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Parametric Forms and the Inductive Response of a Permeable Conducting Sphere

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ABSTRACT

At early time, the time derivative of the response of isolated conductive bodies to step function excitation decays as $t^{-1/2}$ (under a quasi-static approximation). One simple parametric form for the response with correct early time behaviour is $k'(1+t^{1/2}/\alpha^{1/2})^{-\beta}e^{-t/\gamma}$. For a conducting magnetic sphere, parameter values are determined from the high and low frequency limit responses, together with two time scales taken from the form of the analytic solution. Parameters α and γ correspond to transition times, for transition from an early time $t^{-1/2}$ derivative response, and to late time exponential decay. For conducting spheres with high permeability, increasing the permeability moves the transition from early time behaviour earlier in inverse proportion to the relative permeability μ_r, and increases the time constant of the late time decay in proportion to μ_r. Magnitude parameter k' corresponds to the difference between high frequency and low frequency limit responses.

INTRODUCTION

A simple parametric form for approximating the inductive response of conducting objects

$$k(\alpha+t)^{-\beta}e^{-t/\gamma} \tag{1}$$

has been suggested as a means of diagnosing conductive objects as magnetic (e.g., ferrous with $\mu_r \gg 1$) or non-magnetic ($\mu_r \approx 1$), for use in discriminating buried unexploded military ordnance (UXO) from other conductive objects (e.g., Pasion and Oldenberg, 2001). Unfortunately, form (1) does not fit the early time response of such a simple object as a conductive sphere, much less a magnetic (permeable) and conducting sphere, leaving some uncertainty as to its generality. At early time the time derivative of the magnetic field of the response of a isolated conducting non-magnetic body is proportional to $t^{-1/2}$ (Kaufman, 1994, p249), so is inconsistent with form (1) used to model either the magnetic field, or the time derivative of the magnetic field. This can be remedied, by using form (1) to model the magnetic field or magnetic moment m with the first occurrence of t replaced by $t^{1/2}$, or equivalently using the form

$$f(t) = k'\left(1+\frac{t^{1/2}}{\alpha^{1/2}}\right)^{-\beta} e^{-t/\gamma} \tag{2}$$

(for $t>0$). In this form, α represents a time scale at which the nature of the inductive response changes from an early time response with $dB/dt \propto t^{-1/2}$, towards an intermediate time with $B \propto t^{-\beta/2}$.

Values of parameters k', α, β, and γ appropriate for a permeable conducting sphere can be obtained by considering the early time response, the low frequency limit (D.C.) and high frequency limit responses of the sphere, together with two time scales from the analytic solution for the sphere. Consideration of the properties of the response of a permeable conducting sphere is particularly important, as the axial and transverse responses of permeable conducting spheroids have been shown to be reasonably approximated by sphere responses of appropriate diameters, for spheroids of the high permeabilities and moderate aspect ratios appropriate for modelling the responses of most UXO (Smith and Morrison, 2004).

EARLY TIME RESPONSE OF A MAGNETIC CONDUCTING SPHERE

Early time responses are obtained from high frequency behaviour of the inductive response (e.g, Kaufman, 1994, p229, Morse and Feshbach,1953, p462). Early time behaviour of a non-magnetic conducting sphere has been treated by Kaufman (1994). In the frequency domain, the magnetic dipole moment of a conducting permeable sphere of conductivity σ, permeability $\mu_r\mu_o$, and radius R, in a time varying uniform externally applied field H_0 with time dependence $e^{i\omega t}$ is

$$m(\omega) = 2\pi\chi_1(\omega)R^3 H_0 \ , \tag{3}$$

where

$$\chi_1(\omega) = \frac{(2\mu_r+1)(\sinh\alpha - \alpha\cosh\alpha) + \alpha^2\sinh\alpha}{(\mu_r-1)(\sinh\alpha - \alpha\cosh\alpha) - \alpha^2\sinh\alpha} \tag{4}$$

and

$$\alpha = (i\omega\mu_r\mu_o\sigma)^{1/2}R \ , \tag{5}$$

with $re(\alpha) > 0$ (Wait and Spies, 1969, Wait, 1951). At high frequency ($\omega \to \infty$);

$$\chi_1(\omega) \to \frac{3\mu_r}{\alpha} - 1 \ . \tag{6}$$

Early time dB/dt behaviour for a step function turn-off excitation is then given by -1 times the inverse Laplace transform of limiting form (6) with Laplace transform variable $s = i\omega$, so

$$\frac{d\chi_1(t)}{dt} \to \delta(t) - \frac{3\mu_r}{(\pi\mu_r\mu_o\sigma t)^{1/2}R} \tag{7}$$

for t > 0 (Spiegel, 1965, p169, f32.108), where $\delta(t)$ is a Dirac function which vanishes for t > 0 so will be neglected. When $\mu_r = 1$, this agrees with previous results for non-magnetic spheres.

TWO TIME SCALES FROM ANALYTIC SOLUTION FOR A SPHERE

In the time domain, the solution for the magnetic fields arising from step function turn-on excitation of a conducting magnetic sphere is given by equation (3) with t replacing ω, and

$$\chi_1(t) = \frac{2(\mu_r-1)}{\mu_r+2} - \sum_{n=1}^{\infty} \frac{e^{-\delta_n^2 t/\sigma\mu_r\mu_o R^2}}{(\mu_r+2)(\mu_r-1)+\delta_n^2} \tag{8}$$

for $t > 0$, where δ_n are the positive solutions of

$$\tan\delta_n = \frac{(\mu_r-1)\delta_n}{\mu_r - 1 + \delta_n^2} \tag{9}$$

(Wait and Spies, 1969). Coefficients δ_n are spaced roughly π apart, with

$$n\pi \leq \delta_n < (n+1/2)\pi \tag{10}$$

($n > 0$), and δ_n approaching $n\pi$ for $n^2 \gg (\mu_r - 1)/\pi^2$. For step function turn-off excitation, one has the D.C. term,

$$\chi_1(t) = \frac{2(\mu_r-1)}{\mu_r+2} \tag{11}$$

for $t < 0$, and

$$\chi_1(t) = \sum_{n=1}^{\infty} \frac{e^{-\delta_n^2 t/\sigma\mu_r\mu_o R^2}}{(\mu_r+2)(\mu_r-1)+\delta_n^2}, \tag{12}$$

for $t > 0$. The most persistent term, $n = 1$, corresponds to the fundamental mode of the sphere. The fundamental mode time constant

$$\tau_0 \equiv \sigma\mu_r\mu_o R^2/\delta_1^2 \tag{13}$$

is found easily by iterating on

$$\delta_n^{(new)} = n\pi + atan\left(\frac{(\mu_r-1)\delta_n^{(old)}}{\mu_r - 1 + (\delta_n^{(old)})^2}\right) \tag{14}$$

with $n = 1$ and, for example, an initial guess of $\delta_1 = 4.4934$, its value for $\mu_r = \infty$, to evaluate δ_1. Iterates (14) converge to six decimal places in about as many iterations.

For magnetic spheres, a second time scale is manifest in expansion (12). Inverse squared coefficients δ_n^{-2} correspond to non-dimensional time constants, and do not vary a large amount with μ_r (semi-inequalities 10). For $\delta_n^2 > (\mu_r+2)(\mu_r-1)$ the denominator in expansion (12) is dominated by δ_n^2 and the expansion terms approach their values for the non-magnetic case, in non-dimensional time $t/\sigma\mu_r\mu_o R^2$. The transition between magnetic dominated and non-magnetic dominated terms occurs about

$$\delta_n^2 = (\mu_r+2)(\mu_r-1) \ , \tag{15}$$

provided $(\mu_r+2)(\mu_r-1) \geq \delta_1^2$. This yields a characteristic time of

$$\tau_1 \equiv \frac{\sigma\mu_r\mu_o R^2}{(\mu_r+2)(\mu_r-1)} \tag{16}$$

when $(\mu_r+2)(\mu_r-1) \geq \delta_1^2$, that is, when $\mu_r \geq 3.453$. When $\delta_1^2 > (\mu_r+2)(\mu_r-1)$, all terms in expansion (12) have their denominator dominated by δ_n^2, so we set $\tau_1 \equiv \tau_0$, for $\mu_r < 3.453$. For large μ_r,

$$\frac{\tau_1}{\tau_0} \approx \left(\frac{4.5}{\mu_r}\right)^2 \ . \tag{17}$$

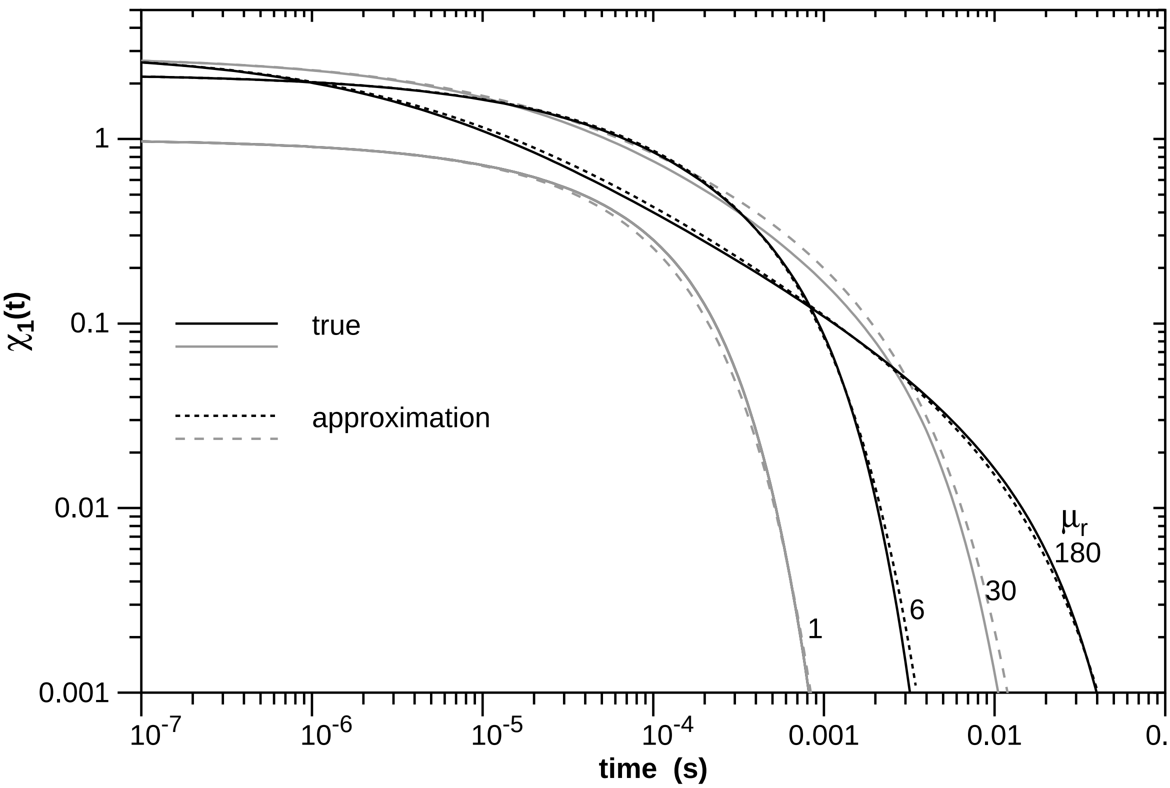

Figure 1: Effective induced magnetization factor χ_1 as a function of time, for 20 mm diameter spheres of varying relative permeability. Solid; full solution. Dashed; approximate form (2) with k', β, α and γ, given by equations (18), (21), (22) and (25) with $a = 1.38$.

PARAMETER VALUES FOR A CONDUCTING PERMEABLE SPHERE

Form (2) has been chosen so that, $k' = f(0^+)$. For step function turn-off excitation, this corresponds to the difference between the low frequency limit value and the high frequency limit value. Using form (2) to approximate the effective induced magnetization factor $\chi_1(t)$;

$$k'_\chi = \chi_1(\omega = 0) - \chi_1(\omega = \infty) = \frac{3\mu_r}{\mu_r + 2} \,. \quad (18)$$

Form (2) has derivative

$$\frac{df(t)}{dt} = -\left[\frac{1}{\gamma} + \frac{\beta}{2(t^{1/2}\alpha^{1/2} + t)}\right] f(t) \,, \quad (19)$$

which reduces to

$$\frac{df(t)}{dt} \approx -\frac{k'\beta}{2t^{1/2}\alpha^{1/2}} \quad (20)$$

at early time, consistent with the $t^{-1/2}$ derivative response of general isolated conductive bodies. Matching early time derivatives (7) and (20) and using (18), requires

$$\beta = \frac{2(\mu_r + 2)\alpha^{1/2}}{(\pi\mu_r\mu_o\sigma)^{1/2}R} \,. \quad (21)$$

Writing α and γ in terms of τ_1 and τ_0,

$$\alpha = a\tau_1 \,, \qquad \gamma = b\tau_0 \quad (22)$$

(for some a and b), this reduces to

$$\beta = \frac{2a^{1/2}}{\pi^{1/2}} min\left[\frac{\mu_r + 2}{\delta_1}, \left(\frac{\mu_r + 2}{\mu_r - 1}\right)^{1/2}\right] \,. \quad (23)$$

We find $a = 1.38$ particularly suitable, giving β values about 1.33 for large μ_r. At late enough time, sphere response (12) is eventually dominated by the fundamental mode which decays as e^{-t/τ_0}, with $d\chi_1/dt/\chi_1 \approx -1/\tau_0$. Given the somewhat different

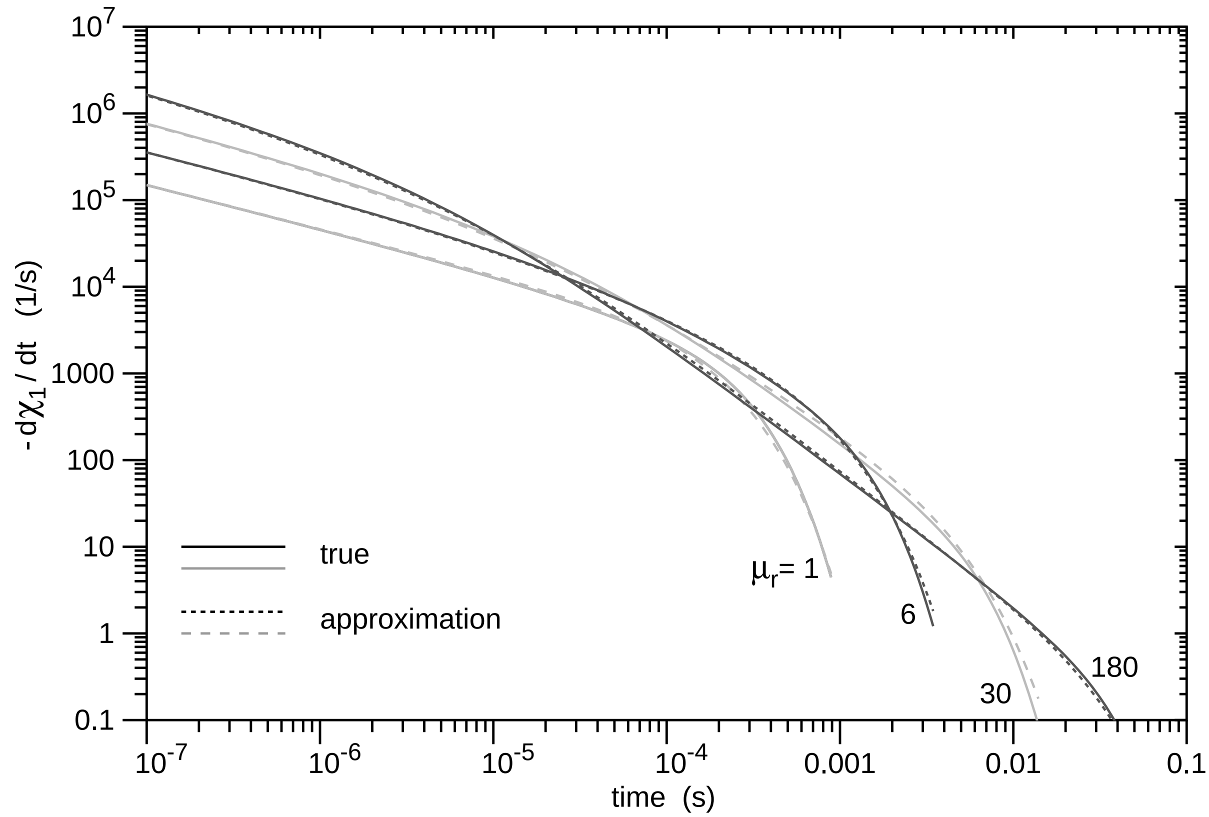

Figure 2: Effective induced magnetization factor derivative $d\chi_1/dt$ as a function of time, for 20 mm diameter spheres of varying relative permeability. Solid; full solution. Dashed; approximate form (2) with k', β, α and γ, given by equations (18), (21), (22) and (25) with $a = 1.38$.

late time dependence of form (2), we restrict ourselves to matching

$$\frac{1}{f}\frac{df}{dt}\bigg|_{t=2\tau_0} = -\frac{1}{\tau_0}\,. \qquad (24)$$

This leads to the choice

$$b = \frac{1+(a\tau_1/2\tau_0)^{1/2}}{1+(a\tau_1/2\tau_0)^{1/2}-\beta/4}\,. \qquad (25)$$

For large μ_r, $\tau_1/\tau_0 \ll 1$, so with our choice of $a = 1.38$, $b \approx 1.50$.

RESULTS

The result of using choices (18), (21) and (22) for k', β, α and γ, in form (2), together with $a = 1.38$, and b given by equation (25), are shown in Figure (1), together with the exact solution, for 0.02 m diameter spheres with relative permeability ranging from 1 to 180, and $\sigma = 10^7\ \Omega^{-1}m^{-1}$. The results of Figure (1) may be shifted to non-dimensional time $t/\sigma\mu_r\mu_o R^2$ by dividing the time scale by $\sigma\mu_r\mu_o R^2 = 0.01257\ \mu_r$ (seconds), or shifted directly to results for other diameter and conductivity spheres by multiplying the time scale by the ratio of σR^2 values. The time derivatives of the results plotted in Figure (1) are plotted in Figure (2). The particularly good fit at relative permeability $\mu_r = 180$ appropriate for many steels, motivated our choice of $a = 1.38$.

CONCLUSION

Parametric form (2) matches the earlier time dB/dt response of isolated conductive bodies correctly. Appropriate values of component parameters k', α, β, and γ for use in form (2) for the response of a conducting magnetic sphere can be deduced from the analytic solution for the sphere. Parameters α and γ correspond to time scales, for transition from the early time $t^{-1/2}$ response, and to late time exponential decay, with the second transition on the order of μ_r^2 times later than the first. Magnitude parameter k' corresponds to the difference between high frequency and low frequency limit responses.

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