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An Explicit Construction for Homotopy Monoidal Structure

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An Explicit Construction for Homotopy Monoidal Structure

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Curriculum Vitae

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Abstract of the Dissertation

An Explicit Construction for Homotopy Monoidal Structure

By

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Doctor of Philosophy in Mathematics
University of California, Irvine, 2015
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In this paper, we begin with the bar construction of a (noncommutative) dg-algebra. We go over the concept of a Hirsch associative algebra, turning the bar construction into a bialgebra. We move on to the bar construction of a module over that algebra. Using the Hirsch algebra, we introduce a twisted tensor product using techniques from [15] in order to construct a tensor product for left modules over our algebra and show that in the case when our algebra is commutative, our tensor product is quasi-isomorphic to the Tor functor. From [19], we introduce the concepts of a dg-nerve of a category and monoidal ∞-categories and use these constructions as guidelines to prove that left modules over a Hirsch associative form a monoidal ∞-category.

Introduction

Drinfeld's approach with Quantum groups, given in [2], is to start with a vector space L that has both a bracket and cobracket. Ignoring the cobracket, we can construct the universal enveloping algebra $U(L)$, which has noncommutative product and cocommutative coproduct. Reintroducing the cobracket, our coproduct should deform to match with this cobracket. However, describing these formulas is messy and Drinfeld introduced the idea of defining the coproduct through monoidal structure on modules. If we are not so lucky and L doesn't have an honest Lie bracket, but only an L-∞ bracket, it's difficult to describe the product on $U(L)$. However, since the A-∞ product on $U(L)$ still agrees with the standard coproduct on $U(L)$, by Drinfeld's approach, it may be worthwhile to look at the homotopy monoidal structure on modules. From this point of view, we aim to construct a homotopy monoidal structure on modules over an algebra.

If A is a commutative algebra over a field k and M_1 and M_2 are left A modules, we can consider, for example, M_1 to actually be a right module by defining $m \cdot a := am$. In this way we can take the tensor product of these modules over A, getting $M_1 \underset{A}{\otimes} M_2$. This object is well-defined and is in fact an A module itself with $a \cdot (m_1 \otimes m_2) = m_1 a \otimes m_2 = m_1 \otimes a m_2$. If A is not commutative, trying to turn a left module M_1

into a right module is not possible. So the best we can do is take our tensor product over k, $M_1 \underset{k}{\otimes} M_2$. In this case we are left with a left A module, where the action is given by $a \cdot (m_1 \otimes m_2) = am_1 \otimes m_2$. As a module, this only utilizes information from M_1, meaning we only use M_2 for its vector space structure.

Our goal is to try to construct a structure that allows us to turn the category of modules over an algebra into a monoidal category as best we can. We know we are unable to do this honestly, so we look for a concession. Instead we create a structure where the tensor product does not obey monoidal properties, such as having identity or being associative, but these properties hold only up to higher homotopies. We begin with a dg algebra and consider its bar construction. On page 7 we introduce the concept of associative Hirsch algebras that turn the bar construction into a bialgebra. There are several examples of these, the easiest being when the algebra is commutative. Kadeishvili offers other examples, specifically when the algebra is cochain complex of a topological space as given in Theorem 1 of [10] and when the algebra is in fact a bialgebra as given in Remark 1 in [11]. Gerstenhaber showed in Corollary 1 of [6] that the algebra may be the Hochschild cohomology of an associative algebra. This fact is often called Deligne's conjecture.

We use left modules over our algebra to create comodules over the bar construction and use these to construct a sort of product on these modules. We show in proposition 2.1 on page 15 that if our algebra is commutative, this product agrees with the normal tensor product using the Tor functor. Also, in proposition 2.2 we show that A acts as the identity on our product, as desired. We show explicitly in proposition 2.3 that we get associativity of this product in the derived category.

The structure that facilitates us generalizing these explicit constructions to all products comes from Lurie in[18], a paper over 1,000 pages that is not explicit. We use his construction of a monodial ∞-category, defined on page 31. A monoidal ∞-category is a structure like a monoidal category where the conditions on our tensor product that should normally hold, such as associativity, hold only up to homotopy. We accomplish this by constructing a simplicial set, called a monoidal dg-nerve, along with a nice map to $\mathbf{\Delta}$, the simplicial complex, on page 46. This all leads to our main result, theorem 3.1, proving that this monoidal dg-nerve is in fact a monoidal ∞-category.

Chapter 1

The Bar Construction as a Bialgebra

1.1 Notation and Foundations

Fix a field k. Unless otherwise stated, all tensor products are over k. The following

A *differential graded algebra*, or dg-algebra, over a field k is a vector space A over k with an associative, though not necessarily commutative, multiplication μ, unit map $\eta_A : k \to A$, differential d_A, and grading $A = \bigoplus_{i \leq 0} A_i$ such that $\mu : A_i \otimes A_j \to A_{i+j}$, $d_A : A_i \to A_{i-1}$, $d_A^2 = 0$ and d_A respects the graded Leibniz Rule, $d_A(a \cdot b) = d_A(a) \cdot b + (-1)^{|a|} a \cdot d_A(b)$, where $|a|$ is the degree of a, i.e. the i such that $a \in A_i$.

Note that $\bar{A}$ denotes $\bigoplus_{i \neq 0} A_i$

A *differential graded coalgebra*, or dg-coalgebra, over a field k is a vector space C over k with a coassociative comultiplication $\Delta : C \to C \otimes C$, counit $\epsilon : C \to k$, differential d_C, and grading $C = \bigoplus_i C_i$ such that $d_C : C_i \to C_{i-1}$, $d_C^2 = 0$ and d_C satisfies the relation $\Delta d_C = (d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_C)\Delta$.

A *differential graded bialgebra*, or dg-bialgebra, over a field k is a vector space C over k that is both a dg-algebra with product μ and dg-coalgebra with coproduct Δ, sharing differential d_C such that $\mu : C \otimes C \to C$ is a morphism of dg-coalgebras, meaning this diagram commutes:

$$\begin{tikzcd}
C \otimes C \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"] & C \arrow[r, "\Delta"] & C \otimes C \\
C \otimes C \otimes C \otimes C \arrow[rr, "\mathrm{id}\otimes T\otimes\mathrm{id}"] & & C \otimes C \otimes C \otimes C \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}$$

where T is the linear map $T(x \otimes y) = (-1)^{|x|\cdot|y|} y \otimes x$.

1.2 The Bar Construction

Let A be a dg-algebra over k. The *bar construction of* A, denoted B, is the set $T(s^{-1}\overline{A}) = \bigoplus_{i\geq 0} \overline{A}^{\otimes i}$, where $\overline{A} = \ker\epsilon_A$, where ϵ_A is the projection $\epsilon_A : A \to k$. By $s^{-1}\bar{A}$ we mean the desuspension of $\overline{A}$, i.e. $(s^{-1}\overline{A})_n = \overline{A}_{n+1}$. An element of the form $s^{-1}a_1 \otimes s^{-1}a_2 \otimes \cdots \otimes s^{-1}a_k$ will be denoted $[a_1, a_2, \cdots, a_k]$. On this set we define a standard comultiplication $\Delta : B \to B \otimes B$,

$$\Delta([a_1, \cdots, a_k]) = 1 \otimes [a_1, \cdots, a_k] + \sum_{i=1}^{k} [a_1, \cdots, a_i] \otimes [a_{i+1}, \cdots, a_k] + [a_1, \cdots, a_k] \otimes 1.$$

We can define a counit $\epsilon : B \to k$ defined as projection onto k. If we define $d : B \to B$,

$$d([a_1, \cdots, a_k]) = \sum_{i=2}^{k} (-1)^i [a_1, \cdots, a_{i-1}a_i, \cdots, a_k] + \sum_{i=1}^{k} [a_1, \cdots, d_A(a_i), \cdots, a_k],$$

then B becomes a dg-coalgebra where the degree of $[a_1, \cdots, a_k]$ is $-k + \sum_{i=1}^{k} |a_i|$.

We would like to endow B with a multiplication so that it becomes a dg-bialgebra. The naive hope that the simple product of $\mu([a_1, \cdots, a_m] \otimes [b_1, \cdots, b_n]) = [a_1, \cdots, a_m, b_1, \ldots, b_n]$ unfortunately does not work. To see this, ignoring sign, consider $[a_1, a_2] \otimes [b_1, b_2]$. If we apply the above μ to this, we are left with $[a_1, a_2, b_1, b_2]$. Using the coproduct on this yields

(1)

$$1 \otimes [a_1, a_2, b_1, b_2] + [a_1] \otimes [a_2, b_1, b_2] + [a_1, a_2] \otimes [b_1, b_2] + [a_1, a_2, b_1] \otimes [b_2] + [a_1, a_2, b_1, b_2] \otimes 1.$$

If instead we first apply $\Delta \otimes \Delta$, we get

$$\big(1 \otimes [a_1, a_2] + [a_1] \otimes [a_2] + [a_1, a_2] \otimes 1\big) \otimes \big(1 \otimes [b_1, b_2] + [b_1] \otimes [b_2] + [b_1, b_2] \otimes 1\big).$$

Then applying $(\mu \otimes \mu)(\mathrm{id} \otimes T \otimes \mathrm{id})$ to this gives us

$$1 \otimes [a_1, a_2, b_1, b_2] + b_1 \otimes [a_1, a_2, b_2] + [b_1, b_2] \otimes [a_1, a_2] + a_1 \otimes [a_2, b_1, b_2] + [a_1, b_1] \otimes [a_2, b_2]$$

$$+[a_1, b_1, b_2] \otimes a_2 + [a_1, a_2] \otimes [b_1, b_2] + [a_1, a_2, b_1] \otimes b_2 + [a_1, a_2, b_1, b_2] \otimes 1. \tag{2}$$

Equations (1) and (2) are not equal. To make B into a bialgebra, we need to find a product that agrees with the standard coproduct.

1.3 Bialgebra Structure on the Bar Construction

1.3.1 Shuffle Product

In the case when A is commutative, we can construct such a product without too much difficulty. The *shuffle product*, an associative product $\mu_0 : B \otimes B \to B$, is defined as

$$\mu_0([a_1, a_2, \cdots, a_k], [a_{k+1}, \cdots, a_{k+l}]) = \sum_{\sigma \in S_{k,l}} sgn(\sigma)[a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \cdots, a_{\sigma^{-1}(k+l)}],$$

where $S_{k,l}$ is the subset of the symmetric group S_{k+l} such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

If we endow B with this multiplication, then we do have that (B, μ_0, Δ) is a dg-bialgebra. When A is commutative, μ_0 respects Leibniz's rule with respect to d, so (B, μ_0, Δ, d) is in fact a dg-bialgebra. Note that if A is not commutative, μ_0 does not respect Leibniz's rule. For instance, (ignoring the differential from A which indeed does cancel out)

$$d(\mu_0([a_1, a_2], a_3)) = d([a_1, a_2, a_3] - [a_1, a_3, a_2] + [a_3, a_1, a_2]) =$$

$$[a_1a_2, a_3] - [a_1, a_2a_3] - [a_1a_3, a_2] + [a_1, a_3a_2] + [a_3a_1, a_2] - [a_3, a_1a_2]$$

$$\mu_0(d([a_1, a_2]), a_3) + \mu_0([a_1, a_2], d(a_3)) = \mu_0(a_1a_2, a_3) + \mu_0([a_1, a_2], 0) = [a_1a_2, a_3] - [a_3, a_1a_2].$$

1.3.2 Associative Hirsch Algebras

From this point on, we will assume A is over $\mathbb{Z}/2\mathbb{Z}$ to avoid signs.

Let (C, Δ, d_C) be a dg-coalgebra and (A, μ, d_A) be a dg-algebra. A *twisting cochain* is a degree 1 morphism $\tau : C \to \overline{A}$ such that the Maurer-Cartan equation is satisfied. That is, that $d_A \circ \tau + \tau \circ d_C + \mu \circ (\tau \otimes \tau) \circ \Delta = 0$.

If C is a dg-coalgebra, then so is $C^{\otimes n}$ with comultiplication $\Delta^{\otimes n} = T_n \circ (\Delta_C \otimes \cdots \otimes \Delta_C)$. If $V_1, \ldots, V_{2n}$ are k-vector spaces, the map $T_n : V_1 \otimes \cdots V_{2n} \to V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(2n)}$, given by $T_n(v_1 \otimes \cdots \otimes v_{2n}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(2n)}$, is defined by the permutation $\sigma \in S_{2n}$,

$$\sigma(i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ \frac{i}{2} + n & \text{if } i \text{ is even.} \end{cases}$$

The differential on $C^{\otimes n}$ is $\sum_{i=1}^{n} \mathrm{id}^{\otimes i} \otimes d_C \otimes \mathrm{id}^{\otimes n-i-1}$. In particular, this makes $B \otimes B$ a coalgebra.

In [11], Kadeishvili shows that to endow B with a multiplication μ that turns B into a dg-bialgebra, it is equivalent to defining a twisting cochain $E : B \otimes B \to A$, which can be defined component-wise as $E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A$ for $p, q = 0, 1, 2, \ldots$, where $E_{p,q}$ acting on $[a_1, \cdots, a_p] \otimes [b_1, \cdots, b_q]$ is denoted $E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$. These maps must satisfy the following conditions:

1. To guarantee that B has a unit element, we require that

$$E_{0,1} = E_{1,0} = \mathrm{id}; E_{0,k} = E_{k,0} = 0 \text{ for } k > 1. \tag{1.1}$$

2. To guarantee that our multiplication is a dg-coalgebra map, we require that on

$A^{\otimes m} \otimes A^{\otimes n}$

(1.2)

$$
\begin{aligned}
dE_{m,n}(a_1, \dots, a_m; b_1, \dots, b_n) + \sum_{i=1}^{m} E_{m,n}(a_1, \dots, d(a_i), \dots, a_m; b_1, \dots, b_n) \\
+ \sum_{i=1}^{n} E_{m,n}(a_1, \dots, a_m; b_1, \dots, d(b_i), \dots, b_n) = \\
a_1 \cdot E_{m-1,n}(a_2, \dots, a_m; b_1, \dots, b_n) + E_{m-1,n}(a_1, \dots, a_{m-1}; b_1, \dots, b_n) \cdot a_m \\
+ b_1 \cdot E_{m,n-1}(a_1, \dots, a_m; b_2, \dots, b_n) + E_{m,n-1}(a_1, \dots, a_m; b_1, \dots, b_{n-1}) \cdot b_n \\
+ \sum_{i=1}^{m-1} E_{m-1,n}(a_1, \dots, a_i a_{i-1}, \dots, a_m; b_1, \dots, b_n) \\
+ \sum_{i=1}^{n-1} E_{m,n-1}(a_1, \dots, a_m; b_1, \dots, b_i b_{i+1}, \dots, b_n) \\
+ \sum_{p=1}^{m-1} \sum_{q=1}^{n-1} E_{p,q}(a_1, \dots, a_p; b_1, \dots, b_q) \cdot E_{m-p,n-q}(a_{p+1}, \dots, a_m; b_{1+q}, \dots, b_n).
\end{aligned}
$$

3. To guarantee that our multiplication is associative, we require that on $A^{\otimes k} \otimes A^{\otimes l} \otimes A^{\otimes m}$

(1.3)

$$
\begin{aligned}
& \sum_{r=1}^{l+m} \sum_{\substack{l_1 + \cdots + l_r = l \\ m_1 + \cdots + m_r = m}} \\
& \qquad E_{k,r}\Big(a_1, \dots, a_k; E_{l_1,m_1}(b_1, \dots, b_{l_1}; c_1, \dots, c_{m_1}), \dots, \\
& \qquad E_{l_r,m_r}(b_{l_1+\cdots+l_{r-1}+1}, \dots, b_{l_r}; c_{m_1+\cdots+m_{r-1}+1}, \dots, c_{m_r})\Big) \\
& = \sum_{s+l}^{k+l} \sum_{\substack{k_1+\cdots+k_s=k, \\ l_1+\cdots+l_s=l}} \\
& \qquad E_{s,m}\Big(E_{k_1,l_1}(a_1, \dots, a_{k_1}; b_1, \dots, b_{l_1}), \dots, \\
& \qquad E_{k_s,l_s}(a_{k_1+\cdots+k_{s-1}+1}, \dots, a_{k_s}; b_{l_1+\cdots+l_{s-1}+1}, \dots, b_{l_s}); c_1, \dots, c_m\Big).
\end{aligned}
$$

If A is an algebra endowed with multioperations E satisfying the above criteria, A is called an *associative Hirsch algebra*.

Kadeishvili further shows that if A is an associative Hirsch algebra, B becomes a dg-bialgebra via the multiplication $\mu_E : B \otimes B \to B$ defined by

$$\mu_E = \sum_i (E \otimes \cdots \otimes E)\Delta^i_{B\otimes B},$$

where $\Delta^i_{B\otimes B} : B \otimes B \to (B \otimes B)^{\otimes i}$ is the i-fold iteration of

$$\Delta_{B\otimes B} = (\mathrm{id} \otimes T \otimes \mathrm{id})(\Delta \otimes \Delta) : B \otimes B \to (B \otimes B)^{\otimes 2}.$$

If we reduce to the case that all $E_{p,q} = 0$ except $E_{0,1}$ and $E_{1,0}$, equation (1.2) on $A \otimes A$ shows that

$$0 = dE_{1,1}(a; b) + E_{1,1}(d(a); b) + E_{1,1}(a; d(b)) = ab + ba,$$

i.e. that A is commutative. In this case, our multiplication μ_E becomes exactly the shuffle product μ_0 described above.

With this in mind, we can see that any product on the bar construction must come from deforming the shuffle product, i.e. adding nontrivial maps of $E_{p,q}$ not all zero for $p, q \geq 1$, so that the product still obeys equations 1.1, 1.2, and 1.3.

Chapter 2

Comodules over the Bar Construction

2.1 Comodule Construction

A *left differential graded comodule*, or left dg-comodule, over a dg-coalgebra C is a vector space N over k along with a coaction $\Delta_N : N \to C \otimes N$ satisfying coassociativity, i.e. that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "\Delta_N"] & C \otimes N \arrow[d, "\mathrm{id}\otimes\Delta_N"] \\ C \otimes N \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes N \end{tikzcd}$$

Also equipped is a differential d_N satisfying the condition $\Delta_N d = (d_C \otimes \mathrm{id} + (-1)^{deg} \mathrm{id} \otimes d_N)\Delta_N$. A right comodule is defined similarly.

Similar to how starting with an algebra A we constructed a bialgebra B, given a left dg-module M over a dg-algebra A, we can construct the *bar construction of a left module* over B, denoted $B(M)$, as follows. $B(M)$ as a set is $B \otimes M$, where the element $[a_1, \ldots, a_k] \otimes m$ will be denoted $[a_1, \ldots, a_k, m]$. The coaction $\Delta_{B(M)}$: $B(M) \to B \otimes B(M)$ is defined by

$$\Delta_{B(M)}([a_1, \ldots, a_k, m]) = 1 \otimes [a_1, \ldots, a_k, m]$$

$$+\sum_{i=1}^{k-1}[a_1,\ldots,a_i]\otimes[a_{i+1},\ldots,a_k,m]+[a_1,\ldots,a_k]\otimes m.$$

Lastly, the differential $d_{B(M)}$ is defined as

$$d_{B(M)}([a_1,\cdots,a_k,m])=\sum_{i=2}^{k}[a_1,\cdots,a_{i-1}a_i,\cdots,a_k,m]+[a_1,\cdots,a_k\cdot m]$$

$$+\sum_{i=1}^{k}[a_1,\cdots,d_A(a_i),\cdots,a_k,m]+[a_1,\cdots,a_k,d_M(m)].$$

In the future, all mention of modules over A will imply left dg-modules over A, unless otherwise stated.

2.1.1 Product of Comodules

Given left dg-comodules L_1 and L_2 over B, $L_1\otimes L_2$ is a comodule over $B\otimes B$ via the action $(\mathrm{id}\otimes T\otimes\mathrm{id})\circ(\Delta_{L_1}\otimes\Delta_{L_2})$ and the differential is given by $d_{L_1\otimes L_2}=d_{L_1}\otimes\mathrm{id}+\mathrm{id}\otimes d_{L_2}$. Thus, given left A dg-modules M and N, we can construct a left dg-comodule $B(M)\otimes B(N)$ over the coalgebra $B\otimes B$.

Let L be a left C dg-comodule with comodule coaction Δ_L and let M be a right A module with right module action m_M. If τ is a twisting cochain $\tau: C\to A$, a *twisted tensor product* $M\underset{\tau}{\otimes}L$, is the chain complex $M\otimes L$ with new differential d_τ defined as

$$d_\tau=d_M\otimes\mathrm{id}+\mathrm{id}\otimes d_L+(m_M\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\otimes(\mathrm{id}\otimes\Delta_L).$$

This last operation $(m_M\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_L)$ is the map

$$M\otimes L\xrightarrow{\mathrm{id}\otimes\Delta_L}M\otimes C\otimes L\xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}}M\otimes A\otimes L\xrightarrow{m_M\otimes\mathrm{id}}M\otimes L.$$

We can equally construct $L\underset{\tau}{\otimes}M$ where L is a right C comodule and M is a left A module, switching from left and right as necessary in the above operations.

Given a twisting cochain $\tau: C\to A$, in Lemma 2.2.2.6 in [15] Lefèvre-Hasegawa showed that, by considering A and C as (co)modules over themselves, respectively, the functors $A\underset{\tau}{\otimes}\_$ and $C\underset{\tau}{\otimes}\_$ are adjoint. Specifically, given an A module map $f\in\mathrm{Hom}_A(A\underset{\tau}{\otimes}N,M)$, we have a C comodule map $\tilde{f}\in\mathrm{Hom}_C(N,C\underset{\tau}{\otimes}M)$, where

$$\tilde{f}=(\mathrm{id}\otimes f)\circ(\mathrm{id}\otimes(\eta_A\otimes\mathrm{id}))\circ\Delta,$$

where $\eta_A : k \to A$ is the unit map. The map $\eta_A \otimes \mathrm{id}$ maps $N(\cong k \otimes N) \to A \otimes N$.

Alternately, given a C comodule map $\tilde{g} \in \mathrm{Hom}_C(N, C \underset{\tau}{\otimes} M)$, we have an A module map $g \in \mathrm{Hom}_A(A \underset{\tau}{\otimes} N, M)$, where

$$g = m_M \circ (\mathrm{id} \otimes \epsilon_C \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tilde{g}),$$

where m_M is module multiplication and $\epsilon_C : C \to k$ is the counit map. The map $\mathrm{id} \otimes \epsilon_C \otimes \mathrm{id}$ maps $A \otimes C \otimes M \to (A \otimes k \otimes M \cong) A \otimes M$. Moreover, in the case where $C = B \otimes \cdots \otimes B$ and we have a C comodule N there is a quasi-isomorphism of C comodules

$$i : N \to C \underset{\tau}{\otimes} A \underset{\tau}{\otimes} N, \tag{2.1}$$

where $i = (\mathrm{id} \otimes \eta_A \otimes \mathrm{id}) \circ \Delta$. Similarly in this case, if M is a left A module, there is a quasi-isomorphism of A modules

$$p : A \underset{\tau}{\otimes} C \underset{\tau}{\otimes} M \to M, \tag{2.2}$$

where $p = m_M \circ (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})$.

We define the map $\tau_1 : B \to \overline{A}$ as projection. This map is a twisting cochain. With this notation, we can now write $B(M)$ as $B \underset{\tau_1}{\otimes} M$.

The map $E : B \otimes B \to A$ is also a twisting cochain. Therefore, considering A as a right dg-A module and given left dg-modules M and N over A, we can, as above, think of $B(M) \otimes B(N)$ as a comodule over $B \otimes B$. Using τ_1, we have a twisting cochain $\tau_1 \otimes \epsilon + \epsilon \otimes \tau_1 : B \otimes B \to A \otimes A$, which we will denote abusing some notation as $\tau_1 \otimes \tau_1$. We can then write $B(M) \otimes B(N)$ as $(B \otimes B) \underset{\tau_1 \otimes \tau_1}{\otimes} M \otimes N$, understanding which coalgebra matches with which module. Then, using our twisting cochain $E : B \otimes B \to A$, we can create a left A dg-module $A \underset{E}{\otimes} (B(M) \otimes B(N))$. We will denote this as $[M, N]$ and, to avoid confusing shuffling of terms for our twisting cochains, we will write $[M, N]$ as

$$A \bigotimes_E \begin{array}{c} B \\ \underset{k}{\otimes} \\ B \end{array} \bigotimes_{\tau_1 \otimes \tau_1} \begin{array}{c} M \\ \underset{k}{\otimes} \\ N \end{array} .$$

The differential works as:

$$d\left(a\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array}\right) = d(a)\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array} + a\otimes\begin{array}{ccc}d([b_1,\dots,b_j]) & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array}$$

$$+a\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ d([c_1,\dots,c_k]) & & n\end{array} + a\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & d(m)\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array} + a\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & d(n)\end{array}$$

$$+ab_1\otimes\begin{array}{ccc}[b_2,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array} + ac_1\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_2,\dots,c_k] & & n\end{array}$$

$$+a\otimes\begin{array}{ccc}[b_1,\dots,b_{j-1}] & & b_jm\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_k] & & n\end{array} + a\otimes\begin{array}{ccc}[b_1,\dots,b_j] & & m\\ \otimes & \otimes & \otimes\\ [c_1,\dots,c_{k-1}] & & c_kn\end{array}.$$

In addition to $[M,N]$ acting as an A module, if A is commutative, it is also a $\hat{B}(A)$ module, where $\hat{B}(A)$ is the double-sided bar construction, $\hat{B}(A) := A \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} A$. The action is given by

$$(a_1\otimes[b_1,\dots,b_i]\otimes a_2)\cdot\left(a\otimes\begin{array}{ccc}[c_1,\dots,c_j] & & m\\ \otimes & \otimes & \otimes\\ [c'_1,\dots,c'_k] & & n\end{array}\right)$$

$$=\sum_{p=0}^{i} a\otimes\begin{array}{ccc}\mu_0([b_p,\dots,b_1],[c_1,\dots,c_j]) & & a_1m\\ \otimes & \otimes & \otimes\\ \mu_0([b_{p+1},\dots,b_i],[c'_1,\dots,c'_k]) & & a_2n\end{array},$$

where μ_0 is the shuffle product.

2.1.2 The Tor Complex

If A is commutative and M is a left A module, we may equally consider it as a right A module via the action $a\cdot m = ma$. Let M and N be left dg-modules over A, where we consider M as a right A module. The *Tor complex* of M and N is the vector space $\mathrm{Tor}(M,N) = M \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} N$. To comply with our layout of $[M,N]$, we will write this as:

$$\mathrm{Tor}(M,N) = \hat{B}(A) \bigotimes_{A\otimes A} \begin{array}{c} M \\ \underset{k}{\otimes} \\ N \end{array} ,$$

where the tensor product $\bigotimes_{A\otimes A}$ applies to the outer A's.

As we have the tensor product over $A \otimes A$, we can consider

$$a \otimes [b_1, \ldots, b_k] \otimes c \otimes \begin{array}{c} a_1 \\ \otimes \\ a_2 \end{array} = 1 \otimes [b_1, \ldots, b_k] \otimes 1 \otimes \begin{array}{c} aa_1 \\ \otimes \\ ca_2 \end{array}$$

and we may define all terms using 1's in these places. Notice in particular that the differential works as:

$$d\left(1 \otimes [b_1, \ldots, b_k] \otimes 1 \otimes \begin{array}{c} m \\ \otimes \\ n \end{array} \right) = 1 \otimes d([b_1, \ldots, b_k]) \otimes 1 \otimes \begin{array}{c} m \\ \otimes \\ n \end{array}$$

$$+1 \otimes [b_1, \ldots, b_k] \otimes 1 \otimes \begin{array}{c} d(m) \\ \otimes \\ n \end{array} + 1 \otimes [b_1, \ldots, b_k] \otimes 1 \otimes \begin{array}{c} m \\ \otimes \\ d(n) \end{array}$$

$$+1 \otimes [b_2, \ldots, b_k] \otimes 1 \otimes \begin{array}{c} b_1 m \\ \otimes \\ n \end{array} + 1 \otimes [b_1, \ldots, b_{k-1}] \otimes 1 \otimes \begin{array}{c} m \\ \otimes \\ b_k n \end{array} .$$

A being commutative is necessary for d to act as a differential.

We can consider $\mathrm{Tor}(M,N)$ as a $\hat{B}(A)$ module, where the module action is given by

$$(a_1 \otimes [b_1, \ldots, b_i] \otimes a_2) \cdot \left(1 \otimes [b'_1, \ldots, b'_j] \otimes 1 \otimes \begin{array}{c} m \\ \otimes \\ n \end{array} \right)$$

$$= 1 \otimes \mu_0 \left([b_1, \ldots, b_i], [b'_1, \ldots, b'_j]\right) \otimes 1 \otimes \begin{matrix} a_1 m \\ \otimes \\ a_2 n \end{matrix} .$$

Consider the map $\psi : \mathrm{Tor}(M, N) \to [M, N]$, where

$$\psi \left(1 \otimes [a_1, \ldots, a_k] \otimes 1 \bigotimes_{A\otimes A} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} \right)$$

$$= \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i, \ldots, a_1] \\ \otimes_k \\ [a_{i+1}, \ldots, a_k] \end{matrix} \bigotimes_{\tau_1 \otimes \tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} .$$

Effectively, this map ψ just applies the coproduct to the B term and reverses the order of the first of the $B \otimes B$ terms. As we are supposing A is commutative, this reversing causes no issues with the differential. Also, as A is commutative in this case, the E map is 0 except for $E_{0,1}$ and $E_{1,0}$.

2.1.3 Proposition 2.1

Proposition 2.1. *The map ψ is an isomorphism of $\hat{B}(A)$ modules in the derived category.*

Proof of Proposition 2.1. We must show that **(1)** ψ is a chain map and **(2)** that ψ is a quasi-isomorphism.

(1) We first show that ψ is a chain map, meaning $d\psi = \psi d$.

Consider first $d\psi \left(1 \otimes [a_1, \ldots, a_k] \otimes 1 \bigotimes_{A\otimes A} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} \right)$

$$= d \left(\sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i, \ldots, a_1] \\ \otimes_k \\ [a_{i+1}, \ldots, a_k] \end{matrix} \bigotimes_{\tau_1 \otimes \tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} \right)$$

$$
= \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} d([a_i,\dots,a_1]) \\ \otimes_k \\ [a_{i+1},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} + \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i,\dots,a_1] \\ \otimes_k \\ d([a_{i+1},\dots,a_k]) \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix}
$$

$$
+ \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i,\dots,a_1] \\ \otimes_k \\ [a_{i+1},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} d(m) \\ \otimes_k \\ n \end{matrix} + \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i,\dots,a_1] \\ \otimes_k \\ [a_{i+1},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ d(n) \end{matrix}
$$

$$
+ \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i,\dots,a_2] \\ \otimes_k \\ [a_{i+1},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} a_1 m \\ \otimes_k \\ n \end{matrix} + \sum_{i=0}^{k} 1 \bigotimes_{E} \begin{matrix} [a_i,\dots,a_1] \\ \otimes_k \\ [a_{i+1},\dots,a_{k-1}] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ a_k n \end{matrix}
$$

$$
+ \sum_{i=1}^{k} E_{1,0}(a_i,1) \bigotimes_{E} \begin{matrix} [a_{i-1},\dots,a_1] \\ \otimes_k \\ [a_{i+1},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix}
$$

$$
+ \sum_{i=0}^{k-1} E_{0,1}(1,a_{i+1}) \bigotimes_{E} \begin{matrix} [a_i,\dots,a_1] \\ \otimes_k \\ [a_{i+2},\dots,a_k] \end{matrix} \bigotimes_{\tau_1\otimes\tau_1} \begin{matrix} m \\ \otimes_k \\ n \end{matrix}
$$

Notice first that the last two rows of the above sum cancels out. When $i = j+1$, we have $a_i = E_{1,0}(a_i,1) = E_{0,1}(1,a_{j+1}) = a_{j+1}$.

Now consider $\psi d \left(1 \otimes [a_1,\dots,a_k] \otimes 1 \bigotimes_{A\otimes A} \begin{matrix} m \\ \otimes_k \\ n \end{matrix} \right)$

$$
= \psi \Bigg(1 \otimes d([a_1,\dots,a_k]) \otimes 1 \otimes \begin{matrix} m \\ \otimes \\ n \end{matrix} + 1 \otimes [a_1,\dots,a_k] \otimes 1 \otimes \begin{matrix} d(m) \\ \otimes \\ n \end{matrix}
$$

$$
+ 1 \otimes [a_1,\dots,a_k] \otimes 1 \otimes \begin{matrix} m \\ \otimes \\ d(n) \end{matrix} + 1 \otimes [a_2,\dots,a_k] \otimes 1 \otimes \begin{matrix} a_1 m \\ \otimes \\ n \end{matrix}
$$

$$+1 \otimes [a_1, \ldots, a_{k-1}] \otimes 1 \otimes \begin{matrix} m \\ \otimes \\ a_k n \end{matrix} \Bigg).$$

That this is equal to $d\psi$ follows since the coproduct and d obey the Leibniz Rule. All remaining terms are identical.

(2) To see that ψ is a quasi-isomorphism, we first show that $\mathrm{Tor}(A, A)$ and $[A, A]$ are quasi-isomorphic.

Consider the map $q_1 : \mathrm{Tor}(A, A) \to A$ defined by

$$q_1 : 1 \otimes [b_1, \ldots, b_i] \otimes 1 \otimes \begin{matrix} a_1 \\ \otimes \\ a_2 \end{matrix} \mapsto a_1 a_2 \epsilon([b_1, \ldots, b_k]).$$

The map q_1 is a chain map as $q_1 = 0$ if $i \geq 1$ and just multiplication otherwise.

Consider the map $q_2 : [A, A] \to A$ defined by

$$q_2 : a \otimes \begin{matrix} [b_1, \ldots, b_i] \\ \otimes \\ [c_1, \ldots, c_j] \end{matrix} \otimes \begin{matrix} a_1 \\ \otimes \\ a_2 \end{matrix} \mapsto a a_1 a_2 \epsilon([b_1, \ldots, b_i]) \epsilon([c_1, \ldots, c_j]).$$

The map q_2 is also a chain map as $q_2 = 0$ if $i + j \geq 1$ and just multiplication otherwise.

As the coproduct maps $A^{\otimes i}$ to $\sum_{p+q=i} A^{\otimes p} \otimes A^{\otimes q}$, we then have that the following diagram commutes

$$\begin{tikzcd}
A \arrow[d, Rightarrow, no head] & \mathrm{Tor}(A, A) \arrow[l, "q_1"'] \arrow[d, "\psi"] \\
A & {[A, A]} \arrow[l, "q_2"]
\end{tikzcd}$$

Both q_1 and q_2 are quasi-isomorphisms due to [15], therefore q_2 can be reversed in the derived category.

That ψ is a quasi-isomorphism of $\hat{B}(A)$ modules follows since ψ is effectively the coaction map Δ on B and (B, Δ, μ_0) is a bialgebra, meaning Δ is a B algebra morphism.

Both $[A,A]$ and $\mathrm{Tor}(A,A)$ are A-A bimodules, where $[M,N] = M \underset{A}{\otimes} [A,A] \underset{A}{\otimes} N$ and $\mathrm{Tor}(M,N) = M \underset{A}{\otimes} \mathrm{Tor}(A,A) \underset{A}{\otimes} N$. This makes ψ a bimodule morphism. As $[A,A]$ and $\mathrm{Tor}(A,A)$ are flat as they are just tensor products of A, we have that the quasi-isomorphism of ψ extends as a quasi-isomorphism from $\mathrm{Tor}(M,N)$ to $[M,N]$.

□

2.2 Left Modules Begin to Form a Monoidal Category

If A is commutative, the category of modules over A is monoidal with tensor product $\underset{A}{\otimes}$. We go back to the case where we are not assuming commutativity and obtain some weaker though similar results in our case.

2.2.1 Proposition 2.2

Proposition 2.2. *A acts quasi-isomorphically as the identity object on $[,]$, i.e. in the derived category M is isomorphic to both $[A,M]$ and $[M,A]$.*

Proof. We will show that M is quasi-isomorphic to $[M,A]$. That M is quasi-isomorphic to $[A,M]$ is virtually identical. In the commutative case the statement becomes $\mathrm{Tor}(M,A)$ is quasi-isomorphic to M. As A is free over itself, we obtain the well-known result that $\mathrm{Tor}(M,A)$ is quasi-isomorphic to $M \underset{A}{\otimes} A \cong M$.

When A is not necessarily commutative, we start by considering k as a comodule over B, where the coaction is $c \mapsto 1 \otimes c = c \otimes 1$. By the quasi-isomorphism i, we have a quasi-isomorphism, $i : k \to B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} k$. As τ_1 is 0 on k, this is the same as $k \to B \underset{\tau_1}{\otimes} A \otimes k \cong B \underset{\tau_1}{\otimes} A$.

As all of these objects (and therefore their tensor products) are vector spaces over

k, they are flat over k, so this extends to a quasi-isomorphism

$$f : A \bigotimes (B \underset{\tau_1}{\otimes} M) \otimes k \to A \bigotimes \begin{array}{c} B \\ \underset{k}{\otimes} \\ B \end{array} \underset{\tau_1 \otimes \tau_1}{\bigotimes} \begin{array}{c} M \\ \underset{k}{\otimes} \\ A \end{array}$$

using the usual tensor product differential $d_A \otimes \mathrm{id} + \mathrm{id} \otimes d$. It remains to show we still have a quasi-isomorphism when we throw in the twisted tensor product $\underset{E}{\bigotimes}$. As the above map is a quasi-isomorphism, the mapping cone $C(f) = (A \otimes B(M))[1] \bigoplus (A \otimes B(M) \otimes (B \otimes A))$ is acyclic. $C(f)$ and its homology are trivially quasi-isomorphic, so we may employ the perturbation lemma between $C(f)$ and its homology using the perturbation made up of the twisting cochain E on both chain complexes, i.e. by the perturbation

$$\Big((\mu_A \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes E \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_{B(M) \otimes k}) \Big) \bigoplus \mathrm{id}$$
$$+ \mathrm{id} \bigoplus \Big((\mu_A \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes E \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_{B(M) \otimes (B \otimes A)}) \Big).$$

As $H(C(f))$ is 0, so too is its perturbation, meaning the perturbed mapping cone is quasi-isomorphic to 0, meaning it's acyclic. Therefore there is a quasi-isomorphism

$$A \underset{E}{\bigotimes} (B \underset{\tau_1}{\otimes} M) \otimes k \to A \underset{E}{\bigotimes} \begin{array}{c} B \\ \underset{k}{\otimes} \\ B \end{array} \underset{\tau_1 \otimes \tau_1}{\bigotimes} \begin{array}{c} M \\ \underset{k}{\otimes} \\ A \end{array}.$$

As $A \underset{E}{\otimes} (B \underset{\tau_1}{\otimes} M) \otimes k \cong A \underset{\tau_1}{\otimes} (B \underset{\tau_1}{\otimes} M)$, we have the quasi-isomorphism

$$A \underset{\tau_1}{\otimes} (B \underset{\tau_1}{\otimes} M) \to A \underset{E}{\bigotimes} \begin{array}{c} B \\ \underset{k}{\otimes} \\ B \end{array} \underset{\tau_1 \otimes \tau_1}{\bigotimes} \begin{array}{c} M \\ \underset{k}{\otimes} \\ A \end{array} = [M, A].$$

Using the map p from diagram 2.2, we have a quasi-isomorphism $p : A \underset{\tau_1}{\otimes} (B \underset{\tau_1}{\otimes} M) \to M$. This leaves us with maps of quasi-isomorphisms

$$M \leftarrow A \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} M \to [M, A].$$

In the derived category the first map is invertible, which gives the desired isomorphism. $\square$

2.2.2 Proposition 2.3

Proposition 2.3. *For left dg-modules L, M, and N over A, $[L,[M,N]]$ and $[[L,M],N]$ are isomorphic A-modules in the derived category.*

To prove this proposition, we require a lemma:

Lemma 2.1. *If M and N are left dg-modules over A, then $B(M)\otimes B(N)$ is a dg-comodule over B via the action $\Delta_{B(M)\otimes B(N)} = (\mu_E \otimes id \otimes id)\circ(id \otimes T \otimes id)\circ(\Delta_{B(M)} \otimes \Delta_{B(N)})$, where the map is*

$$\begin{tikzcd}
B(M)\otimes B(N) \arrow[rr, "\Delta_{B(M)\otimes B(N)}"] \arrow[d, "\Delta_{B(M)}\otimes\Delta_{B(N)}"] & & B\otimes B(M)\otimes B(N) \\
(B_M \otimes B(M))\otimes(B_N\otimes B(N)) \arrow[rr, "id\otimes T\otimes id"] & & B_M\otimes B_N\otimes B(M)\otimes B(N) \arrow[u, "\mu_E\otimes id\otimes id"']
\end{tikzcd}$$

where B_M and B_N are just B and the subscripts are only for bookkeeping.

Proof of Lemma 2.1. We will show 1. $\Delta_{B(M)\otimes B(N)}$ is actually a coaction and 2. that $d_{B(M)\otimes B(N)}$ is a comodule differential.

1. For $\Delta_{B(M)\otimes B(N)}$ to be a coaction, it must satisfy the equation

$$\big(\Delta\otimes \mathrm{id}\big)\Delta_{B(M)\otimes B(N)} = \big(\mathrm{id}\otimes \Delta_{B(M)\otimes B(N)}\big)\Delta_{B(M)\otimes B(N)}.$$

We will apply each of these maps to $[a_1,\dots,a_k,m]\otimes[b_1,\otimes,b_l,n]$ and see we get the same result. We will compare the terms ending in $[a_{i+1},\dots,a_k,m]\otimes[b_{j+1},\dots,b_l,n]$. Applying $(\Delta\otimes\mathrm{id})\Delta_{B(M)\otimes B(N)}$ and looking at the terms in question, we get

$$\Delta(\mu_E([a_1,\dots,a_i],[b_1,\dots,b_j]))\otimes[a_{i+1},\dots,a_k,m]\otimes[b_{j+1},\dots,b_l,n]$$

Since (B,μ_E,Δ) is a bialgebra, we have that $\Delta\mu_E = (\mu_E\otimes\mu_E)\otimes(\mathrm{id}\otimes T\otimes \mathrm{id})(\Delta\otimes\Delta)$. Therefore, our left hand side becomes

$$\sum_{p=1}^{i}\sum_{q=1}^{j}\mu_E([a_1,\dots,a_p],[b_1,\dots,b_q])\otimes\mu_E([a_{p+1},\dots,a_i],[b_{q+1},\dots,b_j])$$
$$\otimes[a_{i+1},\dots,a_k,m]\otimes[b_{j+1},\dots,b_l,n].$$

We apply $(\mathrm{id}\otimes\Delta_{B(M)\otimes B(N)})\Delta_{B(M)\otimes B(N)}$ to $[a_1,\dots,a_k,m]\otimes[b_1,\otimes,b_l,n]$. We first just apply $\Delta_{B(M)\otimes B(N)}$ and since our aim is the terms ending in $[a_{i+1},\dots,a_k,m]\otimes[b_{j+1},\dots,b_l,n]$, we first look at terms

$$\sum_{p=1}^{i}\sum_{q=1}^{j}\mu_E([a_1,\dots,a_p],[b_1,\dots,b_q])\otimes[a_{p+1},\dots,a_k,m]\otimes[b_{q+1},\dots,b_l,n]$$

Now applying $\mathrm{id}\otimes\Delta_{B(M)\otimes B(N)}$ to this and looking only at our desired terms, we see they are

$$\sum_{p=1}^{i}\sum_{q=1}^{j}\mu_E([a_1,\dots,a_p],[b_1,\dots,b_q])\otimes\mu_E([a_{p+1},\dots,a_i],[b_{q+1},\dots,b_j])$$

$$\otimes[a_{i+1},\dots,a_k,m]\otimes[b_{j+1},\dots,b_l,n],$$

as desired.

2. That $d^2_{B(M)\otimes B(N)}=0$ follows since both $d_{B(M)}$, $d_{B(N)}$ are differentials and $(d_{B(M)}\otimes\mathrm{id})\circ(\mathrm{id}\otimes d_{B(N)})=(\mathrm{id}\otimes d_{B(N)})\circ(d_{B(M)}\otimes\mathrm{id})$. It remains to show that

$$\Delta_{B(M)\otimes B(N)}d_{B(M)\otimes B(N)}=(d\otimes\mathrm{id}+\mathrm{id}\otimes d_{B(M)\otimes B(N)})\Delta_{B(M)\otimes B(N)}.$$

Let us consider the left hand side.

$$\Delta_{B(M)\otimes B(N)}d_{B(M)\otimes B(N)}=(\mu_E\otimes(\mathrm{id}\otimes\mathrm{id}))\circ(\mathrm{id}\otimes T\otimes\mathrm{id})$$

$$\circ(\Delta_{B(M)}\otimes\Delta_{B(N)})\circ(d_{B(M)}\otimes\mathrm{id}+\mathrm{id}\otimes d_{B(N)})$$

As $d_{B(M)}$ is a differential on the coalgebra $(B(M),\Delta_{B(M)})$ we have $\Delta_{B(M)}d_{B(M)}=(d\otimes\mathrm{id}+\mathrm{id}\otimes d_{B(M)})\Delta_{B(M)}$ and the same goes for the same operations on $B(N)$. Therefore we can reorder the last two maps to give us

$$=(\mu_E\otimes(\mathrm{id}\otimes\mathrm{id}))\circ(\mathrm{id}\otimes T\otimes\mathrm{id})\circ$$

$$(d\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes d_{B(M)}\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes d\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes d_{B(N)})\circ(\Delta_{B(M)}\otimes\Delta_{B(N)})$$

Now the middle two maps can be reordered simply by applying the involution $(\mathrm{id}\otimes T\otimes\mathrm{id})$

$$=(\mu_E\otimes(\mathrm{id}\otimes\mathrm{id}))\circ(d\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes d_{B(M)}\otimes\mathrm{id}+\mathrm{id}\otimes d\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes d_{B(N)})$$

$$\circ(\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta_{B(M)} \otimes \Delta_{B(N)})$$

Since d is a differential on the algebra (B, μ_E), μ_E obeys Leibniz's rule, i.e. $\mu_E \circ (d \otimes \mathrm{id} + \mathrm{id} \otimes d) = d \circ \mu_E$. Therefore we can reorder the first two maps above to give us

$$= (d \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes d_{B(M)} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes d_{B(N)}) \circ (\mu_E \otimes (\mathrm{id} \otimes \mathrm{id}))$$

$$\circ(\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta_{B(M)} \otimes \Delta_{B(N)}).$$

This is exactly $(d \otimes \mathrm{id} + \mathrm{id} \otimes d_{B(M)\otimes B(N)})\Delta_{B(M)\otimes B(N)}$, as desired.

$\square$

Proof of Proposition 2.3. We construct a quasi-isomorphism

$$q : A \underset{E}{\otimes} ((B(L) \otimes B(M)) \otimes B(N)) \to [[L, M], N],$$

where we consider $B(L) \otimes B(M)$ as a B module via the coaction given in Lemma 2.1. Similarly, we construct a quasi-isomorphism

$$q' : A \underset{E}{\otimes} (B(L) \otimes (B(M) \otimes B(N))) \to [L, [M, N]],$$

where we consider $B(M) \otimes B(N)$ as a B module.

That $A \underset{E}{\otimes} ((B(L) \otimes B(M)) \otimes B(N)) = A \underset{E}{\otimes} (B(L) \otimes (B(M) \otimes B(N)))$ follows from the coassociativity in B and the associativity of E, as in equation 1.3.

In the derived category, our quasi-isomorphism of associativity follows from $q^{-1} \circ q'$. By symmetry it suffices to show that q is a quasi-isomorphism.

Denote by K

$$K = \begin{array}{ccc} B & & L \\ \underset{k}{\otimes} & \underset{\tau_1 \otimes \tau_1}{\bigotimes} & \underset{k}{\otimes} \\ B & & M \end{array}.$$

By construction K is a $B \otimes B$ comodule. By lemma 2.1, K can be thought of as a B comodule by using the coaction $\Delta' = (\mu_E \circ \mathrm{id}) \circ \Delta$. As the following diagram commutes, we have that $A \underset{\tau_1}{\otimes} K$ is equal to $A \underset{E}{\otimes} K$ using the appropriate comodule action:

$$\begin{tikzcd}
 & B \otimes B \otimes K \arrow[r, "E\otimes\mathrm{id}"] \arrow[d, "\mu_E\otimes\mathrm{id}"] & A \otimes K \arrow[d, "="] \\
K \arrow[ur, "\Delta"] \arrow[r, "\Delta'"'] & B \otimes K \arrow[r, "\tau_1\otimes\mathrm{id}"] & A \otimes K
\end{tikzcd}$$

Therefore, by prepending B onto our tensor product using the twisting cochain τ_1, we have that

$$B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} K = B \underset{\tau_1}{\otimes} A \underset{E}{\otimes} K.$$

Diagram 2.1 gives a quasi-isomorphism $i : K \to B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} K$. By what we've just shown, $i' : K \to B \underset{\tau_1}{\otimes} A \underset{E}{\otimes} K$ is then also a quasi-isomorphism.

By an argument identical to that in the proof of Proposition 2.2, the above quasi-isomorphism extends to a quasi-isomorphism

$$A \underset{E}{\otimes} \big(B(L) \otimes B(M)\big) \otimes B(N) = A \underset{E}{\otimes} K \otimes B(N) \to A \underset{E}{\otimes} \big(B \underset{\tau_1}{\otimes} A \underset{E}{\otimes} K\big) \otimes B(N) = [[L, M], N].$$

This is our quasi-isomorphism q. $\square$

Chapter 3

∞-Categories

As the isomorphisms that usually hold for a monoidal category only hold up to quasi-isomorphism in the previous section, it is much easier to deal instead with ∞-categories.

An ∞-category is like a category, except composition is not uniquely defined. Instead of only containing objects, which we call 0-morphisms, and morphisms, which we call 1-morphsims, there are also n-morphsims between $(n-1)$-morphisms for $n \geq 1$. These higher morphisms are all somewhat invertible, in the sense that the maps are invertible up to homotopy.

Formally, let $S = \{S_n\}_{n\geq 0}$ be a simplicial set, where we think of each S_n as the set of all continuous maps from the n-simplex Δ^n into a topological space X. The specifics of X are unimportant. For $i = 0, 1, 2, \ldots, n$, define the i-horn of Δ^n, denoted $\Lambda_i^n(S)$, to be the maps of S_n whose domain is the faces of Δ^n containing the ith vertex. There is a restriction map $S_n \to \Lambda_i^n(S)$, given by the formula $\sigma \mapsto (d_0\sigma, d_1\sigma, \ldots, d_{i-1}\sigma, d_{i+1}\sigma, \ldots, d_n\sigma)$, where d_j is the jth face map.

A simplicial set $S = \{S_n\}$ is a *Kan complex* if the restriction map $S_n \to \Lambda_i^n(S)$ is surjective for all $0 \leq i \leq n$. That is, every map containing the ith vertex must come from an n-simplex of S.

A simplicial set S is called a weak Kan complex, or an ∞-*category* if, for each $0 < i < n$, the map $S_n \to \Lambda_i^n(S)$ is surjective.

To see why we are interested only in this weaker condition, consider a map f :

$x \to y$. We can construct the $\Lambda_2^2(S)$ horn:

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[rr, "\mathrm{id}_y"] & & y. \end{tikzcd}$$

If we were able to complete this horn to a 2-simplex, i.e. there was a 2-simplex that mapped onto this horn, we would be giving f a right inverse, which is too strong an assumption in most cases.

For an ∞-category, in the 2-dimensional case, given maps $f : x \to y$ and $g : y \to z$, we can complete the horn below with $h : x \to z$ to construct a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

As the definition of ∞-category makes no mention of uniqueness, there's no assumption on h being the only way of completing this horn, so there may be another map h' which defines a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h'"] & & z. \end{tikzcd}$$

To see how h and h' are related, notice we also naturally have the 2-simplex

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"] & & y \end{tikzcd}$$

Using these 2-simplices, we can construct the horn $\Lambda_2^3(S)$:

$$\begin{tikzcd} & x \arrow[d, "f"] \arrow[ddr, "h'"] & \\ & y \arrow[dr, "g"] & \\ x \arrow[uur, "\mathrm{id}_x"] \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

which can be filled in to give us the 2-simplex of the x, x, z side, which is a homotopy between h and h'. Now this homotopy is not necessarily unique, which starts to give you an idea of why this is called an ∞-category.

3.1 The $\mathcal{C}^{\otimes}$ Category

To show all the monoidal coherence conditions by dealing explicitly with maps is extremely cumbersome as shown in the proofs of Propositions 2.2 and 2.3. Even in the much simpler case of showing associativity of tensor product of vector spaces for example, to give an explicit isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ requires making choices of bases. It is easier and more straightforward instead to show the existence of a canonical isomorphism. We proceed then using this reasoning. This requires a certain construction.

The following construction comes from Lurie in [16] as an alternate to operads. If $\mathcal{C}$ is a monoidal category, such as complexes of k-vector spaces with usual tensor product over k, we construct a new category $\mathcal{C}^{\otimes}$ as follows:

1. An object of $\mathcal{C}^{\otimes}$ is a finite, possibly empty, sequence of objects of $\mathcal{C}$, denoted $[C_1, \ldots, C_n]$.

2. A morphism from $[C_1, \ldots, C_n]$ to $[C'_1, \ldots, C'_m]$ is given by a nonstrictly order-preserving map $f : [m] \to [n]$, and a collection of morphisms $C_{f(i-1)+1} \otimes \cdots \otimes C_{f(i)} \to C'_i$ for $1 \leq i \leq m$, where $[n] = \{0, \ldots, n\}$ and $[-1]$ is the empty set.

3. Composition in $\mathcal{C}^{\otimes}$ is determined by composition of order preserving maps, composition in $\mathcal{C}$, and the associativity and unit constraints of the monoidal structure on $\mathcal{C}$.

Let $\boldsymbol{\Delta}$ denote the category of simplices, where the objects are the ordered sets $[n]$, with $n \geq 0$ and the morphisms are non-strictly increasing maps of linearly ordered sets. This means that a map $[m] \to [n]$ embeds an m-simplex as a face on an n-simplex. In the category $\boldsymbol{\Delta}^{op}$, which we will focus on, a map $[n] \to [m]$ specifies an m-simplex face from the n-simplex $[n]$.

We then have a forgetful functor $\rho : \mathcal{C}^{\otimes} \to \mathbf{\Delta}^{op}$, which sends an object $[C_1, \ldots, C_n]$ to $[n]$, which has the following properties:

(A) For every object $[C_1, \ldots, C_n]$ in $\mathcal{C}^{\otimes}$ and every morphism $f : [n] \to [m]$ in $\mathbf{\Delta}^{op}$, there exists a morphism $\overline{f} : [C_1, \ldots, C_n] \to [C'_1, \ldots, C'_m]$ which covers f and is universal in composing with $\overline{f}$ to induce a bijection to the fiber product

$$\mathrm{Hom}_{\mathcal{C}^{\otimes}}([C'_1, \ldots, C'_m], [C''_1, \ldots, C''_k]) \to$$

$$\mathrm{Hom}_{\mathcal{C}^{\otimes}}([C_1, \ldots, C_n], [C''_1, \ldots, C''_k]) \times_{\mathrm{Map}_{\Delta}([k],[n])} \mathrm{Map}_{\mathbf{\Delta}}([k], [m])$$

for every $[C''_1, \ldots, C''_k]$ in $\mathcal{C}^{\otimes}$. This is done by choosing $\overline{f}$ so that the maps $C_{f(i-1)+1} \otimes \cdots \otimes C_{f(i)} \to C'_i$ are isomorphisms for $1 \leq i \leq m$.

(B) Let $\mathcal{C}^{\otimes}_{[n]}$ denote the fiber of ρ over the object $[n] \in \mathbf{\Delta}^{op}$. Then $\mathcal{C}^{\otimes}_{[1]}$ is equivalent to $\mathcal{C}$ and $\mathcal{C}^{\otimes}_{[n]}$ is equivalent to an n-fold product of copies of $\mathcal{C}$ using the inclusion $[1] \cong \{i-1, i\} \subset [n]$ for all i.

The category $\mathcal{C}$ is determined up to canonical equivalence by $\mathcal{C}^{\otimes}$. To see this, suppose we have a category $\mathcal{D}$ and there is a functor $\rho : \mathcal{D} \to \mathbf{\Delta}^{op}$ obeying conditions (A) and (B) above. We identify $\mathcal{D}_{[1]} =: \mathcal{C}$. Keeping in mind that maps from $\mathbf{\Delta}^{op}$ reverse, the inclusion $[1] \cong \{0, 2\} \hookrightarrow [2]$,

(3.1)
$$\begin{tikzcd} 0 \arrow[d] & 2(A_1 \otimes A_2) \arrow[dr] & \\ 0 & 1(A_1) & 2(A_2), \end{tikzcd}$$

where elements in parentheses correspond to objects of $\mathcal{C}$ determines a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we denote $\otimes$. Associativity follows from this diagram where all maps are inclusion

$$\begin{tikzcd} \{0,3\} \arrow[r] \arrow[d] & \{0,1,3\} \arrow[d] \\ \{0,2,3\} \arrow[r] & \{0,1,2,3\}. \end{tikzcd}$$

This corresponds to the diagram

$$\begin{tikzcd}
\mathcal{D}_{[1]} & \mathcal{D}_{[2]} \arrow[l] \\
\mathcal{D}_{[2]} \arrow[u] & \mathcal{D}_{[3]}, \arrow[l] \arrow[u]
\end{tikzcd}$$

which, after we use the equivalence $\mathcal{D}_{[n]} \cong \mathcal{C}^n$, we get the isomorphism $(A_1 \otimes A_2) \otimes A_3 \cong A_1 \otimes (A_2 \otimes A_3)$.

By condition (B), $\mathcal{D}_{[0]}$ has a single object, which we denote k. The surjection $[1] \twoheadrightarrow [0]$

(3.2)
$$\begin{tikzcd}
0 \arrow[d] & 1(k) \arrow[dl] \\
0 &
\end{tikzcd}$$

gives a functor $\mathcal{D}_{[0]} \to \mathcal{D}_{[1]} = \mathcal{C}$, which produces our unit in $\mathcal{C}$.

Using diagrams 3.1 and 3.2, we see that $k \otimes A$ and $A \otimes k$ are isomorphic to A by using two different diagrams of the maps: $\{0, 2\} \to \{0, 1, 2\} \to \{0, 2\}$:

$$\begin{tikzcd}
0 \arrow[d] & 2(k \otimes A) \arrow[dr] & \\
0 \arrow[d] & 1(k) \arrow[dl] & 2(A) \arrow[dl] \\
0 & 2(A) &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[d] & 2(A \otimes k) \arrow[dr] & \\
0 \arrow[d] & 1(A) \arrow[d] & 2(k) \arrow[dl] \\
0 & 2(A). &
\end{tikzcd}$$

The triangle identity, meaning the commutativity of the diagram

$$\begin{tikzcd}
(A_1 \otimes k) \otimes A_2 \arrow[rr] \arrow[dr] & & A_1 \otimes (k \otimes A_2) \arrow[dl] \\
& A_1 \otimes A_2 &
\end{tikzcd}$$

is satisfied by these two diagrams

$$\begin{tikzcd}
0 \arrow[d] & 3((A_1 \otimes k) \otimes A_2) \arrow[dr] & & \\
0 \arrow[d] & 2(A_1 \otimes k) \arrow[dr] & 3(A_2) \arrow[dr] & \\
0 \arrow[d] & 1(A_1) \arrow[d] & 2(k) \arrow[dl] & 3(A_2), \arrow[dl] \\
0 & 2(A_1) & 3(A_2) &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[d] & 3(A_1 \otimes (k \otimes A_2)) \arrow[dr] & & \\
0 \arrow[d] & 1(A_1) \arrow[d] & 3(k \otimes A_2) \arrow[dr] & \\
0 \arrow[d] & 1(A_1) \arrow[d] & 2(k) \arrow[dl] & 3(A_2) \arrow[dl] \\
0 & 2(A_1) & 3(A_2) &
\end{tikzcd}$$

along with the fact that the top two rows (including arrows) of the first diagram can be replaced by the first two rows of the second and realizing this is exactly the associativity diagram.

Lastly, that the pentagon axiom, meaning the commutativity of the diagram

$$\begin{tikzcd}
 & (A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[dr] & \\
((A_1 \otimes A_2) \otimes A_3) \otimes A_4 \arrow[ur] \arrow[d] & & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) \\
(A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \arrow[rr] & & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \arrow[u]
\end{tikzcd}$$

is satisfied follows from the commutativity of this inclusion diagram from [1] $\cong$

$\{0,4\} \to [4]$:

$$
\begin{tikzcd}
 & \{0,4\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{0,1,4\} \arrow[d] \arrow[dr] & \{0,2,4\} \arrow[dl] \arrow[dr] & \{0,3,4\} \arrow[dl] \arrow[d] \\
\{0,1,2,4\} \arrow[dr] & \{0,1,3,4\} \arrow[d] & \{0,2,3,4\} \arrow[dl] \\
 & \{0,1,2,3,4\} &
\end{tikzcd}
$$

These coherence conditions show that $\mathcal{D}$ determines $\mathcal{C}$ as a monoidal category. We will attempt to replicate something similar to this to define something called a monoidal ∞-category. Of course we don't expect to have a true tensor product, so our result will not be exactly this. To begin this process, we need to define the dg-nerve of a category.

3.2 The Nerve and dg-Nerve of a Category

The following definitions come from Lurie in [19].

The *nerve of a category* $\mathcal{C}$, denoted $N(\mathcal{C})$, is the collection of sets $\{\mathcal{C}_n\}_{n\geq 0}$ where $\mathcal{C}_n$ is the set of all composable chains of morphisms $\mathcal{C} \to \cdots \to \mathcal{C}$ of length n. This has the structure of a simplicial set. The nerve also determines $\mathcal{C}$ up to isomorphism as the objects of $\mathcal{C}$ are $\mathcal{C}_0$ and the morphisms are the elements of $\mathcal{C}_1$.

If $\mathcal{C}$ is a differential graded category, we can associate to $\mathcal{C}$ a simplicial set $N_{\mathrm{dg}}(\mathcal{C})$ called the *differential graded nerve of* $\mathcal{C}$, or dg-nerve of $\mathcal{C}$. For each nonnegative n, $N_{\mathrm{dg}}(\mathcal{C})_n \cong \mathrm{Hom}(\Delta^n, N_{\mathrm{dg}}(\mathcal{C}))$ to be the set of all ordered pairs $(\{X_i\}_{0\leq i\leq n}, \{f_I\})$, where:

1. For $0 \leq i \leq n$, X_i denotes an object of $\mathcal{C}$.

2. For every subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\}$ with $m \geq 0$, f_I is an element of the group $\mathrm{Map}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$, satisfying the equation

$$d \circ f_I + f_I \circ d = \sum_{1\leq j\leq m} f_{I-\{i_j\}} + f_{\{i_j<\cdots<i_1<i_+\}} \circ f_{\{i_-<i_m<\cdots<i_j\}} \tag{3.3}$$

Note that we have forgone using signs so this is a simplified version.

If $\alpha : [m] \to [n]$ is a nondecreasing function, the induced map $N_{\mathrm{dg}}(\mathcal{C})_n \to N_{\mathrm{dg}}(\mathcal{C})_m$ is given by

$$(\{X_i\}_{0 \le i \le n}, \{f_I\}) \mapsto (\{X_\alpha(j)\}_{0 \le j \le m}, \{g_J\}),$$

where

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha \mid J \text{ is injective,} \\ \mathrm{id} & \text{if } J = \{j, j'\},\ \text{with } \alpha(j) = \alpha(j') = i, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Monoidal ∞-Categories

In [16], with some unwinding coming from [18] and [8], Lurie defines a *monoidal* ∞-*category* is a map of simplicial sets $p : \mathcal{C}^{\otimes} \to N(\mathbf{\Delta}^{op})$ that adhere to the following properties below, making p a *cocartesian fibration.* That the maps below are defined in terms of p^{op} is to avoid further confusion on mapping Δ^n and its subsets to $N(\mathbf{\Delta})$. The properties are:

1. p^{op} has the right lifting property with respect to all inner horn inclusions, that is for all diagrams with $0 < i < n$,

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C}^{\otimes op} \arrow[d, "p^{op}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & N(\mathbf{\Delta}). \end{tikzcd}$$

Figure 3.1: Diagram for an inner fibration

there exists a dotted arrow as indicated which makes the diagram commutative. This condition on p^{op} is the same as saying p^{op} is an *inner fibration.* By Proposition 2.3.1.5., Example 1.1.2.5., and the last paragraph of section 1.2.1 all from [18], this is equivalent to saying that $\mathcal{C}^{\otimes}$ is an ∞-category.

2. For every edge $f' : x' \to y'$ of $N(\mathbf{\Delta})$, i.e. a map $\Delta^1 \to N(\mathbf{\Delta})$, and every vertex y of $\mathcal{C}^{\otimes op}$ with $p^{op}(y) = y'$, there exists an edge $f : x \leftarrow y$, i.e. $f : \Delta^1 \to \mathcal{C}^{\otimes op}$ where $p^{op}(x) = x'$, such that $p^{op}(f) = f'$ and for every $n \geq 2$ and every commutative diagram

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[d, hook] \arrow[dr, "f"] \arrow[ddr, bend left, "f'"] & \\
\Lambda^n_n \arrow[r] \arrow[d, hook] & \mathcal{C}^{\otimes op} \arrow[d, "p^{op}"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & N(\mathbf{\Delta})
\end{tikzcd}$$

Figure 3.2: Diagram for a p-cocartesian edge

there exists a dotted arrow as indicated which makes the diagram commutative. Such a diagram with f exists is equivalent to saying that f is a *p-cocartesian edge.*

3. For each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$ determine an equivalence of ∞-categories

$$\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{0,1\}} \times \cdots \times \mathcal{C}^{\otimes}_{\{n-1,n\}} \cong (\mathcal{C}^{\otimes}_{[1]})^n.$$

Using this equivalence above, considering the three embeddings of [1] into [2], specifically $\{0,1\}, \{1,2\}$, and $\{0,2\}$, we get an equivalence $\mathcal{C}^{\otimes}_{\{0,1\}} \times \mathcal{C}^{\otimes}_{\{1,2\}} \leftarrow \mathcal{C}^{\otimes}_{[2]}$. There is also the map $\mathcal{C}^{\otimes}_{[2]} \to \mathcal{C}^{\otimes}_{\{0,2\}}$. Identifying $\mathcal{C}^{\otimes}_{\{a,b\}}$ as $\mathcal{C}$, where a, b are $0, 1, 2, a < b$ and taking the homotopy inverse of the first map given above, we are left with a map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We have obtained a tensor product. Using the diagrams in Lurie's construction of $\mathcal{C}^{\otimes}$ in section 3.1, we can similarly obtain associativity up to homotopy as well as all other required coherence conditions.

We are nearing our main result. Our first step is introducing what our maps will look like. This will require a few lemmas. We begin by defining new maps

$$\begin{tikzcd}
B(M_1)\otimes\cdots\otimes B(M_k) \arrow[r, "\phi"] \arrow[d, "\Delta^{\otimes k}"] & B(M) \arrow[d, "\Delta"] \\
B^{\otimes k}\otimes B(M_1)\otimes\cdots\otimes B(M_k) \arrow[r, "\mu_E^k\otimes\phi"] & B\otimes B(M)
\end{tikzcd}$$

Figure 3.3: Comodule diagram

$\mu_E^n : B^{\otimes n} \to B$, specifically,

$$\mu_E^n(\beta_1,\ldots,\beta_n) = \mu_E(\beta_1, \mu_E(\beta_2,\ldots,\mu_E(\beta_{n-1},\beta_n)\ldots)),$$

which is unambiguous by the associativity of μ_E. This map is comultiplicative. Note that $\mu_E^2 = \mu_E$ and $\mu_E^1 = \mathrm{id}$. We use this to introduce twisting cochains $\tau_n : B^{\otimes n} \to \overline{A}$, where $\tau_n = \tau \circ \mu_E^n$. Our definition of τ_1 from above agrees with this definition and $\tau_2 = E$. Given left A dg-modules $M_1, \ldots M_n$ over A, we use the twisted tensor product to construct a module

$$A \underset{\tau_n}{\otimes} \big(B(M_1)\otimes\cdots\otimes B(M_n)\big).$$

In the proof of proposition 2.3 we wrote $A \underset{E}{\otimes} \big((B(L)\otimes B(M))\otimes B(N)\big)$ and $A \underset{E}{\otimes} \big(B(L)\otimes(B(M)\otimes B(N))\big)$, explaining that they were equal. With this notation, it is clear that both may be written as $A \underset{\tau_3}{\otimes} \big(B(L)\otimes B(M)\otimes B(N)\big)$.

3.3.1 Lemma 3.1

Lemma 3.1. *Given a map of A modules, $\tilde{\phi} : A \underset{\tau_k}{\otimes} \big(B(M_1)\otimes\cdots\otimes B(M_k)\big) \to M$, where $M_1,\ldots,M_k$ are modules over A, we have a map $\phi : B(M_1)\otimes\cdots B(M_k) \to B(M)$ such that the following diagram commutes:*

Proof of Lemma 3.1. By the adjoint functors that gave us diagrams 2.1 and 2.2, given $\tilde{\phi} \in \mathrm{Hom}_A\big(A \underset{\tau_k}{\otimes} (B(M_1)\otimes\cdots\otimes B(M_k)), M\big)$, we have a corresponding $\tilde{\phi}' \in \mathrm{Hom}_{B^{\otimes k}}\big(B(M_1)\otimes\cdots\otimes B(M_k), B^{\otimes k} \underset{\tau_k}{\otimes} M\big)$, meaning $\tilde{\phi}'$ is a map of comodules over $B^{\otimes k}$.

This yields a commutative diagram

$$\begin{tikzcd}
B(M_1) \otimes \cdots \otimes B(M_k) \arrow[rr, "\tilde{\phi}'"] \arrow[d, "\Delta^{\otimes k}"] & & B^{\otimes k} \underset{\tau_k}{\otimes} M \arrow[d, "\Delta^{\otimes k} \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes B(M_1) \otimes \cdots \otimes B(M_k) \arrow[rr, "\mathrm{id} \otimes \tilde{\phi}'"] & & B^{\otimes k} \otimes B^{\otimes k} \underset{\tau_k}{\otimes} M
\end{tikzcd}$$

Moreover, as (B, μ_E, Δ) forms a bialgebra, we have a commutative diagram

$$\begin{tikzcd}
B^{\otimes k} \underset{\tau_k}{\otimes} M \arrow[r, "\mu_E^k \otimes \mathrm{id}"] \arrow[d, "\Delta^{\otimes k} \otimes \mathrm{id}"] & B \underset{\tau_1}{\otimes} M \arrow[d, "\Delta \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes B^{\otimes k} \underset{\tau_k}{\otimes} M \arrow[r, "\mu_E^k \otimes \mu_E^k \otimes \mathrm{id}"] & B \otimes B \underset{\tau_1}{\otimes} M.
\end{tikzcd}$$

That this is a chain map is perhaps not obvious by the presence of our twisted tensor products. As we have a dg-bialgebra, the only issue that could arise would be from the twisted differential. For this map to be a chain map then, we require that the following diagram commutes:

$$\begin{tikzcd}
B^{\otimes k} \otimes M \arrow[r, "\mu_E^k \otimes \mathrm{id}"] \arrow[d, "\Delta^{\otimes k} \otimes \mathrm{id}"] & B \otimes M \arrow[d, "\Delta \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes B^{\otimes k} \otimes M \arrow[d, "\mathrm{id} \otimes \tau_k \otimes \mathrm{id}"] & B \otimes B \otimes M \arrow[d, "\mathrm{id} \otimes \tau_1 \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes A \otimes M \arrow[d, "\mathrm{id} \otimes m_M"] & B \otimes A \otimes N \arrow[d, "\mathrm{id} \otimes m_M"] \\
B^{\otimes k} \otimes M \arrow[r, "\mu_E^k \otimes \mathrm{id}"] & B \otimes M
\end{tikzcd}$$

These down arrows are exactly the twisted part of the differential. To see that this commutes, the path that goes right and then down is given by:

$$(\mathrm{id} \otimes m_M) \circ (\mathrm{id} \otimes \tau_1 \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ (\mu_E^k \otimes \mathrm{id}).$$

As B is a bialgebra, we can rearrange the two rightmost terms as

$$(\mathrm{id} \otimes m_M) \circ (\mathrm{id} \otimes \tau_1 \otimes \mathrm{id}) \circ (\mu_E^k \otimes \mu_E^k \otimes \mathrm{id}) \circ (\Delta^{\otimes k} \otimes \mathrm{id}).$$

We can commute the left μ_E^k of the $(\mu_E^k \otimes \mu_E^k \otimes \mathrm{id})$ function to the end of this composition to get

$$(\mu_E^k \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m_M) \circ (\mathrm{id} \otimes \tau_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_E^k \otimes \mathrm{id}) \circ (\Delta^{\otimes k} \otimes \mathrm{id}).$$

Using the fact that $\tau_k = \tau_1 \circ \mu_E^k$, this becomes

$$(\mu_E^k \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m_M) \circ (\mathrm{id} \otimes \tau_k \otimes \mathrm{id}) \circ (\Delta^{\otimes k} \otimes \mathrm{id}),$$

which is exactly the path of going down and then right, as desired.

Therefore, we may combine the first two diagrams and, by defining $\phi := (\mu_E^k \otimes \mathrm{id}) \circ \tilde{\phi}'$, we obtain our desired diagram. $\square$

3.3.2 Proposition 3.1

Proposition 3.1. *Given a $B^{\otimes k}$ comodule N, we have a chain map:*

$$\alpha : N \to B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N,$$

such that the following diagram commutes

$$\begin{tikzcd}
N \arrow[r, "\alpha"] \arrow[d, "\Delta^{\otimes k}"] & B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N \arrow[d, "\Delta"] \\
B^{\otimes k} \otimes N \arrow[r, "\mu_E^k \otimes \alpha"] & B \otimes B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N
\end{tikzcd}$$

In particular, given A modules $M_1, \ldots, M_k$, we have a chain map

$$B(M_1) \otimes \cdots \otimes B(M_k) \to B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} \big(B(M_1) \otimes \cdots \otimes B(M_k)\big)$$

satisfying the above diagram. This map in a sense turns a $B^{\otimes k}$ comodule into a B comodule.

Proof of Proposition 3.1. Using the map i from diagram 2.1, we have a chain map such that this diagram commutes:

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[d, "\Delta^{\otimes k}"] & B^{\otimes k} \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N \arrow[d, "\Delta^{\otimes k}"] \\
B^{\otimes k} \otimes N \arrow[r, "\mathrm{id}^{\otimes k} \otimes i"] & B^{\otimes k} \otimes B^{\otimes k} \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N
\end{tikzcd}$$

From the proof of lemma 3.1, we get the following commutative diagram

$$\begin{tikzcd}
B^{\otimes k} \underset{\tau_k}{\otimes} (A \underset{\tau_k}{\otimes} N) \arrow[r, "\mu_E^k \otimes \mathrm{id}"] \arrow[d, "\Delta^{\otimes k} \otimes \mathrm{id}"] & B \underset{\tau_1}{\otimes} (A \underset{\tau_k}{\otimes} N) \arrow[d, "\Delta \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes B^{\otimes k} \underset{\tau_k}{\otimes} (A \underset{\tau_k}{\otimes} N) \arrow[r, "\mu_E^k \otimes \mu_E^k \otimes \mathrm{id}"] & B \otimes B \underset{\tau_1}{\otimes} (A \underset{\tau_k}{\otimes} N).
\end{tikzcd}$$

We define $\alpha = (\mu_E^k \otimes \mathrm{id}) \circ i : N \to B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N$ to obtain the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "(\mu_E^k \otimes \mathrm{id}) \circ i"] \arrow[d, "\Delta^{\otimes k}"] & B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N \arrow[d, "\Delta \otimes \mathrm{id}"] \\
B^{\otimes k} \otimes N \arrow[r, "(\mu_E^k \otimes \mu_E^k \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes k} \otimes i)"] & B \otimes B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} N,
\end{tikzcd}$$

where we identify the map across the top as α and the map across the bottom as $\mu_E^k \otimes \alpha$. $\square$

We will be using the map α often and will not distinguish which coalgebra the comodule is over unless it is unclear. Moreover, we may reinterpret that a map f satisfies figure 3.3 by using the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "(\mu_E^k \otimes \mathrm{id}) \circ \Delta^{\otimes k}"] \arrow[dd, bend right=60, "\alpha"'] & Y \arrow[d, "\Delta"'] \arrow[dd, bend left=60, "\alpha"] \\
B \otimes X \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\mathrm{id} \otimes \eta"] & B \otimes Y \arrow[d, "\mathrm{id} \otimes \eta"'] \\
B \underset{\tau_1}{\otimes} A \underset{\tau_k}{\otimes} X \arrow[r, "\mathrm{id}^{\otimes 2} \otimes f"] & B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} Y
\end{tikzcd}$$

As the map $\mathrm{id} \otimes \eta$ has a left inverse, namely the counit map ϵ_A, if a map f satisfies figure 3.3, it is equivalent to $\alpha \circ f = (\mathrm{id}^{\otimes 2} \otimes f) \circ \alpha$.

Corollary 3.1.

1. *Given comodules N_1, N_2, N_3 over $B^{\otimes k_i}$ for $i = 1, 2, 3$ respectively, and a map $f : N_1 \otimes N_2 \otimes N_3 \to B(M)$ for an A module M so that $\alpha \circ f = (id^{\otimes 2} \otimes f) \circ \alpha$, there exists a map $\tilde{f} : N_1 \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_2}}{\otimes} N_2) \otimes N_3 \to B(M)$ so that $\alpha \circ \tilde{f} = (id^{\otimes 2} \otimes \tilde{f}) \circ \alpha$*

such that the following diagram commutes:

$$
\begin{tikzcd}
N_1 \otimes N_2 \otimes N_3 \arrow[r, "f"] \arrow[d, "id\otimes\alpha\otimes id"'] & B(M) \\
N_1 \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_2}}{\otimes} N_2) \otimes N_3 \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
$$

2. *Given comodules* $N_1, \ldots, N_n$ *over* $B^{\otimes k_i}$ *for* $i = 1, \ldots, n$ *respectively, a map* $f : N_1 \otimes \cdots \otimes N_n \to B(M)$ *where* f *satisfies* $\alpha \circ f = (id^{\otimes 2} \otimes f) \circ \alpha$, *and a sequence of integers* $\{j_1, \ldots, j_l\}$ *such that* $1 < j_1 < j_2 < \cdots < j_l < n$, *there exists a map*

$$\tilde{f}:$$

$$N_1\otimes\cdots\otimes(B\underset{\tau_1}{\otimes}A\underset{\tau_{k_{j_1}}}{\otimes}N_{j_1})\otimes N_{j_1+1}\otimes\cdots\otimes(B\underset{\tau_1}{\otimes}A\underset{\tau_{k_{j_2}}}{\otimes}N_{j_2})\otimes\cdots\otimes(B\underset{\tau_1}{\otimes}A\underset{\tau_{k_{j_l}}}{\otimes}N_{j_l})\otimes\cdots\otimes N_n$$
$$\to B(M),$$

such that $\alpha \circ \tilde{f} = (id^{\otimes 2} \otimes \tilde{f}) \circ \alpha$ *such that the following diagram commutes:*

$$
\begin{tikzcd}
N_1 \otimes \cdots \otimes N_n \arrow[r, "f"] \arrow[d, "id^{j_1-1}\otimes\alpha\otimes id^{j_2-j_1-1}\otimes\cdots\otimes\alpha\otimes id^{n-j_l}"'] & B(M) \\
N_1 \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_1}}}{\otimes} N_{j_1}) \otimes N_{j_1+1} \otimes \cdots \arrow[ur, "\tilde{f}"] &
\end{tikzcd}
$$

$$\otimes(B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_2}}}{\otimes} N_{j_2}) \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_l}}}{\otimes} N_{j_l}) \otimes \cdots \otimes N_n$$

Proof of Corollary 3.1. 1. We make reference to this diagram:

$$
\begin{tikzcd}
N_1 \otimes N_2 \otimes N_3 \arrow[r, "f"] \arrow[d, "\mathrm{id}\otimes\mu_E^2\circ\Delta^{\otimes k_2}\otimes\mathrm{id}"] \arrow[dd, bend right=60, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"'] & B(M) \arrow[d, "\Delta"'] \arrow[dd, bend left=60, "i"] \\
N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[r, "f\circ(\mu_E^{k_1+1+k_3}\circ(\Delta^{\otimes k_1}\otimes\mathrm{id}^{\otimes 2}\otimes\Delta^{\otimes k_3}))"] \arrow[d, "\mathrm{id}^{\otimes 2}\otimes\eta\otimes\mathrm{id}^{\otimes 2}"] & B \otimes B(M) \arrow[d, "\mathrm{id}\otimes\eta\otimes\mathrm{id}"] \\
N_1 \otimes B \otimes A \otimes N_2 \otimes N_3 \arrow[r, "f'"] & B \otimes A \otimes B(M)
\end{tikzcd}
$$

As f satisfies figure 3.3, by the associativity of μ_E, the top rectangle in the above diagram commutes. The second rectangle commutes as we are just inserting 1 in the appropriate places, where we've written f' to avoid the messiness of dealing with a map almost identical to that above it except with some permutation. The composition of the down arrows are given. What we need to finish our proof then, is an inverse of i.

In general, given a B comodule L, there is not a comodule map $B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} L \to L$ (the usual map goes in the opposite direction). However, for a module M, we do have a map $p : A \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} M \to M$. Prepending this with $B \underset{\tau_1}{\otimes}$, we get a map $B \underset{\tau_1}{\otimes} (A \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} M) \to B \underset{\tau_1}{\otimes} M = B(M)$. This map is a dg-comodule map. Moreover, the composition $B(M) \xrightarrow{i} B \underset{\tau_1}{\otimes} A \underset{\tau_1}{\otimes} B \underset{\tau_1}{\otimes} B(M) \xrightarrow{\mathrm{id}\otimes p} B \underset{\tau_1}{\otimes} M = B(M)$ is the identity map. This follows from the fact that in a bialgebra the unit and multiplication compose to the identity as do comultiplication and the counit.

Therefore, by defining $\tilde{f} = (\mathrm{id} \otimes p) \circ f'$, we obtain the desired result.

This proposition is malleable. For instance, by setting $N_1 = k$ or $N_3 = k$, we can apply our α map to the first or last term, respectively.

2. The proof proceeds by induction on l. The case when $l = 1$, $j_1 = 2$, is given as part 1 above. Assuming the result holds for $l - 1$, we start by defining a comodule $N' = N_{j_{l-1}+1} \otimes \cdots \otimes N_n$, which is a comodule over $B^{\otimes j_{l-1}+1+\cdots+n}$. Regrouping our last comodules this way does not change f in any meaningful way. Our tensor product looks like $N_1 \otimes \cdots N_{j_{l-1}} \otimes N'$. We can apply our α action to the applicable $l - 1$ comodules between 1 and j_{l-1} and using the induction hypothesis obtain a commutative diagram

$$\begin{tikzcd}
N_1 \otimes \cdots \otimes N_{j_{l-1}} \otimes N' \arrow[rr, "f"] \arrow[d, "\mathrm{id}^{j_1-1}\otimes\alpha\otimes\mathrm{id}^{j_2-j_1-1}\otimes\cdots\otimes\alpha\otimes\mathrm{id}^{j_{l-1}-j_{l-2}+1}\otimes\alpha\otimes\mathrm{id}"'] & & B(M) \\
N_1 \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_{l-1}}}}{\otimes} N_{j_{l-1}}) \otimes N' \arrow[urr, "\tilde{f}_1"] & &
\end{tikzcd}$$

where $\tilde{f}_1$ satisfies figure 3.3. We now reconsider $N_1 \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_{l-1}}}}{\otimes} N_{j_{l-1}}) \otimes N'$ as $N'' \otimes N_{j_l} \otimes N'''$, where $N'' = N_1 \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_{l-1}}}}{\otimes} N_{j_{l-1}}) \otimes \cdots N_{j_l - 1}$ and $N''' = N_{j_l+1} \otimes \cdots N_n$, where N'' and N''' are comodules over the appropriate tensor products of B with itself. Notice that relabeling our objects this way does not change how $\tilde{f}_1$ works, so we can use part 1 above directly to obtain a map $\tilde{f}$ satisfying figure 3.3 where the following diagram commutes:

$$\begin{tikzcd}
N_1 \otimes \cdots \otimes N_{j_{l-1}} \otimes N' \arrow[r, "f"] \arrow[d, "\mathrm{id}^{j_1-1}\otimes\alpha\otimes\mathrm{id}^{j_2-j_1-1}\otimes\cdots\otimes\alpha\otimes\mathrm{id}^{j_{l-1}-j_{l-2}+1}\otimes\alpha\otimes\mathrm{id}"'] & B(M) \\
N_1 \otimes \cdots \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_{l-1}}}}{\otimes} N_{j_{l-1}}) \otimes N' \arrow[ur, "\tilde{f}_1"] \arrow[d, "\mathrm{id}"] & \\
N'' \otimes N_{j_l} \otimes N''' \arrow[d, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"'] & \\
N'' \otimes (B \underset{\tau_1}{\otimes} A \underset{\tau_{k_{j_l}}}{\otimes} N_{j_l}) \otimes N''' \arrow[uuur, "\tilde{f}"'] &
\end{tikzcd}$$

The composition of down arrows is the same as the one given in the statement of the corollary, thus completing the proof.

□

3.3.3 Lemma 3.2

Lemma 3.2. *For a positive integer n and for all $1 \leq i \leq n$, if $f_i : N_i \to N_i'$ and $g : N_1' \otimes \cdots \otimes N_n' \to N''$ are chain maps such that $\alpha \circ f_i = (id^{\otimes 2} \otimes f_i) \circ \alpha$ and $\alpha \circ g = (id^{\otimes 2} \otimes g) \circ \alpha$, where each N_i is a $B^{\otimes k_i}$ comodule, for some positive k_i and each N_i' and N'' is a B comodule, where we then think of $N_1' \otimes \cdots N_n'$ as a $B^{\otimes n}$ comodule, we have that the map $g \circ (f_1 \otimes \cdots \otimes f_n) : N_1 \otimes \cdots \otimes N_n \to N''$ satisfies the condition that*

$$\alpha \circ (g \circ (f_1 \otimes \cdots \otimes f_n)) = (id^{\otimes 2} \otimes (g \circ (f_1 \otimes \cdots \otimes f_n))) \circ \alpha,$$

where we think of $N_1 \otimes \cdots \otimes N_n$ as a $B^{\otimes k_1 + \cdots + k_n}$ comodule.

Proof of Lemma 3.2. By induction on n, we show that $\Delta^{\otimes n} \circ (f_1 \otimes \cdots \otimes f_n) = (\mu_E^{k_1} \otimes \cdots \mu_E^{k_n}) \otimes (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{\otimes k_1 + \cdots + k_n}$. In the case when $n = 1$, the result is the same as the assumption on f_1. The following diagram is then commutative:

$$\begin{tikzcd}
N_1 \arrow[r, "f_1"] \arrow[d, "\Delta^{\otimes k_1}"] & N_1' \arrow[r, "g"] \arrow[d, "\Delta"] & N'' \arrow[d, "\Delta"] \\
B^{\otimes k_1} \otimes N_1 \arrow[r, "\mu_E^{k_1} \otimes f_1"] & B \otimes N_1' \arrow[r, "\mathrm{id} \otimes g"] & B \otimes N''
\end{tikzcd}$$

where the second rectangle is due to the assumption on g. The composition of across arrows proves our initial claim.

If we assume the result holds for $n-1$, the first rectangle below commutes. By what we are given about N_n and f_n, the second rectangle commutes:

$$\begin{tikzcd}
N_1 \otimes \cdots \otimes N_n \arrow[r, "\Delta^{\otimes k_1 + \cdots + k_{n-1}} \otimes \Delta^{\otimes k_n}"] \arrow[d, "f_1 \otimes \cdots \otimes f_{n-1} \otimes \mathrm{id}"] & B^{\otimes k_1 + \cdots k_{n-1}} \otimes B^{\otimes k_n} \otimes N_1 \otimes \cdots \otimes N_n \arrow[d, "\mu_E^{k_1} \otimes \cdots \otimes \mu_E^{k_{n-1}} \otimes \mathrm{id}^{\otimes k_n} \otimes f_1 \otimes \cdots \otimes f_{n-1} \otimes \mathrm{id}"'] \\
N_1' \otimes \cdots \otimes N_{n-1}' \otimes N_n \arrow[r, "\Delta^{\otimes n-1} \otimes \Delta^{\otimes k_n}"'] \arrow[d, "\mathrm{id}^{\otimes n-1} \otimes f_n"] & B^{\otimes n-1} \otimes B^{\otimes k_n} \otimes N_1' \otimes \cdots N_n' \arrow[d, "\mathrm{id}^{\otimes n-1} \otimes \mu_E^{k_n} \otimes \mathrm{id}^{\otimes n-1} \otimes f_n"'] \\
N_1' \otimes \cdots \otimes N_n' \arrow[r, "\Delta^{\otimes n}"'] & B^{\otimes n} \otimes N_1' \otimes \cdots \otimes N_n'
\end{tikzcd}$$

This proves our induction. We add to the bottom of that diagram

$$\begin{tikzcd}
N_1' \otimes \cdots \otimes N_n' \arrow[r, "\Delta^{\otimes n}"] \arrow[d, "g"] & B^{\otimes n} \otimes N_1' \otimes \cdots \otimes N_n' \arrow[d, "\mu_E^n \otimes g"'] \\
N'' \arrow[r, "\Delta"] & B \otimes N''
\end{tikzcd}$$

By compacting much of this information, we get the following commutative diagram:

$$\begin{tikzcd}
N_1 \otimes \cdots \otimes N_n \arrow[r, "\Delta^{\otimes k_1 + \cdots + k_n}"] \arrow[d, "g \circ (f_1 \otimes \cdots \otimes f_n)"] & B^{\otimes k_1 + \cdots k_{n-1} + k_n} \otimes N_1 \otimes \cdots \otimes N_n \arrow[d, "\mu_E^{k_1 + \cdots + k_n} \otimes g \circ (f_1 \otimes \cdots \otimes f_n)"'] \\
N'' \arrow[r, "\Delta"] & B \otimes N''
\end{tikzcd}$$

proving our lemma. □

Definition 3.1 (Simplicial coaction map)**.** *If we take g to be the identity and we have that a map f defined so that $f = f_1 \otimes \cdots \otimes f_n$, each satisfying the above condition, then we get that $\alpha \circ f = (id^{\otimes 2}) \circ f$. All of the lemmas above are to aid us in defining maps that behave in this way, meaning are a tensor product of maps each from a product of comodules to a comodule, specifically $B(L)$ for some module L. Such a map will be called a simplicial coaction map.*

3.4 Comodules as a Simplicial Set

We now must introduce some more lemmas to define our simplicial set $\mathcal{C}^{\otimes}$ using Lurie's definition of a dg-nerve as a guideline, at which point the map $p : \mathcal{C}^{\otimes} \to N(\mathbf{\Delta})^{op}$ will be clear.

3.4.1 Lemma 3.3

Lemma 3.3. *Given an n simplex where the i^{th} vertex is the tensor product of comodules N_i for $i = 1, \ldots, n$, over the coalgebra $B^{\otimes j_i}$ respectively, and $N_0 = B(L_0)$ for an A module L_0 at the 0^{th} vertex. All maps are simplicial coaction maps and are given by $f_{\{i_1,\ldots,i_k\}} : N_{i_k} \to N_{i_1}$, where $i_1 < i_2 < \ldots, < i_k$, $i_1, \ldots, i_k \in \{0, \ldots, n\}$. If we set $L_i = A \underset{\tau_{j_i}}{\otimes} N_i$ we can create an inner n-simplex using corollary 3.1 where the i^{th} vertex is $B(L_i)$ and maps are now given by $\alpha \circ \widetilde{f_{\{i_1,\ldots,i_k\}}}$ except in the case when $i_1 = 0$ in which case our map is $\tilde{f}_{\{i_1,\ldots,i_k\}}$.*

We fix a set $\{0, i_1, \ldots, i_k\}$. If we assume that $\tilde{f}_{\{0,i_1,\ldots,i_k\}}$ is a homotopy map, meaning that

$$d\tilde{f}_{\{0,i_1,\ldots,i_k\}} = \sum_{1 \leq j \leq k-1} \tilde{f}_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + \tilde{f}_{\{0,i_1,\ldots,i_j\}} \circ \alpha \circ \widetilde{f_{\{i_j,\ldots,i_k\}}}$$

where $df = d \circ f + f \circ d$, then the map $f_{\{0,i_1,\ldots,i_k\}}$ is also a homotopy.

Proof of Lemma 3.3. Before we prove this, consider what this actually looks like in the case of a triangle, where all outer triangles and quadrilateral commute, aside from the maps $f_{\{0,1,2\}}$ and $\tilde{f}_{\{0,1,2\}}$:

$$
\begin{tikzcd}
 & & N_2 \arrow[d, "\alpha"] \arrow[ddddll, bend right, "f_{\{0,1,2\}}"'] \arrow[ddddll, "f_{\{0,2\}}"'] \arrow[ddddrr, "f_{\{1,2\}}"] & & \\
 & & & & \\
 & & B(L_2) \arrow[ddll, bend right, "\tilde{f}_{\{0,1,2\}}"'] \arrow[ddll, "\tilde{f}_{\{0,2\}}"] \arrow[dr, "\widetilde{\alpha\circ f_{\{1,2\}}}"] & & \\
 & & & B(L_1) \arrow[dlll, "\tilde{f}_{\{0,1\}}"'] & \\
B(L_0) & & & & N_1 \arrow[ul, "\alpha"] \arrow[llll, "f_{\{0,1\}}"]
\end{tikzcd}
$$

Figure 3.4: Reduction of $B^{\otimes j_i}$ modules to B comodules

Back to our general case. If a map g that maps to $B(L_0)$ produces $\tilde{g}$, we have $\tilde{g} \circ \alpha = g$ if or g maps to something other than $B(L_0)$ and produces $\widetilde{\alpha \circ g}$, we have $\widetilde{\alpha \circ g} \circ \alpha = \alpha \circ g$. If $\tilde{f}_{\{0,i_1,\ldots,i_k\}}$ behaves as in the lemma, then

$$
\begin{aligned}
d(f_{\{0,i_1,\ldots,i_k\}}) = d(\tilde{f}_{\{0,i_1,\ldots,i_k\}} \circ \alpha) &= \left(\sum_{1 \le j \le k-1} \tilde{f}_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + \tilde{f}_{\{0,i_1,\ldots,i_j\}} \circ \alpha \circ \widetilde{f_{\{i_j,\ldots,i_k\}}} \right) \circ \alpha \\
&= \sum_{1 \le j \le k-1} \tilde{f}_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} \circ \alpha + \tilde{f}_{\{0,i_1,\ldots,i_j\}} \circ \alpha \circ \widetilde{f_{\{i_j,\ldots,i_k\}}} \circ \alpha \\
&= \sum_{1 \le j \le k-1} f_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + (\tilde{f}_{\{0,i_1,\ldots,i_j\}} \circ \alpha) \circ f_{\{i_j,\ldots,i_k\}} \\
&= \sum_{1 \le j \le k-1} f_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + f_{\{0,i_1,\ldots,i_j\}} \circ f_{\{i_j,\ldots,i_k\}}.
\end{aligned}
$$

Therefore $f_{\{0,i_1,\ldots,i_k\}}$ acts as the required homotopy. $\square$

3.4.2 Proposition 3.2

Proposition 3.2. *If we have multiple of the simplices defined in Lemma 3.3, say $\{L_0^i, N_j^i, \{f_I^i\}\}$ for $i = 1, \ldots, k$, having the property that all of the maps f_I^i are homotopies, , in the sense of the definition of Lurie's dg-nerve, then in the simplex $\{\bigotimes_{i=1}^k B(L_1^i), \bigotimes_{i=1}^k N_j^i, \{\bigotimes_{i=1}^k f_I^i\}\}$, for any set I_0, we can find a map f_{I_0} that acts as a homotopy.*

Proof of Proposition 3.2. When we have multiple simplices $\{L_0^i, N_j^i, \{f_I^i\}\}$ for $i = 1, \ldots, n$ such that all maps behave as homotopies, we create a homotopy on the n-fold tensor product iteratively. That is, we show how to do it for two simplices and the rest follows from induction. Let $\{L_0^1, N_j^1, \{f_I^1\}\}$ and $\{L_0^2, N_j^2, \{f_I^2\}\}$ be two simplices and we fix a set $I_0 = \{0, i_1, \ldots, i_k\}$. We then have that

$$d(f^1_{\{0,i_1,\ldots,i_k\}}) = \sum_{1\le j\le k-1} f^1_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + f^1_{\{0,i_1,\ldots,i_j\}} \circ f^1_{\{i_j,\ldots,i_k\}}.$$

$$d(f^2_{\{0,i_1,\ldots,i_k\}}) = \sum_{1\le j\le k-1} f^2_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + f^2_{\{0,i_1,\ldots,i_j\}} \circ f^2_{\{i_j,\ldots,i_k\}}.$$

We want to find a map $f^{1\otimes 2}_{\{0,i_1,\ldots,i_k\}}$ so that

$$d(f^{1\otimes 2}_{\{0,i_1,\ldots,i_k\}}) = \sum_{1\le j\le k-1} f^{1\otimes 2}_{\{0,i_1,\ldots,\widehat{i_j},\ldots,i_k\}} + f^{1\otimes 2}_{\{0,i_1,\ldots,i_j\}} \circ f^{1\otimes 2}_{\{i_j,\ldots,i_k\}},$$

where these other maps are defined similarly.

To simplify a bit, we replace our set $\{0, i_1, \ldots, i_k\}$ with $\{0, 1, \ldots, k\}$. The easiest case is when we are considering an edge, i.e. we have maps $f^1_{\{0,1\}}$ and $f^2_{\{0,1\}}$. Then $f^{1\otimes 2}_{\{0,1\}} = f^1_{\{0,1\}} \otimes f^2_{\{0,1\}}$.

In the next case up, a triangle, meaning we have $f^1_{\{0,1,2\}}$ and $f^2_{\{0,1,2\}}$ along with all edges, we define

$$f^{1\otimes 2}_{\{0,1,2\}} = f^1_{\{0,2\}} \otimes f^2_{\{0,1,2\}} + f^1_{\{0,1,2\}} \otimes f^2_{\{0,1\}} \circ f^2_{\{1,2\}}.$$

This is in no way canonical as we've made a choice and we could have equally defined

$$f^{1\otimes 2}_{\{0,1,2\}} = f^1_{\{0,1\}} \circ f^1_{\{1,2\}} \otimes f^2_{\{0,1,2\}} + f^1_{\{0,1,2\}} \otimes f^2_{\{0,2\}}.$$

For the general case we use induction. We define $\langle k\rangle = \{1, \ldots, k-1\}$. Then our map is

$$f^{1\otimes 2}_{\{0,1,\ldots,k\}} = \sum_{\{j_1,\ldots,j_l\}} f^1_{\{0,j_1,\ldots,j_l,k\}} \otimes f^2_{\{0,\ldots,j_1\}} \circ f^2_{\{j_1,\ldots,j_2\}} \circ \cdots \circ f^2_{\{j_l,\ldots,k\}},$$

where the sum is taken over all subsets $\{j_1, \ldots, j_l\} \subset \langle k\rangle$, $j_1 < \cdots < j_l$, including the empty set and $\langle k\rangle$ itself. This definition agrees with $f^{1\otimes 2}_{\{0,1\}}$ and $f^{1\otimes 2}_{\{0,1,2\}}$ given above.

Once again, here we have made the choice to put the composition of maps on the right of the tensor product when we could just as easily, and just as correctly, made the maps from the first simplex the compositions.

To see that this map behaves as a homotopy on the tensor product, i.e. that

$$d(f^{1\otimes 2}_{\{0,1,\ldots,k\}}) = \sum_{1\le j\le k-1} f^{1\otimes 2}_{\{0,1,\ldots,\widehat{j},\ldots,k\}} + f^{1\otimes 2}_{\{0,1,\ldots,j\}} \circ f^{1\otimes 2}_{\{j,\ldots,k\}},$$

we consider both sides of this equation. The left hand side is:

$$d\big(f^{1\otimes 2}_{\{0,1,\ldots,k\}}\big) = d\Bigg(\sum_{\{j_1,\ldots,j_l\}} f^{1}_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ f^{2}_{\{j_1,\ldots,j_2\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}\Bigg)$$

$$= \sum_{\{j_1,\ldots,j_l\}} df_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ f^{2}_{\{j_1,\ldots,j_2\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$+\sum_{i=0}^{l} f_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ f^{2}_{\{j_1,\ldots,j_2\}} \circ \cdots \circ df^{2}_{\{j_i,\ldots,j_{i+1}\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$(1) \qquad = \sum_{\{j_1,\ldots,j_l\}} \Bigg(\sum_{i=1}^{l} f^{1}_{\{0,\ldots,j_{i-1},j_{i+1},\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$(2) \qquad +f^{1}_{\{0,\ldots,j_i\}} \circ f^{1}_{\{j_i,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$(3) \qquad +\Bigg(\sum_{p=j_i+1}^{j_{i+1}-1} f^{1}_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ \cdots \circ f^{2}_{\{j_i,\ldots,p-1,p+1,\ldots,j_{i+1}\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$(4) \qquad +f^{1}_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ \cdots \circ f^{2}_{\{j_i,\ldots,p\}}, f^{2}_{\{p,\ldots,j_{i+1}\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}\Bigg)\Bigg)$$

$$(5) \qquad +\Bigg(\sum_{q=1}^{j_1-1} f^{1}_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,p-1,p+1,\ldots j_1\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$$

$$(6) \qquad +f^{1}_{\{0,j_1,\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,p\}} \circ f^{2}_{\{p,\ldots,j_1\}} \circ \cdots \circ \circ f^{2}_{\{j_l,\ldots,k\}}\Bigg)$$

Rows (1), (4), and (6) above cancel out. To see why, notice that for a subset $\{j_1,\ldots,j_l\}$ and an i such that $1 \le i \le l$, the element $f^{1}_{\{0,\ldots,j_{i-1},j_{i+1},\ldots,j_l,k\}} \otimes f^{2}_{\{0,\ldots,j_1\}} \circ \cdots \circ f^{2}_{\{j_l,\ldots,k\}}$

from row (1) is the same as the element in row (4) corresponding to the subset $\{j_1,\dots,\widehat{j_i},\dots,j_l\}$ and $p=j_i$. If $i=1$, the the element comes from rows (6). Conversely, given a subset $\{j_1,\dots,j_l\}$ fixing an integer m and an element p, where $j_m < p < j_{m+1}$ corresponding to the element $f^2_{\{0,\dots,j_1\}}\circ\dots\circ f^2_{\{j_m,\dots,p\}}, f^2_{\{p,\dots,j_{m+1}\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}}$ from row (4) (or row (6) if $m=0$), by defining the subset $\{j_1,\dots,j_m,p,j_{m+1},\dots,j_l\}$ along with setting $i=m+1$, we get the same element in row (1). For the left hand side, we are then left with:

$$\sum_{\{j_1,\dots,j_l\}}\Bigg(\sum_{i=1}^{l} f^1_{\{0,\dots,j_i\}}\circ f^1_{\{j_i,\dots,j_l,k\}}\otimes f^2_{\{0,\dots,j_1\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}} \tag{2}$$

$$+\Bigg(\sum_{p=j_i+1}^{j_{i+1}-1} f^1_{\{0,j_1,\dots,j_l,k\}}\otimes f^2_{\{0,\dots,j_1\}}\circ\dots\circ f^2_{\{j_i,\dots,p-1,p+1,\dots,j_{i+1}\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}}\Bigg)\Bigg) \tag{3}$$

$$+\Bigg(\sum_{q=1}^{j_1-1} f^1_{\{0,j_1,\dots,j_l,k\}}\otimes f^2_{\{0,\dots,p-1,p+1,\dots j_1\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}}\Bigg) \tag{5}$$

Now we look at the right hand side:

$$\sum_{1\le j\le k-1} f^{1\otimes 2}_{\{0,1,\dots,\widehat{j},\dots,k\}} + f^{1\otimes 2}_{\{0,1,\dots,j\}}\circ f^{1\otimes 2}_{\{j,\dots,k\}}$$

$$=\sum_{j=1}^{k-1}\Bigg(\sum_{\{j_1,\dots,j_l\}\subset\langle k\rangle-\{j\}} f^1_{\{0,j_1,\dots,j_l,k\}}\otimes f^2_{\{0,\dots,j_1\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}}\Bigg) \tag{a}$$

(b)

$$+\sum_{\substack{\{j'_1,\dots,j'_{l'}\}\subset\langle j\rangle,\\ \{j''_1,\dots,j''_{l''}\}\subset\langle k\rangle-\langle j\rangle}} f^1_{\{0,j'_1,\dots,j'_{l'},j\}}\circ f^1_{\{j,j''_1,\dots,j''_{l''},k\}}\otimes f^2_{\{0,\cdots,j'_1\}}\circ\dots\circ f^2_{\{j'_{l'},\cdots,j\}}\circ f^2_{\{j,\cdots,j''_1\}}\circ\dots\circ f^2_{\{j''_{l''},\cdots,k\}}$$

We first show that rows (2) and (b) are equal. Given a subset $\{j_1,\dots,j_l\}$ of $\langle k\rangle$ and an integer i between 1 and l, this yields the element $f^1_{\{0,\dots,j_i\}}\circ f^1_{\{j_i,\dots,j_l,k\}}\otimes f^2_{\{0,\dots,j_1\}}\circ\dots\circ f^2_{\{j_l,\dots,k\}}$ from row (2). By fixing $j=j_i$ and setting $\{j'_1,\dots,j'_{l'}\}=\{j_1,\dots,j_{i-1}\}\subset\langle j\rangle$ and $\{j''_1,\dots,j''_{l''}\}=\{j_{i+1},\dots,j_l\}\subset\langle k\rangle-\langle j\rangle$, we get the same element in row (b). Conversely, by picking an element j and subsets $\{j'_1,\dots,j'_{l'}\}\subset\langle j\rangle,\{j''_1,\dots,j''_{l''}\}\subset$

$\langle k\rangle - \langle j\rangle$, we get the element $f^1_{\{j,j''_1,\ldots,j''_{l''},k\}} \otimes f^2_{\{0,\cdots,j'_1\}} \circ \cdots \circ f^2_{\{j'_{l'},\cdots,j\}} \circ f^2_{\{j,\cdots,j''_1\}} \circ \cdots \circ f^2_{\{j''_{l''},\cdots,k\}}$ from row (b). By setting our set $\{j_1,\ldots,j_l\} = \{j'_1,\ldots,j'_{l'},j,j''_1,\ldots,j''_{l''}\} \subset \langle k\rangle$ and $i = l'+1$, we get the same element in row (2).

It now remains to show that row (3) + row (5) is equal to row (a). Fixing an integer i, and given a subset $\{j_1,\ldots,j_l\} \subset \langle k\rangle$ and an integer p between j_i+1 and $j_{i+1}-1$ (or between $1 = j_0+1$ and j_1-1), this yields the element $f^1_{\{0,j_1,\ldots,j_l,k\}} \otimes f^2_{\{0,\ldots,j_1\}} \circ \cdots \circ f^2_{\{j_i,\ldots,p-1,p+1,\ldots,j_{i+1}\}} \circ \cdots \circ f^2_{\{j_l,\ldots,k\}}$ from row (3). Then the element from row (a) corresponding to integer $j = p$ between 1 and $k-1$ and the set $\{j_1,\ldots,j_l\} \subset \langle k\rangle - \{p\}$ is exactly the same as that from row (3). Conversely, fixing an integer j and a subset $\{j_1,\ldots,j_l\} \subset \langle k\rangle - \{j\}$, we get the element $f^1_{\{0,j_1,\ldots,j_l,k\}} \otimes f^2_{\{0,\ldots,j_1\}} \circ \cdots \circ f^2_{\{j_l,\ldots,k\}}$ from row (a). Now looking at the element in row (3) corresponding to the subset $\{j_1,\ldots,j_l\} \subset \langle k\rangle$ and $p = j$ is this exact same element.

Therefore this formula for $f^{1\otimes 2}$ is a homotopy. By induction, this result can be extended to n different simplices tensored together.

$\square$

3.5 Modules of an Algebra as a Modoidal ∞-Category

Definition 3.2 ($\mathcal{C}^{\otimes}$)**.** *We can now begin to use these results to construct our simplicial set $\mathcal{C}^{\otimes}$, what we will call a monoidal dg-nerve. We begin by assuming our algebra is an associative Hirsch algebra, making B a bialgebra.*

A vertex of our set is, for every nonnegative integer n, a finite sequence of A modules $(M_1,\ldots,M_n)$.

An edge from $(M_1,\ldots,M_n)$ to $(L_1,\ldots,L_k)$ consists of a $\mathbf{\Delta}$ map $f' : [k] \to [n]$ along with simplicial coaction maps $f_i : B(M_{f'(i-1)+1}) \otimes \cdots \otimes B(M_{f'(i)}) \to B(L_i)$ and a map finally $f = f_1 \otimes \cdots f_k$. If a map f' in $\mathbf{\Delta}$ is not injective, say $f'(j) = f'(j+1)$, then the map f_j would not map to the $B(M'_{j+1})$ component. In this case we artificially insert a $B(A)$ so our map f_{i+1} is the map $B(A) \to B(M'_{i+1})$. This loosely imitates Lurie's construction of a monoidal category $\mathcal{C}$ given a category $\mathcal{C}^{\otimes}$ and a functor to $\mathbf{\Delta}$. Similarly, for a map $f : [m] \to [n]$ if $f(0) > 0$ or $f(m) < m$, there are certain *comodules, specifically* $B(M_1), \ldots, B(M_{f(0)})$ *and* $B(M_{f(n)+1}), \ldots, B_n$ *that just map to* k*. This doesn't change any of our maps, so for simplicity, we will assume* $f(0) = 0$ *and* $f(m) = n$*.*

Vertices and edges are enough to define a category, but we do not expect this to work so nicely. As this is a simplicial set, we must define higher n *simplices.*

Much as a vertex of $\mathcal{C}^{\otimes}$ *has associated to it an object from* $\mathbf{\Delta}$ *and an edge a morphism from* $\mathbf{\Delta}$*, an* n *simplex has associated to a chain of* n *composable morphisms in* $\mathbf{\Delta}$*, meaning* $[m_0] \xrightarrow{f_1'} [m_1] \xrightarrow{f_2'} \cdots \xrightarrow{f_n'} [m_n]$ *is an element of* $N(\Delta)$*. However, our maps will not exactly be over this chain of morphisms. Instead we split this chain into* m_0 *different chains. We do this by splitting up* $[m_0]$ *as* $\{0,1\} \cup \{1,2\} \cup \cdots \{m_0 - 1, m_0\}$ *and splitting up all subsequent sets in the following way. First, we define* $F_i' = f_i' \circ \cdots \circ f_1'$*,* $F_0' = id$*. Then we define our new sets as* $m_i^j = \{F_i'(j-1), \ldots, F_i'(j)\}$*, where* $0 \leq i \leq n$*,* $1 \leq j \leq m_0$*. Then we limit our view to maps* $f_i' \mid_{m_0^j}$ *by fixing* j *and letting* i *go from 1 to* n*. As an example, consider when* $n = 4$*.*

$$
\begin{tikzcd}
m_0 : & 0 \arrow[dr] & 1 \arrow[dr] & 2 \arrow[drr] & 3 \arrow[drr] & & & & & & \\
m_1 : & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[dr] & 3 \arrow[drr] & 4 \arrow[drrr] & 5 \arrow[drrr] & & & & \\
m_2 : & 0 \arrow[d] & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[dr] & 5 \arrow[dr] & 6 \arrow[drr] & 7 \arrow[dr] & 8 \arrow[dr] & \\
m_3 : & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{tikzcd}
$$

This splits up as

$$
\begin{tikzcd}
m_0^1 : & 0 \arrow[dr] & 1 \arrow[dr] & & & \\
m_1^1 : & & 1 \arrow[d] & 2 \arrow[dr] & & \\
m_2^1 : & & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & \\
m_3^1 : & & & 2 & 3 & 4
\end{tikzcd}
$$

$$\begin{tikzcd}
m_0^2 : & 1 \arrow[dr] & 2 \arrow[drr] & & & & & & \\
m_1^2 : & & 2 \arrow[dr] & 3 \arrow[drr] & 4 \arrow[drrr] & & & & \\
m_2^2 : & & & 3 \arrow[dr] & 4 \arrow[dr] & 5 \arrow[dr] & 6 \arrow[drr] & 7 \arrow[dr] & \\
m_3^2 : & & & & 4 & 5 & 6 & 7 & 8
\end{tikzcd}$$

$$\begin{tikzcd}
m_0 : & 2 \arrow[drr] & 3 \arrow[drr] & & & & & & \\
m_1 : & & & 4 \arrow[drrr] & 5 \arrow[drrr] & & & & \\
m_2 : & & & & & & 7 \arrow[dr] & 8 \arrow[dr] & \\
m_3 : & & & & & & & 8 & 9
\end{tikzcd}$$

For convenience we will always relabel so our initial vertex of every m_i^j is 0.

A face of an n simplex, where we have reduced and simplified so our vertices are labeled $\{0, \dots, n\}$, corresponding to the vertices $\{i_0, \dots, i_k\}$ is a simplicial coaction map. This map, denoted $f_{\{i_0,\dots,i_k\}}$ is a map from the i_k-th vertex to the i_0-th vertex, encoded by the appropriate chain of morphisms from $\mathbf{\Delta}$ and having the further property that

$$d \circ f_{\{i_0,\dots,i_k\}} + f_{\{i_0,\dots,i_k\}} \circ d = \sum_{0<j<k} f_{\{i_0,\dots,\hat{i}_j,\dots,i_k\}} + f_{\{i_0,\dots,i_j\}} \circ f_{\{i_j,\dots,i_k\}}$$

In the future we use the notation of differential on maps, so $d \circ f + f \circ d$ will be denoted df.

If we reverse all our maps, there is the forgetful map $p^{op} : \mathcal{C}^{\otimes op} \to N(\mathbf{\Delta})$. We aim to show that p^{op} is a cocartesian fibration.

3.5.1 Theorem 3.1

Theorem 3.1. *The category $\mathcal{C}^{\otimes}$ described above comprised of A modules is a monoidal ∞-category.*

Proof of Theorem 3.1.

1. First we show that p^{op} is an inner fibration. We will look at maps in $C^{\otimes}$, meaning they will be in the opposite direction of maps in $\mathbf{\Delta}$.

   Given a Λ^n_j horn for $1 \leq j \leq n-1$, we may lift this horn into our monoidal dg-nerve which then affords us all maps $f_{\{0,\ldots,\hat{\imath},\ldots,n\}}$ for $0 \leq i \leq n, i \neq j$, along with all maps $f_{\{i_1,\ldots,i_k\}}$, $i_1 \geq 0, i_k \leq n$ and $i_1 < i_2 < \cdots < i_k \in \{0,\ldots,n\}$. To show this horn can be completed to Δ^n, we must find faces $f_{\{0,\ldots,\hat{\jmath},\ldots,n\}}$ along $f_{\{0,\ldots,n\}}$. By the definition of higher simplices in $\mathcal{C}^{\otimes}$, we are able to write $df_{\{0,\ldots,\hat{\jmath},\ldots,n\}}$ as we have all lower maps. Similarly, for any other homotopy g in our j-horn we are able to write dg. As $d^2 = 0$, we have

   $$0 = d^2 f_{\{0,\ldots,n\}} = \sum_{1 \leq i \leq n-1} df_{\{0,\ldots,\hat{\imath},\ldots,n\}} + d\big(f_{\{0,\ldots,i\}} \circ f_{\{i,\ldots,n\}}\big)$$

   Therefore if we set

   $$f_{\{0,\ldots,\hat{\jmath},\ldots,n\}} = \sum_{\substack{1 \leq i \leq n-1 \\ i \neq j}} f_{\{0,\ldots,\hat{\imath},\ldots,n\}} + f_{\{0,\ldots,i\}} \circ f_{\{i,\ldots,n\}} + f_{\{0,\ldots,j\}} \circ f_{\{j,\ldots,n\}},$$

   we get that $f_{\{0,\ldots,\hat{\jmath},\ldots,n\}}$ obeys its necessary homotopy property. Moreover, as it is a sum and composition of simplicial coaction maps, it itself is a simplicial coaction map.

   Moreover, defining $f_{\{0,\ldots,\hat{\jmath},\ldots,n\}}$ in this way allows us to define $f_{\{0,\ldots,n\}} = 0$ as we get

   $$0 = \sum_{1 \leq i \leq n-1} f_{\{0,\ldots,\hat{\imath},\ldots,n\}} + f_{\{0,\ldots,i\}} \circ f_{\{i,\ldots,n\}},$$

   as required.

2. Next we show that p^{op} is a cocartesian fibration. This is less straightforward. What this looks like is we are given an edge $f' : [l] \to [m]$ and modules $M_1, \ldots, M_m$. We lift f' to a map we call $f_{\{n-1,n\}}$ in $\mathcal{C}^{\otimes op}$ from $B(M_1) \otimes \cdots \otimes B(M_m)$ to a particular lift of $[l]$, based on a morphism f' from $\mathbf{\Delta}$. We then fill this out to a lift of Λ^n_n horn, placing it as the edge between the $n-1^{th}$ and n^{th} vertices, hence its name. This horn contains all $n+1$ vertices as well as all maps $f_{\{i_0,\ldots,i_k\}}$ for $0 \leq i_0 < i_1 < \cdots < i_k \leq n$ except for the faces $f_{\{0,\ldots,n-1\}}$ and

$f_{\{0,\ldots,n\}}$. We aim to use what we know about $f_{\{n-1,n\}}$ to finds these last two maps so we have completed our horn to a lift of Δ^n.

Now we come to constructing our cocartesian edge. Given a map $F' : [s] \to [t]$ and a corresponding set of A modules $M_1, \ldots, M_t$, we lift this to the map

$$F : B(M_1) \otimes \cdots \otimes B(M_t) \to$$

$$B(M_1) \otimes \cdots \otimes \left(B \underset{\tau_1}{\otimes} A \underset{\tau_{F'(1)-F'(0)+1}}{\otimes} B(M_{F'(0)+1}) \otimes \cdots \otimes B(M_{F'(1)}) \right) \otimes \cdots$$

$$\otimes \left(B \underset{\tau_1}{\otimes} A \underset{\tau_{F'(s)-F'(s-1)+1}}{\otimes} B(M_{F'(s-1)+1}) \otimes \cdots \otimes B(M_{F'(s)}) \right) \otimes \cdots \otimes B(M_t),$$

which we showed in corollary 3.1 is a simplicial coaction map.

As an example consider the map F':

$$\begin{tikzcd}[column sep=small] 0 & 1 & 2 & 3 & 4 & 5 \\ 0 \arrow[u] & 1 \arrow[ur] & 2 \arrow[ur] & 3 \arrow[urr] & & \end{tikzcd}$$

along with modules M_1, M_2, M_3, M_4, M_5. Then our map F is

$$\begin{tikzcd} B(M_1) \otimes B(M_2) \otimes B(M_3) \otimes B(M_4) \otimes B(M_5) \arrow[d, "F"'] \\ B\big(A \underset{\tau_2}{\otimes} B(M_1) \otimes B(M_2)\big) \otimes B(M_3) \otimes B\big(A \underset{\tau_2}{\otimes} B(M_4) \otimes B(M_5)\big) \end{tikzcd}$$

To see how this process of constructing a Λ^2_2 horn works, let H' be the map

$$\begin{tikzcd}[column sep=small] 0 & 1 & 2 & 3 \\ 0 \arrow[u] & 1 \arrow[ur] & 2 \arrow[ur] & \end{tikzcd}$$

and define $G' = F' \circ H'$. Using this information we can construct our Λ^2_2 horn. It is a lift of

$$\begin{tikzcd} & 1[3] \arrow[dr, "F'"] & \\ 0[2] \arrow[ur, "H'"] \arrow[rr, "G'"] & & 2[5] \end{tikzcd}$$

$$\begin{tikzcd}
 & & B\big(A \underset{\tau_2}{\otimes} B(M_1) \otimes B(M_2)\big) \otimes B(M_3) \otimes B\big(A \underset{\tau_2}{\otimes} B(M_4) \otimes B(M_5)\big) \arrow[dll, dashed, "H"'] \\
B(L_1) \otimes B(L_2) & & B(M_1) \otimes B(M_2) \otimes B(M_3) \otimes B(M_4) \otimes B(M_5) \arrow[ll, "G"] \arrow[u, "F"]
\end{tikzcd}$$

Figure 3.5: Example of a Λ_2^2 horn

Now, letting G be any lift of G', we get the diagram:

We don't have a map H yet, but we know it must be a lift of H'. As we require our maps to be simplicial coaction maps when we've reduced to just a comodule over B in the 0 vertex, by picking $2 \in [2]$, for instance, and following the maps F' and G', we reduce our view to the triangle:

$$\begin{tikzcd}
 & B\big(A \underset{\tau_2}{\otimes} B(M_4) \otimes B(M_5)\big) \arrow[dl, dashed, "h"'] \\
B(L_2) & B(M_4) \otimes B(M_5) \arrow[l, "g"] \arrow[u, "f"]
\end{tikzcd}$$

Figure 3.6: Example of a reduced Λ_2^2 horn

So using this method, we can reduce our work to only the case where there is an element $B(L_i)$ in the 0^{th} vertex.

We are now ready to tackle our Λ_n^n horn. By proposition 2.4.1.1. in [15], given a quasi-isomorphism $B(M_2) \to B(M_1)$, where M_2 and M_1 are modules, this map is in fact a homotopy equivalence. Unfortunately, the maps we are lifting are not so nice so as to let our cocartesian edges act so nicely. We now employ our lemmas. By what we said above about reducing to a single module in the 0^{th} vertex, we may use Lemma 3.3. Our cocartesian edge is a quasi-isomorphsism by an argument similar to the proof of Proposition 2.3 along with lemma 3.1 which lets us deal with comodules instead of modules. So, when looking at the Λ^n_n horn which has been filled in from our map $f_{\{n-1,n\}}$, by using Lemma 3.3 we may consider the map $f_{\{n-1,n\}}$ to be a homotopy equivalence. This means there exists a simplicial coaction map we will call $f_{\{n,n-1\}}$ going in the opposite direction and simplicial coaction maps acting as homotopies which we call $f_{\{n-1,n,n-1\}}$ and $f_{\{n,n-1,n\}}$, such that

$$df_{\{n-1,n,n-1\}} = \mathrm{id} + f_{\{n-1,n\}} \circ f_{\{n,n-1\}}, \qquad df_{\{n,n-1,n\}} = \mathrm{id} + f_{\{n,n-1\}} \circ f_{\{n-1,n\}}.$$

For convenience, for each vertex we will write only the number label of that vertex, not the algebraic object living there. As an example, consider the Λ^2_2 horn with all our added information:

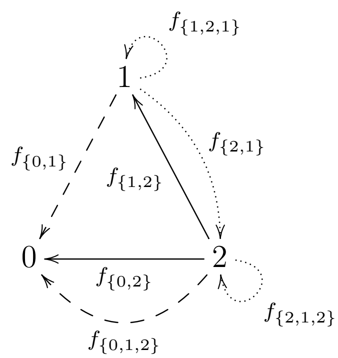

Figure 3.7: Generic Λ^2_2 horn with new maps added

The small dotted maps, $f_{\{1,2,1\}}, f_{\{2,1\}}$, and $f_{\{2,1,2\}}$, come from our homotopy equivalence and the large dotted maps, $f_{\{0,1\}}$ and $f_{\{0,1,2\}}$, are the ones we are hoping to find if we want to show we have a p-cocartesian edge. We define $f_{\{0,1\}} = f_{\{0,2\}} \circ f_{\{2,1\}}$. Both of these maps are chain maps, so the differential is 0, as desired. For the homotopy of the entire triangle, we define

$$f_{\{0,1,2\}} = f_{\{0,2\}} \circ f_{\{2,1,2\}} + f_{\{0,1\}} \circ \big(f_{\{1,2\}} \circ f_{\{2,1,2\}} + f_{\{1,2,1\}} \circ f_{\{1,2\}}\big).$$

We then get

$$df_{\{0,1,2\}} = d\big(f_{\{0,2\}} \circ f_{\{2,1,2\}} + f_{\{0,1\}} \circ (f_{\{1,2\}} \circ f_{\{2,1,2\}} + f_{\{1,2,1\}} \circ f_{\{1,2\}})\big)$$

$$= f_{\{0,2\}} + f_{\{0,2\}} \circ f_{\{2,1\}} \circ f_{\{1,2\}} = f_{\{0,2\}} + (f_{\{0,2\}} \circ f_{\{2,1\}}) \circ f_{\{1,2\}} = f_{\{0,2\}} + f_{\{0,1\}} \circ f_{\{1,2\}},$$

as desired.

In the general case when considering the lift of the Λ_n^n horn, we have an explicit formula for these two maps. First we introduce some notation to ease our equations. Let

$$F = \sum_{1 \le j \le n-2} f_{\{0,\ldots,\hat{j},\ldots,n\}} + f_{\{0,\ldots,j\}} \circ f_{\{j,\ldots,n\}} + f_{\{0,\ldots,\widehat{n-1},n\}}.$$

Recall from when we completed an inner horn, even if we do not have $f_{\{0,\ldots,n-1\}}$ defined, using equation 3.3 we can define $df_{\{0,\ldots,n-1\}}$. We then define

$$f_{\{0,1,\ldots,n-1\}} = F \circ f_{\{n,n-1\}} + df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n,n-1\}}$$

and, using this formula, we define

$$f_{\{0,1,\ldots,n\}} = F \circ f_{\{n,n-1,n\}} + f_{\{0,\ldots,n-1\}} \circ \big(f_{\{n-1,n\}} \circ f_{\{n,n-1,n\}} + f_{\{n-1,n,n-1\}} \circ f_{\{n-1,n\}}\big).$$

These definitions agree with the definitions given above in the Λ_2^2 example. To see why this works, recall from the proof of the inner horn that $dF + df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}} = 0$. Therefore, using our definition above,

$$df_{\{0,1,\ldots,n-1\}} = dF \circ f_{\{n,n-1\}} + df_{\{0,\ldots,n-1\}} \circ df_{\{n-1,n,n-1\}}$$

$$= df_{\{0,\ldots,n-1\}} + \big(dF + df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}}\big) \circ f_{\{n,n-1\}},$$

as desired.

Now checking that our definition of $f_{\{0,1,\ldots,n\}}$ is correct,

$$df_{\{0,1,\ldots,n\}} = dF \circ f_{\{n,n-1,n\}} + F + F \circ f_{\{n,n-1\}} \circ f_{\{n-1,n\}}$$

$$+ df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}} \circ f_{\{n,n-1,n\}} + f_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}}$$

$$+ f_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}} \circ f_{\{n,n-1\}} \circ f_{\{n-1,n\}} + df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n,n-1\}} \circ f_{\{n-1,n\}}$$

$$+ f_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}} + f_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n\}} \circ f_{\{n,n-1\}} \circ f_{\{n-1,n\}}$$

$$= F + \big(F \circ f_{\{n,n-1\}} + df_{\{0,\ldots,n-1\}} \circ f_{\{n-1,n,n-1\}}\big) \circ f_{\{n-1,n\}}$$

$$+\big(dF + df_{\{0,\dots,n-1\}} \circ f_{\{n-1,n\}}\big) \circ f_{\{n,n-1,n\}}.$$

Using that $dF + df_{\{0,\dots,n-1\}} \circ f_{\{n-1,n\}} = 0$, our definition of $f_{\{0,\dots,n-1\}}$ from above, and that we want $df_{\{0,\dots,n\}} = F + f_{\{0,\dots,n-1\}} \circ f_{\{n-1,n\}}$, we see our goal has been reached.

3. Lastly, we must show that for each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$ determine an equivalence of ∞-categories, basically saying is an equivalence of simplicial sets

$$\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{0,1\}} \times \cdots \times \mathcal{C}^{\otimes}_{\{n-1,n\}} \cong (\mathcal{C}^{\otimes}_{[1]})^n.$$

Giving a lift of $[n]$ corresponds to all simplices such that at every vertex is a sequence $(M_1, \dots, M_n)$. Note at different vertices are different modules. The maps lifted from $N(\mathbf{\Delta}^{op})$ are all identity maps $f(x) = x$, meaning that our maps in $\mathcal{C}^{\otimes}$ can all be filtered for each i between 1 and n. For example an edge is the n-fold tensor product of maps $B(M_i) \to B(M_i')$ for $i = 1, \dots, n$. By our definition of our monoidal dg-nerve, looking only at the maps corresponding to a particular i, these maps all are simplicial coaction maps. Thus we have map of simplicial sets $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$. As our maps and vertices are just tensor products of each of these particular i restrictions, we have, better than just an equivalence, a true bijection

$$\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{0,1\}} \times \cdots \times \mathcal{C}^{\otimes}_{\{n-1,n\}} \cong (\mathcal{C}^{\otimes}_{[1]})^n,$$

proving our theorem.

$\square$

Chapter 4

Conclusions

In this dissertation, we generalized monoidal structure on the category of modules over an algebra. In our case, we restricted ourselves by looking at associative Hirsch algebras and our monoidal structure holding only up to homotopy and used Lurie's construction of monoidal ∞-categories to accomplish this.

A *Homotopy Gerstenhaber-algebra*, or hG-algebra, is an algebra with multiplication defined via μ_E given above subject to the further constraints that all $E_{p,q} = 0$ except for $E_{0,1}, E_{1,0}$, and $E_{1,k}, k = 1, 2, \ldots$. If we began with algebras such as the cochain complex of a topological space, the Hochschild cohomology of an associative algebra, and a bialgebra, the bar construction of these can be exhibited as an hG-algebra. Our result then applies to these particular cases.

For future work, consider if we have multiple monoidal structures on a single triangulated category. Then we have a family of products on its periodic cyclic homology. For instance, we may have two varieties that share a derived category, but have different monoidal structures. Ballmer in [1] showed that given the triangulated category of a variety plus the monoidal structure on that variety, we can retrieve the variety, as a sort of generalization of spectrum. The two varieties would have isomorphic cohomologies as vector spaces, but not as rings. This isn't automatic as there is a gap from an algebra to a category. However, given a category of a geometric nature, it is usually the case that there is an associated algebra for which our result could then be applied.

Another possibility of future work is for some derived categories there exists a sheaf A which is called a generator such that this category is equivalent to modules over $C = \mathrm{Ext}(A, A)$, via a sort of Morita equivalence. Unfortunately, the standard monoidal structure of $D^b(X)$ gets lost in this construction, so there may be a way to recover it by looking at the bar construction of C.

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