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Authors

De Loera, JA
La Haye, RN
Oliveros, D
[et al.](#)

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HELLY NUMBERS OF ALGEBRAIC SUBSETS OF \mathbb{R}^d AND AN EXTENSION OF DOIGNON'S THEOREM

J. A. DE LOERA, R. N. LA HAYE, D. OLIVEROS, AND E. ROLDÁN-PENSADO

ABSTRACT. We study S -convex sets, which are the geometric objects obtained as the intersection of the usual convex sets in \mathbb{R}^d with a proper subset $S \subset \mathbb{R}^d$, and contribute new results about their S -Helly numbers. We extend prior work for $S = \mathbb{R}^d$, \mathbb{Z}^d , and $\mathbb{Z}^{d-k} \times \mathbb{R}^k$, and give some sharp bounds for several new cases; low-dimensional situations, sets that have some algebraic structure, in particular when S is an arbitrary subgroup of \mathbb{R}^d or when S is the difference between a lattice and some of its sublattices. By abstracting the ingredients of Lovász method we obtain colorful versions of many monochromatic Helly-type results, including several colorful versions of our own results.

1. INTRODUCTION

Eduard Helly stated his fundamental theorem slightly under a century ago [23]. It says that the members of a finite family of convex sets in \mathbb{R}^d intersect, if every $d + 1$ of them intersect (see [28] for the basic introduction to this part of convexity). Since its discovery Helly's theorem has found many generalizations, extensions and applications in many areas of mathematics (see [11, 13, 18, 33] and references therein). Continuing the work of many authors (see e.g., [4, 17, 25, 26, 27, 32] and the many references therein), our paper presents new versions of Helly's theorem where the intersections in the hypothesis and conclusions of the theorems are restricted to exist in a proper subset S of \mathbb{R}^d . The key challenge is to compute the associated S -Helly number. It is fair to say that applications in optimization have prompted already many papers about S -Helly numbers [4, 8, 9, 10, 20, 25, 30]. The interest in our new S -Helly numbers is motivated not just by old and even new (see e.g., [12]) applications to optimization, but also in relation to algebraic structures for S and new applications to algebraic computation with polynomials [14], where more sophisticated values for the variables appear.

The 1970s and 1980s saw large growth in the research of *abstract convexity* where Helly-type theorems were explored in abstract settings beyond Euclidean spaces. For a proper subset $S \subset \mathbb{R}^d$ the *S -convex sets*, are the geometric objects obtained as the

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intersection of the usual convex sets in \mathbb{R}^d with S . The S -convex sets $\mathcal{K}_S = \{S \cap K : K \subseteq \mathbb{R}^d \text{ is convex}\}$ satisfy the convexity space axioms as stated in [32]. In the literature these are called *convexity spaces* (see [2, 17, 25, 26, 27, 32] and the many references therein for more on convexity spaces). For example, there is a Helly-type theorem about the existence of intersections over the *integer lattice* \mathbb{Z}^d , proved by Doignon [16] (later rediscovered in [8, 25, 30]); it states that a finite family of convex sets in \mathbb{R}^d intersects at a point of \mathbb{Z}^d if every 2^d members of the family intersect at a point of \mathbb{Z}^d . A second example is the work of Averkov and Weismantel [4] who gave a *mixed* version of Helly's and Doignon's theorems which includes them both. This time the intersection of the convex sets is required to be in $\mathbb{Z}^{d-k} \times \mathbb{R}^k$ and this can be guaranteed if every $2^{d-k}(k+1)$ sets intersect in such a point (their formula had been previously stated by A.J. Hoffman in [25]). Note that the size of subfamilies that guarantees an intersection of all members of the family in S depends on S . These are the *Helly numbers of S* from our title, which we now define precisely.

For a nonempty family \mathcal{K} of sets, the *Helly number* $h = h(\mathcal{K}) \in \mathbb{N}$ of \mathcal{K} is defined as the smallest natural number h satisfying the following:

$$(1.1) \quad \forall i_1, \dots, i_h \in [m] : F_{i_1} \cap \dots \cap F_{i_h} \neq \emptyset \quad \implies \quad F_1 \cap \dots \cap F_m \neq \emptyset$$

for all $m \in \mathbb{N}$ and $F_1, \dots, F_m \in \mathcal{K}$. If no such h exists, then $h(\mathcal{K}) := \infty$. E.g., for the traditional Helly's theorem, \mathcal{K} is the family of all convex subsets of \mathbb{R}^d .

Finally for $S \subseteq \mathbb{R}^d$ we define

$$h(S) := h(\{S \cap K : K \subseteq \mathbb{R}^d \text{ is convex}\}).$$

That is, $h(S)$ is the Helly number when the sets are required to intersect at points in S ; we will call this the *S -Helly number*. The problem we study here is, given a set S , to provide bounds for the S -Helly number $h(S)$.

For instance, note that when S is finite then the bound $h(S) \leq \#(S)$ is trivial. The original Helly number is $h(\mathbb{R}^d) = d + 1$ and, interestingly, if \mathbb{F} is any subfield of \mathbb{R} (e.g., $\mathbb{Q}(\sqrt{2})$), then Radon's proof of Helly's theorem directly shows that the S -Helly number of $S = \mathbb{F}^d$ is still $d + 1$. As we mentioned earlier, we also know $h(\mathbb{Z}^{d-k} \times \mathbb{R}^k) = 2^{d-k}(k + 1)$. However, if S is not necessarily a lattice but a general additive subgroup (e.g., $S = \{(\alpha\pi + \beta, \gamma) \in \mathbb{R}^2 : \alpha, \beta, \gamma \in \mathbb{Z}\}$), then the S -Helly number is not covered by prior results. It is known that the S -Helly number may be infinite in some situations.

Now we are ready to state our contributions. The proofs will be presented after preliminaries in Section 2. The preliminary results were priorly obtained by other authors, but we show that together they are a very useful toolkit for computing S -Helly numbers.

Our contributions. We first present bounds on Helly numbers for low dimensions and for algebraic subsets S of \mathbb{R}^d . For simplicity we will always assume that the linear span of S is the whole space \mathbb{R}^d . We begin in Section 3 with results and open questions in dimensions one and two. For the case of dimension $d = 1$ it is easy to see that for any set S , the Helly number $h(S)$ exists and is at most two. In Section 3 we present the following two theorems:

Theorem 1.1. *If S is a dense subset of \mathbb{R}^2 , then $h(S) \leq 4$. This result is sharp.*

Theorem 1.2. *Let L be a proper sublattice of \mathbb{Z}^2 . If $S = \mathbb{Z}^2 \setminus L$, then $h(S) \leq 6$. This result is sharp.*

In Section 4 we explore the situation where S is a set with an algebraic structure. First we consider the case where S is an additive subgroup of \mathbb{R}^d (not necessarily closed) and we propose a conjecture for general subgroups. If S has additional structure—for instance, if S is a G -module, (i.e., an additive Abelian group $(S, +)$ with a compatible scalar multiplication $G \times S \rightarrow S$, with coefficients in a group G)—then we can indeed obtain better bounds for $h(S)$:

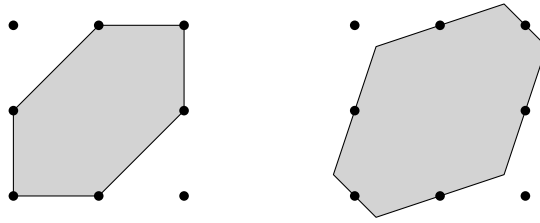
Theorem 1.3. *Let G be a dense subgroup of \mathbb{R} . If $S \subset \mathbb{R}^d$ is a G -module then $h(S) \leq 2d$. Furthermore, this number is best possible.*

We extended the classical Doignon’s theorem for a lattice by proving that the set difference of a lattice and the union of several of its sublattices has bounded Helly number too. We prove the following theorem which has already found applications in stochastic convex discrete optimization (see [12]). In fact, lattices with removed sublattices play also an interesting role in Minkowski’s geometry of numbers, see for example [19, 24] and the references within.

Theorem 1.4. *Let L_1, \dots, L_k be (possibly translated) sublattices of \mathbb{Z}^d . Then the set $S = \mathbb{Z}^d \setminus (L_1 \cup \dots \cup L_k)$ has Helly number $h(S) \leq C_k 2^d$ for some constant C_k depending only on k .*

Note that when no sublattices are removed this is exactly Doignon’s theorem.

The final contribution is about colorful versions of all our Helly theorems. Many Helly-type theorems admit a colorful version. In this setting, each convex set is assigned one of N colors and every N convex sets of distinct colors are required to intersect. The conclusion is that there is a color for which all convex sets intersect. In Section 5 we analyze and abstract a popular method due to Lovász. We show only that very general assumptions are necessary to yield colorful versions for several Helly-type theorems. As a result we recover older colorful Helly-type theorems and find a few new ones.

FIGURE 1. An S -vertex-polytope and an S -facet-polytope.

2. TOOLS TO COMPUTE HELLY NUMBERS

We will be concerned with proving bounds on S -Helly numbers of specific $S \subseteq \mathbb{R}^d$. In this section we discuss some useful methods priorly used to bound the S -Helly number of a subset S of \mathbb{R}^d . They were studied first by Hoffman [25] in the setting of abstract convexity, and later on refined by Averkov and Weismantel in [4], and Averkov in [3].

Definition 2.1. We define an S -vertex-polytope as the convex hull of points $x_1, x_2, \dots, x_k \in S$ in convex position such that no other point of S is in $\text{conv}(x_1, \dots, x_k)$.

Similarly, an S -facet-polytope is defined as the intersection of half-spaces H_1, H_2, \dots, H_k such that $\bigcap_i H_i$ has k facets and contains exactly k points of S , one contained in the relative interior of each facet.

Figure 1 shows an S -vertex-polytope with six vertices and an S -facet-polytope with six sides in \mathbb{R}^2 .

Lemma 2.2 (Theorem 2.1 in [3] and Proposition 3 in [25]). *Assume $S \subset \mathbb{R}^d$ is discrete, then $h(S)$ is equal to the following two numbers:*

- (1) *The supremum $f(S)$ of the number of facets of an S -facet-polytope.*
- (2) *The supremum $g(S)$ of the number of vertices of an S -vertex-polytope.*

The case when S is not discrete is a bit more delicate. Averkov [3] showed that in general $f(S) \leq h(S)$. Recently, new results involving these two quantities and the S -Helly number $h(S)$ have been obtained by Conforti and Di Summa [10]. E.g., they proved that when S is a closed subset of \mathbb{R}^d and $f(S)$ is finite, then $h(S) \leq (d+1)f(S)$. They also showed an example where $f(S) = 1$, yet the S -Helly number is infinite. Fortunately, for the calculation of Helly numbers there is a very general result due to A.J. Hoffman characterizing S -Helly numbers.

Lemma 2.3 (Proposition 2 in [25]). *If $S \subset \mathbb{R}^d$, then $h(S)$ can be computed as the supremum of h such that the following holds: There exists a set $R = \{x_1, \dots, x_h\} \subset S$ such that $\bigcap_i \text{conv}(R \setminus \{x_i\})$ does not intersect S .*

Note that, in particular, the points of R must be in strictly convex position. As a direct application of Lemma 2.3 we have the following proposition.

Proposition 2.4. *If $S_1, S_2 \subseteq \mathbb{R}^d$, then $h(S_1 \cup S_2) \leq h(S_1) + h(S_2)$.*

The Helly number of a product is not always so well-behaved as in the mixed integer case, but Averkov and Weismantel proved the following

Proposition 2.5 (Theorem 1.1 in [4]). *If M is a closed subset of \mathbb{R}^k then $h(\mathbb{R}^d \times M) \leq (d+1)h(M)$.*

Averkov and Weismantel also proved that $h(\mathbb{Z}^d \times M) \geq 2^d h(M)$ for closed $M \subseteq \mathbb{R}^k$, but this is less useful for upper bounds. Similarly, one can prove a general bound in the case of discrete sets.

Theorem 2.6 (Conforti, Di Summa [10]). *If $S_1, S_2 \subseteq \mathbb{R}^d$ are discrete sets, then $h(S_1 \times S_2) \geq h(S_1)h(S_2)$.*

3. S-HELLY NUMBERS IN LOW DIMENSION

We start by looking at the one-dimensional case, here there is a very general result which immediately follows from Lemma 2.3.

Lemma 3.1. *If $S \subseteq \mathbb{R}$, then $h(S) \leq 2$.*

That is, the S -Helly number of any subset of the real line is two, or one in case S consists of a single point. Unfortunately in \mathbb{R}^2 we no longer have such a nice theorem. Consider the example below.

Example 3.2. Let $S_n = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ be a set of n points in strictly convex position. Then any subset of S_n can be expressed as $S_n \cap K$ where $K \subset \mathbb{R}^2$ is convex. Lemma 2.2 implies $h(S_n) = n$. If for each n we take a copy of S_n such that their convex hulls do not intersect, then their union S will have $h(S) = \infty$. A simpler, but non-discrete, example with $h(S) = \infty$ is a circumference.

In spite of Example 3.2, we still have a general result in dimension two.

Proof of Theorem 1.1. We begin by noting that the bound of four is sharp. This can be seen by taking $S = \mathbb{R}^2 \setminus \{0\}$ and convex sets $\{(x, y) \in \mathbb{R}^2 | x \geq 0\}$, $\{(x, y) \in \mathbb{R}^2 | x \leq 0\}$, $\{(x, y) \in \mathbb{R}^2 | y \geq 0\}$, $\{(x, y) \in \mathbb{R}^2 | y \leq 0\}$. In this case $h(S) = 4$.

Now, assume that S is some arbitrary dense set with $h(S) > 4$. Lemma 2.3 provides a set $R = \{x_1, \dots, x_5\}$ such that $\bigcap_i (R \setminus \{x_i\})$ does not intersect S . But since the points of R are in strictly convex position, $\bigcap_i (R \setminus \{x_i\})$ has non-empty interior and must intersect S , a contradiction. \square

Theorem 1.1 does not hold in dimensions three and higher. In fact, we can construct a dense set S in \mathbb{R}^3 with $h(S) = \infty$.

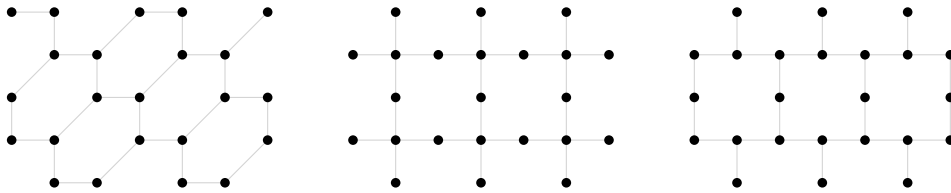


FIGURE 2. Three examples of lattice differences.

Example 3.3. In \mathbb{R}^3 , consider a set of points S_0 on the plane $\{z = 0\}$ with $h(S_0) = \infty$. Let $S = (\mathbb{R}^3 \setminus \{z = 0\}) \cup S_0$. Then S is clearly dense and $h(S) = \infty$.

One very important family of subsets of \mathbb{R}^d are *lattices*, i.e., discrete subgroups of \mathbb{R}^d . As we mention before, Doignon in [16] made an interesting investigation of the Helly number of a lattice S of rank d and showed that $h(S) = 2^d$ (his work was independently rediscovered by D. Bell and H. Scarf [8, 30]). Recently in [1] this result was generalized to force not just a non-empty intersection with the lattice, but to control the number of lattice points in the intersection. Next we state a particular case of the results:

Lemma 3.4 (See Theorem 1 in [1]). *Let $k = 0, 1$ or 2 . Assume that \mathcal{F} is a family of convex sets in \mathbb{R}^2 such that their intersection contains exactly k points of \mathbb{Z}^2 . Then there is a subfamily of \mathcal{F} with at most 6 elements such that their intersection contains exactly k points of \mathbb{Z}^2 .*

The set consisting of the difference between two lattices is a rather rich set of points in \mathbb{R}^d : they have interesting periodic patterns but contain complicated empty regions, and are closely related to tilings of space [22]. See Figure 2 for some examples. In the case of dimension two, Theorem 1.2 deals with finding $h(S)$ for this type of sets, and we achieve a tight result. In contrast, Theorem 1.4 presents a more general theorem for arbitrary dimensions but the bounds are far from tight. It is worth noting that the proof of Theorem 1.4 contains a surprising application of Ramsey’s theorem.

Proof of Theorem 1.2. From Lemma 2.2 it is enough to bound the number of edges of an S -facet-polygon. Assume $k > 6$ and there is an S -facet-polygon K determined by the semi-planes H_1, H_2, \dots, H_k .

Note that if $|\text{int}(K) \cap \mathbb{Z}^2| \leq 2$, (were $\text{int}(K)$ is the interior set of K) then by Lemma 3.4 applied to the interiors of the H_i , we can find H_{i_1}, \dots, H_{i_6} among our original semi-planes such that $\text{int}(\bigcap_j H_{i_j})$ contains no additional points of L . But the relative interior of at least one side of K is contained in $\text{int}(\bigcap_j H_{i_j})$, contradicting that there is a point of S in each side of K . Therefore $|\text{int}(K) \cap \mathbb{Z}^2| \geq 3$, but since the points cannot lie on $\mathbb{Z}^2 \setminus L$ then we may assume $|\text{int}(K) \cap L| \geq 3$. We look at two possible cases:

1. Suppose there is a triplet of non collinear points in $\text{int}(K) \cap L$. Then we may choose $p, q, r \in \text{int}(K) \cap L$ such that $T = \text{conv}(p, q, r)$ contains no other point from L . Consequently, the area of T is equal to $\frac{d(L)}{2}$, where $d(L)$ is the determinant of the lattice L . On the other hand, since $\text{int}(K) \cap S$ is empty, $\text{int}(K) \cap L = \text{int}(K) \cap \mathbb{Z}^2$ and therefore $T \cap \mathbb{Z}^2 = \{p, q, r\}$. Hence the area of T is also equal to $\frac{d(\mathbb{Z}^2)}{2}$. This contradicts the fact that L is a proper sublattice of \mathbb{Z}^2 .

2. Suppose that the points of $\text{int}(K) \cap L$ are collinear. After applying a suitable \mathbb{Z}^2 -preserving linear transformation we may assume that $\text{int}(K) \cap L$ is contained in the x -axis. Since K intersects the x -axis in an interval with at least 3 interior lattice points, its length is greater than 2. Suppose K contains a point with y -coordinate say at least 2. Then by convexity $K \cap \{|y| = 1\}$ has width greater than 1 and contains an interior lattice point p . Because the elements of $\text{int}(K) \cap L$ are collinear, p cannot be in L , thus p must be in S . However, this is impossible as K is S -free. Therefore $K \subseteq \{|y| < 2\}$. Similarly, it can be shown that $K \subseteq \{|y| > -2\}$. Each edge of K contains a point from S in its relative interior and every lattice point in K has y -coordinate $-1, 0$, or 1 . But since a convex set intersect a line in a whole line segment then intersects each of the three lines $\{y = -1\}$, $\{y = 0\}$, $\{y = 1\}$ in at most two points. It follows that K has at most six sides, contradicting the assumption that K has at least seven sides or edges. \square

We conclude with a conjecture that stresses interesting connections to number theory.

Conjecture. *Let \mathbb{P} be the set of prime numbers. Then $h(\mathbb{P}^2) = \infty$.*

We have been able to show that $h(\mathbb{P}^2) \geq 14$ and $h((\mathbb{Z} \setminus \mathbb{P})^2)$ is finite (e.g., through a simple modification of the argument used in Theorem 1.4 and Lemma 2.3). This problem appears to be related to the Gilbreath-Proth conjecture [29] on the behavior of differences of consecutive primes.

4. HELLY NUMBERS FOR SUBGROUPS OF \mathbb{R}^d AND DIFFERENCES OF LATTICES

Now we move to spaces of arbitrary dimension and to sets S with rich algebraic structure.

4.1. Subgroups of \mathbb{R}^d . We look at the case when S is an additive subgroup of \mathbb{R}^d . The most famous examples are lattices. We recall a lattice is defined as a discrete subgroup of \mathbb{R}^d ; it is well known that lattices in \mathbb{R}^d are generated by at most d elements (see [7]). This is precisely the context of Doignon's theorem, which says that $h(\mathbb{Z}^d) = 2^d$.

If a group $S \subset \mathbb{R}^d$ is finitely generated by m elements, then the natural epimorphism between \mathbb{Z}^m and S given by linear combinations yields a linear map from \mathbb{R}^m

onto \mathbb{R}^d . From here Doignon's theorem implies that $h(S) \leq 2^m$. However, there is no dependency on d here, and m could be large compared to d .

On the other hand, it is known in the topological groups literature (see e.g. [15]) that every closed subgroup that linearly spans \mathbb{R}^d is of the form $\phi(\mathbb{Z}^{d-k} \times \mathbb{R}^k)$, where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear bijection. In this case the mixed version of Helly's and Doignon's theorems by Averkov and Weismantel [4], guarantees that $h(\mathbb{Z}^{d-k} \times \mathbb{R}^k) = 2^{d-k}(k+1)$.

The closure \bar{S} of a group S is also a group; therefore we may assume that $\bar{S} = \mathbb{Z}^{d-k} \times \mathbb{R}^k$. This allows us to express S as a product of the form $\mathbb{Z}^{d-k} \times D$, where D is a dense subgroup of \mathbb{R}^k . In this case, we might expect that $h(S) \leq (k+1)2^{d-k}$, but the following example shows that this is not the case.

Example 4.1. Let $k \leq d$ and let $1, \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ be linearly independent over \mathbb{Q} . Let $\{e_1, e_2, \dots, e_d\}$ be the canonical basis of \mathbb{R}^d and consider the group S generated by

$$S := \langle e_1, \dots, e_{k-1}, \alpha_1(e_k + e_1), \dots, \alpha_{k-1}(e_k + e_{k-1}), \alpha_k e_k, e_{k+1}, \dots, e_d \rangle.$$

Note that $S = \mathbb{Z}^{d-k} \times D$, where

$$D = \langle e_1, \dots, e_{k-1}, \alpha_1(e_k + e_1), \dots, \alpha_{k-1}(e_k + e_{k-1}), \alpha_k e_k \rangle$$

is a dense set in \mathbb{R}^d , so the closure of S is $\mathbb{Z}^{d-k} \times \mathbb{R}^k$. Observe that if we intersect S with the space $\{x_k = 0\}$ we obtain a $(d-1)$ -dimensional lattice. Since

$$h(S \cap \{x_k = 0\}) = 2^{d-1},$$

we can construct a family \mathcal{F} of S -convex sets with $2^{d-1} + 2$ elements containing $\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d \mid x_k \geq 0\}$ and $\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d \mid x_k \leq 0\}$ to show that $h(S) \geq 2^{d-1} + 2$. If $k \geq 3$, then $h(S)$ is larger than $(k+1)2^{d-k}$.

In spite of this, we state the following conjecture:

Conjecture. *For any subgroup $G \subseteq \mathbb{R}^d$, the Helly number $h(G)$ is finite.*

Note that this conjecture is true for all finitely generated groups by Doignon's theorem. One might be tempted to conjecture that $h(G) \leq 2^d$. As we see in the next example, this is not always the case.

Example 4.2. Let $G_0 \subset \mathbb{R}$ be the group generated by $1, \pi$ and e , so that $G = \mathbb{Z}^2 \times G_0$ is a subgroup of \mathbb{R}^3 . Consider the points $x_1 = (0, 1, 3)$, $x_2 = (2, 1, e)$ and $x_3 = (2, 3, \pi)$ in \mathbb{R}^3 . The important property these points have is that the 3 midpoints they define are not in G . Let $H \subset \mathbb{R}^3$ be the plane through x_1, x_2, x_3 . Consider three additional points $x_4, x_5, x_6 \in G$ above $(0, 2), (1, 0), (3, 3) \in \mathbb{Z}^2$ and slightly above H and another 3 points x_7, x_8, x_9 above the same three points in \mathbb{Z}^2 but this time slightly below H . Concrete coordinates for these points are

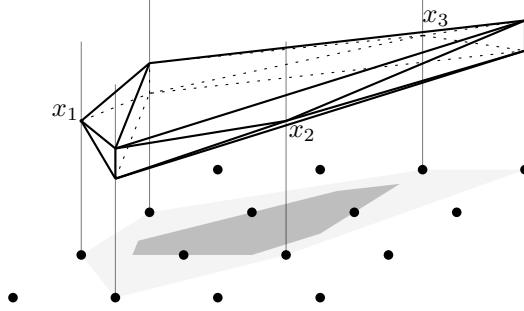


FIGURE 3. The convex hull of 9 points in $\mathbb{Z}^2 \times G_0$, in dark gray is the projection of the intersection described in Lemma 2.3.

$$x_4 = \frac{1}{2}(6 - e + \pi) + \varepsilon, \quad x_5 = \frac{1}{2}(6 - e + \pi) - \varepsilon, \quad x_6 = \frac{1}{2}(3 + 2e - \pi) + \varepsilon,$$

$$x_7 = \frac{1}{2}(3 + 2e - \pi) - \varepsilon, \quad x_8 = \frac{1}{2}(-3 + e + 2\pi) + \varepsilon, \quad x_9 = \frac{1}{2}(-3 + e + 2\pi) - \varepsilon,$$

Where $\varepsilon = 1/5$ or smaller (a bit larger also works). Finally note if $R = \{x_1, \dots, x_9\}$ then the hypothesis of Lemma 2.3 are satisfied (see Figure 3), and therefore we have $h(G) \geq 9$.

If the group G has some additional structure, we are able to bound its Helly number.

Proof of Theorem 1.3. We prove the stronger statement that $h(S) \leq 2d$ for all $S \subseteq \mathbb{R}^d$ with the following property (which is satisfied by G -modules with dense G): $\overline{S \cap A}$ is affine for all affine subspaces $A \subseteq \mathbb{R}^d$.

We have already proved in Lemma 3.1 the case $d = 1$ and in Theorem 1.1 the case $d = 2$. For the general case we use induction over d . Let $d > 1$ and let S be a set such that $\overline{S \cap A}$ is affine for all affine A . Assume that the theorem is false and $h = h(S) > 2d$. This means that there is a family \mathcal{F} of convex sets such that $\bigcap \mathcal{F}$ does not intersect S , but for every subfamily \mathcal{G} with $h - 1$ elements, $\bigcap \mathcal{G}$ intersects S .

If S is not dense in \mathbb{R}^d , it must have dimension less than d ; in this case $h(S) < 2d$ by induction. Thus S must be dense. It follows that $\text{int}(\bigcap \mathcal{F}) = \emptyset$ —that is, $m = \dim(\bigcap \mathcal{F}) < d$. We claim that there exists a subfamily $\mathcal{H} \subset \mathcal{F}$ with $2(d - m)$ elements such that $\dim(\bigcap \mathcal{H}) = \dim(\bigcap \mathcal{F})$, which can be constructed as follows. For simplicity assume that the origin O belongs to the relative interior of $\bigcap \mathcal{F}$ and for every $K \in \mathcal{F}$, consider the cone V_K of vectors v such that

$$K \subseteq \{x \in \mathbb{R}^d : \langle v, x \rangle \geq 0\}.$$

Recall that a family V of vectors *positively spans* a vector space if every element of the vector space can be expressed as a linear combination of elements of V with non-negative coefficients.

Note that $\bigcup_{K \in \mathcal{F}} V_K$ positively spans the orthogonal complement of $\bigcap \mathcal{F}$. In particular $\dim(\bigcup_{K \in \mathcal{F}} V_K) = d - m$. Consequently, by Steinitz's theorem [31, 18], there are vectors $v_1, v_2, \dots, v_{2(d-m)} \in \bigcup_{K \in \mathcal{F}} V_K$ which positively span the orthogonal complement of $\bigcap \mathcal{F}$. For each $i = 1, \dots, 2(d-m)$ choose $K_i \in \mathcal{F}$ such that $v_i \in K_i$ and let $\mathcal{H} = \{K_1, K_2, \dots, K_{2(d-m)}\}$; then $\dim(\bigcap \mathcal{H}) = \dim(\bigcap \mathcal{F})$.

Take $R = \text{aff}(\bigcap \mathcal{H}) \cap S$. Since R satisfies the same property that S does, $h(R) \leq 2m$ by the induction hypothesis. Let $\mathcal{J} = \{K \cap (\bigcap \mathcal{H}) : K \in \mathcal{F} \setminus \mathcal{H}\}$. By the definition of \mathcal{F} , every $(h-1) - 2(d-m)$ elements of \mathcal{J} intersect in R . However, $\bigcap(\mathcal{J})$ does not intersect R , as then $\bigcap \mathcal{F}$ would intersect S . Thus $(h-1) - 2(d-m) < h(R)$, and therefore $h \leq h(R) + 2(d-m) \leq 2d$. This contradicts our choice of h . \square

To show that this is best possible we have the following example.

Example 4.3. Let $a_1, a_2, \dots, a_d, b_1, b_2, \dots, b_d, c_1, c_2, \dots, c_d$ be linearly independent numbers when considered as a vector space over \mathbb{Q} and satisfying $a_i < c_i < b_i$ for every $1 \leq i \leq d$. Consider the \mathbb{Q} -module S generated by the $2d$ vectors of the form

$$\begin{aligned} A_i &= (c_1, \dots, c_{i-1}, a_i, c_{i+1}, \dots, c_d), \\ B_i &= (c_1, \dots, c_{i-1}, b_i, c_{i+1}, \dots, c_d). \end{aligned}$$

For $i = 1, \dots, d$. It is easy to see that $C = (c_1, \dots, c_d) \notin S$.

Let K_i^- and K_i^+ be the two half-spaces with boundary through C orthogonal to e_i so that $A_i \in K_i^-$ and $B_i \in K_i^+$. If \mathcal{F} is the family consisting of these $2d$ half-spaces then $A_i \in \bigcap(\mathcal{F} \setminus \{K_i^+\})$ and $B_i \in \bigcap(\mathcal{F} \setminus \{K_i^-\})$ but $\bigcap \mathcal{F} = \{C\}$ does not intersect S ; therefore $h(S) \geq 2d$.

4.2. Difference of lattices in \mathbb{R}^d . Recall that the Ramsey number

$$R_k = R(\underbrace{3, 3, \dots, 3}_k)$$

is the minimum natural number needed to guarantee the existence of a monochromatic triangle in any edge-coloring with k colors of the complete graph with R_k vertices. We now prove Theorem 1.4 with constant $C_k = R_k - 1$.

Proof of Theorem 1.4. Assume that $h(S) > (R_k - 1)2^d$. Lemma 2.2 implies the existence of an S -vertex-polytope K with $h(S)$ vertices. We say that two elements of \mathbb{Z}^d have the same parity if their difference has only even entries, or equivalently, if their midpoint is in \mathbb{Z}^d . Since the vertex-set of K has more than $(R_k - 1)2^d$ elements, it contains a subset V consisting of R_k points with the same parity.

By definition of K , if $A, B \in V$ then their midpoint cannot be in S ; it must be contained in some L_i . Consider V as the vertex-set of the complete graph with R_k vertices. We call it G and assign to each edge AB a color i so that the midpoint of A and B is in L_i .

Since G has R_k vertices, by Ramsey's theorem it contains a monochromatic triangle. This means that there exist $A_1, A_2, A_3 \in V \subset S$ such that the three midpoints $M_1 = \frac{1}{2}(A_1 + A_2)$, $M_2 = \frac{1}{2}(A_2 + A_3)$ and $M_3 = \frac{1}{2}(A_3 + A_1)$ are in some L_i . But then $A_1 = M_1 - M_2 + M_3 \in L_i$ which contradicts the fact that $A_1 \in S$. \square

5. ANALYSIS OF A POPULAR METHOD TO COLOR HELLY-TYPE THEOREMS

In this final section, we will see the colorful version of Helly's theorem. It can be interpreted as a situation when the constraints are divided into colors, where the colors indicate constraint types or categories. What the theorem guarantees then is a situation in which at least one entire color class has a common solution. We discuss the essential features of a general method to obtain colorful versions of some Helly-type theorems in the sense of Bárány and Lovász. This method has been used in a number of occasions [4, 6] and can be applied to the usual Helly's theorem, Doignon's theorem, Theorem 1 in [1] and our Theorems 1.2 and 1.4 presented in this paper. This idea has been around for some time and it is based on Lovász's proof of Helly's theorem, but as far as we know it has never been abstracted to its essential features.

Let $\mathcal{P}(K)$ be a property of a convex set $K \subset \mathbb{R}^d$. By a property we mean a function from the set of convex sets in \mathbb{R}^d that takes values in $\{\text{true}, \text{false}\}$. As examples of properties $\mathcal{P}(K)$ we have:

- (i) K intersects a fixed set S ,
- (ii) K contains at least k integer lattice points,
- (iii) K is at least k -dimensional,
- (iv) K has volume at least 1.

The following generalizes the definition of Helly number we gave in the introduction to properties shared by sets.

Definition 5.1. For a property \mathcal{P} as above, the *Helly number* $h = h(\mathcal{P}) \in \mathbb{N}$ is defined as the smallest number satisfying the following.

$$(5.1) \quad \forall i_1, \dots, i_h \in [m] : \mathcal{P}(F_{i_1} \cap \dots \cap F_{i_h}) \implies \mathcal{P}(F_1 \cap \dots \cap F_m)$$

for all $m \in \mathbb{N}$ and convex sets F_1, \dots, F_m . If no such h exists, then $h(\mathcal{P}) := \infty$.

Let us look at the four examples of properties above. If \mathcal{P} is property (i) then $h(\mathcal{P}) = h(S)$. For property (ii), the quantitative Doignon theorem in [1] gives bounds for $h(\mathcal{P})$. A result from Grünbaum [21] bounds and determines $h(\mathcal{P})$ for property

(iii) with $0 \leq k \leq d$. In the case of property (iv), it is easy to show that in fact $h(\mathcal{P}) = \infty$, despite the existence of a quantitative Helly theorem [5].

Definition 5.2. We say that property \mathcal{P} is:

1. *Helly* if the corresponding Helly number $h(\mathcal{P})$ is finite.
2. *Monotone* if $K \subset K'$ and $\mathcal{P}(K)$ is true implies that $\mathcal{P}(K')$ is true too.
3. *Orderable* if for any finite family \mathcal{F} of convex sets there is a direction $v \in \mathbb{R}^d \setminus \{0\}$ such that for every $K \in \mathcal{F}$ where $\mathcal{P}(K)$ is true, there is a containment-minimal v -half-space (i.e., a half-space of the form $\{x : v^T x \geq w\}$ for some $w \in \mathbb{R}$) H_K such that:
 - (a) $\mathcal{P}(K \cap H_K)$ is true.
 - (b) There is a unique containment-minimal convex set $K' \subset K \cap H_K$ with $\mathcal{P}(K')$ true.
4. *N-colorable* if for any given finite families $\mathcal{F}_1, \dots, \mathcal{F}_N$ of closed convex sets in \mathbb{R}^d (each a colored family), such that $\mathcal{P}(\bigcap \mathcal{G})$ is true for every *rainbow family* \mathcal{G} (i.e. a family consisting of one element from each \mathcal{F}_i), then $\mathcal{P}(\bigcap \mathcal{F}_i)$ is true for some family \mathcal{F}_i .

It is not too difficult to see that property (i) is Helly, monotone and orderable when S is discrete or the whole space. Property (ii) is also Helly, monotone and orderable, whereas property (iii) fails to be orderable and property (iv) is neither Helly nor orderable. Under this definition we can state the key result:

Theorem 5.3 (Generic Colorful Helly). *A Helly, monotone, orderable property \mathcal{P} is always $h(\mathcal{P})$ -colorable.*

Proof. Set $N = h(\mathcal{P})$ and let $\mathcal{F}_1, \dots, \mathcal{F}_N$ be finite families of closed convex sets in \mathbb{R}^d such that $\mathcal{P}(\bigcap \mathcal{G})$ holds true for every rainbow family \mathcal{G} .

We need an auxiliary construction. Recall that for a family of convex sets \mathcal{H} , $\bigcap \mathcal{H}$ denotes the intersection of all elements in \mathcal{H} (we go from a family to a single set). A family \mathcal{G} of convex sets selected from the \mathcal{F}_i in such a way that for some j , $|\mathcal{G} \cap \mathcal{F}_j| = 0$ and $|\mathcal{G} \cap \mathcal{F}_i| = 1$ for every other $i \neq j$ will be called *almost rainbow*. Note that there are many possible almost rainbow families with respect to $\mathcal{F}_1, \dots, \mathcal{F}_N$ (but there are finitely many almost rainbow families if \mathcal{F}_i are finite). For each almost rainbow family \mathcal{G} we apply the intersection operator $\bigcap \mathcal{G}$. Now, we focus our attention on \mathcal{F} , the family consisting of all the convex sets $\bigcap \mathcal{G}$ for each possible almost rainbow family.

Since property \mathcal{P} is orderable, there is a direction $v \in \mathbb{R}^d \setminus \{0\}$ associated to the family \mathcal{F} as in part 3 of Definition 5.2. By hypothesis and monotonicity, $\mathcal{P}(K)$ is true for every $K \in \mathcal{F}$ and therefore there is a v -half-space H_K and a convex set $K' \subset K \cap H_K$ as in part 3 of Definition 5.2. This set K' will be used a bit later.

Among all elements in \mathcal{F} pick a $K \in \mathcal{F}$ such that the corresponding H_K is containment-maximal, we may assume that $K = K_1 \cap \dots \cap K_{N-1}$ with $K_i \in \mathcal{F}_i$ for $i = 1, \dots, N-1$. For a fixed $L_N \in \mathcal{F}_N$, consider the family $\mathcal{F}' = \{K_1, \dots, K_{N-1}, L_N, H_K\}$ which consists of $N+1$ convex sets. We now show that for every subfamily \mathcal{G}' of \mathcal{F}' , consisting of N elements, $\mathcal{P}(\mathcal{G}')$ is true. There are three possibilities for \mathcal{G}' :

- If $\mathcal{G}' = \mathcal{F}' \setminus \{H_K\}$, then $\mathcal{P}(\bigcap \mathcal{G}') = \mathcal{P}(\bigcap_{i=1}^{N-1} K_i \cap L_N)$ is true by hypothesis.
- If $\mathcal{G}' = \mathcal{F}' \setminus \{L_N\}$, then $\mathcal{P}(\bigcap \mathcal{G}') = \mathcal{P}(H_K \cap (\bigcap_{i=1}^{N-1} K_i))$ is true by the definition of H_K .
- If $\mathcal{G}' = \mathcal{F}' \setminus \{K_j\}$ for some $j < N$, then $\mathcal{P}(\bigcap \mathcal{G}') = \mathcal{P}(H_K \cap (\bigcap_{i \neq j} K_i) \cap L_N)$ is true. Otherwise, the half-space from part 3 of Definition 5.2 corresponding to $\bigcap_{i \neq j} K_i \cap L_N$ exists and is to the right of H_K , contradicting the definition of H_K .

Since $N = h(\mathcal{P})$, by Definition 5.1, $\mathcal{P}(\bigcap \mathcal{F}')$ is true. The minimality of K' then gives $K' \subset \bigcap \mathcal{F}' \subset L_N$. We conclude that K' is contained in every element of \mathcal{F}_N , so by monotonicity $\mathcal{P}(\bigcap \mathcal{F}_N)$ is true. \square

As a corollary we can obtain colored versions of three new Helly-type theorems:

Corollary 5.4. *Let \mathcal{P} be one of the following properties, all of which have an associated Helly-type theorem:*

- (1) K intersects a fixed set $S \subset \mathbb{R}^d$, where $h(S) < \infty$ and S discrete,
- (2) K contains at least k integer lattice points (see [1]),
- (3) K contains at least k points from a set difference of lattices (see [13]),

Then \mathcal{P} is $h(\mathcal{P})$ -colorable.

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(J. A. De Loera, R. N. La Haye) DEPARTMENT OF MATHEMATICS, UC DAVIS
E-mail address: deloera@math.ucdavis.edu, rlahaye@math.ucdavis.edu

(D. Oliveros, E. Roldán-Pensado) INSTITUTO DE MATEMÁTICAS, UNAM CAMPUS JURQUILLA
E-mail address: dolivero@matem.unam.mx, e.roldan@im.unam.mx