdots\|A(2m)\|.$$

Proof. For this proof, we extend the definition of $\widetilde{\pi}$ to all partitions $\pi\in\mathcal{P}(m)$ in the obvious manner. We can rewrite expression above as

$$\sum_{\substack{1\leq i_1,\ldots,i_{4m}\leq N\\ \widetilde{\sigma\wr K(\pi)}\leq\ker\mathbf{i}}} A(1)_{i_1i_2}\cdots A(2m)_{i_{4m-1}i_{4m}},$$

and so the result will follow from

$$\Bigg\|\sum_{\substack{1\leq i_1,\ldots,i_{2m}\leq N\\ \widetilde{\sigma}\leq\ker\mathbf{i}}} A(1)_{i_1i_2}\cdots A(m)_{i_{2m-1}i_{2m}}\Bigg\| \leq N^{|\sigma|}\|A(1)\|\cdots\|A(m)\|$$

for any partition $\sigma\in\mathcal{P}(m)$.

The idea now is to realize this expression as the trace of a larger matrix. For each $V\in\sigma$, let M_N^V be a copy of $M_N(\mathbb{C})$. Consider the algebra

$$\bigotimes_{V\in\sigma} M_N^V \simeq M_{N^{|\sigma|}}(\mathbb{C}),$$

with the natural unital inclusions ι_V of M_N^V for $V\in\sigma$. For $1\leq l\leq m$, let

$$X(l)=\big(\iota_{\sigma(l)}\otimes\mathrm{id}_{\mathcal{B}}\big)A(l)\in\Bigg(\bigotimes_{V\in\sigma}M_N^V\Bigg)\otimes\mathcal{B}\simeq M_{N^{|\sigma|}}(\mathcal{B}),$$

where we have used the notation $\sigma(l)$ for the block of σ which contains l.

In other words, $X(l)$ is the matrix indexed by maps $i : \sigma \to [N] = \{1, \dots, N\}$ such that

$$X(l)_{ij} = A(l)_{i(\sigma(l))j(\sigma(l))} \prod_{\substack{V\in\sigma \\ l\notin V}} \delta_{i(V)j(V)}.$$

Consider now the trace

$$\begin{aligned}(\mathrm{Tr}_{N^{|\sigma|}} \otimes \mathrm{id}_{\mathcal{B}})(X(1)\cdots X(m)) &= \sum_{\substack{i_1,\dots,i_m \\ i_l:\sigma\to[N]}} X(1)_{i_1i_2}\cdots X(m)_{i_mi_1} \\ &= \sum_{\substack{i_1,\dots,i_m \\ i_l:\sigma\to[N]}} A(1)_{i_1(\sigma(1))i_2(\sigma(1))}\cdots A(m)_{i_m(\sigma(m))i_1(\sigma(m))} \prod_{1\le l\le m}\prod_{\substack{V\in\sigma \\ l\notin V}} \delta_{i_l(V)i_{\gamma(l)}(V)},\end{aligned}$$

where $\gamma \in S_m$ is the cyclic permutation $(123\cdots m)$. The nonzero terms in this sum are obtained as follows: for each block $V = (l_1 < \cdots < l_s)$ of σ, choose $1 \le i_{l_1}(V), i_{\gamma(l_1)}(V), \dots, i_{l_s}(V), i_{\gamma(l_s)}(V) \le N$ with the restrictions $i_{\gamma(l_1)}(V) = i_{l_2}(V), \dots, i_{\gamma(l_{s-1})}(V) = i_{l_s}(V)$ and $i_{\gamma(l_s)}(V) = i_{l_1}(V)$. Comparing with the definition of $\widetilde{\sigma}$, it follows that

$$(\mathrm{Tr}_{N^{|\sigma|}} \otimes \mathrm{id}_{\mathcal{B}})(X(1)\cdots X(m)) = \sum_{\substack{1\le i_1,\dots,i_{2m}\le N \\ \widetilde{\sigma}\le\ker\mathbf{i}}} A(1)_{i_1i_2}\cdots A(m)_{i_{2m-1}i_{2m}}$$

is the expression to be bounded. However, $(\mathrm{tr}_{N^{|\sigma|}} \otimes \mathrm{id}_{\mathcal{B}}) = N^{-|\sigma|}(\mathrm{Tr}_{N^{|\sigma|}} \otimes \mathrm{id}_{\mathcal{B}})$ is a contractive conditional expectation onto $\mathcal{B}$ and so

$$\|(\mathrm{Tr}_{N^{|\sigma|}} \otimes \mathrm{id}_{\mathcal{B}})(X(1)\cdots X(m))\| \le N^{|\sigma|}\|X(1)\|\cdots\|X(m)\|.$$

Since $(\iota_V \otimes \mathrm{id}_{\mathcal{B}})$ is a contractive $*$-homomorphism, we have $\|X(l)\| = \|(\iota_{\sigma(l)} \otimes \mathrm{id}_{\mathcal{B}})(A(l))\| \le \|A(l)\|$ and the result follows. □

We are now prepared to prove the main theorem.

Proof of Theorem 7.5.2. Fix $p_1, \dots, p_{2m} \in \mathcal{B}\langle t(i)|i \in I\rangle$, and set $A_N(k) = p_k(D_N)$ for $1 \le k \le 2m$. For notational simplicity, we will suppress the subscript N in our computations.

Let $l_1, \dots, l_{2m} \in \mathbb{N}$, $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$ and consider

$$\begin{aligned}&(\psi_N^{*\infty} \otimes E_N)[U(l_1)^{\epsilon_1}A(1)U(l_2)^{\epsilon_2}\cdots U(l_{2m})^{\epsilon_{2m}}A(2m)] \\ &\quad= (\psi_N^{*\infty} \otimes \mathrm{id}_{\mathcal{B}})N^{-1}\sum_{1\le i_1,\dots,i_{4m}\le N}(U(l_1)^{\epsilon_1})_{i_1i_2}A(1)_{i_2i_3}(U(l_2)^{\epsilon_2})_{i_3i_4}\cdots A(2m)_{i_{4m}i_1} \\ &\quad= \sum_{1\le i_1,\dots,i_{4m}\le N} N^{-1}\psi_N^{*\infty}\left[(U(l_1)^{\epsilon_1})_{i_1i_2}\cdots(U(l_{2m})^{\epsilon_{2m}})_{i_{4m-1}i_{4m}}\right]A(1)_{i_2i_3}\cdots A(2m)_{i_{4m}i_1}.\end{aligned}$$

By Corollary 7.4.4, this is equal to

$$\sum_{1\le i_1,\dots,i_{4m}\le N} N^{-1}\psi_N^{*\infty}\left[U(l_1)^{\epsilon_1}_{i_1i_2}U(l_2)^{\epsilon_2}_{i_4i_3}\cdots U(l_{2m})^{\epsilon_{2m}}_{i_{4m}i_{4m-1}}\right]A(1)_{i_2i_3}\cdots A(2m)_{i_{4m}i_1}.$$

After reindexing, this becomes

$$\sum_{1\le i_1,\dots,i_{2m}\le N}\ \sum_{1\le j_1,\dots,j_{2m}\le N} N^{-1}\psi_N^{*\infty}\left[U(l_1)^{\epsilon_1}_{i_{2m}j_1}U(l_2)^{\epsilon_2}_{i_1j_2}\cdots U(l_{2m})^{\epsilon_{2m}}_{i_{2m-1}j_{2m}}\right]A(1)_{j_1j_2}\cdots A(2m)_{i_{2m-1}i_{2m}}.$$

Applying Corollary 7.4.8, we have

$$\begin{aligned}&\sum_{1\le i_1,\dots,i_{2m}\le N}\ \sum_{1\le j_1,\dots,j_{2m}\le N}\ \sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \widetilde{\pi}\le\overrightarrow{\ker \mathbf{i}}\wedge\ker\mathbf{l}\\ \widetilde{\sigma}\le\ker\mathbf{j}\wedge\ker\mathbf{l}}} N^{-|K(\pi)|-|\sigma|}(\mu_m(\sigma,\pi)+O(N^{-2}))A(1)_{j_1j_2}\cdots A(2m)_{i_{2m-1}i_{2m}}\\ &=\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \widetilde{\pi}\le\ker\mathbf{l}\\ \widetilde{\sigma}\le\ker\mathbf{l}}}(\mu_m(\sigma,\pi)+O(N^{-2}))N^{-|K(\pi)|-|\sigma|}\sum_{\substack{1\le j_1,\dots,j_{2m}\le N\\ \widetilde{\sigma}\le\ker\mathbf{j}}}\ \sum_{\substack{1\le i_1,\dots,i_{2m}\le N\\ \widetilde{K(\pi)}\le\ker\mathbf{i}}} A(1)_{j_1j_2}\cdots A(2m)_{i_{2m-1}i_{2m}}.\end{aligned}$$

By Propositions 7.5.7 and 7.5.9, this is equal to

$$\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \sigma\le\pi\\ \widetilde{\pi}\vee\widetilde{\sigma}\le\ker\mathbf{l}}}\mu_m(\sigma,\pi)E_N^{(\sigma\wr K(\pi))}[A(1),\dots,A(2m)],$$

up to $O(N^{-2})$ with respect to the norm on $\mathcal{B}$. Set $a(k)=p_k(d_N)$ for $1\le k\le 2m$, then by Proposition 7.5.6 we have

$$\begin{aligned}E[u(l_1)^{\epsilon_1}a(1)\cdots u(l_{2m})^{\epsilon_{2m}}a(2m)]&=\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \sigma\le\pi\\ \widetilde{\pi}\vee\widetilde{\sigma}\le\ker\mathbf{l}}}\mu_m(\sigma,\pi)E^{(\sigma\wr K(\pi))}[a(1),\dots,a(2m)]\\ &=\sum_{\substack{\pi,\sigma\in NC^\epsilon(m)\\ \sigma\le\pi\\ \widetilde{\pi}\vee\widetilde{\sigma}\le\ker\mathbf{l}}}\mu_m(\sigma,\pi)E_N^{(\sigma\wr K(\pi))}[A(1),\dots,A(2m)],\end{aligned}$$

and the result now follows immediately. □

Randomly quantum rotated matrices. It follows easily from Theorem 7.5.4 and the definition of asymptotic freeness that under the hypotheses of the theorem, the sets

$$(\{D_N(i):i\in I\},\{U_N(1)D_N(i)U_N(1)^*:i\in I\},\{U_N(2)D_N(i)U_N(2)^*:i\in I\},\dots)$$

are asymptotically (infinitesimally) free with amalgamation over $\mathcal{B}$ as $N\to\infty$. The condition on existence of a limiting joint distribution can be weakened slightly as follows:

Corollary 7.5.10. *Let $\mathcal{B}$ be a unital C*-algebra, and let $(D_N(i))_{i\in I}$ and $(D'_N(j))_{j\in J}$ be two families of matrices in $M_N(\mathcal{B})$ for $N \in \mathbb{N}$. Suppose that there is a finite constant C such that $\|D_N(i)\| \leq C$ and $\|D'_N(j)\| \leq C$ for $N \in \mathbb{N}$, $i \in I$ and $j \in J$. For each $N \in \mathbb{N}$, let U_N be a $N \times N$ Haar quantum unitary random matrix, independent from $\mathcal{B}$.*

(1) If the joint distributions of $(D_N(i))_{i\in I}$ and $(D'_N(j))_{j\in J}$ both converge weakly (in norm or in L^2 with respect to a faithful trace), then $(U_N D_N(i)U_N^*)_{i\in I}$ and $(D'_N(j))_{j\in J}$ are asymptotically free with amalgamation over $\mathcal{B}$ as $N \to \infty$.

(2) If the joint distribution of $(D_N(i))_{i\in I}$ and $(D'_N(j))_{j\in J}$ both converge infinitesimally (in norm or in L^2 with respect to a faithful trace), then $(U_N D_N(i)U_N^*)_{i\in I}$ and $(D'_N(j))_{j\in J}$ are asymptotically infinitesimally free with amalgamation over $\mathcal{B}$.

Proof. The only condition of Theorem 7.5.4 which is not satisfied is that $\{D_N(i) : i \in I\} \cup \{D'_N(j) : j \in J\}$ should have a limiting (infinitesimal) joint distribution as $N \to \infty$. We can see that this is not an issue as follows. Let $p_1, \ldots, p_m \in B\langle t(i)|i \in I\rangle$ and $q_1, \ldots, qm \in B\langle t(j)|j \in J\rangle$ and set $A_N(k) = p_k(D_N)$, $B_N(k) = q_k(D'_N)$ for $1 \leq k \leq m$. From the proof of Theorem 7.5.2, we have

$$(\psi_N \otimes E_N)[UA(1)U^*B(1) \cdots UA(m)U^*B(m)] = \sum_{\substack{\pi,\sigma\in NC(m)\\ \sigma\leq\pi}} \mu_m(\sigma,\pi)E_N^{(\sigma\wr K(\pi))}[A(1), B(1), \ldots, A(m), B(m)],$$

up to $O(N^{-2})$. But the right hand side depends only on the distributions of $(D(i))_{i\in I}$ and $(D'(j))_{j\in J}$, and so the result follows from Theorem 7.5.4. □

Classical Haar unitary random matrices. In the remainder of this section, we will discuss the failure of these results for classical Haar unitaries. First we show that if $\mathcal{B}$ is finite dimensional, then classical Haar unitaries are sufficient.

Proposition 7.5.11. *Let $\mathcal{B}$ be a finite dimensional C*-algebra, and let $(D_N(i))_{i\in I}$ be a family of matrices in $M_N(\mathcal{B})$ for each $N \in \mathbb{N}$. Assume that there is a finite constant C such that $\|D_N(i)\| \leq C$ for all $N \in \mathbb{N}$ and $i \in I$. For each $N \in \mathbb{N}$, let $(U_N(l))_{l\in\mathbb{N}}$ be a family of independent $N \times N$ Haar unitary random matrices, independent from $\mathcal{B}$. Let $(u(l), u(l)^*)_{l\in\mathbb{N}}$ and $(d_N(i))_{i\in I,N\in\mathbb{N}}$ be random variables in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ such that*

(1) $(u(l), u(l)^*)_{l\in\mathbb{N}}$ is free from $(d_N(i))_{i\in I}$ with respect to E for each $N \in \mathbb{N}$.

(2) $(\{u(l), u(l)^*\})_{l\in\mathbb{N}}$ is a free family with respect to E, and $u(l)$ is a Haar unitary, independent from $\mathcal{B}$ for each $l \in \mathbb{N}$.

(3) $(d_N(i))_{i\in I}$ has the same $\mathcal{B}$-valued joint distribution with respect to E as $(D_N(i))_{i\in I}$ has with respect to E_N.

Then for any polynomials $p_1, \ldots, p_{2m} \in \mathcal{B}\langle t(i) : i \in I\rangle$, $l_1, \ldots, l_{2m} \in \mathbb{N}$ and $\epsilon_1, \ldots, \epsilon_{2m} \in \{1, *\}$,

$$\left\|(\psi_N^{*\infty} \otimes E_N)[U_N(l_1)^{\epsilon_1}p_1(D_N) \cdots U_N(l_{2m})^{\epsilon_{2m}}p_{2m}(D_N)] - E[u(l_1)^{\epsilon_1}p_1(d_N) \cdots u(l_{2m})^{\epsilon_{2m}}p_{2m}(d_N)]\right\|$$

is $O(N^{-2})$ as $N \to \infty$.

Proof. Let $e_1, \dots, e_q$ be a basis for $\mathcal{B}$ with $\|e_r\| = 1$ for $1 \le r \le q$. Let $p_1, \dots, p_{2m} \in \mathcal{B}\langle t(i) | i \in I\rangle$, let $A_N(k) = p_k(D_N)$ and let $A_N(k,r) \in M_N(\mathbb{C})$ be the matrix of coefficients of the entries of $A_N(k)$ on e_r for $1 \le k \le 2m$ and $1 \le r \le q$. Let $a_N(k,r)$ and $(u(l), u(l)^*)_{l\in\mathbb{N}}$ be random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$ such that

(1) $\{a_N(k,r) : 1 \le k \le 2m, 1 \le r \le q\}$ and $(u(l), u(l)^*)_{l\in\mathbb{N}}$ are free with respect to φ.

(2) $(a_N(k,r))_{1\le k\le 2m, 1\le r\le q}$ has the same joint distribution as $(A_N(k,r))_{1\le k\le 2m, 1\le r\le q}$.

(3) $(u(l), u(l)^*)_{l\in\mathbb{N}}$ are freely independent with respect to φ and $u(l)$ has a Haar unitary distribution.

For $1 \le k \le 2m$ and $N \in \mathbb{N}$, let $a_N(k) = \sum a_N(k,r) \otimes e_r \in \mathcal{A} \otimes \mathcal{B}$, and note that the family $(a_n(k))_{1\le k\le 2m}$ has the same joint distribution with respect to $E = \varphi \otimes \mathrm{id}_{\mathcal{B}}$ as does $(A_N(k))_{1\le k\le 2m}$ with respect to E_N. Identifying $u(l) = u(l) \otimes 1$ in $\mathcal{A} \otimes \mathcal{B}$, it is also easy to see that $(u(l), u(l)^*)$ and $(a_N(k))_{1\le k\le 2m}$ are freely independent with respect to E.

Now let $\epsilon_1, \dots, \epsilon_{2m} \in \{1, *\}$ and consider

$$(\mathrm{tr}_N \otimes \mathbb{E} \otimes \mathrm{id}_{\mathcal{B}})[U(l_1)^{\epsilon_1} A(1) \cdots A(2m) U(l_{2m})^{\epsilon_{2m}}] = \sum_{1 \le r_1, \dots, r_{2m} \le q} (\mathrm{tr}_N \otimes \mathbb{E})[U(l_1)^{\epsilon_1} A(1, r_1) \cdots A(2m, r_{2m}) U(l_{2m})^{\epsilon_{2m}}] e_{r_1} \cdots e_{r_{2m}}.$$

Since $\|e_r\| = 1$, it follows that

$$\begin{aligned}\Big\|(\mathrm{tr}_N \otimes \mathbb{E} \otimes \mathrm{id}_{\mathcal{B}})[U(l_1)^{\epsilon_1} A(1) \cdots U(l_{2m})^{\epsilon_{2m}} A(2m)] - E[u(l_1)^{\epsilon_1} a(1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m)]\Big\| \\ \le \sum_{1 \le r_1, \dots, r_{2m} \le q} \Big|(\mathrm{tr}_N \otimes \mathbb{E})[U(l_1)^{\epsilon_1} A(1, r_1) \cdots U(l_{2m})^{\epsilon_{2m}} A(2m, r_{2m})] \\ - \varphi[u(l_1)^{\epsilon_1} a(1, r_1) \cdots u(l_{2m})^{\epsilon_{2m}} a(2m, r_{2m})]\Big|.\end{aligned}$$

From standard asymptotic freeness results (see e.g. [19]), this expression is $O(N^{-2})$ as $N \to \infty$. □

Remark 7.5.12. We will now give an example to show that Theorem 7.5.2 may fail for classical Haar unitaries if the algebra $\mathcal{B}$ is infinite dimensional. First we recall the Weingarten formula for computing the expectation of a word in the entries of a $N \times N$ Haar unitary random matrix and its conjugate:

$$\mathbb{E}[U_{i_1 j_1}^{\epsilon_1} \cdots U_{i_{2m} j_{2m}}^{\epsilon_{2m}}] = \sum_{\substack{\pi, \sigma \in \mathcal{P}_2^{\epsilon}(2m) \\ \pi \le \ker \mathbf{i} \\ \sigma \le \ker \mathbf{j}}} W_{\epsilon N}^c(\pi, \sigma),$$

where $\mathcal{P}_2^{\epsilon}(2m)$ is the set of pair partitions for which each pairing connects a 1 with a $*$ in the string $\epsilon_1, \dots, \epsilon_{2m}$, and $W_{\epsilon N}^c$ is the corresponding Weingarten matrix, see [19, 10].

Example 7.5.13. Let $\mathcal{B}$ be a unital C*-algebra, and for each $N \in \mathbb{N}$ let $\{E_{ij}(N,l) : 1 \le i, j \le N, l = 1, 2\}$ be two commuting systems of matrix units in $\mathcal{B}$, i.e.,

(1) $E_{i_1 j_1}(N,1) E_{i_2 j_2}(N,2) = E_{i_2 j_2}(N,2) E_{i_1 j_1}(N,1)$ for $1 \le i_1, j_1, i_2, j_2 \le N$.

(2) $E_{ij}(N,l)^* = E_{ji}(N,l)$ for $1 \le i,j \le N$.

(3) $E_{ik_1}(N,l)E_{k_2j}(N,l) = \delta_{k_1k_2}E_{ij}(N,l)$ for $1 \le i,j,k_1,k_2 \le N$.

(4) $E_{ii}(N,l)$ is a projection for $1 \le i \le N$, and

$$\sum_{i=1}^{N} E_{ii}(N,l) = 1.$$

For $N \in \mathbb{N}$, define $A_N, B_N \in M_N(\mathcal{B})$ by

$$(A_N)_{ij} = E_{ji}(N,1), \qquad (B_N)_{ij} = E_{ji}(N,2).$$

Note that A_N, B_N are self-adjoint and A_N^2, B_N^2 are the identity matrix, indeed

$$(A_N^2)_{ij} = \sum_{k=1}^{N} E_{ki}(N,1)E_{jk}(N,1) = \delta_{ij}\sum_{k=1}^{N} E_{kk}(N,1) = \delta_{ij}\cdot 1,$$

and likewise for B_N. It follows that $\|A_N\| = \|B_N\| = 1$ for $N \in \mathbb{N}$.

For each $N \in \mathbb{N}$, let U_N be a $N \times N$ Haar unitary random matrix, independent from $\mathcal{B}$. Since

$$(\mathrm{tr}_N \otimes \mathrm{id}_{\mathcal{B}})[A_N] = \frac{1}{N}\sum_{i=1}^{N} E_{ii}(N,1) = \frac{1}{N}\cdot 1$$

converges to zero as $N \to \infty$, and likewise for B_N, for asymptotic freeness we should have

$$\lim_{N\to\infty} (\mathrm{tr}_N \otimes \mathbb{E} \otimes \mathrm{id})[(U_N A_N U_N^* B_N)^3] = 0.$$

However, we will show that this limit is in fact equal to 1.

Indeed, suppressing the subindex N we have

$$\begin{aligned}(\mathrm{tr} \otimes \mathbb{E} \otimes \mathrm{id}_{\mathcal{B}})[(UAU^*B)^3] &= \frac{1}{N}\sum_{1\le i_1,\dots,i_{12}\le N} \mathbb{E}[U_{i_1i_2}\overline{U}_{i_4i_3}\cdots\overline{U}_{i_{12}i_{11}}]A_{i_2i_3}B_{i_4i_5}\cdots B_{i_{12}i_1}\\ &= \sum_{1\le i_1,j_1,\dots,i_6,j_6\le N} \mathbb{E}[U_{i_6j_1}\overline{U}_{i_1j_2}\cdots\overline{U}_{i_5j_6}]A_{j_1j_2}A_{j_3j_4}A_{j_5j_6}B_{i_1i_2}B_{i_3i_4}B_{i_5i_6}.\end{aligned}$$

Applying the Weingarten formula, we obtain

$$\sum_{\pi,\sigma\in\mathcal{P}_2^{\epsilon}(6)} N^{-1}W_{\epsilon N}^{c}(\pi,\sigma)\Big(\sum_{\substack{1\le j_1,\dots,j_6\le N\\ \sigma\le\ker\mathbf{j}}} A_{j_1j_2}A_{j_3j_4}A_{j_5j_6}\Big)\Big(\sum_{\substack{1\le i_1,\dots,i_6\le N\\ \overleftarrow{\pi}\le\ker\mathbf{i}}} B_{i_1i_2}B_{i_3i_4}B_{i_5i_6}\Big).$$

Note that $\mathcal{P}_2^{\epsilon}(6)$ consists of the 5 noncrossing pair partitions and $\tau = \{(1,4),(2,5),(3,6)\}$. The noncrossing pair partitions can be expressed as $\widetilde{\sigma}$ for some $\sigma \in NC(3)$, in which case we have

$$\sum_{\substack{1\le j_1,\dots,j_6\le N\\ \widetilde{\sigma}\le\ker\mathbf{j}}} A_{j_1j_2}A_{j_3j_4}A_{j_5j_6} = N^{|\sigma|}E_N^{(\sigma)}[A,A,A].$$

Using $E_N[A] = E_N[A^3] = N^{-1}$ and $E_N[A^2] = 1$, one easily sees that this expression is $O(N)$ for the 5 noncrossing pair partitions. For τ, we have

$$\begin{aligned}\sum_{\substack{1\le j_1,\dots,j_6\le N\\ \tau\le\ker\mathbf{j}}} A_{j_1j_2}A_{j_3j_4}A_{j_5j_6} &= \sum_{1\le j_1,j_2,j_3\le N} A_{j_1j_2}A_{j_3j_1}A_{j_2j_3}\\ &= \sum_{1\le j_1,j_2,j_3\le N} E_{j_2j_1}(N,1)E_{j_1j_3}(N,1)E_{j_3j_2}(N,1)\\ &= \sum_{1\le j_1,j_2,j_3\le N} E_{j_2j_2}(N,1)\\ &= N^2\cdot 1,\end{aligned}$$

and likewise for B_N. Also we have $N^3W^c_{\epsilon N}(\pi,\sigma) = \delta_{\pi\sigma} + O(N^{-1})$. Putting these statements together, we find that the only term remaining in the limit comes from $\pi = \sigma = \tau$, which gives 1.

Remark 7.5.14.

(1) Note that $M_{N^2}(\mathbb{C}) = M_N(\mathbb{C})\otimes M_N(\mathbb{C})$ has a natural pair of commuting systems of matrix units, so this example demonstrates that Theorem 7.5.2 fails for any unital C*-algebra $\mathcal{B}$ which contains $M_{N_k^2}(\mathbb{C})$ as a unital subalgebra for some increasing sequence of natural numbers (N_k).

(2) It is a natural question whether the matrices A_N, B_N in the above example have limiting $\mathcal{B}$-valued distributions, which would demonstrate that Theorem 7.1.1 also fails for classical Haar unitaries. First observe that

$$\lim_{N\to\infty}(\mathrm{tr}_N\otimes\mathrm{id}_{\mathcal{B}})[A_N^k] = \begin{cases}1, & k \text{ is even}\\ 0, & k\text{ is odd}\end{cases},$$

which follows from the case $k = 1$ and the fact that A_N^2 is the identity matrix. However, it is not clear that moments of the form $b_0A_N\cdots A_Nb_k$ will converge for arbitrary $b_0,\dots,b_k\in\mathcal{B}$.

Let us point out a special case in which the limiting distribution does exist. Suppose that there is a dense $*$-subalgebra $\mathcal{F}\subset\mathcal{B}$ such that each element of $\mathcal{F}$ commutes with the matrix units $E_{ij}(N,l)$ for N sufficiently large. Then for any $b_0,\dots,b_k\in\mathcal{B}$ we have

$$\lim_{N\to\infty}(\mathrm{tr}_N\otimes\mathrm{id}_{\mathcal{B}})[b_0A_N\cdots A_Nb_k] = \begin{cases}b_0b_1\cdots b_k, & k\text{ is even}\\ 0, & k\text{ is odd}\end{cases},$$

and likewise for B_N, indeed this holds for $b_0,\dots,b_k\in\mathcal{F}$ by hypothesis and for general $b_0,\dots,b_k$ by density.

In particular, we may take $\mathcal{B}$ to be the C*-algebraic infinite tensor product

$$\mathcal{B} = \bigotimes_{N\in\mathbb{N}} M_N(\mathbb{C})$$

with the obvious systems of matrix units $E(N,l)_{ij}\in M_{N^2} = M_N(\mathbb{C})\otimes M_N(\mathbb{C})\subset\mathcal{B}$, and $\mathcal{F}\subset\mathcal{B}$ to be the image of the purely algebraic tensor product. Note that $\mathcal{B}$ is *uniformly hyperfinite*, in particular *approximately finitely dimensional* in the C*-sense.

(3) Note that if $\mathcal{B}$ is a von Neumann algebra with a non-zero *continuous* projection p, then $p\mathcal{B}p$ contains $M_N(\mathbb{C})$ as a unital subalgebra for all $N \in \mathbb{N}$ and hence (1) applies to $p\mathcal{B}p$. It follows that Theorem 7.5.2 fails also for $\mathcal{B}$. To obtain a contradiction to Theorem 7.1.1 for classical Haar unitaries in the setting of a von Neumann algebra with faithful, normal trace, we may modify the example in (2) by taking $(\mathcal{B}, \tau)$ to be the infinite tensor product

$$(\mathcal{B}, \tau) = \bigotimes_{N \in \mathbb{N}} (M_N(\mathbb{C}), \mathrm{tr}_N)$$

taken with respect to the trace states tr_N on $M_N(\mathbb{C})$, which is the *hyperfinite II_1 factor*.

Chapter 8

Analytic subordination results in free probability

8.1 Introduction

A derivation-comultiplication on a unital algebra A over $\mathbb{C}$ is a linear map

$$\Delta : A \to A \otimes A,$$

which satisfies the product rule $\Delta(ab) = (a \otimes 1)\Delta(b) + \Delta(a)(1 \otimes b)$. Derivation-comultiplications play a prominent role in free probability theory, most notably in Voiculescu's "microstates-free" approaches to free entropy, free Fisher information and free mutual information ([52, 53]). Of particular interest is the free difference quotient, introduced to study free Fisher information and free entropy, and at the center of the "free analysis" of Voiculescu ([56, 57, 58]).

The free difference quotient $\partial_{X:B}$ is the derivation-comultiplication on $B\langle X\rangle$ determined by

$$\partial_{X:B}(X) = 1 \otimes 1,$$
$$\partial_{X:B}(b) = 0, \qquad (b \in B),$$

where B is a unital algebra over $\mathbb{C}$ and X is algebraically free from B. $\partial_{X:B}$ has the additional property of coassociativity, i.e.

$$(\mathrm{id} \otimes \partial_{X:B}) \circ \partial_{X:B} = (\partial_{X:B} \otimes \mathrm{id}) \circ \partial_{X:B}.$$

In considering the corepresentations of this coalgebra, Voiculescu found a natural explanation for the phenomenon of analytic subordination, a powerful tool in free harmonic analysis.

In [51], Voiculescu proved (under some easily removed genericity assumptions) that if X and Y are self-adjoint and freely independent random variables, then the Cauchy transforms of G_{X+Y} and G_X satisfy an analytic subordination relation in the upper half-plane. He used this result to prove certain inequalities on p-norms of densities, free entropies and Riesz energies. It was later discovered by Biane that the subordination extends to the operator-valued resolvents, and that a similar result holds for free multiplicative convolution [15]. He used these results to prove certain Markov-transitions properties for processes with free increments.

In [42], Nica and Speicher showed that for any Borel probability measure μ on $\mathbb{R}$, there is a partially defined continuous free additive convolution semigroup starting at μ, i.e. a family $\{\mu_t : t \geq 1\}$ such that $\mu = \mu_1$, $\mu_{s+t} = \mu_s \boxplus \mu_t$. It was shown by Belinschi and Bercovici in [13] that the analytic subordination for $\mu^{\boxplus n}$ extends to μ_t. This can be used to prove certain regularity results for the free additive convolution semigroup.

Though technically useful, the proofs of these results did little to explain why analytic subordination appears in the context of free convolutions. What Voiculescu observed in [54] is that, roughly speaking, the invertible corepresentations of $\partial_{X:B}$ are the B-resolvents $(b-X)^{-1}$ (and their matricial generalizations). Moreover, if X and Y are B-freely independent, then a certain conditional expectation is a coalgebra morphism from the coalgebra of $\partial_{X+Y:B}$ to the coalgebra of $\partial_{X:B}$. Since coalgebra morphisms preserve corepresentations, one should expect that B-resolvents of $X+Y$ are mapped to B-resolvents of X by this conditional expectation. This approach led to the generalization of the earlier results for free additive convolution to the B-valued context.

In [55], Voiculescu found that he could extend this result by simple operator-valued analytic continuation arguments. Here he found a general subordination result for freely Markovian triples, and gave a B-valued extension of Biane's result for multiplicative convolution of unitaries.

In this chapter we extend the approach of [54] to give a B-valued generalization of the subordination result for free compressions from [13], and to recover the results of Voiculescu from [55] for freely Markovian triples and B-valued multiplicative convolution of unitaries. To recover the results from [55], we will use certain comultiplication-derivations appearing in free probability which are not coassociative. Because of the failure of coassociativity, we cannot expect to find interesting corepresentations for these comultiplications. However, we will see that the resolvents which we are interested in are still characterized by certain relations involving these comultiplications. Moreover, these relations are preserved by certain conditional expectations which are coalgebra morphisms. We should expect then that these resolvents are preserved by these conditional expectations. The technical difficulties that arise are in working with the closures of these unbounded derivations.

This chapter is organized as follows. The next section is purely algebraic. We look at the relationship between derivations and certain resolvents in a general setting. In Section 8.3 we show that certain conditional expectations are coalgebra morphisms for ∂, δ and d. In Section 8.4 we prove the analytic subordination results for free compressions. In Section 8.5 we extend some technical results from [53] to the operator-valued case, which will be needed in the next section. Section 8.6 contains the proof of the analytic subordination result for freely Markovian triples. Section 8.7 covers the analytic subordination result for multiplication of B-freely independent unitaries.

8.2 Derivations and resolvents

Here we discuss the relationship between derivations and certain resolvents in a general algebraic framework.

Let A, B be unital algebras over $\mathbb{C}$, and let $\varphi_1, \varphi_2 : A \to B$ be unital homomorphisms. A linear map $D : A \to B$ is a derivation with respect to the A-bimodule structure defined by φ_1, φ_2 if

$$D(a_1 a_2) = \varphi_1(a_1) D(a_2) + D(a_1)\varphi_2(a_2).$$

It is easy to see that this implies $D(1) = 0$, and if $a \in A$ is invertible then

$$D\left(a^{-1}\right) = -\varphi_1\left(a^{-1}\right) D(a)\varphi_2\left(a^{-1}\right).$$

Proposition 8.2.1. *Let $A, B, \varphi_1, \varphi_2, D$ be as above and let $N = \operatorname{Ker} D$.*

(1) Fix $X \in A$ such that $D(X) = 1$. If $a \in A$ is invertible and satisfies $D(a) = \varphi_1(a)\varphi_2(a)$, then $a = (n - X)^{-1}$ for some $n \in \operatorname{Ker} N$. Conversely, if $n \in N$ is such that $(n - X)$ is invertible, then $a = (n - X)^{-1}$ satisfies $D(a) = \varphi_1(a)\varphi_2(a)$.

(2) Fix $a \in A$. If $\alpha \in A$ is invertible, and $D(\alpha) = -\varphi_1(\alpha)D(a)\varphi_2(\alpha)$, then $\alpha = (a + n)^{-1}$ for some $n \in N$. Conversely, if $n \in N$ is such that $a + n$ is invertible, then

$$D\left((a+n)^{-1}\right) = -\varphi_1\left((a+n)^{-1}\right) D(a)\varphi_2\left((a+n)^{-1}\right).$$

(3) Suppose $U \in A$ is invertible, and $D(U) = \varphi_2(U)$. If $\alpha \in A$ is such that $1 + \alpha$ is invertible and $D(\alpha) = \varphi_1(\alpha + 1)\varphi_2(\alpha)$, then $\alpha = Un(1 - Un)^{-1}$ for some $n \in N$ such that $1 - Un$ is invertible. Conversely, if $n \in N$ is such that $1 - Un$ is invertible, then

$$D\left(Un(1-Un)^{-1}\right) = \varphi_1\left(Un(1-Un)^{-1} + 1\right)\varphi_2\left(Un(1-Un)^{-1}\right).$$

Proof.

(1) Suppose $D(a) = \varphi_1(a)\varphi_2(a)$, then

$$D(a^{-1}) = -\varphi_1(a^{-1})D(a)\varphi_2(a^{-1}) = -1,$$

so that $a^{-1} + X \in N$. Conversely,

$$D((n-X)^{-1}) = -\varphi_1((n-X)^{-1})D(n-X)\varphi_2(n-X)^{-1} = \varphi_1((n-X)^{-1})\varphi_2((n-X)^{-1}).$$

(2) Fix $a \in A$ and suppose $\alpha \in A$ satisfies the hypotheses, then

$$D\left(\alpha^{-1}\right) = -\varphi_1(\alpha^{-1})D(\alpha)\varphi_2(\alpha^{-1}) = D(a).$$

So $\alpha^{-1} - a \in \operatorname{Ker} D$ which proves one direction, the converse is trivial.

(3) Suppose $U \in A$ is invertible, and $\alpha \in A$ satisfies the hypotheses, then

$$\begin{aligned} D\left(U^{-1}(\alpha+1)^{-1}\right) &= -\varphi_1\left(U^{-1}\right)\varphi_1\left((\alpha+1)^{-1}\right)D(\alpha+1)\varphi_2\left((\alpha+1)^{-1}\right) \\ &\quad - \varphi_1\left(U^{-1}\right)D(U)\varphi_2\left(U^{-1}\right)\varphi_2\left((\alpha+1)^{-1}\right) \\ &= -\varphi_1\left(U^{-1}\right)\left[\varphi_2(\alpha)+1\right]\varphi_2\left((\alpha+1)^{-1}\right) \\ &= D\left(U^{-1}\right). \end{aligned}$$

So $n = U^{-1} - U^{-1}(\alpha+1)^{-1} \in \operatorname{Ker} D$, and hence $\alpha = (1 - Un)^{-1} - 1 = Un(1 - Un)^{-1}$. The converse is a simple computation.

□

Remark 8.2.2. In the sequel, we will apply Proposition 8.2.1 to certain completions of ∂, δ and d.

(1) Note that $B \subset \operatorname{Ker} \partial_{X:B}$, so $b \in B$ is such that $b - X$ is invertible in the completion of $B\langle X\rangle$, then $\alpha = (b - X)^{-1}$ satisfies the hypotheses of (1) above. Note that in this case, the relation in (1) becomes the corepresentation relation $\partial_{X:B}(\alpha) = \alpha \otimes \alpha$.

(2) Likewise, $B \subset \operatorname{Ker} \delta_{A:B}$, so if $a \in A$, $b \in B$ are such that $a + b$ is invertible in the completion of $A \vee B$, then $\alpha = (a + b)^{-1}$ satisfies the hypotheses of (2) above.

(3) Likewise, $B \subset \operatorname{Ker} d_{U:B}$, so if $b \in B$ is such that $(1 - Ub)$ is invertible in the completion of $B\langle U, U^*\rangle$, then $\alpha = Ub(1 - Ub)^{-1}$ satisfies the conditions of (3) above.

8.3 Coalgebra morphisms in free probability

In this section we prove that certain conditional expectations arising in the contexts of free compression, free Markovianity, and B-free multiplicative convolution of unitaries, are coalgebra morphisms for the comultiplications ∂, δ and d, respectively. Because we will need these results in the next section, we will work with operator-valued generalizations of δ and d.

Remark 8.3.1. In the remainder of the paper, (M, τ) will denote a tracial W*-probability space. If $A, B \subset M$, $A \vee B$ will denote the algebra generated (algebraically) by $A \cup B$. If $1 \in A \subset M$ is a $*$-subalgebra, $E_A^{(M)}$, or just E_A, will denote the canonical trace preserving conditional expectation of M onto $W^*(A)$.

Definition 8.3.2. Suppose that $1 \in B \subset M$ is a W*-subalgebra, and that $1 \in A_1, A_2 \subset M$ are subalgebras containing B which are algebraically free with amalgamation over B. Letting $A = A_1 \vee A_2$ denote the algebra generated by A_1 and A_2, define

$$\delta_{A_1:A_2;B} : A \to A \otimes_B A$$

to be the derivation into the A-bimodule $A \otimes_B A$, which is determined by

$$\delta_{A_1:A_2;B} = \begin{cases} a \otimes 1 - 1 \otimes a, & \text{if } a \in A_1, \\ 0, & \text{if } a \in A_2. \end{cases}$$

The *B-valued liberation gradient* $j = j(A_1 : A_2; B)$ is then defined by the requirements that $j \in L^2(A)$, and

$$E_B(ja) = (E_B \otimes E_B)(\delta_{A_1:A_2;B}(a)), \quad (a \in A).$$

Except in Section 3, we will be interested only in the case $B = \mathbb{C}$, in which case we recover the definitions of Voiculescu in [53] of $\delta(A_1 : A_2)$ and of the *liberation gradient* $j(A_1 : A_2)$. This B-valued generalization was introduce by Nica, Shlyakhtenko and Speicher in [41] as a method for studying B-freeness of the algebras A_1 and A_2.

Definition 8.3.3. Suppose $1 \in B \subset M$ is a W*-subalgebra, $A \subset M$ is a subalgebra containing B and $U \in M$ is a unitary such that $B\langle U, U^*\rangle$ is algebraically free with amalgamation over B from A. Define

$$d_{U:A;B} : A\langle U, U^*\rangle \to A\langle U, U^*\rangle \otimes_B A\langle U, U^*\rangle$$

to be the derivation determined by

$$\begin{aligned} d_{U:A;B}(U) &= 1 \otimes U, \\ d_{U:A;B}(U^*) &= -U^* \otimes 1, \\ d_{U:A;B}(a) &= 0, \qquad\qquad (a \in A). \end{aligned}$$

The *conjugate of U relative to A with respect to B*, denoted $\xi = \xi(U : A; B)$, is then defined by the requirements that $\xi \in L^2(A\langle U, U^*\rangle)$ and

$$E_B(\xi m) = (E_B \otimes E_B)(d_{U:A;B}(m)), \quad m \in A\langle U, U^*\rangle.$$

We will mostly be interested in the case $B = \mathbb{C}$, in which case we recover the definition of $d_{U:B}$ from [53]. This B-valued generalization was considered by Shlyakhtenko in [45].

Remark 8.3.4. The following lemma is an operator-valued generalization of a result in [54]. The proof is an easy adaptation of the argument found there, we include it here for the convenience of the reader.

Lemma 8.3.5. *Let $1 \in B_1, B \subset M$ be W*-subalgebras in (M, τ) such that $B_1 \subset B$. Let $1 \in A, C \subset M$ be $*$-subalgebras which are B-free in (M, E_B). Let $D : A \vee B \vee C \to (A \vee B \vee C) \otimes_{B_1} (A \vee B \vee C)$ be a derivation such that $D(B \vee C) = 0$ and $D(A \vee B) \subset (A \vee B) \otimes_{B_1} (A \vee B)$. Then*

$$(E_{A\vee B} \otimes_{B_1} E_{A\vee B}) \circ D = D \circ E_{A\vee B}|_{A\vee B\vee C}.$$

Proof. First note that B-freeness implies

$$E_{A\vee B}(A \vee B \vee C) \subset A \vee B.$$

Let $F_1 = (A \vee B) \cap \operatorname{Ker} E_B$, $F_2 = (B \vee C) \cap \operatorname{Ker} E_B$. Since $A \vee B$ and $B \vee C$ are B-free, we have

$$(A \vee B \vee C) \ominus (A \vee B) = F_2 \oplus \bigoplus_{k\geq 2} \bigoplus_{\substack{\alpha_1 \neq \cdots \neq \alpha_k \\ \alpha_i \in \{1,2\}}} F_{\alpha_1} F_{\alpha_2} \cdots F_{\alpha_k},$$

where the orthogonal difference and direct sums are with respect to the B_1-valued inner product defined by E_{B_1}. Now $DF_2 = 0$, and $DF_1 \subset (F_1 + B) \otimes (F_1 + B)$ by hypothesis. If $\alpha_1 \neq \cdots \neq \alpha_k$, $\alpha_i \in \{1, 2\}$, $k \geq 2$, then

$$D(F_{\alpha_1} \cdots F_{\alpha_k}) \subset \sum_{\substack{1\leq i\leq k \\ \alpha_i = 1}} F_{\alpha_1} \cdots F_{\alpha_{i-1}} (F_1 + B) \otimes_{B_1} (F_1 + B) F_{\alpha_{i+1}} \cdots F_{\alpha_k}.$$

If $k \geq 2$, either $i > 1$ or $i < k$ so that either

$$E_{A\vee B}\left(F_{\alpha_1}\cdots F_{\alpha_{i-1}}\left(F_1+B\right)\right)=0$$

or

$$E_{A\vee B}\left(\left(F_1+B\right)F_{\alpha_{i+1}}\cdots F_{\alpha_k}\right)=0.$$

Since also $DF_2 = 0$, we have shown that

$$\left(E_{A\vee B}\otimes_{B_1}E_{A\vee B}\right)\left(D(A\vee B\vee C\ominus A\vee B)\right)=0.$$

Since

$$\left(E_{A\vee B}\otimes_{B_1}E_{A\vee B}\right)\circ D\circ E_{A\vee B}|_{A\vee B\vee C}=D\circ E_{A\vee B}|_{A\vee B\vee C}$$

by hypothesis, the result follows. □

Remark 8.3.6. We will now apply this lemma to ∂, δ and d in the contexts of free compression, free Markovianity and free multiplicative convolution. We begin with the case of free compression, which requires another lemma.

Lemma 8.3.7. *Suppose that $1 \in B \subset M$ is a $*$-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$, which we consider as a Bp-valued random variable in pMp. Define $\psi : pMp \to M$ by $\psi(pmp) = \alpha^{-1}pmp$. Then $\psi(Bp\langle X_p\rangle) \subset B\langle p, X\rangle$ and*

$$(\psi\otimes\psi)\circ\partial_{X_p:Bp}=\partial_{X:B[p]}\circ\psi|_{Bp\langle X_p\rangle}$$

i.e., $\psi|_{B_p\langle X_p\rangle}$ is a coalgebra morphism for the comultiplications $\partial_{X_p:Bp}$ and $\partial_{X:B[p]}$.

Proof. Clearly $\psi(Bp\langle X_p\rangle) \subset B\langle p, X\rangle$, we must show that ψ is comultiplicative. Both sides of the above equation are derivations from $Bp\langle X_p\rangle$ into $M \otimes M$ with respect to the natural $Bp\langle X_p\rangle$ bimodule structure on $M \otimes M$. It is clear that Bp is in the kernel of both derivations, we need only compare them on X_p. We have

$$\partial_{X:B[p]}\circ\psi(X_p)=\alpha^{-2}\partial_{X:B[p]}(pXp)=\alpha^{-2}p\otimes p=(\psi\otimes\psi)(p\otimes p)=(\psi\otimes\psi)\circ\partial_{X_p:Bp}(X_p)$$

□

Proposition 8.3.8. *Suppose that $1 \in B \subset M$ is a W*-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p is B-free with X, p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Define $\Psi : pMp \to M$ by $\Psi = E^{(M)}_{B\langle X\rangle} \circ \psi$. Then*

$$(\Psi\otimes\Psi)\circ\partial_{X_p:Bp}=\partial_{X:B}\circ\Psi|_{Bp\langle X_p\rangle}$$

Proof. Since X and p are B-free in M, $E^{(M)}_{B\langle X\rangle}B[X,p] \subset B\langle X\rangle$ so that

$$\Psi(Bp[X_p]) \subset B\langle X\rangle$$

By the Lemma 8.3.5 applied to $A = \mathbb{C}[X], B = B, B_1 = \mathbb{C}, C = \mathbb{C}[p], D = \partial_{X:B[p]}$ we have

$$\left(E^{(M)}_{B\langle X\rangle} \otimes E^{(M)}_{B\langle X\rangle}\right) \circ \partial_{X:B[p]} = \partial_{X:B} \circ E^{(M)}_{B\langle X\rangle]}\Big|_{B\langle X,p\rangle}$$

The result then follows from composing both sides with $\psi|_{Bp[X_p]}$ and applying Lemma 8.3.7. □

Remark 8.3.9. To attach probabilistic meaning to the map Ψ, it should be unital and preserve trace and expectation onto B. These properties require the additional assumption that p is independent from B with respect to τ.

Proposition 8.3.10. *Let M, B, X, p, Ψ as above and suppose, in addition to the previous hypotheses, that p is independent from B with respect to τ. Then $\Psi(bp) = b$ for $b \in B$, in particular Ψ is unital. Furthermore, Ψ preserves trace and expectation onto B, i.e.*

$$\tau \circ \Psi = \tau_p$$
$$\Psi \circ E^{(pMp)}_{Bp} = E^{(M)}_{B} \circ \Psi$$

Proof. First remark that independence implies $E^{(M)}_B(p) = \alpha$. Since X and p are B-free,

$$E^{(M)}_{B\langle X\rangle}(p) = E^{(M)}_B(p) = \alpha$$

Therefore, for $b \in B$ we have

$$\Psi(bp) = \alpha^{-1}E^{(M)}_{B\langle X\rangle}(bp) = \alpha^{-1}bE^{(M)}_{B\langle X\rangle}(p) = b$$

Next observe that

$$\begin{aligned}\tau\left(\Psi(pmp)\right) &= \alpha^{-1}\tau\left(E^{(M)}_{B\langle X\rangle}(pmp)\right)\\ &= \alpha^{-1}\tau(pmp)\\ &= \tau_p(pmp)\end{aligned}$$

so that Ψ preserves trace. Next we claim that

$$E^{(pMp)}_{Bp}(pmp) = \alpha^{-1}E^{(M)}_B(pmp)p$$

First observe that the right hand side is a conditional expectation from pMp onto $W^*(Bp)$. Since $E^{(pMp)}_{Bp}$ is the unique such conditional expectation which preserves τ_p, it remains only to show that this map is trace preserving. We have

$$\tau_p\left(\alpha^{-1}E^{(M)}_B(pmp)p\right) = \alpha^{-2}\tau\left(E^{(M)}_B(pmp)p\right) = \alpha^{-1}\tau(pmp) = \tau_p(pmp)$$

which proves the claim. We then have

$$\begin{aligned}\left(\Psi \circ E_{Bp}^{(pMp)}\right)(pmp) &= \Psi\left(\alpha^{-1}E_B^{(M)}(pmp)p\right)\\ &= \alpha^{-2}E_{B\langle X\rangle}^{(M)}\left(E_B^{(M)}(pmp)p\right)\\ &= E_B^{(M)}\left(\alpha^{-1}E_{B\langle X\rangle}^{(M)}(pmp)\right)\\ &= \left(E_B^{(M)} \circ \Psi\right)(pmp)\end{aligned}$$

So that Ψ preserves expectation onto B.

□

Remark 8.3.11. If $X = X^* \in M$ is algebraically free from a W*-subalgebra $1 \in B \subset M$, the *conjugate variable* $\mathscr{J}(X : B)$ is defined by the relations $\mathscr{J}(X : B) \in L^2(B\langle X\rangle)$ and

$$\tau(\mathscr{J}(X : B)m) = (\tau \otimes \tau)(\partial_{X:B}(m)), \qquad (m \in B\langle X\rangle).$$

If $X = X^*, Y = Y^* \in M$ are B-free, where $1 \in B \subset M$ is a W*-subalgebra, then it was shown by Voiculescu [52] that if $\mathscr{J}(X : B)$ exists so does $\mathscr{J}(X + Y : B)$ and is obtained from a conditional expectation. This is also true for a free compression:

Corollary 8.3.12. *Suppose that $1 \in B \subset M$ is a W*-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Assume that p and B are independent, and that X and p are B-freely independent. If $\mathscr{J}(X : B)$ exists, then $\mathscr{J}(X_p : Bp)$ exists and is given by*

$$E_{Bp\langle X_p\rangle}^{(pMp)}(p\mathscr{J}(X : B)p)$$

Proof. Let Ψ be as above, then for $pmp \in Bp\langle X_p\rangle$ we have

$$\begin{aligned}(\tau_p \otimes \tau_p)(\partial_{X_p:Bp}(pmp)) &= (\tau \otimes \tau)(\partial_{X:B}\Psi(pmp))\\ &= \alpha^{-1}\tau\left(\mathscr{J}(X : B)E_{B\langle X\rangle}^{(M)}(pmp)\right)\\ &= \alpha^{-1}\tau(\mathscr{J}(X : B)pmp)\\ &= \tau_p((p\mathscr{J}(X : B)p)pmp)\\ &= \tau_p\left(E_{Bp\langle X_p\rangle}^{(pMp)}(p\mathscr{J}(X : B)p)pmp\right)\end{aligned}$$

□

Corollary 8.3.13. *Let $1 \in B_1, B \subset M$ be W*-subalgebras such that $B_1 \subset B$, and let $1 \in A, C \subset M$ be $*$-subalgebras which are B-free in (M, E_B) and such that A is algebraically free from $B \vee C$ with amalgamation over B_1. Then*

$$(E_{A\vee B} \otimes_{B_1} E_{A\vee B}) \otimes \delta_{A:B\vee C;B_1} = \delta_{A:B;B_1} \circ E_{A\vee B}|_{A\vee B\vee C}.$$

Corollary 8.3.14. *Suppose that $j(A : B; B_1)$ exists, then so does $j(A : B \vee C; B_1)$ and*

$$j(A : B \vee C; B_1) = j(A : B; B_1).$$

Proof. For $m \in A \vee B \vee C$, we have

$$\begin{aligned} E_{B_1}\left(j(A : B; B_1)m\right) &= E_{B_1}\left(j(A : B; B_1)E_{A \vee B}(m)\right) \\ &= \left(E_{B_1} \otimes E_{B_1}\right)\delta_{A:B;B_1}\left(E_{A\vee B}(m)\right) \\ &= \left(E_{B_1} \otimes E_{B_1}\right)\delta_{A:B\vee C;B_1}(m). \end{aligned}$$

□

Corollary 8.3.15. *Let $1 \in B_1, B \subset M$ be W*-subalgebras such that $B_1 \subset B$. Let $U, V \in M$ be unitaries which are B-freely independent, and such that U is algebraically free from $B\langle V, V^*\rangle$ with amalgamation over B_1. Then $E_{B\langle U,V,U^*,V^*\rangle} \subset B\langle U, U^*\rangle$, and*

$$\left(E_{B\langle U,U^*\rangle} \otimes_{B_1} E_{B\langle U,U^*\rangle}\right) \circ d_{UV:B;B_1} = d_{U:B;B_1} \circ E_{B\langle U,U^*\rangle}|_{B\langle UV,V^*U^*\rangle}.$$

Proof. Apply Lemma 8.3.5 to find that $E_{B\langle U,U^*\rangle}B\langle U, V, U^*, V^*\rangle \subset B\langle U, U^*\rangle$, and

$$\left(E_{B\langle U,U^*\rangle} \otimes_{B_1} E_{B\langle U,U^*\rangle}\right) \circ d_{U:B\langle V,V^*\rangle;B_1} = d_{U:B;B_1} \circ E_{B\langle U,U^*\rangle}|_{B\langle U,V,U^*,V^*\rangle}.$$

Since $d_{U:B\langle V,V^*\rangle;B_1}|_{B\langle UV,V^*U^*\rangle} = d_{UV:B;B_1}$, the result follows by restricting to $B\langle UV, V^*U^*\rangle$. □

Corollary 8.3.16. *Suppose that $\xi(U : B; B_1)$ exists, then so does $\xi(UV : B; B_1)$, and*

$$\xi(UV : B; B_1) = E_{B\langle UV,V^*U^*\rangle}\left(\xi(U : B; B_1)\right).$$

Proof. The proof is similar to Corollary 8.3.14. □

8.4 Analytic subordination for free compression

In this section we prove the analytic subordination result for a free compression.

8.4.1. Let $1 \in B \subset M$ be a W*-subalgebra, and let $B\langle t\rangle$ denote the algebra of noncommutative polynomials with coefficients in B. Given any $m \in M$, there is a unique homomorphism from $B\langle t\rangle$ into M which is the identity on B and sends t to m, which we will denote by $f \mapsto f(m)$.

8.4.2. Fix a self-adjoint element $X \in M$ which is algebraically free from B. Define $\partial^{(p)}_{X:B} : B\langle X\rangle \to B\langle X\rangle^{\otimes(p+1)}$ recursively by $\partial^{(0)}_{X:B} = \mathrm{id}$ and

$$\partial^{(p+1)}_{X:B} = (\partial_{X:B} \otimes \mathrm{id}^{\otimes p}) \circ \partial^{(p)}_{X:B}.$$

We will work with a certain "smooth" completion of $B\langle t\rangle$. Define a norm $\| \|^{\sim}_{R,X}$ on $B\langle t\rangle$ by

$$\|f\|^{\sim}_{R,U} = \sum_{p\geq 0} \left\|\partial^{(p)}_{X:B}(f(X))\right\|^{\wedge}_{(p+1)} R^p,$$

where $\| \|^{\wedge}_{(s)}$ denotes the projective tensor product norm on $M^{\widehat{\otimes} s}$.

Lemma 8.4.3. $\| \|_{R,X}^{\sim}$ *is a finite norm on* $B\langle t\rangle$*, and if* $f, g \in B\langle t\rangle$ *then*

$$\|fg\|_{R,X}^{\sim} \leq \|f\|_{R,X}^{\sim}\|g\|_{R,X}^{\sim}.$$

Proof. It is clear that $\| \|_{R,X}^{\sim}$ is a finite norm, since the sum appearing in the definition is finite for $f \in B\langle t\rangle$. Since $\partial_{X:B}$ is a derivation, if $f, g \in B\langle t\rangle$ then we have

$$\partial_{X:B}^{(p)}(f(X)g(X)) = \sum_{k=0}^{p}\big(\partial_{X:B}^{(k)}(f(X)) \otimes 1^{\otimes(p-k)}\big)\big(1^{\otimes k} \otimes \partial_{X:B}^{\otimes(p-k)}(g(X))\big),$$

so that

$$\begin{aligned}\|fg\|_{R,X}^{\sim} &= \sum_{p\geq 0}\big\|\partial_{X:B}^{(p)}(f(X)g(X))\big\|_{(p+1)}^{\wedge}R^p \\ &\leq \sum_{p\geq 0}\sum_{k=0}^{p}\big\|\partial_{X:B}^{(k)}(f(X))\|_{(k+1)}^{\wedge}R^k\|\partial_{X:B}^{(p-k)}(g(X))\big\|_{(p-k+1)}^{\wedge}R^{p-k} \\ &= \|f\|_{R,X}^{\sim}\|g\|_{R,X}^{\sim}.\end{aligned}$$

□

8.4.4. Let $B_{R,X}^{\sim}\langle t\rangle$ denote the completion of $B\langle t\rangle$ under $\| \|_{R,X}^{\sim}$, which is a Banach algebra by the previous lemma. It is clear that the evaluation map $f \mapsto f(X)$ extends to a contractive unital homomorphism on $B_{R,X}^{\sim}\langle t\rangle$, which we will still denote by $f \mapsto f(X)$.

The main analytic tool that we will use to control the kernel of $\partial_{X:B}$ is a Taylor series type expansion of $f(X+Y)$ for $Y = Y^* \in M$ with $\|Y\|$ small. First we introduce some notation. Given $m_1, \dots, m_s \in M$, let $\theta_s[m_1, \dots, m_s]$ denote the linear map from $M^{\otimes(s+1)}$ into M determined by

$$\theta_p[m_1, \dots, m_s](m_1' \otimes \dots m_{s+1}') = m_1'm_1m_2' \cdots m_s m_{s+1}'.$$

Note that

$$\|\theta_p[m_1, \dots, m_s](\xi)\| \leq \|m_1\| \cdots \|m_s\|\|\xi\|_{(s+1)}^{\wedge},$$

where $\| \|_{(s+1)}^{\wedge}$ denotes the projective tensor product norm on $M^{\widehat{\otimes}(s+1)}$.

Proposition 8.4.5. *If* $X, Y \in M$ *are self-adjoint operators with* X *algebraically free from* B*, and* $f \in B\langle t\rangle$*, then*

$$f(X+Y) = \sum_{p\geq 0}\theta_p[Y, \dots, Y]\big(\partial_{X:B}^{(p)}(f(X))\big).$$

In particular, if $\|Y\| \leq R$ *then* $f \mapsto f(X+Y)$ *extends to a contractive homomorphism on* $B_{R,X}^{\sim}\langle t\rangle$*.*

Proof. Let $\varphi(f)$ denote the right hand side, it is clear that φ is the identity on B and $\varphi(t) = X+Y$, so it suffices to show that φ is a homomorphism. We have

$$\begin{aligned}\varphi(fg) &= \sum_{p\geq 0}\theta_p[Y,\dots,Y]\big(\partial^{(p)}_{X:B}(f(X)g(X))\big)\\ &= \sum_{p\geq 0}\theta_p[Y,\dots,Y]\sum_{k=0}^{p}\big(\partial^{(k)}_{X:B}(f(X))\otimes 1^{\otimes(p-k)}\big)\big(1^{\otimes k}\otimes\partial^{\otimes(p-k)}_{X:B}(g(X))\big)\\ &= \sum_{p\geq 0}\sum_{k=0}^{p}\theta_k[Y,\dots,Y]\big(\partial^{(k)}_{X:B}(f(X))\otimes 1^{\otimes(p-k)}\big)\theta_{(p-k)}[Y,\dots,Y]\big(1^{\otimes k}\otimes\partial^{\otimes(p-k)}_{X:B}(g(X))\big)\\ &= \varphi(f)\varphi(g).\end{aligned}$$

The second statement then follows from the remark above and the definition of $\| \,\|^{\sim}_{R,X}$. □

Remark 8.4.6. Voiculescu showed in [52] that the existence of the conjugate variable $\mathscr{J}(X : B)$ in $L^2(B\langle X\rangle)$ is a sufficient condition for closability of $\partial_{X:B}$, viewed as an unbounded operator $L^2(B\langle X\rangle) \to L^2(B\langle X\rangle)\otimes L^2(B\langle X\rangle)$. In particular, $\partial_{X:B}$ is then closable in the uniform norm, we denote the closure by $\overline{\partial}_{X:B}$. We will need the following standard result on closable derivations ([18, 54]).

Proposition 8.4.7. *Let K, L be unital C*-algebras, let $\varphi_1, \varphi_2 : K \to L$ be unital $*$-homomorphisms, let $1 \in A \subset K$ be a unital $*$-subalgebra, and let $D : A \to L$ be a closable derivation with respect to the A-bimodule structure on L defined by φ_1, φ_2. The closure $\overline{D}$ is then a derivation, and the domain of definition $\mathfrak{D}(\overline{D})$ is a subalgebra. Moreover, if $a \in A$ is invertible in K, then $a^{-1} \in \mathfrak{D}(\overline{D})$ and*

$$\overline{D}\left(a^{-1}\right) = -\varphi_1\left(a^{-1}\right)D(a)\varphi_2\left(a^{-1}\right).$$

□

Proposition 8.4.8. *Let $1 \in B \subset M$ be a W*-subalgebra, and $X \in M$ a self-adjoint operator which is algebraically free with B. Suppose that $|\mathscr{J}(X : B)|_2 < \infty$. If $R > \|X\|$, then $f(X) \in \mathfrak{D}(\overline{\partial}_{X:B})$ for any $f \in B^{\sim}_{R,X}\langle t\rangle$. Furthermore, if $f \in B^{\sim}_{R,X}\langle t\rangle$ and $\overline{\partial}_{X:B}(f(X)) = 0$, then $f(X) \in B$.*

Proof. The first part is clear from the definition of $\| \,\|^{\sim}_{R,X}$. Suppose then that $f \in B^{\sim}_{R,X}\langle t\rangle$ and $\overline{\partial}_{X:B}(f(X)) = 0$. Let $f_n \in B\langle X\rangle$ such that $f_n \to f$ in $\| \,\|^{\sim}_{R,X}$. Then

$$\lim_{n\to\infty}\partial_{X:B}(f_n(X)) = \overline{\partial}_{X:B}(f(X)) = 0,$$

the limit holding in the projective tensor product norm $\| \,\|^{\wedge}_{(2)}$. Since $(\partial_{X:B}\otimes \mathrm{id})$ is closable, it follows that

$$\lim_{n\to\infty}\|\partial^{(2)}_{X:B}(f_n(X))\|^{\wedge}_{(3)} = 0.$$

Iterating, we have

$$\lim_{n\to\infty}\|\partial^{(p)}_{X:B}(f_n(X))\|^{\wedge}_{(p+1)} = 0$$

for all $p \geq 1$. Let $Y = Y^* \in M$ with $\|Y\| < R$. Since $f_n \to f$ in $\| \|^{\sim}_{R,X}$, it follows that $\|f_n\|^{\sim}_{R,X} \leq C$ for some constant C. Given $\epsilon > 0$, find P such that

$$C \cdot \frac{(\|Y\|/R)^P}{1 - (\|Y\|/R)} < \epsilon.$$

Then find N such that $n \geq N$ implies

$$\sum_{p=1}^{P-1} \|\partial^{(p)}_{X:B}(f_n(X))\|^{\wedge}_{(p+1)} \|Y\|^p < \epsilon.$$

We then have for $n \geq N$,

$$\begin{aligned} \|f_n(X+Y) - f_n(X)\| &= \Big\| \sum_{p \geq 1} \theta_p[Y, \ldots, Y]\big(\partial^{(p)}_{X:B}(f_n(X))\big) \Big\| \\ &\leq \sum_{p=1}^{P-1} \|Y\|^p \|\partial^{(p)}_{X:B}(f_n(X))\|^{\wedge}_{(p+1)} + \sum_{p \geq P} \|Y\|^p C R^{-p} \\ &< 2\epsilon. \end{aligned}$$

It follows that

$$f(X+Y) - f(X) = \lim_{n \to \infty} f_n(X+Y) - f(X) = 0.$$

Applying this to $Y = -X$ we have $f(X) = f(0)$, and since clearly $f(0) \in B$ this completes the proof. □

8.4.9. We recall the following from [54]. If A is a unital C*-algebra, the *upper half-plane* of A is defined as $\mathbb{H}_+(A) = \{T \in A : \text{Im } T \geq \epsilon 1 \text{ for some } \epsilon > 0\}$. Similarly, the *lower half-plane* of A is defined as $\mathbb{H}_-(A) = \{T \in A : \text{Im } A \leq -\epsilon 1 \text{ for some } \epsilon > 0\}$. If $T \in \mathbb{H}_+(A)$, then T is invertible and

$$\|T^{-1}\| \leq \epsilon^{-1} \qquad\qquad \text{Im}(T^{-1}) \leq -\left(\epsilon + \epsilon^{-1}\|T\|^2\right)^{-1},$$

in particular $T^{-1} \in \mathbb{H}_-(A)$.

Proposition 8.4.10. *Let $1 \in B \subset M$ be a W*-subalgebra, and suppose that $X = X^* \in M$ and that $p \in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$ and put $X_p = \alpha^{-1}pXp$. Assume that $\|\mathscr{J}(X : B)\|_2 < \infty$. Then there is an analytic function $F : \mathbb{H}_+(B) \to \mathbb{H}_+(B)$ such that*

$$\alpha^{-1} E_{B\langle X \rangle}(bp - X_p)^{-1} = (F(b) - X)^{-1}$$

for $b \in \mathbb{H}_+(B)$.

Proof. Since $|\mathscr{J}(X : B)|_2 < \infty$, also $|\mathscr{J}(X_p : Bp)|_2 < \infty$ by Corollary 8.3.12. So $\partial_{X:B}$ and $\partial_{X_p:Bp}$ are both closable in the uniform norm. We have $\Psi(\mathfrak{D}(\overline{\partial}_{X_p:Bp})) \subset \mathfrak{D}(\overline{\partial}_{X:B})$ and

$$(\Psi \otimes \Psi) \circ \overline{\partial}_{X_p:Bp} = \overline{\partial}_{X:B} \circ \Psi|_{Bp\langle X_p \rangle}.$$

For $b \in \mathbb{H}_+(B)$, we have $(bp - X_p)^{-1} \in \mathfrak{D}(\partial_{X_p:Bp})$ by Proposition 8.4.7, and

$$\overline{\partial}_{X_p:Bp}(bp - X_p)^{-1} = (bp - X_p)^{-1} \otimes (b_p - X_p)^{-1}$$

by Proposition 8.2.1. Let $\gamma = \Psi((bp - X_p)^{-1})$, then $\gamma \in \mathfrak{D}(\overline{\partial}_{X:B})$ and

$$\overline{\partial}_{X:B}(\gamma) = \gamma \otimes \gamma.$$

Moreover, since Ψ is positive we have $\gamma \in \mathbb{H}_-(M)$ and so γ is invertible. It follows from Proposition 8.2.1 that $\gamma = (n - X)^{-1}$ for some $n \in \ker \overline{\partial}_{X:B}$. Since $\gamma \in \mathbb{H}_-(M)$ we have $n \in \mathbb{H}_+(M)$.

It is clear that n depends analytically on b, it remains to show that $n \in B$. By analytic continuation, it suffices to show this for b in an open subset of $\mathbb{H}_+(B)$.

Choose $R > (1 + \|X\|)$ and $\rho > 4R$, and put

$$\Omega = \{b \in B | \|i\rho - b\| < 1\} \subset \mathbb{H}_+(B).$$

If $b \in \Omega$, then

$$(bp - X_p)^{-1} = (i\rho(p - \Gamma))^{-1} = (i\rho)^{-1} \sum_{m \geq 0} \Gamma^m$$

where $\Gamma = (i\rho)^{-1}(i\rho p - bp + X_p)$.

Since $\partial_{Xp:Bp}(\Gamma) = (i\rho)^{-1} p \otimes p$, it follows that

$$\partial^{(s)}_{Xp:Bp}\Gamma^m = \sum_{\substack{m_1,\ldots,m_{s+1} \\ m_1+\cdots+m_{s+1}=m-s}} (i\rho)^{-s}\Gamma^{m_1} \otimes \Gamma^{m_2} \otimes \cdots \otimes \Gamma^{m_{p+1}}.$$

Since $\|\Gamma\| < 1/4$ we have

$$\|\partial^{(s)}_{X_p:Bp}\Gamma^m\|^{\wedge}_{(s+1)} < \rho^{-s} 2^{-(m-s)} \binom{m}{s}$$

if $m \geq s$, while if $m < s$ then $\partial^{(s)}_{X_p:Bp}\Gamma^m = 0$.

Choose $P_m \in B\langle t \rangle$ with $P_m(X) = \Psi(\Gamma^m)$. Then

$$\partial^{(s)}_{X:B}(P_m) = \Psi^{\otimes(p+1)}(\partial^{(s)}_{X_p:Bp}\Gamma^m),$$

so

$$\|\partial^{(s)}_{X:B}(P_m)\|^{\wedge}_{(s+1)} < \rho^{-s} 4^{-(m-s)} \binom{m}{s}$$

if $m \geq s$ and is zero if $m < s$. Therefore

$$\|P_m\|_{\widetilde{R,X}} \leq \sum_{s=0}^{m} (R/\rho)^s 4^{-(m-s)} \binom{m}{s} = (4^{-1} + (R/\rho))^m < 2^{-m}$$

It follows that $\sum_{m \geq 1} P_m$ converges in $B_{\widetilde{R,X}}\langle t \rangle$ to a limit with norm less than 1. Let

$$P = t + (i\rho)\Big(1 + \sum_{m \geq 1} P_m\Big)^{-1} \in B_{\widetilde{R,X}}\langle t \rangle,$$

then we have

$$n = X + (\Psi((bp - X_p)^{-1}))^{-1} = P(X).$$

Since $\overline{\partial}_{X:B}(n) = 0$, by Proposition 8.4.8 we have $n \in B$, which completes the proof. □

Remark 8.4.11. We can remove the condition on the conjugate variable $\mathscr{J}(X : B)$ using the same method as [54]. We leave the details to the reader, but will give proofs of similar results for freely Markovian triples and free multiplicative convolutions later in the chapter. We also note that our argument works equally well for matricial resolvents, for details see [21].

Theorem 8.4.12. *Let $1 \in B \subset M$ be a W*-subalgebra, and suppose that $X = X^* \in M$ and that $p \in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$ and put $X_p = \alpha^{-1}pXp$. Then there is an analytic function $F : \mathbb{H}_+(B) \to \mathbb{H}_+(B)$ such that*

$$\alpha^{-1}E_{B\langle X\rangle}(bp - X_p)^{-1} = (F(b) - X)^{-1}$$

for $b \in \mathbb{H}_+(B)$. □

8.5 Regularization via unitary conjugation

Our aim in this section is to show that if $1 \in A, B \subset M$ are $*$-subalgebras, then we can can find a unitary U arbitrarily close to the identity such that $W^*(UAU^* \vee B) \cap W^*(A \vee B) = B$, which will be needed in the next section. In the case $B = \mathbb{C}$, this follows easily from the considerations in [53]. Here we extend the necessary results from that paper to the B-valued case by using the B-valued liberation gradient introduced in the previous section.

Remark 8.5.1. The L^2-norm of the B-valued liberation gradient gives a measure of how far the algebras A_1 and A_2 are from being B-free. In particular, it is shown in [41] that A_1 and A_2 are B-free if and only if $j(A_1 : A_2; B) = 0$. In the case $B = \mathbb{C}$, Voiculescu gave some estimates on the "distance" between the algebras A_1 and A_2 when the liberation gradient $j(A_1 : A_2)$ is bounded [53]. We begin by observing that his estimates extend directly to the B-valued case.

Lemma 8.5.2. *Let $1 \in B \subset M$ be a W*-subalgebra, and let $1 \in A_1, A_2 \subset M$ be $*$-subalgebras which contain B, and such that A_1 is algebraically free from A_2 with amalgamation over B. Suppose that $j(A_1 : A_2; B)$ exists. If $m \in A_1 \cap \operatorname{Ker} E_B$, $m' \in A_2 \cap \operatorname{Ker} E_B$ then*

$$E_B(j(A_1 : A_2; B)mm') = -E_B(j(A_1 : A_2; B)m'm) = -E_B(mm')$$

and

$$E_B(j(A_1 : A_2; B)[m, m']) = -2E_B(mm').$$

In particular,

$$\tau(j(A_1 : A_2; B)[m, m']) = -2\tau(mm').$$

□

Proposition 8.5.3. *Suppose that* $\|j(A_1 : A_2; B)\| < \infty$*. If* $m \in A_1 \cap \operatorname{Ker} E_B$*,* $m' \in A_2 \cap \operatorname{Ker} E_B$ *then*

$$|\tau(mm')| \leq \frac{\|j(A_1 : A_2; B)\|}{(1 + \|j(A_1 : A_2; B)\|^2)^{1/2}} |m|_2 |m'|_2.$$

Equivalently,

$$\|(E_{A_1} - E_B)(E_{A_2} - E_B)\| \leq \frac{\|j(A_1 : A_2; B)\|}{(1 + \|j(A_1 : A_2; B)\|^2)^{1/2}}.$$

Proof. Identical to [53, Proposition 7.2]. □

Remark 8.5.4. We now turn to the existence of the B-valued liberation gradient $j(A_1 : A_2; B)$ after conjugating by a unitary in M which commutes with B. As observed in the scalar case by Voiculescu, the key is the relation between δ and d.

Proposition 8.5.5. *Let* $1 \in B \subset M$ *be a* W^**-subalgebra, and* $1 \in A \subset M$ *a* $*$*-subalgebra which contains* B*. If* U *is a unitary in* M *which commutes with* B *and is algebraically free from* A *with amalgamation over* B*, then*

$$d_{U:A;B}|_{A \vee UAU^*} = -\delta_{UAU^*:A;B}.$$

Proof. We have

$$\begin{aligned} d_{U:A;B}\left(a_1 U a_2 U^* \cdots a_{2k-1} U a_{2k} U^*\right) = \sum_{1 \leq p \leq k} \big(& a_1 U a_2 U^* \cdots a_{2p-1} \otimes U a_{2p} U^* \cdots a_{2k-1} U a_{2k} U^* \\ &- a_1 U a_2 U^* \cdots a_{2p-1} U a_{2p} U^* \otimes a_{2p+1} \cdots a_{2k-1} U a_{2k} U^*\big) \\ = &-\delta_{UAU^*:A;B}\left(a_1 U a_2 U^* \cdots a_{2k-1} U a_{2k} U^*\right). \end{aligned}$$

□

Corollary 8.5.6. *If* $\xi(U : A; B)$ *exists, then so does* $j(UAU^* : A; B)$ *and*

$$j(UAU^* : A; B) = -E_{A \vee UAU^*}(\xi(U : A; B)).$$

□

Proposition 8.5.7. *Let* $1 \in B \subset M$ *be a* W^**-subalgebra, and suppose that* $U \in M$ *is a unitary such that* $\mathbb{C}[U, U^*]$ *is independent from* B*. Then if* $\xi(U : \mathbb{C}; \mathbb{C})$ *exists, so does* $\xi(U : B; B)$ *and*

$$\xi(U : B; B) = \xi(U : \mathbb{C}; \mathbb{C}).$$

Proof. Since U commutes with B, we just need to check that

$$E_B(\xi(U : \mathbb{C}; \mathbb{C}) U^n) = (E_B \otimes_B E_B)(d_{U:B;B}(U^n))$$

for all $n \in \mathbb{Z}$. If $n \geq 0$, then by independence we have

$$\begin{aligned} E_B(\xi(U:\mathbb{C};\mathbb{C})U^n) &= \tau(\xi(U:\mathbb{C};\mathbb{C})U^n) \\ &= (\tau \otimes_{\mathbb{C}} \tau)\left(d_{U:\mathbb{C};\mathbb{C}}(U^n)\right) \\ &= \sum_{k=0}^{n-1} \tau(U^k)\tau(U^{n-k}) \\ &= \sum_{k=0}^{n-1} E_B(U^k)E_B(U^{n-k}) \\ &= (E_B \otimes_B E_B)\left(d_{U:B;B}(U^n)\right). \end{aligned}$$

The case $n < 0$ is similar.

□

Proposition 8.5.8. *Let $1 \in B \subset M$ be a W^*-algebra, $1 \in A \subset M$ a $*$-subalgebra containing B, and $U \in M$ a unitary such that A is B-free from $B\langle U, U^*\rangle$ in (M, E_B). If $\xi(U:B;B)$ exists, then so does $\xi(U:A;B)$ and*

$$\xi(U:A;B) = \xi(U:B;B).$$

Proof. Apply Lemma 8.3.5 with D, A, B_1, B, C replaced by $d_{U:A;B}, B\langle U, U^*\rangle, B, B, A$ to find

$$(E_B \otimes_B E_B) \circ d_{U:B;B} \circ E_{B\langle U,U^*\rangle}|_{A\langle U,U^*\rangle} = (E_B \otimes_B E_B) \circ d_{U:A;B}.$$

Now for $m \in A\langle U, U^*\rangle$, we have

$$\begin{aligned} E_B\left(\xi(U:B;B)m\right) &= E_B\left(\xi(U:B;B)E_{B\langle U,U^*\rangle}(m)\right) \\ &= (E_B \otimes_B E_B)\, d_{U:B;B}\left(E_{B\langle U,U^*\rangle}(m)\right) \\ &= (E_B \otimes_B E_B)\, d_{U:A;B}(m). \end{aligned}$$

□

Proposition 8.5.9. *Let S be a $(0,1)$-semicircular random variable in (M, τ). Fix $0 < \epsilon < 1$, and let $U_\epsilon = \exp(\pi i \epsilon S)$. Then $\xi(U_\epsilon : \mathbb{C};\mathbb{C})$ exists, and*

$$\|\xi(U_\epsilon : \mathbb{C};\mathbb{C}) - i(2\pi^2\epsilon)^{-1}S\| \leq \frac{\epsilon(2-\epsilon)}{2\pi(1-\epsilon)}.$$

In particular, $\xi(U_\epsilon : \mathbb{C};\mathbb{C}) \in W^*(U_\epsilon)$.

Proof. The distribution of U_ϵ with respect to τ has density

$$p\left(e^{i\theta}\right) = \chi_{[-\pi\epsilon,\pi\epsilon]}\frac{4}{\pi\epsilon^2}\sqrt{\epsilon^2 - \theta^2/\pi^2}$$

with respect to the normalized Lebesgue measure on $\mathbb{T}$. By [53, Proposition 8.7], $\xi(U_\epsilon : \mathbb{C}; \mathbb{C})$ exists, and is given by $i(Hp)(U_\epsilon)$, where Hp is the circular Hilbert transform of p, i.e. Hp is the a.e. limit of $H_\delta p$ as $\delta \to 0$, where

$$(H_\delta p)\left(e^{i\theta_1}\right) = -\frac{1}{2\pi}\int_{\delta<|\theta|\leq\pi} p\left(e^{i(\theta_1-\theta)}\right)\cot\left(\frac{\theta}{2}\right)\, d\theta.$$

For $x \neq 0$, we have the expansion ([65])

$$\frac{1}{2}\cot\left(\frac{x}{2}\right) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{x+2\pi n} - \frac{1}{2\pi n}.$$

It follows that for $0 < |\theta| \leq 2\pi\epsilon$, we have

$$\begin{aligned}\left|\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{1}{\theta}\right| &\leq \sum_{n\geq 1} \frac{|\theta|}{|\theta+2n\pi|2n\pi} \\ &\leq (2\pi\epsilon)\left(\frac{1}{2\pi(1-\epsilon)2\pi} + \frac{1}{2\pi}\sum_{n\geq 2}\frac{1}{2\pi(n-1)} - \frac{1}{2\pi n}\right) \\ &= \frac{\epsilon(2-\epsilon)}{2\pi(1-\epsilon)}.\end{aligned}$$

Hence if $|\theta_1| \leq \pi\epsilon$, then

$$\left|(H_\delta p)\left(e^{i\theta_1}\right) + \frac{1}{\pi}\int_{\delta<|\theta|\leq\pi} \frac{p\left(e^{i(\theta_1-\theta)}\right)}{\theta}\, d\theta\right| \leq \frac{\epsilon(2-\epsilon)}{2\pi(1-\epsilon)},$$

since $p(\exp(i(\theta_1-\theta))) = 0$ if $|\theta| > 2\pi\epsilon$. But

$$-\frac{1}{\pi}\int_{\delta<|\theta|\leq\pi} \frac{p\left(e^{i(\theta_1-\theta)}\right)}{\theta}\, d\theta$$

converges as $\delta \to 0$ to the Hilbert transform of the semicircular law of radius $\pi\epsilon$ evaluated at θ_1. By the results in [52, Section 3], this is equal to $\theta_1/(2\pi^3\epsilon^2)$. So for $|\theta_1| \leq \pi\epsilon$, we have

$$\left|(Hp)\left(e^{i\theta_1}\right) - \frac{\theta_1}{2\pi^3\epsilon^2}\right| \leq \frac{\epsilon(2-\epsilon)}{2\pi(1-\epsilon)}.$$

It follows that

$$\|\xi(U_\epsilon : \mathbb{C}; \mathbb{C}) - i(2\pi^2\epsilon)^{-1}S\| \leq \frac{\epsilon(2-\epsilon)}{2\pi(1-\epsilon)}.$$

□

Corollary 8.5.10. *Let $1 \in B \subset M$ be a W*-subalgebra, $1 \in A \subset M$ a $*$-subalgebra containing B, and S a $(0,1)$-semicircular element in (M, τ) which is independent from B and B-freely independent from A. Then for $0 < \epsilon < 1$, we have*

$$W^*(A \vee B) \cap W^*(U_\epsilon A U_\epsilon^* \vee B) = B,$$

where $U_\epsilon = \exp(\pi i \epsilon S)$.

Proof. By Propositions 8.5.7 and 8.5.8, $\xi(U_\epsilon : A; B)$ exists and

$$\xi(U_\epsilon : A; B) = \xi(U_\epsilon : B; B) = \xi(U_\epsilon : \mathbb{C}; \mathbb{C}).$$

Applying Corollary 8.5.6, we see that $j(U_\epsilon A U_\epsilon^* : A; B)$ exists and

$$j(U_\epsilon A U_\epsilon^* : A; B) = -E_{A \vee U_\epsilon A U_\epsilon^*} \left[\xi(U_\epsilon : \mathbb{C}; \mathbb{C}) \right].$$

By Proposition 8.5.9, $\xi(U_\epsilon : \mathbb{C}; \mathbb{C})$ is bounded and so $j(U_\epsilon A U_\epsilon^* : A; B)$ is as well. The result then follows from Proposition 8.5.3. □

8.6 Analytic subordination for freely Markovian triples

In this section we use the derivation $\delta_{A:B}$ to prove the analytic subordination result for a freely Markovian triple (A, B, C). The main difficulty is in showing that certain "smooth" elements in the kernel of the closure of $\delta_{A:B}$ actually lie in B.

8.6.1. Let $1 \in A, B \subset M$ be $*$-subalgebras which are algebraically free. Let $A * B$ denote the $*$-algebra free product of A and B (with amalgamation over $\mathbb{C}$). Given an invertible $S \in M$, there is a unique $*$-homomorphism $\rho_S : A * B \to M$ determined by

$$\begin{aligned} \rho_S(a) &= SaS^{-1}, \quad (a \in A), \\ \rho_S(b) &= b, \qquad\quad (b \in B). \end{aligned}$$

We will denote by ρ the isomorphism of $A * B$ onto $A \vee B$.

8.6.2. For $p \geq 0$, define $\delta_{A:B}^{(p)} : A \vee B \to (A \vee B)^{\otimes (p+1)}$ recursively by $\delta_{A:B}^{(0)} = \mathrm{id}_{A \vee B}$ and

$$\delta_{A:B}^{(p+1)} = \left(\delta_{A:B} \otimes \mathrm{id}^{\otimes p} \right) \circ \delta_{A:B}^{(p)}.$$

Given $0 < R < 1$, define $\| \, \|_R^{\sim}$ on $A * B$ by

$$\|f\|_R^{\sim} = \sum_{p \geq 0} \|\delta_{A:B}^{(p)}(\rho(f))\|_{(p+1)}^{\wedge} R^p,$$

where $\| \, \|_{(s)}^{\wedge}$ denotes the norm on the projective tensor product $M^{\widehat{\otimes} s}$.

Lemma 8.6.3. $\| \, \|_R^{\sim}$ *is a finite norm on $A * B$, and if $f, g \in A * B$ then*

$$\|fg\|_R^{\sim} \leq \|f\|_R^{\sim} \|g\|_R^{\sim}.$$

Proof. The proof of $\|fg\|_R^{\sim} \leq \|f\|_R^{\sim}\|g\|_R^{\sim}$ is the same as Lemma 8.4.3. Since $\|\ \|_R^{\sim}$ is easily seen to be finite when restricted to A and to B, it follows that $\|\ \|_R^{\sim}$ is a finite norm on $A * B$. □

Let $A \mathbin{\widetilde{*}_R} B$ denote the Banach algebra obtained by completing $A * B$ under $\|\ \|_R^{\sim}$. It is clear that ρ extends to a contractive homomorphism $\widetilde{\rho} : A \mathbin{\widetilde{*}_R} B \to C^*(A \vee B)$, note however that $\widetilde{\rho}$ need not be injective.

The main analytic tool we have for studying $\delta_{A:B}$ is its relation to $\rho_{(1-m)}$, $m \in M$, $\|m\| < 1$.

Proposition 8.6.4. *If $f \in A * B$ and $m \in M$, $\|m\| < 1$, then*

$$\rho_{(1-m)}(f) = \sum_{p\geq 0} \theta_p[m, \dots, m]\left(\delta_{A:B}^{(p)}(\rho(f))\right),$$

where the series converges absolutely in the uniform norm on M. In particular, $\rho_{(1-m)}$ extends to a contractive homomorphism $\widetilde{\rho}_{(1-m)} : A \mathbin{\widetilde{*}_R} B \to M$.

Proof. First we will check that the series converges absolutely. Indeed, we have

$$\sum_{p\geq 0} \|\theta_p[m, \dots, m]\left(\delta_{A:B}^{(p)}(\rho(f))\right)\| \leq \sum_{p\geq 0} \|m\|^p \|\delta_{A:B}^{(p)}(\rho(f))\|^{\wedge}_{(p+1)} = \|f\|^{\sim}_{\|m\|},$$

which is finite by 8.6.2.

Now let $\varphi(f)$ denote the right hand side, it suffices to show that φ is a homomorphism from $A * B$ into M which agrees with $\rho_{(1-m)}$ when restricted to A or B. The proof that φ is a homomorphism is the same as Proposition 8.4.5. Clearly $\varphi(b) = b = \rho_{(1-m)}(b)$. For $a \in A$, we have

$$\begin{aligned}
\varphi(a) &= \sum_{p\geq 0} \theta_p[m, \dots, m]\left(a \otimes 1^{\otimes p} - 1 \otimes a \otimes 1^{\otimes (p-1)}\right) \\
&= \sum_{p\geq 0} (am^p - mam^{p-1}) \\
&= (1-m)a \sum_{p\geq 0} m^p \\
&= (1-m)a(1-m)^{-1} \\
&= \rho_{(1-m)}(a).
\end{aligned}$$

Now if $\|m\| \leq R < 1$, then we have

$$\|\rho_{(1-m)}(f)\| \leq \|f\|^{\sim}_{\|m\|} \leq \|f\|^{\sim}_R,$$

so that $\rho_{(1-m)}$ extends by continuity to a contractive homomorphism $\widetilde{\rho}_{(1-m)} : A \mathbin{\widetilde{*}_R} B \to M$. □

8.6.5. Voiculescu has shown [53] that the existence of $j(A : B)$ in $L^2(W^*(A \vee B))$ is a sufficient condition for the closability of $\delta_{A:B}$, viewed as an unbounded operator

$$\delta_{A:B} : L^2(W^*(A \vee B)) \to L^2(W^*(A \vee B) \otimes W^*(A \vee B)).$$

In particular, $|j(A : B)|_2 < \infty$ implies that $\delta_{A:B}$ is closable in the uniform norm, we will denote this closure by $\overline{\delta}_{A:B}$.

Proposition 8.6.6. *Let $1 \in B \subset M$ be a W*-subalgebra, and $1 \in A \subset M$ a $*$-subalgebra such that A and B are algebraically free. Suppose also that $|j(A:B)|_2 < \infty$. If $0 < R < 1$, then $\widetilde{\rho}(A \mathbin{\widetilde{*}_R} B) \subset \mathfrak{D}(\overline{\delta}_{A:B})$. Furthermore, if $f \in A \mathbin{\widetilde{*}_R} B$ and $\overline{\delta}_{A:B}(\widetilde{\rho}(f)) = 0$, then $\widetilde{\rho}(f) \in B$.*

Proof. It is clear from the definition of the norm $\| \, \|_{\widetilde{R}}$ that $\widetilde{\rho}$ maps $A \mathbin{\widetilde{*}_R} B$ into $\mathfrak{D}(\overline{\delta}_{A:B})$. Suppose then that $f \in A \mathbin{\widetilde{*}_R} B$, and $\overline{\delta}_{A:B}(\widetilde{\rho}(f)) = 0$. Let $f_n \in A * B$ s.t. $f_n \to f$ in $A \mathbin{\widetilde{*}_R} B$. Combining Proposition 8.6.4 with the argument given in Proposition 8.4.8, we have

$$\begin{aligned}\widetilde{\rho}_{(1-m)}(f) - \widetilde{\rho}(f) &= \lim_{n\to\infty} \rho_{(1-m)}(f_n) - \rho(f_n) \\ &= \lim_{n\to\infty} \sum_{p\geq 1} \theta_p[m,\ldots,m]\big(\delta^{(p)}_{A:B}(\rho(f_n))\big) \\ &= 0,\end{aligned}$$

for any $m \in M$ with $\|m\| < R$.

Now let S be a $(0,1)$-semicircular element in M which is independent from B and B-freely independent from A. Take $\epsilon > 0$ sufficiently small so that $\|U_\epsilon - 1\| < R$, where $U_\epsilon = \exp(i\pi\epsilon S)$. Then $\widetilde{\rho}_{U_\epsilon}(f) = \widetilde{\rho}(f)$, in particular $\widetilde{\rho}(f) \in C^*(A \vee B) \cap C^*(U_\epsilon A U_\epsilon^* \vee B)$. By Corollary 8.5.10, we have $\widetilde{\rho}(f) \in B$. □

Proposition 8.6.7. *Let $1 \in B \subset M$ be a W*-subalgebra, and let $1 \in A, C \subset M$ be $*$-subalgebras. Assume A and C are B-free in (M, E_B). Suppose also that $|j(A:B)|_2 < \infty$. Then there is a holomorphic function $F : \mathbb{H}_+(A) \times \mathbb{H}_+(C) \to B$ such that*

$$E_{A\vee B}(a + c)^{-1} = (a + F(a,c))^{-1}$$

for $a \in \mathbb{H}_+(A)$, $c \in \mathbb{H}_+(C)$.

Proof. Let $a \in \mathbb{H}_+(A)$, $c \in \mathbb{H}_+(C)$, and let $\alpha = (a+c)^{-1}$. By Proposition 8.3.14, $|j(A : B \vee C)|_2 < \infty$, so $\delta_{A:B}$ and $\delta_{A:B\vee C}$ are closable in norm. By Proposition 8.4.7, $\alpha \in \mathfrak{D}(\overline{\delta}_{A:B\vee C})$. By Proposition 8.2.1,

$$\overline{\delta}_{A:B}(\alpha) = -\alpha(a \otimes 1 - 1 \otimes a)\alpha.$$

It follows from Proposition 8.3.13 that $\gamma = E_B(\alpha) \in \mathfrak{D}(\overline{\delta}_{A:B})$ and

$$\overline{\delta}_{A:B}(\gamma) = -\gamma(a \otimes 1 - 1 \otimes a)\gamma.$$

Since $\alpha \in \mathbb{H}_-(M)$, it follows that also $\gamma \in \mathbb{H}_-(M)$, in particular γ is invertible. By Proposition 8.2.1, $\gamma = (a+n)^{-1}$ for some $n \in \operatorname{Ker} \overline{\delta}_{A:B}$.

Setting $F(a,c) = n$, it is clear that $F(a,c)$ depends analytically on (a,c), it remains only to show that $F(a,c) \in B$. Fix $a \in \mathbb{H}_+(A)$ and denote $F_a(c) = F(a,c)$ for $c \in \mathbb{H}_+(C)$. Since $F_a : \mathbb{H}_+(C) \to M$ is holomorphic, it suffices to show that $F_a(c) \in B$ for c in some open subset of $\mathbb{H}_+(C)$.

Fix $0 < R < 1$ and choose x sufficiently large so that $2\|a\|(1-R)^{-1}x^{-1} < 1/2$. Let

$$\Omega = \{c \in \mathbb{H}_+(C) : \|c - ix\| < \|a\|\}.$$

Given $c \in \Omega$, we have

$$(a+c)^{-1} = ((ix)(1-\Gamma))^{-1} = (ix)^{-1} \sum_{k \geq 0} \Gamma^k,$$

where $\Gamma = (ix)^{-1}(ix - a - c)$. Note that $\|\Gamma\| < 2\|a\|x^{-1}$. For $p \geq 1$ we have

$$\delta^{(p)}_{A:B\vee C}(\Gamma) = (ix)^{-1} \left(1 \otimes a \otimes 1^{\otimes(p-1)} - a \otimes 1^{\otimes p}\right),$$

so that

$$\|\delta^{(p)}_{A:B\vee C}(\Gamma)\|^{\wedge}_{(p+1)} \leq 2\|a\|x^{-1}.$$

Letting $P \in A * B$ such that $\rho(P) = \Gamma$, it follows that $f \in A *_R (B \vee C)$ and

$$\|P\|_{A*_R(B\vee C)} < 2\|a\|x^{-1}(1-R)^{-1} < 1/2.$$

Since $A *_R (B \vee C)$ is a Banach algebra, we have

$$\|P^k\|_{A*_R(B\vee C)} < 2^{-k}$$

for $k \geq 1$. Let $f_k \in A * B$ be such that $\rho(f_k) = E_{A\vee B}(\Gamma^k)$, by Proposition 8.3.13 we have

$$\|f_k\|^{\sim}_R < 2^{-k}$$

for $k \geq 1$. It follows that $\sum_{k\geq 1} f_k$ converges in $A \mathbin{\widetilde{*}_R} B$ to a limit f with $\|f\|^{\sim}_R < 1$. Let $g = (ix)^{-1}(1+f)^{-1} - a \in A \mathbin{\widetilde{*}_R} B$, then

$$F_a(c) = \widetilde{\rho}(f),$$

so that $\overline{\delta}_{A:B}(\widetilde{\rho}(f)) = \overline{\delta}_{A:B}(F_a(c)) = 0$. By Proposition 8.6.6, $F_a(c) \in B$.

□

We may now remove the condition on the liberation gradient.

Theorem 8.6.8. *Let $1 \in B \subset M$ be a W^*-subalgebra, and let $1 \in A, C \subset M$ be $*$-subalgebras. Assume A and C are B-free in (M, E_B). Then there is a holomorphic function $F : \mathbb{H}_+(A) \times \mathbb{H}_+(C) \to B$ such that*

$$E_{A\vee B}(a+c)^{-1} = (a + F(a,c))^{-1}$$

for $a \in \mathbb{H}_+(A)$, $c \in \mathbb{H}_+(C)$.

Proof. Let $a \in \mathbb{H}_+(A), c \in \mathbb{H}_+(C)$, and set

$$F(a,c) = \left(E_{A\vee b}(a+c)^{-1}\right)^{-1} - a,$$

we must show that $F(a,c) \in B$. Clearly $F(a,c)$ depends analytically on (a,c), hence it suffices to show that $F(a,c) \in B$ for (a,c) in some open subset of $\mathbb{H}_+(A) \times \mathbb{H}_+(C)$. Let

$$\Omega = \{(a,c) \in \mathbb{H}_+(A) \times \mathbb{H}_+(C) : \|a - i\| < 1/2, \|c - Ki\| < 1/2\},$$

where $K \gg 0$.

Now let S be a $(0,1)$-semicircular element in M which is freely independent from $A \vee B \vee C$. For $0 < \epsilon < 1$ let $U_\epsilon = \exp(i\pi\epsilon S)$. By Proposition 8.5.9, $|\xi(U_\epsilon : \mathbb{C})|_2 < \infty$. Hence by Proposition 8.5.6, $|j(U_\epsilon A U_\epsilon^* : B)|_2 < \infty$.

So fix $(a,c) \in \Omega$, by the proposition there are $b_\epsilon \in B$ for $0 < \epsilon < 1$ such that

$$E_{U_\epsilon A U_\epsilon^* \vee B}(U_\epsilon a U_\epsilon^* + c)^{-1} = (a + b_\epsilon)^{-1}.$$

Now since $(a,c) \in \Omega$, we have

$$\|U_\epsilon a U_\epsilon^* + c\| \leq K + 2 \qquad \operatorname{Im}(U_\epsilon a U_\epsilon^* + c) \geq K.$$

It follows from 8.4.9 that

$$\|(U_\epsilon a U_\epsilon^* + c)^{-1}\| \leq K^{-1} \qquad \operatorname{Im}(U_\epsilon a U_\epsilon^* + C)^{-1} \leq -(K + (K+2)^2/K)^{-1}.$$

Therefore

$$\|(U_\epsilon a U_\epsilon^* + b_\epsilon)^{-1}\| \leq K^{-1} \qquad \operatorname{Im}(U_\epsilon a U_\epsilon^* + b_\epsilon)^{-1} \leq -(K + (K+2)^2/K)^{-1}.$$

Applying 8.4.9 once more, we see

$$\operatorname{Im}(U_\epsilon a U_\epsilon^* + b_\epsilon) \geq \left((K + (K+2)^2/K)^{-1} + (K + (K+2)^2/K)K^{-2}\right)^{-1} = \frac{K + \frac{(K+2)^2}{K}}{3 + \frac{K^2}{(K+2)^2} + \frac{(K+2)^2}{K^2}}.$$

For K sufficiently large, this is greater than 2, from which it follows that $\operatorname{Im}(b_\epsilon) > 1/2$ for $0 < \epsilon < 1$. In this case, it follows from 8.4.9 that

$$\|(a + b_\epsilon)^{-1}\| \leq C,$$

for some finite constant C which does not depend on ϵ. Hence

$$\lim_{\epsilon \to 0} \|(U_\epsilon a U_\epsilon^* + b_\epsilon)^{-1} - (a + b_\epsilon)^{-1}\| = 0,$$

and therefore

$$\lim_{\epsilon \to 0} \|(a + b_\epsilon)^{-1} - E_{A \vee B}(U_\epsilon a U_\epsilon^* + b_\epsilon)^{-1}\| = 0.$$

An application of [53, Lemma 3.3] shows that $A \vee B$, C and S are B-free, and another application shows that $A \vee B$, $U_\epsilon A U_\epsilon^* \vee B$, $U_\epsilon A U_\epsilon^* \vee B \vee C$ is a freely Markovian triple. By [53, Lemma 3.7], we have

$$E_{A \vee B} E_{U_\epsilon A U_\epsilon^* \vee B} E_{U_\epsilon A U_\epsilon^* \vee B \vee C} = E_{A \vee B} E_{U_\epsilon A U_\epsilon^* \vee B \vee C}.$$

We therefore have

$$\begin{aligned} E_{A \vee B}(a + c)^{-1} &= \lim_{\epsilon \to 0} E_{A \vee B}(U_\epsilon a U_\epsilon^* + c)^{-1} \\ &= \lim_{\epsilon \to 0} E_{A \vee B} E_{U_\epsilon A U_\epsilon^* \vee B}(U_\epsilon a U_\epsilon^* + c)^{-1} \\ &= \lim_{\epsilon \to 0} E_{A \vee B}(U_\epsilon a U_\epsilon^* + b_\epsilon)^{-1} \\ &= \lim_{\epsilon \to 0} (a + b_\epsilon)^{-1}. \end{aligned}$$

It follows that b_ϵ converges as $\epsilon \to 0$ to

$$F(a,c) = \left(E_{A\vee B}(a+c)^{-1}\right)^{-1} - a,$$

hence $F(a,c) \in B$ which completes the proof. □

8.7 Analytic subordination for free multiplicative convolution

In this section we use the derivation $d_{U:B}$ to prove the analytic subordination result for multiplication of B-freely independent unitaries, where B is a general W*-algebra of constants.

8.7.1. Fix a unitary $U \in M$ which is algebraically free from B. Define $d^{(p)}_{U:B} : B\langle U, U^*\rangle \to (B\langle U, U^*\rangle)^{\otimes(p+1)}$ recursively by $d^{(0)}_{U:B} = \mathrm{id}$,

$$d^{(p+1)}_{U:B} = (d_{U:B} \otimes \mathrm{id}^{\otimes p}) \circ d^{(p)}_{U:B}.$$

Define a norm $\| \|^{\sim}_{R,U}$ on $B\langle t\rangle$ by

$$\|f\|^{\sim}_{R,U} = \sum_{p\geq 0} \|d^{(p)}_{U:B}(f(U))\|^{\wedge}_{(p+1)},$$

where $\| \|^{\wedge}_{(s)}$ denotes the projective tensor product norm on $M^{\widehat{\otimes} s}$.

Lemma 8.7.2. *$\| \|^{\sim}_{R,U}$ is a finite norm on $B\langle t\rangle$, and if $f, g \in B\langle t\rangle$ then*

$$\|fg\|^{\sim}_{R,U} \leq \|f\|^{\sim}_{R,U}\|g\|^{\sim}_{R,U}.$$

The proof is the same as the argument for $\partial_{X:B}$.

8.7.3. Let $B^{\sim}_{R,U}\{t\}$ denote the completion of $B\langle t\rangle$ under $\| \|^{\sim}_{R,U}$. The map sending $f \in B\langle t\rangle$ to $f(U)$ extends to a contractive homomorphism from $B^{\sim}_{R,U}$ into M, which we will still denote by $f \mapsto f(U)$.

Remark 8.7.4. Similarly to ∂, δ, $d_{U:B}$ is related to the homomorphism $f \mapsto f((1+m)U)$, $f \in B\langle t\rangle$, where $m \in M$ is fixed.

Proposition 8.7.5. *Fix $m \in M$, then for $f \in B\langle t\rangle$ we have*

$$f((1+m)U) = \sum_{p\geq 0} \theta_p[m,\ldots,m]\left(d^{(p)}_{U:B}(f(U))\right).$$

In particular, if $\|m\| \leq R$ then $f \mapsto f((1+m)U)$ extends to a contractive homomorphism from $B_{R,U}\{t\}$ into M, which we will also denote by $f \mapsto f((1+m)U)$.

Proof. First observe that the right hand side has only finitely many nonzero terms, so convergence is not an issue. Let $\varphi(f)$ denote the right hand side. Repeating the argument from Proposition 8.4.5, we see that φ is a homomorphism from $B\langle t\rangle$ into M. Since $\varphi(b) = b$ for $b \in B$, and

$$\varphi(t) = (1+m)U,$$

it follows that $\varphi(f) = f((1+m)U)$ as claimed. For $f \in B\langle t\rangle$ and $\|m\| \leq R$, we then have

$$\begin{aligned}\|f((1+m)U)\| &\leq \sum_{p\geq 0} \|\theta_p[m,\ldots,m]\left(d^{(p)}_{U:B}f(U)\right)\| \\ &\leq \sum_{p\geq 0} \|m\|^p \|d^{(p)}_{U:B}f(U)\|^{\wedge}_{(p+1)} \\ &\leq \|f\|^{\sim}_{R,U}.\end{aligned}$$

So $f \mapsto f((1+m)U)$ extends to a contractive homomorphism on $B^{\sim}_{R,U}\{t\}$ as claimed. ☐

8.7.6. Recall that $\xi(U:B)$ is determined by $\xi(U:B) \in L^1(W^*(B\langle U,U^*\rangle))$ and

$$\tau\left(\xi(U:B)m\right) = (\tau\otimes\tau)\left(d_{U:B}(m)\right) \quad m \in B\langle U,U^*\rangle.$$

Voiculescu has proved that the existence of $\xi(U:B) \in L^2(B\langle U,U^*\rangle)$ is a sufficient condition for the closability of $d_{U:B}$ when viewed as an unbounded operator

$$d_{U:B} : L^2(W^*(B\langle U,U^*\rangle)) \to L^2(W^*(B\langle U,U^*\rangle)\otimes W^*(B\langle U,U^*\rangle)).$$

In particular, $|\xi(U:B)|_2 < \infty$ implies that $d_{U:B}$ is closable in the uniform norm, we will denote this closure by $\overline{d}_{U:B}$.

Proposition 8.7.7. *Suppose that $|\xi(U:B)|_2 < \infty$. If $f \in B_{R,U}\{t\}$, then $f(U) \in \mathfrak{D}(\overline{d}_{U:B})$. Furthermore, if $R > 2$ and if $\overline{d}_{U:B}(f(U)) = 1\otimes f(U)$, then $f(U) = Ub$ for some $b \in B$.*

Proof. The first statement is clear, so let $f \in B^{\sim}_{R,U}\langle t\rangle$ and take $f_n \in B\langle t\rangle$ with $f_n \to f$ in $\|\ \|^{\sim}_{R,U}$. Combining Proposition 8.7.5 with the argument in Proposition 8.4.8, we have

$$\begin{aligned}f((1+m)U) - (1+m)f(U) &= \lim_{n\to\infty} f_n((1+m)U) - (1+m)f_n(U) \\ &= \lim_{n\to\infty}\sum_{p\geq 2}\theta_p[m,\ldots,m]\big(d^{(p)}_{U:B}(f_n(U))\big) \\ &= 0,\end{aligned}$$

for any $m \in M$ with $\|m\| < R$.

If $R > 2$, we can apply this to $m = U^* - 1$ to find

$$f(1) = U^*f(U).$$

Since $f(1) \in B$, the result follows. ☐

We will also use the following technical lemma from [55]:

Lemma 8.7.8. *If $x \in A$, where A is a unital C^*-algebra, the following are equivalent:*

(i) $\|x\| < 1$.

(ii) $1 - x$ *is invertible and* $2Re(1-x)^{-1} \geq (1+\epsilon)$ *for some* $\epsilon > 0$.

Proposition 8.7.9. *Let $1 \in B \subset M$ be a W^*-subalgebra, and let $U, V \in M$ be unitaries such that $B\langle U, U^*\rangle$ is B-freely independent from $B\langle V, V^*\rangle$ in (M, E_B). Suppose also that $|\xi(U : B)|_2 < \infty$. Then there is a holomorphic map $F : \mathbb{D}(B) \to \mathbb{D}(B)$ such that*

$$E_{B\langle U,U^*\rangle} UVb(1-UVb)^{-1} = UF(b)(1-UF(b))^{-1}$$

and $\|F(b)\| \leq b$ for $b \in \mathbb{D}(B)$.

Proof. Since $|\xi(U : B)|_2 < \infty$, also $|\xi(UV : B)|_2 < \infty$ by 8.3.16. So $d_{U:B}$ and $d_{UV:B}$ are both closable in the uniform norm. Let $b \in \mathbb{D}(B)$, and set $\alpha = UVb(1-UVb)^{-1}$. Then $\alpha \in \mathfrak{D}(\overline{d}_{UV:B})$ by Proposition 8.4.7, and by Proposition 8.2.1 we have

$$\overline{d}_{UV:B}(\alpha) = (\alpha + 1) \otimes \alpha.$$

It follows from Corollary 8.3.15 that $\gamma = E_{B\langle U,U^*\rangle}(\alpha) \in \mathfrak{D}(\overline{d}_{U:B})$, and

$$\overline{d}_{U:B}(\gamma) = (\gamma + 1) \otimes \gamma.$$

Now

$$\gamma + 1 = E_{B\langle U,U^*\rangle}(1-UVb)^{-1},$$

so to show that $\gamma+1$ is invertible, it suffices to show that 0 is not in the convex hull of the spectrum of $(1-UVb)^{-1}$. Let $z \in \mathbb{C}$, then

$$(1-UVb)^{-1} - z = (1 - z + zUVb)(1-UVb)^{-1}$$

is invertible if $|z|\|b\| < |1-z|$, in particular if $\mathrm{Re}(z) < 1/2$. So $\gamma+1$ is invertible, and by Proposition 8.2.1 we have $\gamma = Un(1-Un)^{-1}$ for some $n \in \mathrm{Ker}\ \overline{d}_{U:B}$ such that $1 - Un$ is invertible.

It is clear that n depends analytically on b, it remains to show that $n \in \mathbb{D}(B)$, and that $\|n\| \leq \|b\|$. First we claim that $\|n\| < 1$. Since U is unitary, it suffices to show that $\|Un\| < 1$. By Lemma 8.7.8, it suffices to show that $1 - Un$ is invertible, and $2\mathrm{Re}(1-Un)^{-1} \geq (1+\epsilon)$ for some $\epsilon > 0$. But we have

$$(1-Un)^{-1} = \gamma + 1 = E_{B\langle U,U^*\rangle}(1-UVb)^{-1},$$

and since $\|UVb\| < 1$, applying Lemma 8.7.8 again shows that $2\mathrm{Re}(1-UVb)^{-1} \geq (1+\epsilon)$ for some $\epsilon > 0$. So $\|b\| < 1$, and it then follows from analyticity that in fact $\|F(b)\| \leq \|b\|$. Indeed, let $b \in \mathbb{D}(B)$, and let ψ a bounded linear functional on M, then $z \mapsto \psi(F(z(b/\|b\|)))$ is an analytic function $\mathbb{D}(\mathbb{C}) \to \mathbb{D}(\mathbb{C})$. By Schwarz's lemma, $|\psi(F(z(b/\|b\|)))| \leq |z|$ for $z \in \mathbb{D}(\mathbb{C})$. Taking $z = \|b\|$, we have $|\psi(F(b))| \leq \|b\|$, since ψ is arbitrary we have $\|F(b)\| \leq \|b\|$.

Finally we claim that $F(b) \in B$ for $b \in \mathbb{D}(B)$. By analytic continuation, it suffices to show this for $\|b\|$ sufficiently small. Let $R > 2$, $0 < \epsilon < 1/2$ and let $b \in B$, $\|b\|(1+R) < \epsilon$. We have

$$UVb(1 - UVb)^{-1} = \sum_{n\geq 1} (UVb)^n.$$

Now

$$d^{(p)}_{UV:B}(UVb) = \begin{cases} UVb & p = 0 \\ 1 \otimes UVb & p = 1 \\ 0 & p \geq 2 \end{cases}.$$

In particular, setting $f = tb \in B\langle t\rangle$ we have

$$\|f\|^{\sim}_{R,UV} < \epsilon.$$

It follows that

$$\|f^n\|^{\sim}_{R,UV} < \epsilon^n.$$

Now since U and V are B-free, it follows that

$$E_{B\langle U,U^*\rangle}(UVb)^n \in B\langle U\rangle,$$

so let $P_n \in B\langle t\rangle$ be such that

$$P_n(U) = E_{B\langle U,U^*\rangle}(UVb)^n.$$

By Corollary 8.3.15,

$$d^{(p)}_{U:B} P_n(U) = (E_{B\langle U,U^*\rangle})^{\otimes(p+1)} d^{(p)}_{UV:B}(UVb)^n.$$

In particular,

$$\|P_n\|^{\sim}_{R,U} \leq \|f^n\|^{\sim}_{R,UV} < \epsilon^n.$$

So $\sum_{n\geq 1} P_n$ converges in $B^{\sim}_{R,U}\{t\}$ to some limit h with $\|h\| < 1$. It follows that $1 + h$ is invertible in $B^{\sim}_{R,U}\{t\}$, and

$$UF(b) = g(U),$$

where $g = 1 - (1 + h)^{-1}$. But $g \in B^{\sim}_{R,U}\{t\}$ and $\overline{d}_{U:B}(g(U)) = 1 \otimes g(U)$, so by Proposition 8.7.7, $g(U) = Ub$ for some $b \in B$. Since U is invertible, we have $F(B) = b \in B$, which completes the proof.

□

We may now remove the condition on the conjugate $\xi(U : B)$.

Theorem 8.7.10. *Let $1 \in B \subset M$ be a W*-subalgebra, and let $U, V \in M$ be unitaries such that $B\langle U,U^*\rangle$ is B-freely independent from $B\langle V,V^*\rangle$ in (M, E_B). Then there is a holomorphic map $F : \mathbb{D}(B) \to \mathbb{D}(B)$ such that*

$$E_{B\langle U,U^*\rangle} UVb(1 - UVb)^{-1} = UF(b)(1 - UF(b))^{-1}$$

and $\|F(b)\| \leq b$ for $b \in \mathbb{D}(B)$.

Proof. Let S be a $(0,1)$-semicircular element in (M,τ) which is freely independent from the algebra $B\langle U,V,U^*,V^*\rangle$. Applying [53, Lemma 3.3] twice, we see that $B\langle U,U^*\rangle$, $B\langle U_\epsilon U, U^*U_\epsilon^*\rangle$, $B\langle U_\epsilon UV, V^*U^*U_{\epsilon^*}\rangle$ is a freely Markovian triple, where $U_\epsilon = \exp(\pi i\epsilon S)$. By [53, Lemma 3.4], we have

$$E_{B\langle U,U^*\rangle}E_{B\langle U_\epsilon U,U^*U_\epsilon^*\rangle}E_{B\langle U_\epsilon UV,V^*U^*U_\epsilon^*\rangle} = E_{B\langle U,U^*\rangle}E_{B\langle U_\epsilon UV,V^*U^*U_\epsilon^*\rangle}.$$

Now $B\langle U_\epsilon U, U^*U_\epsilon^*\rangle$ and $B\langle V,V^*\rangle$ are B-free, and $|\xi(U_\epsilon U : B)|_2 < \infty$ by Corollary 8.5.9. So given $b \in \mathbb{D}$, we may apply the proposition to find $n_\epsilon \in B$, $0<\epsilon<1$, such that $\|n_\epsilon\| \le \|b\|$ and

$$E_{B\langle U_\epsilon U,U^*U_\epsilon^*\rangle}U_\epsilon UVb(1-U_\epsilon UVb)^{-1} = U_\epsilon Un_\epsilon(1-U_\epsilon Un_\epsilon)^{-1}.$$

It follows that

$$E_{B\langle U,U^*\rangle}U_\epsilon UVb(1-U_\epsilon UVb)^{-1} = E_{B\langle U,U^*\rangle}U_\epsilon Un_\epsilon(1-U_\epsilon Un_\epsilon)^{-1}.$$

Now since $U_\epsilon UVb$ tends to UVb as $\epsilon \to 0$, and $(1-UVb)^{-1}$ is invertible, it follows that

$$\begin{aligned}\lim_{\epsilon\to 0} U_\epsilon UVb(1-U_\epsilon UVb)^{-1} &= \lim_{\epsilon\to 0}(1-U_\epsilon UVb)^{-1} - 1\\ &= (1-UVb)^{-1} - 1\\ &= UVb(1-UVb)^{-1},\end{aligned}$$

with convergence in norm. Since $\|n_\epsilon\| \le \|b\| < 1$ for $0<\epsilon<1$, it follows that

$$\lim_{\epsilon\to 0}\|U_\epsilon Un_\epsilon(1-U_\epsilon Un_\epsilon)^{-1} - Un_\epsilon(1-Un_\epsilon)^{-1}\| = 0.$$

Hence,

$$\begin{aligned}E_{B\langle U,U^*\rangle}UVb(1-UVb)^{-1} &= \lim_{n\to\infty} E_{B\langle U,U^*\rangle}U_\epsilon UVb(1-U_\epsilon UVb)^{-1}\\ &= \lim_{n\to\infty} E_{B\langle U,U^*\rangle}U_\epsilon Un_\epsilon(1-U_\epsilon Un_\epsilon)^{-1}\\ &= \lim_{n\to\infty} Un_\epsilon(1-Un_\epsilon)^{-1}.\end{aligned}$$

By the argument in the previous proposition, $E_{B\langle U,U^*\rangle}(1-UVb)^{-1}$ is invertible, so that

$$\lim_{\epsilon\to 0} 1 - Un_\epsilon = \left(E_{B\langle U,U^*\rangle}(1-UVb)^{-1}\right)^{-1}.$$

From this it follows that n_ϵ converges to a limit $n \in B$, such that $\|n\| \le \|b\|$ and

$$E_{B\langle U,U^*\rangle}UVb(1-UVb)^{-1} = Un(1-Un)^{-1}.$$

Since the analytic dependence is clear, this completes the proof.

□

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